Quantum Calculus on Finite Intervals and Applications to Impulsive Difference Equations

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ABSTRACT

In Mathematics, quantum calculus is a version of calculus in which limits are not taken. This type of calculus plays important role both in theoretical and practical areas of mathematics. In quantum calculus, derivatives are differences and anti-derivatives are sums. Quantum calculus is a theory where smoothness is no more needed. In this work, we study finite intervals in quantum calculus. We review and study the q_j-derivative and q_j-integral of a function and demonstrate their properties. We apply this concept to provide existence and uniqueness results for the initial value problems, namely for first and second order impulsive q_j-difference equations.

Keywords: q_j-derivative, q_j- integral, impulsive q_j-difference equation, existence, uniqueness.

ÖZ

Matematikte q-Kalkülüs, Kalkülüsda limitlerin alınmadığı bir versiyonudur. Bu tür matematik birçok teorik ve pratik alanda önemli rol oynamaktadır. Kuantum Kalkülüsda türevler fark ve integral ise toplam olarak tanımlanır. Kuantum Kalkülüs düzgünlüğün gerekli olmadığı bir teoridir. Bu çalışmada, kuantum Kalkülüsın sınırlı aralıkları dikkate alınmıştır. Ayrıca, bu tezde bir fonksiyonun q_j −türevini ve q_j −integralini inceleyip özellikleri verilmiştir. Bu kavram, baslangıç-değer problemlerinin varlık ve teklik sonuçları üzerinde uygulanmıştır. Özelde birinci ve ikinci dereceden impulsif q_j-fark denklemleri dikkate alınmıştır.

Anahtar Kelimeler: varlık, teklik, q_j-türev, q_j- integral, impulsive q_j-fark denklemi

To My Family

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Chapter 1

INTRODUCTION

1.1 Historical Background

For a long time, studying, investigating and developing calculus had based on using limits. Later it had appeared a calculus without limits called $q-$calculus. The quantum calculus started with F.H. Jackson in the beginning of last century as emerging area of mathematics, although it had been discovered already and vigorously studied by Euler and Jacobi.

We can separate the evolution of quantum calculus from historical perspective into two parts:

1. Development of quantum calculus in the period 1893-1950.

1893-1895 Rogers had some work on orthogonal polynomial that is possible to write using $q-$hypergeometric series. Rogers demonstrated the proof of two Rogers-Ramanujan identities that represent an infinite sum as a quotient of infinite product. During this time, the big battlefield came to Europe. Beginning from 1904, the English reverend Jackson had a lot of mathematics works intended completely to quantum calculus that lasted until 1951. Jackson worked on elliptic functions, and special functions.

Jackson started to find q-analogues of trigonometric functions, Bessel functions, Legendre polynomials and the gamma function.

He elaborated the link between the q-gamma function and elliptic functions.

In 1910, Watson [19] constructed the proof a q-analogue of Barnes contour integral expression for a hypergeometric series. Most of the people do not remember the extraordinary proof of the R-Ramanujan identities.

2. Development of quantum calculus in the second half of the 20th century.

Many parts of quantum calculus evolved in the second half of the 20^{th} century. In the 1950's, many of the great developers of the subject were Lucy J. Slater and D.B. Sears (1918-1999). L.J. Slater participated at Bailey's classes on q −hypergeormetric series in 1947-50 at Bedford College, London University and in 1966 presented the book [17], which explains the extraordinary developments made in the subject from 1936 when Bailey's book [7] was published. The so-called Sturm-Liouville q-difference equation had been worked in [7]. This equation is a q-analogue of the Sturm-Liouville differential equation.

.1.2 Significance and Importance of Quantum Calculus

For the last decades, it had attracted attention of researchers and scientists after the emergence of huge interest of mathematics that is used in modeling quantum computing. For further details, a distinct works can be found in the papers [4, 11, 12] and the references had been described there.

Quantum calculus worked in from the beginning of this century represented as a link between mathematics and physics. Most of the scientific community, which benefit from quantum calculus, is physicists.

The area witnessed a great expansion, because of using foundations of hypergeometric series to the different subjects of combinatorics, quantum theory, number theory, statistical mechanics that are continuously discovered.

One of the most important works in quantum calculus is the book written in last decade by Kac and Cheung [18], which studies a lot of the foundational basis of quantum calculus.

It is widely understood that quantum calculus is a branch of broader mathematical area of time scales calculus. Time scales gives a generalized basis for working on dynamic equations on both discrete and continuous domains. The text by Bohner and Peterson [13] brought together important contribution in the calculus of time scales. Working out in quantum calculus focuses on a special time scale, called the q-time scale, described below:

$$\mathrm{Z} := q^{N_0} := \{q^z : \mathrm{z} \in N_0\}, \text{where } \mathrm{q} > 1.$$

In our work, we study finite intervals in quantum calculus. We describe the q_j-derivative of a function $f: K_j := [z_j, z_{j+1}] \to R$ and demonstrate some properties, for instance as derivative of a sum, of a product or a quotient of two functions. In addition, it is useful to describe q_j-integral and provide its properties. We apply this concept to present existence and uniqueness results for initial value problem of first and second order impulsive q −difference equations.

The early type of quantum calculus is not possible to be used in areas with impulses because if an impulse point z_j, $j \in N$ lies between the points z and qz, then the definition of q −derivative does not hold. Nevertheless, we do not see this in impulsive problems on q-time scale for the reason that the points z and $qz = \rho(z)$ are consecutive. Finite intervals in quantum calculus the points z and $q_j z + (1 - q_j)z_j$ are considered only in an interval $[z_j, z_{j+1}]$. Therefore, using q_j-calculus it is possible to solve systems with impulses at fixed times.

The remaining part of our work arranged as follows. In chapter 2, we revise the basics of q-calculus. In chapter 3, we present the concept of finite intervals in q_j-derivatives and q_j-integrals and demonstrate some properties which are basic. In chapter 4, we use the outcome of chapter 3 to impulsive q_j-difference equations and provide existence and uniqueness results. We also discuss illustrations that demonstrate the results.

Chapter 2

BASICS OF QUANTUM CALCULUS

Definition 2.1

Let f be a function defined on a q −geometric set L, i.e.$qz \in L$, $z \in L$.

For $0 < q < 1$, the q −derivative of f is;

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \; z \in L/\{0\} \qquad D_q f(0) = \lim_{z \to 0} D_q f(z).$$

We see that $\lim_{q \to 1} D_q f(z) = \lim_{q \to 1} \frac{f(qz) - f(z)}{(q-1)z} = \frac{df(z)}{dz}$.

if f(z) is differentiable.

Some Properties of q −Derivative

It is clear that q-derivative of a function has a linearity property. For constants c and d we have:

$$D_q\{cf(z) + dh(z)\} = \frac{\left(cf(z) + dh(z)\right) - \left(cf(qz) + dh(qz)\right)}{(1-q)z},$$

$$= \frac{\left(cf(z) - cf(qz)\right) + \left(dh(z) - dh(qz)\right)}{(1-q)z},$$

$$= c\frac{f(z) - f(qz)}{(1-q)z} + d\frac{h(z) - h(qz)}{(1-q)z},$$

$$= cD_q\{f(z)\} + dD_q\{h(z)\}.$$

Derivative of a Product of Functions

Derivative of a product of functions in quantum calculus is the same as in classic calculus (calculus with limits). Therefore, by definition 2.1 it can be stated as follows:

$$D_q\{f(z)h(z)\} = f(qz)D_q h(z) + h(z)D_q f(z),$$
$$= f(z)D_q h(z) + h(qz)Dqf(z). \qquad (2.1)$$

This can be proved as follows:

$$D_q\{f(z)h(z)\} = \frac{f(z)h(z) - f(qz)h(qz)}{(1-q)z},$$

$$= \frac{f(z)h(z) - f(z)h(qz) + f(z)h(qz) - f(qz)h(qz)}{(1-q)z},$$

$$= \frac{f(z)h(z) - f(z)h(qz)}{(1-q)z} + \frac{f(z)h(qz) - f(qz)h(qz)}{(1-q)z},$$

$$= f(z)\frac{h(z) - h(qz)}{(1-q)z} + h(qz)\frac{f(z) - f(qz)}{(1-q)z},$$

$$= f(z)D_q h(z) + h(qz)D_q f(z).$$

Derivative of a Quotient of Functions

$$D_q\left\{\frac{f(z)}{h(z)}\right\} = \frac{h(z)D_q f(z) - f(z)D_q h(z)}{h(qz)h(z)}. \qquad (2.2)$$

For the proof, the following steps are can be done.

