The Exact Solutions in Quantum Mechanics

Mohammed Noor Sedeeq Rammoo

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Prof. Dr. Elvan Yılmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. Mustafa Halilsoy
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Physics.

Assoc. Prof. S. Habib Mazharimousavi
Supervisor

Examining Committee

1. Prof. Dr. Özay Gurtug
2. Prof. Dr. Mustafa Halilsoy
3. Assoc. Prof. S. Habib Mazharimousavi

ABSTRACT

In this study we consider two different and particular quantum systems. The first system is a one-dimensional particle confined within an infinite well with the left wall at rest and the right wall moving with or without acceleration. This is a quantum configuration with time-dependent Hamiltonian where the conservation of energy does not hold anymore. Our task in this problem is to solve the correspondence time dependent Schrödinger equation and find the energy eigenvalues and eigenfunctions. The second system is a particle which undergoes a modified radial attraction force in two dimensions. The modification is such that the corresponding potential depends on not only the radial coordinate but also the polar angle. We find the energy spectrum of the particle and using the concept of coherent states, we show that the classical trajectory of a classical particle directed by a similar radial potential is almost the same as the predicted probability density of the quantum particle by the coherent states.

Keywords: One-dimensional infinite well, Time-dependent potential, Exact solution, Coherent state.

ÖZ

Bu tezde iki farklı özel kuantum sistemi incelenmektedir. Birinci sistemde sol duvarı sabit , sağ duvarı ivmeli /ivmesiz hareketli bir kutu içerisindeki tek boyutlu partikül ele alınıyor. Zamana bağımlı Hamilton fonksiyonu nedeniyle enerji korunumu sağlanmıyor.Zamana bağımlı Schrödinger denklemini kullanarak enerji uygun değer ve fonksiyonları bulunuyor.

İkinci sistem ise iki boyutlu modife edilmiş çekim kuvveti için partikül incelenmektedir. Burada potansiyel hem radial hemde açısal bağımlılık içermektedir. Düzgün durumlar kullanarak partikül enerji spektrumu bulunmuştur.Klasik partikül yörüngesi ile düzgün durum kuantum olasılık dağılımı arasındaki sıkı benzerlik ortaya konmaktadır.

Anahtar kelimeler : Bir boyutlu sonsuz kuyu, zamana bağlı potansiyel , düzgün durum fonksiyonu

DEDICATION

TO MY FAMILY

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INTRODUCTION

The infinite square well in one dimension is the simplest example in quantum mechanics. When studying quantum mechanics, it gives a solid description as well as a way of approaching and solving many quantum mechanics problems. The one dimension infinite square well is a particular choice of the Hamiltonian, with the usual kinetic energy, but have a particular potential term that is imposed to a particle. The particle can be considered as an electron, a proton or a hydrogen atom. We consider a particle inside a box, moving freely with in the box, but when it comes to one of the walls of the box, it cannot move further but to bounce back therefore $0 \le x \le L$.

The Schrödinger equation of a particle of mass m inside the potential $V(x)$ is given by

$$-\frac{\hbar^2}{2m}\frac{\partial\psi(x)}{\partial x}+V(x)\psi(x)=E\psi(x). \tag{1.1}$$

in which E denotes the energy of the particle, $V(x)$ denotes potential energy of the particle and $\psi(x)$ stands for the wave function. It is important to note that $V(x)$ is known before we solve the equation. Let's consider only $V(x)$ the one dimensional infinite square well potential function, defined by

$$V(x)=\begin{cases}0 & 0<x<L\\ \infty & \text{otherwise}\end{cases} \tag{1.2}$$

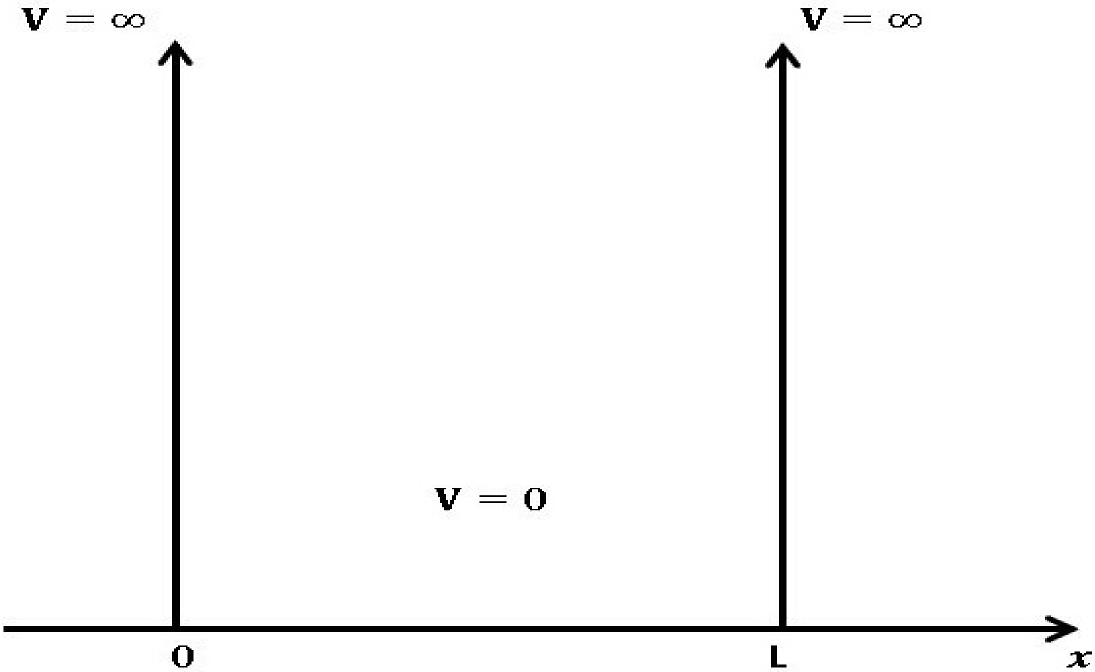

Figure 1.1: Infinite Potential wall

Having $V(x)=\infty,$ outside the well, forces the wave function $\psi(x)$ to vanish in that intervals, i.e. $\psi(x)=0$ for $x\le 0$ and $L\le x$. Hence we solve Eq. (1.1) for $V(x)=0$, inside the well, where the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{\partial\psi(x)}{\partial x}=E\,\psi(x). \qquad (1.3)$$

Let's introduce $k^2=\frac{2mE}{\hbar^2}$ and then the latter becomes

$$\frac{\partial\psi(x)}{\partial x}=-k^2\psi(x). \qquad (1.4)$$

We note that, E (energy of the particle) can not be negative. In fact a negative energy can not satisfy the boundary conditions.

A general solution to Eq. (1.4) is given by

$$\psi(x) = A\sin kx + B\cos kx. \tag{1.5}$$

in which A and B are two integration constants.

Imposing boundary conditions at $x = 0$ and $x = L$ on the general solution we find

$$\psi(0) = 0 \Rightarrow B = 0, \tag{1.6}$$

$$\psi(L) = 0 \Rightarrow kL = n\pi. \tag{1.7}$$

in which $n = 1, 2, 3, \ldots$.

Having the wave number k one finds $\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2}$ which consequently the energy of the particle is obtained as

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}. \tag{1.8}$$

Furthermore the wave function $\psi(x)$ becomes

$$\psi(x) = A\sin\frac{n\pi}{L}x, \tag{1.9}$$

in which A is the normalization constant.

Next, we determine the normalization constant A by imposing

$$\int_0^L \left|\psi(x)\right|^2 dx = 1. \tag{1.10}$$

This, in turn, gives

$$|A|^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx = 1. \tag{1.11}$$

and finally the simplest/real normalization constant is found to be $A = \sqrt{\frac{2}{L}}$.

Hence the normalized energy eigenfunctions are found to be

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x, \tag{1.12}$$

with the energy eigenvalues given in Eq. (1.8).

In Figs. (1.2) and (1.3) we plot the eigenfunctions from Eq. (1.12) in terms of x, for the specific value of $L = 1$ and $n = 1$ and $n = 2$. In Figs. (1.4) and (1.5) we plot $|\psi_n|^2$ or probability density with respect to x for the same values of L and n as Figs. (1.2) and (1.3).

