On Fractional Differential Equations

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
September 2015
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

In this thesis, we collect some results on sufficient conditions for the existence and unique of solutions for various classes of initial and boundary value problem for fractional differential equations involving the Caputo fractional derivative. Although the tools of fractional calculus have been available and applicable to various fields of study, the investigation of the theory of fractional differential equations has only been started quite recently. The differential equations involving Caputo differential operators of fractional order, appear to be important in modeling several physical phenomena and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

In this thesis, we shall study systematically the basic theory of fractional differential equations involving Caputo differential operators. We follow the method of deducing the basic existence and uniqueness results from the fixed point theory.

Keywords: Boundary Value Problems, Fractional Differential Equation, Fractional Calculus

ÖZ

Bu tezde, Caputo fraksiyonel türevli fraksiyonel diferansiyel denklemler için başlangıç ve sınır değer probleminin çeşitli sınıflar için varlığı ve tekliği araştırılmıştır. Kesirli analizin araçları, çalışmanın çeşitli alanlarda kullanılabilir ve uygulanabilir olmasına rağmen, fraksiyonel diferansiyel denklemlerin teorisi sadece çok yakın zamanda araştırılmaya başlanmıştır. Fraksiyonel düzenin Caputo diferansiyel operatörleri kapsayan diferansiyel denklemler, çeşitli fiziksel olguları modelleme de önemli gibi görünmektedir ve bu nedenle adi diferansiyel denklemlerin tanınmış teoriye kendi teorisi paralel bağımsız bir çalışma yı haketmekte gibi görünüyor.

Bu tezde, sistematik olarak Caputo diferansiyel operatörleri kapsayan fraksiyonel diferansiyel denklemlerin temel teorisini incelenecektir.

Anahtar Kelimeler: Sınır değer problemi, Fraksiyonel diferansiyel denklemler, Fraksiyonel kalkulus

DEDICATION

This study is respectfully dedicated to my family, specifically my parents who provided me with sufficient help and encouragement that let me to reach this level.

ACKNOWLEDGMENT

First of all, I would like to thank Almighty and Gracious God for giving me the ability and patience to complete this work.

My heartfelt thanks due to my research supervisor Prof. Dr. Nazim Mahmudov for his guidance, help and encouragement.

Many thanks are presented to my family, specifically my parents who provided me with sufficient help and encouragement that let me to reach this level.

Finally, I am in debt to all of my friends for their encouragement.

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Chapter 1

INTRODUCTION

This thesis collects recent results for different classes of initial value problems and boundary value problems (BVP) for fractional differential equations. Fractional differential equations (FDE) have recently proved to be valuable instruments in the modeling of many phenomena in different fields of engineering and science.

There has been a considerable development in differential equations involving Caputo fractional derivatives in recent years; see the monographs of Kilbas et al. , Kiryakova , Miller and Ross , Samko et al. and the papers in the references.

On the other hand BVP with nonlocal boundary conditions define an important class of Fractional Boundary Value Problems. This class include multipoint initial value problems and Boundary Value Problem as special cases.

Chapter 2

FIXED POINT THEOREMS

2.1 Fractional Calculus

Definition 2.1[28, 29]: Let $\alpha \in \mathbb{R}_+$ and $h \in L^1([a,b], \mathbb{R}_+)$. The fractional order integral of h of order α is introduced as follows:

$$I_a^\alpha h(s) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(t)}{(s-t)^{1-\alpha}} ds,$$

while $a = 0$, $I^\alpha h(s) = h(s) * \varphi_\alpha(s)$, where $\varphi_\alpha(s) = \frac{1}{\Gamma(\alpha) s^{1-\alpha}}$ for s > 0, and $\varphi_\alpha(s) = 0$ for $s \leq 0$, and $\varphi_\alpha \to \gamma(s)$ case $\alpha \to 0$, where γ is the gamma function.

Definition 2.2 [28,29]: Let h is a function given on the interval [a, b], the αth Riemann-Liouville fractional-order derivative of function is defined by

$$(D_{a+}^\alpha h)(s) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{ds}\right)^n \int_a^s \frac{h(t)}{(s-t)^{\alpha-n+1}} dt,$$

where n= $[\alpha]$ + 1 and $[\alpha]$ represents the integer parts of α.

Definition 2.3 [28]: Let h be a function given on the interval [a, b]. The Caputo fractional-order derivative of h, is defined by

$$({}^c D_{a+}^\alpha h)(s) = \int_a^s \frac{h(t)}{\Gamma(n-\alpha)(s-t)^{\alpha-n+1}} dt ,$$

where $n = \alpha + 1$.

Lemma 2.1 [32]: Assume that α is positive. Consider the following FDE

$$^{c}D^{\alpha}h(s) = 0 .$$

This equation has solutions in the following form:

$$h(s) = \sum_{i=0}^{n-1} c_i s^i \quad , c_i \in \mathbb{R} , \; n = [\alpha]+1 .$$

Lemma 2.2 [32]: Assume that α is positive, then

$$I^{\alpha c}D^{\alpha}h(s) = h(s) + \sum_{i=0}^{n-1} c_i s^i ,$$

for some $c_i \in \mathbb{R}$, $n = [\alpha]+1$.

We will utilize the result which is an outcome of Lemma2.2.

Lemma 2.3 [27]: v is a function and it is a solution of the fractional integral equation. Let $\alpha \in (0,1)$ and assume h : C$[0, \bar{S}\,] \rightarrow \mathbb{R}$. is defined as follows

$$v(s) = v_0 + \int_0^s \frac{h(t)}{\Gamma(\alpha)(s-t)^{1-\alpha}} dt, \tag{2.1}$$

if and only if v is a solution of the IVP for then FDE

$$^{c}D^{\alpha}\, v(s) = \mathrm{h}(s),\; s \in [0, \bar{S}\,] , \tag{2.2}$$

$$v(0) = v_0. \tag{2.3}$$

2.2 Some Fixed Point Theorems

Theorem 2.1 [27]: (*Non-linear alternative of Leray-Schauder type*): Let B be a nonempty convex subset of Banach space X. Assume Z is a nonempty open set of B with $0 \in Z$ and X: $Z \rightarrow B$ continuous and compact. Then either

(1) X has a fixed point

(2) $\exists \;\; z \in \partial Z$ and $\lambda \in [0,1]$ with $z = \lambda X(z)$.

Theorem 2.2 [22] (*The Schaefer Fixed Point Theorem***):** Assume *Y* is a Banach space and $M\colon Y \to Y$ is completely continuous. If the sets

$$E(M) = \{y \in Y : y = yMy \;\; for \;\; any \;\; \lambda \in [0,1]\}$$

is bounded, then *M* has fixed point.

Theorem 2.3 [19, 22]: Assume (ξ_1 and ξ_2) are two operators and $\xi_1, \xi_2 : X \to X$. If X is a Banach space, ξ_1 is a contraction and ξ_2 is completely continuous, then either

(1) equality $y = \xi_1(y) + \xi_2(y)$ has a solution, or

(2) the set E = $\{ z \in X : z = \lambda \xi_1(z/\lambda) + \lambda \xi_2(z)\}$ is bounded four $\lambda \in (0,1)$.

Chapter 3

BOUNDARY VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATION

3.1 Introduction

In this chapter, we study the existence and uniqueness of solutions of some classes of BVP for FDE. More accurately, we investigate the following BVPs.

$$ {}^{c}D^{\alpha}\, v(s) = \mathrm{f}(\, s, v) \text{ for all } s \in J = [0, \bar{S}\,], \qquad 0 < \alpha < 1 \tag{3.1}$$

$$ a\, v\,(0) + \mathrm{b}\, v(\bar{S}\,) = c\,, \tag{3.2}$$

where $f : j \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, a, b, c are real constants with a + b $\neq 0$,

$$ {}^{c}D^{\alpha}v(s) = f(s,v), \qquad for\ all \quad \mathrm{s} \in J = [0, \bar{S}],\ 2 < \alpha \le 3 \tag{3.3}$$

$$ v(0) = v_0, \qquad v'(0) = v_0^{*}, \qquad y''(\bar{S}) = v_{\bar{S}} \tag{3.4}$$

where $f : j \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, $v_0, v_0^{*}\ and\ v_{\bar{S}}$ are real constants,

$$ {}^{c}D^{\alpha}v(s) = f(s,v), \quad for\ each\ s \in J = [0, \bar{S}\,], 0 < \alpha < 1 \tag{3.5}$$

$$ v(0) + g(v) = v_0, \tag{3.6}$$

where $f : j \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, and continuous function. $g : C\,(\left[0, \bar{S}\right], \mathbb{R}) \to \mathbb{R}$,

$$ {}^{c}D^{\alpha}v(s) = f(s,v), \quad for\ each\ s \in J = [0, \bar{S}\,], 1 < \alpha < 2 \tag{3.7}$$

$$ v(0) = g(v)\,, \qquad v(\bar{S}\,) = v_{\bar{S}} \tag{3.8}$$

where $f : j \times \mathbb{R} \to \mathbb{R}$ is a continuous function, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative and continuous function $g : \mathbb{R} \to \mathbb{R}$ and $v_{\bar{S}} \in \mathbb{R}$,

$$ {}^{c}D^{\alpha}v(s) = f(s,v), \quad for\ each\ s \in J = [0, \bar{S}\,], 1 < \alpha \leq 2 \tag{3.9} $$

$$ v(0) = \int_0^S g(t,v)ds, \tag{3.10} $$

$$ v(\bar{S}\,) = \int_0^{\bar{S}} h(t,v)ds, \tag{3.11} $$

where $f : j \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, and $g, h : J \times \mathbb{R} \to \mathbb{R}$ are continuous,

$$ {}^{c}D^{\alpha}v(s) = f(s,v), \quad for\ each\ s \in J = [0, \bar{S}\,], 1 < \alpha \leq 2 \tag{3.12} $$

$$ v(0) - \acute{v}(0) = \int_0^{\bar{S}} g(t,v)ds, \tag{3.13} $$

$$ v(\bar{S}\,) - \acute{v}\left(\bar{S}\right) = \int_0^{\bar{S}} h(t,v)ds, \tag{3.14} $$

where $f : j \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ${}^{c}D^{\alpha}$ is the Caputo fractional derivative and g, h : $J \times \mathbb{R} \to \mathbb{R}$ are continuous.

3.2 Boundary Value Problem of Order $\alpha \in (0,1]$

The following definitions are used while solving the problem (3.1)-(3.2).

Definition 3.1 [1]: Suppose that v is a continuously differentiable function on an open interval J, then v is a solution of (3.1)-(3.2) if v satisfies

$$ {}^{c}D^{\alpha}v(s) = f(s,v), \quad for\ each\ s \in J = [0, \bar{S}\,], 0 < \alpha < 1 \tag{3.1} $$

$$ av(0) + bv(\bar{S}\,) = c \tag{3.2} $$

The Lemma 3.1 will be used to solve the problem (3.1)-(3.2).

Lemma 3.1: Assume that $0 < \alpha < 1$, $g: [0, \bar{S}\,] \to \mathbb{R}$ is a continuous function. A function *v* is a solution of Fractional Integral Equation (FIE)

$$v(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} g(t)dt$$

$$-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^{\bar{S}} (\bar{S} - t)^{\alpha-1}\, g(t)dt - c\right] \tag{3.15}$$

if and only if *v* is a solution of the following FBVP.

$${}^{c}D^{\alpha}v(s) = g(s), \quad for\ each\ s \in [0, \bar{S}\,], \tag{3.16}$$

$$a\, v(0) + b\, v\left(\bar{S}\right) = c\,. \tag{3.17}$$

Proof: Let *v* be a solution of (3.16). Integrating (3.16) we get

$$v(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} g(t)dt + d, \quad for\ each\ t \in [0, \bar{S}\,], \tag{3.16}$$

where *d* is constant. To find *d*, we use boundary condition (3.17),

$$ad + b\left(\frac{1}{\Gamma(\alpha)} \int_0^{\bar{S}} \left(\bar{S} - t\right)^{\alpha-1} g(t)dt + d\right) = c\ .$$

It follows that

$$d = -\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^{\bar{S}} (\bar{S} - t)^{\alpha-1}\, g(t)dt - c\right].$$

Inserting the value of *d* into (3.16), we get the desired formula.

Banach Fixed Point Theorem is used to prove the Theorem3.1.

Theorem 3.1: Suppose that

(A1) $\exists L > 0$ such that

$$|f(s,z) - f(s,\bar{z})| \leq L|z - \bar{z}|, \quad for\ each\ s \in J, and\ all\ z, \bar{z} \in \mathbb{R}.$$

Moreover, assume that

$$\frac{L\ \bar{S}^{\alpha}(1 + \frac{|b|}{|a+b|})}{\Gamma(\alpha+1)} < 1. \tag{3.18}$$

Then the FBVP (3.1)-(3.2) has one solution on $[0, \bar{S}\,]$.

