# Riemann-Liouville Type Fractional Differential Equations

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# ABSTRACT

This work is dedicated to investigate the existence and uniqueness of solutions for nonlinear fractional differential equations with boundary conditions involving the Riemann-Liouville fractional derivative. After introducing some basic preliminaries and the important concepts of fractional calculus, we considered the model of boundary value problems of Riemann-Liouville fractional derivative. The existence and uniqueness of solution are obtained via Banach'fixed point theorem and Schauder'fixed point theorem for the two models. In addition, both results are provided by the illustrative examples to support them.

**Keywords**: Fractional integrals and derivatives, Fractional differential equations, Existence, Uniqueness, Fixed point theorems.

# ÖZ

Bu çalışma Caputo kesirli türevi içeren sınır koşulları ile doğrusal olmayan fraksiyonel diferansiyel denklemlerin çözümleri varlığını ve tekliğini araştırmak için adamıştır.bazı temel öncüller ve Kesirli analizin önemli kavramları tanıttıktan sonra biz Caputo kesirli türevi sınır değer problemlerinin iki model düşündü. İlki yerel olmayan dört nokta fraksiyonel sınır koşulları ile doğrusal olmayan fraksiyonel diferansiyel denklemdir.İkinci denklem kesirli yerel olmayan dört nokta fraksiyonel sınır koşulları ile desteklenmiş çoklu siparişlerin doğrusal olmayan dürtüsel sınır değer problemidir.çözümün varlığı ve tekliği iki model için Banach'fixed nokta teoremi ve Schauder'fixed nokta teoremi ile elde edilir. Buna ek olarak, her iki sonuç da, onları desteklemek için açıklayıcı örnekler tarafından sağlanmaktadır.

Anahtar Kelimeler: Fraksiyonel integraller ve türevler, Fraksiyonel diferansiyel denklemler, Varlık, Teklik, Sabit nokta teoremleri, Impulse.

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# DEDICATION

**To My Family**

# ACKNOWLEDGMENT

The last words I would like to write on EMU paper for my graduation would not have any value if I do not thank God and then my beloved father who passed away in 2005, but still helped to build the person who I am today. I would also like to express my heartfelt thanks to all my family members, and I will not forget to appreciate my supervisor Prof. Dr. Nazim Mahmudov for this support and patient toward my research who works tearlessly to make this thesis to its completion, all my supportive instructors, friends and everyone who encouraged my success. It is indeed sad to leave EMU, However, I am grateful for all the experience I lived. I had the chance to study in a multicultural atmosphere where both my knowledge, skills and social life increased. All I get to say is "Thank you EMU" and "Thank you Cyprus"

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# LIST OF ABBREVIATION

| | |
|---|---|
| R-L | Riemann-Liouville |
| RLFDE | Riemann-Liouville Fractional Differential Equations |
| FP | Fixed Point |
| BFP | Banach's Fixed Point |
| FDE | Fractional Differential Equations |
| BVP | Boundary Value Problems |
| A-A | Arzela-Ascoli |
| HFD | Hadamard Fractional Derivative |
| HFI | Hadamard Fractional Integral |
| KFP | Krasnoselskii's Fixed Point |
| FICs | Fractional Integral Conditions |
| RHS | Right Hand Side |

# Chapter 1

# INTRODUCTION

In this Chapter we want to provide a concise history of fractional calculus. The theory of fractional calculus emanated from the origin of classical calculus itself. Historically, classical calculus was developed by Isaac Newton and Gottfried Wihelm Leibniz in the 17th century and the latter (Leibniz) first brought out the conception of a symbolic method, more precisely his notation.

$\frac{d^n y}{dx^n} = D^{\,n} y,$

for the nth derivative of function $y(x)$, where *n* is a non-negative integer.

In [1] L'Hospital had written a letter to Leibniz in 1695, and asked about the likelihood of *n* being a fraction "What does $(\frac{d^n f(x)}{dX^{\,n}})$ mean if $n$=$\frac{1}{2}$?". Leibniz ascertains that "It will lead a paradox". But predictably "from this apparent paradox, some day it would lead to useful consequences"[1]. In view of the increasing interest in the development of fractional calculus by means of many mathematicians, it can be extended to the *n*th derivative of $\mathrm{D^n y}$ to any number , where *n* may be rational , irrational or complex number.