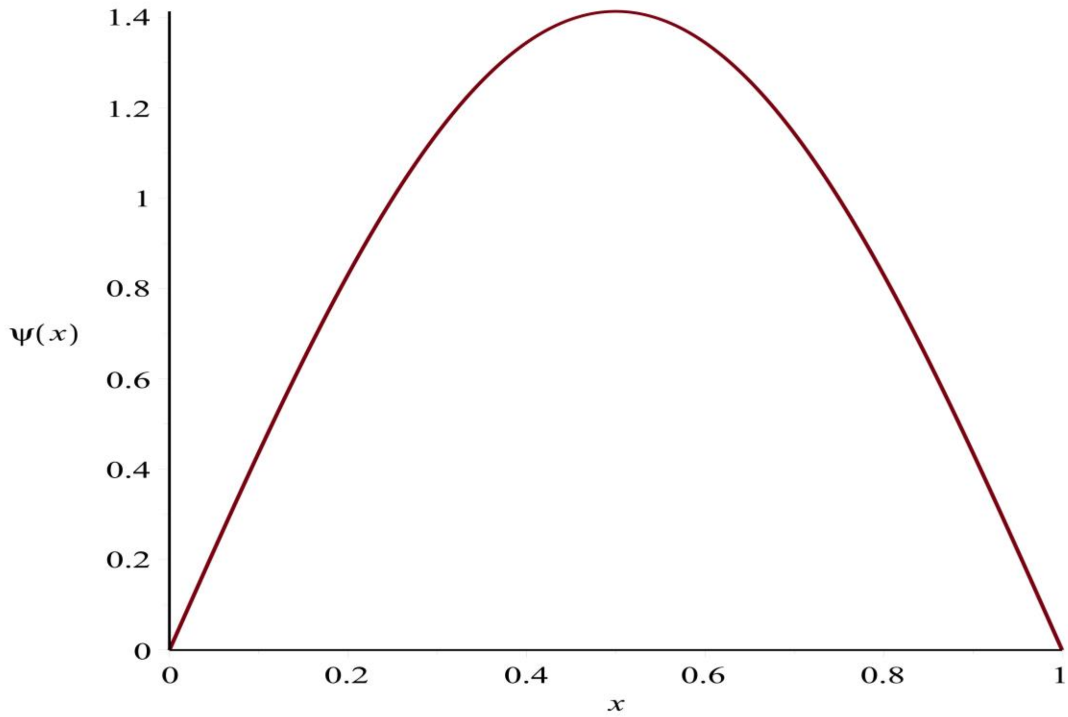

Figure 1.2: A plot of $\psi_n(x)$ with respect to x for L = 1 and n = 1 from Eq. (1.12). This state is the ground state of the particle in the well.

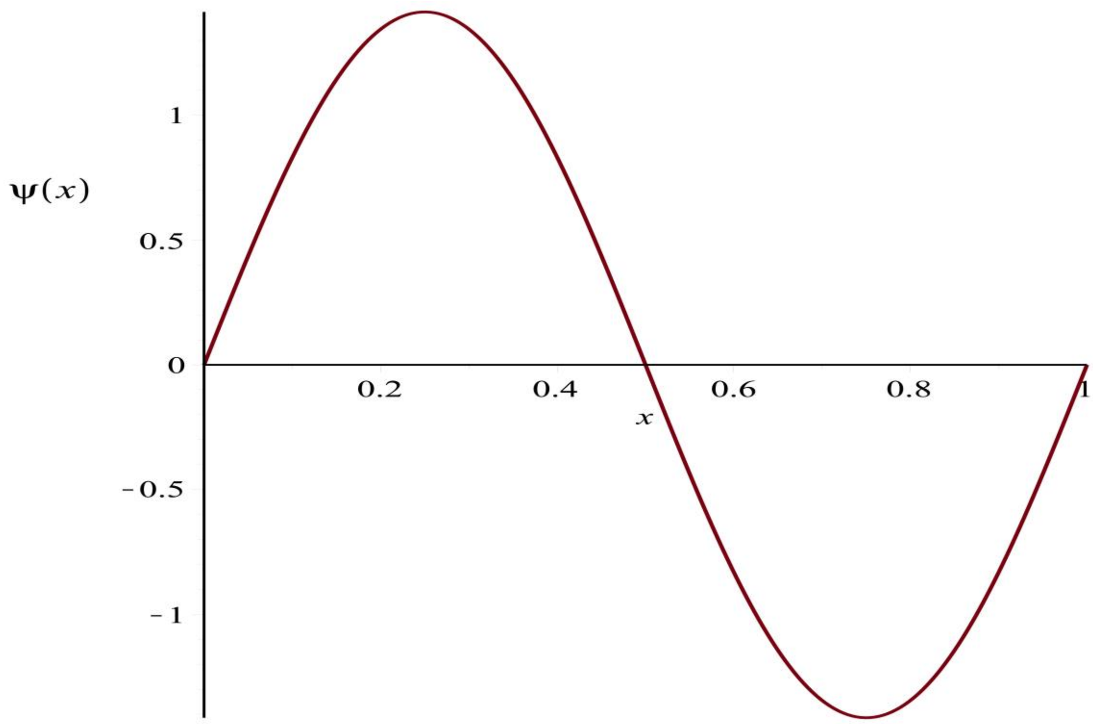

Figure 1.3: A plot of $\psi_n(x)$ with respect to x for L = 1 and n = 2 from Eq. (1.12). This state is the first excited state of the particle in the well.

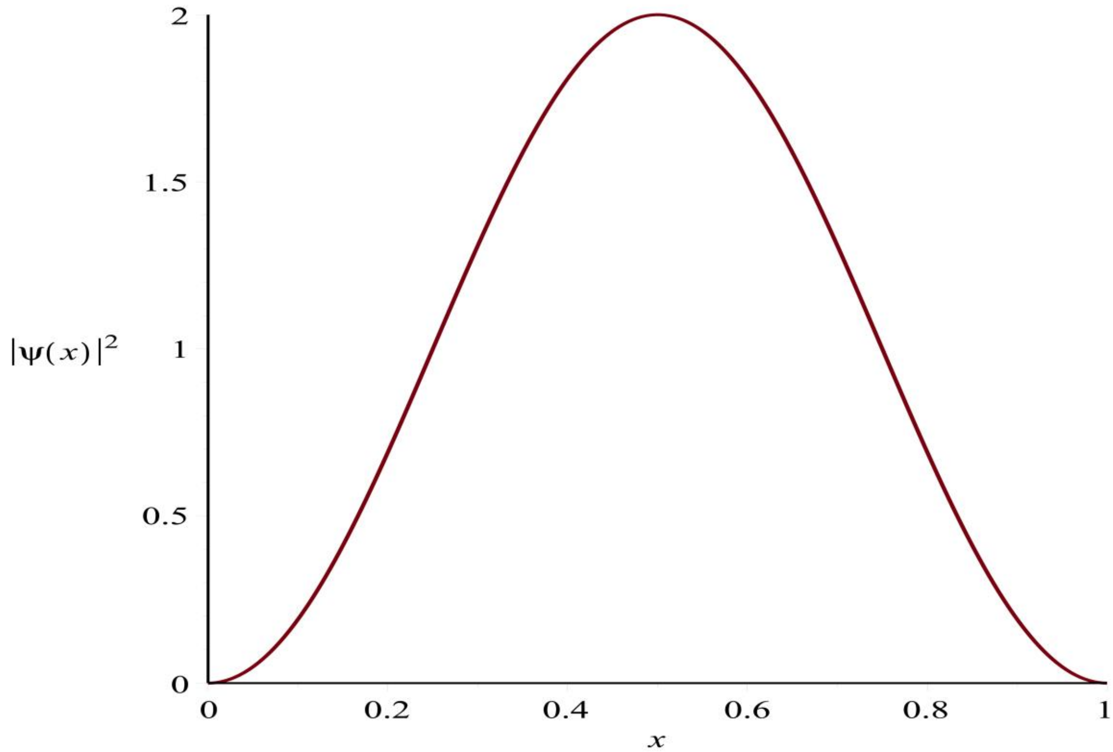

Figure 1.4: Probability density $|\psi_n(x)|^2$ for n = 1 from Eq. (1.12).

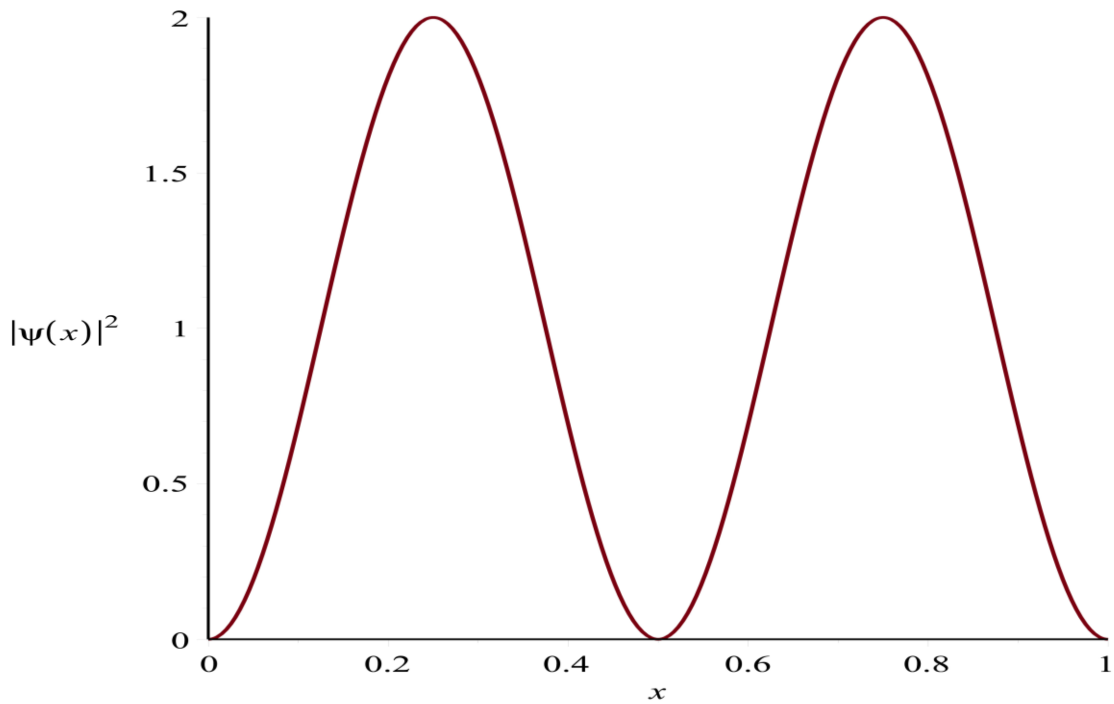

Figure 1.5: Probability density $|\psi_n(x)|^2$ for n = 2 from Eq. (1.12).