Proof: To start to prove the theorem we transform the problem (3.1)-(3.2) into a fixed point problem. To this end we introduce the following operator

$$F : \mathrm{C}\,([0, \bar{S}\,], \mathbb{R}) \to \mathrm{C}\,([0, \bar{S}\,], \mathbb{R}),$$

where F is defined by

$$F(v)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t\,, v(t))dt$$

$$-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^{\bar{S}} (\bar{S} - t)^{\alpha-1} f\big(t, v(t)\big)dt - c\right] . \tag{3.19}$$

We are aimed to show the existence of a unique fixed point of F in $\mathrm{C}[0, \bar{S}\,]$. To do this we need to show that F is contraction mapping. Indeed, for every $s \in [0, \bar{S}]$ we have

$$\left|(Fx)(s) - (Fv)(s)\right| = \left| I_{0+}^{\alpha} f\left(s, x(s)\right) - I_{0+}^{\alpha} f\left(s, v(s)\right) + \frac{b}{a+b} I_{0+}^{\alpha} f\left(\bar{S}, v(\bar{S})\right) - \frac{b}{a+b} I_{0+}^{\alpha} f\left(\bar{S}, x(\bar{S})\right)\right|$$

$$\leq I_{0+}^{\alpha}\left| f\left(s, x(s)\right) - f\left(s, v(s)\right)\right| + \frac{|b|}{|a+b|} I_{0+}^{\alpha}\left| f\left(\bar{S}, x(\bar{S})\right) - f\left(\bar{S}, v(\bar{S})\right)\right|$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (s-t)^{\alpha-1}\left| f\left(t, x(t)\right) - f\left(t, v(t)\right)\right| dt + \frac{|b|}{|a+b|}\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1}\left| f\left(t, x(t)\right) - f\left(t, v(t)\right)\right| dt.$$

By (A1),

$$
\begin{aligned}
&\left|Fx(s)-Fv(s)\right| \\
&\le \frac{L}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}\left|x(t)-v(t)\right|dt+\frac{|b|}{|a+b|}\frac{L}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}\left|x(t)-v(t)\right|dt \\
&\le \frac{L}{\Gamma(\alpha)}\left\|x-v\right\|_\infty\int_0^s (s-t)^{\alpha-1}dt+\frac{|b|}{|a+b|}\frac{L}{\Gamma(\alpha)}\left\|x-v\right\|_\infty\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}dt \\
&= \frac{L}{\alpha\Gamma(\alpha)}s^\alpha\left\|x-v\right\|_\infty+\frac{|b|}{|a+b|}\frac{L}{\alpha\Gamma(\alpha)}\bar{S}^\alpha\left\|x-v\right\|_\infty dt \\
&\le \frac{L\bar{S}^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\left\|x-v\right\|_\infty
\end{aligned}
$$

By (3.18)

$$\frac{L\bar{S}^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)<1$$

Then, $F:C[0,\bar{S}]\to C[0,\bar{S}]$ is a contraction mapping and by the Banach Fixed Point Theorem F has a unique fixed point in $C\left[0,\bar{S}\right]$, which is a unique solution of FBVP.

Schaefer’s fixed point theorem is used in Theorem3.2 given below.

Theorem 3.2 Assume that the following assumptions hold:

(A2) The function $f : [0, \bar{S}\ ] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A3) $\exists$ a non-negative constant M such that

$$|\,f\ (\,s\,,z)| \le\ M, for\ each\ s \in [0,\bar{S}\,] and\ all\ \ z\ \in\ \mathbb{R}\,.$$

Then the FBVP (3.1)–(3.2) has one or more solution on $[0,\bar{S}]$.

Proof: Firstly, it needs to be shown that $F\colon C([0,\bar{S}\ ],\mathbb{R}) \to C([0,\bar{S}\ ],\mathbb{R})$ is completely continuous. Secondly, the set D defined below is bounded

$$D\ =\ \{v\ \in\ C(J,\mathbb{R})\colon v\ =\ \lambda F(v) for\ some\ \ \lambda \in (0,1)\,.$$

F described as in (3.19) has a fixed point by Schaefer's Fixed Point Theorem. Hence it is the solution of BVP (3.1)–(3.2).

The proof is based on the Schaefer's FPT, $F:Y\to Y$

i- F is continuous and compact operator.

ii- $\varepsilon(F)=\{y\in Y; y=\lambda Fy \text{ for some } \lambda\in[0,1]\}$ is bounded ,

So that F has one or more fixed points.

Step 1: F is continuous in C$[0,\bar{S}]$;

Step 2: F maps bounded sets into bounded set in $C\left[0,\bar{S}\right]$. i.e $F:[0,\bar{S}]\to[0,\bar{S}]$;

Step 3: F maps bounded sets into equicontinuous sets of $C[0,\bar{S}]$;

Step 4: $\varepsilon(F)$ is bounded.

Hint: Step 2 and 3 together is Arzela Ascoli Theorem then F is compact.

Step 1: Let $\{v_n\}\subset C[0,\bar{S}], v\in C[0,\bar{S}]$

$$\lim_{n\to\infty}\|v_n-v\|_\infty=0\Rightarrow\lim_{n\to\infty}\|Fv_n-Fv\|_\infty=0\ .$$

Indeed,

$$\begin{aligned}|F(v_n)(s)-F(v)(s)|&\le\frac{1}{\Gamma(\alpha)}\int_0^t(s-t)^{\alpha-1}\left|f(t,v_n(t))-f(t,v(t))\right|dt\\&\quad+\frac{|b|}{|a+b|}\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}\left|f(t,v_n(t))-f(t,v(t))\right|dt\\&\le\left(\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}dt+\frac{|b|}{|a+b|}\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}dt\right)\\&\quad\times\sup_{0\le t\le S}\left|f(t,v_n(t))-f(t,v(t))\right|\end{aligned}$$

$$\le \left(\frac{\bar{S}^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{|b|}{|a+b|}\frac{\bar{S}^{\alpha}}{\alpha\Gamma(\alpha)} \right) \sup_{0\le t\le S} \left| f(t, v_n(t)) - f(t, v(t)) \right| \ .$$

take $\lim_{n\to\infty} \to 0$, f is continuous

$\Rightarrow \|Fv_n - Fv\|_{\infty} \to 0$ when, n $\to \infty$ $\Rightarrow$ F is continuous.

Step 2: It needs to be shown that for any $N > 0$, $\exists$ a non-negative constant P such that for all $v \in B(0,N) = \left\{ v \in C[0,\bar{S}] : \|v\|_{\infty} \le P \right\}$, we have $\|F(v)\|_{\infty} \le P$, then

$$\|Fv(s)\| \le \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1} \left| f(t,v(t)) \right| dt + \frac{|b|}{|a+b|}\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} \left| f(t,v(t)) \right| dt + \frac{|c|}{|a+b|} \ .$$

By (A3)

$$\|Fv(s)\| \le \frac{M}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1} dt + \frac{|b|}{|a+b|}\frac{M}{\Gamma(\alpha)}\int_0^{S} (\bar{S}-t)^{\alpha-1} dt + \frac{|c|}{|a+b|}$$

$$\le \frac{M}{\alpha\Gamma(\alpha)}\bar{S}^{\alpha} + \frac{|b|}{|a+b|}\frac{M}{\alpha\Gamma(\alpha)}\bar{S}^{\alpha} + \frac{|c|}{|a+b|} = P$$

$$\Rightarrow \|Fy\|_{\infty} \le P \ .$$

Step 3: Let $s_1, s_2 \in C[0,\bar{S}],\ \ s_1 < s_2\ ,\ v \in B(0,N)$. Then

$$|Fv(s_2) - Fv(s_1)| = \left| \frac{1}{\Gamma(\alpha)}\int_0^{s_2} (s_2-t)^{\alpha-1} f(t,v(t))dt - \frac{1}{\Gamma(\alpha)}\int_0^{s_1} (s_1-t)^{\alpha-1} f(t,v(t))dt \right|$$

$$\le \left| \frac{1}{\Gamma(\alpha)}\int_0^{s_1} \left((s_2-t)^{\alpha-1} - (s_1-t)^{\alpha-1} \right) f(t,v(t))dt \right| + \left| \frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2} (s_2-t)^{\alpha-1} f(t,v(t))dt \right|$$

$$\le \frac{M}{\alpha\Gamma(\alpha)}\left((s_2-s_1)^{\alpha} + (s_1^{\ \alpha} - s_2^{\ \alpha}) \right) + \frac{M}{\alpha\Gamma(\alpha)}(s_2-s_1)^{\alpha}$$

$$= \frac{M}{\Gamma(\alpha+1)}(s_1^{\ \alpha} - s_2^{\ \alpha}) \to 0 \quad \text{as} \quad s_2 \to s_1 \ .$$

Step 1, 2 and 3 together are Arzela Ascoli theorem, then F is continuous and compact.

Step 4:

$$\varepsilon(F)=\left\{v\in C[0,\bar{S}]:v=\lambda F(v),0<\lambda<1\right\} \quad \text{is bounded.}$$

Assume $v\in\varepsilon(F)$, then $v=\lambda F(v)$, We need to show that $\exists L>0$ such that $\|v\|_\infty \le P$.

Indeed (see step 2),

$$\|v\|_\infty=\lambda\|F(v)\|_\infty \le \|F(v)\|_\infty \le P=\frac{M}{\alpha\Gamma(\alpha)}\bar{S}^{\alpha}+\frac{|b|}{|a+b|}\frac{M}{\alpha\Gamma(\alpha)}\bar{S}^{\alpha}+\frac{|c|}{|a+b|} \ .$$

By the Schaefer's Theorem F has at least one fixed point in $C\left[0,\bar{S}\right]$.

Remark 3.1: The results of the BVP (3.1)–(3.2) are applied for IVP (a = 1, b = 0), terminal value problems (a = 0, b = 1) and the anti-periodic solutions (a = 1,b = 1, c = 0).

Example 3.1: As an application of Theorem3.1, we consider the following FBVP

$$^{c}D^{\alpha}v(t)=\frac{e^{-s}|v(s)|}{(9+e^{s})(1+|v(s)|)} \quad , \quad s\epsilon[0,1], \tag{3.20}$$

$$v(0)+v(1)=0 \ . \tag{3.21}$$

Assume that

$$f(s,x)=\frac{e^{-s}x}{(9+e^{s})(1+x)}, \quad (s,x)\in[0,1]\times[0,\infty] \ .$$

For $0\le x\,,v\ <\infty$ *and* $s\ \epsilon\ [0,1]$ such that

$$|f(s,x)-f(s,v)|=\frac{e^{-s}}{(9+e^{s})}\left|\frac{x}{1+x}-\frac{v}{1+v}\right|$$

$$= \frac{e^{-s}|x-v|}{(9+e^s)(1+x)(1+v)}$$

$$\leq \frac{e^{-s}}{(9+e^s)}|x-v|$$

$$\leq \frac{1}{10}|x-v|.$$

Thus (A1) satisfied with $L = \frac{1}{10}$. It is clear that a = b = S = 1. Then inequality (3.18) is satisfied if

$$\frac{3L}{2\Gamma(\alpha+1)} < 1 \Leftrightarrow \Gamma(\alpha+1) > \frac{3L}{2} = 0{,}15. \quad (3.22)$$

Applying Theorem 3.1, the FBVP (3.20)-(3.21) has one solution on [0, 1] for values of α satisfying (3.22). For example

- $If\ \alpha = \frac{1}{5}\ then\ \Gamma(\alpha+1) = \Gamma\left(\frac{6}{5}\right) = 0.92\ and$

$$\frac{3L}{2}\frac{1}{\Gamma(\alpha+1)} = \frac{0.15}{0.92} = 0.16 < 1.$$

- $If\ \alpha = \frac{2}{3}\ then\ \Gamma(\alpha+1) = \Gamma(\frac{5}{3}) = 0.89\ and$

$$\frac{3L}{2}\frac{1}{\Gamma(\alpha+1)} = \frac{0.15}{0.89} = 0.168 < 1.$$

3.3 Boundary Value Problem of Orders α ∈ (2,3]

From this part we study the following fractional BVP.

$$^{c}D^{\alpha}v(s) = f(s,v), \qquad for\ all \quad s \in [0,\bar{S}], 2 < \alpha \leq 3 \quad (3.23)$$

$$v(0)=v_0, \qquad v'(0)=v_0^*, \qquad v''(\bar{S})=v_{\bar{S}} \quad (3.24)$$

where $f:[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function, and $^{c}D^{\alpha}$ is the Caputo fractional derivative, $v_0, v_0^*\ and\ v_{\bar{S}}$ are real constants.

Definition 3.2[1]: Suppose that v has a three times continuously differentiable function on $[0, \bar{S}\,]$ with its α-derivatives exists on $[0, \bar{S}]$. We say that v is a solution of (3.3),(3.4) if v satisfies ${}^{c}\mathrm{D}^{\alpha}\, v(s) = f\,(s, v(s))$, with the boundary conditions $v(0) = v_0$, $\acute{v}(0) = v_0^*$, $v''(\bar{S}\,) = v_{\bar{S}}$.

To solve (3.3)–(3.4), we need to use the following lemma.

Lemma 3.2: Assume $2<\alpha\leq 3$ and assume that $g: J \to \mathbb{R}$ is a continuous. So that v is a solution of the FIE.

$$v(s)=\int_0^s \frac{g(t)}{\Gamma(\alpha)(s-t)^{1-\alpha}}\,ds$$

$$-\frac{s^2}{\Gamma(\alpha-2)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-3}g(t)\,dt+v_0+v_0^* s+\frac{v_{\bar{S}}}{2}s^2, \qquad (3.23)$$

if and only if v is the solution of the FBVP.

$${}^{c}D^{\alpha}v(s)=g(s), \qquad s\in J, \qquad (3.24)$$

$$v(0) = v_0, \qquad v'(0) = y_0^*, \qquad v''(\bar{S})=v_{\bar{S}} \;. \qquad (3.25)$$

The Banach fixed point theorem is used to prove unique and unique result.

