Integral Type Fractional Gronwall Inequalities

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We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

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ABSTRACT

The current research involves the ideas and principles about integral inequalities of Gronwall type. It deals with the possibilities that we mathematicians use in order to solve equations in various ways. The first case we adopted to solve equations is Linear Generalization. The latter deals with equations that are different from those treated with Non-Linear Generalization.

The research we conduct overlaps to study the relation between fractional and Gronwall inequalities by analyzing how Gronwall inequalities are included and used in fractional inequalities.

Keywords: Gronwall inequalities, Fractional inequalities, Linear generalizations and Non-Linear generalizations.

ÖZ

Mevcut araştırma Gronwall Çeşidi integral eşitsizlikler hakkında fikir ve ilkeleri içermektedir. Biz matematikçiler çeşitli şekillerde denklemleri çözmek için kullanmak olasılıklar ile ilgilenir. Biz denklemleri çözmek için kabul edilen ilk vaka Doğrusal Genelleme olduğunu. Doğrusal Olmayan Genelleme ile tedavi farklıdır denklemler ile ikinci fırsatlar.

Yaptığımız araştırmalar Gronwall eşitsizlikler dahil ve fraksiyonel eşitsizliklerin nasıl kullanıldığını analiz ederek fraksiyonel ve Gronwall eşitsizlikler arasındaki ilişkiyi incelemek için örtüşür.

Anahtar Kelimeler: Gronwall eşitsizlikler, Fraksiyonel eşitsizlikler, lineer genellemeler ve Doğrusal Olmayan genellemeler.

DEDICATION

To my family and my country

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Chapter 1

INTRODUCTION

Gronwall inequalities are an important tool in the study of existence, boundedness, uniqueness, stability, invariant manifolds and other qualitative properties of solution of differential equation and integral equation.

As R. Bellman pointed out in 1953 in his book “Stability Theory of Differential Equations”, McGraw Hill, New York, the Gronwall type integral inequalities of one variable for real functions play a very important role in the Qualitative Theory of Differential Equations. The main aim of the present thesis is to present (fractional) Gronwall inequality and some natural applications of (fractional) Gronwall inequalities to certain fractional integral equations. The work begins by presenting a number of classical facts in the domain of Gronwall type inequalities. We collected in a reorganized manner most of the above inequalities from the book “Inequalities for Functions and Their Integrals and Derivatives”, Kluwer Academic Publishers, 1994, by D.S. Mitrinovic, J.E. Pecaric and A.M. Fink. Chapter 2 contains some nonlinear generalization of the Gronwall inequalities. Chapter 3 contains some fractional generalization of the Gronwall inequalities. These results are then employed in this chapter to study some properties of fractional Volterra Integral Equations.

Chapter 2

LINEAR INEQUALITY

In the qualitative theory of differential and Volterra integral equations, the Gronwall type inequalities of one variable for the real functions play a very important role.

The first use of the Gronwall inequality to establish boundedness and stability is due to R. Bellman. For the ideas and the methods of R. Bellman, see [R. BELLMAN, Stability Theory of Differential Equations, McGraw Hill, New York, 1953.] where further references are given.

In 1919, T.H. Gronwall [T.H. GRONWALL, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. Math., 20(2) (1919), 293-296.] proved a remarkable inequality which has attracted and continues to attract considerable attention in the literature.

Also we will present some other inequalities of Gronwall type that are known in the literature and we will give various generalizations of Gronwall's inequality involving an unknown function of a single variable, by the recent reference [34].

Theorem 2.1(see [3]) Let ϕ, ψ and χ be a continuous mappings on $[\alpha,\beta]$ and $\chi(s)\geq 0$, $\forall s\in[\alpha,\beta]$.

Moreover, assume that

$$\varphi(s)\leq\psi(s)+\int_{\alpha}^{s}\chi(t)\varphi(t)\,dt \quad \forall s\in[\alpha,\beta] \tag{2.1}$$

Then

$$\varphi(t)\leq\psi(s)+\int_{\alpha}^{s}\chi(t)\psi(t)\exp\left[\int_{t}^{s}\chi(u)du\right]dt \quad \forall s\in[\alpha,\beta]. \tag{2.2}$$

Proof Assume that $y(s)=\int_{\alpha}^{s}\chi(u)\varphi(u)du,\ s\in[\alpha,\beta]$.

Then clearly $y(\alpha)=0$ and

$$y'(s)=\chi(s)\varphi(s).$$

From (2.1) we have

$$y'(s)\leq\chi(s)\left[\psi(s)+\int_{\alpha}^{s}\chi(t)\varphi(t)\,dt\right]$$

$$=\chi(s)\psi(s)+\chi(s)\int_{\alpha}^{s}\chi(t)\varphi(t)dt$$

$$=\chi(s)\psi(s)+\chi(s)y(s), \quad s\in(\alpha,\beta).$$

Multiply both sides with $\exp\left(-\int_{\alpha}^{s}\chi(t)dt>0\right)$, we get

$$y'(s)\exp\left(-\int_{\alpha}^{s}\chi(t)dt\right)\leq\chi(s)\psi(s)\exp\left(-\int_{\alpha}^{s}\chi(t)dt\right)+\chi(s)y(s)\exp\left(-\int_{\alpha}^{s}\chi(t)dt\right)$$

or

$$y'(s)\exp\left(-\int_{\alpha}^{s}\chi(t)dt\right)-\chi(s)y(s)\exp\left(-\int_{\alpha}^{s}\chi(t)\,dt\right)\leq\chi(s)\psi(s)\exp\left(-\int_{\alpha}^{s}\chi(t)dt\right)$$

and

$$\frac{d}{ds}\left[y(s)\exp\left(-\int_{\alpha}^{s}\chi(t)dt \right)\right] \le \chi(s)\psi(s)\,\exp\left(-\int_{\alpha}^{s}\chi(t)dt \right).$$

Integrating on $[\alpha, s]$, gives

$$y(s)\,\exp\left(-\int_{\alpha}^{s}\chi(t)dt \right) \le \int_{\alpha}^{s}\psi(u)\,\chi(u)\,\exp\left(-\int_{\alpha}^{u}\chi(t)dt \right)du.$$

Multiply both sides by $\left(\exp(\int_{\alpha}^{s}\chi(t)dt) \right)$, we get

$$y(s) \le \int_{\alpha}^{s}\psi(u)\,\chi(u)\,\exp\left(\int_{u}^{s}\chi(t)dt \right)du\,,\quad s\in[\alpha,\beta]\,.$$

Since $\phi(s) \le \psi(s) + y(s)$, then

$$\varphi(t) \le \psi(s) + \int_{\alpha}^{s}\chi(t)\psi(t)\exp\left[\int_{t}^{s}\chi(u)du \right]dt\,,$$

which completes the proof.

Corollary 1 Let ψ be differentiable, by the inequality (2.1),

$$\varphi(s) \le \psi(\alpha)\left(\int_{\alpha}^{s}\chi(u)\,du \right) + \int_{\alpha}^{s}\exp\left(\int_{t}^{s}\chi(u)du \right)\psi'(t)dt,\quad \forall s\in[\alpha,\beta]. \tag{2.3}$$

Proof: It is clear that,

$$-\int_{\alpha}^{s}\psi(t)\frac{d}{ds}\left(\exp\left(\int_{t}^{s}\chi(u)du \right)\right)dt$$

$$= -\psi(t)\exp\left(\int_{t}^{s}\chi(u)du \right)\Bigg|_{\alpha}^{\beta} + \int_{\alpha}^{s}\exp\left(\int_{t}^{s}\chi(u)du \right)\,\psi'(t)dt$$

$$= -\psi(s) + \psi(\alpha)\exp\left(\int_\alpha^s \chi(u)\,du\right) + \int_\alpha^s \exp\left(\int_t^s \chi(u)du\right)\psi'(t)dt,\ \forall s \in [\alpha,\beta].$$

Hence

$$\psi(s) + \int_\alpha^s \psi(u)\,\chi(u)\exp\left(\int_u^s \chi(t)dt\right)du$$

$$= \psi(\alpha)\exp\left(\int_\alpha^s \chi(u)du\right) + \int_\alpha^s \exp\left(\int_t^s \chi(u)du\right)\psi'(t)dt,\ s \in [\alpha,\beta].$$

Then we get the desired inequality.

Corollary 2 If $\psi \in \mathbb{R}$, then from

$$\varphi(s) \le \psi + \int_\alpha^s \chi(t)\varphi(t)dt, \tag{2.4}$$

it follows that

$$\varphi(s) \le \psi \exp\left(\int_\alpha^s \chi(u)du\right). \tag{2.5}$$

Theorem 2.2(see [7], [3]) Assume that $\varphi:[\alpha,\beta] \to \mathbb{R}_+$ is a continuous mapping, satisfying the following inequality:

$$\varphi(s) \le \mathrm{M} + \int_\alpha^s \psi(t)\omega(\varphi(t))dt, \qquad s \in [\alpha,\beta], \tag{2.6}$$

where $\mathrm{M} \ge 0$, $\psi:[\alpha,\beta] \to \mathbb{R}_+$ and $\omega:\mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions and ω is increasing. Then the inequality

$$\varphi(s) \le \mathrm{X}^{-1}\left(\mathrm{X}(\mathrm{M}) + \int_\alpha^s \psi(t)dt\right), \qquad s \in [\alpha,\beta] \tag{2.7}$$

holds, where $\mathrm{X}:\mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{X}(v)=\int_{v_0}^{v}\frac{dt}{\omega(t)}\ ,\qquad v\in\mathbb{R}. \tag{2.8}$$

Proof Let

$$y(s)=\int_{\alpha}^{s}\psi(t)\,\omega(\varphi(t))\,dt,\ \ s\in[\alpha,\beta],$$

clearly $y(\alpha)=0$ and from (2.6), we get

$$y'(s)\le\psi(s)\omega(\mathrm{M}+y(s)),\qquad s\in[\alpha,\beta].$$

Integrating both sides on $[\alpha,s]$, we get

$$\int_{0}^{y(s)}\frac{dt}{\omega(M+t)}\le\int_{\alpha}^{s}\psi(t)\,dt+\mathrm{X}(M),\quad s\in[\alpha,\beta]$$

that is,

$$\mathrm{X}(y(s)+M)\le\int_{\alpha}^{s}\psi(t)\,dt+\mathrm{X}(M),\ \ s\in[\alpha,\beta],$$

apply X^{-1} to both sides, we have

$$y(s)+M\le\mathrm{X}^{-1}\left(\int_{\alpha}^{s}\psi(t)\,dt+\mathrm{X}(M)\right),$$

or

$$y(s)\le\mathrm{X}^{-1}\left(\int_{\alpha}^{s}\psi(t)\,dt+\mathrm{X}(M)\right)-M,$$

sine $\varphi(s)\le\mathrm{M}+y(s)$, then we get the proof.

Theorem 2.3 (see [9])Assume $\varphi:[\alpha,\beta]\to\mathbb{R}$ be a continuous mapping and satisfies the inequality:

$$\frac{1}{2}\varphi^2(s)\le\frac{1}{2}\varphi_0^2+\int_{\alpha}^{s}\psi(t)\varphi(t)\,dt,\quad s\in[\alpha,\beta]\,, \tag{2.9}$$

Where $\phi_0 \in \mathbb{R}$ and $\psi \geq 0$ are continuous non-negative. Then the inequality

$$|\varphi(s)| \leq |\varphi_0| + \int_{\alpha}^{s} \psi(t)\,dt, \quad s \in [\alpha, \beta] \tag{2.10}$$

holds true.

Proof Let

$$y_\delta(s) = \frac{1}{2}\left(\varphi_0^2 + \delta^2\right) + \int_{\alpha}^{s} \psi(t)\varphi(t)\,dt, \;\; s \in [\alpha, \beta],$$

where $\delta > 0$. From (2.9), we have

$$\varphi^2(s) \leq y_\delta(s), \quad s \in [\alpha, \beta]. \tag{2.11}$$

Because $y'_\delta(s) = \psi(s)|\varphi(s)|,\; s \in [\alpha, \beta]$, we get

$$y'_\delta(s) \leq \sqrt{2y_\delta(\alpha)} + \int_{\alpha}^{s} \psi(t)\,dt, \;\; s \in [\alpha, \beta].$$

Integrating on $[\alpha, s]$, we can deduce that

$$\sqrt{2y_\delta(s)} \leq \sqrt{2y_\delta(\alpha)} + \int_{\alpha}^{s} \psi(t)\,dt, \;\; s \in [\alpha, \beta].$$

From (2.11), gets

$$|\varphi(s)| \leq |\varphi_0| + \delta + \int_{\alpha}^{s} \psi(t)\,dt, \; s \in [\alpha, \beta].$$

Hence $\forall \delta > 0$, (2.10) holds.

Theorem 2.4 (see [16]) Suppose that $\varphi(s) \geq 0$ is a continuous function such that

$$\varphi(s) \leq \alpha + \int_{s_0}^{s} \left[\beta + \lambda\varphi(t)\right] dt, \quad \text{for } \; s \geq s_0,$$

where $\alpha \geq 0,\; \beta \geq 0,\; \lambda \geq 0$. Then for $s \geq s_0$, $\varphi(s)$ satisfies

$$\varphi(s) \le \left(\frac{\beta}{\lambda}\right)\exp\left(\lambda\left(s-s_0\right)\right)-1)+\alpha\exp\lambda\left(s-s_0\right).$$

Theorem 2.5 (see [55) Assume $\varphi(s)$ be a continuous function and satisfy

$$\left|\varphi(s)\right| \le \left|(s_0)\right|\exp\left(-a\left(s-s_0\right)\right)+\int_{s_0}^{s}\left(\alpha\left|\varphi(t)\right|+\beta\right)\exp(-a(s-t))dt,$$

where $\alpha > 0,\ \beta > 0,\ a > 0,$ are any real numbers. Then

$$\left|\varphi(s)\right| \le \left|\varphi(s_0)\right|\exp\left(-\mathrm{\alpha}\left(s-\mathrm{s}_0\right)\right)+\beta\left(a-\alpha\right)^{-1}\left(1-\exp\left(-\left(a-\alpha\right)\left(\mathrm{s}-s_0\right)\right)\right).$$

Theorem 2.6(see [55]) Assume $\varphi(s)$ is a continuous mapping satisfying

$$\varphi(s) \le \varphi(T)+\int_{T}^{s}x(t)\varphi(t)\,dt,$$

$\forall\, s, T \in (\alpha,\beta)$, where $x(s) \ge 0$ and continuous, then

$$\varphi(s_0)\exp\left(-\int_{s_0}^{s}x(t)\,dt\right) \le \varphi(s) \le \varphi(s_0)\exp\left(\int_{s_0}^{s}x(t)\,dt\right), \quad \forall s \ge s_0.$$

Theorem 2.7 (see [11]) Assume $\varphi(s) \ge 0$ be a continuous on $[0, v]$, and satisfy the following inequality:

$$\varphi(s) \le x(s)+\int_{0}^{s}\left[x_1(t)\varphi(t)+y(t)\right]dt,$$

where $x_1(s) \ge 0$ and $y(s) \ge 0$ are integrable mapping on $\left[0, v\right]$ and $x(s)$ is a bounded there. Then, on $\left[0, v\right]$ we have

$$\varphi(s) \le \int_{0}^{s}y(t)\,dt+\sup_{0\le s\le h}\left|x(s)\right|\exp\left(\int_{0}^{s}x_1(t)\,dt\right).$$

Theorem 2.8(see [11]) Assume that $\varphi(s) \in C[0,\infty)$ and nonnegative such that

$$\varphi(s) \le \lambda s^{\alpha} + m s^{\beta} \int_0^s \frac{\varphi(t)}{t} dt ,$$

where $\lambda > 0,\ \alpha \ge 0,\ \beta \ge 0.$ Then

$$\varphi(s) \le \lambda s^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{m^n s^{n\beta}}{\alpha(\alpha+\beta)+\cdots+(\alpha+(n-1)\beta)}\right).$$

Theorem 2.9 (see [4]) Let φ and ψ be a continuous and x and y be a Riemann integrable mappings on $I=[a,b]$ with $y \ge 0$ and $\psi \ge 0$.