Chapter 2

QUANTUM PARTICLE IN ONE DIMENSIONAL INFINITE WELL WITH ONE WALL MOVING.

In previous chapter we studied the non-relativistic quantum particle inside a square well (or infinite box) in one dimension. We determined the energy eigenfunctions and energy eigenvalues of the particle in the box. In this chapter, we shall study a one-dimensional non-relativistic quantum particle confined inside an infinite box with one wall at rest and the other wall moving as shown in the Fig. (2.1). [1-16]

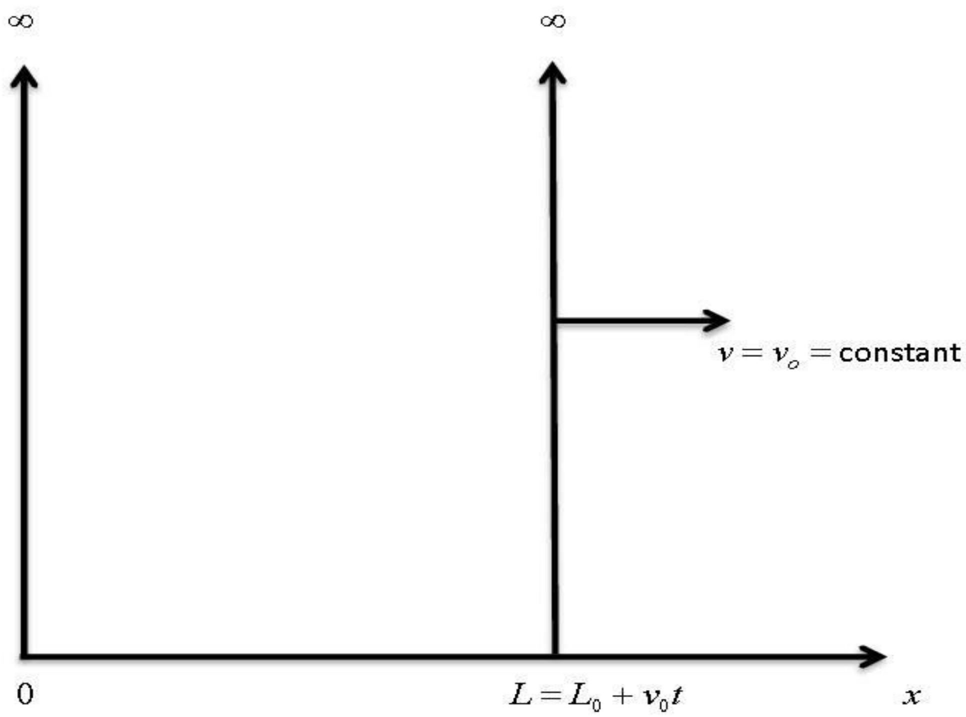

Figure 2.1: Infinite box with one wall at rest and the other wall moving.

This new problem is not as simple as what we found in chapter one. Simply, the Hamiltonian is time-dependent and the Schrödinger equation may not be solved easily.

2.1 Classical Approach

We start with the following one-dimensional classical Hamiltonian given by [3,6,14]

$$H\left(x, p, \ell(t)\right) = \frac{p^2}{2m} + \alpha\left(\ell(t)\right) V\left(\frac{x}{\ell}\right) \tag{2.1}$$

in which x and p are the coordinate and the momentum of the particle, $\ell(t)$ is a time dependent parameter, $\alpha(\ell)$ is a function of ℓ and $V\left(\frac{x}{\ell}\right)$ is a potential which is a function of $\frac{x}{\ell}$.

We note that beeing time-dependent, the Hamiltonian does not represent a conserved-energy system.

Hamiltonian in Eq. (2.1) is for a particle in the field of an expanding potential. Since ℓ is a time dependent function, when ℓ increases / decreases in time the domain of the potential increases / decreases too. As an specific case when the potential is an infinite square well i.e.

$$V = \begin{cases} 0 & 0 < \frac{x}{\ell} < 1 \\ \infty & \text{elsewhere} \end{cases}, \tag{2.2}$$

an increasing ℓ causes the right wall of the potential i.e. $x = \ell$ to move. The parameter α, controls the strongness or weakness of the potential.

Using Hamiltonian equations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \tag{2.3}$$

and

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \tag{2.4}$$

one finds from Eq. (2.3)

$$\frac{dp}{dt} = -\alpha \frac{\partial V\left(\frac{x}{\ell}\right)}{\partial x}, \tag{2.5}$$

and from Eq. (2.4)

$$\frac{dx}{dt} = \frac{p}{m}. \tag{2.6}$$

Therefore from Eq. (2.6) $p = m\,\dfrac{dx}{dt}$ and imposing in Eq. (2.5) one finds

$$m\frac{d^2x}{dt^2} = -\alpha \frac{\partial V\left(\frac{x}{\ell}\right)}{\partial x}. \tag{2.7}$$

We note that both x and ℓ are time-dependent i.e. $x = x(t)$ and $\ell = \ell(t)$. Let's introduce a new variable of $y(t) = \dfrac{x(t)}{\ell(t)}$ and a new time variable $\tau = \tau(t)$. These change the main equation as

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt}, \tag{2.8}$$

and

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{d\tau}\frac{d\tau}{dt}\right). \tag{2.9}$$

After simplification, latter becomes

$$\frac{d^2x}{dt^2} = \frac{d^2x}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 + \frac{dx}{d\tau}\frac{d^2\tau}{dt^2}. \tag{2.10}$$

From the other side, also one gets:

$$\frac{\partial V\left(\frac{x}{\ell}\right)}{\partial x} = \frac{dV\left(y\right)}{dy}\frac{dy}{dx}. \tag{2.11}$$

Therefore the Newton's equation of motion becomes

$$m\left(\frac{d^2x}{d\tau}\left(\frac{d\tau}{dt}\right)^2 + \frac{dx}{d\tau}\frac{d^2\tau}{dt^2}\right) = -\alpha\frac{dV}{dy}\frac{dy}{dx}, \tag{2.12}$$

which further simplification gives

$$m\left(\frac{d^2x}{d\tau^2}\tau'^2 + \tau''\frac{dx}{d\tau}\right) = -\alpha\frac{\frac{dV}{dy}}{\ell}, \tag{2.13}$$

and finally

$$\frac{m\ell}{\alpha}\left[\frac{d^2x}{d\tau^2}\tau'^2 + \tau''\frac{dx}{d\tau}\right] = -\frac{dV}{dy}. \tag{2.14}$$

Next, we transform x in the left side into y.