Theorem 3.3: Suppose that (A1) holds. Moreover, assume that

$$L\bar{S}^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right]<1. \qquad (3.26)$$

Then there exists unique solution of the FBVP (3.3)–(3.4).

Proof: We transform FBVP (3.3)-(3.4) into a Fixed Points Problem. To this end, we introduce F_1

$$F_1 = C(J, R) \longrightarrow C(J, R),$$

defined by

$$F_1(v)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t, v(t))dt$$

$$-\frac{s^2}{2\Gamma(\alpha-2)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-3} f(t, v(t))dt + v_0 + v_0^* s + \frac{v_{\bar{S}}}{2} s^2.$$

Repeating the proof of Theorem 3.1 we can see that F_1 is a contraction. Then the fixed point of the operator F_1 is the solution of the FBVP (3.3)–(3.4).

The Schaefer fixed point theorem is used to prove existence result.

Theorem 3.4 FBVP (3.3)–(3.4) has at least one solution on $\left[0, \bar{S}\right]$, provided that (A2)-(A3) hold.

Proof: It is clear that $F_1 : C(\left[0, \bar{S}\right], \mathbb{R}) \to C(\left[0, \bar{S}\right], \mathbb{R})$ is continuous and compact and the set

$$\mathrm{E} = \{v \in C(J, \mathbb{R}): v = \lambda F_1(v) \text{ for some } 0 < \lambda < 1\}$$

is bounded. By the Schaefer Fixed Point Theorem, FBVP (3.3)–(3.4) has at least one solution on $C(\left[0, \bar{S}\right], \mathbb{R})$.

In the Theorem 3.5 we apply the non-linear alternative of Leray-Schauder type in which (A3) is debilitated.

Theorem 3.5: Suppose (A2) is satisfied. Moreover the conditions (A4) and (A5) hold.

(A4) $\exists \ \varnothing_f \in L^1(J,\mathbb{R}+)$ and $\psi:[0,\infty)\to(0,\infty)$ is continuous and non-decreasing function such that

$$\left|f\left(t,w\right)\right| \leq \varnothing_f\left(t\right)\psi\left(\left|w\right|\right) \quad , \quad \forall s\in J \quad \text{and} \quad \forall w\in\mathbb{R}.$$

(A5) $\exists$ a non-negative constant $M>0$ such that

$$\frac{M}{\left\|I^{\alpha}\varnothing_f\right\|_{L^1}+\frac{\bar{S}^2}{2}\left(I^{\alpha-2}\varnothing_f\right)\left(\bar{S}\right)\psi\left(\mathrm{M}\right)+\left|y_0\right|+\left|y_0^*\right|\bar{S}+\frac{\left|y_S\right|}{2}\bar{S}^2}>1 \ . \tag{3.27}$$

Then there is a one or more solution on J for FBVP (3.3)-(3.4).

Proof: define the operator

$$F_1 = C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R}),$$

as

$$F_1(v)(s)=\frac{1}{\Gamma\left(\alpha\right)}\int_0^s\left(s-t\right)^{\alpha-1} f\left(\ t, \ v\left(t\right)\right)dt$$

$$-\frac{s^2}{2\Gamma\left(\alpha-2\right)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-3} f(t, \ v(t))dt \ +v_0+v_0^*t+\frac{v_{\bar{S}}}{2}t^2.$$

We may show that F_1 is continuous and compact. For $\lambda\in[0,1]$, for *all s* ∈ *J*, we have

$$v(\mathrm{s})=\lambda(Fy)(s),$$

then (A4) and (A5) hold for all $t\in J$ and we get,

$$\frac{\left\|v\right\|_\infty}{\psi\left(\left\|v\right\|_\infty\right)\left\|I^{\alpha}\varnothing_f\right\|_{L^1}+\frac{\bar{S}^2}{2}\left(I^{\alpha-2}\varnothing_f\right)\left(\bar{S}\right)\psi\left(\left\|v\right\|_\infty\right)+\left|v_0\right|+\left|v_0^*\right|\bar{S}+\frac{\left|v_{\bar{S}}\right|}{2}\bar{S}^2}\leq 1.$$

So that via case (3.27), $\exists$ non-negative M such that $\|y\|_\infty \neq M$.

Assume that

$$Z = \{v \in C(J,\mathbb{R}) : v_{\infty} < M\}.$$

In the definition of Z, there is no $v \in \partial Z$, such that $v = \lambda F_1(v)$ for some $\lambda \in (0,1)$. Now, we may apply the non-linear alternative of Leray-Schauder [27]. Applying this, we deduce that F_1 has a fixed point v in Z. This fixed point is a solution of the BVP (3.3)–(3.4). This finishes the proof.

Example 3.12: Assume that equation (3.20), where $\alpha \in (2,3]$ with the boundary condition (3.28)

$$^{c}D^{\alpha}v(t) = \frac{e^{-s}|v(s)|}{(9+e^{s})(1+|v(s)|)} \quad , \quad s \epsilon [0,1], \tag{3.20}$$

$$v(0) = 0 \quad , v'(0) = 1 \quad , v''(1) = 0 \tag{3.28}$$

Suppose $s \in J$ and $0 < v, x < \infty$, Then we will get

$$|f(s,x) - f(s,v)| \leq \frac{1}{10}|x - v| .$$

Since then (A1) applies with L = $\frac{1}{10}$. We find out the case (3.26) which is fulfilled with $\bar{S} = 1$, Indeed

$$L\bar{S}^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right] < 1 \Leftrightarrow \frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} < 10 , \tag{3.29}$$

then

$$\frac{1}{6} \leq \frac{1}{\Gamma(\alpha+1)} < \frac{1}{2} , \tag{3.30}$$

$$\frac{1}{2} \leq \frac{1}{2\Gamma(\alpha-1)} < c , \tag{3.31}$$

to fix constant c , case (3.29)– (3.31) imply that

$$\frac{1}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)} < \frac{1}{2}+ c < 10 . \qquad (3.32)$$

In case (36), we get

$$c < \frac{19}{2}.$$

From (35), we find out

$$\Gamma(\alpha-1) > \frac{1}{19} \cong 0. \qquad (3.33)$$

By Theorem 3.3, Equation (3.20) and boundary condition, (3.28) has a unique solution on [0,1] which is fulfilled for some $\alpha \in (2,3]$, for values of α satisfying (3.33).

3.4 Boundary Value Problem of Orders with Nonlocal Condition

The following definitions are used while solving the FBVP (3.5)-(3.6).

Definition 3.3: Suppose that v is a continuously differentiable function on $[0, \bar{S}\,]$ with its α-derivatives exists one $[0, \bar{S}\,]$. Then v is a solution of the FBVP (3.5)-(3.6), if v satisfies the equation ${}^{c}D^{\alpha} v(s)= f(s, v(s))$ with the nonlocal BVP $v(0)+g(v)=v_0$.

We need to give some properties of the function g.

(A6) There exist a non-negative constant $\bar{M} > 0$ such t hat

$$| g(v)| \leq \bar{M} \quad \text{four each}\, u \in C([0, \bar{S}\,], \mathbb{R}).$$

(A7) There exist a non-negative constant $\bar{a} > 0$ such that

$$|g(\bar{v})-g(v)| \le \bar{a}\ |\bar{v}-v|, \quad \text{for each } \bar{v}, v \in C([0,\bar{S}],\ \mathbb{R}).$$

Theorem 3.6: Let (A1) and (A7) hold. If

$$\bar{a}+\frac{k\bar{S}^{\alpha}}{\Gamma(\alpha+1)}<1 \ , \tag{3.34}$$

then the nonlocal problem (3.5)-(3.6) has one solution on $[0,\bar{S}\ ]$.

Proof: We transform BVP (3.5),(3.6) into the fixed point problem. To this end, we consider

$$F_2 : \mathrm{C}([0,\bar{S}],\mathbb{R}) \to \mathrm{C}([0,\bar{S}],\mathbb{R}),$$

defined by

$$F_2(v)(s) = v_o - g(v) + \int_0^s \frac{f(t,v(t))}{\Gamma(\alpha)(s-t)^{1-\alpha}}\, dt \ .$$

Then it is easily seen that F_2 is a contraction mapping, so that Banach fixed point theorem can be applied.

Theorem 3.7: Let (A2), (A3) and (A6) hold. Then the nonlocal FBVP (3.5), (3.6) has one or more than one solution on $[0,\bar{S}\ ]$.

Proof: Since $F_2 : C([0,\bar{S}],\mathbb{R}) \to C([0,\bar{S}],\mathbb{R})$ is completely continuous, then the set

$$D = \{v \in C(J,\mathbb{R}) : v = \lambda F_2(v) \quad \text{for some } 0<\lambda<1\}$$

is bounded.

Then Definition 3.4 will be used while solving the problem (3.7)-(3.8).

Definition 3.4: Suppose that v has a continuous second derivative on the open interval $(J, \mathbb{R})$ with its α-derivatives exists one J , if v holds then equation ${}^{c}D^{\alpha}\, v(t)$ $= f(s, v(s))$ *a.e.* J , with the boundary value problem $v\left(\bar{S}\right) = v_{\bar{S}}$ and $v\left(0\right) = g\left(v\right)$.

To solve the BVP (3.7),(3.8), we can use the results of Lemma 2.1 and Lemma 2.2.

Lemma 3.3: Assume that $1 < \alpha < 2$ and let h: $[0, S] \to \mathbb{R}$ be continuous then v is a solution of the FIE given as

$$(v)(s) = \int_0^s \frac{h(t)}{\Gamma(\alpha)(s-t)^{1-\alpha}}\, dt$$

$$-\frac{s}{\Gamma(\alpha) S} \int_0^{\bar{S}} (\bar{S} - t)^{\alpha-1} h(t)\, dt + \left(\frac{s}{\bar{S}} - 1\right) \mathrm{g}(v) + \frac{s}{\bar{S}}\, y_{\bar{S}} \tag{3.35}$$

if and only if v is a solution of the FBVP.

$${}^{c}D^{\alpha}\, v(s) = h(v(s)), \quad where \quad s \in \left[0, \bar{S}\right], \tag{3.36}$$

$$v(0) = g(v)\,, \qquad\qquad v(\ \bar{S}\ ) = v_{\bar{S}} \tag{3.37}$$

Then Banach Fixed Point Theorem is a base for first consequence.

Theorem 3.8: Apply (A1 and (A7). If

$$\frac{2k\bar{S}^{\alpha-1}\,\bar{S}^{\alpha}}{\Gamma(\alpha+1)} + \bar{k} < 1\,, \tag{3.38}$$

then, there is one solution on [0, $\bar{S}$] for the BVP (3.7)-(3.8).

Proof: We transform BVP (3.7)-(3.8) into the fixed point problem. To this end, we consider

$$F_3 : C([0,\bar{S}],\mathbb{R}) \to C([0,\bar{S}],\mathbb{R})$$

defined by

$$F_3 v\,(s) = \int_0^s \frac{f\left(t, v(t)\right)}{\Gamma(\alpha)(s-t)^{1-\alpha}}\,dt$$

$$-\frac{s}{\Gamma(\alpha)}\int_0^{\bar{S}} \frac{f\left(t, v(t)\right)}{\bar{S}(\bar{S}-t)^{\alpha-1}}\,dt + \left(\frac{s}{\bar{S}} - 1\right)\mathrm{g}(v) + \frac{s}{\bar{S}} v_{\bar{S}} \ .$$

The operator F_3 is contraction and Schaefer's FPT is used for second consequence. Then the fixed point of the operator F_3 is the solution of the BVP (3.7)-(3.8).

Theorem 3.9: Assume that (A2),(A3) and (A6) hold. Then there is one or more than one solution on $[0,\bar{S}\,]$ for the BVP (3.7)-(3.8).

Proof: we can show that F_3: $C([0,\bar{S}],\mathbb{R}) \to C([0,\bar{S}],\mathbb{R})$ is compact and continuous and we show also the set

$$D = \{v \in C(J,\mathbb{R}) : \ = \lambda F_3(v) \ for\, some\, 0 < \lambda < 1\}$$

is bounded.

3.5 Boundary Value Problem of Orders with Integral Problem

The following definitions are used while solving the problem (3.9)-(3.11)

Definition 3.5: Suppose v has a continuous second derivative on the open interval $(J,\mathbb{R})$ with its α-derivatives exists on J is said to be a solution of (3.9)-(3.11) if v holds the equation ${}^cD^{\alpha}v(s) = \mathrm{f}(\mathrm{s},\ v(\mathrm{s}))\ a.e.\ J,$ with the boundary value problem

$v(\bar{S}) = \int_0^{\bar{S}} h(t, v(t))\mathrm{dt}$ and $v(0) = \int_0^{\bar{S}} \mathrm{g}(t, v(t))\mathrm{dt}$

To solve the BVP (3.9)–(3.11), we need to use Lemma 3.4.