Take $f(z) = h(z)\frac{f(z)}{h(z)}$.

Differentiating q −derivative on both sides of above we get:

$$D_q f(z) = D_q\left\{h(z)\frac{f(z)}{h(z)}\right\}, \text{ using product rule 2.1}$$

$$D_q f(z) = h(qz) D_q\left\{\frac{f(z)}{h(z)}\right\} + \frac{f(z)}{h(z)} D_q h(z),$$

$$D_q\left\{\frac{f(z)}{h(z)}\right\} = \frac{D_q f(z) - \frac{f(z)}{h(z)} D_q h(z)}{h(qz)},$$

$$D_q\left\{\frac{f(z)}{h(z)}\right\} = \frac{h(z) D_q f(z) - f(z) D_q h(z)}{h(z) h(qz)}.$$

Definition 2.2

The higher order q −derivative is expressed as

$$D_q^0 f(z) = f(z), \qquad D_q^{\alpha} f(z) = D_q D_q^{\alpha-1} f(z), \qquad \alpha \in N.$$

***q* −Integral of a Function (Jackson integral)**

Suppose f(z) is arbitrary function. Define the operator $E_q^{\wedge}(f(z)) = f(qz)$.

Construct q −antiderivative of f(z). By definition 2.1 we have:

$$f(z) = \frac{F(z) - F(qz)}{(1-q)z} = \frac{1}{(1-q)}(1 - E_q^{\wedge})F(z).$$

Formally writing q −antiderivative,

$$(1 - E_q^{\wedge})F(z) = (1-q) z f(z),$$

$$F(z) = \frac{(1-q)}{(1 - E_q^{\wedge})} z f(z),$$

$$=(1\text{-}q)\sum_{m=0}^{\infty} E_q^{\wedge n}(zf(z)),$$

$$=(1\text{-}q)\sum_{m=0}^{\infty} q^m zf(q^m z)$$

Using geometric series expansion we get

$\int f(z)d_q z = (1-q)\sum_{m=0}^{\infty} q^m zf(q^m)$ Provided the series converges.

N.B. For convergence of the series see next theorem.

Theorem 2.1 [18]

Suppose that $0 < q < 1$. I f $\left|f(z)z^{\alpha}\right|$ is bounded on (0, A] for some $0 \leq \alpha < 1$ then the above integral converges to a function F(z) on (0, A] which is q −antiderivative of f(z). Moreover, F(z) is continuous at z=0 with F(0)=0

Proof.

Suppose that

$\left|f(z)z^{\alpha}\right| \leq N$, $\forall\, z \in (0, A]$. Then $\forall\, m \geq 0$,

$$\left|(q^m z)^{\alpha} f(z)z^{\alpha}\right| \leq N,$$

$$\left|f(q^m z)\right| \leq N(q^m z)^{-\alpha}, \qquad z \in (0, A]$$

$$q^m\left|f(q^m z)\right| \leq N(q^m)^{1-\alpha}\frac{1}{z^{\alpha}}, \qquad z \in (0, A],\ 1-\alpha > 0,$$

Since $1-\alpha > 0$, $0 < q < 1$,

$$\frac{N}{z^{\alpha}}\sum_{m=0}^{\infty}(q^m)^{1-\alpha} < \infty, \quad z \in (0, A].$$

By M-weierstrass test $\sum_{m=0}^{\infty} q^m f(q^m z)$ is point wise convergent and

$$F(z) = (1-q)z\sum_{m=0}^{\infty} q^m f(q^m z).$$

We now show that F(z) is continuous at z = 0. Indeed,

$$|f(z)| = \left|(1-q)z\sum_{m=0}^{\infty} q^m f(q^m z)\right|,$$

$$\leq (1-q)z\sum_{m=0}^{\infty} N(q^m)^{1-\alpha}\frac{1}{z^{\alpha}},$$

$$= N(1\text{-}q)z^{1-\alpha}\sum_{m=0}^{\infty}(q^{1-\alpha})^m,$$

$$= N\frac{(1-q)}{(1-q^{1-\alpha})}z^{1-\alpha}.$$

$\lim_{z\to o}|F(z)| = 0$ and F(0)=0.

F(z) is continuous at Z=0 and F(0)=0.

Now we show that F(z) is q −antiderivative of f(z).

$$D_q F(z) = \frac{F(z)-F(qz)}{(1-q)z} = \frac{1}{(1-q)z}\left((1-q)z\sum_{m=0}^{\infty} q^m f(q^m z) - (1-q)qz\sum_{m=0}^{\infty} q^m f(q^{m+1}z)\right),$$

$$= \frac{1}{(1-q)z}(1-q)z\left(\sum_{m=0}^{\infty} q^m f(q^m z) - \sum_{m=0}^{\infty} q^{m+1} f(q^{m+1}z)\right),$$

$$= \sum_{m=0}^{\infty} q^m f(q^m z) - \sum_{m=1}^{\infty} q^m f(q^m z),$$

$$= f(z).$$

Remark 2.1

If f is continuous at z = 0, then

$$I_q D_q f(z) = f(z) - f(0).$$

This can be verified as follows:

$$I_q D_q f(z) = (1-q)z \sum_{m=0}^{\infty} q^m D_q f(q^m z),$$

Since a partial sum of q −integral (Jackson's integral)

$$= (1-q)z \sum_{m=0}^{N} q^m \frac{f(q^m z) - f(q^{m+1} z)}{(1-q)q^m z},$$

$$= \sum_{m=0}^{N} f(q^m z) - f(q^{m+1} z),$$

$$= f(z) - f(q^{N+1} z),$$

Which tends to f (z) - f (0) as N→ ∞ by the continuity of f (z) at z = 0.

Definition 2.3

For $z \geq 0$ we set $K_z = zq^{\propto}$:∝∈ $N \cup \{0\}$and express the definite q −integral of a function $f: K_z \to R$ by

$$I_q f(z) = \int_0^z f(l) d_q l = \sum_{\alpha=0}^{\infty} z(1-q) q^{\alpha} f(q^{\alpha} z).$$

The series must converge for integral to exist (we stated convergence in theorem 2.1).

For ∈ K_z , we expand

$$\int_c^d f(l) d_q l = I_q f(d) - I_q f(c) = (1-q) \sum_{n=0}^{\infty} q^{\alpha} [df(dq^{\alpha}) - cf(cq^{\alpha})].$$

Note that for $c, d \in k_z$, we have $c = zq^{\alpha_1}$, $d = zq^{\alpha_2}$ for some α_1, $\alpha_2 \epsilon N$. The definite integral $\int_c^d f(l) d_q l$ is just a finite sum and it is clear that it converges.

Corollary 2.1 [18]

If $f'(z)$ exist in a neighborhood of z = 0, is continuous at z = 0, where $f'(z)$ denotes the ordinary derivative of f (z), we have

$$\int_a^b D_q f(z) d_q z = f(b) - f(a).$$

Proof.

We use L'Hopital's rule to get

$$\lim_{z\to 0} D_q f(z) = \lim_{z\to 0} \frac{f(z) - f(qz)}{(1-q)z} \Rightarrow \lim_{z\to 0} \frac{f'(z) - qf'(qz)}{(1-q)} = \frac{f'(0) - qf'(0)}{(1-q)} = f'(0).$$

$D_q f(z)$ can be made continuous at z = 0 if

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \\ f'(0) & z=0 \end{cases}$$

Integration by Parts

In q −calculus, the integration by parts formula is given by

$$\int_0^z f(t) D_q h(t) d_q t = [f(t)h(t)]_0^z - \int_0^z D_q f(t) h(qt) d_q t.$$

This is proved as follows:

From corollary 2.1 we get

$$\int_0^z f(t) D_q h(t) d_q t = f(z)h(z) - f(0)h(0). \qquad (2.3)$$

Using product rule we have

$$D_q(f(t)h(t)) = f(t)D_q h(t) + h(qt)D_q f(t).$$

q −integrating both sides of above we get

$$\int_0^z D_q(f(t)h(t))\,d_q t = \int_0^z f(t)D_q h(qt)\,d_q t + \int_0^z h(qt)D_q f(t)\,d_q t.$$

Using (2.3) we get

$$\left[f(t)h(t)\right]_0^z - \int_0^z h(qt)D_q f(t)\,d_q t = \int_0^z f(t)D_q h(t)\,d_q t.$$

Chapter 3

FINITE INTERVALS IN $q-$ CALCULUS

Now we are studying the main part of our work, which is the concept of $q-$derivative and $q-$integral of finite intervals in quantum calculus.