To do so we use

$$\frac{dx}{d\tau} = \frac{d}{d\tau}\left(\ell y\right), \tag{2.15}$$

which becomes

$$\frac{dx}{d\tau} = \ell \frac{dy}{d\tau} + y\,\frac{d\ell}{d\tau}. \tag{2.16}$$

Consequently

$$\frac{d^2x}{d\tau^2} = \frac{d}{d\tau}\left(\ell \frac{dy}{d\tau} + y\,\frac{d\ell}{d\tau}\right), \tag{2.17}$$

or simply

$$\frac{d^2x}{d\tau^2} = 2\frac{d\ell}{d\tau}\frac{dy}{d\tau} + \ell\frac{d^2y}{d\tau^2} + y\,\frac{d^2\ell}{d\tau^2}. \tag{2.18}$$

Taking these results back to the main equation one finds

$$\frac{\ell m}{\alpha}\left[\tau'^2\left(2\frac{d\ell}{d\tau}\frac{dy}{d\tau} + \ell\frac{d^2y}{d\tau^2} + y\,\frac{d^2\ell}{d\tau^2}\right) + \tau''\left(\ell\frac{dy}{d\tau} + y\,\frac{d\ell}{d\tau}\right)\right] = -\frac{dV}{dy}. \tag{2.19}$$

Let's rearrange the terms as

$$\begin{aligned}&\left[\frac{\ell^2 m}{\alpha}\tau'^2\,\frac{d^2y}{d\tau^2} + \left(\frac{2\ell m}{\alpha}\tau'^2\,\frac{d\ell}{d\tau} + \frac{\ell^2 m}{\alpha}\tau''\right)\frac{dy}{d\tau} + \right.\\ &\left.\tau'^2\,\frac{\ell m}{\alpha}y\,\frac{d^2\ell}{d\tau^2} + \frac{\ell m}{\alpha}y\,\frac{d\ell}{d\tau}\tau''\right] = -\frac{dV}{dy},\end{aligned} \tag{2.20}$$

where the first term becomes

$$\frac{\ell^2 m}{\alpha}\left(\frac{d\tau}{dt}\right)^2\frac{d^2y}{d\tau^2}, \tag{2.21}$$

and the second term reads as

$$\left(\frac{2\ell m}{\alpha}\left(\frac{d\tau}{dt}\right)^2\frac{d\ell}{d\tau} + \frac{\ell^2 m}{\alpha}\frac{d^2\tau}{dt^2}\right)\frac{dy}{d\tau}. \tag{2.22}$$

Latter can be simplified more, as

$$\frac{m\ell}{\alpha}\left(2\tau'\frac{d\ell}{dt}+\tau''\right)\frac{dy}{d\tau}, \tag{2.23}$$

which can be written as

$$\frac{m}{\alpha}\left(\tau'\ell^2\right)'\frac{dy}{d\tau}, \tag{2.24}$$

where a prime $'$ implies derivative with respect to t. The last term in left hand side of Eq. (2.20) reads

$$\frac{\ell m}{\alpha}\left(\tau'^2\frac{d^2\ell}{d\tau^2}+\frac{d\ell}{d\tau}\tau''\right)y, \tag{2.25}$$

which is equivalent to

$$\frac{m\ell}{\alpha}\left(\frac{d^2\ell}{dt^2}\right)y. \tag{2.26}$$

After all this simplifications, we are left with the following equation of motion,

$$\frac{m\ell^2}{\alpha}\left(\tau'\right)^2\frac{d^2y}{d\tau^2}+\frac{m}{\alpha}\left(\ell^2\tau'\right)'\frac{dy}{d\tau}+\frac{m\ell\ell''}{\alpha}y=-\frac{dV\left(y\right)}{dy}. \tag{2.27}$$

Next, we impose the constraint

$$\ell^2\tau'=1, \tag{2.28}$$

which helps us to eliminate the first order term in the target equation Eq. (2.27), and it becomes more like a Newton's equation of motion.

Upon Eq. (2.28) one finds

$$d\tau = \frac{dt}{\ell^2(t)} \Rightarrow \tau = \int_0^t \frac{dt}{\ell^2(t)}. \tag{2.29}$$

Putting into Eq. (2.27) we determine

$$\frac{m}{\alpha}\tau' \frac{d^2 y}{d\tau^2} + \frac{m\ell\ell''}{\alpha} y = -\frac{dV(y)}{dy}, \tag{2.30}$$

which after knowing $\alpha = \frac{1}{\ell^2}$ it becomes

$$m\frac{d^2 y}{d\tau^2} + m\ell^3\ell'' y = -\frac{dV(y)}{dy}. \tag{2.31}$$

Next we impose another constraint of the form $m\ell^3\ell'' = k$ in which k is a constant.

Upon that, the equation of motion Eq. (2.31) becomes

$$m\frac{d^2 y}{d\tau^2} + ky = -\frac{dV(y)}{dy}. \tag{2.32}$$

Furthermore, the constraint itself admits

$$m\ell^3\ell'' = k \Rightarrow \frac{d^2\ell}{dt^2}\ell^3 = \frac{k}{m}, \tag{2.33}$$

which upon integration it yields

$$\ell^2 = \frac{1}{mc_1}\left(c_1^2 t^2 + 2c_1^2 c_2 t + c_1^2 c_2^2 + mk\right), \tag{2.34}$$

where c_1 and c_2 are two integration constants. Lets simplify the solution as

$$\ell^2 = \frac{c_1}{m}t^2 + 2\frac{c_1 c_2}{m}t + \frac{c_1 c_2^2}{m} + \frac{k}{c}, \tag{2.35}$$

or

$$\ell^2 = at^2 + 2bt + c, \tag{2.36}$$

in which $ac - b^2 = \frac{k}{m}$.

The main equation of motion, i.e. Eq. (2.32) can be written as

$$m\frac{d^2y}{d\tau^2} = -\frac{d}{dy}\left(V + \frac{1}{2}ky^2\right). \tag{2.37}$$

Eq. (2.37) in $\tau - y$ space is found by using the following Hamiltonian

$$H = \frac{\bar{p}^2}{2m} + \bar{V}, \tag{2.38}$$

in which $\bar{V} = \frac{1}{2}ky^2 + V$, and $\bar{p} = m\frac{dy}{dt}$

We used $\bar{p}$ to distinguish from the momentum in $t - x$ space. What we found is a Hamiltonian which is time-independent and the corresponding energy is conserved. The price we paid is an extra term in the potential which is a harmonic oscillatory potential.

2.2 Quantum Approach

In the following we use the results found in previous section to study the quantum particle in an infinite well with one wall moving.

We again start from the Hamiltonian given in $t - x$ space in Eq. (2.1), i.e.

$$H = \frac{p^2}{2m} + \alpha V\left(\frac{x}{\ell}\right). \tag{2.39}$$

The Schrödinger equation of a particle with such Hamiltonian, then becomes

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi+\frac{1}{\ell^2}V\left(\frac{x}{\ell}\right)\Psi=i\hbar\frac{d\Psi}{dt}. \tag{2.40}$$

Now, without going through the detail, we can use the results we found in

previous section. this system is equivlant to a system in $\tau-y$ space with Hamiltonian given by Eq. (2.38). Therefore the Schrödinger equation reads in $\tau-y$ coordinates as

$$-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\Phi+\left(V(y)+\frac{1}{2}ky^2\right)\Phi=i\hbar\frac{d\Phi}{d\tau}, \tag{2.41}$$

in which a direct substitution shows that

$$\Psi(\mathrm{x},\mathrm{t})=\frac{1}{\sqrt{\ell}}\Phi(y,\tau)\mathrm{e}^{\left[\frac{i}{2\hbar}m\ell\ell' y^2\right]}. \tag{2.42}$$

2.2.1 For $k\neq 0$

To find the potential of an infinite square well we set $V=0$ in Eq. (2.41) and we get,

$$-\frac{\hbar^2}{2m}\frac{d^2\Phi}{dy^2}+\frac{1}{2}ky^2\Phi=i\hbar\frac{d\Phi}{d\tau}. \tag{2.43}$$

This is the simple harmonic oscillator in one-dimension.

To solve this equation we assume

$$\Phi(y,\tau)=U(y)e^{\left(\frac{-iE\tau}{\hbar}\right)}. \tag{2.44}$$

Which upon a substitution in Eq. (2.43) it becomes

$$-\frac{\hbar^2}{2m}e^{\frac{-iE\tau}{\hbar}}\frac{d^2U(y)}{dy^2}+\frac{1}{2}ky^2U(y)e^{\frac{-iE\tau}{\hbar}}=i\hbar\left(\frac{-iE}{\hbar}\right)U(y)e^{\frac{-iE\tau}{\hbar}}, \tag{2.45}$$

which a simplification implies

$$\frac{d^2U(y)}{dy^2}+\left(\frac{2mE}{\hbar^2}-\frac{mky^2}{\hbar^2}\right)U(y)=0. \tag{2.46}$$

We introduce $z^2=\frac{km}{\hbar}y^2$ and $\bar{E}=\frac{2mE}{\hbar^2}$, which yields

$$\frac{d^2U(z)}{dz^2}+\left(\bar{E}-z^2\right)U(z)=0. \tag{2.47}$$

Next, we consider the solution of the form

$$U(z)=H(z)e^{\frac{-z^2}{2}}, \tag{2.48}$$

whose first derivative is given by

$$U'(z)=H'(z)e^{\frac{-z^2}{2}}-H(z)ze^{\frac{-z^2}{2}}, \tag{2.49}$$

and second derivative reads as

$$U''(z)=\left[H''(z)e^{\frac{-z^2}{2}}+H'(z)\left(-ze^{\frac{-z^2}{2}}\right)- H'(z)ze^{\frac{-z^2}{2}}-H(z)e^{\frac{-z^2}{2}}+H(z)z^2e^{\frac{-z^2}{2}}\right]. \tag{2.50}$$