Many other mathematicians such as Euler, Laplace, and Fourier have investigated fractional calculus in order to answer L'Hospital's question. Each of them had unique notations and methodology and also proposed many divergent concepts of non-integer order integral or derivative. The first discussion of a derivative of fractional order in calculus was written by Lacroix in 1819 [2]. Lacroix expressed the precise formula for the nth derivative which is defined by

$$D^n x^m = \frac{m!}{(m-n)!} x^{m-n}, \text{ where } n(\leq m) \text{is integer,} \qquad (1.1)$$

he replaced the discrete factorial function with Euler's continuous Gamma function and obtained the following formula

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \qquad (1.2)$$

where α and β are fractional numbers

In particular, he computed

$$D^{\frac{1}{2}} x = \frac{\Gamma(2)}{\Gamma(3/2)} x^{\frac{1}{2}} = 2\frac{x}{\pi}. \qquad (1.3)$$

The first application of fractional calculus was made by Niels Henrik Abel in[3] at the beginning of the nineteenth century. He used mathematical tool to solve an integral equation which arose from the tautochrone problem. This problem simply deals with the determination of curve on the (x, y) plane through the origin in vertical plane such that the required time for a particle with a total mass (m) will be released at a time which is absolutely independent of the origin.

In this situation the physical law states that “the potential energy lost during the descent of the particle is equal to the kinetic energy the particle gains”:

$$\frac{1}{2} m \left(\frac{ds}{dt}\right)^2 = mg\left(y_{0-y}\right), \qquad (1.4)$$

where (m) is defined as the mass of the particle, s is the distance of the particle from origin along the curve and g implies acceleration due to gravity. The formula above can be solved by separating the variables which yields

$$\frac{-ds}{\sqrt{y_{0-y}}} = \sqrt{2g}\,dt$$

and integration from when time $t = 0$ to $t = T$

$$\sqrt{2g}T = \int_0^{y_0} (y_0 - y)^{-\frac{1}{2}} ds \qquad (1.5)$$

Assuming that the time a particle needs to reach the lowest point of the curve is constant. So the left hand side must be a constant, say k. If we denoted the path length s as a function of height $s = F(y), then, \frac{ds}{dy} = F'(y)$.

By changing the variables $y_0$with x and y with t and putting $F' = f$ the tautochrone integral equation becomes

$$k = \int_0^x (x - t)^{-\frac{1}{2}} f(t) dt, \qquad (1.6)$$

Where f is the function to be determined.

After multiplying both sides of the integral equation with $\frac{1}{\Gamma(\frac{1}{2})}$, Abel got on the right hand side a fractional integral of order $\frac{1}{2}$

$$\frac{k}{\Gamma(\frac{1}{2})} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x - t)^{-\frac{1}{2}} f(t) dt = \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} f(x) \qquad (1.7)$$

Or , equivalently,

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \frac{k}{\Gamma(\frac{1}{2})} = \frac{d^{1/2}}{dx^{1/2}} \frac{d^{-1/2}}{dx^{-1/2}} f(x) = \frac{d^0}{dx^0} f(x) = f(x). \qquad (1.8)$$

So , we have the tautochrone solution given as follows

$$f(x) = \frac{1}{\Gamma(\frac{1}{2})} \frac{d^{1/2}}{dx^{1/2}} K = \frac{K}{\pi\sqrt{x}} \ , \qquad (1.9)$$

where the Abel problem has a solution which ıs subjected to the condition that derivative constant k is not zero always.

Here, It is necessary to note that Abel not only give a solution to the tautochrone problem, but also gave the solution for more general integral equation

$$f(x) = \int_0^x \frac{f(t)}{(x-t)^\alpha} dt, x > a, 0 < \alpha < 1. \quad (1.10)$$

After Abel application of fractional operators to a problem in physics, the first series of papers were stated by Liouville (see e.g. [1-3]). Liouville extended the known integer order derivatives $D^n e^{ax} = a^n e^{ax}$ to a derivative of arbitrary order α (formally replacing n∈N with α∈ℂ ) as follows:

$$D^\alpha e^{ax} = a^\alpha e^{ax} \quad (1.11)$$

Liouville developed two definitions for fractional derivatives. The first definition of a derivative of arbitrary order α for certain class of functions involved an infinite series. Here the series must be convergent for some α. Based on the Gamma function, Loiuville formulated the second definition as follows:

$$\Gamma(\beta)x^{-\beta} = \int_0^\infty t^{\beta-1} e^{-xt} dt, \quad \beta > 0. \quad (1.12)$$

$$D^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}, \ \beta > 0. \quad (1.13)$$

This definition is useful only for rational function.