(i) If

$$\varphi(s) \le x(s) + y(s)\int_a^s \psi(t)\varphi(t)dt, \qquad s \in I, \tag{2.12}$$

then

$$\varphi(s) \le x(s) + y(s)\int_a^s x(t)\psi(t)\exp\left(\int_t^s y(\kappa)\psi(\kappa)d\kappa\right)dt, \quad s \in I. \tag{2.13}$$

Furthermore, equality holds in (2.13) for $I_1 = [a,b_1] \in I$ if equality holds in (2.12) for $s \in I_1$.

(ii) In both (2.12) and (2.13) the result remainders valid if $\le$ is changed by $\ge$.

(iii) Together (i) and (ii) still useable if $\int_a^s$ is changed by $\int_s^b$ and $\int_t^s$ by $\int_s^t$.

Proof Suppose that

$\mathrm{E}(s) = \int_a^s \psi(t)\varphi(t)dt$ such that $\mathrm{E}(a) = 0,$

where

$$\mathrm{E}'(s) = \psi(s)\varphi(s).$$

Since

$$E'(t) \le x(t)\psi(t) + y(t)\psi(t)E(t)\,.$$

Multiplying by $\exp\left(\int_t^s y(\kappa)\psi(\kappa)\,d\kappa\right)$ and integrating on (a,t) we get

$$E(s) \le \int_\alpha^s x(t)\psi(t)\,\exp\left(\int_s^t y(\kappa)\psi(\kappa)\,d\kappa\right)dt, \qquad s \in I. \tag{2.14}$$

Since $y \ge 0$, substituting of (2.14) into (2.12) leads to (2.13). The equality requirements are clear and proof of the equation (ii) can be written by transformation of variables s → −s.

Theorem 2.10 (see [17]) If

$$\varphi(s) \le x(s) + x_1(s)\int_{s_1}^{s} y_1(t)\varphi(t)\,dt + x_2(s)\sum_{n=2}^{m}\lambda_n\int_{s_1}^{s_n} y_n(t)\varphi(t)\,ds,$$

where $s \in [\alpha, \beta]$, $\alpha = s_0 < \bullet\ \bullet\ \bullet < s_n = \beta$, $\lambda_n \in \mathbb{R}$ and the generated functions are all continuous, nonnegative and real and if the following inequality holds

$$\sum_{n=2}^{m}\lambda_n\int_{s_1}^{s_n} y_n(s)\left[x_2(s) + x_1(s)\int_{s_1}^{s} y_1(t)x_2(t)\left(\int_{s_1}^{s} x_1(\kappa)y_1(\kappa)\,d\kappa\right)dt\right]ds < 1,$$

then

$$\varphi(s) \le K_1(s) + MK_2(s)\,,$$

where

$$K_1(s) = x(s) + x_1(s)\int_{s_1}^{s} y_n(t)x(t)\exp\left(\int_t^s x_1(\kappa)y_1(\kappa)\,d\kappa\right)dt,$$

$$K_2(s) = x_2(s) + x_1(s)\int_{s_1}^{s} y_1(t)x_2(t)\exp\left(\int_t^s x_1(\kappa)y_1(\kappa)\,d\kappa\right)dt,$$

and

$$M = \left(\sum_{n=2}^{m} \lambda_n \int_{s_1}^{s_n} y_n(t) K_1(t)\, dt \right) \left(1 - \sum_{n=2}^{m} \lambda_n \int_{s_1}^{s_n} y_n(t) K_2(t)\, dt \right)^{-1}.$$

Theorem 2.11 (see [35]) Let $\varphi(s)$ be continuous, real, and non-negative such that for $s > s_0$

$$\varphi(s) \le \lambda + \int_{s_0}^{s} \psi(s,t)\varphi(t)\, dt, \quad \lambda > 0$$

where $\psi(s,t)$ is a continuously differentiable function in s and continuous in t with $\psi(s,t) \ge 0$ for $s \ge t \ge s_0$. Then

$$\varphi(s) \le \lambda \exp\left(\int_{s_0}^{s} \left(\psi(t,t) + \int_{s_0}^{t} \frac{\partial \psi}{\partial t}(t,r)\, dr \right) dt \right).$$

Theorem 2.12 (see [17]) Suppose that $\varphi(s)$ be continuous, nonnegative and real on $[\alpha,\beta]$, such that

$$\varphi(s) \le x(s) + y(s) \int_{\alpha}^{s} \psi(s,t)\varphi(t)\, dt,$$

where $x(s) \ge 0,\ y(s) \ge 0,\ \psi(s,t) \ge 0$ and are continuous mappings for $\alpha \le t \le s \le \beta$

then

$$\varphi(s) \le X(s) \exp\left(Y(s) \int_{\alpha}^{s} \Psi(s,t)\, dt \right),$$

where $X(s) = \sup_{\alpha \le t \le s} x(t),\ Y(s) = \sup_{\alpha \le t \le s} y(t),\ \Psi(t,s) = \sup_{t \le \sigma \le s} \psi(s,\sigma).$

Theorem 2.13(see [14]) Assume $\varphi,\ x \in C[a,b]$ and let $I = [a,b]$ furthermore let ψ be a non-negative continuous function on $\tau : a \le t \le s \le b.$ If

$$\varphi(s) \le x(s) + \int_a^s \psi(s,t)\varphi(t)\,dt, \quad s \in I, \tag{2.15}$$

then

$$\varphi(s) \le x(s) + \int_a^s \mu(s,t)x(t)\,dt, \quad s \in I, \tag{2.16}$$

where $\mu(s,t) = \sum_{n=1}^{\infty} \Psi_n(s,t)$ with $(t,s) \in \tau,$ is the resolving kernel of $\psi(s,t)$ and $\Psi_n(s,t)$ are repeated kernels of $\psi(s,t)$.

Remark: If we take $\psi(s,t) = y(s)\psi(t)$ and $\psi(s,t) = \sum_{n=1}^{m} y_n(s)\psi_n(t)$ we have the results of D. Willett [14].

Chapter 3

NONLINEAR INEQUALITY

We can consider various nonlinear generalizations of Gronwall's inequality. The following theorem is proved in [43].

Theorem 3.1(see [43]) Assume $\varphi(s) \geq 0$ be a function satisfy

$$\varphi(s) \leq \lambda + \int_{s_0}^{s} \left(x(t)\varphi(t) + y(t)\varphi^{a}(t) \right) dt, \ a \geq 0, \ \lambda \geq 0, \tag{3.1}$$

where $x(s) \geq 0$ and $y(0) \geq 0$ are continuous mappings for $s \geq s_0$. When $0 \leq a < 1$ we have

$$\varphi(s) \leq \left\{ \lambda^{1-a} \exp\left((1-a) \int_{s_0}^{s} x(t) dt \right) + (1-a) \int_{s_0}^{s} y(t) \exp\left((1-a) \int_{t}^{s} x(\kappa) d\kappa \right) dt \right\}^{\frac{1}{1-a}}; \tag{3.2}$$

for $a = 1$

$$\varphi(s) \leq \lambda \exp\left\{ \int_{s_0}^{s} \left[x(t) + y(t) \right] dt \right\}, \tag{3.3}$$

and for $a > 1$ with the additional hypothesis

$$\lambda < \left\{ \exp\left((1-a) \int_{s_0}^{s_0+v} x(t) dt \right) \right\}^{\frac{1}{a-1}} \left\{ (a-1) \int_{s_0}^{s_0+v} y(t) dt \right\}^{\frac{-1}{a-1}} \tag{3.4}$$

we also get for $s_0 \leq s \leq s_0 + v$, for $v > 0$, we have

$$\phi(s) \le \lambda \left\{ \exp\left((1-a)\int_{s_0}^{s} x(t)\,dt \right) \right.$$
$$\left. -\lambda^{-1}(a-1)\int_{s_0}^{s} y(t)\exp\left((1-a)\int_{t}^{s} x(\kappa)\,d\kappa \right) dt \right\}^{\frac{1}{a-1}}. \qquad (3.5)$$

proof

If $a=1$ we have linear inequality so that (3.2) is valid.

Now let $0<a<1$. h is a solution of the integral equation

$$h(s)=\lambda+\int_{s_0}^{s}\left[x(t)h(t)+y(t)h^{a}(t)\right]dt,\ s\ge s_0.$$

In differential system this is the Bernoulli equation

$$h'(s)=x(s)h(s)+y(s)h^{a}(s),\ h(0)=\lambda.$$

This is linear in the variable h^{1-a} so can readily be integrated to create

$$h(s)=\left\{ \lambda^{1-a}\exp\left((1-a)\int_{s_0}^{s} x(t)dt \right) \right.$$
$$\left. +(1-a)\int_{s_0}^{s} y(t)\exp\left((1-a)\int_{t}^{s} x(\kappa)\,d\kappa \right) dt \right\}^{\frac{1}{1-a}}.$$

This equals the right side of the equation (3.2).

For $a>1$ the equation is an equation of Bernoulli type. For the proof we need the additional condition (3.4) if this condition is to holds on bounded interval $s_0 \le s \le s_0+v.$

Theorem 3.2 (see [17]) If

$$\phi(s)\le g(s)+\lambda\int_{0}^{s} u(t)\phi^{a}(t)\,dt,\quad 0<a<1,$$

where all mappings are non-negative and continuous on $\left[0,v\right]$, $\lambda \geq 0.$ Then

$$\phi(s) \leq g(s) + \lambda \delta_0^a \left(\int_0^s u^{\frac{1}{1-a}}(t)dt \right)^{1-a},$$

where δ_0 is the unique root of $\delta = a + \beta\delta^a$.

Theorem 3.3 (see [17]) Assume $\phi(s),\ u(s) \in C[0,v]$ be non-negative functions if

$$\phi(s) \leq \lambda_1 + \lambda_2 \int_0^s u(t)\phi^a(t)\,dt + \lambda_3 \int_0^v u(t)\phi^a(t)\,dt,$$

where $\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 > 0$, then for $0 < a < 1$ we have

$$\phi(s) \leq \left(\delta_0^{1-a} + \lambda_2 (1-a) \int_0^s u(t)\,dt \right)^{\frac{1}{1-a}}$$

where δ_0 is the uniquesolution of the equality

$$\left(\frac{\lambda_2 + \lambda_3}{\lambda_3} .\delta + \frac{\lambda_1 \lambda_2}{\lambda_3} \right)^{1-a} - \delta^{1-a} - \lambda_2 (1-a) \int_0^v u(t)\,dt = 0.$$

If $\lambda_2 (a-1) \int_0^s u(t)\,dt < \lambda_1^{1-a}$ and $a > 1$ there exists an interval $[0,\varepsilon] \subset [0,v]$ where

$$\phi(s) \leq \left(\lambda_1^{a-1} - \lambda_2 (a-1) \int_0^s u(t)\,dt \right)^{\frac{1}{(1-a)}}.$$

Theorem 3.4(see [48]) Assume $\varphi,\ x,\ y$ and ψ be non-negative and continuous of $I = [a,b]$ and let $\frac{x}{y}$ be a non-decreasing function. If

$$\varphi(s) \leq x(s) + y(s) \int_a^s \psi(t)\varphi^n(t)\,dt,\quad s \in I,\quad n \geq 2, \tag{3.6}$$

then

$$\varphi(s) \le x(s)\left\{1-(n-1)\int_a^s \psi(t)\,y(t)\,x^{n-1}(t)dt\right\}^{\frac{1}{1-n}},\ a \le s \le b_n, \quad (3.7)$$

where

$$b_n = \sup\left\{s \in I : (n-1)\int_a^s \psi y x^{n-1} dt < 1\right\}.$$

Theorem 3.5 (see [30]) Suppose that $\phi(s) \in C[a,b],\ \psi(s) \in C[a,b]$ are positive functions, $\alpha \ge 0,\ \beta \ge 0$ and $f(z) > 0$ be a non-decreasing mapping for $z \ge 0$. If

$$\phi(s) \le \alpha + \beta\int_a^s \psi(t)\, f(\phi(t))\,dt, \quad s \in [a,b],$$

then

$$\phi(s) \le F^{-1}\left(F(\alpha) + \beta\int_a^s \psi(t)\,dt\right),\ a \le s \le b_1 \le b,$$

where

$$F(\lambda) = \int_\delta^s \frac{dt}{f(t)} \quad (\delta > 0, \lambda > 0)$$

and b_1 is defined such that

$F(\alpha) + \beta\int_\alpha^s \psi(t)\,dt \in$ the domain of (F^{-1}) for $s \in [a, b_1]$.

Theorem 3.6 (see [22]) Assume $\phi(s) \ge 0,\ y(s) \ge 0$ be continuous functions on $[s_0, \infty)$. Moreover let $g(s),\ f(\phi)$ and $x(s)$ be differentiable mappings with g non-negative, $f > 0$ non-decreasing, and $fx \ge 0$ non-increasing. Let

$$\phi(s) \le g(s) + x(s)\int_{s_0}^{s} y(t)\, f(\phi(t))\,dt. \tag{3.8}$$

If

$$g'(\mathrm{s})\left(\frac{1}{f(\eta(s))} - 1\right) \le 0, \text{on } [s_0, \infty) \tag{3.9}$$

for each non-negative continuous mapping η, then

$$\phi(s) \le F^{-1}\left\{F(g(s_0)) + \int_{s_0}^{s}[y(t)x(t) + g'(s)]\,ds\right\}, \tag{3.10}$$

where

$$F(\varepsilon) = \int_{\delta}^{\varepsilon}\frac{dt}{f(t)}, \quad \delta > 0,\ \varepsilon > 0, \tag{3.11}$$

and (3.10) holds for all values of s which make the function

$$\varepsilon(s) = F[g(s_0)] + \int_{s_0}^{s}[y(t)x(t) + g'(t)]\,dt$$

belongs to the domain of the inverse mapping F^{-1}.

Proof Let

$$K(t) = g(s) + \int_{s_0}^{s} x(t)\,y(t)\, f(\phi(t))\,dt.$$

Since f is non-decreasing and x is non-increasing, from (3.8) we get that

$f(\phi(s)) \le f(K(s))$. As of this we get

$$g'(s) + x(s)y(s)\,f[\phi(s)] \le x(s)y(s)\,f[K(s)] + g'(s),$$

this may be written as

$$\frac{K'(s)}{f[K(s)]} \le x(s)y(s) + \frac{g'(s)}{f[K(s)]}.$$

By (3.9), we have,

$$\frac{K'(s)}{f\left[K(s)\right]} \le x(s)\,y(s)+g'(s)\ .$$

Integrating from both sides we get,

$$F\left[K(s)\right] \le F\left[g(s_0)\right]+\int_{s_0}^{s}\left[x(t)\,y(t)+g'(t)\right]dt.$$

If we assume that $\varepsilon(s) \in$ the domain of (F^{-1}), then we get the inequality (3.10) since $\phi(s) \le K(s)$.