Latter can be simplified as

$$U''(z)=H''(z)e^{\frac{-z^2}{2}}-H(z)e^{\frac{-z^2}{2}}-2zH'(z)e^{\frac{-z^2}{2}}+H(z)z^2e^{\frac{-z^2}{2}}, \tag{2.51}$$

and upon substituting into Eq. (2.47), it becomes

$$\left[H''(z)e^{\frac{-z^2}{2}} - H(z)e^{\frac{-z^2}{2}} - 2zH(z)e^{\frac{-z^2}{2}} + \right.$$
$$\left. H(z)z^2e^{\frac{-z^2}{2}} + \left(\bar{E} - z^2\right)H(z)e^{\frac{-z^2}{2}}\right] = 0. \qquad (2.52)$$

Further simplification, then, implies

$$H''(z) - H(z) - 2zH'(z) + \bar{E}H(z)e^{\frac{-z^2}{2}} = 0, \qquad (2.53)$$

or consequently

$$H''(z) - 2zH'(z) + \left(\bar{E} - 1\right)H(z) = 0. \qquad (2.54)$$

Let's consider $\left(\bar{E} - 1\right) = 2n$, in Eq. (2.53) which yields

$$H''(z) - 2zH'(z) + 2nH(z) = 0. \qquad (2.55)$$

This equation is called Hermite differential equation and its convergence

solution are called Hermite Polynomials

$$H_n(z) = (-1)^n e^{z^2} \frac{\partial^n}{\partial z^n}\left(e^{-z^2}\right), \qquad (2.56)$$

in which $n = 1, 2, 3....$

Therefore the energy eigenvalue is given by

$$\bar{E} = 2n + 1 \Rightarrow E = \frac{\hbar^2}{m}\left(\frac{1}{2} + n\right) \qquad n = 0, 1, 2, ... \qquad (2.57)$$

Furtheremore, the solution for the wavefunction of Eq. (2.47) reads

$$U(z) = e^{\frac{-z^2}{2}} (-1)^n e^{z^2} \frac{\partial^n}{\partial z^n}\left(e^{-z^2}\right). \tag{2.58}$$

We recall that $z = \sqrt{\frac{km}{\hbar}} y$ which yields

$$U(y) = e^{\frac{-y^2}{2}} (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n}\left(e^{-y^2}\right). \tag{2.59}$$

Considering time (τ) one finds the full wavefunction as:

$$\Phi_n(y,\tau) = e^{-\left(\frac{iE\tau}{\hbar}\right)} e^{\frac{-y^2}{2}} (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n}\left(e^{-y^2}\right). \tag{2.60}$$

After, we found $\Phi_n(y,\tau)$ in $y-\tau$ space, its time to go back into the original space of $x-t$ and work out $\Psi_n(x,t)$ which is given by

$$\Psi_n(x,t) = \frac{1}{\sqrt{\ell(t)}} \Phi_n(y,\tau) e^{\left(\frac{i}{2\hbar} m\ell\ell' y^2\right)}. \tag{2.61}$$

More explicity one gets

$$\Psi_n(x,t) = \frac{1}{\sqrt{\ell(t)}} e^{\left(\frac{i}{2\hbar} m\ell\ell' y^2\right)} U_n(y) e^{\frac{-iE_n\tau}{\hbar}}, \tag{2.62}$$

with energy eigenvalues given by Eq. (2.57).

2.2.2 For $k = 0$

Now in order to get some exact answer, Let's $k = 0$ in Eq. (2.41). The Schrödinger equation becomes

$$-\frac{\hbar^2}{2m} \frac{\partial^2 U_n}{\partial y^2} = E_n U_n, \tag{2.63}$$

and after simplification

$$\frac{\partial^2 U_n}{\partial y^2} + \frac{2mE_n}{\hbar^2} U_n = 0. \tag{2.64}$$

Following the standard method, we write the latter equation as

$$\frac{\partial^2 U_n}{\partial y^2} + q^2 U_n = 0, \tag{2.65}$$

in which $\frac{2mE_n}{\hbar^2} = q^2.$

The solution to Eq. (2.65) is given by

$$U_n = A \sin qy + B \cos qy. \tag{2.66}$$

with the boundary conditions $U_n(0) = U_n(1) = 0.$ The boundary conditions imply $B = 0$ and $q = n\pi$ which also yield

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2m}. \tag{2.67}$$

The wavefunction after normalization becomes

$$U_n = \sqrt{\frac{2}{\ell}} \sin n\pi y. \tag{2.68}$$

Letting $k = 0,$ nevetheless, yields $\ell'' = 0$ which implies

$$\ell = \ell_0 + v_0 t. \tag{2.69}$$

Herein ℓ_0 is ℓ at $t = 0$ and v_0 is the speed of the wall. Considering

$$y = \frac{x}{\ell}, \tag{2.70}$$

and

$$d\tau = \frac{dt}{\ell^2} \Rightarrow \tau = \frac{t}{\ell_0 \left(\ell_0 + v_0 t\right)}. \qquad (2.71)$$

The wave function becomes

$$\Psi_n\left(x,t\right) = \frac{\sqrt{2}}{\ell} \sin\left(n\pi \frac{x}{\ell}\right) e^{-i\frac{E_n}{\hbar}\left(\frac{t}{\ell_0\left(\ell_0+v_0 t\right)}\right)} \mathrm{e}^{\frac{im}{2\hbar} v_0\left(\ell_0+v_0 t\right) y^2}. \qquad (2.72)$$

After some manipolation it reads as

$$\Psi_n\left(x,t\right) = \frac{\sqrt{2}}{\ell_0 + v_0 t} \sin\left(\frac{n\pi x}{\ell_0 + v_0 t}\right) e^{\left[\frac{i}{\hbar}\left(\frac{-E_n t}{\ell_0\left(\ell_0+v_0 t\right)} + \frac{m v_0\left(\ell_0+v_0 t\right)}{2} y^2\right)\right]}, \qquad (2.73)$$

in which E_n is given in Eq. (2.67).

Chapter 3

ENERGY ZERO SYSTEM

In this chapter we consider the energy of the quantum particle equal to zero $E = 0$ and solve the two dimensional Schrödinger equation. In 2-dimensions polar coordinates, the time-independent Schrödinger equation is given by [17-22]

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)+V\left(r,\phi\right)\right]\psi\left(r,\phi\right)=E\psi\left(r,\phi\right), \qquad (3.1)$$

in which $V\left(r,\phi\right)$ is the potential.

When we set $E = 0$, Eq. (3.1) becomes

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)+V\left(r,\phi\right)\right]\psi\left(r,\phi\right)=0, \qquad (3.2)$$

and in the sequal we shall try to find exact solutions for the specific potentials.

3.1 Complex Potential

Let's start with a complex potential of the form

$$V\left(r,\phi\right)=-\frac{\Gamma}{r^4}-\frac{\Lambda}{r^2}e^{i\phi}, \qquad (3.3)$$

in which Γ and Λ are two real costants, and ϕ is the azimuthal angle.

Applying the separation method, we set

$$\psi\left(r,\phi\right)=R\left(r\right)\Phi\left(\phi\right), \qquad (3.4)$$

and upon the seperation, the radial part of the equation becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{\ell^2}{r^2}+\frac{2m\Gamma}{\hbar^2 r^4}\right]R\left(r\right)=0, \tag{3.5}$$

where ℓ^2 is just an integration constant.

The angular part of the equation, however, becomes

$$\frac{\partial\Phi\left(\phi\right)}{\partial\phi^2}+\frac{2m\Lambda}{\hbar^2}e^{i\phi}\Phi\left(\phi\right)=-\ell^2\Phi\left(\phi\right), \tag{3.6}$$

To make our equation simpler, we define the following new variable and

parameters,

$$\rho=\frac{r}{\alpha_\circ},\ \gamma^2=\frac{2\mathrm{m}\Gamma}{\hbar^2\alpha_\circ^{\ 2}}\text{ and }\lambda^2=\frac{2\mathrm{m}\Lambda}{\hbar^2}. \tag{3.7}$$

Upon using these, Eq. (3.5) and (3.6) becomes

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)-\frac{\ell^2}{\rho^2}+\frac{2m\Gamma}{\hbar^2\rho^4}\right]R\left(r\right)=0, \tag{3.8}$$

and

$$\frac{\partial\Phi\left(\phi\right)}{\partial\phi^2}+\lambda^2 e^{i\phi}\Phi\left(\phi\right)=-\ell^2\Phi\left(\phi\right), \tag{3.9}$$

respectively.

