Approximation Properties of Schurer Type q-Bernstein Operators

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ABSTRACT

This thesis consist of five chapters. In the first chapter, the introduction is given. In the second chapter, we consider the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. We state the Korovkin type approximation theorem and obtain the error of approximation by using modulus of continuity and Lipschitz-type functionals. Moreover, we obtain the rate of approximation in terms of the first derivative of the function and we examine the generalization of the operators.

In the third chapter, we define Chlodowsky type q-Bernstein-Stancu-Kantorovich operators. Many properties and results of these polynomials, such as Korovkin type approximation and the rate of convergence of these operators in terms of Lipschitz class functional are given.

In the fourth chapter, we introduce and study Chlodowsky-Durrmeyer type q-Bernstein-Schurer-Stancu operators. We state the Korovkin-type approximation theorem and obtain the order of convergence of the operators.

In the last chapter, we define two dimensional Chlodowsky type of q-Bernstein-Schurer-Stancu operators. We study Korovkin-type approximation theorem and state the error of approximation by using full and partial modulus of continuity. Finally, we define the generalization of the operators and investigate their approximation properties in weighted space.

Keywords: Chlodowsky variant of q-Bernstein-Schurer-Stancu operators, Chlodowsky

type q-Bernstein-Stancu-Kantorovich, Chlodowsky-type q-Durrmeyer operators.

ÖZ

Bu tez beş bölümden oluşmaktadır. Birinci bölüm giriş kısmına ayrılmıştır. İkinci bölümde, Chlodowsky tipli q-Bernstein-Schurer-Stancu Operatörleri tanımlanmıştır. Korovkin tipli teorem yaklaşımı ispatlanmış ve fonksiyonun yakınsaklığındaki hatalar süreklilik modülü yardımıyla ve Lipschitz sınıfındaki yakınsaklığı incelenmiştir.

Üçüncü bölümde Chlodowsky tipli q-Bernstein-Stancu-Kantorovich Operatörleri tanımlanmıştır. Bu operatörlerin Korovkin tipli yaklaşım teoremi ve Lipschitz tipli fonksiyonların yakınsaklık hızları gibi özellikler incelenmiştir.

Dördüncü bölümde, Chlodowsky-Durrmeyer tipli q-Bernstein-Schurer-Stancu Operatörleri tanımlanmıştır. Korovkin tipli yakınsaklık teoremi verilmiş ve yakınsamanın yakınsaklık derecesi incelenmiştir.

Beşinci bölümde, iki değişkenli Chlodowsky tipli q-Bernstein-Schurer-Stancu Operatörleri tanımlanmıştır. Korovkin tipli yakınsaklık teoremi verilmiş, fonksiyonun süreklilik modülü ve kısmi süreklilik modülü yardımıyla yakınsama hızları hesaplanmıştır. Son olarak, operatörlerin bir genelleştirilmesi verilmiş ve onların ağırlıklı uzaydaki yaklaşım özellikleri inclenmiştir.

Anahtar Kelimeler: Chlodowsky tip q-Bernstein-Schurer-Stancu Operatörleri, Chlodowsky tip q-Bernstein-Stancu-Kantorovich Operatörleri, Chlodowsky Tip q-Durrmeyer Operatörleri.

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LIST OF SYMBOLS

$\mathbb{N}$	the set of natural number
$\mathbb{N}_0$	the set of natural number including zero
$\mathbb{R}$	the set of real numbers
(a,b)	an open interval
[a,b]	a closed interval
C[a,b]	the set of all real-valued and continuous functions defined on the compact interval [a,b].
C_ρ	the space of all continuous functions space
(f,δ)	the first modulus of continuity
$L(f;x)$	linear operator
$B_n(f;x)$	Bernstein polynomials
$B_n(f;q;x)$	q-Bernstein polynomials
$B_n^c(f;x)$	Bernstein Chlodowsky polynomials
$C_n(f;x)$	q-Bernstein Chlodowsky polynomials
$B_n^p(f;q;x)$	q-Bernstein Schurer operators
$C_n^p(f;q;x)$	q-Bernstein-Schurer- Chlodowsky polynomials
$K_n^p(f;q;x)$	Schurer type q-Bernstein Kantorovich operators

$C_{m,p}^{(\alpha,\beta)}(f;q;x)$	Chlodowsky variant of q-Bernstein-Schurer-Stancu operators
$K_{m,p}^{(\alpha,\beta)}(f;q;x)$ Stancu-	Chlodowsky-type q-Bernstein-Kantorovich operators
$C_{n,m,p}^{(\alpha,\beta)}(f;q;x)$	Two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators
$D_{m,p}^{(\alpha,\beta)}(f;q;x)$ Bernstein-	Chlodowsky-Durrmeyer type q-Schurer-Stancu operators

Chapter 1

INTRODUCTION

In 1912, the Bernstein operators were introduced by Bernstein [21] as

$$B_m(f;x) = \sum_{r=0}^{m} f\left(\frac{r}{m}\right)\binom{m}{r} x^r (1-x)^{m-r},$$

where the function f is defined on $[0,1]$.

In 1937, Chlodowsky [7] defined the Bernstein-Chlodowsky operators by

$$C_m(f;x) = \sum_{r=0}^{m} f\left(\frac{r}{m}b_m\right)\binom{m}{r}\left(\frac{x}{b_m}\right)^r\left(1-\frac{x}{b_m}\right)^{m-r}, \quad 0 \le x \le b_m$$

where $f \in [0,\infty)$ and the positive increasing sequence $\{b_m\}$ satisfies conditions $b_m \to \infty$ and $\lim_{m\to\infty} \frac{b_m}{m} = 0$.

In 1968, Schurer operators were proposed by Schurer [35]

$$B_m^p(f;x) = \sum_{r=0}^{m+p} f\left(\frac{r}{m}\right)\binom{m+p}{r} x^r (1-x)^{m+p-r}, \quad 0 \le x \le 1$$

where $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is fixed.

In 1969, the Bernstein-Stancu operators were proposed and studied by Stancu [33] as

$$P_m^{\alpha,\beta}(f;x) = \sum_{k=0}^{m} f\left(\frac{k+\alpha}{m+\beta}\right)\binom{m}{k} x^k (1-x)^{m-k}$$

where α and β are real numbers such that $0 \leq \alpha \leq \beta$. Note that, if we choose $\alpha = \beta = 0$ in $P_m^{\alpha,\beta}(f;x)$, it reduces to $B_m(f;x)$.

On the other hand, the q-Bernstein operators were defined firstly by Lupaş [22] as

$$R_{m,q}(f;x) = \sum_{r=0}^{m} f\left(\frac{[r]_q}{[m]_q}\right) \begin{bmatrix} m \\ r \end{bmatrix}_q \frac{q^{\frac{r(r-1)}{2}} x^r (1-x)^{m-r}}{(1-x+qx)\dots(1-xq^{m-1}x)}, \quad 0 \leq x \leq 1,$$

where $0 < q < 1$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and the definition of q-integer $[r] = [r]_q$ is

$$[r]_q := \begin{cases} 1+q+\dots+q^{r-1}; & r \neq 0, \\ 0 & ; \ r = 0, \end{cases}$$

the definition of q-factorial $[r]! := [r]_q!$ is

$$[r]_q! := \begin{cases} [1]_q [2]_q \dots [m]_q; & r \neq 0, \\ [0]_q! = 1 & ; \ r = 0, \end{cases}$$

and the definition of q-binomial coefficient $\begin{bmatrix} m \\ r \end{bmatrix} = \begin{bmatrix} m \\ r \end{bmatrix}_q$ is

$$\begin{bmatrix} m \\ r \end{bmatrix}_q := \frac{[m]_q!}{[r]_q! [m-r]_q!}, \quad 0 \leq r \leq m.$$

After this work, Phillips [31] proposed another type of q-based Bernstein operators for $0 < q < 1$ as

$$B_{m,q}(f,x) = \sum_{r=0}^{m} f\left(\frac{[r]_q}{[m]_q}\right) \begin{bmatrix} m \\ r \end{bmatrix}_q x^r \prod_{s=0}^{m-r-1} (1-q^s x), \quad 0 \leq x \leq 1.$$

In 2011, for $p \in \mathbb{N}_0$, Muraru [28] defined the q-analogue of Bernstein-Schurer operators by

$$B_m^p(f;q,x) = \sum_{r=0}^{m+p} f\left(\frac{[r]_q}{[m]_q}\right) \begin{bmatrix} m+p \\ r \end{bmatrix}_q x^r \prod_{s=0}^{m+p-r-1} (1-q^s x), \quad 0 \leq x \leq 1.$$

Additionally, Vedi and Özarslan [37] investigated some properties of the q-Bernstein-Schurer operators.

Recently, Agrawal et al. [2] proposed the q-Bernstein-Schurer operators as

$$S_{m,p}^{\alpha,\beta}(f;q,x)=\sum_{r=0}^{m+p} f\left(\frac{[r]_q+\alpha}{[m]_q+\beta}\right)\begin{bmatrix} m \\ r \end{bmatrix}_q x^r \prod_{s=0}^{m+p-r-1}(1-q^s x),$$

where p is a positive integer and $0 \leq \alpha \leq \beta$.

On the other hand, several authors studied and investigated the q-Kantorovi ch type operators in the papers [29], [24], [25].

In 2008, q-analogue of Chlodowsky operators were defined in [20] by

$$C_m(f;q,x)=\sum_{r=0}^{m} f\left(\frac{[r]_q}{[m]_q}b_m\right)\begin{bmatrix} m \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m-r-1}\left(1-q^s\frac{x}{b_m}\right), \quad 0 \leq x \leq b_m$$

where also $\{b_m\}$ satisfies the same properties of $C_m(f;x)$. Later, Büyükyazıcı introduced the q-analogue of two dimensional Bernstein-Chlodowsky operators [4] as

$$\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{[r]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right)\Omega_{k,n,q_n}\left(\frac{x}{\alpha_n}\right)\Omega_{k,m,q_m}\left(\frac{y}{\beta_n}\right)$$

where $\Omega_{k,m,q_m}(u)=\begin{bmatrix} m \\ r \end{bmatrix} u^r \prod_{s=0}^{m+p-r-1}(1-q_m^s x)$ and $\{\alpha_n\}$ and $\{\beta_m\}$ satisfy the similar properties with the sequence $\{b_m\}$ as mentioned above.

This thesis is organized as follows:

In chapter 2, Chlodowsky variant of q-Bernstein-Schurer-Stancu operators has been defined. Several approximation properties of these operators are also investigated.

Note that, Chapter two and three are reflected from our papers [38] and [39], respectively. In chapter 3, Chlodowsky type q-Bernstein-Stancu-Kantorovich operators are defined and systematically investigated.

In chapter 4 , approximation properties of Chlodowsky-Durrmeyer type q-Bernstein-Schurer-Stancu operators are introduced. We state the Korovkin-type approximation theorem and obtain the order of convergence of the operators.

In chapter 5, two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators are defined. Korovkin-type approximation theorem is studied and the error of approximation is stated by using full and partial modulus of continuity. Finally, the generalization of the operators is defined and its approximation properties in weighted space are given.

Chapter 2

CHLODOWSKY VARIANT OF q-BERNSTEIN-SCHURER-STANCU OPERATORS

2.1 Construction of the operators

For fixed $p \in \mathbb{N}_0$, Chlodowsky variant of q-Bernstein-Schurer-Stancu operators [38] are proposed as

$$\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x) := \sum_{r=0}^{m+p} f\left(\frac{[r]_q+\alpha}{[m]_q+\beta} b_m\right) \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s \frac{x}{b_m}\right), \tag{2.1.1}$$

where $0 \leq x \leq b_m$, $0<q<1$, α and β are real numbers with $0 \leq \alpha \leq \beta$, $m \in \mathbb{N}$. Notice that if $q \to 1$ and $p=0$ in (2.1.1), $\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)$ reduces to the Stancu-Chlodowsky polynomials [5].

Now, let us calculate some moments of the operator $\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)$:

Lemma 2.1.1 *([38]) The operators $\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)$ satisfy the followings:*

(i) $\mathscr{C}_{m,p}^{(\alpha,\beta)}(1;q,x)=1$,

(ii) $\mathscr{C}_{m,p}^{(\alpha,\beta)}(u;q,x)=\dfrac{[m+p]_q x+\alpha b_m}{[m]_q+\beta}$,

(iii) $\mathscr{C}_{m,p}^{(\alpha,\beta)}(u^2;q,x)=\frac{1}{\left([m]_q+\beta\right)^2}\Big\{[m+p-1]_q [m+p]_q q x^2$

$+(2\alpha+1)[m+p]_q b_m x+\alpha^2 b_m^2\Big\}$,

(iv) $\mathscr{C}_{m,p}^{(\alpha,\beta)}(u-x;q,x)=\left(\frac{[m+p]_q}{[m]_q+\beta}-1\right)x+\frac{\alpha b_m}{[m]_q+\beta}$,

(v) $\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)=\left(\frac{[m+p-1]_q[m+p]_q q}{\left([m]_q+\beta\right)^2}-2\frac{[m+p]_q}{[m]_q+\beta}+1\right)x^2$

$$+\left(\frac{(2\alpha+1)[m+p]_q}{\left([m]_q+\beta\right)^2}-\frac{2\alpha}{[m]_q+\beta}\right)b_m x+\frac{\alpha^2 b_m^2}{\left([m]_q+\beta\right)^2}.$$

Proof. With the help of Lemma 1.1 [2, p.7755] and the equalities

$$\begin{aligned}
\mathscr{C}_{m,p}^{\alpha,\beta}(1;q,x) &= S_{m,p}^{\alpha,\beta}\left(1;q,\frac{x}{b_m}\right)\\
\mathscr{C}_{m,p}^{\alpha,\beta}(u;q,x) &= b_n S_{m,p}^{\alpha,\beta}\left(u;q,\frac{x}{b_m}\right)\\
\mathscr{C}_{m,p}^{\alpha,\beta}\left(u^2;q,x\right) &= b_n^2 S_{m,p}^{\alpha,\beta}\left(u^2;q,\frac{x}{b_m}\right)
\end{aligned}$$

the assertions (i), (ii) and (iii) are proved. From linearity, we get

$$\begin{aligned}
\mathscr{C}_{m,p}^{(\alpha,\beta)}(u-x\,;q,x) &= \mathscr{C}_{m,p}^{(\alpha,\beta)}(u;q,x) - x\mathscr{C}_{m,p}^{(\alpha,\beta)}(1;q,x)\\
&= \left(\frac{[m+p]_q}{[m]_q+\beta}-1\right)x + \frac{\alpha b_m}{[m]_q+\beta}.
\end{aligned}$$

So, we have (iv).In a similar way, we obtain

$$\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right) = \mathscr{C}_{m,p}^{(\alpha,\beta)}\left(u^2;q,x\right) - 2x\mathscr{C}_{m,p}^{(\alpha,\beta)}(u;q,x) + x^2\mathscr{C}_{m,p}^{(\alpha,\beta)}(1;q,x).$$

Finally, we get

$$\begin{aligned}
\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right) &= \left(\frac{[m+p-1]_q\,[m+p]_q}{\left([m]_q+\beta\right)^2}q - 2\frac{[m+p]_q}{[m]_q+\beta}+1\right)x^2\\
&+\left(\frac{(2\alpha+1)\,[m+p]_q}{\left([m]_q+\beta\right)^2} - 2\frac{\alpha}{[m]_q+\beta}\right)b_m x + \frac{\alpha^2 b_m^2}{\left([m]_q+\beta\right)^2}.
\end{aligned}$$

Whence the result. ■

Lemma 2.1.2 *([38]) For each fixed* $0<q<1$*, we get*

$$\frac{[m+p-1]_q\,[m+p]_q}{\left([m]_q+\beta\right)^2}q - 2\frac{[m+p]_q}{[m]_q+\beta}+1 \le \frac{\left(q^m\,[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2}.$$