Theorem 3.7 (see [2]) Assume that $\phi(s)$ is a continuous mapping on $[s_0,\tau]$ such that

$$0 \le \phi(s) \le g(s)+\int_{s_0}^{u(s)}\psi(s,t)\,f\left(\phi(t)\right)dt,$$

where

1) $g(s) \ge 0$ is continuous, and non-increasing;

2) $u(s)$ is differentiable, and $u'(s) \ge 0,\ u(s) \le s,\ u(s_0) = s_0$;

3) $f(\phi) > 0$ is non-decreasing on $\mathbb{R}$;

4) $\psi(s,t) \in C[s_0,\tau]\times[s_0,\tau]$ is non-negative with $\frac{\partial\psi}{\partial s}(s,t) \ge 0$ is continuous.

Then for F defined by (3.11) we get

$$\phi(s) \le g(s)-g(s_0)+F^{-1}\left\{F\left(g(s_0)\right)+\int_{s_0}^{s}G(t)\,dt\right\},$$

where

$$G(s)=\psi\left(s,u(s)\right)u'(s)+\int_{s_0}^{u(s)}\frac{\partial\psi}{\partial s}(s,t)\,dt.$$

Theorem 3.8(see [26]) Assume $\phi(s)$ and $x(s)$ be non-negative and continuous functions on $[\alpha,\beta]$, $f(\phi)>0$ is non-decreasing for $\phi>0$, and let for each t in $[\alpha,s]$

$$\phi(t)\leq\phi(s)+\int_t^s x(r)\,f\left(\phi(r)\right)dr.$$

Then for each s in $[\alpha,\beta]$ we have

$$\phi(s)\geq F^{-1}\left\{F\left(\phi(\alpha)\right)-\int_\alpha^s x(r)\,dr\right\},$$

where F is defined in (3.11) and let $\left\{F\left(\phi(\alpha)\right)-\int_\alpha^s x(r)\,dr\right\}\in$ the domain of (F^{-1}).

Proof Let

$$H(t)=\int_t^s x(r)\,f\left(\phi(r)\right)dr$$

then we have

$$\phi(t)\leq\phi(s)+H(t).$$

Since f is non-decreasing, we get

$$f\left[\phi(t)\right]\leq f\left[\phi(s)+H(t)\right],$$

this can be written as,

$$\frac{d\left(\phi(s)+H(t)\right)}{f\left[\phi(s)+H(t)\right]}\geq -x(t)\,dt.$$

By integrating from s to t $(t\leq s)$ we have,

$$-F\left(\phi(s)+H(t)\right)+F\left(\phi(s)\right)\geq-\int_t^s x(t)\,dt.$$

However, F is non-decreasing thus $F\left[\phi(t)\right]\leq F\left[\phi(s)+H(t)\right]$, combining the last two inequalities and reorganizing, we get

$$F\left(\phi(s)\right) > F\left(\phi(t)\right) - \int_t^s x(t)\,dt.$$

Apply F^{-1} to both sides we get the result.

Theorem 3.9 (see [2])Suppose that the positive functions $\phi(s)$ and $g(s)$ are continuous on $[s_0,\infty)$, moreover assume

$$\phi(s) < g(s) + \sum_{n=1}^{m} x_n(s) \int_{s_0}^{s} y_n(t)\, f[\phi(t)]\, dt, \quad s \ge s_0,$$

where $y_n(s) > 0$ be a continuous on $[s_0,\infty)$, $x_n(t) > 0$, while $x_n'(t) \ge 0$, and f is a non-decreasing function that satisfies $f(\omega) > \omega$, where $\omega > 0$.

Then

$$\phi(s) \le g(s) - \bar{g} + F^{-1}\left[F(\bar{g}) + \ln \prod_{n=1}^{m} x_n(s) - \ln \prod_{n=1}^{m} x_n(s_0) + \int_{s_0}^{s} \sum_{n=1}^{m} x_n(t) y_n(t)\,dt \right],$$

where $\bar{g} = \max g(s)$ and F is defined in (3.11).

Theorem 3.10(see [2])Suppose that $\phi(s) > 0$ is a continuous function and satisfy

$$\phi(s) \le g(s) + \int_{s_0}^{u_1(s)} x_1(t) G_1\left(\phi(t)\right) dt + \int_{s_0}^{u_2(s)} x_2(t) G_2\left(\phi(t)\right) dt,$$

with

1) $g(s)$ is a non-increasing mapping on $[s_0,\tau]$,

2) $x_1 \in C[s_0,\tau]$, $x_2 \in C[s_0,\tau]$ are nonnegative on $[s_0,\tau]$;

3) u_1 and u_2 are non-decreasing and continuously differentiable mapping with $u_n(s_0) = s_0$, $n = 1,2$, and $u_1(s) \le s$;

4) $G_1(\omega)$ and $G_2(\omega)$ are non-decreasing, continuous functions, and satisfy $G_2(\omega)>0$, for all ω and

$$\frac{d}{d\omega}\left[\frac{G_1(\omega)}{G_2(\omega)}\right]=\frac{\lambda}{G_2(\omega)},$$

where λ is any constant. Then

$$\phi(s)\le g(s)-g(s_0)+\omega(s),$$

where

$$\omega(s)=F^{-1}\left\{\exp\left(\lambda\int_{s_0}^{u_1(s)}x_1(t)\,dt\right)\left[F\left(g\left(s_0\right)\right)\right.\right.$$
$$\left.\left.+\lambda\int_{s_0}^{s}x_2\left(u_2(t)\right)u_2'(t)\exp\left(-\lambda\int_{s_0}^{u_1(t)}x_1(r)\,dr\right)dt\right]\right\},$$

is a continuous solution of the initial value problem

$$\omega'(s)=y_1\left(u_1(s)\right)u_1'(s)G_1(\omega)+y_2\left(u_2(s)\right)u_2'(s)G_2(\omega)$$
$$\omega(s_0)=g(s_0),$$

F^{-1} is the inverse of F, and

$$F(\omega)=\frac{G_1(\omega)}{G_2(\omega)}=F(\omega_0)+\int_{\omega_0}^{\omega}\frac{\lambda}{G_2(t)}\,dt.$$

Theorem 3.11(see [2])Assume $\phi(s)$ is a continuous function and satisfy

$$\phi(s)\le g(s)+\sum_{n=1}^{m}x_n(s)\int_{s_0}^{u_n(s)}y_n(t)g\left(\phi(t)\right)dt,\quad \text{on } [s_0,\tau],$$

with the following condition

1) $x_n\ge 0$ is bounded, non-increasing functions ;

2) $y_n\ge 0$ is continuous functions;

3) $u_n(s_0)=s_0,\ u_n(s)\le s,\ u_n'(s)>0$;

4) $g(s)$ is a non-increasing continuous function ;

5) $f(\omega)>0$ is a non-decreasing function defined on $\mathbb{R}$.

Then

$$\phi(s)\le g(s)-g(s_0)+\omega(s)$$

where F is defined by (3.11) and

$$\omega(s)=F^{-1}\left[F\left(g\left(s_0\right)\right)+\sum_{n=1}^{m}\int_{s_0}^{u_n(s)}x_n(t)\,y_n(t)\,dt\right],$$

is a continuous solution of the initial value problem $\omega(s_0)=g(s_0)$ and

$$\omega'(s)=\sum_{n=1}^{m}x_n\left(u_n(s)\right)y_n\left(u_n(s)\right)u_n'(s)\,f(\omega).$$

Theorem 3.12 (see [8])Assume that $\phi(s)>0,\ x(s)>0$ and $y(s)>0$ are bounded on $[a,b]$; $\psi(s,t)\ge 0$ is bounded for $a\le t\le s\le b$; $\phi(s)$ and $\psi(\bullet,t)$ are measurable functions. Let $g(\varphi)$ be strictly increasing and $f(\varphi)$ be non-decreasing. If $X(s)=\sup\limits_{a\le t\le s}x(t),\ Y(s)=\sup\limits_{a\le t\le s}y(t)$ and $\Psi(s,t)=\sup\limits_{t\le\sigma\le s}\psi(\sigma,t),$ then from

$$g\left(\phi(s)\right)\le x(s)+y(s)\int_a^s\psi(s,t)\,f\left(\phi(t)\right)dt,\quad s\in[a,b],$$

it follows that

$$\phi(s)\le g^{-1}\left[F^{-1}\left\{F\left(X(s)\right)+Y(s)\int_a^s\Psi(s,t)\,dt\right\}\right],\ s\in[a,b'],$$

where

$$F(\phi)=\int_\delta^\phi\frac{dz}{f\left(g^{-1}(z)\right)}\quad(\delta>0,\ \phi>0)\ ,$$

and

$$b' = \max\left\{a \le \kappa \le b:\ F\left(X(\kappa) + Y(\kappa)\int_a^{\kappa}\Psi(\kappa,t)\,dt \le F\left(g(\infty)\right)\right)\right\}.$$

Theorem 3.13 (see [22])Assume that the functions $g(s)$, $x(s)$, $y(s)$ and $f(\phi)$ satisfy the conditions of Theorem 3.6 and the function $G(\varphi) > 0$ is monotone decreasing for $\phi > 0$. Let

$$G(\phi(s)) \le g(s) + x(s)\int_{s_0}^{s} y(t)\, f(\phi(t))\,dt\,.$$

Then, on $[s_0, b']$

$$\phi(s) \le G^{-1}\left\{F^{-1}\left(F\left(g(s_0)\right) + \int_{s_0}^{s} x(t)\,y(t) + g'(t)\,dt\right)\right\},$$

where

$$F(\omega) = \int_{\delta}^{s} \frac{dt}{f\left[G^{-1}(t)\right]}, \qquad \omega > \delta \ge 0$$

and b' is defined such that the mapping $\varepsilon(s)$ obtained in Theorem 3.6 belongs to the domain of the mapping $G^{-1} \circ F^{-1}$.

Theorem 3.14 (see[50]) Assume $\varphi(s) \ge 0,\ x(s) \ge 0,\ y \ge 0$ and $\psi \ge 0$ be continuous functions on $I = [a,b]$, and

$$\varphi(s) \le x(s) + y(s)\left(\int_a^s \psi(t)\,\varphi^q(t)\,dt\right)^{\frac{1}{q}}, \quad s \in I, \quad 1 \le q < \infty\,.$$

Then

$$\varphi(s) \le x(s) + y(s)\frac{\left(\int_a^s \psi(t)z(t)x^q(t)dt\right)^{\frac{1}{q}}}{1-\left[1-z(s)\right]^{\frac{1}{q}}}, \quad s \in I,$$

and

$$z(s) = \exp\left(-\int_a^s \psi(t)x^q(t)dt\right).$$

Theorem 3.15(see [15], [3])Assume

1) $\phi(s) > 0,\ g(s) > 0$ and $G(s,t) > 0$ are continuous functions on $\mathbb{R}$, and $t \le s$;

2) $\dfrac{\partial G(s,t)}{\partial s} \ge 0$ is continuous;

3) $f(\phi) > 0$ is continuous, additive and non-decreasing on $(0, \infty)$;

4) $v(\omega)$ is a positive, non-decreasing and continuous function on $(0, \infty)$.

If

$$\phi(s) \le g(s) + v\left(\int_0^s G(s,t)\, f(\phi(t))dt\right),$$

for $s \in J$, then we have

$$\phi(s) \le g(s) + v\left\{F^{-1}\left(F\left[\int_0^s G(s,t)\, f(g(t))dt\right] + \int_0^s u(t)dt\right)\right\},$$

while

$$F(\phi) = \int_\delta^\phi \frac{dt}{f(v(t))}, \quad \phi > 0,\ \delta > 0,$$
$$u(s) = G(s,s) + \int_0^s \frac{\partial G}{\partial s}(s,t)dt,$$

and

$$J=\left\{ s\in(0,\infty):\,F(\infty)\ge F\left(\int_0^s G(s,t)\,f\left(g(t)\right)dt)\right)+\int_0^s u(t)\,dt\right\}.$$

Proof Since the function f is additive and $G(s,t)$ in S is non-decreasing we have

$$\phi(s)-g(s)\le v\left(h(s)\right),$$

where

$$h(s)=\int_0^s G(s,t)\,f\left(\phi(t)-g(t)\right)dt+\int_0^\tau G(\tau,t)\,f\left(g(t)\right)dt,$$

$s\in(0,\tau)$ and $\tau>0.$ Moreover since f is non-decreasing, we find that

$$f\left(\phi(s)-g(s)\right)\le f\left(v\left(h(s)\right)\right). \tag{3.12}$$

Multiplyingboth sides by $\dfrac{\partial G(s,t)}{\partial s}$ and integrating from 0 to s, we get

$$\int_0^s \frac{\partial G}{\partial s}(s,t)\,f\left(\phi(t)-g(t)\right)dt\le\int_0^s \frac{\partial G}{\partial s}(s,t)\,f\left(v\left(h(t)\right)\right)dt.$$

Conversely, if we multiply (3.12) by $G(s,s)$ and using this previous

inequality, we get

$$h'(s)\le G(s,s)\,f\left(v\left(h(s)\right)\right)+\int_0^s \frac{\partial G}{\partial s}(s,t)\,f\left(v\left(h(t)\right)\right)dt,$$

that is,

$$\frac{d}{ds}F\left(h(\tau)\right)\le G(s,s)+\int_0^s \frac{\partial G}{\partial s}(s,t)\,dt.$$

Then, by integrating from 0 to τ we have

$$F\left(h(\tau)\right)-F\left(h(0)\right)\le\int_0^\tau u(s)\,ds,$$

and since $\phi(\tau)-g(\tau)<v\left(h(\tau)\right)$ we have

$$\phi(\tau)-g(\tau)\le v\left\{F^{-1}\left(F\left[\int_0^{\tau}G(\tau,t)\,f\left(g(t)\right)dt\right]+\int_0^{\tau}u(s)\,ds\right)\right\}.$$

Since τ is arbitrary, we get the result.

Theorem 3.16 (see [15])Assume $I=(0,\infty)$ and

1) $\phi(s)>0,\ g(s)>0$ and $G(s)>0$ are continuous on I ;

2) $f(\phi)>0$ is additive, continuous and non-decreasing on I ;

3) $v(\omega)$ is continuous, positive and non-decreasing.