In the angular part of the Schrödinger equation Eq. (3.9) we consider a change of variable given by

$$x=e^{i\phi}. \tag{3.10}$$

Differentiating Eq. (3.10), we have

$$\frac{dx}{d\phi} = ie^{i\phi} \Rightarrow \frac{d^2x}{d\phi^2} = -e^{i\phi}. \tag{3.11}$$

Hence, using transformation one obtains

$$\frac{d}{d\phi} = \frac{dx}{d\phi}\frac{d}{dx} = ie^{i\phi}\frac{d}{dx}. \tag{3.12}$$

Consequently

$$\frac{d^2}{d\phi^2} = \left(\frac{dx}{d\phi}\right)^2 \frac{d^2}{dx^2} + \frac{d^2x}{d\phi^2}\frac{d}{dx}, \tag{3.13}$$

or equivalently

$$\frac{d^2}{d\phi^2} = -e^{2i\phi}\frac{d^2}{dx^2} - e^{i\phi}\frac{d}{dx}. \tag{3.14}$$

Considering Eq. (3.14) and Eq. (3.10) in Eq. (3.9), we find

$$-e^{2i\phi}\frac{d^2\Phi}{dx^2} - e^{i\phi}\frac{d\Phi}{dx} + \lambda^2 e^{i\phi}\Phi = -\ell^2\Phi, \tag{3.15}$$

or after simplification

$$-x^2\frac{d^2\Phi}{dx^2} - x\frac{d\Phi}{dx} + \lambda^2 x\,\Phi = -\ell^2\Phi. \tag{3.16}$$

By rearranging this equation, we get a second order differential equation given by

$$x^2\frac{d^2\Phi}{dx^2} + x\frac{d\Phi}{dx} - \left(\ell^2 + \lambda^2 x\right)\Phi = 0. \tag{3.17}$$

Next, we consider the radial equation Eq. (3.8). To solve Eq. (3.8) we change the variable as

$$\xi = \frac{1}{\rho}, \tag{3.18}$$

and using the chain rule

$$\frac{d\xi}{d\rho} = -\frac{1}{\rho^2} = -\xi^2, \qquad \frac{d^2\xi}{d\rho^2} = \frac{2}{\rho^3} = 2\xi^3. \tag{3.19}$$

Using these transformation in Eq. (3.18) and Eq. (3.19) we obtain

$$\frac{d}{d\rho} = \frac{d\xi}{d\rho}\frac{d}{d\xi} = -\xi^2 \frac{d}{d\xi}, \tag{3.20}$$

and

$$\frac{d^2}{d\rho^2} = \left(\frac{d\xi}{d\rho}\right)^2 \frac{d^2}{d\xi^2} + \frac{d^2\xi}{d\rho^2}\frac{d}{d\xi}. \tag{3.21}$$

The latter expressions are simplified as

$$\frac{d^2}{d\rho^2} = \xi^4 \frac{d^2}{d\xi^2} + 2\xi^3 \frac{d}{d\xi}, \tag{3.22}$$

which upon that, Eq. (3.8) becomes

$$\left[\xi^3 \frac{d}{d\xi}\left(\xi \frac{d}{d\xi}\right) - \ell^2\xi^2 + \gamma^2\xi^4\right] R = 0. \tag{3.23}$$

After an expansion of the first term one finds

$$\left[\xi^3 \frac{d}{d\xi} + \xi^4 \frac{d^2}{d\xi^2} - \ell^2\xi^2 + \gamma^2\xi^4\right] R = 0, \tag{3.24}$$

Rearranging Eq. (3.24) and dividing by ξ^4 we find

$$\xi^2 \frac{d^2R}{d\xi^2} + \xi \frac{dR}{d\xi} + \left(\gamma^2\xi^2 - \ell^2\right)R = 0, \tag{3.25}$$

This is the Bessel differential equation.

Hence, the solution to Eq. (3.25) for $R(\xi)$ can be written in terms of the Bessel and Neumann functions

$$R(\xi) = C_1 J_1(\gamma\xi) + C_2 Y_1(\gamma\xi). \tag{3.26}$$

The angular equation (3.17) is also Bessel modified differential equation and the function $\Phi(\phi)$ is given by

$$\Phi(\phi) = \bar{C}_1 I_{21}\left(2\lambda e^{\frac{i\phi}{2}}\right) + \bar{C}_2 K_{21}\left(2\lambda e^{\frac{i\phi}{2}}\right). \tag{3.27}$$

To have a regular radial solution at the origin one must set $\ell > 1$ and $C_2 = 0$. Therefore

$$R(\xi) = N_\ell J_\ell(\gamma\xi), \tag{3.28}$$

in which N_ℓ is a normalization constant which is found to be

$$N_\ell = \frac{2}{a_\circ \gamma}\sqrt{\frac{(\ell+1)!}{(\ell-2)!}}. \tag{3.29}$$

For the same reason (i.e, regular solution) $\bar{C}_2 = 0$ and then

$$\Phi(\phi) = C_{\ell\lambda} I_{2\ell}\left(2\lambda e^{\frac{i\phi}{2}}\right), \tag{3.30}$$

in which $C_{\ell\lambda}$ is also a normalization constant.

Hence, the general solution to the Schrödinger equation Eq. (3.2) becomes

$$\psi(r,\phi)=R(r)\Phi(\phi). \tag{3.31}$$

Substituting Eq. (3.28), (3.29) and (3.30) into Eq. (3.31) we find the complete solution as

$$\psi_\ell(r,\phi)=N_\ell J_\ell(\gamma\xi)C_{\ell\lambda}I_{2\ell}\left(2\lambda e^{\frac{i\phi}{2}}\right). \tag{3.32}$$

Next we expand $I_{2\ell}$ and the wavefunction becomes,

$$\psi_\ell(r,\phi)=C_{\ell\lambda}\frac{2}{a_\circ\gamma}\sqrt{\frac{(\ell+1)!}{(\ell-2)!}}J_\ell\left(\frac{\gamma a_\circ}{r}\right)\sum_s\frac{\lambda^{2(S+1)}}{\Gamma(S+2\ell+1)S!}e^{i\phi(S+\ell)}, \tag{3.33}$$

İn whch $C_{\ell\lambda}$ is the normalization constant

3.2 Real Potential

In this section we set the potential to be a real function of r only

$$V\pm(r)=-\frac{\Gamma}{r^4}\pm\frac{\Lambda}{r^2}, \tag{3.34}$$

in which Γ and Λ are two constants.

The separation method, brings

$$\psi(r,\phi)=R(r)\Phi(\phi). \tag{3.35}$$

and upon Substituting Eq. (3.35) and Eq. (3.34) into Eq. (3.2) we have

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)-\frac{\tilde{\ell}^2}{\rho^2}+\frac{\gamma^2}{\rho^4}\right]R\left(r\right)=0, \tag{3.36}$$

and

$$\frac{\partial\Phi(\phi)}{\partial\phi^2}=-\ell^2\Phi(\phi). \tag{3.37}$$

From Eq. (3.38), using change of variable, one finds

$$\xi^2\frac{d^2R}{d\xi^2}+\xi\frac{dR}{d\xi}+\left(\gamma^2\xi^2-\tilde{\ell}^2\right)R=0, \tag{3.38}$$

where

$$\tilde{\ell}^2=\ell^2\pm\lambda^2. \tag{3.39}$$

Hence Eq. (3.38) admits a regular solution which is given as Bessel function

$$R_\ell\left(\xi\right)=N_\ell J_\ell\left(\gamma\xi\right), \tag{3.40}$$

in which N_ℓ is the normalization constant.

The angular part reads

$$\frac{\partial\Phi(\phi)}{\partial\phi^2}+\ell^2\Phi(\phi)=0, \tag{3.41}$$

whose solution is simply

$$\Phi(\phi)=Ae^{i\ell\phi}+Be^{-i\ell\phi}, \tag{3.42}$$

in which A and B are integration constants.

Considering $\ell=0,\pm1,...$ one writes

$$\Phi_{\ell}(\phi)=Ae^{i\ell\phi}. \quad (3.43)$$

To find the normalization constant A we write the condition

$$\int_0^{2\pi}|\Phi|^2 d\phi=1 \Rightarrow \int_0^{2\pi}|A|^2 d\phi=1, \quad (3.44)$$

and

$$|A|^2\int_0^{2\pi}d\phi=1 \qquad \Rightarrow A=\frac{1}{\sqrt{2\pi}}. \quad (3.45)$$

Upon substituting the normalization constant to Eq.(3.45) we have

$$\Phi_{\ell}(\phi)=\frac{1}{\sqrt{2\pi}}e^{i\ell\phi}. \quad (3.46)$$

We also find the normalization constant for the radial equation in Eq. (3.40) which is found to be

$$N_{\ell}=\frac{2}{a_{\circ}\gamma}\sqrt{\frac{(\tilde{\ell}+1)!}{(\tilde{\ell}-2)!}}. \quad (3.47)$$

Finally, the complete wave function is written as

$$\psi_{\ell\gamma\lambda}(r,\phi)=\frac{1}{\sqrt{2\pi}}\frac{2}{a_{\circ}\gamma}\sqrt{\frac{(\tilde{\ell}+1)!}{(\tilde{\ell}-2)!}}e^{i\ell\phi}J_{\tilde{\ell}}\left(\frac{1}{r}\right). \quad (3.48)$$