Let $J \in N \cup \{0\}$, $K_j := [z_j, z_{j+1}] \subset R$, $0 < q < 1$ be a constant. We define the q_j-derivative of a function $f : K_j \to R, z \in K_j$ as follows:

Definition 3.1

Let $f: K_j \to R$ is a continuous function, $z \in K_j$. Then

$$D_{q_j} f(z) = \frac{f(z) - f(q_j z + (1-q_j) z_j)}{(1-q_j)(z - z_j)}, \qquad z \neq z_j, \tag{3.1}$$

$D_{q_j} f(z_j) = \lim_{z \to z_j} D_{q_j} f(z),$ is called the $q_j -$ derivatvie of f at z.

We say that f is $q_j - d$ifferentiable on K_j provided $D_{q_j} f(z)$ exist for all z $\in K_j$.

We see that if $z_j = 0$ and q_j = q in (3.1) and $D_{q_j} f(z) = D_q f(z)$, where D_q is the $q-$derivative of the function f (z) defined in definition 2.1.

Example 3.1

Let $f(z) = z^2$ for $z \in [1, 4]$ $\qquad q_j = \frac{1}{2}$ then,

$$D_{q_j} f(z) = \frac{z^2 - (q_j z + (1-q_j) z_j)^2}{(1-q_j)(z-z_j)},$$

$$= \frac{(1+q_j) z^2 - 2 q_j z_j z - (1-q_j) z_j^2}{z - z_j},$$

$$= \frac{3z^2 - 2z - 1}{2(z-1)}, \quad z \in (1,4],$$

and $\lim_{z \to z_j} D_{q_j} f(z) = 2$, if $z = 1$. $D_{\frac{1}{2}} f(3) = 5$ has another way to write as difference quotient $\frac{f(3) - f(2)}{3-2}$.

Example 3.2

In quantum calculus, we have $D_q z^{\alpha} = [\alpha]_q z^{\alpha - 1}$ where $[\alpha]_q = \frac{1-q^{\alpha}}{1-q}$. However, $q_j -$ calculus gives $D_{q_j} (z - z_j)^{\alpha} = [\alpha]_{q_j} (z - z_j)^{\alpha - 1}$. Indeed,

$$D_{q_j} f(z) = \frac{(z - z_j)^{\alpha} - (q_j z + (1-q_j) z_j - z_j)^{\alpha}}{(1-q_j)(z - z_j)},$$

$$= [\alpha]_{q_j} (z - z_j)^{\alpha - 1},$$

and $[\alpha]_{q_j} = \frac{1 - q_j^{\alpha}}{1 - q_j}$.

Theorem 3.1 [12]

Let f, h : $K_j \to R$ have $q_j -$ derivative on K_j.

(1) The sum f + h : $K_j \to R$ has $q_j -$ derivative on K_j,

$$D_{q_j}(f(z) + h(z)) = D_{q_j} f(z) + D_{q_j} h(z).$$

(2) For constant c $cf: K_j \to R$ has $q_j -$ derivative on K_j with,

$$D_{q_j}(cf)(z) = cD_{q_j}f(z).$$

(3) The product $fh: K_j \to R$ has $q_j - derivative$ on K_j,

$$D_{q_j}(fh)(z) = f(z)D_{q_j}h(z) + h\left(q_jz + (1-q_j)z_j\right)D_{q_j}f(z)$$
$$= h(z)D_{q_j}f(z) + f\left(q_jz + (1-q_j)z_j\right)D_{q_j}h(z).$$

(4) If $h(z)h(q_jz + (1-q_j)z_j) \neq 0$ then $\frac{f}{h}$ is has a q_j-derivative on K_j with

$$D_{q_j}\left(\frac{f}{h}\right)(z) = \frac{h(z)D_{q_j}f(z) - f(z)D_{q_j}h(z)}{h(z)h(q_jz + (1-q_j)z_j)}.$$

Proof.

(1) $$D_{q_j}\left(f(z) + h(z)\right) = \frac{f(z) + h(z) - \left[f(q_jz + (1-q_j)z_j) + h(q_jz + (1-q_j)z_j)\right]}{(1-q_j)(z-z_j)},$$

$$= \frac{f(z) - f(q_jz + (1-q_j)z_j)}{(1-q_j)(z-z_j)} + \frac{h(z) - h(q_jz + (1-q_j)z_j)}{(1-q_j)(z-z_j)},$$

$$= D_{q_j}f(z) + D_{q_j}h(z).$$

(2) $$D_{q_j}(cf)(z) = \frac{cf(z) - cf(q_jz + (1-q_j)z_j)}{(1-q_j)(z-z_j)},$$

$$= c\frac{f(z) - f(q_jz + (1-q_j)z_j)}{(1-q_j)(z-z_j)},$$

$$= cD_{q_j}f(z).$$

(3) $D_{q_j}(fh)(z)=\dfrac{f(z)h(z)-f(q_jz+(1-q_j)z_j)h(q_jz+(1-q_j)z_j)}{(1-q_j)(z-z_j)}.$

$$=f(z)h(z)-f(z)h(q_jz+(1-q_j)z_j)+f(z)h(q_jz+(1-q_j)z_j)$$
$$-f(q_jz+(1-q_j)z_j)h(q_jz+(1-q_j)z_j)/(1-q_j)(z-z_j),$$

$$=f(z)\left(\frac{h(z)-h(q_jz+(1-q_j)z_j)}{(1-q_j)(z-z_j)}\right)+h(q_jz+(1-q_j)z_j)\left(\frac{f(z)-f(q_jz+(1-q_j)z_j)}{(1-q_j)(z-z_j)}\right),$$

$$=f(z)D_{q_j}h(z)+h(q_jz+(1\text{-}q_j)z_j)D_{q_j}f(z).$$

(4) $D_{q_j}\left(\dfrac{f}{h}\right)(z)\ =\dfrac{\dfrac{f(z)}{h(z)}-\dfrac{f(q_jz+(1-q_j)z_j)}{h(q_jz+(1-q_j)z_j)}}{(1-q_j)(z-z_j)},$

$$=\frac{f(z)h(q_jz+(1-q_j)z_j)-h(z)f(q_jz+(1-q_j)z_j)}{h(z)h(q_jz+(1-q_j)z_j)(1-q_j)(z-z_j)},$$

$$=\frac{h(z)\left(\frac{f(z)-f(q_jz+(1-q_j)z_j)}{(1-q_j)(z-z_j)}\right)-f(z)\left(\frac{h(z)-h(q_jz+(1-q_j)z_j)}{(1-q_j)(z-z_j)}\right)}{h(z)h(q_jz+(1-q_j)z_j)},$$

$$=\frac{h(z)D_{q_j}f(z)-f(z)D_{q_j}h(z)}{h(z)h(q_jz+(1-q_j)z_j)}.$$

Remark 3.1

In Example 2.2 we see that in q −difference, if $f(z) = z^{\alpha}$ then $D_q z^{\alpha} = [\alpha] z^{\alpha-1}$ it is not possible to get easy formula for q_j −difference. Using the derivative of a product, it is possible to write it as follows:

$$D_{q_j} z = 1,$$

$$D_{q_j} z^2 = D_{q_j}(z \cdot z) = (1+q_j)z + (1-q_j)z_j$$

$$D_{q_j} z^3 = D_{q_j}(z^2.z) = (1+q_j+q_j^2)z^2 + (1+q_j-2q_j^2)zz_j + (1-q_j)^2 z_j{}^2,$$

$$\begin{aligned} D_{q_j} z^4 &= D_{q_j}(z^3 \cdot z) \\ &= (1+q_j+q_j^2+q_j^3)z^3 + (1+q_j+q_j^2-3q_j^3)z_j z^2 + (1+q_j-5q_j^2+3q_j^3)z_j^2 z + (1-q_j)^3 z^3{}_j. \end{aligned}$$