Lemma 3.4: Assume that $1 < \alpha \le 2$ and let $\sigma, \rho_1, \rho_2 : J \to \mathbb{R}$ be continuous. The function v is a solution of the FIE,

$$(v(s)) = \int_0^s \frac{\sigma(t)}{\Gamma(\alpha)(s-t)^{1-\alpha}}\,dt$$

$$-\frac{s}{\Gamma(\alpha)}\int_0^{\bar{S}} \frac{\sigma(t)}{\bar{S}(\bar{S}-t)^{1-\alpha}}\,dt$$

$$-\left(\frac{s}{\bar{S}}-1\right)\int_0^{\bar{S}} \rho_1(t)\,dt + \frac{s}{\bar{S}}\int_0^{\bar{S}} \rho_2(t)\,dt \ , \tag{3.39}$$

if and only if v is a solution of the FBVP.

$$^{c}D^{\alpha} v(s) = \omega(s) . \quad s \in J \quad , \tag{3.40}$$

$$v(0) = \int_0^{S} \rho_1(t)\,dt \tag{3.41}$$

$$v(\bar{S}) = \int_0^{\bar{S}} \rho_2(t)\,dt \ . \tag{3.42}$$

The Banach Fixed Point Theorem is used for the consequence.

Theorem 3.10: Let (A1) hold and the below case applied;

(A8) $\exists\ k^* > 0$ such that;

$$|g(s,w) - g(s,\bar{w})| \le k^* |w - \bar{w}|, \text{ for all } s \in J, \text{and } \forall w, \bar{w} \in R.$$

(A9) $\exists\ k^{**} > 0$ such that;

$$|h(s,w) - h(s,\bar{w})| \le k^{**} |w - \bar{w}|, \text{ for each } s \in J, \text{ and } \forall w, \bar{w} \in R\ .$$

If

$$\frac{2k\bar{S}^{\alpha}}{\Gamma(\alpha+1)} + \bar{S}(k^* + k^{**}) < 1 \ , \tag{3.43}$$

then there is one solution on J for the BVP (3.9)–(3.11).

Proof: We transform BVP (3.9)-(3.11) into the fixed point problem To this end, we consider

$$F_4 : C(J,\mathbb{R}) \to C(J,\mathbb{R})$$

which is defined by

$$F_4(v)(s) = \int_0^s \frac{\mathrm{f}\left(t, v(t)\right)}{\Gamma(\alpha)(s-t)^{1-\alpha}}\,dt$$

$$-\frac{s}{\Gamma(\alpha)}\int_0^{\bar{S}} \frac{\mathrm{f}\left(t, v(t)\right)}{\bar{S}(\bar{S}-t)^{1-\alpha}}\,dt$$

$$-\left(\frac{s}{\bar{S}}-1\right)\int_0^{\bar{S}} \mathrm{g}\left(t, v(t)\right)dt + \frac{s}{\bar{S}}\int_0^{\bar{S}} h\left(t, v(t)\right)dt.$$

The fixed point of F_4 is the solution of the BVP (3.9)-(3.11). Then F_4 is contraction mapping.

The Schaefer's Fixed Point Theorem is a base for secondary consequence.

Theorem 3.11: Let (A1), (A2) hold and the below case applied:

(A10) $\exists$ a non-negative constant $N_1>0$ such that:

$$\left|g(s,v)\right| \le N_1 \quad, \forall s \in J \;, \; \forall v \in \mathbb{R}.$$

(A11) $\exists$ non-negative constant N_2 such that

$$|\, h(s,v) \,| \le N_2 \;, \forall s \in J \;, \forall v \in \mathbb{R}.$$

Then the BVP (3.9)-(3.11) has one or more than one solution on J .

Consider the Theorem(3.12), the settings (A2), (A10), (A11) are debilitated.

Theorem 3.12: Let (A1) hold then

(A12) $\exists \quad \varnothing_f \in L^1(J, \mathbb{R}+)$ and the continuous and increasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$|f(s,w)| \leq \varnothing_f(s)\psi(|w|) \quad \text{for each} \quad s \in J \quad \text{and for all} \quad w \in \mathbb{R}.$$

(A13) $\exists \ \varnothing_g \in L^1(J, \mathbb{R}+)$ and the continuous and increasing function $\psi^* : [0,\infty) \to (0,\infty)$ such that

$$|g(s,w)| \leq \varnothing_g(s)\psi^*(|w|) \quad \text{for each}\, s \in J \ \text{ and } \ \text{for all}\, w \in \mathbb{R}.$$

(A14) $\exists \ \varnothing_h \in L^1(J, \mathbb{R}+)$ and the continuous and increasing function $\bar{\psi} : [0,\infty) \to (0,\infty)$ such that

$$|h(s,w)| \leq \varnothing_h(s)\bar{\psi}(|w|) \ , \forall s \in J \ , \ \forall w \in \mathbb{R}.$$

(A15) $\exists$ a non-negative constant $N_1 > 0$ such that

$$1 < \frac{N_1}{\left\|I^{\alpha}\varnothing_f\right\|_{L^1} \psi(N_1) + \left(I^{\alpha}\varnothing_f\right)\left(\bar{S}\right)\psi(N_1) + a\psi^*(N_1) + b\bar{\psi}(N_1)} \tag{3.44}$$

where

$$a = \int_0^{\bar{S}} \varnothing_g(t)\,dt \ , \qquad b = \int_0^{\bar{S}} \varnothing_h(t)\,dt \ .$$

Then the FBVP (3.9) - (3.11) has at least one solution on J.

Proof: F_4 is determined in Theorems (3.10) and (3.11). Clearly F_1 is continuous and compact. For $\lambda \in [0,1]$, for *all* $s \in J$, we have $v(s) = \lambda(F_4 v)(s)$, then (A12) and (A15) hold for *all* $s \in J$, and we get,

$$\frac{\|v\|_{\infty}}{\psi\left(\|v\|_{\infty}\right)\left\|I^{\alpha}\varnothing_f\right\|_{L^1} + \left(I^{\alpha}\varnothing_f\right)\left(\bar{S}\right)\psi\left(\|v\|_{\infty}\right) + a\psi^*\left(\|v\|_{\infty}\right) + b\bar{\psi}\left(\|v\|_{\infty}\right)} \leq 1.$$

Then by case (3.44), $\exists$ a non-negative constant N_1 such that $\| v \|_{\infty} \neq N_1$.

Then

$$Z_1 = \{ v \in C(J,\mathbb{R}) : \|v\|_{\infty} < M_1 \} .$$

In the definition of Z_1, there is no $v \in \partial Z_1$ such that $v = \lambda F_4(v)$ for some $\lambda \in (0,1)$. Now, we may apply the non-linear alternative of Leray-Schauder, applying this, we deduce that F_4 has a fixed point v in Z_1. This fixed point is a solution of the BVP (3.9)–(3.11). This finishes the proof.

Example 3.3: Assume the equation (3.20) with the boundary conditions (3.45), (3.46)

$$y(0) = \sum_{i=0}^{\infty} c_i v(s_i) , \qquad 0 < s_1 < s_2 < s_3 < \ldots < 1 \tag{3.45}$$

$$y(1) = \sum_{j=0}^{\infty} d_j v(\tilde{s}_j) , \tag{3.46}$$

where, $0 < \tilde{s}_0 < \tilde{s}_1 < \tilde{s}_2 < \ldots < 1$, c_i, d_j , i , $j=0,1,2\ldots$ are given non-negative constants with

$$\sum_{i=0}^{\infty} c_i < \infty \qquad , \sum_{j=0}^{\infty} d_j < \infty ,$$

also

$$\sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j = \frac{4}{5},$$

where $\alpha \in (1,2]$ and $0 \le v, x < \infty$ and $s \in J$, Then

$$\left| f(s,x) - f(s,v) \right| \le \frac{1}{10} |x - v| .$$

Since (A1) occurred with $k = \frac{1}{10}$. if α ∈ (0, 1] with $\bar{S} = 1$, $k^* = \sum_{1=0}^{\infty} c_i$ and $k^{**} = \sum_{i=0}^{\infty} d_j$,

then equation (3.44) will be

$$\frac{2k\bar{S}^{\alpha}}{\Gamma(\alpha+1)} + \bar{S}\left(k^* + k^{**}\right) = \frac{1}{5\Gamma(\alpha+1)} + \sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j < 1 \iff \Gamma\left(\alpha+1\right) > 1 \,. \quad (3.47)$$

It is applied for each α ∈ (1, 2]. Then via Theorem (3.10), there is one solution on [0,1] for the equation (20) and boundary conditions (3.45), (3.46).

Remark 3.2: One can select the constants c_i and d_j as

$$c_i = \frac{2}{5}\left(\frac{1}{3}\right)^i \quad , \qquad d_j = \frac{2}{5}\left(\frac{1}{3}\right)^j$$

and then

$$\sum_{i=0}^{\infty} c_i = \frac{3}{5} \quad , \qquad \sum_{j=0}^{\infty} d_j = \frac{1}{5}.$$

The following definitions are used while solving the problem (3.12)–(3.14).

Definition 3.6: Suppose that v has a continuous second derivative on the open interval $(J, \mathbb{R})$ with its α-derivatives exists one J, if v holds the equation ${}^{c}D^{\alpha}v(s) = f(s, v(s))$ *a.e.* J, with the BVP, $v(\bar{S}) + v'(\bar{S}) = \int_0^{\bar{S}} h(t, v(t))dt$ and $v(0) - v'(0) = \int_0^{\bar{S}} g(t, v(t))dt$. Then v is a solution of (3.12)-(3.14).

To solve the problem (3.12)–(3.14), we use the results of Lemma3.5.

Lemma 3.5: Assume $1 < \alpha \le 2$ and let $\omega, \rho_1,\ \rho_2\ :\ J\ \to \mathbb{R}$ be continuous. Then v be the solution of the FIE.

$$v(s)=\delta(s)+\int_0^{\bar{S}} G(s,t)\,\omega(t)\,dt \quad , \tag{3.48}$$

when

$$\delta(s)=\left(\frac{\bar{S}+1-s}{\bar{S}+2}\right)\int_0^{\bar{S}} \rho_1(t)\,dt+\frac{(s+1)}{\bar{S}+2}\int_0^{\bar{S}} \rho_2(t)\,dt \quad , \tag{3.49}$$

And the Green function

$$G(s,t)=\begin{cases} \dfrac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}-\dfrac{(1+s)\left(\bar{S}-t\right)^{\alpha-1}}{\left(\bar{S}+2\right)\Gamma(\alpha)}-\dfrac{(1+s)\left(\bar{S}-t\right)^{\alpha-2}}{\left(\bar{S}+2\right)\Gamma(\alpha-1)}\ , & 0\le t\le s \\ -\dfrac{(1+s)\left(\bar{S}-t\right)^{\alpha-1}}{\left(\bar{S}+2\right)\Gamma(\alpha)}-\dfrac{(1+s)\left(\bar{S}-t\right)^{\alpha-2}}{\left(\bar{S}+2\right)\Gamma(\alpha-1)}\ , & s\le t\le \bar{S} \end{cases} \tag{3.50}$$

If and only if v is a solution of the FBVP.

$${}^{c}D^{\alpha}v(s)=\omega(s)\ , \qquad s\in J\ , \tag{3.51}$$

$$v(0)-v'(s)=\int_0^{\bar{S}} \rho_1(t)\,dt \tag{3.52}$$

$$v\left(\bar{S}\right)+v'(\bar{S})=\int_0^{\bar{S}} \rho_2(t)\,dt\ . \tag{3.53}$$

***Proof*:** By Lemma3.1, we get

$$\left(v(s)\right)=c_0+c_1 s+\frac{1}{\Gamma(\alpha)}\int_0^{s} (s-t)^{\alpha-1}\,\omega(t)\,dt\ . \tag{3.54}$$

From (3.52) and (3.53), we have

$$c_0-c_1=\int_0^{\bar{S}} \rho_1(t)\,dt \tag{3.55}$$

$$c_0+c_1\left(\bar{S}+1\right)+\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}} \left(\bar{S}-t\right)^{\alpha-1}\omega(t)\,dt$$

$$+\frac{1}{\Gamma(\alpha-1)}\int_0^{\bar{S}} \left(\bar{S}-t\right)^{\alpha-2}\omega(t)\,dt=\int_0^{\bar{S}} \rho_2(t)\,dt \tag{3.56}$$

Then, we get

$$c_1 = \frac{1}{T+2}\int_0^{S}\rho_2(t)\,dt - \frac{1}{\bar{S}+2}\int_0^{\bar{S}}\rho_1(t)\,dt$$

$$-\frac{1}{(\bar{S}+2)\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}\,\omega(t)\,dt$$

$$-\frac{1}{(\bar{S}+2)\Gamma(\alpha-1)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-2}\,\omega(t)\,dt\ , \tag{3.57}$$

while

$$c_0 = \frac{\bar{S}+1}{\bar{S}+2}\int_0^{\bar{S}}\rho_1(t)\,dt + \frac{1}{\bar{S}+2}\int_0^{\bar{S}}\rho_2(t)\,dt$$

$$-\frac{1}{(\bar{S}+2)\Gamma(\alpha)}\int_0^{\bar{S}}\frac{1}{(\bar{S}-t)^{1-\alpha}}\,\omega(t)\,dt$$

$$-\frac{1}{(\bar{S}+2)\Gamma(\alpha-1)}\int_0^{\bar{S}}\frac{1}{(\bar{S}-t)^{2-\alpha}}\,\omega(t)\,dt\ . \tag{3.58}$$

In (3.54), (3.57), (3.58) and apply this fact $\int_0^{\bar{S}} = \int_0^{s} + \int_s^{\bar{S}}$ *we* get

$$(v)(s) = \delta(s) + \int_0^{\bar{S}} G(s,t)\,\sigma(t)\,dt, \tag{3.59}$$

where

$$\delta(s) = \frac{(\bar{S}+1-s)}{\bar{S}+2}\int_0^{\bar{S}}\rho_1(t)\,dt + \frac{(s+1)}{\bar{S}+2}\int_0^{\bar{S}}\rho_2(t)\,dt\ \ , \tag{3.60}$$

and

$$G(t,s) = \begin{cases} \dfrac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(1+s)(\bar{S}-t)^{\alpha-1}}{(\bar{S}+2)\Gamma(\alpha)} - \dfrac{(1+s)(\bar{S}-t)^{\alpha-2}}{(\bar{S}+2)\Gamma(\alpha-1)}\ , & 0 \le t \le s \\ -\dfrac{(1+s)(\bar{S}-t)^{\alpha-1}}{(\bar{S}+2)\Gamma(\alpha)} - \dfrac{(1+s)(\bar{S}-t)^{\alpha-2}}{(\bar{S}+2)\Gamma(\alpha-1)}\ , & s \le t \le \bar{S} \end{cases} \tag{3.61}$$

since we gain (3.48). If *v* applies (3.48), then (3.51)–(3.53) hold.