Proof. The following inequality

$$\frac{[m+p-1]_q\,[m+p]_q}{\left([m]_q+\beta\right)^2}q-2\frac{[m+p]_q}{[m]_q+\beta}+1\leq\left(\frac{[m+p]_q}{[m]_q+\beta}-1\right)^2$$

is satisfied when $[m+p-1]_q\,[m+p]_q\,q\leq[m+p]_q^2$. Using the above inequality, we obtain

$$\begin{aligned}
&\frac{[m+p-1]_q\,[m+p]_q}{\left([m]_q+\beta\right)^2}q-2\frac{[m+p]_q}{[m]_q+\beta}+1\\
\leq\ &\frac{1}{\left([m]_q+\beta\right)^2}\left\{\frac{(1-q^{m+p})^2}{(1-q)^2}-2\frac{(1-q^{m+p})(1-q^m)}{(1-q)^2}-2\beta\frac{(1-q^{m+p})}{1-q}\right.\\
&\left.+\frac{(1-q^m)^2}{(1-q)^2}+2\beta\frac{(1-q^m)}{1-q}+\beta^2\right\}\\
&=\frac{1}{\left([m]_q+\beta\right)^2}\left\{\frac{q^{2m}\left(q^{2p}-2q^p+1\right)}{(1-q)^2}+2\beta\frac{q^m\left(q^p-1\right)}{1-q}+\beta^2\right\}\\
&=\frac{1}{\left([m]_q+\beta\right)^2}\left\{\frac{q^{2m}\left(1-q^p\right)^2}{(1-q)^2}-2\beta\frac{q^m\left(1-q^p\right)}{1-q}+\beta^2\right\}\\
&=\frac{1}{\left([m]_q+\beta\right)^2}\left\{q^{2m}\,[p]_q^2-2\beta q^m\,[p]_q+\beta^2\right\}=\frac{\left(q^m\,[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2}.
\end{aligned}$$

The proof is completed. ■

Remark 2.1.3 *([38]) As a result of Lemma 2.1.1 and Lemma 2.1.2, we get*

$$\begin{aligned}
&\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)2.1.2 \qquad (2.1.1)\\
\leq\ &\frac{\left(q^m\,[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2}x^2+\left(\frac{(2\alpha+1)\,[m+p]_q}{\left([m]_q+\beta\right)^2}\right)b_mx+\frac{\alpha^2b_m^2}{\left([m]_q+\beta\right)^2}.
\end{aligned}$$

Lemma 2.1.4 *([38]) For the second central moment, the following estimation*

$$
\begin{aligned}
&\sup_{0\leq x\leq b_m} \mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right) \\
\leq\ & \frac{\left(q^m[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2} b_n^2+\left(\frac{(2\alpha+1)[m+p]_q}{\left([m]_q+\beta\right)^2}\right) b_m^2+\frac{\alpha^2 b_m^2}{\left([m]_q+\beta\right)^2}
\end{aligned}
$$

is satisfied.

Proof. If we take supremum over $[0,b_n]$ in (2.1.2), we obtain the desired result. ■

2.2 Korovkin-Type Approximation Theorem

In this subsection, Korovkin-type approximation theorem is proved for the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. Denoting by C_μ the space of all continuous functions f, provide the following condition

$$
|f(x)|\leq L_f\mu(x), \qquad -\infty<x<\infty.
$$

Clearly, C_μ is a linear normed space with the norm

$$
\|f\|_\mu=\sup_{-\infty<x<\infty}\frac{|f(x)|}{\mu(x)}.
$$

Theorem 2.2.1 *(See [12]) There exists a sequence of positive linear operators $\mathscr{T}_m^{(\alpha,\beta)}$, acting from C_μ to C_μ, satisfying the conditions*

$$
\lim_{m\to\infty}\|\mathscr{T}_m(1,x)-1\|_\mu=0 \tag{2.2.1}
$$

$$
\lim_{m\to\infty}\|\mathscr{T}_m(\phi,x)-\phi\|_\mu=0 \tag{2.2.2}
$$

$$
\lim_{m\to\infty}\left\|\mathscr{T}_m\left(\phi^2,x\right)-\phi^2\right\|_\mu=0 \tag{2.2.3}
$$

where $\phi(x)$ is a continuous and increasing function on $(-\infty,\infty)$ such that $\lim_{x\to\pm\infty}\phi(x)=\pm\infty$ and $\mu(x)=1+\phi^2$ and there exists a function $f^*\in C_\mu$ for which

$$\overline{\lim_{m\to\infty}}\|\mathscr{T}_m f^* - f^*\|_\mu > 0.$$

Let C_μ^0 denote the subset of C_μ such that $\lim\limits_{|x|\to\infty}\frac{f(x)}{\mu(x)}$ is satisfied .

Theorem 2.2.2 *(See [12]) The conditions (2.2.1), (2.2.2), (2.2.3) imply*

$$\lim_{m\to\infty}\|\mathscr{T}_m f - f\|_\mu = 0$$

for any function f belonging to C_μ^0.

Particularly, let us take $\mu(x) = 1 + x^2$ and apply Theorem 2.2.2 to the operators

$$\mathscr{T}_m^{(\alpha,\beta)}(f;q,x) = \begin{cases} \mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x) & \text{if } \ 0 \le x \le b_m \\ f(x) & \text{if } \ b_m < x < \infty \end{cases}.$$

Note that, the operators $\mathscr{T}_m^{(\alpha,\beta)}(f;q,x)$ act from C_{1+x^2} to C_{1+x^2} . For all $f \in C_{1+x^2}$, we get

$$\begin{aligned} \left\|\mathscr{T}_m^{(\alpha,\beta)}(f;q,\cdot)\right\|_{1+x^2} &\le \sup_{x\in[0,b_m]}\frac{\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)\right|}{1+x^2} + \sup_{b_m<x<\infty}\frac{|f(x)|}{1+x^2} \\ &\le \|f\|_{1+x^2}\left[\sup_{x\in[0,\infty)}\frac{\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}\left(1+t^2;q,x\right)\right|}{1+x^2} + 1\right]. \end{aligned}$$

Hence, using Lemma 2.1.1

$$\left\|\mathscr{T}_m^{(\alpha,\beta)}(f;q,\cdot)\right\|_{1+x^2} \le M\|f\|_{1+x^2}$$

is satisfied for $q := (q_m)$ with $0 < q_m < 1$, $\lim\limits_{m\to\infty} q_m = 1$ and $\lim\limits_{m\to\infty}\frac{b_m}{[m]_q} = 0$ as $m \to \infty$.

Theorem 2.2.3 *([38]) For all $f \in C_{1+x^2}^0$ we get*

$$\left\| \mathscr{T}_m^{(\alpha,\beta)}(f;q_m,\cdot) - f(\cdot) \right\|_\mu = 0$$

provided that $q := (q_m)$ *with* $0 < q_m < 1$, $\lim_{m\to\infty} q_m = 1$ *and* $\lim_{m\to\infty} \frac{b_m}{[m]_q} = 0$ *as* $m \to \infty$.

Proof. Using Theorem 2.2.2 and Lemma 2.1.1 (i), (ii) and (iii) we have the following results:

$$\begin{aligned}
\sup_{0\le x\le\infty} \frac{\left|\mathscr{T}_m^{(\alpha,\beta)}(u;q_m,x) - f(x)\right|}{1+x^2} &= \sup_{0\le x\le b_m} \frac{\left|\mathscr{C}_m^{(\alpha,\beta)}(u;q_m,x) - x\right|}{1+x^2} \\
&\le \sup_{0\le x\le b_m} \frac{\left|\left(\frac{[m+p]_q}{[m]_q+\beta} - 1\right)\right| x + \frac{\alpha b_m}{[m]_q+\beta}}{(1+x^2)} \\
&\le \left|\frac{[m+p]_q}{[m]_q+\beta} - 1\right| + \frac{\alpha b_m}{[m]_q+\beta} \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{0\le x\le\infty} \frac{\left|\mathscr{T}_m^{(\alpha,\beta)}(u^2;q_m,x) - x^2\right|}{1+x^2} \\
&= \sup_{0\le x\le b_m} \frac{\left|\mathscr{C}_m^{(\alpha,\beta)}(u^2;q_m,x) - x^2\right|}{1+x^2} \\
&\le \sup_{0\le x\le b_m} \frac{\left|[m+p-1]_q [m+p] q_m - \left([m]_q+\beta\right)^2\right| x^2 + (2\alpha+1)[m+p]_q b_m x + \alpha^2 b_m^2}{\left([m]_q+\beta\right)^2 (1+x^2)} \\
&\le \frac{\left|[m+p-1]_q [m+p]_q q_m - \left([m]_q+\beta\right)^2\right| + (2\alpha+1)[m+p]_q b_m + \alpha^2 b_m^2}{\left([m]_q+\beta\right)^2} \to 0
\end{aligned}$$

provided that when $\lim_{m\to\infty} q_m = 1$ and $\frac{b_m}{[m]_q} \to 0$ as $m \to \infty$. ■

Lemma 2.2.4 *([38]) Let* f *be a continuous function which vanishes on* $[A,\infty)$ *for which* $A \in \mathbb{R}^+$ *is a positive real number independent of m. Suppose that* $q := (q_m)$ *with* $0 < q_m < 1$, $\lim_{m\to\infty} q_m^m = M < \infty$ *and* $\lim_{m\to\infty} \frac{b_m^2}{[m]_q} = 0$. *Then we get*

$$\lim_{m\to\infty}\sup_{0\le x\le b_m}\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q_m,x)-f(x)\right|=0.$$

Proof. By the assumption on f, one can write $|f(x)|\le L\ \ (L>0)$. For arbitrary small $\varepsilon>0$, we have

$$\left|f\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m\right)-f(x)\right|<\varepsilon+\frac{2M}{\delta^2}\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right)^2,$$

where $x\in[0,b_m]$ and $\delta=\delta(\varepsilon)$ are independent of m. Using the following equality

$$\sum_{r=0}^{m+p}\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right)^2\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)$$
$$=\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q_m,x\right),$$

we have from Remark 2.1.3 that

$$\sup_{0\le x\le b_m}\left|C_{m,p}^{(\alpha,\beta)}(f;q_m,x)-f(x)\right|$$
$$\le\varepsilon+\frac{2M}{\delta^2}\left[\frac{\left(q_m^m[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2}b_m^2+\left(\frac{(2\alpha+1)[m+p]_q}{\left([m]_q+\beta\right)^2}\right)b_m^2+\frac{\alpha^2b_m^2}{\left([m]_q+\beta\right)^2}\right].$$

We have the desired result under the conditions stated in the hypothesis of lemma. ■

Theorem 2.2.5 *([38]) Let f be a continuous function on $[0,\infty)$ and*

$$\lim_{x\to\infty}f(x)=L_f<\infty.$$

Suppose that $q:=(q_m)$ with $0<q_m<1$, $\lim_{m\to\infty}q_m=1$, $\lim_{m\to\infty}q_m^m=L<\infty$ and $\lim_{m\to\infty}\frac{b_m^2}{[m]_q}=0$. Then

$$\lim_{x\to\infty} \sup_{0\le x\le b_m} \left| \mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q_m,x) - f(x) \right| = 0.$$

Proof. Using the proof of Theorem 2.5 in [14]. Obviously, it is enough to prove the theorem for the condition $L_f = 0$. By virtue of $\lim_{x\to\infty} f(x) = 0$, given any $\varepsilon > 0$ there exists $x_0 > 0$ such that

$$|f(x)| \le \varepsilon, \qquad x \ge x_0 \, . \tag{2.2.4}$$

For any fixed $a > 0$, let us define an auxiliary function as

$$g(x) = \begin{cases} f(x) & , \quad 0 \le x \le x_0 \\ f(x_0) - \dfrac{f(x_0)}{a}(x - x_0) & , \quad x_0 \le x \le x_0 + a \\ 0 & , \quad x \ge x_0 + a. \end{cases}$$

Then for m large enough in such a way that $b_m \ge x_0 + a$ and on account of $\sup_{x_0 \le x \le x_0+a} |g(x)| = |f(x_0)|$, we get

$$\begin{aligned} \sup_{0\le x\le b_m} |f(x) - g(x)| &\le \sup_{x_0\le x\le x_0+a} |f(x) - g(x)| + \sup_{b_m\ge x\ge x_0+a} |f(x)| \\ &\le 2 \sup_{x_0\le x\le x_0+a} |f(x)| + \sup_{b_m\ge x\ge x_0+a} |f(x)| \, . \end{aligned}$$

We get by (2.2.4)

$$\sup_{0\le x\le b_m} |f(x) - g(x)| \le 3\varepsilon.$$

Now, we obtain

$$\sup_{0\le x\le b_m}\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q_m,x)-f(x)\right|$$

$$\le \sup_{0\le x\le b_m}\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(|f-g|;q_m,x)\right|+\sup_{0\le x\le b_m}\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(g;q_m,x)-g(x)\right|$$

$$+\sup_{0\le x\le b_m}|f(x)-g(x)|$$

$$\le 6\varepsilon+\sup_{0\le x\le b_m}\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(g;q_m,x)-g(x)\right|,$$

where $g(x)=0$ for $x_0+a\le x\le b_m$. From Lemma 2.2.4, we get the result. ■

2.3 Order of Convergence

We give the error of approximation of the operators $\mathscr{C}_{m,p}^{(\alpha,\beta)}$ in terms of the Lipschitz class $Lip_M(\mu)$, for $0<\mu\le 1$. Let $C_B[0,\infty)$ denote the space of bounded continuous functions on $[0,\infty)$. A function $f\in C_B[0,\infty)\subset Lip_M(\mu)$ if

$$|f(t)-f(x)|\le M|t-x|^{\mu}\qquad (t,x\in[0,\infty)) \tag{2.3.1}$$

is satisfied.

Theorem 2.3.1 *([38]) Let $f\in Lip_M(\mu)$. Then, we have*

$$|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)|\le M\left(\delta_{m,q}(x)\right)^{\mu/2}$$

where $\delta_{m,q}(x)=\left(\frac{\left(q_m^m[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2}\right)x^2+\left(\frac{(2\alpha+1)[m+p]_q}{\left([m]_q+\beta\right)^2}-\frac{2\alpha}{[m]_q+\beta}\right)b_m x+\frac{\alpha^2 b_m^2}{\left([m]_q+\beta\right)^2}.$

Proof. From the monotonicity and the linearity of the operators, for $f\in Lip_M(\mu)$, we get

$$
\begin{aligned}
&|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)| \\
&=|\sum_{r=0}^{m+p}(f(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m)-f(x)\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)| \\
&\leq \sum_{r=0}^{m+p}\left|f(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m)-f(x)\right|\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \\
&\leq M\sum_{r=0}^{m+p}|\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x|^\mu\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right).
\end{aligned}
$$

By Hölder's inequality with $p=\dfrac{2}{\gamma}$ and $q=\frac{2}{2-\gamma}$, we have from (2.1.2)

$$
\begin{aligned}
&|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)| \\
&\leq M\sum_{r=0}^{m+p}\left\{[(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x)^2\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)]^{\frac{\mu}{2}}\right. \\
&\left.\times[\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)]^{\frac{2-\mu}{2}}\right\} \\
&\leq M\left[\{\sum_{r=0}^{m+p}[(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x)^2\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)]\}^{\frac{\mu}{2}}\right. \\
&\left.\times\{\sum_{r=0}^{m+p}\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)]\}^{\frac{2-\mu}{\mu}}\right] \\
&=M[\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)]^{\frac{\mu}{2}} \\
&\leq M(\delta_{m,q}(x))^{\frac{\mu}{2}}.
\end{aligned}
$$

Whence the result. ■

Now we state the rate of convergence of the operators by means of the modulus of continuity which is represented by $\omega(f;\delta)$. Let $f\in C_B[0,\infty)$ and $x\geq 0$. Then let us give the definition of the modulus of continuity of f as

$$\omega(f;\delta)=\max_{\substack{|t-x|\leq\delta \\ t,x\in[0,\infty)}}|f(t)-f(x)|.$$

For any $\delta > 0$ the following property of modulus of continuity

$$|f(x) - f(y)| \leq \omega(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right) \tag{2.3.2}$$

is satisfied ([8]).