Definition 3.2

Let $f: K_j \to R$ as a continuous function. We call the second order q_j −derivative $D_{q_j}^2 f$ provided $D_{q_j} f$ has a q_j −derivatives on K_j with $D_{q_j}^2 f = D_{q_j}(D_{q_j} f): K_j \to R.$

In a similar way, it is possible to define the higher order q_j −derivative $D_{q_j}^{\alpha}: K_j \to R.$ For instance, $f: K_j \to R$, then

$$\begin{aligned} D_{q_j}^2 f(z) &= D_{q_j}(D_{q_j} f(z)), \\ &= \frac{D_{q_j} f(z) - D_{q_j} f(q_j z + (1-q_j)z_j)}{(1-q_j)(z-z_j)}, \\ &= \frac{\dfrac{f(z) - f(q_j z + (1-q_j)z_j)}{(1-q_j)(z-z_j)} - \dfrac{f(q_j z + (1-q_j)z_j) - f(q_j^2 z + (1-q_j^2)z_j)}{(1-q_j)(z-z_j)}}{(1-q_j)(z-z_j)}, \\ &= \frac{f(z) - 2f(q_j z + (1-q_j)z_j) + f(q_j^2 z + (1-q_j^2)z_j)}{(1-q_j)^2(z-z_j)^2}, \qquad z \neq z_j \end{aligned}$$

and $D_{q_j}^2 f(z_j) = \lim_{z \to z_j} D_{q_j}^2 f(z).$

To demonstrate q_j −antiderivative $F(x)$, it is useful to describe a shifting operator by $E_{q_j}F(z) = F(q_j z + (1-q_j)z_j)$.

It can be verified by mathematical induction that

$E^m_{q_j}F(z) = E_{q_j}\left(E^{m-1}_{q_j}F\right)(z) = F\left(q^m_j z + \left(1-q^m_j\right)z_j\right)$, where $m \in N$ and

$E^0_{q_j}F(z) = F(z)$. Then by definition 3.1, we get

$$\frac{F(z) - F(q_j z + (1-q_j)z_j)}{(1-q_j)(z-z_j)} = \frac{1-E_{q_j}}{(1-q_j)(z-z_j)}F(z) = f(z).$$

Therefore, q_j −antiderivative becomes as follows:

$F(z) = \frac{1}{1-E_{q_j}}\left((1-q_j)(z-z_j)f(z)\right)$. By expanding the geometric series, we get:

$$\begin{aligned} F(z) &= (1-q_j)\sum_{m=0}^{\infty}E^m_{q_j}(z-z_j)f(z) \\ &=(1-q_j)\sum_{m=0}^{\infty}\left(q^m_j z + \left(1-q^m_j\right)z_j - z_j\right)f\left(q^m_j z + \left(1-q^m_j\right)z_j\right) \\ &= = (1-q_j)(z-z_j)\sum_{m=0}^{\infty}q^m_j f\left(q^m_j z + \left(1-q^m_j\right)z_j\right). \end{aligned} \tag{3.2}$$

It is obvious that the above calculus is true if the series in the last part converges.

Definition 3.3

Let $f: K_j \to R$ is a continuous function. Then the q_j − integral is defined by

$$\int_{z_j}^{z} f(l)\,d_{q_j}l = (1-q_j)(z-z_j)\sum_{m=0}^{\infty}q^m_j f\left(q^m_j z + \left(1-q^m_j\right)z_j\right) \qquad \text{for } z \in K_j \tag{3.3}$$

Moreover, if $c \in (z_j, z)$, then the definite q_j −integral is defined by

$$\int_c^z f(l)\, d_{q_j} l = \int_{z_j}^{z} f(l)\, d_{q_j} l - \int_{z_j}^{c} f(l)\, d_{q_j} l$$

$$=(1-q_j)(z-z_j)\sum_{m=0}^{\infty} q_j^m f\left(q_j^m z + \left(1-q_j^m\right) z_j\right) - (1-q_j)(c-z_j)\sum_{m=0}^{\infty} q_j^m f\left(q_j^m c + \left(1-q_j^m\right) z_j\right).$$

Note that if $z_j = 0$ and $q_j = q$ then (3) becomes q −integral of a function $f(z)$ defined by

$$\int_0^z f(l)\, d_q l = (1-q_j) z \sum_{m=0}^{\infty} q^m f(q^m z), \text{ for } z \in [0, \infty).$$

Example 3.3

Take $f(z) = z$ for $z \in K_j$.

$$\int_{z_j}^z f(l) d_{q_j} l = \int_{z_j}^z l d_{q_j} l = (1-q_j)(z-z_j)\sum_{m=0}^{\infty} q_j^m (q_j^m + (1-q_j^m) z_j),$$

$$= (1-q_j)(z-z_j)\left[\sum_{m=0}^{\infty} q_j^{2m} z + \sum_{m=0}^{\infty} q_j^m z_j - \sum_{m=0}^{\infty} q_j^{2m} z_j\right],$$

$$= (1-q_j)(z-z_j)\left[\frac{z}{1-q_j^2} + \frac{z_j}{1-q_j} - \frac{z_j}{1-q_j^2}\right],$$

$$= (1-q_j)(z-z_j)\left[\frac{z + z_j(1+q_j) - z_j}{(1-q_j^2)}\right],$$

$$= (1-q_j)(z-z_j)\left[\frac{(z+q_j z_j)}{(1-q_j)(1+q_j)}\right],$$

$$= \frac{(z-z_j)(z+q_j z_j)}{(1+q_j)}.$$

Theorem 3.2 [12]

For $z \in K_j$,

(1) $D_{q_j} \int_{z_j}^{z} f(l)\, d_{q_j} l = f(z)$;

(2) $\int_{z_j}^{z} D_{q_j} f(l)\, d_{q_j} l = f(z)$

(3) $\int_{c}^{z} D_{q_j} f(l)\, d_{q_j} l = f(z) - f(c)$ for $c \in (z_j, z)$.

Proof.

(1) Applying definitions 3.1 and 3.3, we get

$$D_{q_j} \int_{z_j}^{z} f(l) d_{q_j} l = D_{q_j} \left[(1-q_j)(z-z_j) \sum_{m=0}^{\infty} q_j^m f(q_j^m z + (1-q_j^m) z_j) \right],$$

$$= \frac{(1-q_j)}{(1-q_j)(z-z_j)} \left[\begin{array}{l} (z-z_j) \sum_{m=0}^{\infty} q_j^m f(q_j^m z + (1-q_j^m) z_j) - (q_j z + (1-q_j) z_j - z_j) \\ \times \sum_{m=0}^{\infty} q_j^m f(q_j^m (q_j z + (1-q_j) z_j) + (1-q_j^m) z_j) \end{array} \right],$$

$$= \frac{1}{(z-z_j)} \left[(z-z_j) \sum_{m=0}^{\infty} q_j^m f(q_j^m z + (1-q_j^m) z_j) - q_j (z-z_j) \sum_{n=0}^{\infty} q_j^m f(q_j^{m+1} z + (1-q_j^{m+1}) z_j) \right],$$

$$= \sum_{m=0}^{\infty} q_j^m f(q_j^m z + (1-q_j^m) z_j) - \sum_{m=0}^{\infty} q_j^{m+1} f(q_j^{m+1} z + (1-q_j^{m+1}) z_j),$$
$$= f(z).$$

(2) $$\int_{z_j}^{z} D_{q_j} f(l) d_{q_j} l = \int_{z_j}^{z} \frac{f(l) - f(q_j l + (1-q_j) z_j)}{(1-q_j)(l-z_j)} d_{q_j} l,$$

$$= (1-q_j)(z-z_j) \sum_{m=0}^{\infty} q_j^m \frac{f(q_j^m z + (1-q_j^m) z_j) - f(q_j (q_j^m z + (1-q_j^m) z_j) + (1-q_j) z}{(1-q_j)(q_j^m z + (1-q_j^m) z_j - z_j)},$$

$$=(z-z_j)\sum_{m=0}^{\infty} q_j^m \frac{f(q_j^m z+(1-q_j^m)z_j)-f((q_j^{m+1}z+(1-q_j^{m+1})z_j)}{q_j^m(z-z_j)},$$

$$=\sum_{m=0}^{\infty} f(q_j^m z+(1-q_j^m)z_j)-f((q_j^{m+1}z+(1-q_j^{m+1})z_j),$$
$$=f(z).$$

(3) Second part of this theorem leads to

$$\int_c^z D_{q_j} f(l)\, d_{q_j} l=\int_{z_j}^z D_{q_j} f(l)\, d_{q_j} l-\int_{z_j}^c D_{q_j} f(l)\, d_{q_j} l=f(z)-f(c).$$

Theorem 3.3 [12]

Let $f, h: K_j \to R$ are continuous function $\gamma \epsilon R$.Then for $z \in K_j$

(1) $\int_{z_j}^z [f(l)+h(l)]\, d_{q_j} l=\int_{z_j}^z f(l)\, d_{q_j} l+\int_{z_j}^z h(l)\, d_{q_j} l$

(2) $\int_{z_j}^z (\alpha f)(l)\, d_{q_j} l=\alpha\int_{z_j}^z f(l)\, d_{q_j} l$

(3) $\int_{z_j}^z f(l) D_{q_j} h(l)\, d_{q_j} l=(fh)(z)-\int_{z_j}^z h(q_j l+(1-q_j)z_j) D_{q_j} f(l)\, d_{q_j} l$

Proof.