The Banach Fixed Point Theorem is a base for the consequence.

Theorem 3.13: Let (A1), (A8) and (A9) hold. If

$$\left[\frac{\bar{S}(\bar{S}+1)}{\bar{S}+2}k^{*}+\frac{\bar{S}(\bar{S}+1)}{\bar{S}+2}k^{**}+\bar{S}k\chi\right]<1\,, \tag{3.62}$$

where

$$\chi=\sup_{(s,t)\in J\times J}\left|G(s,t)\right|$$

Then there is a one solution on J for the BVP (3.12)–(3.14).

Proof: We transform BVP (3.12)-(3.14) into the fixed point BVP. To this end, we consider

$$F_5\ :C(J,\mathbb{R})\to C(J,\mathbb{R}),$$

which is given by

$$\left(F_5 v\right)\left(s\right)=\delta\left(s\right)+\int_0^{\bar{S}}G(s,t)\,f(t,v(t))dt,$$

where

$$\delta\left(s\right)=\frac{(\bar{S}+1-s)}{\bar{S}+2}\int_0^{\bar{S}}\mathrm{G}\left(t,v(t)\right)dt+\frac{(s+1)}{\bar{S}+2}\int_0^{\bar{S}}h\left(t,v(t)\right)dt\ .$$

The function $\{G(s,t)\}$ is defined by (3.50). Then fixed point of the operator F_5 is the solution of the BVP (3.12)–(3.14), then F_5 is contraction mapping.

So that Schaefer's Fixed Point Theorem is a base for the second consequence.

Theorem 3.14 Let (A2), (A3), (A10) and (A11) hold, then there is one or more solution on J for the BVP (3.12.)–(3.14.).

Consider the Theorem(3.15), the settings (A10), (A11) are debilitated.

Theorem 3.15 Let (A2), (A12), (A13), (A14) hold and the below case holds:

(A16) $\exists N_2 > 0$ such that

$$\frac{N_2}{a\psi^*(N_2)+b\overline{\psi}(N_2)+c\chi\psi(N_2)} > 1, \tag{3.63}$$

where $a = \frac{\overline{S}+1}{\overline{S}+2}\int_0^{\overline{S}} \varnothing_g(t)\,dt,$ $\qquad b = \frac{\overline{S}+1}{\overline{S}+2}\int_0^{\overline{S}} \varnothing_h(t)\,dt$

and

$$c = \int_0^{\overline{S}} \varnothing_f(t)\,dt.$$

Then the BVP (3.12) – (3.14) has one or more solution on *J*.

Proof: F_5, which is determined in Theorems (3.13) and (3.14) clearly F_5 is continuous and compact. for $\lambda \in [0,1]$, for *all s* ∈ *J*, we have $v(s) = \lambda(F_5 v)(s)$. Then (A13) and (A14) hold for *all s* ∈ *J* we get,

$$\frac{\|v\|_\infty}{a\psi^*(\|v\|_\infty)+b\overline{\psi}(\|v\|_\infty)+c\chi\psi(\|v\|_\infty)} \leq 1.$$

For all $s \in J$ so that (6.8), $\exists M_2$ such that $\| v \|_\infty \neq M_2$.

Assume

$$Z_2 = \{v \in C(J,\mathbb{R}) : \|v\|_\infty < M_2\}.$$

In the definition of Z_2, there is no $v \in \partial Z_2$ such that $v = \lambda F_5(v)$ for some $\lambda \in (0,1)$. Now, we may apply the non-linear alternative of Leray-Schauder [27], applying this, we deduce that F_5 has a fixed point *v* in Z_2. This fixed point is a solution of the BVP (3.12)–(3.14). This finishes the proof.

There is another existence consequence for the BVP (3.12)–(3.14) depend on the Burton and Kirk FPT [19].

Theorem 3.16 Let (A8), (A9**.**) and (A12) hold.

$$(k^{**}+k^{*})\frac{\left(\bar{S}^{2}+\bar{S}\right)}{\bar{S}+2}<1 \tag{3.64}$$

Since

$$\limsup_{u\to+\infty}\frac{\left(1-\frac{\left(\bar{S}^{2}+\bar{S}\right)(k^{*}+k^{**})}{\bar{S}+2}\right)u}{c\chi\psi(u)+\frac{\bar{S}\left(\bar{S}+1\right)(g^{*}+h^{*})}{\bar{S}+2}}>1 \tag{3.65}$$

where $g^{*}=\sup_{t\in J}|g(t,0)|$ and $h^{*}=\sup_{t\in J}|h(t,0)|$, so that BVP (3.12)-(3.14) has at least one solution on J .

Proof: Suppose the operators ξ_1,ξ_2 ;$C(J,\mathbb{R})\to C(J,\mathbb{R})$ given by

$$(\xi_1 v)(s)=\left(\frac{\bar{S}+1-s}{\bar{S}+2}\right)\int_0^{\bar{S}}g(t,v(t))\,dt+\frac{(s+1)}{\bar{S}+2}\int_0^{\bar{S}}h(t,v(t))\,dt$$

and

$$(\xi_2 v)(s)=\int_0^{\bar{S}}G(s,t)\,f(t,v(t))dt$$

G(s,t) is given as formula (3.50). From (3.62) it is shown that ξ_1 is a contraction mapping. the operator ξ_2 is continuous and completely continuous by (A12). By Theorem 2.3 the set D is bounded.

Assume $v\in D$, so that $\forall s\in J$,

$$v(s) = \lambda\xi_1\left(\frac{u}{\lambda}\right)(s) + \lambda\xi_2(u)(s).$$

In (A8), (A9), (A12) we get

$$|v(s)| \le \frac{\lambda(\bar{S}+1)}{\bar{S}+2}\int_0^{\bar{S}}\left|g\left(t,\frac{v(t)}{\lambda}\right)\right|dt + \frac{\lambda(\bar{S}+1)}{\bar{S}+2}\int_0^{\bar{S}}\left|h\left(t,\frac{v(t)}{\lambda}\right)\right|dt$$

$$+\lambda\int_0^{\bar{S}}|G(s,t)||f(t,v(t))|\,dt$$

$$\le \frac{(\bar{S}+1)}{\bar{S}+2}\int_0^{\bar{S}}k^*|v(t)|\,dt + \frac{(\bar{S}+1)}{\bar{S}+2}\int_0^{\bar{S}}|g(t,0)|\,dt$$

$$+\frac{(\bar{S}+1)}{\bar{S}+2}\int_0^{\bar{S}}k^{**}|v(t)|\,dt + \frac{(\bar{S}+1)}{\bar{S}+2}\int_0^{\bar{S}}|h(t,0)|\,dt$$

$$i\ +\chi\int_0^{\bar{S}}\varnothing_f(t)\psi(|v(t)|)\,dt$$

$$\le \frac{\bar{S}(\bar{S}+1)(k^*+k^{**})}{\bar{S}+2}\|y\|_\infty + \frac{\bar{S}(\bar{S}+1)(g^*+h^*)}{\bar{S}+2} + c\chi\psi(\|v\|_\infty)\ .$$

Thus,

$$\frac{\left(1-\frac{\bar{S}(\bar{S}+1)(k^*+k^{**})}{\bar{S}+2}\right)\|v\|_\infty}{c\chi\psi(\|v\|_\infty)+\frac{\bar{S}(\bar{S}+1)(g^*+h^*)}{\bar{S}+2}} \le 1. \tag{3.66}$$

In (3.65), $\exists R > 0$ such that $\forall v \in D$ and $\| v\|_\infty > \mathbb{R}$ by (3.64). Therefor $\| v\|_\infty \le R$, for all $v \in D.$, Clearly D is bounded.

Example 3.4 [1]: Assume the equation (3.20) with boundary conditions (3.67), (3.68)

$${}^cD^\alpha v(t) = \frac{e^{-s}|v(s)|}{(9+e^s)(1+|v(s)|)} \quad , \quad s\epsilon[0,1], \tag{3.20}$$

$$v(0)-v'(0)=\sum_{i=0}^{\infty}c_i v(s_i)\ , \tag{3.67}$$

$$v(1)+v'(1)=\sum_{j=0}^{\infty}d_j v(\tilde{s}_j)\ , \tag{3.68}$$

where

$0<s_0<s_1<s_2<\cdot\cdot\cdot<1,\quad 0<\tilde{s}_0<\tilde{s}_1<\tilde{s}_2<\cdot\cdot\cdot<1, c_i\ , d_j\ , i, j=0, \ldots,$ are given non-negative constants with

$$\sum_{i=0}^{\infty}c_i<\infty\ , \qquad\qquad \sum_{j=0}^{\infty}c_j<\infty$$

Where $\alpha\in(1,2]$, $0\le v, x<\infty$ and $s\in J$. So that

$$\left|f(s,x)-f(s,v)\right|\le \frac{1}{10}|x-v|\ .$$

Since The condition (A1) occur with $k=\frac{1}{10}$. We need to show the problem (3.54), it is applied with $\bar{S}=1, k^*=\sum_{i=0}^{\infty}c_i\ , k^{**}=\sum_{j=0}^{\infty}d_j$ and by (3.50) G is defined by

$$G(s,t)=\begin{cases}\dfrac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}-\dfrac{(1+s)(1-t)^{\alpha-1}}{3\Gamma(\alpha)}-\dfrac{(1+s)(1-t)^{\alpha-2}}{3\Gamma(\alpha-1)}\ , & 0\le t\le s\\ -\dfrac{(1+s)(1-t)^{\alpha-1}}{(\bar{S}+2)\Gamma(\alpha)}-\dfrac{(1+s)(1-t)^{\alpha-2}}{3\Gamma(\alpha-1)}\ , & s\le t\le 1\end{cases} \tag{3.69}$$

Additionally by (3.69),

$$\chi<\frac{4}{3\Gamma(\alpha)}+\frac{2}{3\Gamma(\alpha-1)},$$

then

$$\frac{(\bar{S}^2+\bar{S})}{\bar{S}+2}k^*+\frac{(\bar{S}^2+\bar{S})}{\bar{S}+2}k^{**}+\bar{S}kG^*$$

$$=\frac{2}{3}\left(\sum_{i=0}^{\infty}c_i+\sum_{j=0}^{\infty}d_j\right)+\frac{2}{15\Gamma(\alpha)}+\frac{1}{15\Gamma(\alpha-1)}<1. \tag{3.70}$$

It is applied for fit figures of $c_i,\ d_j$ and $\alpha \in (1,\ 2]$. Then by Theorem (3.13) the equation (3.20) with boundary conditions (3.67)–(3.68) has a unique solution on $[0,\ 1]$ for such figures of $\alpha \in (1,\ 2]$.

Remark 3.3: We select the constants c_i and d_j as

$$c_i = \frac{2}{15}\left(\frac{1}{3}\right)^i \quad , \qquad d_i = \frac{2}{45}\left(\frac{1}{3}\right)^j .$$

Therefore

$$\sum_{i=0}^{\infty} c_i = \frac{3}{15}, \qquad \sum_{j=0}^{\infty} d_j = \frac{1}{15} .$$

By (3.62),

$$\frac{2}{15\Gamma(\alpha)} + \frac{1}{15\Gamma(\alpha-1)} < \frac{37}{45}.$$

It is applied for $\forall\ \alpha \in (1,\ 2]$. By numerical calculations

$$\tilde{g}(\alpha) = \frac{2}{15\Gamma(\alpha)} + \frac{1}{15\Gamma(\alpha-1)} - \frac{37}{45}$$

takes negative values on the interval (1, 2].