Theorem 2.3.2 *([38]) If $f \in C_B[0,\infty)$, we get*

$$\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x)\right| \leq 2\omega\left(f;\sqrt{\delta_{m,q}(x)}\right)$$

where $\delta_{m,q}(x)$ be defined in Theorem 2.3.1 and $\omega(f;\cdot)$ is the modulus of continuity.

Proof. From triangular inequality, we have

$$\begin{aligned}
&\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x)\right| \\
&= \left|\sum_{r=0}^{m+p} f\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m\right)\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) - f(x)\right| \\
&\leq \sum_{r=0}^{m+p}\left|\left(f\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m\right) - f(x)\right|\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right).
\end{aligned}$$

Then from (2.3.2) and Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right| \\
&\leq \sum_{r=0}^{m+p}\left(\frac{\left|\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right|}{\delta}+1\right)\omega(f;\lambda)\begin{bmatrix} m+p \\ r \end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \\
&=\omega(f;\delta)\sum_{r=0}^{m+p}\begin{bmatrix} m+p \\ r \end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \\
&+\frac{\omega(f;\delta)}{\delta}\sum_{r=0}^{m+p}\left|\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right|\begin{bmatrix} m+p \\ r \end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \\
&=\omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\left\{\sum_{r=0}^{m+p}\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right)^2\begin{bmatrix} m+p \\ r \end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\right\}^{1/2} \\
&=\omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\left\{\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2\right);q,x\right\}^{1/2}.
\end{aligned}
$$

Finally, let us choose $\delta_{m,q}(x)$ as in Theorem 2.3.1, then we get

$$
\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|\leq 2\omega\left(f;\sqrt{\delta_{m,q}(x)}\right).
$$

■

Theorem 2.3.3 *([38]) If $f(x)$ has a continuous bounded derivative $f'(x)$ and $\omega\left(f';\delta\right)$ in $[0,A]$, then*

$$
\begin{aligned}
&|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)| \\
&\leq M\left(\left|\frac{[m+p]_q}{[m]_q+\beta}-1\right|A+\frac{\alpha b_m}{[m]_q+\beta}\right)+2\left(B_{m,q}(\alpha,\beta)\right)^{1/2}\omega\left(f';\left(B_{m,q}(\alpha,\beta)\right)^{1/2}\right),
\end{aligned}
$$

where $\omega\left(f';\delta\right)$ is the modulus of continuity of $f'(x)$, M is a positive constant such that $|f'(x)|\leq M$, and $B_{m,q}(\alpha,\beta)=\frac{\left(q^m[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2}A^2+\left|\frac{(2\alpha+1)[m+p]_q}{\left([m]_q+\beta\right)^2}-\frac{2\alpha}{[m]_q+\beta}\right|Ab_m+\frac{\alpha^2b_m^2}{\left([m]_q+\beta\right)^2}$.

Proof. From the mean value theorem, we get

$$f\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m\right)-f(x)=\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right)f^{'}(\eta)$$
$$=\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right)f^{'}(x)+\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right)\left(f^{'}(\eta)-f^{'}(x)\right),$$

where η is a point between x and $\frac{[r]_q+\alpha}{[m]_q+\beta}b_m$. With the help of the above equality, we obtain

$$\begin{aligned}
&\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\\
&=f^{'}(x)\sum_{r=0}^{m+p}\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_n-x\right)\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\\
&+\sum_{k=0}^{n+p}\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_n-x\right)\left(f^{'}(\eta)-f^{'}(x)\right)\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_n}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|\\
&\leq\left|f^{'}(x)\right|\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}((u-x);q,x)\right|\\
&+\sum_{r=0}^{m+p}\left|\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right|\left|f^{'}(\eta)-f^{'}(x)\right|\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\\
&\leq M\left(\left|\frac{[m+p]_q}{[m]_q+\beta}-1\right|A+\frac{\alpha b_m}{[m]_q+\beta}\right)\\
&+\sum_{k=0}^{n+p}\left|\frac{[r]_q+\alpha}{[m]_q+\beta}-x\right|\left|f^{'}(\xi)-f^{'}(x)\right|\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\\
&\leq M\left(\left|\frac{[m+p]_q}{[m]_v+\beta}-1\right|A+\frac{\alpha b_m}{[m]_q+\beta}\right)\\
&+\sum_{k=0}^{n+p}\omega\left(f';\delta\right)\left(\frac{\left|\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right|}{\delta}+1\right)\left|\frac{[r]_q+\alpha}{[m]_q+\beta}b_m-x\right|\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\\
&\times\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right),
\end{aligned}$$

since

$$|\eta - x| \leq \left| \frac{[r]_q + \alpha}{[m]_q + \beta} b_m - x \right|.$$

From above inequality, we get

$$\begin{aligned}
&\left| \mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x) \right| \\
&\leq M \left(\left| \frac{[m+p]_q}{[m]_q + \beta} - 1 \right| A + \frac{\alpha b_m}{[m]_q + \beta} \right) \\
&+ \omega(f';\delta) \sum_{r=0}^{m+p} \left| \frac{[r]_q + \alpha}{[m]_q + \beta} b_m - x \right| \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m} \right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m} \right) \\
&+ \frac{\omega(f';\delta)}{\delta} \sum_{r=0}^{m+p} \left(\frac{[r]_q + \alpha}{[m]_q + \beta} b_m - x \right)^2 \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m} \right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m} \right).
\end{aligned}$$

Therefore, performing Cauchy-Schwarz inequality for the second term, we have

$$\begin{aligned}
&\left| \mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x) \right| \\
&\leq M \left(\left| \frac{[m+p]_q}{[m]_q + \beta} - 1 \right| A + \frac{\alpha b_m}{[m]_q + \beta} \right) \\
&+ \omega(f';\delta) \left(\sum_{r=0}^{m+p} \left(\frac{[r]_q + \alpha}{[m]_q + \beta} b_m - x \right)^2 \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m} \right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m} \right) \right)^{1/2} \\
&+ \frac{\omega(f';\delta)}{\delta} \sum_{r=0}^{m+p} \left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_m - x \right)^2 \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m} \right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m} \right). \\
&= M \left(\left| \frac{[m+p]_q}{[m]_q + \beta} - 1 \right| A + \frac{\alpha b_m}{[m]_q + \beta} \right) \\
&+ \omega(f';\delta) \sqrt{\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)} + \frac{\omega(f';\delta)}{\delta} \mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right).
\end{aligned}$$

Hence, by (2.2.2), we can write that

$$\sup_{0\leq x\leq A}\mathscr{C}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)\leq\sup_{0\leq x\leq A}\frac{\left(q^m[p]_q-\beta\right)^2}{\left([m]_q+\beta\right)^2}x^2$$
$$+\left(\frac{(2\alpha+1)[m+p]_q}{\left([m]_q+\beta\right)^2}-\frac{2\alpha}{[m]_q+\beta}\right)b_mx+\frac{\alpha^2b_m^2}{\left([m]_q+\beta\right)^2}$$
$$=B_{m,q}(\alpha,\beta).$$

Therefore,

$$\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right|\leq M\left(\left|\frac{[m+p]_q}{[m]_q+\beta}-1\right|A+\frac{\alpha b_m}{[m]_q+\beta}\right)$$
$$+\omega\left(f';\delta\right)\left[\left(B_{m,q}(\alpha,\beta)\right)^{1/2}+\frac{1}{\delta}B_{m,q}(\alpha,\beta)\right].$$

By choosing $\delta:=\delta_{m,q}(p)=\left(B_{m,q}(\alpha,\beta)\right)^{1/2}$, we complete the proof. ■

2.4 Generalization of the operators

In this section, we introduce generalization of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. The generalized operators help us to approximate continuous functions on more general weighted spaces. Note that this kind of generalization was considered earlier for the Bernstein-Chlodowsky polynomials [12] and q-Bernstein-Chlodowsky polynomials [4].

For $x\geq 0$, consider any continuous function $\omega(x)\geq 1$ and define

$$G_f(t)=f(t)\frac{1+t^2}{\omega(t)}.$$

Let us take into consideration the generalization of the $\mathscr{C}_{m,p}^{(\alpha,\beta)}(f;q,x)$ by

$$\mathscr{L}_{n,p}^{(\alpha,\beta)}(f;q,x)$$
$$= \frac{\omega(x)}{1+x^2}\sum_{r=0}^{m+p} G_f\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m\right)\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right),$$

where $0 \leq x \leq b_m$ and (b_m) has the same properties of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators.

Theorem 2.4.1 *[38] For the continuous functions satisfying*

$$\lim_{x\to\infty}\frac{f(x)}{\omega(x)} = N_f < \infty,$$

we have

$$\lim_{n\to\infty}\sup_{0\leq x\leq b_m}\frac{\left|\mathscr{L}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|}{\omega(x)} = 0.$$

Proof. Clearly,

$$\mathscr{L}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)$$
$$= \frac{\omega(x)}{1+x^2}\left(\sum_{r=0}^{m+p} G_f\left(\frac{[r]_q+\alpha}{[m]_q+\beta}b_m\right)\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)-G_f(x)\right),$$

thus

$$\sup_{0\leq x\leq b_m}\frac{\left|\mathscr{L}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|}{\omega(x)} = \sup_{0\leq x\leq b_m}\frac{\left|\mathscr{C}_{m,p}^{(\alpha,\beta)}\left(G_f;q,x\right)-G_f(x)\right|}{1+x^2}.$$

By using $|f(x)| \leq N_f\omega(x)$ and continuity of the function f, we get that $\left|G_f(x)\right| \leq$

$N_f\left(1+x^2\right)$ for $x \geq 0$ and $G_f(x)$ is a continuous function on $[0,\infty)$. Hence, by Theorem 2.2.2 we get the result. ■

Lastly, notice that, if we take $\omega(x) = 1+x^2$, then the operators $\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)$ reduces to $\mathscr{C}_{m,p}^{(\alpha,\beta)}\left(G_f;q,x\right)$.

Chapter 3

CHLODOWSKY TYPE q-BERNSTEIN-STANCU-KANTOROVICH OPERATOR

3.1 Construction of the operators

For fixed $p \in \mathbb{N}_0$, Chlodowsky type q-Bernstein-Stancu-Kantorovich operators [39] are defined by

$$\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) := \sum_{r=0}^{m+p} \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m}\right)$$
$$\times \int_0^1 f\left(\frac{\left(1+(q-1)[r]_q\right)t + [r]_q + \alpha}{[m+1]_q + \beta} b_m\right) dt, \tag{3.1.1}$$

where $0 \leq x \leq b_m$, $0 < q < 1$, α and β are real numbers with $0 \leq \alpha \leq \beta$, $m \in \mathbb{N}$. We should mention that, if we take $p = \alpha = \beta = 0$ in (3.1.1), the operator $\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x)$ reduces to the Chlodowsky-variant q-Bernstein Kantorovich operator. Note that, the q-Bernstein-Kantorovich operators were defined in [25].

Lemma 3.1.1 *([39]) For the operator $\mathscr{K}_{n,p}^{(\alpha,\beta)}(f;q,x)$ which is given in (3.1.1), we calculate the following few moments:*

(i) $\mathscr{K}_{m,p}^{(\alpha,\beta)}(1;q,x) = 1$,

(ii) $\mathscr{K}_{m,p}^{(\alpha,\beta)}(u;q,x) = \frac{[m+p]_q[2]_q x + (2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}$,

(iii) $\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(u^2;q,x\right) = \frac{1}{\left([m+1]_q+\beta\right)^2}\left\{\frac{[3]_q}{3}[m+p-1]_q[m+p]_q q x^2\right.$
$\left. + \left(\frac{q^2+3q+2}{3} + [2]_q \alpha\right)[m+p]_q b_n x + \left(\alpha^2+\alpha+\frac{1}{3}\right)b_m^2\right\}$,

(iv) $\mathscr{K}_{m,p}^{(\alpha,\beta)}(u-x\,;q,x) = \left(\frac{[m+p]_q[2]_q}{2\left([m+1]_q+\beta\right)} - 1\right)x + \frac{(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}$,

(v) $\mathscr{K}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)=\left(\frac{[3]_q[m+p-1]_q[m+p]_q q}{3\left([m+1]_q+\beta\right)^2}-\frac{[2]_q[m+p]_q}{[m+1]_q+\beta}+1\right)x^2$
$+\left(\frac{\left(q^2+3q+2+3[2]_q\alpha\right)[m+p]_q}{3\left([m+1]_q+\beta\right)^2}-\frac{(2\alpha+1)}{[m+1]_q+\beta}\right)b_m x+\frac{\left(3\alpha^2+3\alpha+1\right)b_m^2}{3\left([m+1]_q+\beta\right)^2}.$

Proof. (i) From (3.1.1) and the fact that $C_{n,q}(1,x)=1$ we have,

$$\mathscr{K}_{m,p}^{(\alpha,\beta)}(1;q,x)=\sum_{r=0}^{m+p}\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \tag{3.1.2}$$
$$=\mathscr{C}_{n,p}^{(\alpha,\beta)}(1;q,x)=1.$$

(ii) Direct calculations yield,

$$\begin{aligned}
&\mathscr{K}_{m,p}^{(\alpha,\beta)}(u;q,x)\\
&=\sum_{r=0}^{m+p}\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\\
&\times\int_0^1\left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m\right)dt\\
&=\sum_{r=0}^{m+p}\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\\
&\times\left(\frac{[r]_q+\alpha}{[m+1]_q+\beta}b_m+\frac{1+(q-1)[r]_q}{2\left([m+1]_q+\beta\right)}b_m\right)\\
&=\frac{[m+p]_q[2]_q x+(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}.
\end{aligned}$$

(iii) From (3.1.2) we have

$$\begin{aligned}
&\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(u^2;q,x\right)\\
&=\sum_{r=0}^{m+p}\begin{bmatrix} m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\\
&\times\int_0^1\left(\frac{\left(1+(q-1)\,[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m\right)^2dt\\
&=\sum_{r=0}^{m+p}\begin{bmatrix} m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\frac{b_m^2}{\left([m+1]_q+\beta\right)^2}\\
&\times\left\{\left(\frac{2\,(q-1)^2}{3}+q\right)[r]_q^2+\left(1+[2]_q\,\alpha\right)[k]_q+\alpha^2+\alpha+\frac{1}{3}\right\}.
\end{aligned}$$

If we continue our calculations as in the above assertions, we get the desired conclusion.

(iv) By (i) and (ii), we get

$$\begin{aligned}
\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(u-x\,;q,x\right)&=\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(u;q,x\right)-x\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(1;q,x\right)\\
&=\left(\frac{[m+p]_q\,[2]_q}{2\left([m+1]_q+\beta\right)}-1\right)x+\frac{(2\alpha+1)\,b_m}{2\left([m+1]_q+\beta\right)}.
\end{aligned}$$

(v) Using the following well known property we can obtain our result as

$$\mathscr{K}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)=\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(u^2;q,x\right)-2x\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(u;q,x\right)+x^2\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(1;q,x\right).$$

The proof is completed by (i), (ii) and (iii). ■

Lemma 3.1.2 *([39]) If we take supremum on* $[0,b_m]$ *for* $\mathscr{K}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)$*, we obtain the following estimate:*

$$\mathscr{K}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right) \le b_m^2 \left\{ \left| \frac{[3]_q [m+p-1]_q [m+p]_q q}{3\left([m+1]_q+\beta\right)^2} - \frac{[2]_q [m+p]_q}{[m+1]_q+\beta} + 1 \right| \right.$$
$$+ \left| \frac{\left(q^2+3q+2+3[2]_q \alpha\right)[m+p]_q}{3\left([m+1]_q+\beta\right)^2} - \frac{(2\alpha+1)}{[m+1]_q+\beta} \right|$$
$$\left. + \frac{\left(3\alpha^2+3\alpha+1\right)}{3\left([m+1]_q+\beta\right)^2} \right\}.$$

3.2 Korovkin-Type Approximation Theorem

Let us choose $\mu(x) = 1+x^2$ and take into consideration the operators:

$$\mathscr{U}_m^{(\alpha,\beta)}(f;q,x) = \begin{cases} \mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) & \text{if } x \in [0,b_m] \\ f(x) & \text{if } x \in [0,\infty) \ / \ [0,b_m] \end{cases}.$$

Notice that, the operators $\mathscr{U}_m^{(\alpha,\beta)}(f;q,x)$ act from C_{1+x^2} to C_{1+x^2} . Indeed, for all $f \in C_{1+x^2}$, we get

$$\left\| \mathscr{U}_m^{(\alpha,\beta)}(f;q,\cdot) \right\|_{1+x^2} \le \sup_{x\in[0,b_m]} \frac{\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x)\right|}{1+x^2} + \sup_{b_m<x<\infty} \frac{|f(x)|}{1+x^2}$$
$$\le \|f\|_{1+x^2} \left[\sup_{x\in[0,\infty)} \frac{\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(1+t^2;q,x\right)\right|}{1+x^2} + 1 \right].$$

Hence, using Lemma 3.1.1, we have

$$\left\| \mathscr{U}_m^{(\alpha,\beta)}(f;q,\cdot) \right\|_{1+x^2} \le M \|f\|_{1+x^2}$$

provided that $q := (q_m)$ with $0 < q_m < 1$, $\lim_{m\to\infty} q_m = 1$, $\lim_{m\to\infty} q_m^m = N < \infty$ and $\lim_{m\to\infty} \frac{b_m}{[m]} = 0$.