3.3 The Coherent State

In this section we introduce the Coherent state of the system which we have studied. For the case with complex potential the wave function is given

$$\psi_{\ell\gamma\lambda}(r,\phi)=C_{\ell\lambda}\frac{2}{a_{\circ}\gamma}\sqrt{\frac{(\ell+1)!}{(\ell-2)!}}J_{\ell}\left(\frac{\gamma a_{\circ}}{r}\right)I_{2\ell}\left(2\lambda e^{\frac{i\phi}{2}}\right), \quad (3.49)$$

whose normalization constant $C_{\ell\lambda}$ is written by

$$C_{\ell\lambda}=\left[\int_0^{2\pi}d\phi\left|I_{2\ell}\left(2\lambda e^{\frac{i\phi}{2}}\right)\right|^2\right]^{\frac{-1}{2}}. \tag{3.50}$$

Next, we introduce the coherent state as [23-25]

$$\Psi_N=\frac{1}{\sqrt{2\pi}\left(1+|\tau|^2\right)^{\frac{N}{2}}}\sum_{k=0}^{N}\binom{N}{k}^{\frac{1}{2}}\tau^k\psi_{k,\gamma,\lambda}(r,\phi). \tag{3.51}$$

For N=7, $|\tau|=1,$ and $a_{\circ}\gamma=1;$

$$\Psi_7=\left[\binom{7}{0}^{\frac{1}{2}}\psi_{0,\gamma,\lambda}+\binom{7}{1}^{\frac{1}{2}}\psi_{1,\gamma,\lambda}+\binom{7}{2}^{\frac{1}{2}}\psi_{2,\gamma,\lambda}+\binom{7}{3}^{\frac{1}{2}}\psi_{3,\gamma,\lambda}+\right.$$
$$\left.\binom{7}{4}^{\frac{1}{2}}\psi_{4,\gamma,\lambda}+\binom{7}{5}^{\frac{1}{2}}\psi_{5,\gamma,\lambda}+\binom{7}{6}^{\frac{1}{2}}\psi_{6,\gamma,\lambda}+\binom{7}{7}^{\frac{1}{2}}\psi_{7,\gamma,\lambda}\right]. \tag{3.52}$$

Substituting Eq.(3.51) into Eq.(3.54) where $\ell=k+2$

$$\Psi_7=\frac{1}{\sqrt{2\pi}}\left[\binom{7}{0}^{\frac{1}{2}}2C_{2\lambda}\sqrt{6}J_2\left(\frac{1}{r}\right)I_4\left(2\lambda e^{\frac{i\phi}{2}}\right)+\binom{7}{1}^{\frac{1}{2}}2C_{3\lambda}\sqrt{24}J_3\left(\frac{1}{r}\right)I_6\left(2\lambda e^{\frac{i\phi}{2}}\right)+\right.$$
$$\binom{7}{2}^{\frac{1}{2}}2C_{4\lambda}\sqrt{60}J_4\left(\frac{1}{r}\right)I_8\left(2\lambda e^{\frac{i\phi}{2}}\right)+\binom{7}{3}^{\frac{1}{2}}2C_{5\lambda}\sqrt{120}J_5\left(\frac{1}{r}\right)I_{10}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$\binom{7}{4}^{\frac{1}{2}}2C_{6\lambda}\sqrt{210}J_6\left(\frac{1}{r}\right)I_{12}\left(2\lambda e^{\frac{i\phi}{2}}\right)+\binom{7}{5}^{\frac{1}{2}}2C_{7\lambda}\sqrt{336}J_7\left(\frac{1}{r}\right)I_{14}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$\left.\binom{7}{6}^{\frac{1}{2}}2C_{8\lambda}\sqrt{504}J_8\left(\frac{1}{r}\right)I_{16}\left(2\lambda e^{\frac{i\phi}{2}}\right)+\binom{7}{7}^{\frac{1}{2}}2C_{9\lambda}\sqrt{720}J_9\left(\frac{1}{r}\right)I_{18}\left(2\lambda e^{\frac{i\phi}{2}}\right)\right]. \tag{3.53}$$

By using Eq.(3.52) for $\lambda=0.1$ we get,

$$\Psi_7 = \frac{2}{\sqrt{2\pi}}\Bigg[(478.728275)J_2\left(\frac{1}{r}\right)I_4\left(2\lambda e^{\frac{i\phi}{2}}\right)+(138753.658353)J_3\left(\frac{1}{r}\right)I_6\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(28435874.12066)J_4\left(\frac{1}{r}\right)I_8\left(2\lambda e^{\frac{i\phi}{2}}\right)+(49.249741*10^8)J_5\left(\frac{1}{r}\right)I_{10}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(74.855304*10^{10})J_6\left(\frac{1}{r}\right)I_{12}\left(2\lambda e^{\frac{i\phi}{2}}\right)+(9.89436*10^{13})J_7\left(\frac{1}{r}\right)I_{14}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(10.838759*10^{15})J_8\left(\frac{1}{r}\right)I_{16}\left(2\lambda e^{\frac{i\phi}{2}}\right)+(8.565303*10^{17})J_9\left(\frac{1}{r}\right)I_{18}\left(2\lambda e^{\frac{i\phi}{2}}\right)\Bigg]. \quad (3.54)$$

For $\lambda = 1$ we have,

$$\Psi_7 = \frac{2}{\sqrt{2\pi}}\Bigg[(4.763523)J_2\left(\frac{1}{r}\right)I_4\left(2\lambda e^{\frac{i\phi}{2}}\right)+(138.389733)J_3\left(\frac{1}{r}\right)I_6\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(2839.256842)J_4\left(\frac{1}{r}\right)I_8\left(2\lambda e^{\frac{i\phi}{2}}\right)+(538971.433264)J_5\left(\frac{1}{r}\right)I_{10}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(749513.24267)J_6\left(\frac{1}{r}\right)I_{12}\left(2\lambda e^{\frac{i\phi}{2}}\right)+(9889320)J_7\left(\frac{1}{r}\right)I_{14}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(108346012.28324)J_8\left(\frac{1}{r}\right)I_{16}\left(2\lambda e^{\frac{i\phi}{2}}\right)+(856235149.94422)J_9\left(\frac{1}{r}\right)I_{18}\left(2\lambda e^{\frac{i\phi}{2}}\right)\Bigg]. \quad (3.55)$$

Also for $\lambda = 10$ we obtain,

$$\Psi_7 = \frac{2}{\sqrt{2\pi}}\Bigg[(4.264072*10^4)J_2\left(\frac{1}{r}\right)I_4\left(2\lambda e^{\frac{i\phi}{2}}\right)+(0.002888)J_3\left(\frac{1}{r}\right)I_6\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(0.011142)J_4\left(\frac{1}{r}\right)I_8\left(2\lambda e^{\frac{i\phi}{2}}\right)+(0.031434)J_5\left(\frac{1}{r}\right)I_{10}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(0.070255)J_6\left(\frac{1}{r}\right)I_{12}\left(2\lambda e^{\frac{i\phi}{2}}\right)+(0.126689)J_7\left(\frac{1}{r}\right)I_{14}\left(2\lambda e^{\frac{i\phi}{2}}\right)+$$
$$(0.178945)J_8\left(\frac{1}{r}\right)I_{16}\left(2\lambda e^{\frac{i\phi}{2}}\right)+(0.174556)J_9\left(\frac{1}{r}\right)I_{18}\left(2\lambda e^{\frac{i\phi}{2}}\right)\Bigg]. \quad (3.56)$$

In Fig. (3.1), (3.2) and (3.3) we plot Ψ_7 for the different cases of λ. As one can see the maximum probability, corresponds to the classical limit which we study in our next chapter.