Chapter 4

NEW CLASS OF FRACTIONAL BOUNDARY VALUE PROBLEM

4.1 Boundary Value Problem of Order α ∈ (0, 1]

Definition 4.1 [28,29]: For all $h \in L^1[a,b]$, $\alpha \in \mathbb{R}_+$

$$I_{a_+}^{\alpha} h(s) = \int_a^t \frac{h(t)}{\Gamma(\alpha)(s-t)^{1-\alpha}} dt \ .$$

Definition 4.2 [28,29]: Let h is a function given on the interval [a, b], the α th Riemann-Liouville fractional-order derivative of function is defined by

$$(D_{a+}^{\alpha} h)(s) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{ds}\right)^n \int_a^s \frac{h(t)}{(s-t)^{\alpha-n+1}} dt,$$

where n= $[\alpha]$ + 1 and $[\alpha]$ represents the integer parts of α.

Definition 4.3 [28]: Let h be a function given on the interval [a, b]. The Caputo fractional-order derivative of h, is defined by

$$({}^c D_{a+}^{\alpha} h)(s) = \int_a^s \frac{h(t)}{\Gamma(n-\alpha)(s-t)^{\alpha-n+1}} dt \ ,$$

where $n = \alpha + 1$.

Lemma 4.1: Assume that $\alpha > 0$, the differential equation ${}^c D_{a_+}^{\alpha} h(s) = 0$ has a solution

$h(s) = c_0 + c_1 s + c_2 s^2 + ... + c_{n-1} s^{n-1}$, $c_i \in \mathbb{R}$, i=0,1,2,…,n-1 ; n = 1+ α .

Lemma 4.2: Assume α > 0, so that

$$I^{\alpha}\left({}^{c}D_{a_+}^{\alpha}h\right)(s)=h(s)+\sum_{0}^{n-1}c_i s^i$$

For some $c_i \in \mathbb{R}$, i=0,1,2,…,n-1 ; n=[α]+1.

We consider $\left\{\begin{array}{l} {}^{c}D_{a_+}^{\alpha}v(s)=f(s,v(s)), \quad s\in[0,\bar{S}], 0\prec\alpha\prec 1 \\ av(0)+bv(\bar{S})=c, \quad \text{a+b} \neq 0, \\ \qquad\qquad\qquad \text{a,b,c} \in \mathbb{R} \end{array}\right\}$

Lemma 4.3: Let $0<\alpha<1$, $h\in C\left[0,\bar{S}\right]$. The solution of the Fractional Integral Equation is given as follows,

$$v(s)=v_0+\frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}h(t)\,dt,$$

if and only if v is a solution of the IVP.

$$\begin{array}{l} {}^{c}D_{0_+}^{\alpha}v(s)=h(s); \quad s\in[0,\bar{S}] \\ v(0)=v_0 \ . \end{array}$$

Lemma 4.4: Let $0<\alpha<1$, $h\in C\left[0,\bar{S}\right]$. The solution of the Fractional Differential Equation is given as follows,

$$v(s)=\frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}h(t)dt-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}h(t)dt-c\right],$$

if and only if v is a solution of the BVP.

$$\left\{\begin{array}{l} {}^{c}D_{0_+}^{\alpha}v(s)=h(s), \quad s\in[0,\bar{S}] \\ av(0)+bv(\bar{S})=c \end{array}\right\}$$

Proof: Let v be a solution of

$${}^{c}D_{0_+}^{\alpha}v(s)=h(s). \qquad\qquad \text{take } I_{0+}^{\alpha}$$

Then $$I_{0+}^{\alpha}\left({}^{c}D_{0_+}^{\alpha}v(s)\right)=I_{0+}^{\alpha}h(s)$$

$$\Rightarrow v(s)+k=I_{0+}^{\alpha}h(s) \qquad \text{where k is constant ;}$$

By Lemma 4.3,

$$v(s)+k=\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}h(t)\,ds$$

$$\Rightarrow v(s)=-k+\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}h(t)\,dt, \qquad k\in\mathbb{R} \tag{4.1}$$

We need to find k. By using $av(0)+bv(\bar{S})=c$, let's find $v(0)$ and $v(\bar{S})$ in such a way that

$$v(0)=(-k) \quad \text{and} \quad v(\bar{S})=-k+\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}h(t)\,dt\ .$$

Now, we have

$$a(-k)+b\left(-k+\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}h(t)\,dt\right)=c\ ,$$

$$-k(a+b)=c-\frac{b}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}h(t)\,dt$$

$$-k=\frac{c}{a+b}-\frac{b}{a+b}\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}h(t)\,dt \qquad \text{....(4.2)}$$

Substitute the value of k, to equation (4.1) and (4.2)

To get

$$v(s)=\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}h(t)dt-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}h(t)dt-c\right].$$

Theorem 4.1: Let

(H1) f: $[0,\bar{S}]\times\mathbb{R}\rightarrow\mathbb{R}$ is continuous.

(H2) $\exists$ L > 0 such that $\left| f(s,u) - f(s,v) \right| \le L \left| u - v \right|$; $s \in [0, \bar{S}]$; $u, v \in \mathbb{R}$.

(H3) Let $\frac{L\bar{S}^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) < 1$, so that the BVP has one solution in C[0,T] .

Proof: To start to prove the theorem we transform the problem (3.1)-(3.2) into a fixed point problem. To this end we introduce the following operator

$$F : C\,([0,\bar{S}\,],\mathbb{R}) \to C\,([0,\bar{S}\,],\mathbb{R}),$$

where F is defined by

$$\left(Fv\right)(s) = I_{0+}^{\alpha} f\left(s, v(s)\right) - \frac{b}{a+b} I_{0+}^{\alpha} f\left(\bar{S}, v(\bar{S})\right) + \frac{c}{a+b} \ .$$

If $v \in C[0,\bar{S}]$ then $Fv \in C[0,\bar{S}]$, then $F : C[0,\bar{S}] \to C[0,\bar{S}]$ is Banach space and complete. We need to show F is a contraction mapping, To show this, let $x, v \in C[0,\bar{S}]$. Then for every $s \in [0,\bar{S}]$ we have,

$$\left|\left(Fx\right)(s) - \left(Fv\right)(s)\right| = \left| I_{0+}^{\alpha} f\left(s, x(s)\right) - I_{0+}^{\alpha} f\left(s, v(s)\right) + \frac{b}{a+b} I_{0+}^{\alpha} f\left(\bar{S}, v(\bar{S})\right) - \frac{b}{a+b} I_{0+}^{\alpha} f\left(\bar{S}, x(\bar{S})\right) \right|$$

$$\le I_{0+}^{\alpha} \left| f\left(s, x(s)\right) - I_{0+}^{\alpha} f\left(s, v(s)\right) \right| + \frac{|b|}{|a+b|} I_{0+}^{\alpha} \left| f\left(\bar{S}, x(\bar{S})\right) - f\left(\bar{S}, v(\bar{S})\right) \right|$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} \left| f\left(t, x(t)\right) - f\left(t, v(t)\right) \right| dt + \frac{|b|}{|a+b|} \frac{1}{\Gamma(\alpha)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} \left| f\left(t, x(t)\right) - f\left(t, v(t)\right) \right| dt$$

By (H2),

$$\le \frac{L}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \left| x(t) - v(t) \right| dt + \frac{|b|}{|a+b|} \frac{L}{\Gamma(\alpha)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} \left| x(t) - v(t) \right| dt$$

$$\le \frac{L}{\Gamma(\alpha)} \left\| x - v \right\|_{\infty} \int_0^s (s-t)^{\alpha-1} dt + \frac{|b|}{|a+b|} \frac{L}{\Gamma(\alpha)} \left\| x - v \right\|_{\infty} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} dt$$

$$= \frac{L}{\alpha\Gamma(\alpha)} s^{\alpha} \left\| x - v \right\|_{\infty} + \frac{|b|}{|a+b|} \frac{L}{\alpha\Gamma(\alpha)} \bar{S}^{\alpha} \left\| x - v \right\|_{\infty} dt$$

$$\le \frac{L\bar{S}^{\alpha}}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \left\| x - v \right\|_{\infty} \ .$$

By (H3)

$$\frac{L\bar{S}^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)<1.$$

Then, $F: C[0,\bar{S}] \to C[0,\bar{S}]$ is a contraction mapping and by the Banach Fixed Point Theorem F has a one fixed point in $C\left[0,\bar{S}\right]$, which is a unique solution of BVP.

Theorem 4.2: Let

(H1) f: $[0,\bar{S}] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) $\exists$ $P > 0$ such that $|f(s,z) - f(s,v)| \le P|z-v|$; $s \in [0,\bar{S}]$; $z, v \in \mathbb{R}$.

(H3) $\frac{P\bar{S}^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)<1$

(H4) there exist M > 0 such that $|f(s,z)| \le M$, $\forall s \in [0,\bar{S}]$, $\forall z \in \mathbb{R}$.

Then there is one or more than one solution in $C[0,\bar{S}]$ for the BVP.

Proof: The proof is created on the Schauder Fixed Point Theorem, $F: X \to X$,

where X is Banach space, and

iii-F is continuous and compact function.

iv- $\varepsilon(F) = \{x \in X; x = \lambda Fx \text{ for some } \lambda \in [0,1]\}$ is bounded ,

Then F has at least one fixed point.

Step 1: F is continuous in C$[0,\bar{S}]$.

Step 2: F maps bounded sets into bounded set in $C\left[0,\bar{S}\right]$. i.e $F:[0,\bar{S}] \to [0,\bar{S}]$.

Step 3: F maps bounded sets into equicontinuous sets of $C[0,\bar{S}]$.

Step 4: $\varepsilon(F)$ is bounded.

Hint: Step 2 and 3 are Arzela Ascoli theorem then F is compact.

Step 1: Let $\{v_n\} \subset C[0,\bar{S}], v \in C[0,\bar{S}]$

$$\lim_{n\to\infty} \|v_n - v\|_\infty = 0 \Rightarrow \lim_{n\to\infty} \|Fv_n - Fv\|_\infty = 0.$$

Indeed

$$|F(v_n)(s) - F(v)(s)| \le \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} |f(t,v_n(t)) - f(t,v(t))| dt$$

$$+ \frac{|b|}{|a+b|} \frac{1}{\Gamma(\alpha)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} |f(t,v_n(t)) - f(t,v(t))| dt$$

$$\le \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} dt + \frac{|b|}{|a+b|} \frac{1}{\Gamma(\alpha)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} dt \right) \sup_{0\le t\le \bar{S}} |f(t,v_n(t)) - f(t,v(t))|$$

$$\le \left(\frac{\bar{S}^\alpha}{\alpha\Gamma(\alpha)} + \frac{|b|}{|a+b|} \frac{\bar{S}^\alpha}{\alpha\Gamma(\alpha)} \right) \sup_{0\le t\le \bar{S}} |f(t,v_n(t)) - f(t,v(t))|$$

Taking limit as $n \to \infty$, the above expression tends to zero and continuity of *f*,

Thus $\|Fv_n - Fv\|_\infty \to 0$ as n $\to \infty$ $\Rightarrow$ F is continuous.

Step 2: We need to show that for any $T > 0$, $\exists$ a non-negative constant P such that for all $v \in B(0,T) = \{v \in C[0,\bar{S}] : \|v\|_\infty \le P\}$, we have $\|F(v)\|_\infty \le P$, then

$$\|Fv(s)\| \le \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} |f(t,v(t))| dt + \frac{|b|}{|a+b|} \frac{1}{\Gamma(\alpha)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} |f(t,v(t))| dt + \frac{|c|}{|a+b|} .$$

By (H4)

$$\|Fv(s)\| \le \frac{M}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} dt + \frac{|b|}{|a+b|} \frac{M}{\Gamma(\alpha)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} dt + \frac{|c|}{|a+b|}$$

$$\le \frac{M}{\alpha\Gamma(\alpha)} \bar{S}^\alpha + \frac{|b|}{|a+b|} \frac{M}{\alpha\Gamma(\alpha)} \bar{S}^\alpha + \frac{|c|}{|a+b|} = P$$

That is $\|Fv\|_{\infty} \leq P.$

Step 3: Let $s_1, s_2 \in C[0,\bar{S}],\ \ s_1 < s_2\ ,\ v \in B(0,N)$. Then

$$|Fv(s_2) - Fv(s_1)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{s_2} (s_2 - t)^{\alpha-1} f(t, v(t))dt - \frac{1}{\Gamma(\alpha)} \int_0^{s_1} (s_1 - t)^{\alpha-1} f(t, v(t))dt \right|$$

$$\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{s_1} \left((s_2 - t)^{\alpha-1} - (s_1 - t)^{\alpha-1} \right) f(t, v(t))dt \right| + \left| \frac{1}{\Gamma(\alpha)} \int_{s_1}^{s_2} (s_2 - t)^{\alpha-1} f(t, v(t))dt \right|$$

$$\leq \frac{M}{\alpha\Gamma(\alpha)} \left((s_2 - s_1)^{\alpha} + (s_1{}^{\alpha} - s_2{}^{\alpha}) \right) + \frac{M}{\alpha\Gamma(\alpha)} (s_2 - s_1)^{\alpha}$$

$$= \frac{M}{\Gamma(\alpha+1)} (s_1{}^{\alpha} - s_2{}^{\alpha}) \to 0 \quad \text{as} \quad s_2 \to s_1.$$

Step 1, 2 and 3 and using the Arzela Ascoli theorem, we prove that F is continuous and compact.