If

$$\phi(s)\le g(s)+v\left(\int_0^{s}G(t)\,f\left(\phi(t)\right)dt\right),\quad s\in I,$$

then for $s\in I_1$, we have

$$\phi(s)\le g(s)+v\left\{F^{-1}\left[F\left(\int_0^{s}G(t)\,f\left(g(t)\right)dt\right)+\int_0^{s}G(t)\,dt\right]\right\},$$

where F is well-defined such as in Theorem 3.15 and

$$I_1=\left\{s\in I:F(\infty)\ge F\left(\int_0^{s}G(t)\,f\left(g(t)\right)dt\right)+\int_0^{s}G(t)\,dt\right\}.$$

Theorem 3.17 (see [15])Assume $I=(0,\infty)$ and let

1) $\phi(s)>0,\ g(s)>0$ and $G(s)>0$ are continuous on I ;

2) $f(\phi)>0$ is additive, continuous and non-decreasing on I ;

3) $v(\omega)$ is continuous, positive and non-decreasing.

and assume $f(\varphi)$ be an even function on $\mathbb{R}$. If

$$\phi(s) \geq g(s) - v\left(\int_0^s G(t)\, f(\phi(t))\, dt\right), \quad s \in (0,\infty),$$

then for t ∈ I_1 we have

$$\phi(s) \geq g(s) - v\left\{F^{-1}\left(F\left[\int_0^s G(t)\, f(g(t))\, dt\right] + \int_0^s G(t)\, dt\right)\right\},$$

where

$$I_1 = \left\{s \in I : F(\infty) \geq F\left(\int_0^s G(t)\, f(g(t))\, dt\right) + \int_0^s G(t)\, dt\right\}.$$

Theorem 3.18 (see [13]) Assume $\varphi \geq 0,\ x \geq o,\ \psi \geq 0$ and $\psi_1 \geq 0$ be continuous functions on $I = [a,b]$, and let $x(s)$ be non-decreasing on I. Suppose $f > 0$ and v are non-decreasingcontinuous functions on [0, ∞) such that f is sub-additive and sub-multiplicative on [0, ∞), moreover assume that $v(\phi)$ is positive for $\phi > 0$. Let $g \in C[0,\infty)$ be a strictly increasing function with $g(\phi) \geq \phi$ for $\phi \geq 0$ and $\mathrm{g}(0) = 0.$

If

$$g(\varphi(s)) \leq x(s) + v\left(\int_a^s \psi(t)\, f(\varphi(t))\, dt\right) + \int_a^s \psi_1(t)\varphi(t)\, dt, \quad s \in I,$$

then

$$\begin{aligned}\varphi(s) \leq (g^{-1} \circ G^{-1})\Bigg\{&\int_a^s \psi_1(t) dt \\ &+ G\Bigg(x(s) + (v \circ F^{-1})\Bigg\{\int_a^s \psi(t)\, f(Z(t))\, dt \\ &+ F\left(\int_a^s \psi(t)\, f(x(t)Z(t))\, dt\right)\Bigg\}\Bigg)\Bigg\}, \quad \text{for } a \leq s \leq b_1,\end{aligned}$$

where

$$Z(s)=\exp\left(\int_a^s \psi_1(t)dt\right), \; G(\phi)=\int_{y_0}^{\phi}\frac{dy}{g^{-1}(y)}, \quad y_0>0, \; (y>0)$$

and

$$F(\phi)=\int_{\phi_0}^{\phi}\frac{dy}{f(v(y))}, \; \phi>0, \; (\phi_0>0)$$

while

$$b_1=\sup\left\{s\in I: F\left(\int_a^s \psi(t)\, f\left(x(t)Z(t)\right)dt\right) + \int_a^s \psi(t)\, f\left(Z(t)\right)dt \in F\left(\mathbb{R}^+\right)\right\}.$$

If $x(s)=s$ we may drop the condition that the function f is sub-additive and considering $a\le s\le b_2$ leads us to

$$\varphi(s)\le (g^{-1}\circ G^{-1})\left\{\int_a^s \psi_1(t)dt + G\left(x+(v\circ F_x^{-1})\left[\int_a^s \psi(t)\, f\left(Z(t)\right)dt\right]\right)\right\},$$

where

$$F_x(\phi)=\int_0^{\phi}\frac{dy}{f\left(\left|x+v(y)\right|\right)}, \quad \phi>0$$

and

$$b_2=\sup\left\{t\in I: \int_a^s \psi(t)\, f\left(Z(t)\right)dt \in F_x\left(\mathbb{R}^+\right)\right\}.$$

Theorem 3.19(see [34]) Let $u(s,\phi)$ be continuous and non-decreasing in ϕ on $[0,\tau]\times(-\varepsilon,\varepsilon)$ where $\varepsilon\le\infty$. If $h(s)$ is continuous and satisfies

$$h(s)\le \phi_0+\int_0^{\tau} u\left(s,h(t)\right)dt,$$

where $\phi_0 \in \mathbb{R}$ be any a constant, then

$$h(s) \le \phi(s)$$

where $\phi(s)$ is the maximal solution of the problem

$$\phi'(s) = u(s,\phi),\ \ \phi(0) = \phi_0,$$

defined on $[0,\tau]$.

Proof Consider the function

$$\omega(s) = \phi_0 + \int_0^s u(t,h(t))\,dt,$$

then $h(s) \le \omega(s)$ and

$$\omega'(s) = u(s,h(s)) \le u(s,\omega(s)),\ \text{ with }\ \omega(0) = \phi_0.$$

From Theorem 2 of Chapter XI, [34] we have $\omega(s) \le \phi(s)$ then we get the proof.

Theorem 3.20(see [34]) Assume $\phi_0(s) \in C[0,\tau]$. Suppose $u(s,t,\phi)$ be continuous and non-decreasing in ϕ for $0 \le s,t \le \tau$ and $|\phi| \le \varepsilon$. If $h(s)$ is considered to be a continuous mapping which satisfies the following inequality $(\text{on}[0,\ \tau])$

$$h(s) < \phi_0(s) + \int_0^s u(s,t,h(t))\,dt \tag{3.13}$$

then

$$h(s) < \phi(s)\ \text{ on}[0,\tau], \tag{3.14}$$

where $\phi(s)$ is a solution of the equation

$$\phi(s) = \phi_0(s) + \int_0^s u(s,t,\phi(t))\,dt\ \text{ on } [0,\ \tau]. \tag{3.15}$$

Proof

From (3.13) and (3.15) we get (3.14) at $s = 0$. Based on the continuity of the mappings used, we have (3.14) holding on a number of nontrivial interval. In case the last deduction is not holding on the interval $[0,\tau]$ then there is s_0 such that $h(s) < \phi(s)$ on $[0, s_0)$ however $h(s_0) = \phi(s_0)$. From (3.13) and (3.15) we get

$$h(s_0) = \phi_0(s_0) + \int_0^s u(s_0, t, h(t))\,dt$$

$$\leq \phi_0(s_0) + \int_0^s u(s_0, t, \phi(t))\,dt = \phi(s_o).$$

This contradiction proves the theorem.

In what following we say that the mapping $u(s,t,\phi)$ is a solution of the condition (?) if the equation

$$w(s) = \phi_0(s_0) + c + \int_0^s u(s,t,w(t))\,dt$$

has a solution defined on $[0,\tau]$, $\forall c \in [0, \mu]$.

Theorem 3.21(see [34]) Assume that $u(s,t,\phi)$ is defined for $0 \leq s,t \leq \tau$, $|\phi| \leq \varepsilon$, and is continuous and non-decreasing in ϕ satisfying condition (μ). If the continuous mapping $h(s)$ satisfies

$$h(s) \leq \phi_0(s_0) + \int_0^s u(s,t,h(t))\,dt \tag{3.16}$$

on $[0,\tau]$, then

$$h(s) \leq \phi(s) \text{ on } [0,\tau],$$

where $\phi(s)$ satiates (3.15) on the same interval

Proof For all fixed m, we denote by $w_m(s)$ a solution of the integral equation

$$w_m(s) = \frac{\delta}{m} + \phi_0(s) + \int_0^s u(s,t,w_m(t))\,dt$$

defined on $[o,\tau]$ for δ small enough, we may employ Theorem 3.20 to arrange that

$$\phi(s) < w_{m+1}(s) < w_m(s) < w_1(s)$$

in addition to $h(s) < w_m(s)$.Letting m approaches ∞, we get the result.

Theorem 3.22(see [33]) Assume $u(s,t,\phi)$ be continuous and non-decreasing function in ϕ for $0 \le s,\ t \le \tau,\ |\phi| \le \varepsilon.$ Let $\phi_0(s) \in C[0,\tau]$ and either

1) Considering any continuous function $\omega_0(s)$ which is fixed over $|\phi| \le \varepsilon$ on $[o,\tau]$ and any $c > 0$ which is small enough, the equation

$$w(s) = c + \phi_0(s) + \int_0^s u_1(s,t,w(t))\,dt + \int_0^\tau u(s,t,w_0(t))\,dt$$

has a continuous solution on $[o,\tau]$; or

2)

$$|\phi(s)| + \int_0^\tau \max_{0\le s\le\tau} |u_1(s,t,\varepsilon) + u_2(s,t,\varepsilon)|\,dt \le \varepsilon.$$

Moreover if $h(s)$ satisfies

$$h(s) \le \phi_0(s) + \int_0^s u_1(s,t,h(t))\,dt + \int_0^\tau u_2(s,t,h(t))\,dt,$$

where $h(s)$ is a continuous function, then

$$h(s) < \phi(s) \quad \text{on } [0,\tau],$$

where

$$\phi(s)=\phi_0(s)+\int_0^s u_1(s,t,\phi(t))\,dt+\int_0^\tau u_2(s,t,\phi(t))\,dt.$$

Theorem 3.23(see [32]) Assume $u(s,t,\phi)\in C[0,\infty)$ for $t\in[0,\infty)$ and ϕ with $|\phi|<\infty$. Let for fixed s and $\forall\ \phi(t)\in C[0,\infty)$ the function $u(s,t,\phi(t))$ is measurable in t on $[0,\infty)$. Additionally, suppose u be a non-decreasing in ϕ and $\phi_0(s)\in C[0,\infty]$. If

$$h(s)<\phi_0(s)+\int_s^\infty u(s,t,h(t))\,dt \quad \text{on } [0,\infty), \tag{3.17}$$

while $h(s)$ is any continuous function, then

$$h(s)<\phi(s) \quad \text{on } [0,\infty), \tag{3.18}$$

where $\phi(s)$ is a solution of the equation

$$\phi(s)=\phi_0(s)+\int_s^\infty u(s,t,\phi(t))\,dt \ \text{ on } [0,\infty).$$

Theorem 3.24(see [32]) Suppose that $g(\varphi)\in C(J)$ is strictly monotone functionon an interval J, and let the function $R(s,h)$ be continuous on $I\times K$ where $I=[a,b]$ and K is an interval containing zero, and furthermore assume that the mapping R is monotone with the respect to the variable h.Let $\tau_1=\{(s,t):a\le t\le s\le b\}$ and assume that $w(s,t,\phi)$ is continuous and either positive or negative on $\tau_1\times J$, monotone in the variable ϕ, and monotone in the variable s. Letalso that the mapping φ and the mapping χ are all continuous on I with $\varphi(I)\subset J$ and

$$x(s)+R(s,h)\in g(J) \quad \textit{for } s\in I \ \textit{and } |h|\le\beta, \tag{3.19}$$

where $\beta>0$ is constant. Assume

$$g(\varphi(t)) \le x(t) + R\left(s, \int_a^s w(s,t,\varphi(t))\,dt\right), \qquad s \in I, \tag{3.20}$$

and assume $\kappa = \kappa(s,\tau,a)$ is the maximal (minimal) solution of the initial value problem

$$\begin{aligned} &\kappa' = w\left(\tau, s, g^{-1}\left[x(s) + R(s,\kappa)\right]\right), \\ &\kappa(a) = 0, \;\; a \le s \le \tau \le b_1 \;\; (b_1 < b), \end{aligned} \tag{3.21}$$

if the functions $w(s,t,\bullet)$ and g are monotonicin the same sense, where $b_1 > x$ is chosen such a way that the maximal (minimal) solution can be computed in the given interval. Then, if the function $w(\bullet,t,\phi)$ and $R(s,\bullet)$ are monotonic in the same sense,

$$\varphi(s) \le (\ge)\, g^{-1}\left[x(s) + R(s,\tilde{\kappa}(s))\right], \;\; a \le s \le b_1, \tag{3.22}$$

where $\tilde{\kappa}(s) = \kappa(s,s,a)$ if

1) $R(s,\bullet)$ and $w(s,t,\bullet)$ are monotonic in the same sense and g is increasing; if

2) g is decreasing and $R(s,\bullet)$, $w(s,t,\bullet)$ are monotonic in the opposite sense, then the previous inequality is reversed in (3.22).

ProofThe mapping

$$G(\tau,s,\kappa) = w\left(\tau, s, g^{-1}\left[x(s) + R(s,\kappa)\right]\right)$$

is continuous on the compact set $\tau_1 \times [-\beta, \beta]$, so it is bounded there, say by the constant N. By [34]there exists a, independent of s, such that $a < b$ (indeed $b_1 > a + \min\ (b - a, \beta N^{-1})$) such that the maximal(minimal) solution of the initial value problem (3.21) has a solution on $[a,b]$. If $\tau \in (a,b]$ is fixed, and assume $s \in [a,\tau]$. We define

$$h(s,\phi)=\int_a^s w(\phi,t,\varphi(t))\,dt$$

we have

$$h(s,s)=\int_a^s w(s,t,\varphi(t))\,dt \le (\ge) \int_a^s w(\tau,t,\varphi(t))\,dt = h(s,\tau) \tag{3.23}$$

if $w(\bullet,t,\phi)$ is increasing (decreasing). Note that (3.20) implies that $h(s,s)\in K$ for $s\in I$. Since $0\in K$, it follows that $h(s,\tau)\in K$ in both sense of (3.23).

From (3.20) we get

$$\varphi(s)\le(\ge)\,g^{-1}\left[x(s)+R(s,h(s,s))\right], \tag{3.24}$$

if g is increasing (decreasing). As $h'(s,\tau)=w(\tau,s,\varphi(s))$ for $a\le s\le\tau\le b$, we have

$$h'(s,\tau)\le(\ge)\,w\left(\tau,s,g^{-1}\left[x(s)+R(s,h(s,s))\right]\right), \tag{3.25}$$

if $w(s,t,\bullet)$ and g are monotone in the same (opposite) sense. In additional, using (3.23) leads us to

$$R(s,h(s,s))\le(\ge)\,R(s,h(s,\tau)),\quad a\le s\le\tau, \tag{3.26}$$

if (i) $R(s,\bullet)$ and $w(s,t,\bullet)$ are monotonic in the same ((ii)opposite) sense. Therefore

$$g^{-1}\left[x(s)+R(s,h(s,s))\right]\le(\ge)\,g^{-1}\left[x(s)+R(s,h(s,\tau))\right],$$

on $a\le s\le\tau$ if (i’): g is an increasing function and (i) or g is a decreasing function and (ii) ((ii’) g is an increasing function and (ii) or g is a decreasing function and (i)). Then this implies that

$$\begin{aligned} &w\left(\tau,s,g^{-1}\left[x(s)+R(s,h(s,s))\right]\right)\\ &\qquad\le(\ge)\,w\left(\tau,s,g^{-1}\left[x(s)+R(s,h(s,\tau))\right]\right), \end{aligned} \tag{3.27}$$

if (i”): $w(s,t,\bullet)$ is an increasing function and (i’) or $w(s,t,\bullet)$ is a decreasing function and (ii’) ((ii”): $w(s,t,\bullet)$ is an increasing function and (ii’) or $w(s,t,\bullet)$ is a decreasing function and (i’)). Joining this and (3.25), gives us that to, if $w(s,t,\bullet)$ and $R(s,\bullet)$ are monotonic in the same sense, then

$$h'(s,\tau)\le(\ge)w\left(\tau,s,g^{-1}\left[x(s)+R\left(s,h(s,\tau)\right)\right]\right),\ a\le s\le\tau<b, \quad (3.28)$$

If $w(s,t,\bullet)$ and g are monotone with same (opposite) sense.