Theorem 3.2.1 *([39]) For all $f \in C^0_{1+x^2}$, we get*

$$\lim_{n\to\infty} \left\| \mathscr{U}_m^{(\alpha,\beta)}(f;q_m,\cdot) - f(\cdot) \right\|_{1+x^2} = 0$$

provided that $q := (q_m)$ with $0 < q_m < 1$, $\lim_{m\to\infty} q_m = 1$, $\lim_{m\to\infty} q_m^m = N < \infty$ and $\lim_{m\to\infty} \frac{b_m}{[m]_q} =$ 0.

Proof. Using Theorem 3.2.2 and Lemma 2.1.1 (i), (ii) and (iii), we get

$$\sup_{x\in[0,\infty)} \frac{\left|\mathscr{U}_m^{(\alpha,\beta)}(1;q,x)-1\right|}{1+x^2} = \sup_{0\le x\le b_m} \frac{\left|\mathscr{K}_m^{(\alpha,\beta)}(1;q,x)-1\right|}{1+x^2} = 0,$$

$$\begin{aligned}
&\sup_{x\in[0,\infty)} \frac{\left|\mathscr{U}_m^{(\alpha,\beta)}(u;q,x)-u\right|}{1+x^2} \\
&= \sup_{0\le x\le b_m} \frac{\left|\mathscr{K}_m^{(\alpha,\beta)}(u;q,x)-x\right|}{1+x^2} \le \sup_{0\le x\le b_m} \frac{\left|\frac{[m+p]_q[2]_q}{2\left([m+1]_q+\beta\right)}-1\right| x + \frac{(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}}{(1+x^2)} \\
&\le \left|\frac{[m+p]_q[2]_q}{2\left([m+1]_q+\beta\right)}-1\right| + \frac{(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)} \to 0
\end{aligned}$$

and

$$
\begin{aligned}
\sup_{x\in[0,\infty)} \frac{\left|\mathscr{U}_m^{(\alpha,\beta)}\left(u^2;q,x\right)-u^2\right|}{1+x^2} &= \sup_{0\le x\le b_m} \frac{\left|\mathscr{K}_m^{(\alpha,\beta)}\left(u^2;q,x\right)-x^2\right|}{1+x^2} \\
&\le \sup_{0\le x\le b_m} \frac{1}{1+x^2}\left\{\left|\frac{[3]_q}{3}\frac{[m+p-1]_q\,[m+p]\,q}{\left([m+1]_q+\beta\right)^2}-1\right|x^2\right. \\
&+\left(\frac{q^2+3q+2}{3}+[2]_q\,\alpha\right)\frac{[m+p]_q\,b_m}{\left([m+1]_q+\beta\right)^2}x+\left.\left(\alpha^2+\alpha+\frac{1}{3}\right)\frac{b_m^2}{\left([m+1]_q+\beta\right)^2}\right\} \\
&\le \left\{\left|\frac{[3]_q}{3}\frac{[m+p-1]_q\,[m+p]_q\,q}{\left([m+1]_q+\beta\right)^2}-1\right|\right. \\
&+\left(\frac{q^2+3q+2}{3}+[2]_q\,\alpha\right)\frac{[m+p]_q\,b_m}{\left([m+1]_q+\beta\right)^2} \\
&+\left(\alpha^2+\alpha+\frac{1}{3}\right)\frac{b_m^2}{\left([m+1]_q+\beta\right)^2}\to 0
\end{aligned}
$$

whenever $m\to\infty$, since $q=q_m$ with $\lim\limits_{m\to\infty} q_m=1$ and $\frac{b_m}{[m]}=0$ as $m\to\infty$. ■

Lemma 3.2.2 *([39]) Let $A\in\mathbb{R}^+$ be independent of m and f be a continuous function which vanishes on $[A,\infty)$. Suppose that $q:=(q_m)$ with $0<q_m<1$, $\lim\limits_{m\to\infty} q_m=1$, $\lim\limits_{m\to\infty} q_m^m=N<\infty$ and $\lim\limits_{m\to\infty}\frac{b_m^2}{[m]_q}=0$. Then we get*

$$
\lim_{m\to\infty}\sup_{0\le x\le b_m}\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(f;q_m,x\right)-f(x)\right|=0.
$$

Proof. By the hypothesis on f, one can write $|f(x)|\le M$ $(M>0)$. For arbitrary small $\varepsilon>0$, we have

$$
\begin{aligned}
&\left| f\left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta} b_m \right) - f(x) \right| \\
< \quad & \varepsilon + \frac{2M}{\delta^2}\left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta} b_m - x \right)^2 ,
\end{aligned}
$$

for $x \in [0, b_m]$ and $\delta = \delta(\varepsilon)$. With the help of the following equality

$$
\begin{aligned}
&\sum_{r=0}^{m+p} \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1} \left(1-q^s \frac{x}{b_m}\right) \\
&\times \int_0^1 \left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta} b_m - x \right)^2 dt \\
&= \mathscr{K}_{m,p}^{(\alpha,\beta)}\left((t-x)^2; q_m, x\right),
\end{aligned}
$$

we have from Lemma 4.2.2 that

$$
\begin{aligned}
&\sup_{0\le x\le b_m} \left| \mathscr{K}_{m,p}^{(\alpha,\beta)}(f; q_m, x) - f(x) \right| \\
&\le \varepsilon + \frac{2M}{\delta^2}\left[\left| \frac{[3]_q [m+p-1]_q [m+p]_q q_m}{3\left([m+1]_q+\beta\right)^2} - \frac{[2]_q [m+p]_q}{[m+1]_q+\beta} + 1 \right| b_m^2 \right. \\
&\left. + \left| \frac{\left(q_m^2+3q_m+2+3[2]_q\alpha\right)[m+p]_q}{\left([m+1]_q+\beta\right)^2} - \frac{(2\alpha+1)}{[m+1]_q+\beta} \right| b_m^2 + \frac{\left(3\alpha^2+3\alpha+1\right)b_m^2}{3\left([m+1]_q+\beta\right)^2} \right].
\end{aligned}
$$

Since $0<q_m<1$, $\lim\limits_{m\to\infty} q_m = 1$, $\lim\limits_{m\to\infty} q_m^m = N < \infty$ and $\lim\limits_{m\to\infty} \frac{b_m^2}{[m]_q} = 0$, we get the desired result. ■

Theorem 3.2.3 *([39]) Let f be a continuous function on $[0,\infty)$ and*

$$\lim_{x\to\infty} f(x) = N_f < \infty.$$

Suppose that $q := (q_m)$ with $0<q_m<1$, $\lim\limits_{m\to\infty} q_m = 1$, $\lim\limits_{m\to\infty} q_m^m = K < \infty$ and $\lim\limits_{m\to\infty} \frac{b_m^2}{[m]_q} =$

0. *Then*

$$\lim_{x\to\infty}\sup_{0\le x\le b_m}\left|\mathcal{K}_{m,p}^{(\alpha,\beta)}(f;q_m,x)-f(x)\right|=0.$$

Proof. Applying the same methods as in the proof of Theorem 3.2.5 in [38] and using Lemma 3.2.4, we get the desired conclusion. ■

3.3 Order of Convergence

In this subsection, the error of approximation of operators $\mathcal{K}_{m,p}^{(\alpha,\beta)}(f;q,x)$ are given.

Theorem 3.3.1 *([39]) Let $f\in Lip_M(\gamma)$. Then we have*

$$\left|\mathcal{K}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|\le M\left(\lambda_{m,q}(x)\right)^{\gamma/2}$$

where

$$\begin{aligned}\lambda_{m,q}(x)=&\left(\frac{[3]_q[m+p-1]_q[m+p]q}{3\left([m+1]_q+\beta\right)^2}-\frac{[2]_q[m+p]_q}{[m+1]_q+\beta}+1\right)x^2\\&+\left(\frac{\left(q^2+3q+2+3[2]_q\alpha\right)[m+p]_q}{\left([m+1]_q+\beta\right)^2}-\frac{(2\alpha+1)}{[m+1]_q+\beta}\right)b_mx\\&+\frac{\left(3\alpha^2+3\alpha+1\right)b_m^2}{3\left([m+1]_q+\beta\right)^2}.\end{aligned}$$

Proof. From the monotonicity and the linearity of the operators, we have for $f\in Lip_M(\gamma)$ that

$$
\begin{aligned}
&|\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)| \\
&=\left|\sum_{r=0}^{m+p}\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^k \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\right. \\
&\left.\times\int_0^1 \left(f\left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m\right)-f(x)\right)dt\right| \\
&\leq \sum_{r=0}^{m+p}\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^k \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \\
&\times\int_0^1 \left|f\left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m\right)-f(x)\right|dt \\
&\leq M\sum_{r=0}^{m+p}\begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^k \prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \\
&\times\int_0^1 \left|\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m-x\right|^\gamma dt.
\end{aligned}
$$

Performing Hölder's inequality with $p=\frac{2}{\gamma}$ and $q=\frac{2}{2-\gamma}$, we have the following inequalities by (3.1.2)

$$
\begin{aligned}
&\int_0^1 \left|\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m-x\right|^\gamma dt \\
&\leq \left\{\int_0^1 \left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m-x\right)^2 dt\right\}^{\frac{\gamma}{2}} \left\{\int_0^1 dt\right\}^{\frac{2-\gamma}{2}} \\
&= \left\{\int_0^1 \left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m-x\right)^2 dt\right\}^{\frac{\gamma}{2}}.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
&\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
\leq\ & M\sum_{r=0}^{m+p}\left\{\int_0^1 \left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta}b_m-x\right)^2 dt\right\}^{\frac{\gamma}{2}} p_{m,r}(q;x)
\end{aligned}
$$

where $p_{m,r}(q;x) = \sum_{r=0}^{m+p} \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m}\right)$. Again using the Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we have

$$\begin{aligned}
&\left| \mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x) \right| \\
&\leq M \left\{ \sum_{r=0}^{m+p} \int_0^1 \left(\frac{\left(1 + (q-1)[r]_q\right) t + [r]_q + \alpha}{[m+1]_q + \beta} b_m - x \right)^2 dt \; p_{m,r}(q,x) \right\}^{\frac{\gamma}{2}} \\
&\times \left\{ \sum_{r=0}^{m+p} p_{m,r}(q,x) \right\}^{\frac{2-\gamma}{2}} \\
&= M \left\{ \sum_{r=0}^{m+p} p_{m,r}(q,x) \int_0^1 \left(\frac{\left(1 + (q-1)[r]_q\right) t + [r]_q + \alpha}{[m+1]_q + \beta} b_m - x \right)^2 dt \right\}^{\frac{\gamma}{2}} \\
&= M \left(\lambda_{m,q}(x)\right)^{\gamma/2},
\end{aligned}$$

where $\lambda_{m,q}(x) := \mathscr{K}_{m,p}^{(\alpha,\beta)}\left((u-x)^2; q, x\right)$. ■

Theorem 3.3.2 *([39]) If $f \in C_B[0,\infty)$, we have*

$$\left| \mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x) \right| \leq 2\omega\left(f; \sqrt{\lambda_{m,q}(x)}\right)$$

where $\lambda_{m,q}(x)$ is the same as in Theorem 3.3.1.

Proof. From monotonicity, we get

$$\begin{aligned}
\left| \mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x) \right| &\leq \sum_{r=0}^{m+p} \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m}\right) \\
&\times \int_0^1 \left| f\left(\frac{\left(1 + (q-1)[r]_q\right) t + [r]_q + \alpha}{[m+1]_q + \beta} b_n \right) - f(x) \right| dt
\end{aligned}$$

$$\leq \sum_{r=0}^{m+p} \int_0^1 \left(\frac{\left| \frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta} b_m - x \right|}{\delta} + 1 \right)$$

$$\times \omega(f;\delta) \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1} \left(1-q^s \frac{x}{b_m}\right) dt$$

$$= \omega(f;\delta) \sum_{r=0}^{m+p} \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-k-1} \left(1-q^s \frac{x}{b_m}\right)$$

$$+ \frac{\omega(f;\delta)}{\delta} \sum_{r=0}^{m+p} \int_0^1 \left| \frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta} b_m - x \right| \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^k$$

$$\times \prod_{s=0}^{m+p-r-1} \left(1-q^s \frac{x}{b_m}\right) dt.$$

After that, from Cauchy-Schwarz inequality we get

$$\left| \mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x) \right| \leq \omega(f;\delta)$$

$$+ \frac{\omega(f;\delta)}{\delta} \left\{ \sum_{r=0}^{m+p} p_{m,r}(q,x) \int_0^1 \left(\frac{\left(1+(q-1)[r]_q\right)t+[r]_q+\alpha}{[m+1]_q+\beta} b_m - x \right)^2 dt \right\}^{\frac{1}{2}}$$

$$\times \left\{ \sum_{r=0}^{m+p} p_{m,r}(q,x) \right\}^{\frac{1}{2}}$$

$$= \omega(f;\delta) + \frac{\omega(f;\delta)}{\delta} \left\{ \mathscr{K}_{m,p}^{(\alpha,\beta)}\left((u-x)^2\right);q,x \right\}^{1/2}.$$

Finally, let us choose $\delta_{m,q}(x)$ the same as in Theorem 3.3.1. Then we get

$$\left| \mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x) \right| \leq 2\omega\left(f;\sqrt{\delta_{m,q}(x)}\right).$$

■

Now let us denote, $C_B^2[0,\infty)$, the space of all functions $f \in C_B[0,\infty)$ such that f', $f'' \in C_B^2[0,\infty)$. Let $\|f\|$ denote the usual supremum norm of f. The classical Peetre's

K-functional and the second modulus of continuity of the function $f \in C_B[0,\infty)$ are described respectively by

$$\mathscr{K}(f,\delta) := \inf_{g \in C_B^2[0,\infty)} \left[\|f-g\| + \delta \left\|g''\right\| \right]$$

and

$$\omega_2(f,\delta) := \sup_{\substack{0<h<\delta, \\ x,x+h \in I}} |f(x+2h) - 2f(x+h) + f(x)|$$

where $\delta > 0$. For $A > 0$, the following well known property [8, p.177]

$$\mathscr{K}(f,\delta) \leq A\omega_2\left(f,\sqrt{\delta}\right) \tag{3.3.3}$$

is satisfied.