By Theorem 3.1 part (3), we get $f(z) D_{q_j} h(z)=D_{q_j}(fh)(z)-h(q_j z+(1-q_j)z_j)\, D_{q_j} f(z)$.

q_j −integrating the above equation and using second part of theorem 3.2 we obtain the outcome in (3) as desired.

Theorem 3.4 [12] (reversing the order of q_j −integration)

Take$f \in C(K_j, R)$, then

$$\int_{z_j}^z \int_{z_j}^l f(v)\, d_{q_j} v\, d_{q_j} l=\int_{z_j}^z \int_{q_j v+(1-q_j)z_j}^z f(v)\, d_{q_j} l\, d_{q_j} v.$$

Proof.

Using definition 3.3 we get

$$\int_{z_j}^{z}\int_{z_j}^{l} f(v)\,d_{q_j}v\,d_{q_j}l = \int_{z_j}^{z}(1-q_j)(l-z_j)\sum_{m=0}^{\infty}[q_j^m f(q_j^m l+(1-q_j^m)z_j)]d_{q_j}l$$

$$=(1-q_j)\sum_{m=0}^{\infty}q_j^m\left[\int_{z_j}^{z}(l-z_j)f(q_j^m l+(1-q_j^m)z_j)d_{q_j}l\right]$$

$$=(1-q_j)\sum_{m=0}^{\infty}\int_{z_j}^{z}[(q_j^m l+(1-q_j^m)z_j)f(q_j^m l+(1-q_j^m)z_j)-z_j f(q_j^m l+(1-q_j^m)z_j)d_{q_j}l]d_{q_j}l$$

Since

$$\int_{z_j}^{zq_j^m+(1-q_j^m)z_j} f(y)\,dy = (1-q_j)q_j^m(z-z_j)\sum_{n=0}^{\infty}q_j^n f\left(zq_j^{n+m}+\left(1-q_j^{n+m}\right)z_j\right)$$

$$\int_{z_j}^{z}\int_{z_j}^{l} f(v)\,d_{q_j}v\,d_{q_j}l$$

$$=(1-q_j)^2(z-z_j)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_j^n f\left(q_j^{m+n}z+\left(1-q^{m+n}\right)z_j\right)\times\left[q^{m+n}z+\left(1-q_j^{m+n}\right)z_j-z_j\right]$$

$$=(1-q_j)^2(z-z_j)^2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_j^{m+2n}f\left(q_j^{m+n}z+\left(1-q^{m+n}\right)z_j\right).$$

Now,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_j^{m+2n}f(q_j^{m+n}z+(1-q_j^{m+n})z_j)=\sum_{m=0}^{\infty}[q_j^m f(q_j^m z+(1-q_j^m)z_j)$$
$$+q_j^{m+2}f(q_j^{m+1}z+(1-q_j^{m+1})z_j+q_j^{m+4}f(q_j^{m+2}z+(1-q_j^{m+2})z_j+q_j^{m+6}f(q_j^{m+3}z+(1-q_j^{m+3})z_j+\ldots]$$

$$=f(z)+q_j^2 f(q_j z+(1-q_j)z_j)+q_j^4 f(q_j^2 z+(1-q_j^2)z_j)+\ldots$$
$$+q_j f(q_j z+(1-q_j)z_j)+q_j^3 f(q_j^2 z+(1-q_j^2)z_j)+q_j^5 f(q_j^3 z+(1-q_j^3)z_j)+\ldots$$
$$+q_j^2 f(q_j^2 z+(1-q_j^2)z_j)+q_j^4 f(q_j^3 z+(1-q_j^3)z_j+q_j^6 f(q_j^4 z+(1-q_j^4)+\ldots$$

$$=f(z)+q_j(1+q_j)f(q_j z+(1-q_j)z_j)+q_j^2(1+q_j+q_j^2)f(q_j^2 z+(1-q_j^2)z_j)+\ldots$$
$$=\sum_{m=0}^{\infty}q_j^m\left(\frac{1-q_j^{m+1}}{1-q_j}\right)f(q_j^m z+(1-q_j^m)z_j).$$

$$\int_{z_j}^{z}\int_{z_j}^{l} f(v)\, d_{q_j}v\, d_{q_j}l = (1-q_j)(z-z_j)^2\sum_{m=0}^{\infty} q_j^m(1-q_j^{m+1}) f(q_j^m z + (1-q_j^m)z_j)$$

$$=(1-q_j)(z-z_j)\sum_{m=0}^{\infty} q_j^m(1-q_j^{m+1})(z-z_j) f(q_j^m z + (1-q_j^m)z_j)$$

$$=\int_{z_j}^{z}\int_{z_j}^{z}(z-qv-(1-qz_j) f(v)\, d_{q_j}v$$

$$=\int_{z_j}^{z}\int_{q_j v+(1-q_j)z_j}^{z} f(v) d_{q_j} l d_{q_j}.$$

This completes the proof.

Chapter 4

IMPULSIVE q_j-DIFFERENCE EQUATIONS

Let $k=[0,z]$, $K_0=[z_0, z_1]$, $K_j=(z_j, z_{j+1}]$ for $J=1,2,...,n$. Let continuous vector space VC defined $V:K \to R=\{t:K \to R:t(z)\}$ be continuous everywhere except z_j at which $t(z_J^+)$ and $t(z_J^-)$ exist and $t(z_j^-) = t(z_J\ )$, $j = 1,2, \dots, n$. $VC(K,R)$ is a Banach space with norms $\|t\|_{VC} = \sup\{|t(z)|; x \in K\}$.

4.1 First Order Impulsive q_j-difference Equation

In this section, we discuss the existence and uniqueness of solutions for the following initial value problem for the first impulsive q_j −difference equation.

$$\begin{aligned} & D_{q_j} t(z) = f\left(z, t(z)\right), \quad z \in K, z \neq z_j, \\ & \Delta t(z_j) = I_j\left(t(z_j)\right), \quad j = 1,2,\dots,n, \\ & t(0) = t_0, \qquad t_0 \in R\ ,\ 0 = z_0 < z_1 < z_2 < \cdots < z_j < \cdots < z_n < z_{n+1} = Z \end{aligned} \tag{4.1}$$

$f:\ K \times R \to R$ is a continuous function

$I_j \in S(R,R), \Delta t(z_j) = t(z_j^+) - t(z_j),\ j = 1,2,\dots,n$ and $0 < q_j < 1$ for $j = 0,1,2,...n$

Lemma 4.1

If $t \in VC(K,R)$ is a solution of (4.1), then for any $x \in K_j, j = 0,1,2,\dots,n$

$$t(z) = t_0 + \sum_{0<z<z_j} \int_{z_{j-1}}^{z_j} f(l, z(l)) d_{q_{j-1}} l + \sum_{0<z_j<z} I_j(t(z_j)) + \int_{z_j}^{z} f(l, z(l)) d_{q_j} l \tag{4.2}$$

with $\sum_{0<0}(.) = 0$, a solution of (4.1).

Proof.