Since $h(a,\tau)=0$, and from [34] leads us to, if $w(s,t,\bullet)$ and $R(s,\bullet)$ are monotonicin the same sense and if $\kappa(s,\tau,a)$ is the extreme solution of (3.19) as mentioned, then

$$h(s,\tau)\le(\ge)\kappa(s,\tau,a) \text{ for } a\le s\le T\le b_1,$$

which we get in particular that this hold when $s=\tau$. Since τ is a randomly chosen element from the interval $(a,b_1]$, it follows that

$$h(s,s)\le(\ge)\tilde{\kappa}(s) \text{ on } [a,b_1] \quad (3.29)$$

provided that (I): $w(s,t,\bullet)$ and g are monotonicin the same sense ((II): $w(s,t,\bullet)$ and g are monotonicin the opposite sense).

From (3.23) and (3.26), it follows on $[a,b_1]$ that

$$R\left(s,h(s,s)\right)\le(\ge)R\left(s,\tilde{\kappa}(s)\right)$$

if (I’): $R(s,\bullet)$ is an increasing function and (I) or $R(s,\bullet)$ is a decreasing function and (II), ((II’): $R(s,\bullet)$ is an increasing function and (II) or $R(s,\bullet)$ is a decreasing function and (I)). Now, if (I”): g is an increasing function and (I’) or g is a decreasing function and (II’) ((II”): g is an increasing function and (I’) or g is a decreasing function and (II') ((II"): g is an increasing function and (II') or g is a decreasing function and (I')) then

$$g^{-1}\left[x(s)+R(s,h(s,s))\right]\leq(\geq)g^{-1}\left[x(s)+R(s,\tilde{\kappa}(s))\right].$$

From the different cases, we can conclude that if $w(\bullet,t,\phi)$ and $R(s,\bullet)$ are monotone in the same sense and $w(s,t,\bullet)$ also $R(s,\bullet)$ are monotone in the same (opposite) sense, then

$$g^{-1}\left[x(s)+R(s,h(s,s))\right]\leq(\geq)g^{-1}\left[x(s)+R(s,\tilde{\kappa}(s))\right] \tag{3.30}$$

on $[a,b_1]$. The conclusion (3.22) now follows in cases (1) or (2) from (3.24) and (3.19).

Similarly we can prove the following theorem.

Theorem 3.25(see [33]) Addition to the hypotheses of last theorem, assume

$$g(\varphi(s))\geq x(s)+R\left(s,\int_a^s w(s,t,\varphi(t))\,dt\right),\quad s\in I$$

and that $w(\bullet,t,\phi)$ and $R(s,\bullet)$ are monotone in the opposite sense. Suppose $\tilde{\kappa}=\kappa(s,s,a)$, where $\kappa(s,\tau,a)$ is the maximal (minimal) solution of problem (3.21) and assume that $w(s,t,\bullet)$ and g are monotone with opposite (same) sense. Then

$$\varphi(s)\leq(\geq)g^{-1}\left[x(s)+R(s,\kappa(s))\right] \text{ on } [a,b_1]$$

provided that conditions (1) or (2) of Theorem 3.24 hold.

Remark If $K=[0,s_0]$, then $w>0$ holds (since $h(s,s)\in K$), so by (3.19) $|h|\leq\beta$ can be replaced with $0\leq h\leq\beta$.

Theorem 3.26 (see [27]) Consider φ, x , y and z to be continuous non-negative functions on $I=[a,b]$ and g, v are continuous non-negative mappings on $\mathbb{R}^+$ with g strictly increasing and v non-decreasing. Furthermore, assume that $\psi(s,t)$ is continuous and non-negative on $\tau=\{(s,t): a\le t\le s\le b\}$, and $W(s,\phi)$ is continuous and nonnegative on $I\times\mathbb{R}^+$, with $W(s,\bullet)$ non-decreasing on $\mathbb{R}^+$. Define

$$Z(s)=\max_{a\le t\le s} z(t), \text{ and } \Psi(s,t) = \max_{t\le\sigma\le s}\psi(\sigma,t), \text{ for } a\le t\le s\le b.$$

If

$$g(\varphi(s))\le x(s)+y(s)v\left(z(s)+\int_a^s\psi(s,t)W(t,\varphi(t))\,dt\right), \quad s\in I, \tag{3.31}$$

then

$$\varphi(s)\le g^{-1}[x(s)+y(s)v(\tilde{\kappa}_1(s,Z(s)))], \quad s\in I_0, \tag{3.32}$$

where $\tilde{\kappa}_1(s,z(a))=\kappa(s,s,z(a))$, with $\kappa=\kappa(s,b,z(a))$ is the maximal solution on $I=[a,b_0]$ of

$$\kappa'=\Psi(b_0,s)W(s,g^{-1}[x(s)+y(s)v(\kappa)]), \quad \kappa(a)=z(a).$$

Now consider the following inequality of Gollwitzer[20]

$$\varphi(s)\le x+f^{-1}\left(\int_a^s\psi(s,t)\,f(\varphi(t))\,dt\right), \quad s\in I=[a,b].$$

Using Theorem 3.24 with: $g(\varphi)=\varphi,\ R(s,h)=f^{-1}(h),$

$W(s,t,\phi)=\psi(t)f(\phi),\ \Psi=f(J)$ and J is an interval defined such that $\varphi(I)\subset J$.

The comparison equation is

$$\kappa'=\psi(s)\,f\left(x+f^{-1}(\kappa)\right), \quad \kappa(a)=0 \ . \tag{3.33}$$

From Theorem 3.24 we have

$$\varphi(s) \leq x + f^{-1}(\kappa(s)), \quad a \leq s \leq b_1, \tag{3.34}$$

with $\kappa(s)$ being the unique solution of the problem (3.33) on the interval $[a,b_1]$, if we define now the function F as

$$F(\phi) = \int_0^{\phi} \frac{d\kappa}{f\left[x + f^{-1}(\kappa)\right]}, \qquad \phi \in f(J) = K,$$

then from (3.34) we get

$$\varphi(s) \leq x + f^{-1}\left[F^{-1}\left(\int_a^s \psi \, dt\right)\right], \quad a \leq s \leq b, \tag{3.35}$$

where

$$b_1 = \sup\left\{s \in I : \int_a^s \psi \, dt \in F(K)\right\}.$$

Consider φ and ψ be continuous mapping on the interval $I = [a,b]$ with $\psi \geq 0$, and assume the mapping f be continuous and monotone function in the interval J such that $\varphi(I) \subset I$ and $f \neq 0$ on J except perhaps at an endpoint of J. Let choose v be continuous and monotone on an interval K such that $0 \in K$, and assume that x, y are constantssuch that $x + v(h) \in J^0$ for $h \in K$, $|h| \leq y$.

If the mappings f and v are monotonic in the same sense and

$$\varphi(s) \leq x + v\left(\int_a^s \psi(s,t) \, f(\varphi(t)) \, dt\right), \; s \in I,$$

then

$$\varphi(s) \leq x + v\left[F^{-1}\left(\int_a^s \psi \, dt\right)\right], \quad a \leq s \leq b_1,$$

where

$$F(\phi)=\int_0^{\phi}\frac{d\kappa}{f\left[x+v(\kappa)\right]}, \qquad \phi\in K$$

and

$$b_1=\sup\left\{s\in I:\int_a^s \psi dt\in F(K)\right\}.$$

Theorem 3.27(see [41]) Assume that the functions $\varphi(s)$, $x(s)$, $y(s)$, $w(s)$ and $z(s)$ are non-negative, real and continuous defined on $\mathbb{R}^+$ such that for $s\in\mathbb{R}^+$,

$$\varphi(s)\le x(s)+y(s)\left(\int_0^s w(t)\varphi(t)dt+\int_0^s w(t)y(t)\left(\int_0^t z(v)\varphi(v)dv\right)dt\right).$$

Then on the same interval we have

$$\varphi(s)\le x(s)+y(s)\left(\int_0^s w(t)\left(x(t)+y(t)\exp\left(-\int_0^t y(r)\left(w(r)+z(r)\right)dr\right)\right.\right.$$
$$\left.\left.\times\int_0^t x(r)\left(w(r)+z(r)\right)exp\left(-\int_0^s y(v)\left(w(v)+z(v)\right)dv\right)dr\right)dt\right).$$

Theorem 3.28(see [18]) Assume $v(s)\ge 0,\ u(s)\ge 0,\ h(s,r)\ge 0$ and $H(s,r,x)\ge 0$ for $s\ge r\ge x\ge\alpha$ and $\lambda_1\ge 0,\ \lambda_2\ge 0$ and $\lambda_3\ge 0$ be a constants not all zero. If

$$v(s)\le\lambda_1+\lambda_2\int_\alpha^s\left[u(t)v(t)+\int_\alpha^t h(t,r)v(r)dr\right]dt$$
$$+\lambda_3\int_\alpha^s\int_\alpha^r\int_\alpha^t H(t,r,x)v(x)dtdrds, \qquad (3.36)$$

then for $s\ge\alpha$

$$v(s) \le \lambda_1 \exp\left\{\lambda_2 \int_\alpha^s \left(u(t) + \int_\alpha^t h(t,r)\,dr \right) dt + \lambda_3 \int_\alpha^s \int_\alpha^r \int_\alpha^t H(t,r,x)\,dx\,dr\,dt \right\}. \tag{3.37}$$

Proof Suppose the right hand side of (3.36) be denoted by $\beta(s)$.

Then $\beta(t) \le \beta(s)$ for $t \le s$ since all termsare nonnegative, we get

$$\frac{b'(s)}{b(s)} = \lambda_2 u(s)\frac{v(s)}{b(s)} + \lambda_2 \int_\alpha^s \frac{h(s,r)v(r)}{b(s)}dr + \int_\alpha^s \int_\alpha^r \frac{H(s,r,x)v(x)}{b(s)}dxdr$$
$$\le \lambda_2 u(s) + \lambda_2 \int_\alpha^s h(s,r)\,dr + \lambda_3 \int_\alpha^s \int_\alpha^r H(s,r,x)\,dx\,dr.$$

Integrationfrom a to t we getg

$$\log b(s) - \log \lambda_1$$
$$\le \lambda_2 \int_0^b \left[u(t) + \int_\alpha^t h(t,r)\,drdt \right] + \lambda_3 \int_\alpha^s \int_\alpha^t \int_\alpha^r H(t,r,x)\,dxdrdt.$$

Writing this in terms of $b(s)$ and from $v(s) \le b(s)$ complete the proof.

Theorem 3.29 (see [3])Assume $\varphi(s) \ge 0,$ on $[s_0, \infty)$ and satisfy the inequality

$$\varphi(s) \le \lambda + \int_{s_0}^s \psi(s,t)\varphi(t)\,dt + \int_{s_0}^s \int_{s_0}^t F(s,t,r)\varphi(r)\,drds,$$

Where $\psi(s,t)$ and $F(s,t,r)$ are continuously differentiable non-negative

mappings for $s \ge t \ge r \ge s_0,$ and $\lambda > 0$. Then

$$\varphi(s) \le \lambda \exp\left\{ \int_{s_0}^s \left[\psi(t,t) + \int_{s_0}^t \left(\frac{\partial \psi(t,r)}{\partial t} + F(t,t,r) \right) dr + \int_{s_0}^t \int_{s_0}^r \frac{\partial F(t,r,\kappa)}{\partial t} d\kappa dr \right] dt \right\}.$$

Theorem 3.30(see [18]) Assume $\varphi(s)\ge 0,\ x(s)\ge 0,\ y(s)\ge 0$ and $\omega(s,r)\ge 0$ be continuous functions for $\alpha\le r\le s$, and assume λ_1, λ_2 and λ_3 are all non-negative. If for $s\in[\alpha,\infty)$,

$$\varphi(s)\le \lambda_1 + x(s)\left\{\lambda_2+\lambda_3\int_\alpha^s\left[y(t)\varphi(t)+\int_\alpha^t\omega(t,r)x(r)drdt\right]dt\right\},$$

then for $s\in[\alpha,\infty)$,

$$\varphi(s)\le \lambda_1 + x(s)\left\{\lambda_2\exp\left[\lambda_3\int_\alpha^s\left(y(t)x(t)+\int_\alpha^t\omega(t,r)x(r)dr\right)dt\right]\right.$$
$$+\lambda_1\lambda_3\int_\alpha^s\left(y(t)+\int_\alpha^t\omega(t,r)dr\right)$$
$$\left.\times\exp\left[\lambda_3\int_t^s\left(y(\varepsilon)x(\varepsilon)+\int_\alpha^\varepsilon\omega(\varepsilon,r)x(r)dr\right)d\varepsilon\right]dt\right\}.$$

Theorem 3.31 (see [51]) Assume that $\varphi(s)\in C[0,h)$ is non-negative and suppose that $q(s)\in C[0,h)$ is positive and non-decreasing. Let $g_n(s,t)\ge 0,\ n=1,2,\ldots,m,$ be continuous functions on $[0,h)\times[0,h)$, and non-negative in s. If for $s\in[0,h)$

$$\varphi(s)\le q(s)+\int_0^s g_1(s,s_1)\int_0^{s_0} s_2(s_1,s_2)\ .\ .\ .\int_0^{s_{m-1}} g_m(s_{m-1},m)\varphi(s_m)ds_m\ .\ .\ .ds_1$$

then

$$\varphi(s)\le q(s)V(s)\ ,\ \ s\in[0,h),$$

where $V(s)=U_i(s,s)$ and $U_i(\tau,s)$ is defined by

$$U_1(\tau\ ,s)=\exp\left\{\int_0^s\sum_{j=1}^s g_j(\tau,t)dt\right\}$$

$$U_K(\tau,s)=G_{i-k+1}(\tau,s)\left\{1+\int_0^s g_{i-k+1}(\tau,s)\frac{U_{k-1}(\tau_1,t)}{g_{i-k+1}(\tau,s)}dt\right\},$$

$$G_n(\tau,s)=\exp\left\{\int_0^s\left[\sum_{j=1}^{n-1}g_j(\tau,t)-g_n(\tau,t)\right]dt\right\}.$$

Remark Considering the particular case when $n=2$ in the previous theorem leads us to that $\forall s\in J$

$$\varphi(s)\le q(s)\exp\left(-\int_0^s g_1(s,t)\,dt\right)$$
$$\times\left\{1+\int_0^s g_1(s,t)\exp\left\{\int_0^t\left[2g_1(t,r)-g_2(t,r)\right]dr\right\}dt\right\}.$$

From the last theorems we have took linear inequalities.