Step 4: $\varepsilon(F) = \left\{ v \in C[0,\bar{S}] : v = \lambda F(v), 0 < \lambda < 1 \right\}$ is bounded.

Assume $v \in \varepsilon(F)$, then $v = \lambda F(v)$, we need to show that

$\exists L > 0$ such that $\|v\|_{\infty} \leq P.$

Indeed (see step 2)

$$\|v\|_{\infty} = \lambda \|F(v)\|_{\infty} \leq \|F(v)\|_{\infty} \leq P = \frac{M}{\alpha\Gamma(\alpha)} \bar{S}^{\alpha} + \frac{|b|}{|a+b|} \frac{M}{\alpha\Gamma(\alpha)} \bar{S}^{\alpha} + \frac{|c|}{|a+b|},$$

By the Schauder Fixed Point Theorem F has at least one fixed points in $C\left[0,\bar{S}\right]$.

Remark 4.1: Consider ${}^{c}D_{0+}^{\alpha} v(s) = f(s, v(s))$ Assume that $a + b \neq 0$

$$av(0) + bv(\bar{S}) = c.$$

1- *If* a=1 , b=0, then initial value problem (IVP)

2- *If* $a = 0,\ b = 1 \Rightarrow$ boundary value problem BVP

3- $If\ a=b=1,\ c=0 \Rightarrow v(0)=-v(\bar{S})$ is called anti periodic BVP

4- $If\ a=1, b=-1 \Rightarrow v(0)=v(\bar{S})$ is called periodic BVP.

4.2 Non-Linear Fractional Differential of Order α ∈ (1,2]

Throughout this section. We assume the following;

- ${}^{c}D^{\alpha}x(s)= f\left(s, x(s), {}^{c}D^{r}x(s)\right),\ \ 1<\alpha\le 2,\ \ 0\le \mathrm{r}\le 1,$

- $x(0)+M_1x(\bar{S})=\sigma_1; \quad 0<p<1$

- ${}^{c}D^{p}x(0)+M_2\ {}^{c}D^{p}x(\bar{S})=\sigma_2; \quad \mathrm{s}\in[0,\bar{S}]$ boundary condition (1)

- $f:[0,\bar{S}]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.

- $1+M_1\neq 0, M_2\neq 0, \quad {}^{c}D^{\alpha}$ is caputo derivative.

Lemma 4.5:

For each $g(s)\in C[0,\bar{S}],$ the unique solution of the linear FBVP.

${}^{c}D^{\alpha}x(s)=g(s)$

$x(0)+M_1x(\bar{S})=\sigma_1$ boundary condition (2)

${}^{c}D^{p}x(0)+M_2\ {}^{c}D^{p}x(\bar{S})=\sigma_2$

$x(s)=\int_0^{\bar{S}}G(s,t)g(t)dt+w_1+w_2s;$ where G(s,t) is the Green funtion.

$$G(s,t)=\begin{cases}\dfrac{(s-t)^{\alpha-1}-\dfrac{M_1}{1+M_1}\left(\bar{S}-t\right)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\Gamma(2-\alpha)\left[M_1(\bar{S}-t)-s\right]\left(\bar{S}-t\right)^{\alpha-p-1}}{\left(1+M_1\right)\bar{S}^{1+p}\ \Gamma(\alpha-p)};0\le t\le s\le\bar{S}\\ -\dfrac{M_1\left(\bar{S}-t\right)^{\alpha-1}}{(1+M_1)\Gamma(\alpha)}+\dfrac{\Gamma(2-\alpha)\left[M_1(\bar{S}-s)-s\right]\left(\bar{S}-t\right)^{\alpha-p-1}}{\left(1+M_1\right)\bar{S}^{1+p}\ \Gamma(\alpha-p)};0\le s\le t\le\bar{S}\end{cases},$$

$$w_1=\frac{M_2\sigma_1-M_1\sigma_2\bar{S}^{p}(2-p)}{M_2(1+M_1)}\ ; \qquad w_2=\frac{\sigma_2\bar{S}(2-p)}{M_2\bar{S}^{1-p}}.$$

Proof: We know that

$$ {}^{c}D^{\alpha}x(s)=g(s)\Rightarrow apply\ I_{0+}^{\alpha}\Rightarrow x(s)=I_{0+}^{\alpha}g(s)-b_1-b_2 s $$

$$ x(s)=\frac{1}{\Gamma(\alpha)}\int_0^s \frac{g(t)}{(s-t)^{1-\alpha}}dt-b_1-b_2 s. $$

To find b_1 and b_2 we use the boundary conditions in (2)

$$ {}^{c}D^{p}x(s)=I_{0+}^{\alpha-p}g(s)-b_2\frac{\bar{S}^{1-p}}{\Gamma(2-p)} \tag{4.3} $$

Since

$$ x(0)=-b_1, $$

and

$$ x(T)=I_{0+}^{\alpha}g(\bar{S})-b_1-b_2\bar{S}, $$

$$ {}^{c}D^{p}x(0)=0, $$

$$ {}^{c}D^{p}x(\bar{S})=I_{0+}^{\alpha-p}g(s)-b_2\frac{\bar{S}^{1-p}}{\Gamma(2-p)}, $$

Insert the above equation in (2)

$$ \left\{ \begin{array}{l} -b_1+M_1(I_{0+}^{\alpha}g(\bar{S})-b_1-b_2\bar{S})=\sigma_1 \\ M_2 I_{0+}^{\alpha-p}g(\bar{S})-M_2\sigma_2\dfrac{\bar{S}^{1-p}}{\Gamma(2-p)}=\sigma_2 \end{array} \right\} \tag{4.4} $$

By (4.3)

$$ b_2=\frac{\Gamma(2-p)}{M_2\bar{S}^{1-p}}(M_2 I_{0+}^{\alpha}g(\bar{S})-\sigma_2) $$

$$ \Rightarrow b_2=\frac{\Gamma(2-p)}{\bar{S}^{1-p}}\left(I^{\alpha-p}g(\bar{S})-\frac{\sigma_2}{M_2}\right). $$

And by (4.4)

$$b_1 = -\frac{b_2\bar{S}M_1 - M_1 I_{0+}^{\alpha} g(\bar{S}) + \sigma_1}{1+M_1}$$

$$\Rightarrow b_1 = -\frac{\bar{S}M_1}{1+M_1} b_2 + \frac{M_1}{1+M_1} I_{0+}^{\alpha} g(\bar{S}) - \frac{\sigma_1}{1+M_1}$$

$$\Rightarrow b_1 = -\frac{\bar{S}M_1}{1+M_1} \frac{\Gamma(2-p)}{\bar{S}^{1-p}} \left(I^{\alpha-p} g(\bar{S}) - \frac{\sigma_2}{M_2} \right) + \frac{M_1}{1+M_1} I_{0+}^{\alpha} g(\bar{S}) - \frac{\sigma_1}{1+M_1}.$$

Then

$$\begin{aligned} x(s) &= \frac{1}{\Gamma(\alpha)} \int_0^s \frac{g(t)}{(s-t)^{1-\alpha}} dt - b_1 - b_2 s \\ &= I_{0+}^{\alpha} g(s) + \frac{\bar{S}M_1}{1+M_1} \frac{\Gamma(2-p)}{\bar{S}^{1-p}} I^{\alpha-p} g(\bar{S}) - \frac{M_1\sigma_2}{M_2(1+M_1)} \bar{S}^{p} \Gamma(2-p) \\ &- \frac{M_1}{1+M_1} I_{0+}^{\alpha} g(\bar{S}) + \frac{\sigma_1}{1+M_1} - \frac{\Gamma(2-p)}{\bar{S}^{1-p}} \left(I_{0+}^{\alpha-p} g(\bar{S}) - \frac{\sigma_2}{M_2} \right) s\ . \end{aligned}$$

The Green function becomes

$$G(t,s) = \begin{bmatrix} \frac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1} - \frac{M_1}{1+M_1}\frac{1}{\Gamma(\alpha)}(\bar{S}-t)^{\alpha-1} + \frac{\bar{S}M_1}{1+M_1}\frac{\Gamma(2-p)}{\bar{S}^{1-p}}\frac{1}{\Gamma(\alpha-p)}(\bar{S}-t)^{\alpha-p-1} \\ -\frac{M_1}{1+M_1}\frac{s}{\bar{S}^{1-p}}(\bar{S}-s)^{\alpha-p-1}, t \le s \\ -\frac{M_1}{\Gamma(\alpha)(1+M_1)}(\bar{S}-t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1} - \frac{M_1}{1+M_1}\frac{1}{\Gamma(\alpha)}(\bar{S}-t)^{\alpha-1} + \\ \frac{\bar{S}M_1}{1+M_1}\frac{\Gamma(2-p)}{\bar{S}^{1-p}}\frac{1}{\Gamma(\alpha-p)}(\bar{S}-t)^{\alpha-p-1} - \frac{M_1}{1+M_1}\frac{s}{\bar{S}^{1-p}}(\bar{S}-t)^{\alpha-p-1}, s \le t \end{bmatrix}$$

We find out,

$$x(s) = \int_0^{\bar{S}} G(s,t) g(t) dt + w_1 + w_2 s, \qquad \text{where } w_1 \text{ and } w_2 \text{ are constant.}$$

Lemma 4.6: Existence and uniqueness.

Assume

$$0 < r \le 1; \quad C_r\left[0,\bar{S}\right] := \left\{ x \in C\left[0,\bar{S}\right] : \ {}^{c}D^{r} x \in C\left[0,\bar{S}\right] \right\}$$

$$\|x\|_r = \sup_{0\le t\le T} |x(s)| + \sup_{0\le s\le \bar{S}} |\,{}^cD^r x(s)|$$

$$\left(C_r\left[o,\bar{S}\right];\|.\|_r\right) \approx Banach\ space$$

$$(Fx)(s) := \int_0^s \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} f\left(t, x(t),\ {}^cD^r x(t)\right)dt$$

$$-\frac{M_1}{1+M_1}\int_0^{\bar{S}} \frac{\left(\bar{S}-t\right)^{\alpha-1}}{\Gamma(\alpha)} f\left(t, x(t),\ {}^cD^r x(t)\right)dt +$$

$$\frac{\Gamma(2-p)\left[M_1\left(\bar{S}-s\right)+s\right]}{(1+M_1)\,\bar{S}^{\,1-p}}\int_0^{\bar{S}} \frac{\left(\bar{S}-t\right)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\left(t, x(t),\ {}^cD^r x(t)\right)dt + w_1 + w_2 s.$$

Then there is one solution on $C_r[0,\bar{S}]$ for the BVP (1), If and only if

$F: C_r[0,\bar{S}] \to C_r[0,\bar{S}]$ has a unique fixed point. Let $\rho = \rho_1 + \rho_2$

$$\rho_1 = \frac{\bar{S}^{\,\alpha}(|M_1| + |1+M_1|)[\Gamma(\alpha-p+1) + \Gamma(\alpha+1)\Gamma(2-p)]}{(1+M_1)\Gamma(\alpha+1)\Gamma(\alpha-p+1)}$$

$$\rho_2 = \frac{\bar{S}^{\,\alpha-r}[\Gamma(\alpha-p+1)\Gamma(2-r) + \Gamma(\alpha-r+1)\Gamma(2-p)]}{\Gamma(\alpha-r+1)\Gamma(2-r)\Gamma(\alpha-p+1)}.$$

Theorem 4.3: Let

i) $f:[0,\bar{S}]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous.

ii) $|f(s,x,x') - f(s,v,v')| \le L(|x-v| + |x'-v'|), \quad \forall L\in[0,\bar{S}];\ x,v,x',v\in\mathbb{R}$.

iii) $L\rho < 1$, where ($\rho = \rho_1 + \rho_2$)

Then BVP (1) has a unique solution.