For $z \in K_0, q_0-$integrating (4.1), it follows

$$t(z) = t_0 + \int_0^z f\left(l, t(l)\right) d_{q_0} l,$$

which leads to $t(z_1) = t_0 + \int_0^{z_1} f\left(l, t(l)\right) d_{q_0} l.$

For $z \in K_1$, taking q_1-integral of (4.1), it becomes $t(z) = t\left(z_1^+\right) + \int_{z_1}^{z} f\left(l, t(l)\right) d_{q_1} l.$

Since $t(z_1^+) = t(z_1) + I_1(t(z_1))$, then we have

$$t(z) = t_0 + \int_0^{z_1} f\left(l, t(l)\right) d_{q_0} l + \int_{z_1}^{z} f\left(l, t(l)\right) d_{q_1} l + I_1\left(t(z_1)\right).$$

Again q_2-integrating of (4.1) from z_2 to z, where $z \in K_2$, then

$$t(z)=t\left(z_2^+\right)+\int_{z_2}^{z} f\left(l, t(l)\right) d_{q_2} l$$

$$=t_0+\int_0^{z_1} f\left(l,t(l)\right) d_{q_0} l+\int_{z_1}^{z_2} f\left(l,t(l)\right) d_{q_1} l+\int_{z_2}^{z} f\left(l,t(l)\right) d_{q_2} l++I_1\left(t(z_1)\right)+I_2\left(t(z_2)\right).$$

Repeating the process considered above for $z \in K$, we get (4.2).

Now, let $z(t)$ be a solution of (4.1). Using q_j $-$derivative of (4.2) for $\in K_j$,

$j = 0,1,2, \dots, n$ it becomes the following:

$$D_{q_j} t(z) = f\left(z, t(z)\right).$$

It is clear that

$\Delta t(z_j) = I_j(t(z_j)), j = 1, 2, \dots, n$ and $t(0) = t_0$. This gives the proof.

Theorem 4.1 [12]

Let the following hold.

(H_1) $f: K \times R \to R$ is a continuous function and satisfies

$$\left|f(z,t) - f(z,u)\right| \le S\,|t-u|, \quad S > 0, \forall z \in K, t, u \in R;$$

(H_2) $I_j : R \to R$ $j = 1, 2, \ldots, n$ are continuous function and satisfy

$$\left|I_j(t) - I_j(u)\right| \le N\,|\,t-u\,|, \quad N > 0, \forall t, u \in R.$$

If $SZ + nN \le \zeta < 1$,

Then the nonlinear impulsive q_j −difference initial value problem on (4.1) has unique solution on K.

Proof.

We define the operator $P : VC(K, R) \to VC(K, R)$ by

$$(Pt)(z) = t_0 + \sum_{0<z_j<z} \int_{z_{j-1}}^{z_j} f(l, t(l))d_{q_{j-1}} l + \sum_{0<z_j<z} I_j(t(z_j)) + \int_{z_j}^{z} f(l, t(l))d_{q_j} l.$$

Given $\sum_{0<0}(.) = 0$. Let $\sup_{z\in k} |\,f(z,0)\,| = A_1$ and $\max\{|\,I_j(0)\,| : j = 1, 2, \ldots, n\} = A_2$;

Let us take a constant W such that

$$w \ge \frac{1}{1-\varepsilon}\left[|\,t_0\,| + A_1 Z + nA_2\right], \qquad \text{where } \zeta \le \varepsilon < 1.$$

Now we will state that $PD_w \subset D_w$ where a ball $D_w = \{t \in VC(K, R) : \|t\| \le w\}$. For any $t \in D_w$ and for each $z \in K$,

$$\begin{aligned}
\left|(Pt)(z)\right| &\le \left|t_0\right| + \sum_{0<z_j<z} \int_{z_{j-1}}^{z_j} \left|f(l, t(l))\right| d_{q_{j-1}} l + \sum_{0<z_j<z} \left|I_j(t(z_j)\right| + \int_{z_j}^{z} \left|f(l, t(l))\right| d_{q_j} l \\
&\le \left|t_0\right| + \sum_{0<z_j<z} \int_{z_{j-1}}^{z_j} \left(\left|f(l, t(l) - f(l,0))\right| + \left|f(l,0)\right|\right) d_{q_{j-1}} l \\
&\quad + \sum_{0<z_j<z} \left(\left|I_j(t(z_j)) - I_j(0)\right| + \left|I_j(0)\right|\right) + \int_{z_n}^{z} \left(\left|f(l, t(l) - f(l,0))\right| + \left|f(l,0)\right|\right) d_{q_n} l
\end{aligned}$$

$$\begin{aligned}
&\le \left|t_0\right| + (Sw + A_1) \sum_{0<z_j<z} \int_{z_{j-1}}^{z_j} d_{q_{j-1}} l + \sum_{0<z_j<z} (Nw + A_2) + (Sw + A_1) \int_{z_n}^{z} d_{q_n} l \\
&\le \left|t_0\right| + (Sw + A_1)Z + n(Nw + A_1) \\
&\le (\zeta + 1 - \varepsilon)w \le w.
\end{aligned}$$

This implies that $PD_w \subset D_w$

For $t, u \in VC(K, R)$ and for each $z \in K$ we have

$$\left|(Pt)(z) - (Pu)(z)\right| \leq \sum_{0<z_j<Z} \int_{z_{j-1}}^{z_j} (\left|f(l, t(l) - f(l, u(l))\right| d_{q_{j-1}} l$$

$$+ \sum_{0<z_j<z} \left|I_j(t(z_j)) - I_j(u(z_j))\right| + \int_{z_j}^{z} \left|f(l, t(l) - f(l, u(l))\right| d_{q_j} l$$

$$\leq \sum_{0<z_j<z} \int_{z_{j-1}}^{z_j} \left(S\left|t(l) - u(l)\right|\right) d_{q_{j-1}} l + \sum_{0<z_j<Z} N\left|t(z_j) - u(z_j)\right| + \int_{z_n}^{z} (S\left|t(l) - u(l)\right|) d_{q_n} l$$

$$\leq (SZ + nN)\left\|t - u\right\|.$$

Then it becomes as follows:

$\|Pt - Pu\| \leq (SZ + nN)\|t - u\|$

As SZ + nN < 1, by Banach contraction mapping principle, P is a contraction. Thus P is a fixed point which is a unique solution of (4.1) on J.

Example 4.1

Let us solve this first order impulsive q_j −difference initial value problem.

$$D_{\frac{1}{2+j}} t(z) = \frac{e^{-z} |t(z)|}{(z + \sqrt{5})^2 (1 + |t(z)|)}, \quad z \in K = [0,1], z \neq z_j = \frac{j}{10},$$

$$\Delta t(z_j) = \frac{|t(z_j)|}{12 + |t(z_j)|}, \quad j = 1, 2, \ldots, 9, \tag{4.3}$$

$$t(0) = 0.$$

Here, $q_j = 1/(2 + j), j = 0, 1, 2, \ldots, 9, n = 9$

$Z = 1, f(z,t) = (e^{-z} |t|) / ((z + \sqrt{5})^2 (1 + |t|))$ and $I_j(t) = |t| / (12 + |t|)$.

$\text{Since} \; |f(z,t) - f(z,u)| \le (1/5)\,|t - u$ and $|I_j(t) - I_j(u)| \le (1/12)\,|t - u|$,

then (H_1), (H_2) are satisfied with

$S = (1/5), N = (1/12)$. We can show that

$$SZ + nN = \frac{1}{5} + \frac{9}{12} = \frac{19}{20} < 1.$$

Hence, by theorem (4.1), the initial value problem (4.3) has a unique solution on [0, 1].

4.2 Second Order Impulsive q_j −Difference Equations

In this section, we discuss the second order initial value problem of impulsive q_j −difference equation of the form

$$
\begin{aligned}
&D_{q_j}^2 t(z) = f\left(z, t(z)\right), \quad z \in K, z \neq z_j, \\
&\Delta t(z_j) = I_j\left(t(z_j)\right), \quad j = 1, 2, \ldots, n, \qquad (4.4) \\
&D_{q_j} t\left(z_j^+\right) - D_{q_{j-1}} t(z_j) = I_j^*\left(t(z_j)\right), \quad j = 1, 2, \ldots, n, \\
&t(0) = \gamma, \qquad D_{q_0} t(0) = \phi,
\end{aligned}
$$

Where $\gamma, \phi \in R$, $0 = z_0 < z_1 < z_2 < \cdots < z_j \cdots < z_n < z_{n+1} = Z$, $f : K \times R \to R$ is a continuous function, $I_j, I_j^* \in C(R, R)$ $\Delta t(z_j) = t(z_j^+) - t(z_j)$ for $j = 1, 2, \ldots, n$ and $0 < q_j < 1$ for j = 0,1,2,3,…,n.