Theorem 3.3.3 *Let $q \in (0,1)$, $x \in [0,b_m]$ and $f \in C_B[0,\infty)$. Then for fixed $p \in \mathbb{N}_0$, we have*

$$\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x) - f(x)\right| \leq C\omega_2\left(f,\sqrt{\alpha_{m,q}(x)}\right) + \omega\left(f,\beta_{m,q}(x)\right)$$

for some positive constant C, where

$$\begin{aligned}\alpha_{m,q}(x) := \Bigg[& \left(\left(\frac{[3]_q}{3} + \frac{[2]_q^2}{4} \right) \frac{[m+p]_q^2}{\left([m+1]_q+\beta\right)^2} - 2\frac{[2]_q[m+p]_q}{[m+1]_q+\beta} + 2 \right) x^2 \\ & + \left(\frac{q^2+3q+2+3[2]_q\alpha}{3\left([m+1]_q+\beta\right)^2} + \frac{(2\alpha+1)[2]_q[m+p]_q}{2\left([m+1]_q+\beta\right)} - \frac{2(2\alpha+1)}{[m+1]_q+\beta} \right) b_m x \\ & + \left(\frac{24\alpha^2+24\alpha+7}{12} \right) \frac{b_m^2}{\left([m+1]_q+\beta\right)^2} \Bigg]\end{aligned} \tag{3.3.4}$$

and

$$\beta_{m,q}(x) := \left| \frac{[2]_q [m+p]_q}{2\left([m+1]_q + \beta\right)} - 1 \right| x + \frac{(2\alpha+1) b_m}{2\left([m+1]_q + \beta\right)}. \tag{3.3.5}$$

Proof. Define an auxiliary operator $\mathscr{K}^*_{n,p}(f;q,x) : C_B[0,\infty) \to C_B[0,\infty)$ by

$$\mathscr{K}^*_{m,p}(f;q,x) := \mathscr{K}^{(\alpha,\beta)}_{m,p}(f;q,x) - f\left(\frac{[2]_q [m+p]_q x + (2\alpha+1) b_m}{2\left([m+1]_q + \beta\right)}\right) + f(x). \tag{3.3.6}$$

Then, by Lemma 3.1.1, we get

$$\mathscr{K}^*_{m,p}(1;q,x) = 1$$

$$\mathscr{K}^*_{m,p}(u-x;q,x) = 0. \tag{3.3.7}$$

For a given $g \in C_B^2[0,\infty)$, it follows by the Taylor formula that

$$g(y) - g(x) = (y-x) g'(x) +_x^y (y-\tau) g''(\tau) d\tau.$$

Taking into account (3.3.5) and using (3.3.7) we get

$$\begin{aligned} & \left|\mathscr{K}^*_{m,p}(g;q,x) - g(x)\right| = \left|\mathscr{K}^*_{m,p}(g(y) - g(x);q,x)\right| \\ = & \left| g'(x) \mathscr{K}^*_{m,p}((\tau - x);q,x) + \mathscr{K}^*_{m,p}\left(\int_x^y (y-\tau) g''(\tau) d\tau;q,x\right)\right| \\ = & \left|\mathscr{K}^*_{m,p}\left(\int_x^y (y-\tau) g''(\tau) d\tau;q,x\right)\right|. \end{aligned}$$

Then by (3.3.6),

$$
\begin{aligned}
&\left|\mathscr{K}_{m,p}^{*}\left(g;q;x\right)-g\left(x\right)\right| \\
=\ &\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(\int_{x}^{y}\left(y-\tau\right)g''\left(\tau\right)d\tau;q,x\right)\right. \\
&\left.-\int_{x}^{\frac{[2]_q[m+p]_qx+(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}}\left(\frac{[2]_q\left[m+p\right]_q x+\left(2\alpha+1\right)b_m}{2\left(\left[m+1\right]_q+\beta\right)}-u\right)g''\left(\tau\right)d\tau\right| \\
\leq\ &\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(\int_{x}^{y}\left(y-\tau\right)g''\left(\tau\right)d\tau;q,x\right)\right| \\
&+\left|\int_{x}^{\frac{[2]_q[m+p]_qx+(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}}\left(\frac{[2]_q\left[m+p\right]_q x+\left(2\alpha+1\right)b_m}{2\left(\left[m+1\right]_q+\beta\right)}-u\right)g''\left(\tau\right)d\tau\right|.
\end{aligned}
$$

Since

$$
\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(\int_{x}^{y}\left(y-\tau\right)g''\left(\tau\right)d\tau;q;x\right)\right|\leq\left\|g''\right\|K_{m,p}^{(\alpha,\beta)}\left(\left(y-x\right)^{2};q;x\right)
$$

and

$$
\begin{aligned}
&\left|\int_{x}^{\frac{[2]_q[m+p]_qx+(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}}\left(\frac{[2]_q\left[m+p\right]_q x+\left(2\alpha+1\right)b_m}{2\left(\left[m+1\right]_q+\beta\right)}-u\right)g''\left(\tau\right)d\tau\right| \\
\leq\ &\left\|g''\right\|\left(\left(\frac{[2]_q\left[m+p\right]_q}{2\left(\left[m+1\right]_q+\beta\right)}-1\right)x+\frac{\left(2\alpha+1\right)b_m}{2\left(\left[m+1\right]_q+\beta\right)}\right)^{2}
\end{aligned}
$$

we get

$$
\begin{aligned}
&\left|\mathscr{K}_{m,p}^{*}\left(g;q,x\right)-g\left(x\right)\right|\leq\left\|g''\right\|\mathscr{K}_{m,p}^{(\alpha,\beta)}\left(\left(y-x\right)^{2};q,x\right) \\
&+\left\|g''\right\|\left(\left(\frac{[2]_q\left[m+p\right]_q}{2\left(\left[m+1\right]_q+\beta\right)}-1\right)x+\frac{\left(2\alpha+1\right)b_m}{2\left(\left[m+1\right]_q+\beta\right)}\right)^{2}.
\end{aligned}
$$

Hence Lemma 3.1.1 implies that

$$\begin{aligned}
&\left|\mathscr{K}_{m,p}^{*}(g;q,x)-g(x)\right| \\
&\leq \left\|g''\right\| \left[\left(\frac{[3]_q [m+p-1]_q [m+p]_q q}{3\left([m+1]_q+\beta\right)^2}-\frac{[2]_q [m+p]_q}{\left([m+1]_q+\beta\right)}+1\right)x^2\right. \\
&+\left(\frac{\left(q^2+3q+2+3[2]_q\alpha\right)[m+p]_q}{3\left([m+1]_q+\beta\right)^2}-\frac{(2\alpha+1)}{\left([m+1]_q+\beta\right)}\right)b_n x \\
&+\frac{\left(3\alpha^2+3\alpha+1\right)b_m^2}{3\left([m+1]_q+\beta\right)^2} && (3.3.1) \\
&\left.+\left(\left(\frac{[2]_q [m+p]_q}{2\left([m+1]_q+\beta\right)}-1\right)x+\frac{(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}\right)^2\right]. && (3.3.8)
\end{aligned}$$

Because of the fact that $||\mathscr{K}_{m,p}^{*}(f;q,\cdot)|| \leq 3\|f\|$, taking into account that (3.3.4) a nd (3.3.5), for all $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$, we have from (3.3.8) that

$$\begin{aligned}
&\left|\mathscr{K}_{n,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right| \\
&\leq \left|\mathscr{K}_{m,p}^{*}(f-g;q,x)-(f-g)(x)\right| \\
&+\left|\mathscr{K}_{m,p}^{*}(g;q,x)-g(x)\right|+\left|f\left(\frac{[2]_q [m+p]_q x+(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}\right)-f(x)\right| \\
&\leq 4\|f-g\|+\alpha_{m,q}(x)\left\|g''\right\|+\left|f\left(\frac{[2]_q [m+p]_q x+(2\alpha+1)b_m}{2\left([m+1]_q+\beta\right)}\right)-f(x)\right| \\
&\leq 4\left(\|f-g\|+\alpha_{m,q}(x)\left\|g''\right\|\right)+\omega\left(f,\beta_{m,q}(x)\right)
\end{aligned}$$

which yields that

$$\begin{aligned}
\left|\mathscr{K}_{n,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right| &\leq 4\mathscr{K}\left(f,\alpha_{m,q}(x)\right)+\omega\left(f,\beta_{m,q}(x)\right) \\
&\leq C\omega_2\left(f,\sqrt{\alpha_{m,q}(x)}\right)+\omega\left(f,\beta_{m,q}(x)\right),
\end{aligned}$$

where

$$\beta_{m,q}(x) := \left| \frac{[2]_q [m+p]_q}{2\left([m+1]_q+\beta\right)} - 1 \right| x + \frac{(2\alpha+1)\, b_m}{2\left([m+1]_q+\beta\right)}.$$

and

$$\begin{aligned}\alpha_{m,q}(x) := \Bigg[& \left(\left(\frac{[3]_q}{3} + \frac{[2]_q^2}{4} \right) \frac{[m+p]_q^2}{\left([m+1]_q+\beta\right)^2} - 2\frac{[2]_q [m+p]_q}{[m+1]_q+\beta} + 2 \right) x^2 \\ & + \left(\frac{q^2+3q+2+3[2]_q \alpha}{3\left([m+1]_q+\beta\right)^2} + \frac{(2\alpha+1)[2]_q [m+p]_q}{2\left([m+1]_q+\beta\right)} - \frac{2(2\alpha+1)}{[m+1]_q+\beta} \right) b_m x \\ & + \left(\frac{24\alpha^2+24\alpha+7}{12} \right) \frac{b_m^2}{\left([m+1]_q+\beta\right)^2} \Bigg]\end{aligned}$$

Hence we get the result. ■

3.4 Generalization of the operators

In this subsection, the generalization of Chlodowsky type q-Bernstein-Stancu-Kantorovich operators are introduced in a similar manner as in Subsection 2.4.

Now, we consider the generalization of the $\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x)$ as

$$\begin{aligned}\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x) = & \frac{\omega(x)}{1+x^2} \sum_{r=0}^{m+p} \begin{bmatrix} m+p \\ r \end{bmatrix}_q \left(\frac{x}{b_m}\right)^r \prod_{s=0}^{m+p-r-1} \left(1 - q^s \frac{x}{b_m}\right) \\ & \times \int_0^1 G_f\left(\frac{[r]_q+\alpha}{[m+1]_q+\beta} b_m + \frac{1+(q-1)[r]_q}{[m+1]_q+\beta} t b_m \right) dt,\end{aligned}$$

where $0 \le x \le b_m$ and $\{b_m\}$ has the same properties Chlodowsky variant of q-Bernstein-Schurer-Stancu operators.

Theorem 3.4.1 *([39]) For the continuous functions satisfying*

$$\lim_{x\to\infty} \frac{f(x)}{\omega(x)} = N_f < \infty,$$

we have

$$\lim_{m\to\infty}\sup_{0\le x\le b_m}\frac{\left|\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)-f(x)\right|}{\omega(x)}=0$$

provided that $q:=(q_m)$ *with* $0<q_m<1$, $\lim_{m\to\infty} q_m=1$ *and* $\lim_{m\to\infty}\frac{b_m}{[m]_q}=0$ *as* $m\to\infty$.

Proof. Obviously,

$$\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)-f(x)=\frac{\omega(x)}{1+x^2}\left(\sum_{r=0}^{m+p}\begin{bmatrix}m+p\\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)\right.$$
$$\left.\times\int_0^1 G_f\left(\frac{[r]_q+\alpha}{[m+1]_q+\beta}b_m+\frac{1+(q-1)[r]_q}{[m+1]_q+\beta}tb_m\right)dt-G_f(x)\right),$$

hence

$$\sup_{0\le x\le b_m}\frac{\left|\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)-f(x)\right|}{\omega(x)}=\sup_{0\le x\le b_m}\frac{\left|\mathscr{K}_{m,p}^{(\alpha,\beta)}(G_f;q,x)-G_f(x)\right|}{1+x^2}.$$

From $|f(x)|\le N_f\omega(x)$ and the continuity of the function f, we have $\left|G_f(x)\right|\le N_f\left(1+x^2\right)$ for $x\ge0$ and $G_f(x)$ is a continuous function on $[0,\infty)$. Using Theorem 3.2.3, we get the desired result. ■

Lastly, note that, taking $\omega(x)=1+x^2$, the operators $\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)$ reduces to $\mathscr{K}_{m,p}^{(\alpha,\beta)}(f;q,x)$.

Chapter 4

CHLODOWSKY-DURRMEYER TYPE q-BERNSTEIN-SCHURER-STANCU OPERATORS

4.1 Construction of the operators

In 2005, Derriennic defined the q-Durrmeyer operators [9] by

$$D_{m,q}(f,x) = [m+1]_q \sum_{k=0}^{n} q^{-k} p_{m,k}(q;x)_0^1 f(t)\, p_{m,k}(q;qt)\, d_q t, \quad 0 \le x \le 1$$

where $p_{n,k}(q;v) = \left[{m \atop k}\right]_q v^k \prod_{s=0}^{m-k-1}(1-q^s v)$ and the definition of q-Jackson integral of $f(t)$ as

$$\int f(t)\, d_q t = (1-q)\, t \sum_{i=0}^{\infty} q^i f\left(q^i t\right).$$

Let us introduce the Chlodowsky-Durrmeyer type q-Bernstein-Schurer-Stancu operators as follows:

$$\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)\,4.1.1 \qquad (4.1.1)$$
$$: \;= \frac{[m+p+1]_q}{b_m} \sum_{r=0}^{m+p} q^{-r} p_{m+p,r}\left(q;\frac{x}{b_m}\right) \int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right) f\left(\frac{[r]_q t + \alpha b_m}{[m]_q + \beta}\right) d_q t$$

where m, α, $\beta \in \mathbb{N}$, $p \in \mathbb{N}_0$ with $0 \le \alpha \le \beta$, $0 \le x \le b_m$, $0 < q < 1$. Note that, in the case $p = 0$, the operators in (4.1.1) reduce to the Stancu-Chlodowsky polynomials [5] when $q \to 1^-$.

Definition 4.1.1 *([18]) The q-Beta function is defined as*

$$B_q(t,s)=\int_0^1 x^{t-1}(1-qx)_q^{s-1}\,d_qx$$

where t, s > 0.

Lemma 4.1.2 *For s = 0, 1, ..., we have the following relation*

$$\begin{bmatrix} m \\ r \end{bmatrix}_q \int_0^1 t^{k+s}(1-qt)_q^{n-k}\,d_qt=\frac{[m]_q!\,[r+s]_q!}{[m+s+1]_q\,[r]_q!}.$$

Proof. Using Definition 4.1.1, the proof is completed. ■

Lemma 4.1.3 *Let $D_{m,p}^{(\alpha,\beta)}(f;q,x)$ is given in (5.1.1). Then we have*

$(i)D_{m,p}^{(\alpha,\beta)}(1;q,x)=1,$

$(ii)\mathscr{D}_{m,p}^{(\alpha,\beta)}(u;q,x)=\frac{q[m]_q[m+p]_qx}{[m+p+2]_q\left([m]_q+\beta\right)}+\left(\frac{[m]_q}{[m+p+2]_q}+\alpha\right)\frac{b_m}{[m]_q+\beta},$

$(iii)\mathscr{D}_{m,p}^{(\alpha,\beta)}\left(u^2;q,x\right)$

$$=\frac{1}{\left([m]_q+\beta\right)^2}\left\{\frac{[m]_q^2q^3[m+p]_q^2x^2}{[m+p+3]_q[m+p+2]_q}\right.$$
$$+\left[\frac{\left(q[2]_q+q^2\right)[m]_q^2}{[m+p+3]_q[m+p+2]_q}+\frac{2q[m]_q}{[m+p+2]_q}\right][m+p]_q\,b_mx$$
$$\left.\left[\frac{[m]_q^2[2]}{[m+p+3]_q[m+p+2]_q}+\alpha^2\right]b_m^2\right\},$$

$(iv)\mathscr{D}_{m,p}^{(\alpha,\beta)}(u-x\,;q,x)$

$$=\left(\frac{q[m]_q[m+p]_q}{[m+p+2]_q\left([m]_q+\beta\right)}-1\right)x+\left(\frac{[m]_q}{[m+p+2]_q}+\alpha\right)\frac{b_m}{[m]_q+\beta},$$

$(v)\mathscr{D}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)$

$$=\left(\frac{[m]_q^2q^3[m+p]_q^2}{[m+p+3]_q[m+p+2]_q\left([m]_q+\beta\right)^2}-2\frac{q[m]_q[m+p]_q}{[m+p+2]_q\left([m]_q+\beta\right)}+1\right)x^2$$
$$+\left(\left[\frac{\left(q[2]_q+q^2\right)[m]_q^2}{[m+p+3]_q[m+p+2]_q}+\frac{2q[m]_q}{[m+p+2]_q}\right]\frac{[m+p]_q}{\left([m]_q+\beta\right)^2}-2\frac{[m]_q}{[m+p+2]_q}-2\alpha\right)\frac{b_m}{[m]_q+\beta}x$$
$$+\left[\frac{[m]_q^2[2]_q}{[m+p+3]_q[m+p+2]_q}+\alpha^2\right]\frac{b_m^2}{\left([m]_q+\beta\right)^2}.$$

Proof. (i) From Definition 4.1.1 and Lemma 4.1.2 we have

$$\mathscr{D}_{m,p}^{(\alpha,\beta)}(1;q,x)=\sum_{r=0}^{m+p}\begin{bmatrix} m+p \\ r\end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right)=1.$$

Using Lemma 2.1.1, Definition 4.1.1 and Lemma 4.1.2 we have (ii) and (iii). The proofs of (iv) and (v) follow from the linearity of the operator $\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q;x)$. ■

4.2 Korovkin-Type Approximation Theorem

In this subsection, we prove Korovkin-type approximation theorems for Chlodowsky-Durrmeyer type q-Bernstein-Schurer-Stancu operators in the space C_{1+x^2} which is defined in Section 2.3.