Theorem 3.32(see [40], [38]) Assume that $\varphi(s)$, $x(s)$ and $y(s)$ are continuous, real and non-negative mappings on $[0,\infty)$ such that

$$\varphi(s)\le\varphi_0+\int_0^s x(t)\varphi(t)\,dt$$
$$+\int_0^s x(t)\left(\int_0^t y(r)\varphi^q(r)\,dr\right)dt,\quad s\in[0,\infty),\quad 0\le q\le 1,$$

where $\varphi_0\ge 0$ is constant. Then for $s\in[0,\infty)$

$$\varphi(s)\le\varphi_0+\int_0^s x(t)\exp\left(\int_0^t x(r)\,dr\right)$$
$$\times\left\{\varphi_0^{1-q}+(1-q)\int_0^t y(r)\times\exp\left(-(1-q)\int_0^r x(v)\,dv\right)dr\right\}^{\frac{1}{1-q}}dt.$$

Theorem 3.33(see [40], [38])Assume $\varphi(s)\geq 0,\ x(s)\geq,\ y(s)\geq 0$ and $z(s)\geq 0$ be continuous on $\mathbb{R}$ such that for $s\in[0,\infty)$

$$\varphi(s)\leq\varphi_0+\int_0^s x(t)\left(\varphi(t)+\int_0^t x(r)\left(\int_0^r y(v)\varphi(v)+z(v)\varphi^q(v)dv\right)dr\right)dt,\ \ 0\leq q\leq 1$$

where $\varphi_0\geq 0$ is constant. Then for $s\in[0,\infty)$

$$\varphi(s)\leq\varphi_0+\int_0^s x(t)\left(\varphi_0+\int_0^t x(r)\exp\left(\int_0^r (x(v)+y(v))dv\right)\right.$$
$$\left.\times\left\{\varphi_0^{1-q}+(1-q)\int_0^r z(v)\times\exp\left(-(1-q)\int_0^r (x(u)+y(u))du\right)dv\right\}^{\frac{1}{1-p}}\right)dt.$$

Theorem 3.34(see [54]) Assume $\phi(s),\ x(s),\ g(s,t),\ f_n(s,t)$ and $h_n(s,t)$, $n=1,\ ...,m$ be non-negative continuous mappings defined on $J=[0,h)$ and $J\times J$. Let $x(s)$ be non-decreasing and $g(s,t),\ f_n(s,t)$ *and* $h_n(s,t)$ be non-decreasing in s. If $0<q\leq 1$ and

$$\varphi(s)\leq x(s)+\int_0^s g(s,t)\varphi(t)dt+\sum_{n=1}^{m}\int_0^s f_n(s,t)\left[\int_0^t h_n(s,r)\varphi^q(r)dr\right]dt,$$

then

1) for $0<q<1$ and $s\in J$ we have

$$\varphi(s)\leq\left\{[x(s)G(s)]^{1-q}+(1-q)\sum_{n=1}^{m}F_n(s)G(s)\int_0^s h_n(s,t)dt\right\}^{\frac{1}{1-q}}$$

2) for $q=1$ and $s\in J$ we have

$$\varphi(s)\leq x(s)\exp\left\{\int_0^s\left[g(s,t)+\sum_{n=1}^{m}F_n(s)G(s)h_n(s,t)\right]dt\right\},$$

where

$$G(s)=\exp\int_0^s g(s,t)\,dt$$

also

$$F_n(s)=\int_0^s f_n(s,t)\,dt,\ n=1,2,\ .\ .\ .\ ,m.$$

Theorem 3.35 (see [39], [37]) Assume that $\varphi(s)$, $x(s)$ and $y(s)$ are continuous and non-negative mappings on $J=[\alpha,\beta]$, and suppose that $f(v)>0$ is continuous, sub-additive and strictly increasing function for $v>0$ and $f(0)=0$. If for $s\in J$

$$\varphi(s)\le x(s)+\int_\alpha^s y(t)\left(\varphi(t)+\int_\alpha^t y(r)\,f(\varphi(r))\,dr\right)dt,$$

then for $s\in J_0$ we have

$$\varphi(s)\le x(s)+\int_\alpha^\beta y(t)\left(\varphi(t)+\int_\alpha^t y(r)\,f(x(r))\,dr\right)dt$$
$$+\int_a^s y(t)F^{-1}\left(F\left(\int_\alpha^\beta y(r)\big(x(r)+y(v)\,f(x(v))\,dv\big)dr\right)+\int_\alpha^t y(r)\,dr\right)dt,$$

where

$$F(v)=\int_{v_0}^v\frac{dt}{(t+f(t))},\qquad v\ge v_0>0,$$

and

$$J_0=\left\{s\in[\alpha,\beta]:\,F(\infty)\ge F\left(\int_\alpha^\beta y(t)\big(x(t)+y(r)\,f(x(r))\,dr\big)dt+\int_\alpha^s y(t)\,dt\right)\right\}.$$

Theorem 3.36(see [39], [37]) Assume that $\varphi(s)$, $x(s)$, $y(s)$, $z(s)$ and $\psi(s)$ are continuous mappings on $J=[\alpha,\beta]$ and $g(v)>0$ is strictly increasing, continuous, sub-multiplicative, and sub-additive mapping for $v > 0$, with $g(0) = 0$. If $s \in J$ we have

$$\varphi(s) \le x(s) + y(s) \int_{\alpha}^{s} \left(z(t) g(\varphi(t)) + y(t) \int_{0}^{s} \psi(r) g(\varphi(r)) dr \right) dt,$$

then for $s \in J_0$ we also have

$$\varphi(s) \le x(s) + y(s) \Bigg\{ b + \int_{\alpha}^{s} z(t) g(y(t) F^{-1}$$

$$\times \left(F(b) + \int_{\alpha}^{t} g(y(r))(z(r) + \psi(r)) dr \right) dt \Bigg\},$$

where

$$b = \int_{\alpha}^{\beta} z(t) g \left(x(t) + y(t) \int_{\alpha}^{t} \psi(r) g(x(r)) dr \right) dt,$$

$$F(v) = \int_{v_0}^{v} \frac{dt}{g(t)}, \quad v \ge v_0 > 0,$$

and

$$J_0 = \left\{ s \in [\alpha, \beta] : F(\infty) \ge F(b) + \int_{\alpha}^{\beta} g(y(r))(z(r) + \psi(r)) dr \right\}.$$

Theorem 3.37 (see [39], [37])Suppose that φ, x, y, z, ψ and g are the functions that satisfy the hypotheses of the last theorem. If for $s \in J$ we have

$$\varphi(s) \le x(s) + y(s) g^{-1} \Bigg(\int_{\alpha}^{s} z(t) g(\varphi(t)) dt$$

$$+ \int_{\alpha}^{s} z(t) g(y(t)) \left(\int_{\alpha}^{t} \psi(r) g(\varphi(r)) dr \right) dt \Bigg),$$

then we also have for $s \in J$

$$\varphi(s) \le x(s) + y(s) g^{-1}\left(\int_{\alpha}^{s} z(t) g\left(x(t)\right) + g\left(y(t)\right) \right.$$

$$\times \left(\exp\left(\int_{\alpha}^{\beta} g\left(y(r)\right)\left(z(r)+\psi(r)\right) dr \right) \int_{a}^{t} g\left(x(r)\right)\left(z(r)-\psi(r)\right) \right.$$

$$\left. \left. \times \exp\left(-\int_{\alpha}^{r} g\left(y(v)\right)\left(z(v)+\psi(v)\right) dv \right) dr \right) \right) dt.$$

Theorem 3.38(see [39], [37])Assuming that the mappings is defined as in Theorem 3.36, for $\alpha \le t \le s \le \beta$,

we have

$$\varphi(s) \ge \varphi(t) - y(s) g^{-1}\left(\int_{t}^{s} z(r) g\left(\varphi(r)\right) dr - \int_{t}^{s} z(r)\left(\int_{r}^{s} \psi(u) g\left(\varphi(u)\right) du \right) dr \right),$$

then for the same range of values we have

$$\varphi(s) \ge \varphi(t)\Big(g^{-1}\big(1 + g\left(y(s)\right)\big)$$

$$\left. \int_{t}^{s} z(r) \exp\left(\int_{r}^{s} (z(u) g(y(s)) + \psi(u)) du \right) dr \right) \Bigg)^{-1}.$$

Theorem 3.39 (see [10]) Suppose that the mappings $\phi(s)$, $x(s)$ and $\psi(s,t)$ are non-negative for $0 < t < s < \beta$ and $u(t)$ is positive, non-decreasing and continuous for $t > 0$. If

$$\phi(s) \le \lambda + \int_{\alpha}^{s} \left(x(r) u\left(\phi(r)\right) + \int_{\alpha}^{r} \psi(r,t) u\left(\phi(t)\right) dt \right) dr$$

$$\equiv \lambda + \int_{\alpha}^{s} \mathrm{P} u(\phi)\, dr = \beta(s),$$

where $\lambda > 0$ is constant, then for $s \in (\alpha, \beta)$ we have

$$\int_{\alpha}^{\varphi(s)} \frac{dy}{u(y)} \leq \int_{\alpha}^{s} \left[x(r) + \int_{\alpha}^{r} \psi(r,t)\,dt \right] dr = L(s).$$

Proof From the hypotheses, it follows that

$$\frac{\beta'(s)}{u(\beta(s))} = x(t)\frac{u(\phi(s))}{u(\beta(s))} + \int_{\alpha}^{s} \psi(s,t)\frac{u(\phi(t))}{u(\beta(s))}\,dt \leq x(s) + \int_{\alpha}^{s} \psi(s,t)\,dt.$$

Apply integration operator to both sides we obtain the result.

Theorem 3.40 (see [46]) Let

1) $u(\phi) \geq 0$ and continuous non-decreasing mapping on $[0,\infty)$;

2) $R(\phi) = \int_{\phi_0}^{\phi} \frac{dt}{u(t)}$, $(0 < \phi < \infty,\ \phi_0 \in (0,\infty)$ is fixed),

and R^{-1} is the inverse of R;

3) $\phi(s) \in C[0,\infty)$;

4) $M(s) \geq 0$ is a non-decreasing mapping.

If

$$\phi(s) \leq M(s) + \int_{\alpha}^{s} Pu(\phi)\,dr,$$

with the operator P defined in Theorem 3.39, then

$$\phi(s) \leq R^{-1}\left\{ R\left[M(s) + L(s) \right] \right\}$$

where $L(s)$ is also defined in the previous Theorem.

ProofSuppose $\tau > \alpha$ is fixed. Then for $s \in (\alpha, \tau]$ we get

$$\phi(s) \leq M(\tau) + \int_{0}^{s} Pu(\phi)\,dr,$$

since $M(\tau) \geq M(s)$. On the basis of in Theorem 3.39, we have

$$\phi(s) \leq R^{-1}\left\{R[M(\tau)+L(s)]\right\} \text{ for } s \leq \tau.$$

Setting $s=\tau$ we get the result.

Chapter 4

GRONWALL TYPE INEQUALITY AND ITS APPLICATION TO A FRACTIONAL INTEGRAL EQUATIONS

In recent years, an increasing number of Gronwall inequality generalizations have been discovered to address difficulties encountered in differential equations. Among these generalizations, we focus on the works of Ye, Gao and Qian, Gong, Li, the generalized Gronwall inequality with Riemann-Liouville fractional derivative are presented as follows.In thischapter we will show that fractional Gronwall inequality is useful in investigating the dependence of the solution on the order and the initial condition to a certain fractional differential equation with Riemann–Liouville fractional derivatives.

4.1 Fractional Integral Inequality

In this section, we wish to establish an integral inequality which can be used in a fractional differential equation.

Theorem 4.1 (see [42]) Assume $\beta > 0, x(s)$ is a non-negative function locally integrable on $0 \le s < T$ $(\text{some } T \le +\infty)$ and $f(s) \ge 0$ is a non-decreasing continuous function well-defined on $0 \le s < T$, $f(s)$ $M \in \mathbb{R}$ and let $\phi(s)$ be non-negative and locally integrable on $0 \le s < T$ by the way of

$$\phi(s) \le x(s) + f(s) \int_0^s (s-t)^{\beta-1} \phi(t)\,dt,$$

on this interval then

$$\phi(s) \le x(s) + \int_0^s \left[\sum_{n=1}^{\infty} \frac{\left(f(s)\Gamma(\beta)\right)^n}{\Gamma(n\beta)} (s-t)^{n\beta-1} x(t) \right] dt, \; 0 \le s < \tau.$$

Proof

Suppose $Bu(s) = f(s) \int_0^s (s-t)^{\beta-1} u(t)\, dt, \; s \ge 0,$ for locally integrable functions u.

Then

$$\phi(s) \le x(s) + B\phi(s)$$

implies

$$\phi(s) \le \sum_{k=0}^{n-1} B^k x(s) + B^n \phi(s).$$

Let's show that

$$B^n \phi(s) \le \int_0^s \frac{\left(f(s)\Gamma(\beta)\right)^n}{\Gamma(n\beta)} (s-t)^{n\beta-1} \phi(t) dt \tag{4.1}$$

and $B^n \phi(s) \to 0$ as $n \to +\infty$ for each $s \in [0, T)$.

Clearly for n=1 the relative (4.1) is true.

Let (4.1) be true for some n = k, then we have to prove that it is true for n = k+1.

Then the induction hypothesis equals

$$B^{k+1}\phi(s) = B\left(B^k \phi(s)\right) \le f(s) \int_0^s (s-t)^{\beta-1} \left[\int_0^t \frac{\left(f(t)\Gamma(\beta)\right)^k}{\Gamma(k\beta)} (t-T)^{k\beta-1} \phi(T)\, dT \right] dt.$$

Since $f(s)$ is non-decreasing, then

$$B^{k+1}\phi(s) \le \left(f(s)\right)^{k+1} \int_0^s (s-t)^{\beta-1} \left[\int_0^t \frac{\left(\Gamma(\beta)\right)^k}{\Gamma(k\beta)} (t-T)^{k\beta-1} \phi(T)\, dT \right] dt.$$

Via exchanging the order of integration, we get

$$B^{k+1}\phi(s) \le \left(f(s)\right)^{k+1} \int_0^s \left[\int_T^s \frac{\left(\Gamma(\beta)\right)^k}{\Gamma(k\beta)} (s-t)^{\beta-1}(t-T)^{k\beta-1} dt \right] \phi(T)\, dT.$$

$$= \int_0^s \frac{(f(s)\Gamma(\beta))^{k+1}}{\Gamma((k+1)\beta)} (s-t)^{(k+1)\beta-1} \phi(t) dt,$$

where the integral

$$\int_T^s (s-t)^{\beta-1}(s-T)^{k\beta-1} dt = (s-T)^{k\beta+\beta-1} \int_0^1 (1-z)^{\beta-1} z^{k\beta-1} dz$$

$$= (s-T)^{(k+1)\beta-1} B(k\beta, \beta)$$

$$= \frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)} (s-T)^{(k+1)\beta-1}$$

is calculated by the substitution $t = T + z(s-T)$ and the definition of the beta function, the inequality (4.1) proved

since $B^n\phi(s) \le \int_0^s \frac{\left(M\Gamma\ (\beta)\right)^n}{\Gamma(n\beta)}\ (s-t)^{n\beta-1}\phi(t)dt \to 0$ as $n \to +\infty$ for $s \in [0, \tau)$,

we get the result.

Corollary 4.1 Let $\lambda \ge 0,\ \beta > 0$ and $x(s)$ be a non-negative function locally integrable on $0 \le s < T$ (some $T \le +\infty$), and suppose $\phi(s) \ge 0$ is locally integrable on $0 \le s < T$, and satisfy

$$\phi(s) \le x(s) + \lambda \int_0^s (s-t)^{\beta-1} \phi(t) dt$$

on this interval; then

$$\phi(s) \le x(s) + \int_0^s \left[\sum_{n=1}^{\infty} \frac{(\lambda\Gamma(\beta))^n}{\Gamma(n\beta)} (s-t)^{n\beta-1} x(t) \right] dt, \qquad 0 \le s < T.$$

Corollary 2 Under the hypothesis of Theorem 4.1 let $x(s)$ be a non-decreasing function on $[0,T)$. Then

$$\phi(s) \le x(s) E_{\beta}\,(f(s)\Gamma(\beta)s^{\beta}),$$

where E_{β} is the Mittag- Leffler function defined by $E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

Proof The hypothesis imply

$$\phi(s) \le x(s)\left[1+\int_{o}^{s}\sum_{n=0}^{\infty}\frac{(f(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(s-t)^{n\beta-1}dt\right] = x(s)\sum_{n=0}^{\infty}\frac{(f(s)\Gamma(\beta)s^{\beta})^n}{\Gamma(n\beta+1)}$$
$$= x(s)E_{\beta}(f(s)\Gamma(\beta)s^{\beta}).$$

The proof is complete.