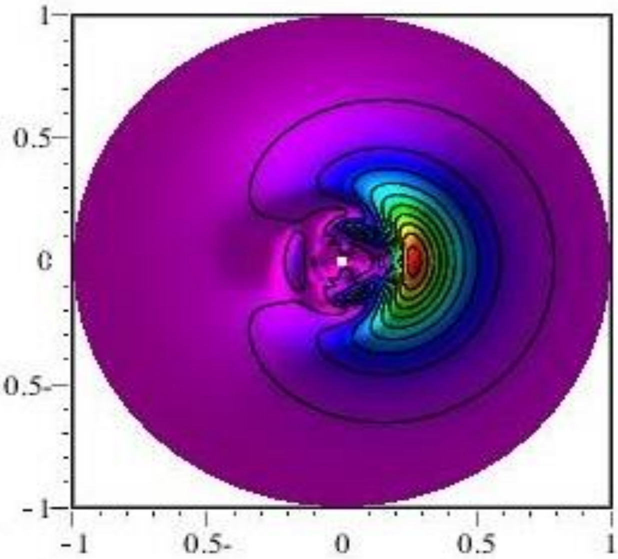

Figure 3.1: Probability density $|\psi(r,\phi)|^2$ for $N=7$ and $\lambda=1$

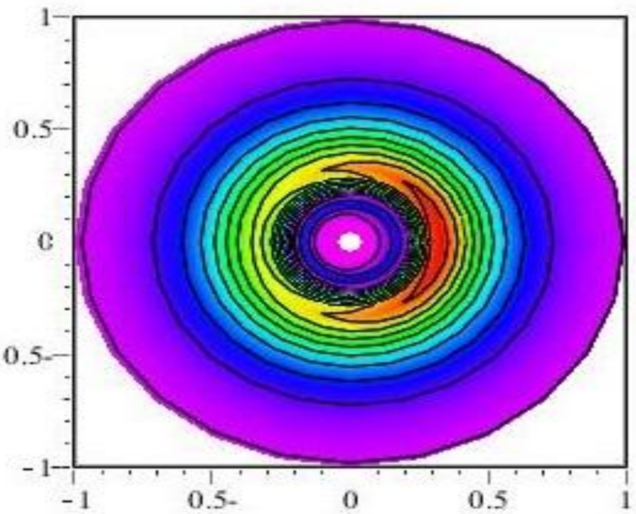

Figure 3.2: Probability density $|\psi(r,\phi)|^2$ for $N=7$ and $\lambda=0.1$

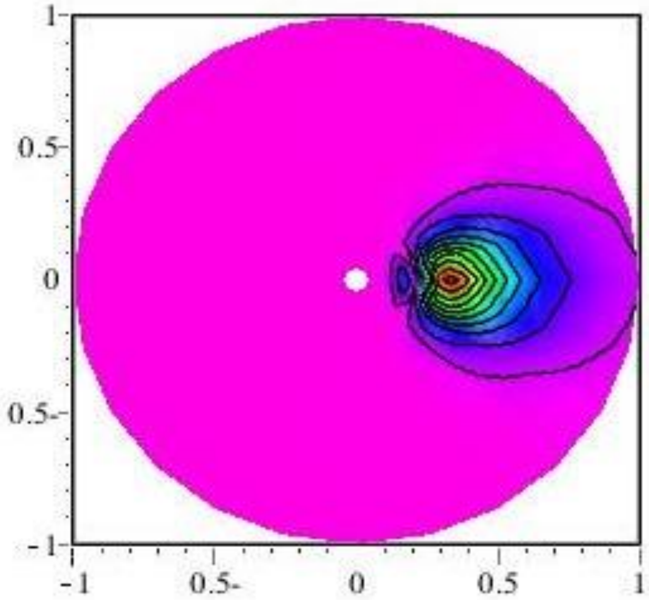

Figure 3.3: Probability density $|\psi(r,\phi)|^2$ for $N=7$ and $\lambda=10$

Chapter 4

CLASSICAL PARTICLE

In this chapter we consider a classical particle under a potential of $V=-\frac{\Gamma}{r^4}-\frac{\Lambda}{r^2}e^{i\phi}$, and we try to find the correspondance solution with the quantum particle given in previous chapter.

The Lagrangian of such particle is given by

$$L=T-V \tag{4.1}$$

which after considering a polar coordinate and expansion it becomes

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)-\left(\frac{\Gamma}{r^4}+\frac{\Lambda}{r^2}e^{i\phi}\right). \tag{4.2}$$

For the case of zero energy and $\Lambda=0$ we find

$$H=T+V \tag{4.3}$$

or consequently

$$\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)=\frac{\Gamma}{r^4}. \tag{4.4}$$

The angular part of the Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)=\frac{\partial L}{\partial\phi}, \tag{4.5}$$

implies

$$mr^2\dot{\phi} = \text{constant} = \alpha, \tag{4.6}$$

or equialently

$$\dot{\phi} = \frac{\alpha}{mr^2} \Rightarrow \dot{\phi} = \frac{\mu}{r^2}. \tag{4.7}$$

in which $\mu = \frac{\alpha}{m}$. Considering $r = r(\phi)$ and using the chain rule in Eq. (4.4),

$$\frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) = \frac{\Gamma}{r^4}, \tag{4.8}$$

admits

$$\frac{1}{2}m\left(\left(\frac{dr}{d\phi}\right)^2\left(\frac{d\phi}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) = \frac{\Gamma}{r^4}, \tag{4.9}$$

which is simplified as

$$\frac{1}{2}m\left(\frac{d\phi}{dt}\right)^2\left(\left(\frac{dr}{d\phi}\right)^2 + r^2\right) = \frac{\Gamma}{r^4}. \tag{4.10}$$

Substituting Eq. (4.7) into Eq. (4.10) we find

$$\frac{1}{2}m\,\frac{\mu^2}{r^4}\left(\left(\frac{dr}{d\phi}\right)^2 + r^2\right) = \frac{\Gamma}{r^4}, \tag{4.11}$$

and after further manipulation, one finds

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 = \frac{2\Gamma}{m\mu^2}. \tag{4.12}$$

Let's define $\frac{2\Gamma}{m\mu^2}=a,$ in Eq. (4.12) which becomes

$$\left(\frac{dr}{d\phi}\right)^2+r^2=a. \tag{4.13}$$

This equation adimit a solution of the form

$$r=A\cos\left(\phi-\phi_0\right), \tag{4.14}$$

in which A and ϕ_0 are two integration constants.

Differentiating Eq. (4.14) it gives

$$\frac{dr}{d\phi}=-A\sin\left(\phi-\phi_0\right), \tag{4.15}$$

and putting Eq. (4.15) and Eq. (4.14) into Eq. (4.13) one gets

$$A^2\sin^2\left(\phi-\phi_0\right)+A^2\cos^2\left(\phi-\phi_0\right)=a, \tag{4.16}$$

which is satisfied if $A=\sqrt{a}$.

Finally, a one parameter solution is found to the Eq. (4.13) which is

$$r=\sqrt{a}\cos\left(\phi-\phi_0\right). \tag{4.17}$$

In Fig. (4.1) we plot r in terms of ϕ when $\phi_0=0$ and $a=1$.

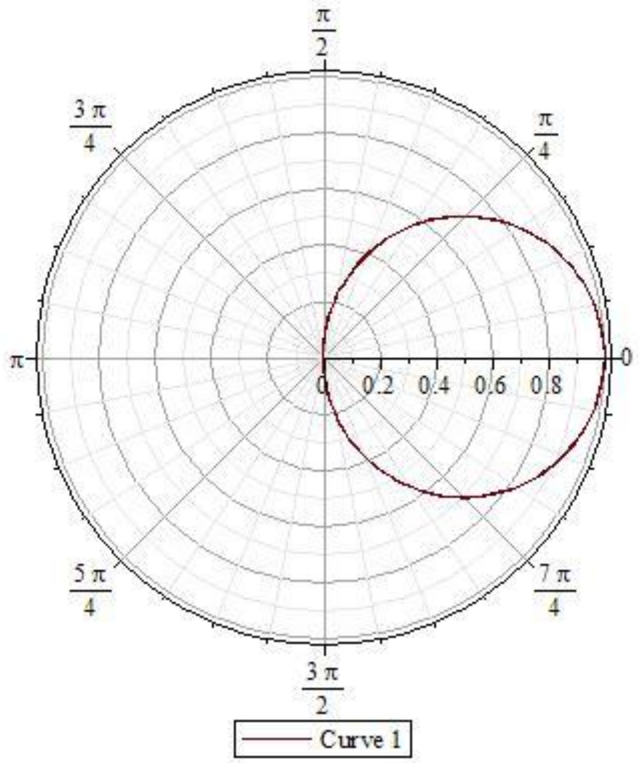

Figure 4.1: Classical trajectory of a particle in the potential given in Eq. (4.2). Here we used $a = 1$ and $\phi_0 = 0$.

Chapter 5

CONCLUSION

In the first part of this thesis we have studied a quantum particle confined in an infinite square well in one dimension. The well which we have considered had the left wall fixed at $x = 0$ and the right wall initially at $x = L_0$ and moving with a constant velocity v_0. The problem is not any more time independent and in order to solve the Schrödinger equation one cannot use the separation method as it is always used for time independent quantum systems. Hence, we introduced a classical time-dependent Hamiltonian with a scale parameter $\ell(t)$ which varies in time. After some manipulation we could transform the time-dependent classical Hamiltonian originally depend on x and t, into a new coordinates of y and τ but independent of τ. The transformed potential in target space is just a function of y plus an additional term proportional to y^2. After we have successfully transformed the classical Hamiltonian we have used the transformed Hamiltonian for the quantum particle inside the infinite square well with one wall moving. Choosing the potential $V = 0$ we could find the exact solution for the system including the energy spectrum and finally we have transformed inversely to the reference space.

In the second part of the thesis we have studied a two dimensional quantum system with a radial potential of complex function form. This kind of quantum systems is called non-Hermitian quantum system. We found the exact energy spectrum together with the eigenfunctions of the system. Finally we have constructed the so-called coherent state of the system and schematically we have shown that the probability density is in agreement with the classical trajectory of the particle.

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