Now, let us consider the operators

$$Q_m^{(\alpha,\beta)}(f;q,x)=\begin{cases}\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x) & \text{if } 0\leq x\leq b_m \\ f(x) & \text{if } \quad x>b_m\end{cases}.$$

Operators $Q_m^{(\alpha,\beta)}(f;q,x)$ acts from C_{1+x^2} to C_{1+x^2}. Indeed, for all $f\in C_{1+x^2}$, we have

$$\begin{aligned}\left\|Q_m^{(\alpha,\beta)}(f;q,\cdot)\right\|_{1+x^2} &\leq \sup_{x\in[0,b_m]}\frac{\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)\right|}{1+x^2}+\sup_{b_m<x<\infty}\frac{|f(x)|}{1+x^2} \\ &\leq \|f\|_{1+x^2}\left[\sup_{x\in[0,\infty)}\frac{\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}\left(1+u^2;q,x\right)\right|}{1+x^2}+1\right].\end{aligned}$$

Hence, with the help of Lemma 4.1.1, there exist a positive constant M such that

$$\left\|Q_m^{(\alpha,\beta)}(f;q,\cdot)\right\|_{1+x^2}\leq M\|f\|_{1+x^2}$$

is satisfied for $q:=(q_m)$ with $0<q_m<1$, $\lim\limits_{m\to\infty}q_m=1$, $\lim\limits_{m\to\infty}q_m^m=N<\infty$ and $\lim\limits_{m\to\infty}\frac{b_m}{[m]_q}=0$.

Theorem 4.2.1 *For all $f \in C^0_{1+x^2}$, we get*

$$\lim_{m\to\infty} \left\| Q_m^{(\alpha,\beta)}(f;q_m,\cdot) \right\|_{1+x^2} = 0$$

provided that $q := (q_m)$ with $0 < q_m < 1$, $\lim_{m\to\infty} q_m = 1$, $\lim_{m\to\infty} q_m^m = N < \infty$ and $\lim_{m\to\infty} \frac{b_m}{[m]_q} =$

0.

Proof. In the proof, we directly use Theorem 2.2.2 Clearly, by Lemma 4.1.1 (i), (ii) and (iii) we get,

$$\sup_{x\in[0,\infty)} \frac{\left| Q_m^{(\alpha,\beta)}(1;q_m,x) - 1 \right|}{1+x^2} = \sup_{0\le x\le b_m} \frac{\left| \mathscr{D}_{m,p}^{(\alpha,\beta)}(1;q_m,x) - 1 \right|}{1+x^2} = 0,$$

$$\sup_{x\in[0,\infty)} \frac{\left| Q_m^{(\alpha,\beta)}(u;q_m,x) - u \right|}{1+x^2} = \sup_{0\le x\le b_m} \frac{\left| \mathscr{D}_{m,p}^{(\alpha,\beta)}(u;q_m,x) - x \right|}{1+x^2}$$

$$\le \sup_{0\le x\le b_m} \frac{\left| \frac{q[m]_q[m+p]_q}{[m+p+2]_q\left([m]_q+\beta\right)} - 1 \right| x + \left(\frac{[m]_q}{[m+p+2]_q} + \alpha \right) \frac{b_m}{[m]_q+\beta}}{(1+x^2)}$$

$$\le \left| \frac{q\,[m]_q\,[m+p]_q}{[m+p+2]_q\left([m]_q+\beta\right)} - 1 \right| + \left(\frac{[m]_q}{[m+p+2]_q} + \alpha \right) \frac{b_m}{[m]_q+\beta} \to 0$$

and

$$\sup_{x\in[0,\infty)} \frac{\left|Q_m^{(\alpha,\beta)}\left(u^2;q_m,x\right)-u^2\right|}{1+x^2} = \sup_{0\le x\le b_m} \frac{\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}\left(u^2;q_m,x\right)-x^2\right|}{1+x^2}$$

$$\le \sup_{0\le x\le b_m} \frac{\left|\frac{[m]_q^2 q^3 [m+p]_q^2}{[m+p+3]_q[n+p+2]_q} - \left([m]_q+\beta\right)^2 [m+p+3]_q [m+p+2]_q\right| x^2}{\left([m]_q+\beta\right)^2(1+x^2)}$$

$$+\frac{\left[\frac{\left(q[2]_q+q^2\right)[m]_q^2}{[m+p+3]_q[m+p+2]_q}+\frac{2q[m]_q}{[m+p+2]_q}\right][m+p]_q b_n x+\left[\frac{[m]_q^2[2]_q}{[m+p+3]_q[m+p+2]_q}+\alpha^2\right]b_m^2}{\left([m]_q+\beta\right)^2(1+x^2)}$$

$$\le \frac{\left|\frac{[m]_q^2 q^3 [m+p]_q^2}{[m+p+3]_q[m+p+2]_q} - \left([m]_q+\beta\right)^2 [m+p+3]_q [m+p+2]_q\right|}{\left([m]_q+\beta\right)^2}$$

$$+\frac{\left[\frac{\left(q[2]_q+q^2\right)[m]_q^2}{[m+p+3]_q[m+p+2]_q}+\frac{2q[m]_q}{[m+p+2]_q}\right][m+p]_q b_n+\left[\frac{[m]_q^2[2]_q}{[m+p+3]_q[m+p+2]_q}+\alpha^2\right]b_m^2}{\left([m]_q+\beta\right)^2} \to 0$$

as $m\to\infty$, since $q:=(q_m)$ with $0<q_m<1$, $\lim_{m\to\infty} q_m = 1$, $\lim_{m\to\infty} q_m^m = N<\infty$ and $\lim_{m\to\infty}\frac{b_m}{[m]_q}=0$. ■

Lemma 4.2.2 *Let $A\in\mathbb{R}^+$ be independent of n and f be a continuous function which vanishes on $[A,\infty]$. Assume that $q:=(q_m)$ with $0<q_m<1$, $\lim_{m\to\infty} q_m=1$, $\lim_{m\to\infty} q_m^m = N<\infty$ and $\lim_{m\to\infty}\frac{b_m^2}{[m]_q}=0$. Then we get*

$$\lim_{m\to\infty}\sup_{0\le x\le b_m}\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}\left(f;q_m,x\right)-f(x)\right|=0.$$

Proof. From the hypothesis on f, we can write $|f(x)|\le M$ $(M>0)$. Thus, by continuity and boundedness of f, we can write for arbitrary small $\varepsilon>0$ that

$$\left|f\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}\right)-f(x)\right|<\varepsilon+\frac{2M}{\delta^2}\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right)^2,$$

where $x\in[0,b_m]$ and $\delta=\delta(\varepsilon)$ are independent of m. From the following equality

$$\begin{aligned}
& \sum_{r=0}^{m+p}\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right)^2\begin{bmatrix} m+p \\ r \end{bmatrix}_q\left(\frac{x}{b_m}\right)^r\prod_{s=0}^{m+p-r-1}\left(1-q^s\frac{x}{b_m}\right) \\
= \ & \mathscr{D}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q_m,x\right),
\end{aligned}$$

we have by Lemma 4.1.1 that

$$\begin{aligned}
& \sup_{0\leq x\leq b_m}\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q_m,x)-f(x)\right| \\
& \leq \varepsilon+\frac{2M}{\delta^2}\left[\left|\frac{[m]_q^2 q^3[m+p]_q^2}{[m+p+3]_q[m+p+2]_q\left([m]_q+\beta\right)^2}-2\frac{q\,[m]_q\,[m+p]_q}{[m+p+2]_q\left([m]_q+\beta\right)}+1\right|b_m^2\right. \\
& +\left|\left[\frac{\left(q[2]_q+q^2\right)[m]_q^2}{[m+p+3]_q[m+p+2]_q}+\frac{2q[m]_q}{[m+p+2]_q}\right]\frac{[m+p]_q}{\left([m]_q+\beta\right)^2}-2\frac{[m]_q}{[m+p+2]_q}-2\alpha\right|\frac{b_m^2}{[m]_q+\beta} \\
& \left.+\left[\frac{[m]_q^2\,[2]_q}{[m+p+3]_q\,[m+p+2]_q}+\alpha^2\right]\frac{b_m^2}{\left([m]_q+\beta\right)^2}\right].
\end{aligned}$$

Because of $q:=(q_m)$ with $0<q_m<1$, $\lim_{m\to\infty}q_m=1$, $\lim_{m\to\infty}q_m^m=N<\infty$ and $\lim_{m\to\infty}\frac{b_m^2}{[m]_q}=0$, we get the desired conclusion. ■

From, Lemma 4.2.2, we can state the following theorem:

Theorem 4.2.3 *Let f be a continuous function on the semi-axis $[0,\infty)$ and*

$$\lim_{x\to\infty}f(x)=k_f<\infty.$$

Assume that $q:=(q_m)$ with $0<q_m<1$, $\lim_{m\to\infty}q_m=1$, $\lim_{m\to\infty}q_m^m=N<\infty$ and $\lim_{m\to\infty}\frac{b_m^2}{[m]_q}=0$. Then

$$\lim_{m\to\infty}\sup_{0\leq x\leq b_m}\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q_m,x)-f(x)\right|=0.$$

Proof. As in the proof of Theorem 2.2.5, it is sufficient to prove the theorem for the case $k_f = 0$. Since $\lim_{x\to\infty} f(x) = 0$, given any $\varepsilon > 0$ we can write a point x_0 such that

$$|f(x)| \leq \varepsilon, \qquad x \geq x_0. \tag{5.2.4}$$

For any fixed $x_0 > 0$, let us define an auxiliary function as follows:

$$s(x) = \begin{cases} f(x), & 0 \leq x \leq x_0 \\ f(x_0) - f(x_0)(x - x_0), & x_0 \leq x \leq x_0 + 1 \\ 0, & x \geq x_0 + 1. \end{cases}$$

Then, for sufficiently large m in such a way that $b_m \geq x_0 + 1$ and in view of the fact that $\sup_{x_0 \leq x \leq x_0+1} |s(x)| = |f(x_0)|$, we have

$$\begin{aligned} \sup_{0\leq x\leq b_m} |f(x) - s(x)| &\leq \sup_{x_0\leq x\leq x_0+1} |f(x) - s(x)| + \sup_{x_0+1\leq x\leq b_m} |f(x)| \\ &\leq 2 \sup_{x_0\leq x\leq x_0+1} |f(x)| + \sup_{x_0+1\leq x\leq b_m} |f(x)|. \end{aligned}$$

Therefore, we have by (4.2.4)

$$\sup_{0\leq x\leq b_m} |f(x) - s(x)| \leq 3\varepsilon.$$

Now, we can write

$$\begin{aligned} &\sup_{0\leq x\leq b_m} \left| \mathscr{D}_{m,p}^{(\alpha,\beta)}(f; q_m, x) - g(x) \right| \\ &\leq \sup_{0\leq x\leq b_m} \left| \mathscr{D}_{m,p}^{(\alpha,\beta)}(|f - s|; q_m, x) \right| + \sup_{0\leq x\leq b_m} \left| \mathscr{D}_{m,p}^{(\alpha,\beta)}(s; q_m, x) - s(x) \right| \\ &+ \sup_{0\leq x\leq b_m} |f(x) - s(x)| \\ &\leq 6\varepsilon + \sup_{0\leq x\leq b_m} \left| \mathscr{D}_{m,p}^{(\alpha,\beta)}(s; q_m, x) - s(x) \right|. \end{aligned}$$

Because of $s(x)=0$ for $x_0+1 \leq x$, we obtain the result directly by Lemma 4.2.4 . ■

4.3 Order of Convergence

We stated to this subsection with the following theorem:

Theorem 4.3.1 *Let $f \in Lip_M(\gamma)$. Then we have*

$$|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)| \leq M(\lambda_{m,q}(x))^{\gamma/2}$$

where

$$\lambda_{m,q}(x) = \left(\frac{[m]_q^2 q^3 [m+p]_q^2}{[m+p+3]_q[m+p+2]_q\left([m]_q+\beta\right)^2} - 2\frac{q[m]_q[m+p]_q}{[m+p+2]_q\left([m]_q+\beta\right)}+1\right)x^2$$
$$\left(\left[\frac{\left(q[2]_q+q^2\right)[m]_q^2}{[m+p+3]_q[m+p+2]_q}+\frac{2q[m]_q}{[m+p+2]_q}\right]\frac{[m+p]_q}{\left([m]_q+\beta\right)^2}-2\frac{[m]_v}{[m+p+2]_q}-2\alpha\right)\frac{b_m}{[m]_q+\beta}x$$
$$+\left[\frac{[m]_q^2[2]_q}{[m+p+3]_q[m+p+2]_q}+\alpha^2\right]\frac{b_m^2}{\left([m]_q+\beta\right)^2}.$$

Proof. From the monotonicity and the linearity of the operators and by (2.3.1), we get

$$\begin{aligned}
&|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)| \\
= &\left|\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p} q^{-r} p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left(f\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}\right)-f(x)\right)d_qt\right| \\
\leq &\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p} q^{-r} p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left|f\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}\right)-f(x)\right|d_qt \\
\leq &M\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p} q^{-r} p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left|\frac{[r]_q t+\alpha b_m}{[m]_v+\beta}-x\right|^{\gamma}d_qt.
\end{aligned}$$

Applying Hölder's inequality with $p=\frac{2}{\gamma}$ and $q=\dfrac{2}{2-\gamma}$, we have the following

$$\frac{[m+p+1]_q q^{-r}}{b_m}\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left|\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right|^{\gamma} d_qt$$

$$\leq \left\{\frac{[m+p+1]_q q^{-r}}{b_m}\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right)^2 d_qt\right\}^{\frac{\gamma}{2}}$$

$$\times\left\{\frac{[m+p+1]_q q^{-r}}{b_m}\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right) d_qt\right\}^{\frac{2}{2-\gamma}}$$

$$= \left\{\frac{[m+p+1]_q q^{-r}}{b_m}\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right)^2 d_qt\right\}^{\frac{\gamma}{2}}.$$

Then, one more application of Hölder's inequality yields

$$|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)|$$

$$\leq M\left\{\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p} q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right)^2 d_qt\right\}^{\frac{\gamma}{2}}$$

$$\times\left\{\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p} q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right) d_qt\right\}^{\frac{2-\gamma}{2}}$$

$$= M\left\{\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p} q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m} p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right)^2 d_qt\right\}^{\frac{\gamma}{2}}$$

$$= M\left[\mathscr{D}_{m,p}^{(\alpha,\beta)}\left((u-x)^2;q,x\right)\right]^{\frac{\gamma}{2}}$$

$$= M(\lambda_{m,q}(x))^{\frac{\gamma}{2}}.$$

■

Theorem 4.3.2 *If $f\in C_B[0,\infty)$, we have*

$$\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right|\leq 2\omega\left(f;\sqrt{\lambda_{m,q}(x)}\right).$$