4.2 Application

Consider the following Riemann-Liouville fractional derivatives problem with initial value problem

$$D^{\alpha}x(s) = f(s, x(s)), \tag{4.2}$$

$$D^{\alpha-1}x(s)\big|_{s=0} = x_0, \tag{4.3}$$

with $0<\alpha<1,\ 0\le s<\tau\le+\infty,\ f:[0,\tau)\times\mathbb{R}\to\mathbb{R}$ and D^{α} stand for Riemann – Liouville derivative operator.

The existence and uniqueness of the initial value problem are studied in what follow. The problem stated by the equations (4.2)-(4.3) is first reduced into fractional integral equation

$$x(s) = \frac{\eta}{\Gamma(\alpha)}s^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{0}^{s}(s-T)^{\alpha-1} f(T, x(T))dT, \tag{4.4}$$

this is called Volterra integral equation of order α .

Obviously, the initial value problem (4.2)-(4.3) and equation (4.4) are equivalent.

Theorem 4.2: (see [44]) Let $\varepsilon > 0$ and $\alpha > 0$ such that $0 < \alpha - \varepsilon < \alpha \leq 1$. Consider f to be a Lipschitz continuous mapping with respect to second variable.

$$\left| f(s,x) - f(s,z) \right| \leq L|x - z|$$

with L being a constant independent of s, x and z in $\mathbb{R}$. For $0 \leq s \leq h < \tau$, assuming that the solutions of initial value problems (4.2)-(4.3) are x and z and

$$D^{\alpha-\varepsilon} z(s) = f(s, z(s)), \tag{4.5}$$

$$D^{\alpha-\varepsilon-1} z(s)\big|_{t=0} = \overline{\eta} \ , \tag{4.6}$$

the following holds for $0 \leq s \leq h$

$$\left| z(s) - x(s) \right| \leq A(s) + \int_0^s \left[\sum_{n=1}^{\infty} \left(\frac{L}{\Gamma(\alpha)} \Gamma(\alpha - \varepsilon) \right)^n \frac{(s-t)^{n(\alpha-\varepsilon)-1}}{\Gamma(n(\alpha-\varepsilon))} A(t) \right] dt,$$

where

$$A(s) = \left| \frac{\overline{\eta}}{\Gamma(\alpha-\varepsilon)} s^{\alpha-\varepsilon-1} - \frac{\eta}{\Gamma(\alpha)} s^{\alpha-1} \right| + \left| \frac{s^{\alpha-\varepsilon}}{(\alpha-\varepsilon)\Gamma(\alpha)} - \frac{s^{\alpha}}{\Gamma(\alpha+1)} \right| . \| f \|$$
$$+ \left| \frac{s^{\alpha-\varepsilon}}{(\alpha-\varepsilon)} \left[\frac{1}{\Gamma(\alpha-\varepsilon)} - \frac{1}{\Gamma(\alpha)} \right] \right| . \| f \|$$

and

$$\| f \| = \max_{0 \leq s \leq h} \left| f(s,x) \right| .$$

Proof: Initial value problem stated by (4.2)-(4.3) and (4.5)-(4.6) have solutions given by

$$x(s) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^s (s-T)^{\alpha-1} f\left(T, x(T)\right) dT$$

and

$$z(s) = \frac{\bar{\eta}}{\Gamma(\alpha-\varepsilon)} s^{\alpha-\varepsilon-1} + \frac{1}{\Gamma(\alpha-\varepsilon)} \int_0^s (s-T)^{\alpha-\varepsilon-1} f\left(T, x(T)\right) dT,$$

it follows that

$$\begin{aligned}
|z(s)-x(s)| &\le \left| \frac{\bar{\eta}}{\Gamma(\alpha-\varepsilon)} s^{\alpha-\varepsilon-1} - \frac{\eta}{\Gamma(\alpha)} s^{\alpha-1} \right| \\
&+ \left| \frac{1}{\Gamma(\alpha-\varepsilon)} \int_0^s (s-T)^{\alpha-\varepsilon-1} f\left(T, z(T)\right) dT - \frac{1}{\Gamma(\alpha)} \int_0^s (s-T)^{\alpha-\varepsilon-1} f\left(T, z(T)\right) dT \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_0^s (s-T)^{\alpha-\varepsilon-1} f\left(T, z(T)\right) dT - \frac{1}{\Gamma(\alpha)} \int_0^s (s-T)^{\alpha-\varepsilon-1} f\left(T, x(T)\right) dT \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_0^s (s-T)^{\alpha-\varepsilon-1} f\left(T, x(T)\right) dT - \frac{1}{\Gamma(\alpha)} \int_0^s (s-T)^{\alpha-1} f\left(T, x(T)\right) dT \right| \\
&\le A(s) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-T)^{\alpha-\varepsilon-1} L |z(T)-x(T)| dT
\end{aligned}$$

where

$$\begin{aligned}
A(s) = \left| \frac{\bar{\eta}}{\Gamma(\alpha-\varepsilon)} s^{\alpha-\varepsilon-1} - \frac{\eta}{\Gamma(\alpha)} s^{\alpha-1} \right| + \left| \frac{s^{\alpha-\varepsilon}}{(\alpha-\varepsilon)\Gamma(\alpha)} - \frac{s^{\alpha}}{\Gamma(\alpha+1)} \right| . \|f\| \\
+ \left| \frac{s^{\alpha-\varepsilon}}{(\alpha-\varepsilon)} \left[\frac{1}{\Gamma(\alpha-\varepsilon)} - \frac{1}{\Gamma(\alpha)} \right] \right| . \|f\|
\end{aligned}$$

using theorem 4.1 leads us to

$$|z(s)-x(s)| \le A(s) + \int_0^s \left[\sum_{n=1}^{\infty} \left(\frac{L}{\Gamma(\alpha)} \Gamma(\alpha-\varepsilon) \right)^n \frac{(s-t)^{n(\alpha-\varepsilon)-1}}{\Gamma\left(n(\alpha-\varepsilon)\right)} A(t) \right] dt.$$

The theorem is therefore proved.

Corollary 3 Based on the hypothesis of theorem 4.2, if $\varepsilon = 0$, it follows that

$$|z(s)-x(s)| \le |\bar{\eta}-\eta| s^{\alpha-1} E_{\alpha,\alpha}\left(L s^{\alpha}\right),$$

for $0<s\leq h$, with the Mittag-Leffer function $E_{\alpha,\alpha}$ defined

$$E_{\alpha,\alpha}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\alpha)},\ \ \alpha>0$$

Proof if $\varepsilon=0$, then

$$A(s)=\left|\frac{s^{\alpha-1}}{\Gamma(\alpha)}(\bar{\eta}-\eta)\right|.$$

By Theorem 4.2, we have

$$\begin{aligned}|z(s)-x(s)|\leq A(s)+\int_0^s\left[\sum_{n=1}^{\infty}L^n\frac{(s-t)^{n\alpha-1}}{\Gamma(n\alpha)}A(t)\right]dt &=|\bar{\eta}-\eta|s^{\alpha-1}\sum_{n=1}^{\infty}\frac{(Ls^{\alpha})^n}{\Gamma(n\alpha+\alpha)}\\ &=|\bar{\eta}-\eta|s^{\alpha-1}E_{\alpha,\alpha}(Ls^{\alpha})\end{aligned}$$

with $0<s\leq h$.

The corollary is then proved.

Theorem 4.3: (see [53]) Suppose $w,v\in c_q(J,\mathbb{R})$ be such that

$$w(s)\geq\frac{w_a}{\Gamma(p)}(s-a)^{p-1}+D_s^{-p}f(s,w)$$

and

$$v(s)\leq\frac{v_a}{\Gamma(p)}(s-a)^{p-1}+D_s^{-p}f(s,v)$$

with $w_a=\Gamma(p)w(s)(s-a)^{1-p}|_{s=a}$ and $v_a=\Gamma(p)v(s)(s-a)^{1-p}|_{s=a}$. If any of the above inequalities come to be strict, and if $w_a>v_a$ thenit follows that $w(s)>v(s)$ on J.

Theorem 4.4 (see [12])Consider as defined in the previous theorem w, v with none of them is having strict inequality. Let's assume also that the right-sided Lipschitz condition is satisfied by f .

$$f(s,y) - f(s,x) \le L(y-x),\ y \ge x.$$

If $w_a \ge v_a$, then $w(s) \ge v(s)$ on J .

Proof Consider $\delta > 0$ and let a function w_δ be defined as $w_\delta(s) = w(s) + \delta\lambda(s)$, $s \in J$, with

$$\lambda(s) = (s-a)^{p-1} E_{p,p}\left(2L(s-a)^p\right)$$

note that

$$\begin{aligned} w_{\delta_a} &= \Gamma(p) w_\delta(s)(s-a)^{1-p}\big|_{s=a} \\ &= w_a + \delta \\ &> w_a \ge v_a \end{aligned}$$

furthermore, by taking $\Lambda(s) = \lambda(s) - \frac{1}{\Gamma(p)}(s-a)^{p-1}$ and applying the Lipschitz condition on f , we get what follows

$$\begin{aligned} w_\delta(s) &\ge \frac{w_a+\delta}{\Gamma(p)}(s-a)^{p-1} + D_s^{-p}\left(f(s,w) - f(s,w_\delta)\right) + D_s^{-p} f(s,w_\delta) + \delta\Lambda(\mathrm{s}) \\ &\ge \frac{w_a+\delta}{\Gamma(p)}(s-a)^{p-1} - \delta L D_s^{-p}\lambda(s) + D_s^{-p} f(s,w_\delta) + \delta\Lambda(\mathrm{s}) \\ &= \frac{w_a+\delta}{\Gamma(p)}(s-a)^{p-1} + D_s^{-p} f(s,w_\delta) + \frac{\delta}{2}\Lambda(\mathrm{s}) \\ &= \frac{w_{\delta_a}}{\Gamma(p)}(s-a)^{p-1} + D_s^{-p} f(s,w_\delta) + \frac{\delta}{2}\bar{\lambda}(s) \\ &> \frac{w_{\delta_a}}{\Gamma(p)}(s-a)^{p-1} + D_s^{-p} f(s,w_\delta) \end{aligned}$$

with

$$\bar{\lambda}(s)=(s-a)^{1-p}\sum_{k=1}^{\infty}\frac{(2L)^{k}(s-a)^{pk}}{\Gamma(pk+p)}.$$

In the previous set of relations, the inequalities come from that $\delta>\frac{\delta}{\Gamma(p)}$. We also have that

$$w_{\delta}(s)(s-a)^{q}=w(s)(s-a)^{1-q}+\delta E_{p,p}\left(2L(s-a)^{p}\right).$$

The previous function is continuous on J, since $w\in c_{q}(J,\mathbb{R})$. Therefore $w_{\delta}\in c_{q}(J,\mathbb{R})$ and using the Theorem 4.3, $w_{\delta}(s)>v(s),\ \forall s\in J$. Now if $\delta\to 0$, we then have on both sides $w(s)>v(s),\ \forall s\in J,$ this finishes the proof.

What will follow now is the solving of linear fractional integral equation, which has variable coefficients. Combined with theorem 4.3, this equation give us the Gronwall type inequality.

Theorem 4.5 (see [28])Let $x\in c(J,\mathbb{R})$, The fractional integral equation

$$y(s)=\frac{y_{a}}{\Gamma(p)}(s-a)^{p-1}+D_{s}^{-p}x(s)y(s) \tag{4.7}$$

with $s\in J$ and $y_{a}=\Gamma(p)y(s)(s-a)^{1-p}|_{s=a}$ has $y\in c_{q}(J,\mathbb{R})$ as solution defined

$$y(s)=\frac{y_{a}}{\Gamma(p)}\sum_{k=0}^{\infty}\tau_{x}^{k}(s-a)^{p-1} \tag{4.8}$$

with $y(s)(s-a)^{q}$ which converges uniformly on J and the operator τ_{x} is defined by

$$\tau_{x}\phi=D_{s}^{-p}x(s)\phi(s).$$

Proof The following corollaries are required to prove the previous theorem

Corollary 4 If the function $x(s)$ is identically equals to a constant λ , then

$$\tau_x^n (s-a)^{p-1} = \tau_\lambda^n (s-a)^{p-1} = \frac{\Gamma(p)\lambda^n}{\Gamma(np+p)}(s-a)^{np+p-1}, \quad \forall n \geq 1 \tag{4.9}$$

Proof Induction is used for the proof.

The equation (4.9) is true for $n=1$ since

$$\tau_\lambda (s-a)^{p-1} = \frac{\Gamma(p)\lambda}{\Gamma(2p)}(s-a)^{2p-1}.$$

Consider this to be the basis step and suppose that the equation (4.9) is true up to an index $k>1$. It follows that

$$\begin{aligned}
\tau_\lambda^{k+1}(s-a)^{p-1} &= \tau_\lambda \tau_\lambda^k (s-a)^{p-1} \\
&= \tau_\lambda \frac{\Gamma(p)\lambda^k}{\Gamma(kp+p)}(s-a)^{kp+p-1} \\
&= \lambda \frac{1}{\Gamma(p)}\int_a^s (s-a)^{p-1} \frac{\Gamma(p)\lambda^k}{\Gamma(kp+p)}(s-a)^{kp+p-1}\, dt \\
&= \frac{\Gamma(p)\lambda^{k+1}}{\Gamma((k+1)p+p)}(s-a)^{(k+1)p+p-1},
\end{aligned}$$

it follows by induction that the equation (4.9) is true $\forall n \geq 1$.

Corollary 5 Let $\lambda > 0$ be defined such that $|x(s)| \leq \lambda,\ \forall s \in J$. Then

$$\left|\tau_x^n (s-a)^{p-1}\right| \leq \tau_\lambda^n (s-a)^{p-1},\ \forall n \geq 1 \tag{4.10}$$

Proof Induction is used for the proof.