Lemma 4.2

The unique solution of problem (4) is given by

$$
\begin{aligned}
t(z) = \gamma + \phi z &+ \sum_{0<z_j<z} \left(\int_{z_{j-1}}^{z_j} (z_j - q_{j-1} l - (1-q_{j-1}) z_{j-1}) f(l, t(l)) d_{q_{j-1}} l + I_j(t(z_j)) \right) \\
&+ z\left[\sum_{0<z_j<z} \int_{z_{j-1}}^{z_j} f(l, t(l)) d_{q_{j-1}} l + I_j^*(t(z_j)) \right] - \sum_{0<z_j<z} z_j \left(\int_{z_{j-1}}^{z_j} f(l, t(l)) d_{q_{j-1}} l + I_j^*(t(z_j)) \right) \\
&+ \int_{z_j}^{z} (z - q_j l - (1-q_j) z_j) f(l, t(l)) d_{q_j} l, \qquad (4.5)
\end{aligned}
$$

with $\sum_{0<0} (.) = 0.$

Proof.

For $z \in K_0$ and q_0 −integrating for the first equation of (4.4), we obtain

$$
D_{q_0} t(z) = D_{q_0} t(0) + \int_0^z f\left(l, t(l)\right) d_{q_0} l = \phi + \int_0^z f\left(l, t(l)\right) d_{q_0} l, \qquad (4.6)
$$

which gives us

$$D_{q_0} t(z_1) = \phi + \int_0^{z_1} f\left(l, t(l)\right) d_{q_0} l. \tag{4.7}$$

For $z \in K_0$. By q_0 $-$integrating (6) it becomes

$$t(z) = \gamma + \phi z + \int_0^z \int_0^l f\left(\varphi, z(\varphi)\right) d_{q_0} \varphi \, d_{q_0} l,$$

reversing the order of the q_0 $-$integral, we get

$$t(z) = \gamma + \phi z + \int_0^z (z - q_0 l) f\left(l, t(l)\right) d_{q_0} l. \tag{4.8}$$

In special case for z = z_1

$$t(z_1) = \gamma + \phi z_1 + \int_0^{z_1} (z_1 - q_0 l) f\left(l, t(l)\right) d_{q_0} l. \tag{4.9}$$

For $z \in K_1 = (z_1, z_2], q_1 -$integrating (4.4) it becomes

$$D_{q_1} t(z) = D_{q_1} t\left(z_1^+\right) + \int_{z_1}^z f\left(l, t(l)\right) d_{q_1} l.$$

Using the third condition of (4.4) with (4.7) the result becomes

$$D_{q_1} t(z) = \phi + \int_0^{z_1} f\left(l, t(l)\right) d_{q_0} l + I_1^*\left(t(z_1)\right) + \int_{z_1}^z f\left(l, t(l)\right) d_{q_1} l. \tag{4.10}$$

For $x \in K_1$, q_1-integrating (4.10) then converting order of $q_1 -$integral we get

$$t(z) = t\left(z_1^+\right) + \left[\phi + \int_0^{z_1} f\left(l, t(l)\right) d_{q_0} l + I_1^*\left(t(z_1)\right)\right](z - z_1)$$
$$+ \int_{z_1}^{z} \left(z - q_1 l - (1 - q_1) z_1\right) f\left(l, t(l)\right) d_{q_1} l. \tag{4.11}$$

Using second equation of (4.4) with (4.9) and (4.11), we obtain

$$t(z) = \gamma + \phi z_1 + \int_0^{z_1} (z_1 - q_0 l) f\left(l, t(l)\right) d_{q_0} l + I_1\left(t(z_1)\right)$$
$$+\left[\phi + \int_0^{z_1} f\left(l, t(l)\right) d_{q_1} l + I_1^*\left(t(z_1)\right)\right](z - z_1)$$
$$+\int_{z_1}^{z} \left(z - q_1 l - (1-q_1) z_1\right) f\left(l, t(l)\right) d_{q_1} l.$$

$$= \gamma + \phi z_1 + \int_0^{z_1} (z_1 - q_0 l) f\left(l, t(l)\right) d_{q_0} l + I_1\left(t(z_1)\right)$$
$$+\left[\int_0^{z_1} f\left(l, t(l)\right) d_{q_1} l + I_1^*\left(t(z_1)\right)\right](z - z_1) + \int_{z_1}^{z} \left(z - q_1 l - (1-q_1) z_1\right) f\left(l, t(l)\right) d_{q_1} l.$$

Repeating the above steps, for $x \in K$, we get (4.5) as needed. Now, it is useful to prove the existence and uniqueness of a solution to the initial value problem (4.4). We will apply Banach fixed point theorem to do this.

Theorem 4.2 [12]

Let the assumptions (H_1) and (H_2) hold. Furthermore, assume that $(H_3)\ I_j^* : R \to R$, $j = 1, 2, \ldots, n$ are continuous functions and hold

$$\left|I_j^*(t) - I_j^*(u)\right| \le N^* \, |\, t - u \,|, \quad N^* > 0, \forall t, u \in R.$$

if $\sigma := S(e_1 + Ze_2 + e_3) + nN + (nZ + e_4) N^* \le \zeta < 1$,

where

$$e_1 = \sum_{j=1}^{n+1} \frac{(z_j - z_{j-1})^2}{1 + q_{j-1}}, \qquad e_2 = \sum_{j=1}^{n} (z_j - z_{j-1}),$$
$$e_3 = \sum_{j=1}^{n} z_j (z_j - z_{j-1}), \qquad e_4 = \sum_{j=1}^{n} z_j,$$

then the initial value problem (4.4) has a unique solution on J.

Proof.

First in light of Lemma 2, we establish $F : VC(K,R) \to VC(K,R)$ as

$$(Ft)(z) = \gamma + \phi z + \sum_{0<z_j<z} \left(\int_{z_{j-1}}^{z_j} (z_j - q_{j-1}l - (1-q_{j-1})z_{j-1}) f(l,t(l)) d_{q_{j-1}} l + I_j(t(z_j)) \right)$$
$$+ z \left[\sum_{0<z_j<z} \int_{x_{j-1}}^{z_j} f(l,t(l)) d_{q_{j-1}} l + I_j^*(t(z_j)) \right]$$
$$- \sum_{0<z_j<z} z_j \left(\int_{z_{j-1}}^{z_j} f(l,t(l)) d_{q_{j-1}} l + I_j^*(t(z_j)) \right)$$
$$+ \int_{z_j}^{z} (z - q_j l - (1-q_j) z_j) f(l,t(l)) d_{q_j} l,$$

Given $\sum_{0<0} (.) = 0.$

Let $\sup_{z\in K} |f(z,0)| = \omega_1, \max\{I_j(0) : j = 1,2,\ldots,n\} = \omega_2$, and $\max\{I_j^*(0) : j = 1,2,\ldots,n\} = \omega_3.$

We will verify that $F D_W \subset D_W$, where $D_W = \{t \in VC(J,\mathbb{R}) : \|t\| \le W\}$ and constant W

satisfies $$W \ge \frac{|\gamma| + |\phi| Z + \omega_1(e_1 + Ze_2 + e_3) + n\omega_2 + (nZ + e_4)\omega_3}{1-\varepsilon},$$

where $\zeta \le \varepsilon < 1.$ For $t \in D_W$, from example 3.3, we have

$$|(Ft)(z)| \le |\gamma| + |\phi| z + \sum_{0<z_j<z} \left(\int_{z_{j-1}}^{z_j} (z_j - q_{j-1}l - (1-q_{j-1})z_{j-1}) |f(l,t(l))| d_{q_{j-1}} l + |I_j(t(z_j))| \right)$$
$$+ z \left[\sum_{0<z_j<z} \int_{z_{j-1}}^{z_j} f(l,t(l)) d_{q_{j-1}} l + |I_j^*(t(z_j)|) \right]$$
$$+ \sum_{0<z_j<z} z_j \left(\int_{z_{j-1}}^{z_j} |f(l,t(l))| d_{q_{j-1}} l + |I_j^*(t(z_j))| \right)$$
$$+ \int_{z_j}^{z} (z - q_j l - (1-q_j) z_j) |f(l,t(l))| d_{q_j} l,$$