Proof: It is shown that $F: C_r[0,\bar{S}] \to C_r[0,\bar{S}]$. We need to show that

$\|Fx - Fv\|_r \le L\rho\|x-v\|_r$. Take derivative of (Fx)(s)

$$\Rightarrow {}^{c}D^{r}\left(Fx\right)(s)=\frac{1}{\Gamma(\alpha-r)}\int_{0}^{s}\frac{f(t,x(t),\ {}^{c}D^{r}x(t))}{(s-t)^{\alpha-r-1}}dt+w_{2}\frac{s^{1-r}}{\Gamma(2-r)}\ ,$$

then

$$\left\|Fx-Fv\right\|_{r}=\max_{0\le s\le \bar{S}}\left|(Fx)(s)-(Fv)(s)\right|+\max_{0\le s\le \bar{S}}\left|{}^{c}D^{r}(Fx)(s)-{}^{c}D^{r}(Fv)(s)\right|$$

$$\left|(Fx)(t)-(Fv)(t)\right|$$

$$\le\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{1}{(s-t)^{1-\alpha}}\left|f(t,x(t),\ {}^{c}D^{r}x(t))-f(t,v(t),\ {}^{c}D^{r}v(t))\right|dt$$

$$+\frac{|M_1|}{|1+M_1|}\frac{1}{\Gamma(\alpha)}\int_{0}^{\bar{S}}\frac{1}{(\bar{S}-t)^{1-\alpha}}\left|f(t,x(t),\ {}^{c}D^{r}x(t))-f(t,v(t),\ {}^{c}D^{r}v(t))\right|dt$$

$$+\frac{\Gamma(2-p)\left|M_1(\bar{S}-s)+s\right|}{|1+M_1|\bar{S}^{(1-p)}\Gamma(\alpha-p)}\int_{0}^{\bar{S}}\frac{1}{(\bar{S}-t)^{1-\alpha+p}}\left|f(t,x(t),\ {}^{c}D^{r}x(t))-f(t,v(t),\ {}^{c}D^{r}v(t))\right|dt$$

$$\le L\left\|x-v\right\|_{r}\left(\begin{array}{l}\frac{1}{\Gamma(\alpha)}\int_{0}^{s}\frac{1}{(s-t)^{1-\alpha}}dt+\frac{|M_1|}{|1+M_1|}\frac{1}{\Gamma(\alpha)}\int_{0}^{\bar{S}}\frac{1}{(\bar{S}-t)^{1-\alpha}}dt\\+\frac{\Gamma(2-p)\left|M_1(\bar{S}-s)+s\right|}{|1+M_1|\bar{S}^{(1-p)}\Gamma(\alpha-p)}\int_{0}^{\bar{S}}\frac{1}{(\bar{S}-t)^{1-\alpha+p}}dt\end{array}\right)$$

$$\le L\left\|x-v\right\|_{r}\left(\frac{\bar{S}^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{|M_1|}{|1+M_1|}\frac{\bar{S}^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{\Gamma(2-p)\left|M_1(\bar{S}-s)+s\right|\bar{S}^{\alpha-p}}{|1+M_1|\bar{S}^{(1-p)}\Gamma(\alpha-p+1)}\right)=L\rho_1\left\|x-v\right\|_{r}\quad(4.5)$$

In a like manner

$$\left|{}^{c}D^{r}(Fx)(s)-{}^{c}D^{r}(Fv)(s)\right|\le L\rho_2\left\|x-v\right\|_{r}\ .\qquad(4.6)$$

Now from (1) and (2) we get the following,

$$\Rightarrow\left\|Fx-Fv\right\|_{r}\le L(\rho_1+\rho_2)\left\|x-v\right\|_{r}$$

$$\Rightarrow\left\|Fx-Fv\right\|_{r}=\max_{0\le t\le T}\left|(Fx)(s)-(Fv)(s)\right|+\max_{0\le s\le \bar{S}}\left|{}^{c}D^{r}(Fx)(s)-{}^{c}D^{r}(Fv)(s)\right|$$

$$\le L\rho_1\left\|x-v\right\|_{r}+L\rho_2\left\|x-v\right\|_{r}=L(\rho_1+\rho_2)\left\|x-v\right\|_{r}=L\rho\left\|x-v\right\|_{r}$$

Thus F is a contraction.

Theorem 4.4: Let,

i- $f:\left[0,\bar{S}\right]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous.

ii- $\exists\ V(s)\in C\left(\left[0,\bar{S}\right];\mathbb{R}^{+}\right)$ such that

$\left|f(s,x,v)\right|\le V(s);\forall s\in\left[0,\bar{S}\right],\forall x,v\in\mathbb{R}$

Then there is at least one solution for the (BVP) (1).

Proof:

Step 1: $F:C_r\left[0,\bar{S}\right]\to C_r\left[0,\bar{S}\right]$ is continuous.

Step 2: F is compact ; for any bounded set $A\subset C_r\left[0,\bar{S}\right]$ $\overline{F(A)}$ is compact in

$C_r\left[0,\bar{S}\right]$.

Step 3: The set $y=\left\{x\in C_r\left[0,\bar{S}\right];x=\lambda F(x);0<\lambda<1\right\}$.

Now start by Step 1:

step 1: *{x_n}*, $x\in C_r\left[0,\bar{S}\right]$, assume that $\left\|x_n-x\right\|_r\to 0$ as $n\to\infty$

Define $\{x_n\}$

$\Rightarrow\left\|Fx_n-Fx\right\|_r\to 0$ as $n\to\infty$

$$\left|(Fx_n)(s)-(Fx)(s)\right|\le\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}\left|f(t,x_n(t),\ {}^{c}D^{r}x_n(t))-f(t,x(t),\ {}^{\alpha}D^{r}x(t))\right|dt$$

$$+\frac{\left|M_1\right|}{\left|1+M_1\right|}\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}}(\bar{S}-t)^{\alpha-1}\left|f(t,x_n(t),\ {}^{c}D^{r}x_n(t))-f(t,x(t),\ {}^{c}D^{r}x(t))\right|dt$$

$$+\left|M(s)\right|\frac{1}{\Gamma(\alpha-p)}\int_0^{\bar{S}}\frac{1}{(\bar{S}-t)^{1-\alpha+p}}\left|f(t,x(t),\ {}^{c}D^{r}x(t))-f(t,y(t),\ {}^{c}D^{r}y(t))\right|dt$$

where $M(t):=\dfrac{\Gamma(2-p)\left[M_1(\bar{S}-s)+s\right]}{\left(1+M_1\right)\bar{S}^{(1-p)}\Gamma(\alpha-p)}$

Take max $s=0; M(t)\le M(0)$,

F is continuous and $\lim\limits_{n\to\infty}\left\|x_n - x\right\|_r = 0$.

$\Rightarrow \lim\limits_{n\to\infty}\left\| f(s,x_n(s),\ {}^cD^r x_n(s)) - f(s,x(s),\ {}^cD^r x(s))\right\|_\infty = 0$ take max of x then $r\to\infty$

It follows that

$$\left|(Fx_n)(s)-(Fx)(s)\right| \le$$

$$\le \left\| f(s,x_n(s),\ {}^cD^r x_n(s)) - f(s,x(s),\ {}^cD^r x(s))\right\|_\infty \left(\begin{array}{l} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^s (s-t)^{\alpha-1}dt \\ +\dfrac{|M_1|}{|1+M_1|}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1}dt \\ +\dfrac{|M(s)|}{\Gamma(\alpha-p)}\displaystyle\int_0^{\bar{S}} \dfrac{1}{(\bar{S}-t)^{1-\alpha+p}}dt \end{array} \right)$$

$\Rightarrow \left\|Fx_n - Fx\right\|_\infty \to 0$ as $n\to\infty$

Similarly,

$\Rightarrow \left\|{}^cD^r Fx_n - {}^cD^r Fx\right\|_\infty \to 0$ as $n\to\infty$

$\Rightarrow \left\|Fx_n - Fx\right\|_r \to 0$ as $n\to\infty$

Step 1: $A\subset C_r[0,\bar{S}]$ is bounded $\left\{\begin{array}{l} 1-F(A) \text{ is bounded} \\ 2-F(A) \text{ is equicontinuous} \end{array}\right\}$

By Arzela Ascoli theorem

$\overline{F(A)}$ is compact (F(A) is relatively compact)

1- *F(A)* is bounded $\Leftrightarrow \exists M>0$ such that $\forall x\in A, \left\|Fx\right\|_r \le M$

$$\begin{aligned} \left|(Fx)(s)\right| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}\left| f(t,x(t),\ {}^cD^r x(t))\right|dt \\ &+\frac{|M_1|}{|1+M_1|}\frac{1}{\Gamma(\alpha)}\int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1}\left| f(t,x(t),\ {}^cD^r x(t))\right|dt \\ &+|M(s)|\frac{1}{\Gamma(\alpha-p)}\int_0^{\bar{S}} (\bar{S}-t)^{\alpha-p-1}\left| f(t,x(t),\ {}^cD^r x(t))\right|dt \end{aligned}$$

$$\leq \|V\|_\infty \left(\frac{\bar{S}^\alpha}{\Gamma(\alpha+1)} + \frac{|M_1|}{|1+M_1|} \frac{\bar{S}^\alpha}{\Gamma(\alpha+1)} + |M(0)| \frac{\bar{S}^{\alpha-p}}{\Gamma(\alpha-p+1)} \right) =: M_1.$$

$\forall x \in A,$

$\Rightarrow \|Fx\|_\infty \leq M_1$

Similarly,

$$\left|{}^c D^r (Fx)(s)\right| \leq \frac{1}{\Gamma(\alpha-r)} \|V\|_\infty \int_0^s (s-t)^{\alpha-r-1} dt + \left|{}^c D^r M(s)\right| \|V\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^{\bar{S}} (\bar{S}-t)^{\alpha-1} dt$$

$$+ \left| w_2 \frac{\bar{S}^{1-r}}{\Gamma(2-r)} \right| \leq M_2.$$

$\exists M_2 > 0$ such that $\forall x \in A$

$$\left\|{}^c D^r Fx\right\|_\infty \leq M_2 \qquad (2) \qquad \text{holds}$$

From (1) and (2) we get,

$\|Fx\|_r \leq M_1 + M_2 =: M$ then F is bounded.

2- F(A) is equicontinuous if and only if $\forall x \in A, \left((Fx)(s),\ {}^c D^r (Fx)(s)\right)$ are uniformly continuous.

Now, for $x \in A; s_1, s_2 \in [0, \bar{S}]$ and $s_1 < s_2$, we have

$$\left|(Fx)(t_2)-(Fx)(t_1)\right| = \left| \frac{1}{\Gamma(\alpha)}\int_0^{s_2}(s_2-t)^{\alpha-1} f(t,x(t),\ {}^cD^r x(t))dt \right.$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^{s_1}(s_1-t)^{\alpha-1} f\left(t,x(t),\ {}^cD^r x(t)\right)dt$$

$$\left. -\frac{\Gamma(2-p)(s_2-s_1)}{\bar{S}^{1-p}}\int_0^{\bar{S}}\frac{(\bar{S}-t)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\left(t,x(t),\ {}^cD^r x(t)\right)dt + w_2(s_2-s_1) \right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-t)^{\alpha-1}-(s_1-t)^{\alpha-1}\right]\left| f(t,x(t),\ {}^cD^r x(t))\right| dt$$

$$+\frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2-t)^{\alpha-1}\left| f\left(t,x(t),\ {}^cD^r x(t)\right)\right| dt$$

$$+\frac{\Gamma(2-p)}{\bar{S}^{1-p}}\left|s_2-s_1\right|\int_0^{\bar{S}}\frac{(\bar{S}-t)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left| f\left(t,x(t),\ {}^cD^r x(t)\right)\right| dt + w_2\left|s_2-s_1\right|$$

where $\int_0^{t_2} f \to \int_0^{t_1} f + \int_{t_1}^{t_2} f$, $0 \le t_1 \le t_2$.

$$\left|(Fx)(t_2)-(Fx)(t_1)\right| \le \|V\|_\infty \left(\frac{\left|s_2^\alpha - s_1^\alpha\right| + 2\left|s_2-s_1\right|^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-p)\left|s_2-s_1\right|\bar{S}^{\alpha-1}}{\Gamma(\alpha-p+1)} + \left|w_2\right|\left|s_2-s_1\right| \right) \tag{4.7}$$

Similarly

$$\left|{}^cD^r(Fx)(s_2) - {}^cD^r(Fx)(s_1)\right| \le \|V\|_\infty \left[\frac{\left|s_2^{\alpha-r}-s_1^{\alpha-r}\right| + 2\left|s_2-s_1\right|^{\alpha-r}}{\Gamma(\alpha-r+1)} + \frac{\Gamma(2-p)\left|s_2^{1-r}-s_1^{1-r}\right|\bar{S}^{\alpha-1}}{\Gamma(\alpha-p+1)} \right.$$
$$\left. + \frac{\Gamma(2-p)\left|s_2^{1-r}-s_1^{1-r}\right|\left|\sigma_1\right|}{\bar{S}^{1-p}\Gamma(2-r)\left|M_1\right|} \right] \tag{4.8}$$

Since the functions

$$s, s^\alpha, s^{\alpha-r}, s^{1-r} \ \ (1<\alpha\le 2,\ \alpha-r>0)$$

are uniformly continuous on $[0,\bar{S}\ ]$, from (1) and (2)

$\Rightarrow \left|Fx(s_2)-Fx(s_1)\right| \to 0$

$\Rightarrow \left|{}^cD^rFx(s_2) - {}^cD^rFx(s_1)\right| \to 0$ as $s_2 \to s_1$

Since, Via Arzela Ascoli Theorem the sets F(A) and ${}^{c}D^{r}F(A)=\left\{{}^{c}D^{r}Fx : x\in A\right\}$ are relatively compact in C[0, $\bar{S}$], then is F(A) is relatively compact in $C_r[0,\bar{S}]$.

Step 3: $V=\left\{x\in C_r\left[0,\bar{S}\right] : x=\lambda Fx, 0<\lambda<1\right\}$ is bounded $\forall x\in C_r[0,\bar{S}]$, we have

$$|x(s)|=\lambda|Fx(s)|<|Fx(s)|\le M_1$$

$$\left|{}^{c}D^{r}x(s)\right|=\lambda\left|{}^{c}D^{r}Fx(s)\right|<\left|{}^{c}D^{r}Fx(s)\right|\le M_2$$

$$\Rightarrow \|x\|_r \le M_1+M_2$$

$\Rightarrow V$ is bounded

Therefore, by Leray-Schauder Theorem *F* has at least one fixed point in $C_r[0,\bar{S}]$.

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