Another scholar who had contributed to the fractional calculus is Riemann[1]. Riemann developed the definition for fractional integral of order α of a given function *f(x)*. The most important definition which is known as Riemann-Liouville fractional integral and formulated as follows:

$$_c D_\alpha^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt \ , Re(\alpha) > 0. \quad (1.14)$$

When $c = 0$, expression (1.14) is the definition of Riemann integral, and when

$c = -\infty$, expression (1.14) represents the Liouville definition. In this regard, it can be shown that

$$ {}_cD_x^{\alpha} f(x) = {}_cD_x^{n-\beta} f(x) = {}_cD_x^{n}\, {}_cD_x^{-\beta} f(x) $$

$$ = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\beta)}\int_c^x (x-t)^{\beta-1} f(t)dt\right) \tag{1.15} $$

holds, which is known today as the Riemann-Liouville fractional derivative, where $n$=$[Re(\alpha)] + 1$ and 0$< \beta = n - \alpha < 1$ .

On the other hand, Grünwald [4] and Letnikov [5] generated the concept of fractional derivative which is the limit of a sum given by

$$ {}^{GL}D_{d+}^{\alpha} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(x-kh)\, , \alpha > 0, \tag{1.16} $$

where$\binom{\alpha}{k}$is the generalized binomial coefficient . At this point in time, it is enough for mentioning the historical development of fractional calculus.

In the twentieth century, the generalization of fractional calculus has been subjected of several approaches. That is why there are various definitions that are proved equivalent, and their use is encouraged by researchers in different scientific fields. Although a great number of results of fractional calculus were presented in this century but the most interesting one was introduced by M.Caputo in [6] and was used extensively. Caputo defined a fractional derivative by

$$ {}^{c}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n f(s)ds \tag{1.17} $$

Where f a function with an (n−1) absolutely continuous derivative and n=[α]+1 . Nowadays, expression (1.17) named Caputo fractional derivative. This derivative (1.17) is strongly connected with Riemann-Liouville fractional derivative and is frequently used in fractional differential equations with initial conditions $X^{(k)}(0) = b_k, k = 0,\ 1, \dots, n - 1$.

Fractional calculus has grown and come to light in the late twentieth century. In 1974, the commencing conference related with the application and theory of fractional calculus was successfully showcased in the New Haven [7]. And a number of books on fractional calculus have appeared in the same year .Finally in 2004 the huge conference on fractional differentiation and its application was held in Bordeaux.

From its birth (simple question from L'Hospital to Leibniz) to its today's wide use in numerous scientific areas fractional calculus has come a long way. Although it's as old as integer calculus, it has still proved good applicability on models describing complex real life problems.

After a review of the historical development of the fractional calculus this work will give a brief investigation to its main goal and form a cornerstone in the application that arise in engineering and other sciences. It is fractional differential equation which has played an important role in mathematical modeling of different specialization such as physics, bio-chemistry, economics, and engineering etc. We will be interested in the boundary conditions of fractional differential equation which involves Caputo derivative.

Recently, problems with boundary value for non-linear FDEs draw many researchers attention. For instance Ahmad,B. et al [8], investigated non-linear

FDEs with fractional separated boundary conditions. Also in [9] , Ahmad,B. and Sivasundaram,S. studied the existence of solutions for impulsive integral boundary condition of non-linear fractional differential condition. By following this technique, I do consider two types of non-linear FDEs which are not the same with boundary value problems.

The first one is concerned with FDEs with four points non-local fractional boundary condition; the second is associated with non-linear impulsive fractional differential equation with four points non-local boundary condition. In each of these we will obtain the existence solutions by means of fixed point theorems. Both results will be illustrated by examples.

# Chapter 2

# RIEMANN-LIOUVILLE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITHFRACTIONAL NONLOCAL INTEGRAL BOUNDARY CONDITIONS

## 2.1 Introduction

Differential equations are an important part of mathematical sciences. The applications in many other sciences such as Physics, chemistry, etc. the theoretically deep studies on differential equation are brought the mathematicians and researchers to the idea of evaluating various types of differential equations that might exist. In this work we study nonlinear fractional integro-differential equation defined by (1.1). Our major interest is to investigate existence and uniqueness of the solution of the problem (1.1).