$$\leq |\gamma|+|\phi|z$$
$$+\sum_{0<z_j<Z}\left(\int_{z_{j-1}}^{z_j}(z_j-q_{j-1}l-(1-q_{j-1})z_{j-1})(|f(l,t(l))-f(l,0)|+|f(l,0)| \quad d_{q_{j-1}}l \atop +|I_j(t(z_j))-I_j(0)|+|I_j(0)|)\right)$$

$$+Z\left[\sum_{0<z_j<z}\int_{z_{j-1}}^{z_j}(|f(l,t(l))-f(l,0)|+|f(l,0)|)d_{q_{j-1}}l+(|I_j^*(t(z_j)-I_j^*(0)|+|I_j^*(0)|))\right]$$
$$+\sum_{0<z_j<Z}z_j\left(\int_{z_{j-1}}^{z_j}|f(l,t(l))-f(l,0)|+|f(l,0)|)d_{q_{j-1}}l+(|I_j^*(t(z_j))-I_j^*(0))|\right)$$
$$+\int_{z_n}^{Z}(Z-q_nl-(1-q_n)t_n)|f(l,t(l))-f(l,0)|+|f(l,0)|d_{q_n}l$$

$$\leq |\gamma|+|\phi|z$$
$$+\sum_{0<z_j<Z}\left(\int_{z_{j-1}}^{z_j}(z_j-q_{j-1}l-(1-q_{j-1})z_{j-1})(|f(l,t(l))-f(l,0)|+|f(l,0)| \quad d_{q_{j-1}}l \atop +|I_j(t(z_j))-I_j(0)|+|I_j(0)|)\right)$$
$$+Z\left[\sum_{0<z_j<z}\int_{z_{j-1}}^{z_j}(|f(l,t(l))-f(l,0)|+|f(l,0)|)d_{q_{j-1}}l+(|I_j^*(t(z_j)-I_j^*(0)|+|I_j^*(0)|))\right]$$
$$+\sum_{0<z_j<Z}z_j\left(\int_{z_{j-1}}^{z_j}|f(l,t(l))-f(l,0)|+|f(l,0)|)d_{q_{j-1}}l+(|I_j^*(t(z_j))-I_j^*(0))|\right)$$
$$+\int_{z_n}^{Z}(Z-q_nl-(1-q_n)t_n)|f(l,t(l))-f(l,0)|+|f(l,0)|d_{q_n}l$$

$$\leq |\gamma|+|\phi|Z+\sum_{j=1}^{n}\left(\frac{(z_j-z_{j-1})^2(S+\omega_1)}{(1+q_{j-1})}+(NW+\omega_2)\right)$$
$$+Z\left[\sum_{j=1}^{n}((SW+\omega_1)(z_j-z_{j-1})+(N^*W+\omega_3))\right]$$
$$+\sum_{j=1}^{n}z_j((SW+\omega_1)(z_j-z_{j-1})+(N^*W+\omega_3))+\frac{(SW+_1)(Z-z_n)^2}{1+q_n}$$
$$=|\gamma|+|\phi|Z+(SW+\omega_1)(e_1+Ze_2+e_3)+(N\ \ w+\omega_3)(nZ+e_4)+n(NW+\omega_2)$$
$$\leq(\zeta+1-\varepsilon)W\leq W.$$

Then we obtain $FD_W\subset D_W$.

For any $t, u \in VC(K,R)$, we have

$$
\begin{aligned}
&\left|(Ft)(z)-(Fu)(z)\right| \\
&\leq \sum_{j=1}^{n}\left(\int_{z_{j-1}}^{z_j}(z_j-q_{j-1}l-(1-q_{j-1})z_{j-1})\left|f(l,t(l))-f(l,u(l))\right|d_{q_{j-1}}l+\left|I_j(t(z_j))-I_j(u(z_j))\right|\right) \\
&\quad +Z\left[\sum_{j=1}^{n}\left(\int_{z_{j-1}}^{z_j}\left|f(l,t(l))-f(l,u(l))\right|d_{q_{j-1}}l+\left|I_j^*(t(z_j)-I_j^*(u(t_j))\right|\right)\right] \\
&\quad +\sum_{j=1}^{n}z_j\left(\int_{z_{j-1}}^{z_j}\left|f(l,t(l))-f(l,u(l))\right|d_{q_{j-1}}l+\left|I_j^*(t(z_j))-I_j^*(u(z_j))\right|\right) \\
&\quad +\int_{z_n}^{Z}(z-q_nl-(1-q_n)z_n)\left|f(l,t(l))-f(l,u(l))\right|d_{q_n}l,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{n}\left(\frac{(z_j-z_{j-1})^2}{(1+q_{j-1})}S+N\right)\|t-u\|+Z\left[\sum_{j=1}^{n}(S(z_j-z_{j-1})+N*)\right]\|t-u\| \\
&\qquad +\sum_{j=1}^{n}(S(z_j-z_{j-1})+N*)\|t-u\|+S\frac{(Z-z_n)^2}{1+q_n}\|t-u\| \\
&=\sigma\|t-u\|,
\end{aligned}
$$

which lead to $\|Ft-Fu\| \leq \sigma\|t-y\|$. Since $\sigma < 1$, by the Banach contraction mapping principle F has a fixed point, which is a unique solution of (4.4) on K.

Example 4.2

Consider this second order impulsive q_j-difference initial value problem:

$$
\begin{aligned}
&D^2_{\frac{2}{3+j}}t(z)=\frac{e^{-\sin^2 z}\left|t(z)\right|}{(7+z)^2(1+\left|t(z)\right|}, \; z\in K=[0,1], z\neq z_j=\frac{j}{10} \\
&\Delta t(z_j)=\frac{\left|t(z_j)\right|}{5(6+\left|t(z_j)\right|}, \; j=1,2,\ldots,9 \qquad (4.12) \\
&D_{\frac{2}{3+j}}t(z_j^+)-D_{\frac{2}{3+j-1}}t(z_j)=\frac{1}{9}\tan^{-1}\left(\frac{1}{5}t(z_j)\right), \; j=1,2,\ldots,9 \\
&t(0)=0, \qquad D_{\frac{2}{3}}t(0)=0
\end{aligned}
$$

Here,

$q_j = 2/(3+j), j = 0,1,2,\ldots,9, n = 9, Z = 1,$

$f(z,t) = (e^{-\sin^2 z}|t|)/((7+z)^2(1+|t|))$

$I_j(t) = |t|/(5(6+|t|)),$ and $I_j^*(t) = (1/9)\tan^{-1}(z/5).$

Since $|f(z,t) - f(z,u)| \le (1/49)|t-u|,$

$|I_j(t) - I_j(u)| \le (1/30)|t-u|,$ and $|I_j^*(t) - I_j^*(u)| \le (1/45)|t-u|.$

Then (H_1), (H_2) and (H_3) holding with

$S = (1/49), N = (1/30), N^* = (1/45).$

The results becomes the following:

$$e_1 = \sum_{j=1}^{n+1} \frac{(z_j - z_{j-1})^2}{1+q_{j-1}} = \frac{1,380,817}{180,180}, \qquad e_2 = \sum_{j=1}^{n} (z_j - z_{j-1}) = \frac{9}{10},$$

$$e_3 = \sum_{j=1}^{n} z_j (z_j - z_{j-1}) = \frac{45}{100}, \qquad e_4 = \sum_{j=1}^{n} z_j = \frac{45}{10}.$$

Clearly, $Z(e_1 + Ze_2 + e_3) + nN + (nZ + e_4)N^* = 0.7839 < 1.$

Hence, by theorem 4.2, the initial value problem (4.12) has a unique solution on [0, 1].

Chapter 5

CONCLUSION

As we say earlier in the introduction, q_j-calculus plays an important role in getting solutions to the systems with impulses at fixed times.

The solution we obtained from the examples in chapter 4 shows how q_j-calculus is beneficial to the application related to impulses, which is important for finding solution for some problems in life such as shocks, harvesting, natural disasters, and population dynamics.

Chapter 6

FURTHER STUDIES

I would like to suggest a further and broader study and research in q_j-calculus in order to get a broader area for research and application. For example, studying exponential, trigonometrical areas, fundamental theorem, and mean value theorem of q_j--calculus.

I would also like to suggest a deep study and research in impulse problems in order to get a solution to some practical applications.

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