Proof. Using the monotonicity, we have

$$\begin{aligned}
&\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right| \\
&=\left|\tfrac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m}p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left(f\left(\tfrac{[r]_qt+\alpha b_m}{[m]_q+\beta}\right)-f(x)\right)d_qt\right| \\
&\leq \tfrac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m}p_{m+p,r}\left(q;\frac{qt}{b_m}\right)\left|f\left(\frac{[r]_qt+\alpha b_m}{[m]_q+\beta}\right)-f(x)\right|d_qt.
\end{aligned}$$

Now using (2.3.3), we write

$$\begin{aligned}
&\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right| \\
&\leq \frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}\int_0^{b_m}\left(\frac{\left|\frac{[r]_qt+\alpha b_m}{[m]_q+\beta}-x\right|}{\lambda}+1\right)\omega(f;\lambda)\,p_{m+p,r}\left(q;\frac{x}{b_m}\right) \\
&\times p_{m+p,r}\left(q;\frac{qt}{b_m}\right)d_qt \\
&=\omega(f;\lambda)\left(\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)_0^{b_m}p_{m+p,r}\left(q;\frac{qt}{b_m}\right)d_qt\right) \\
&+\frac{\omega(f;\lambda)}{\lambda}\left(\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\right. \\
&\left.\times\int_0^{b_m}\left|\frac{[r]_qt+\alpha b_m}{[m]_q+\beta}-x\right|p_{m+p,r}\left(q;\frac{qt}{b_m}\right)d_qt\right)
\end{aligned}$$

Using the Cauchy-Schwarz inequality first to the q-integral and then to the summation, we have

$$\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|\leq\omega(f;\lambda)$$

$$+\frac{\omega(f;\lambda)}{\lambda}\left\{\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\right.^{1/2}$$

$$\left.{}^{b_m}_{0}\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}-x\right)^2 p_{m+p,r}\left(q;\frac{qt}{b_m}\right)d_q t\right\}^{1/2}$$

$$\times\left\{\sum_{r=0}^{m+p}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\right\}^{\frac{1}{2}}$$

$$=\omega(f;\lambda)+\frac{\omega(f;\lambda)}{\lambda}\left\{\mathscr{D}_{m,p}^{(\alpha,\beta)}\left((u-x)^2\right);q;x\right\}^{1/2}.$$

Now, choosing $\lambda_{m,q}(x)$ the same as in Theorem 5.3.1, we get

$$\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|\leq 2\omega\left(f;\sqrt{\lambda_{m,q}(x)}\right).$$

■

4.4 Generalization of operators

In a similar manner as Chapter two and three we will give a generalizations of Chlodowsky-Durrmeyer type q-Bernstein-Schurer-Stancu operators. Let $x\geq 0$, consider any continuous function $\omega(x)\geq 1$ and we define

$$G_f(t)=f(t)\frac{1+t^2}{\omega(t)}$$

and we introduce a generalize operator

$$\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)$$

$$=\frac{\omega(x)}{1+x^2}\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\int_0^{b_m}p_{m+p,r}\left(q;\frac{qt}{b_m}\right)G_f\left(\frac{[r]_q t+\alpha b_m}{[m]_q+\beta}\right)d_q t,$$

where $0\leq x\leq b_n$ and (b_n) has the same properties of Chlodowsky variant of q-

Bernstein-Schurer-Stancu operators.

Theorem 4.4.1 *For the continuous functions satisfying*

$$\lim_{x\to\infty}\frac{f(x)}{\omega(x)}=N_f<\infty,$$

we have

$$\lim_{n\to\infty}\sup_{0\le x\le b_m}\frac{\left|\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)-f(x)\right|}{\omega(x)}=0$$

provided that $q:=(q_m)$ *with* $0<q_m<1$, $\lim\limits_{m\to\infty}q_m=1$, $\lim\limits_{m\to\infty}q_m^m=N<\infty$ *and* $\lim\limits_{m\to\infty}\frac{b_m^2}{[m]_q}=0$.

Proof. Clearly,

$$\begin{aligned}&\mathscr{L}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\\&=\frac{\omega(x)}{1+x^2}\left(\frac{[m+p+1]_q}{b_m}\sum_{r=0}^{m+p}q^{-r}p_{m+p,r}\left(q;\frac{x}{b_m}\right)\right.\\&\left.\times\int_0^{b_m}p_{m+p,r}\left(q;\frac{qt}{b_m}\right)G_f\left(\frac{[r]_qt+\alpha b_m}{[m]_q+\beta}\right)d_qt-G_f(x)\right)\end{aligned}$$

thus

$$\sup_{0\le x\le b_m}\frac{\left|\mathscr{L}_{m,p}^{(\alpha,\beta)}(f;q,x)-f(x)\right|}{\omega(x)}=\sup_{0\le x\le b_m}\frac{\left|\mathscr{D}_{m,p}^{(\alpha,\beta)}\left(G_f;q,x\right)-G_f(x)\right|}{1+x^2}.$$

By using $|f(x)|\le N_f\omega(x)$ and continuity of the function f, we get that $\left|G_f(x)\right|\le N_f\left(1+x^2\right)$ for $x\ge0$ and $G_f(x)$ is a continuous function on $[0,\infty)$. Thus, by the Theorem 4.2.3 we get the result. ■

Last, notice that, taking $\omega(x)=1+x^2$, then the operators $\mathscr{L}_{m,p}^{\alpha,\beta}(f;q,x)$ reduces $\mathscr{D}_{m,p}^{(\alpha,\beta)}(f;q,x)$.

Chapter 5

TWO DIMENSIONAL CHLODOWSKY TYPE OF q-BERNSTEIN-SCHURER-STANCU OPERATORS

5.1 Construction of the operators

Let $\{a_n\}$ and $\{b_m\}$ be increasing sequences of real numbers satisfying

$$\lim_{n\to\infty} a_n = \lim_{m\to\infty} b_m = \infty.$$

Let, D_{a_n,b_m} denotes

$$D_{a_n,b_m} = \{(x,y) : 0 \le x \le a_n,\, 0 \le y \le b_m\}. \tag{5.1.1}$$

For $(x,y) \in D_{a_n,b_m}$, we introduce two dimensional Chlodowsky type of q-Bernstein-Schurer-Stancu operators as

$$\begin{aligned}&\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)\\ &:= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p} f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\end{aligned} \tag{5.1.2}$$

where $n \in \mathbb{N},\ p \in \mathbb{N}_0,\ 0 \le \alpha \le \beta$. $\Phi_{k,n,q_n}(v) = \begin{bmatrix} n+p \\ k \end{bmatrix}_{q_n} v^k_{s=0}{}^{n+p-k-1}(1-q_n^s v)$. We also let $0 < q_n < 1$ $(n \in \mathbb{N})$ for the positivity of the operators. It is easy to show that $\mathscr{C}_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ is a linear and positive operator.

Lemma 5.1.1 *Let $\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ be given in (5.1.2). Then we have*

(i) $\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(1;q_n,q_m;x,y) = 1,$

$(ii)\ \mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_1;q_n,q_m;x,y\right)=\dfrac{[n+p]_{q_n}x+\alpha a_n}{[n]_{q_n}+\beta},$

$(iii)\ \mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_2;q_n,q_m;x,y\right)=\dfrac{[m+p]_{q_m}y+\alpha b_m}{[m]_{q_m}+\beta}$

$(iv)\ \mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_1^2+u_2^2;q_n,q_m;x,y\right)$

$=1\dfrac{}{\left([n]_{q_n}+\beta\right)^2}\left\{[n+p-1]_{q_n}[n+p]_{q_n}q_nx^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$

$+\dfrac{1}{\left([m]_{q_m}+\beta\right)^2}\left\{[m+p-1]_{q_m}[m+p]_{q_m}q_my^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$

Proof. Using Lemma 2.1.1 and the linearity of the operators, the proof is easily obtained. ■

5.2 Korovkin-type Approximation Theorem

For fixed $\nu\geq 0$ consider the space C_{ρ^ν} which consists of all continuous functions f, satisfying the condition

$$|f(x,y)|\leq M_f\rho^\nu(x,y),\quad (x,y)\in[0,\infty)\times[0,\infty):=\mathbb{R}_+^2 \text{ and } \rho(x,y)=1+x^2+y^2.$$

Clearly, C_{ρ^ν} is a linear normed space with the following norm

$$\|f\|_{\rho^\nu}=\sup_{0\leq x,y<\infty}\frac{|f(x,y)|}{\rho^\nu(x,y)}.$$

Theorem 5.2.1 *Let the numbers A and B be any fixed positive real numbers. Let* $D_{A,B}=\{(x,y):0\leq x\leq A,\ 0\leq y\leq B\}$, $q:=\{q_n\}$ *with* $0<q_n<1$, $\lim_{n\to\infty}q_n=1$ *and* $\{a_n\}$ *and* $\{b_m\}$ *be increasing sequences of positive real numbers that satisfy the following properties:*

$$\lim_{n\to\infty}a_n=\lim_{m\to\infty}b_m=\infty \text{ and } \lim_{n\to\infty}\frac{a_n}{[n]_{q_n}}=\lim_{m\to\infty}\frac{b_m}{[m]_{q_m}}=0.$$

Then, for all $f\in C(D_{A,B})$, *we have*

$$\lim_{n,m\to\infty}\max_{(x,y)\in D_{A,B}}\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|=0.$$

Proof. Using Lemma 5.1.1, we get

$$\left\|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(1;q_n,q_m;\cdot,\cdot)-1\right\|_{C(D_{A,B})}=0$$

$$\left\|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(u_1;q_n,q_m;\cdot,\cdot)-x\right\|_{C(D_{A,B})}\leq A\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|+\frac{\alpha a_n}{[n]_{q_n}+\beta}$$

$$\left\|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(u_2;q_n,q_m;\cdot,\cdot)-y\right\|_{C(D_{A,B})}\leq B\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|+\frac{\alpha b_m}{[m]_{q_m}+\beta}.$$

Again by Lemma 4.1.1, we have

$$\mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_1^2+u_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)=\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$

$$\times\left\{\left([n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right)x^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$$

$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$

$$\times\left\{\left([m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right)y^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$$

Finally, from the above equality we obtain

$$\left\|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_1^2+u_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})}$$

$$\leq\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$

$$\times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$

$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$

$$\times\left\{\left|[m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right|B^2+(2\alpha+1)[m+p]_{q_m}b_mB+\alpha^2b_m^2\right\}.$$

Therefore, from the hypothesis of the theorem, we have

$$\begin{aligned}
\left\| \mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_1;q_n,q_m;\cdot,\cdot\right)-x\right\|_{C(D_{A,B})} &\to 0 \\
\left\| \mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_2;q_n,q_m;\cdot,\cdot\right)-y\right\|_{C(D_{A,B})} &\to 0 \\
\left\| \mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(u_1^2+u_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})} &\to 0
\end{aligned}$$

when n and $m \to \infty$.

Hence, the proof is completed by the two dimensional Korovkin theorem. ■

Theorem 5.2.2 *(See [15]) There exists a sequence of positive operators $\mathscr{T}_{n,m}$, acting from $C_\rho\left(\mathbb{R}_+^2\right)$ to $C_\rho\left(\mathbb{R}_+^2\right)$, satisfying the conditions*

$$\begin{aligned}
\lim_{n,m\to\infty}\left\|\mathscr{T}_{n,m}\left(1;\cdot,\cdot\right)-1\right\|_\rho &= 0 \\
\lim_{n,m\to\infty}\left\|\mathscr{T}_{n,m}\left(u_1;\cdot,\cdot\right)-x\right\|_\rho &= 0 \\
\lim_{n,m\to\infty}\left\|\mathscr{T}_{n,m}\left(u_2;\cdot,\cdot\right)-y\right\|_\rho &= 0 \\
\lim_{n,m\to\infty}\left\|\mathscr{T}_{n,m}\left(u_1^2+u_2^2;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_\rho &= 0
\end{aligned}$$

and there exists a function $f^* \in C_\rho$ for which

$$\lim_{n,m\to\infty}\left\|T_{n,m}f^*-f^*\right\|_\rho \geq \frac{1}{4}$$

where $\rho = 1+x^2+y^2$.

Now, consider the following operator

$$\mathscr{T}_{n,m}(f;q_n,q_m;x,y)=\begin{cases}\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y), & (x,y)\in D_{a_n,b_n}\\ f(x,y), & \mathbb{R}_+^2\backslash D_{a_n,b_n}\end{cases}.$$

Theorem 5.2.3 *Let $f\in C_\rho\left(\mathbb{R}_+^2\right)$. Then for any $\gamma>0$*

$$\lim_{n,m\to\infty}\|\mathscr{T}_{n,m}(f;q_n,q_m;\cdot,\cdot)-f(\cdot)\|_{C_{\rho^{1+\gamma}}}=0$$

where $\{a_n\}$, $\{b_m\}$, $\{q_n\}$ and $\{q_m\}$ satisfy the same conditions as in Theorem 5.2.1.

Proof. For all $\varepsilon>0$, there exist sufficiently large positive real numbers A and B such that

$$\left(1+x^2+y^2\right)^{-\gamma}<\varepsilon \tag{5.2.1}$$

when $x>A$ and $y>B$.

Let n, m be sufficiently large so that $D_{A,B}\subset D_{a_n,b_m}$

$$\begin{aligned}&\|\mathscr{T}_{n,m}(f;q_n,q_m;\cdot,\cdot)-f(\cdot)\|_{C_{\rho^{1+\gamma}}}\\ &\leq\sup_{(x,y)\in D_{A,B}}\frac{\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|}{(1+x^2+y^2)^{1+\gamma}}\\ &+\sup_{(x,y)\in D_{a_n,b_n}\backslash D_{A,B}}\frac{\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|}{(1+x^2+y^2)^{1+\gamma}}\\ &=y'_{n,m}+y''_{n,m}.\end{aligned}$$

By Theorem 5.2.1, $\lim_{n,m\to\infty}y'_{n,m}=0$ and for the proof of the second term we have

$$y''_{n,m} \leq \left(1+x^2+y^2\right)^{-\gamma}\left(\frac{\left|\mathscr{C}^{(\alpha,\beta)}_{n,m,p}\left(f;q_n,q_m;x,y\right)\right|}{1+x^2+y^2}+\frac{|f(x,y)|}{1+x^2+y^2}\right).$$

Finally, since $f \in C_\rho\left(\mathbb{R}^2_+\right)$, the term $\frac{|f(x,y)|}{1+x^2+y^2}$ is bounded. Furthermore, because of the fact that

$$\left|\mathscr{C}^{(\alpha,\beta)}_{n,m,p}\left(f;q_n,q_m;x,y\right)\right| \leq \left|\mathscr{C}^{(\alpha,\beta)}_{n,m,p}\left(1+u_1^2+u_2^2;q_n,q_m;x,y\right)\right|,$$

using Lemma 5.1.1, the term $\frac{\left|\mathscr{C}^{(\alpha,\beta)}_{n,m,p}\left(f;q_n,q_m;x,y\right)\right|}{1+x^2+y^2}$ is bounded for sufficiently large n and m. Hence, we get by (5.2.1) that

$$y''_{n,m} \leq \varepsilon(1+M)$$

Since $\varepsilon > 0$ is arbitrary, then $\lim\limits_{n,m\to\infty} y''_{n,m} = 0$. This completes the proof. ■

Now, consider the subspace C^0_ρ of C_ρ which is defined by

$$C^0_\rho := \left\{f \in C_\rho : \lim_{x,y\to 0}\frac{|f(x,y)|}{1+x^2+y^2} = 0\right\}.$$

Theorem 5.2.4 *Let the sequences* $\{q_n\}$, $\{a_n\}$ *and* $\{b_m\}$ *satisfy the same properties as in Theorem 4.2.1. Then for all* $f \in C^0_\rho\left(\mathbb{R}^2_+\right)$, *we obtain*

$$\lim_{n,m\to\infty}\left\|\mathscr{T}_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right)-f(\cdot)\right\|_{C^0_\rho} = 0.$$

Proof. For all $f \in C^0_\rho\left(\mathbb{R}^2_+\right)$, observe that

$$\lim_{x,y\to\infty}\frac{|f(x,y)|}{1+x^2+y^2} = 0, \qquad \lim_{n,m\to\infty}\frac{\left|f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\right|}{1+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)^2} = 0.$$

Therefore, for all $\varepsilon > 0$, we can find sufficiently large numbers A and B such that

$$|f(x,y)| < \varepsilon\left(1+x^2+y^2\right) \tag{5.2.2}$$

for $x > A$ and $y > B$ and there exists natural numbers n_0 and m_0 such that

$$\left|f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\right| < \varepsilon\left(1+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)^2\right) \tag{5.2.3}$$

for all $n > n_0$ and $m > m_0$.