Consider the following equation.

$$D^{\alpha}w(k)=\varphi\big(k,w(k),(\delta w)(k),(\mathrm{O}w)(k)\big),\ \ k\in[0,K],\ \ \alpha\in(1,2], \tag{1.1}$$

the equation (1.1) is subject to a fractional boundary condition defined as follows:

$$D^{\alpha-1}w(0^{+})=0, \tag{1.2}$$

$$D^{\alpha-2}w(0^{+})=\nu I^{\alpha-1}w(\eta),\ 0<\eta<K,\ \text{ the value } \nu \text{ is constant}, \tag{1.3}$$

with $D^{\alpha}$ being the R-L Fractional derivative that has the order $\alpha$, $\varphi$: $[0,K]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is continuous, and

$$(\delta x)(k)=\int_0^k \chi(k,\tau)x(\tau)d\tau,\ \ (\mathrm{O}x)(k)=\int_0^k \gamma(k,\tau)x(\tau)d\tau,$$

with the functions $\chi$ and $\xi$ being continuous on the intervals given by $[0,K]\times[0,K]$.

The nonlinear differentials equation of fractional type have been under investigation recently by many researchers. Most of their results are designed fractional derivatives as a necessary tool to solve the boundary values problems.

## 2.2 Preliminaries

Let us remind the definitions as follows:

**Def 2.2.1:** (Riemann-Liouville). The Riemann-Liouville Fractional integral (R-L fractional integral) of order $\alpha > 0$ for a continuous function $w:(0,\infty)\to\mathbb{R}$ is given by the equation:

$$I^{\alpha}w(k)=\frac{1}{\Gamma(\alpha)}\int_0^k (k-\tau)^{\alpha-1}w(\tau)d\tau,$$

under the assumption that mentioned integral is defined.

**Def 2.2.2:** Consider a continuous function defined as follows $w:(0,\infty)\to\mathbb{R}$, the RLFD of order $\alpha > 0$, $\mathrm{m}=[\alpha]+1$( $[\alpha]$ is considered to be the integer part from the truncation of the real number $\alpha$ ) is given as follows:

$$D^{\alpha}w(k)=\frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dk}\right)^m\int_0^k (k-\tau)^{m-\alpha-1}w(\tau)d\tau=\left(\frac{d}{dk}\right)^m I^{m-\alpha}w(k),$$

provide it exists.

For $\alpha < 0$, let us assume that $D^{\alpha}w=I^{-\alpha}w$. Considering also that for $\beta\in[0,\alpha)$, the following holds $D^{\beta}I^{\alpha}w=I^{\alpha-\beta}w$.

Considering $\omega > 1,\ \omega \neq \alpha\text{-}1, \alpha\text{-}2, ..., \alpha\text{-m},$ lead us to

$$D^{\alpha} k^{\omega} = \frac{\Gamma(\omega+1)}{\Gamma(\omega-\alpha+1)} k^{\omega-\alpha},$$

and

$$D^{\alpha} k^{\alpha-j} = 0, \ \ \text{j} = 1,2,...,m.$$

For the particular case where a constant function is defined by $w(k) = 1,$ we obtain

$$D^{\alpha} 1 = \frac{1}{\Gamma(1-\alpha)} k^{-\alpha}, \ \ \alpha \notin \mathbb{N}.$$

For $\alpha \in \mathbb{N}$, we surely get, $D^{\alpha} 1 = 0$ this because of the numerical computation value of gamma function given at the points defined by the following integers $0, -1, -2, \ldots$.

Considering $\alpha > 0,$ the homogeneous equation has general solution given by

$$D^{\alpha} w(k) = 0,$$

in $C(0, K) \cap L(0, K)$ is

$$w(k) = c_0 k^{\alpha-m} + c_1 k^{\alpha-m-1} + ... + c_{m-2} k^{\alpha-2} + c_{m-1} k^{\alpha-1},$$

where $c_j,\ j = 1, 2, ..., m-1,$ are real numbers randomly selected.

The following relation always holds $D^{\alpha} I^{\alpha} w = w$, as well as

$$I^{\alpha} D^{\alpha} w(k) = w(k) + c_0 k^{\alpha-m} + c_1 k^{\alpha-m-1} + ... + c_{m-2} k^{\alpha-2} + c_{m-1} k^{\alpha-1}.$$

To solve the nonlinear problem which is defined by (1.1) and (1.2)-(1.3) we first consider the linear equations given below by

$$D^{\alpha}w(k)=\rho(k),\ \alpha\in(1,2],\ \mathrm{k}\in[0,K],\ K>0, \qquad (2.1)$$

where $\rho\in C(0,K)$.