The equation (4.10) is true for n=1 since

$$\begin{aligned}
\left|\tau_x (s-a)^{p-1}\right| &\leq \frac{1}{\Gamma(p)}\int_a^s (s-a)^{p-1} |x(t)| (t-a)^{p-1}\, dt \\
&\leq \frac{1}{\Gamma(p)}\int_a^s (s-a)^{p-1} \lambda (t-a)^{p-1}\, dt \\
&= \tau_\lambda (s-a)^{p-1}.
\end{aligned}$$

Consider this to be the basis step and suppose that the equation (4.10) is true up to an index $k > 1$. It follows that

$$\begin{aligned}\left|\tau_x^{k+1}(s-a)^{p-1}\right| &= \left|\tau_x \tau_x^k (s-a)^{p-1}\right| \\ &\leq \frac{1}{\Gamma(p)}\int_a^s (s-t)^{p-1}\left|q(t)\right|\left|\tau_x^k (t-a)^{p-1}\right| dt \\ &\leq \frac{1}{\Gamma(p)}\int_a^s (s-t)^{p-1}\lambda \tau_x^k (t-a)^{p-1}\, dt \\ &= \tau_x^{k+1}(s-a)^{p-1}.\end{aligned}$$

It follows by induction that the equation (4.10) is true $\forall n \geq 1$. What follows now is the rest of the proof of the theorem 4.5,

let the sequence of functions

$$y_n(s) = \frac{y_a}{\Gamma(p)}(s-a)^{p-1} + D_s^{-p} x(s) y_{n-1}(s), \ \forall n \geq 1 \tag{4.11}$$

with $y_0(s) = \frac{y_a}{\Gamma(p)}(s-a)^{p-1}$. Our aim is to show that $\left\{y_n(s)(s-a)^p\right\}$ is uniformly convergent on $\boldsymbol{J}$. The proof is done by induction. Actually, by induction, we can show that

$$y_n(s) = \frac{y_a}{\Gamma(p)}\sum_{k=0}^{n} \tau_x^k (s-a)^{p-1}, \ \ \forall n \geq 1. \tag{4.12}$$

First of all, let consider,

$$\begin{aligned} y_1(s) &= \frac{y_a}{\Gamma(p)}(s-a)^{p-1} + D_s^{-p} x(s) y_0(s) \\ &= \frac{y_a}{\Gamma(p)}\tau_x^0 (s-a)^{p-1} + \frac{y_a}{\Gamma(p)}\tau_x (s-a)^{p-1}.\end{aligned}$$

The equation (4.12) is true for $n = 1$, taking this as the basis step of the induction and assuming that the equation (4.12) is always true up to a certain $k > 1$. Then

$$
\begin{aligned}
y_{k+1}(s) &= y_0(s) + \frac{1}{\Gamma(p)}\int_a^s (s-t)^{p-1} x(t)\, y_k(t)\,dt \\
&= \frac{y_a}{\Gamma(a)}(s-a)^{p-1} + \frac{1}{\Gamma(p)}\int_a^s (s-t)^{p-1} x(t) \frac{y_a}{\Gamma(p)} \sum_{j=0}^{k} \tau_x^j (t-a)^{p-1}\,dt \\
&= \frac{y_a}{\Gamma(a)}\tau_x^0 (s-a)^{p-1} + \frac{y_a}{\Gamma(p)}\sum_{j=0}^{k}\frac{1}{\Gamma(p)}\int_a^s (s-t)^{p-1} p(t)\, \tau_x^j (t-a)^{p-1}\,dt \\
&= \frac{y_a}{\Gamma(a)}\tau_x^0 (s-a)^{p-1} + \frac{y_a}{\Gamma(p)}\sum_{j=0}^{k}\tau_x^{j+1}(s-a)^{p-1} \\
&= \frac{y_a}{\Gamma(p)}\sum_{j=0}^{k+1}\tau_x^j (s-a)^{p-1}.
\end{aligned}
$$

By induction, we are leading to the conclusion that the equation (4.12) is true $\forall n \geq 1$ and $\forall s \in J$. Our aim is now to show that

$$
\lim_{n\to\infty} y_n(s)(s-a)^q = \frac{y_a}{\Gamma(p)}\sum_{k=0}^{\infty}(s-a)^q\, \tau_x^k (s-a)^{p-1}
$$

uniformly on J. First, note that $x \in c(J,\mathbb{R})$, therefore, one may choose $\lambda \in (0,\infty)$ such a way that $|x(s)| \leq \lambda,\ \forall s \in J$. Using the corollary 4 and 5 it follows that $\forall n \geq 1$ and $\forall s \in J$,

$$
\begin{aligned}
\left|(s-a)^q\, \tau_x^n (s-a)^{p-1}\right| &\leq (s-a)^q\, \tau_\lambda^n (s-a)^{p-1} \\
&= \frac{\Gamma(p)\lambda^n}{\Gamma(np+p)}(s-a)^{np} \\
&\leq \frac{\Gamma(p)\lambda^n}{\Gamma(np+p)}(b-a)^{np},
\end{aligned}
$$

also note that

$$
\sum_{n=0}^{\infty}\frac{\Gamma(p)\lambda^n}{\Gamma(np+p)}(b-a)^{np} = \Gamma(p) E_{p,p}\left(\lambda(b-a)^p\right).
$$

The previous equation converges. It follows by the Wieirstrass M-Test that $y(s)(s-a)^q = \frac{y_a}{\Gamma(p)}\sum_{k=0}^{\infty}(s-a)^q\, \tau_x^k (s-a)^{p-1}$ is uniformly convergent on J.

Let's finally show that the considered y satisfies the equation (4.7)

$$y_0(s)+D_s^{-p}x(s)y(s)=\frac{y_a}{\Gamma(p)}(s-a)^{p-1}+\frac{y_a}{\Gamma(p)}D_s^{-p}\left(x(t)\sum_{k=0}^{\infty}\tau_x^k(t-a)^{p-1}\right)$$
$$=\frac{y_a}{\Gamma(p)}\tau_x^0(s-a)^{p-1}+\frac{y_a}{\Gamma(p)}\lim_{n\to\infty}\sum_{k=0}^{\infty}\tau_x^{k+1}(s-a)^{p-1}$$
$$=\frac{y_a}{\Gamma(p)}\lim_{n\to\infty}\sum_{k=0}^{\infty}\tau_x^{k+1}(s-a)^{p-1}$$
$$=\frac{y_a}{\Gamma(p)}\sum_{k=0}^{\infty}\tau_x^k(s-a)^{p-1}=y(s).$$

Then we get the proof.

Remark : To establish the previous result, the requirement is that $x(s)$ is a continuous function. However, to prove our next result, the requirement is that $x(s)$ is nonnegative. And finally were leading to the following Gronwall type inequality.

Theorem 4.6 (see [28])Let $u\in c_q(J,\mathbb{R}_+)$ and $x\in c_q(J,\mathbb{R}_+)$ be such that

$$u(s)\le\frac{u_a}{\Gamma(p)}(s-a)^{p-1}+D_s^{-p}u(s)x(s),$$

then

$$u(s)\le\frac{u_a}{\Gamma(p)}\sum_{k=0}^{\infty}\tau_x^k(s-a)^{p-1}.$$

The proof of this theorem 4.6 is directly established using Theorems 4.4 and 4.5.

Actually, if the theorem 4.6 is considered with the initial condition $\frac{u_a}{\Gamma(p)}(t-a)^{p-1}$ is identically constant say u_a. In which case, when the integer $p=1$, the Theorem4.6becomes

$$u(s) \le u_a + \int_a^s x(t)u(t)\,dt,$$

then

$$\begin{aligned} u(s) &\le u_a \sum_{k=0}^{\infty} \tau_x^k 1 \\ &= u_a \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_a^s x(t)dt \right)^k \\ &= u_a \exp\left(\int_a^s x(t)dt \right), \end{aligned}$$

which is the Gronwall Inequality well known.

Another application of Gronwall Inequality

Theorem 4.7 (see[28]) Let $f \in C(\mathbb{R}_0, \mathbb{R})$ be a function such that $|f(s,y)| \le M$ on R_0 where

$$R_0 = \left\{ (s,y) \,|\, a < s \le a + r \ and \ \left| y - \frac{y_a}{\Gamma(p)} (s-a)^{p-1} \right| \le t \right\}.$$

Suppose that f is Lipschitz. Then we have the following successive approximations

$$y_{n+1}(s) = \frac{y_a}{\Gamma(p)} (s-a)^{p-1} + D_s^{-p} f(s, y_n), \quad \forall n \ge 0$$

exist on $I = [a, a+\eta]$, with $\eta = \min\left\{ r, \left(\frac{t\Gamma(1+p)}{M} \right)^{1/p} \right\}$.

Theorem 4.8 Let $f \in c(D, \mathbb{R})$, with D being a domain, $D \subseteq \mathbb{R} \times \mathbb{R}$. Assuming that the function f is a Lipschitz function on the domain D with respect to y and with Lipchitz constant L. Let $(a, y_a) \in D$. Then $\exists \delta > 0$, $\exists \varepsilon > 0$ all constants such that $\forall x_a \in B_\delta(y_a)$, the equation (4.4) defined by

$y(s)=\frac{y_a}{\Gamma(p)}(s-a)^{p-1}+D_s^{-p}f(s,y(s))$ has a unique solution $y(s,a,x_a)$ on the interval $I=[a,a+\varepsilon]$.

Theorem 4.9 (see [29])Assuming that all the assumptions of the previous theorem are verified. Then $\exists\delta>0$ and $\exists\varepsilon>0$ such that $y(s,a,x_a)$ is continuous with respect to $x_a\in B_\delta(y_a)$ based on the following conditions : If $x,x_a\in B_\delta(y_a)$ then $y(s,a,x)$ and $y(s,a,x_a)$ are solutions of the equation (4.4) and also

$$\lim_{x\to x_a} y(s,a,x)(s-a)^q = y(s,a,x_a)(s-a)^q$$

uniformly on $I=[a,a+\varepsilon]$.

Proof Based on theorem 4.8, it follows that

$$y(s,a,x_a)=\frac{x_a}{\Gamma(p)}(s-a)^{p-1}+D_s^{-p}f(s,y(s,a,x_a)),$$

and

$$y(s,a,x)=\frac{x}{\Gamma(p)}(s-a)^{p-1}+D_s^{-p}f(s,y(s,a,x)),$$

are the unique solutions of the equation (4.4) that exist on the interval $I=[a,a+\varepsilon]$. If the Lipchitz condition is applied on f , it follows that

$$|y(s,a,x)-y(s,a,x_a)|\le\frac{|x-x_a|}{\Gamma(p)}(s-a)^{p-1}+D_s^{-p}L|y(s,a,x)-y(s,a,x_a)|,$$

by the theorem 4.8, it follows that

$$|y(s,a,x)-y(s,a,x_a)|\le|x-x_a|(s-a)^{p-1}E_{p,p}\left(\lambda(s-a)^p\right).$$

Therefore $\forall s \in I$,

$$\left|y(s,a,x)-y(s,a,x_a)\right| \le \left|x-x_a\right|(s-a)^{p-1}E_{p,p}\left(\varepsilon^p\right),$$

which leads us to

$$\lim_{x\to x_a} y(s,a,x)(s-a)^q = y(s,a,x_a)(s-a)^q,$$

uniformly on $I=[a,a+\varepsilon]$.

Theorem 4.10 (see [36]) Let $f \in C(D,\mathbb{R})$, with D being a domain, $D \subseteq \mathbb{R}\times\mathbb{R}$. Assuming that the function f is a Lipschitz function on the domain D with respect to y and with Lipchitz constant L. Let $(a,y_a)\in D$. Then $\exists\delta>0$, $\exists\varepsilon>0$ and $\exists\varepsilon'>0$ all constantssuch taht $\forall s_0 \in [a,a+\varepsilon']$ and $\forall x_0 \in B_\delta(y_a)$, the equation (4.4) has a unique solution on the interval $I=[a,a+\varepsilon]$.

Theorem 4.11 (see [53]) Assuming that all the assumptions of the previous theorem are verified. Then $\exists\delta>0$, $\exists\varepsilon>0$ and $\exists\varepsilon'>0$ such that $y(s,a,x_a)$ is continuous with respect to $(s_0,x_0)\in\Omega=[a,a+\varepsilon']\times B_\delta(y_a)$ based on the following conditions:

If $(T,x),(s_0,x_0)\in\Omega$, then

$$\lim_{(T,x)\to(s_0,x_0)} y(s,T,x)(s-s_0)^q = y(s,s_0,x_0)(s-s_0)^q$$

uniformly on $I=[a,a+\varepsilon]$.

Proof Based on the theorem 4.10, it follows that

$$y(s,s_0,x_0)=\frac{x_0}{\Gamma(p)}(s-s_0)^{p-1}+\frac{1}{\Gamma(p)}\int_{s_0}^{s}(s-t)^{p-1}f\left(t,y(s,s_0,x_0)\right)dt$$

and

$$y(s,T,x)=\frac{x}{\Gamma(p)}(s-T)^{p-1}+\frac{1}{\Gamma(p)}\int_T^s (s-t)^{p-1} f\left(t,y(t,T,x)\right)dt$$

are the unique solutions of the equation (4.4) on $I=[a,a+\varepsilon]$. Suppose now that $a+\varepsilon \geq s_0 \geq T \geq a$. Because of $f\in c(\Omega,\mathbb{R})$ and $(s_0,x_0)\in\Omega$ is fixed, we may choose $M>0$ such that $\left|f\left(s,y(s,s_0,x_0)\right)\right|\leq M,\ \forall s\in I.$ Let $u(s)=\left|y(s,T,x)-y(s,s_0,x_0)\right|$ and $\bar{M}=M\varphi^q$, with $\varphi=a+\varepsilon-s_0$. We have in this case $(s-T)^{p-1}\leq(s-s_0)^{p-1}$ applying the Lipschitz condition on f, it follows that

$$\begin{aligned}
u(s)&\leq\frac{|x-x_0|}{\Gamma(p)}(s-s_0)^{p-1}+\frac{1}{\Gamma(p)}\int_T^{s_0}(s-t)^{p-1}\left|f\left(t,y(t,s_0,x_0)\right)\right|dt+\frac{1}{\Gamma(p)}\int_{s_0}^s(s-t)^{p-1}u(t)\,dt\\
&\leq\frac{|x-x_0|}{\Gamma(p)}(s-s_0)^{p-1}+\frac{\bar{M}}{\Gamma(p)}(s-s_0)^{p-1}\int_T^{s_0}(s-t)^{p-1}dt+\frac{1}{\Gamma(p)}\int_{s_0}^s(s-t)^{p-1}u(t)\,dt\\
&=\frac{|x-x_0|}{\Gamma(p)}(s-s_0)^{p-1}+\frac{\bar{M}}{\Gamma(p+1)}(s-s_0)^{p-1}(s_0-T)^p+\frac{L}{\Gamma(p)}\int_{s_0}^s(s-t)^{p-1}u(t)\,dt\\
&=\frac{1}{\Gamma(p)}\left(|x-x_0|+M^*|T-s_0|\right)(s-s_0)^{p-1}+\frac{L}{\Gamma(p)}\int_{s_0}^s(s-t)^{p-1}u(t)\,dt
\end{aligned}$$

with $M^*=\dfrac{\bar{M}}{p}$, using theorem 4.6, it follows that

$$u(s)\leq\left(|x-x_0|+M^*|T-s_0|\right)(s-s_0)^{p-1}E_{p,p}\left(L(s-s_0)^p\right)$$

There fore $\forall s\in I$, we have

$$\begin{aligned}
u(s)(s-s_0)^q&\leq\left(|x-x_0|+M^*|T-s_0|\right)E_{p,p}\left(L(s-s_0)^p\right)\\
&\leq\left(|x-x_0|+M^*|T-s_0|\right)E_{p,p}\left(L\varphi^q\right)
\end{aligned}$$

which leads us to $\lim\limits_{(T,x)\to(s_0,x_0)} y(s,T,x)(s-s_0)^q=y(s,s_0,x_0)(s-s_0)^q$ uniformly on $I=[a,a+\varepsilon]$, end of proof.

Chapter 5

CONCLUSION

In this thesis, we have studied generalizations of the Gronwall inequality using several mathematical techniques. In addition, we have listed the initial value problems and studied the uniqueness of solutions to these problems by applying the generalized Gronwall inequalities.

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