Hence, for large n and m, we have

$$\begin{aligned}
&\|\mathscr{T}_{n,m}(f;q_n,q_m;\cdot,\cdot)-f(\cdot)\|_{C_\rho}\\
&\leq \sup_{(x,y)\in D_{A,B}} \frac{\left|\mathscr{C}_{n,m}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|}{1+x^2+y^2}\\
&+\sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left|\mathscr{C}_{n,m}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|}{1+x^2+y^2} = z'_{n,m}+z''_{n,m}.
\end{aligned}$$

By Theorem 5.2.1 it is sufficient to show that $z''_{n,m} \to 0$ as $n \to \infty$.

Using (5.2.2) and (5.2.3), we get

$$\begin{aligned}
z''_{n,m} &\leq \varepsilon + \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left|\mathscr{C}_{n,m}^{(\alpha,\beta)}(f;q_n,q_m;x,y))\right|}{1+x^2+y^2}\\
&\leq \varepsilon+\varepsilon \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} t_{n,m}(q_n,q_m;x;y)\\
&= \varepsilon\left(1+\sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y)\right)
\end{aligned}$$

where $t_{n,m}(q_n,q_m;x;y) := \frac{\mathscr{C}_{n,m}^{\alpha,\beta}(1;q_n,q_m;x,y))+\mathscr{C}_{n,m}^{\alpha,\beta}\left(u_1^2;q_n,q_m;x,y)\right)+\mathscr{C}_{n,m}^{\alpha,\beta}\left(u_2^2;q_n,q_m;x,y)\right)}{1+x^2+y^2}$.

By Lemma 5.1.1, it is clear that there exist a number K independent of n and m such that

$$\sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y) \leq K.$$

Therefore, for $n > n_0$ and $m > m_0$, we have

$$z''_{n,m} < (1+K)\varepsilon.$$

This completes the proof. ■

5.3 Order of Convergence

In this subsection, we compute the error of approximation of the operators in terms of the full modulus of continuity and partial modulus of continuities.

Let $f \in D_{A,B}$ and $x \geq 0$. Then the definition of the modulus of continuity of f for two variables is defined by

$$\omega(f;\delta) = \max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \delta \\ x,y \in C(D_{A,B})}} |f(x_1,y_1) - f(x_2,y_2)|. \tag{5.3.1}$$

It is known that for any $\delta > 0$ we can write that

$$|f(x_1,y_1) - f(x_2,y_2)| \leq \omega(f,\delta)\left(\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\delta} + 1\right) \tag{5.3.2}$$

and its partial modulus of continuities are defined by

$$\begin{aligned} \omega^{(1)}(f;\delta) &= \max_{0 \leq y \leq A} \max_{|x_1-x_2| \leq \delta} |f(x_1,y) - f(x_2,y)| \\ \omega^{(2)}(f;\delta) &= \max_{0 \leq x \leq B} \max_{|y_1-y_2| \leq \delta} |f(x,y_1) - f(x,y_2)|. \end{aligned}$$

Also, for any $\delta > 0$ we have

$$\begin{aligned} |f(x_1,y_1) - f(x_2,y_2)| &\leq \omega^{(1)}(f,\delta)\left(\frac{|x_1-x_2|}{\delta} + 1\right), \\ |f(x_1,y_1) - f(x_2,y_2)| &\leq \omega^{(2)}(f,\delta)\left(\frac{|y_1-y_2|}{\delta} + 1\right). \end{aligned}$$

Theorem 5.3.1 *For any $f \in C(D_{A,B})$, the following inequalities*

$$\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \leq 2\left[\omega^{(1)}(f;\delta_m)+\omega^{(2)}(f;\delta_n)\right] \tag{5.3.3}$$

$$\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \leq 2\omega\left(f;\sqrt{\delta_m^2+\delta_n^2}\right) \tag{5.3.4}$$

are satisfied; where

$$\delta_n^2 := \frac{1}{\left([n]_{q_n}+\beta\right)^2} \tag{5.3.5}$$
$$\times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$

and

$$\delta_m^2 := \frac{1}{\left([m]_{q_m}+\beta\right)^2} \tag{5.3.6}$$
$$\times\left\{\left|[m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right|B^2+(2\alpha+1)[m+p]_{q_m}b_mB+\alpha^2b_m^2\right\}.$$

Proof. We directly have,

$$\begin{aligned}
&\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\\
&=\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(x,y)\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&=\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y)\right.\\
&\left.+f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y)-f(x,y)\right]\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}$$

By linearity and positivity of the operators, we get

$$\begin{aligned}
&\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y)\right| \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y)-f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(1)}\left(f;\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&=\Omega_1(x,y)+\Omega_2(x,y).
\end{aligned}$$

Using Lemma 5.1.1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\quad \Omega_1(x,y) \\
&= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&= \sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&\leq \omega^{(2)}(f;\delta_m)\left\{1+\frac{1}{\delta_m}\left[\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\right]^{1/2}\right\}.
\end{aligned}$$

Finally, using Lemma 5.1.1, we get

$$\Omega_1(x,y)\leq 2\omega^{(2)}(f;\delta_m) \tag{5.3.7}$$

where we choose δ_m as in (5.3.6).

In the same way, we obtain

$$\Omega_2(x,y) \leq 2\omega^{(1)}(f;\delta_n) \tag{5.3.8}$$

where δ_n is given in (5.3.5). Combining (5.3.7) and (5.3.8), we obtain (5.3.3) .

Now, by using linearity and the monotonicity of the operators, and taking into account (5.3.1), we have

$$\begin{aligned}
&\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega\left(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m)-f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&\leq 1+\frac{1}{\delta}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2}) \\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned} \tag{5.3.9}$$

Using (4.3.2) and the Cauchy-Schwarz inequality, we get (5.3.4). ■

Theorem 5.3.2 *Let $f(x,y)$ have continuous partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, let $\omega^1(f_x;.)$ and $\omega^2(f_y;.)$ denote the partial moduli of $\partial f/\partial x$ and $\partial f/\partial y$, respectively on $D_{A,B}$. Then the inequality*

$$\begin{aligned}
&\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right] \\
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right]
\end{aligned}$$

holds true, where δ_n and δ_m are the same as in Theorem 5.3.1 and $\left|\frac{\partial f}{\partial x}\right| \leq N$, $\left|\frac{\partial f}{\partial y}\right| \leq M$ on $D_{A,B}$.

Proof. By the mean value theorem, we can write

$$\begin{aligned}
&f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(x,y)\\
&=f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)-f(x,y)+f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\\
&-f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)\\
&=\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right]\\
&+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\frac{\partial f(x,y)}{\partial y}+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\\
&\times\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right] \qquad (5.3.10)
\end{aligned}$$

for any fixed $y\in[0,B]$ and $x\in[0,A]$, where

$$x<\psi_1<\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n$$

and

$$y<\psi_2<\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m.$$

Applying the operator $\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ to (4.3.10)

$$
\begin{aligned}
&\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\\
&=\frac{\partial f(x,y)}{\partial x}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,qq_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\frac{\partial f(x,y)}{\partial y}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}
$$

Hence, taking $\left|\frac{\partial f}{\partial x}\right|\leq N$ and $\left|\frac{\partial f}{\partial y}\right|\leq M$, we get

$$
\begin{aligned}
&\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\\
&\leq\left|\frac{\partial f(x,y)}{\partial x}\right|\left|C_{n,m,p}^{(\alpha,\beta)}(t_1-x;q_n,q_m;x,y)\right|\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left|\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right|\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\left|\frac{\partial f(x,y)}{\partial y}\right|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\left|\frac{\partial f(x,y)}{\partial y}\right|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|\\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&\leq N\left|C_{n,m,p}^{(\alpha,\beta)}\left(t_1-x;q_n,q_m;x,y\right)\right|\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left|\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right|\\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+M\left|C_{n,m,p}^{(\alpha,\beta)}\left(t_2-x;q_n,q_m;x,y\right)\right|\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|\\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}
$$

Then using the properties of partial modulus of continuities, we have

$$
\begin{aligned}
&\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right|\\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)\\
&+\omega^{(1)}\left(f_x;\delta_n\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left(\frac{\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|}{\delta_n}+1\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\\
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)\\
&+\omega^{(2)}\left(f_y;\delta_m\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left(\frac{\left|\frac{[j]_{q_m}+\alpha}{[m]_{qq_m}+\beta}b_m-y\right|}{\delta_n}+1\right)\Phi_{j,m,qq_m}\left(\frac{y}{b_m}\right)
\end{aligned}
$$

since

$$|\psi_1 - x| \leq \left| \frac{[k]_{q_n} + \alpha}{[n]_{q_n} + \beta} a_n - x \right|, \quad |\psi_2 - y| \leq \left| \frac{[j]_{q_m} + \alpha}{[m]_{q_m} + \beta} b_m - y \right|.$$

Applying the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left| \mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y) - f(x,y) \right| \\
&\leq N \left(\left| \frac{[n+p]_{q_n}}{[n]_{q_n} + \beta} - 1 \right| A + \frac{\alpha a_n}{[n]_{q_n} + \beta} \right) \\
&+ \omega^{(1)}(f_x; \delta_n) \left(\sum_{k=0}^{n+p} \left(\frac{[k]_{q_n} + \alpha}{[n]_{q_n} + \beta} a_n - x \right)^2 \Phi_{k,n,q_n}\left(\frac{x}{a_n} \right) \right)^{1/2} \\
&+ \frac{\omega^{(1)}(f_x; \delta_n)}{\delta_n} \sum_{k=0}^{n+p} \left(\frac{[k]_{q_n} + \alpha}{[n]_{q_n} + \beta} a_n - x \right)^2 \Phi_{k,n,q_n}\left(\frac{x}{a_n} \right) \\
&+ M \left(\left| \frac{[m+p]_{q_m}}{[m]_{q_m} + \beta} - 1 \right| B + \frac{\alpha b_m}{[m]_{q_n} + \beta} \right) \\
&+ \omega^{(2)}(f_y; \delta_m) \left(\sum_{j=0}^{m+p} \left(\frac{[j]_{q_m} + \alpha}{[m]_{q_m} + \beta} b_m - y \right)^2 \Phi_{j,m,q_m}\left(\frac{y}{b_m} \right) \right)^{1/2} \\
&+ \frac{\omega^{(2)}(f_y; \delta_m)}{\delta_m} \sum_{j=0}^{m+p} \left(\frac{[j]_{q_m} + \alpha}{[m]_{q_m} + \beta} b_m - y \right)^2 \Phi_{j,m,q_m}\left(\frac{y}{b_m} \right).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\left| \mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y) - f(x,y) \right| \leq N \left(\left| \frac{[n+p]_{q_n}}{[n]_{q_n} + \beta} - 1 \right| A + \frac{\alpha a_n}{[n]_{q_n} + \beta} \right) \\
&+ \omega^{(1)}(f_x; \delta_n) \left(\left(\sqrt{\mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left((u_1 - x)^2; q_n, q_m; x, y \right)} \right) \right. \\
&\left. + \frac{\omega^{(1)}(f_x; \delta_n)}{\delta_n} \left(\mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left((u_1 - x)^2; q_n, q_m; x, y \right) \right) \right) \\
&+ M \left(\left| \frac{[m+p]_{q_m}}{[m]_{q_m} + \beta} - 1 \right| B + \frac{\alpha b_m}{[m]_{q_n} + \beta} \right) \\
&+ \omega^{(2)}(f_y; \delta_m) \sqrt{C_{n,m,p}^{(\alpha,\beta)}\left((u_2 - y)^2; q_n, q_m; x, y \right)} \\
&+ \frac{\omega^{(2)}(f_y; \delta_m)}{\delta_m} \mathscr{C}_{n,m,p}^{(\alpha,\beta)} \left(\mathscr{C}_{n,m,p}^{(\alpha,\beta)}\left((u_2 - y)^2; q_n, q_m; x, y \right) \right).
\end{aligned}$$

Now using Lemma 4.1.1 and choosing δ_n and δ_m as in (4.3.5) and (4.3.6), respectively, we get

$$\begin{aligned}
&\left|\mathscr{C}_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right] \\
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right].
\end{aligned}$$

Whence the result. ■

5.4 Generalization of the operators

In this subsection, we introduce generalization of two dimensional Chlodowsky type of q-Bernstein-Schurer-Stancu operators. The generalized operators help us to approximate continuous functions defined on more general weighted spaces.

For $x \geq 0$, consider any continuous function $\omega(x,y) \geq 1$ and define

$$G_f(t,s) = f(t,s)\frac{1+t^2+s^2}{w(t,s)}.$$

Let us consider the generalization of the two dimensional Chlodowsky type of q-Bernstein-Schurer-Stancu operators as follows

$$\begin{aligned}
&\mathscr{L}_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) \\
&=\begin{cases}
\begin{array}{c}\frac{w(x,y)}{1+x^2+y^2}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}G_f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\\ \times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\end{array}, & (x,y)\in D_{a_n,b_n} \\
f(x,y), & \mathbb{R}_+^2\backslash D_{a_n,b_n}
\end{cases}
\end{aligned}$$

where $(x,y)\in D_{a_n,b_m}$ and $\{a_n\}$ and $\{b_m\}$ have the same properties of two dimensional Chlodowsky type of q-Bernstein-Schurer-Stancu operators.

Theorem 5.4.1 *For all continuous functions f satisfying $|f(x,y)| \leq M_f w(x,y)$, $x,y \geq 0$, and $\lim\limits_{x,y\to\infty} \frac{f(x,y)}{w(x,y)} = 0$, we have*

$$\lim_{n,m\to\infty} \left\| \mathscr{L}_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;\cdot,\cdot) - f(\cdot,\cdot) \right\|_w = 0$$

where $\rho(x,y) = 1+x^2+y^2$.

Proof. Clearly,

$$\begin{aligned}
&\left| \mathscr{L}_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y) \right| \\
= \quad & \frac{w(x,y)}{1+x^2+y^2} \left| \sum_{k=0}^{n+p} \sum_{j=0}^{m+p} G_f \left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right) \right. \\
& \left. \times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right) \Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) - G_f(x,y) \right|,
\end{aligned}$$

thus

$$\begin{aligned}
&\left\| \mathscr{L}_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;\cdot,\cdot) - f(\cdot,\cdot) \right\|_w \\
= \; & \sup_{x,y\in\mathbb{R}_+^2} \frac{\left| \mathscr{L}_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y) \right|}{w(x,y)} \\
= \; & \sup_{x,y\in\mathbb{R}_+^2} \frac{\left| \mathscr{T}_{n,p}^{(\alpha,\beta)}(G_f;q_n,q_m;x,y) - G_f(x,y) \right|}{1+x^2+y^2}.
\end{aligned}$$

Since $|f(x,y)| \leq M_f w(x,y)$, then $\left|G_f(x,y)\right| \leq M_f \rho(x,y)$ for $x,y \geq 0$ and $G_f(x,y)$ is continuous function on R_+^2. Furthermore, from $\lim\limits_{x,y\to\infty} \frac{f(x,y)}{w(x,y)} = 0$, we have

$$\lim_{x,y\to\infty} \frac{G_f(x,y)}{\rho(x,y)} = 0.$$

Hence, by Theorem 5.2.4 we get the desired result. ■

Finally, note that, taking $w(x,y) = 1+x^2+y^2$, then the operators $\mathscr{L}_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$

reduces $\mathscr{T}_{n,p}^{(\alpha,\beta)}\left(G_f;q_n,q_m;x,y\right)$.

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