We define

$$\mathrm{E}=\upsilon\int_0^{\eta}\frac{\tau^{\alpha-1}(\eta-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau=\frac{\upsilon\,\Gamma(\alpha)\eta^{2\alpha-2}}{\Gamma(2\alpha-1)}, \qquad (2.2)$$

such that $\mathrm{E}\neq\Gamma(\alpha)$.

The general solution of equation (2.1) is given as follows

$$w(k)=c_1k^{\alpha-1}+c_0k^{\alpha-2}+I^{\alpha}\rho(k), \qquad (2.3)$$

with $I^{\alpha}$ the usual Riemann-Liouville fractional integral of order $\alpha$.

From (2.3)

$$D^{\alpha-1}w(k)=c_1\Gamma(\alpha)+I^{1}\rho(k), \qquad (2.4)$$

$$D^{\alpha-2}w(k)=c_1\Gamma(\alpha)k+c_0\Gamma(\alpha-1)+I^{2}\rho(k). \qquad (2.5)$$

By using the condition (1.2) and (1.3) in (2.4) and (2.5), we find that $c_0=0$ and

$$c_1=\frac{\upsilon}{[\Gamma(\alpha)-\mathrm{E}]}\int_0^{\eta}\frac{(\eta-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_0^{\tau}\frac{(\tau-x)}{\Gamma(\alpha)}\rho(x)dx\right)d\tau,$$

with E given by (2.2).

The substitution of the values of $c_0$ as well as the value $c_1$ in the equation (2.3), lead to get the unique solution of equation (2.1) provided the boundary conditions (1.2)-(1.3) which are defined above are satisfied as follows:

$$w(k)=\int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)}\rho(\tau)d\tau$$

$$+\frac{\upsilon k^{\alpha-1}}{[\Gamma(\alpha)-\mathrm{E}]}\int_0^\eta \frac{(\eta-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_0^\tau \frac{(\tau-x)}{\Gamma(\alpha)}\rho(x)dx\right)d\tau$$

$$=\int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)}\rho(\tau)d\tau+\frac{\upsilon k^{\alpha-1}}{[\Gamma(\alpha)-\mathrm{E}]}k^{2\alpha-1}\rho(\eta). \qquad (2.6)$$

## 2.3 Main Results

Let $C=C([0,K],\mathbb{R})$, denoted the Banach space is defined on all continuous function defined from $[0,K]$ to $\mathbb{R}$ endow with the following norm $\|w\|=\sup\{|w(k)|,$ $k\in[0,K]\}$.

If $w$ is a solution of (1.1) and (1.2)-(1.3), then

$$\mathrm{w}(k)=\int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))d\tau$$

$$+\nu_1 k^{\alpha-1}\int_0^\eta \frac{(\eta-\tau)^{\alpha-1}}{\Gamma(2\alpha-1)}\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))d\tau,$$

where

$$\upsilon_1=\frac{\upsilon}{[\Gamma(\alpha)-\mathrm{E}]}.$$

We define an operator $P:b\to b$ as

$$(Pw)(k)=\int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))d\tau$$

$$+\upsilon_1 k^{\alpha-1}\int_0^\eta \frac{(\eta-\tau)^{\alpha-1}}{\Gamma(2\alpha-1)}\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))d\tau, \quad \mathrm{k}\in[0,K].$$

It is clear that the problems given by the equations (1.1) ; (1.2)-(1.3) have a solution for the unique condition that the operator defined by the equation $Pw = w$ admitted a fixed point.

**Lemma 2.3.1:** Operator $P$ is compact.

***Proof :***

(i) Let set **B** be bounded in the set $C[0,K]$. There is a real value constant $M$ s.t: $\left|\varphi\left(k, w(k), (\delta w)(k), (\mathrm{O}w)(w)\right)\right| \le M$, $\forall w \in \mathbf{B}$, $k \in [0,K]$. Thus

$$\begin{aligned}
\left|(Pw)(k)\right| &\le M\int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau + M\,|\upsilon_1|\,k^{\alpha-1}\int_0^{\eta}\frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)}d\tau \\
&\le MK^{\alpha-1}\left(\frac{K}{\Gamma(\alpha+1)}+\frac{|\upsilon_1|\eta^{2\alpha-1}}{\Gamma(2\alpha)}\right), \\
\Rightarrow \|(Pw)\| &\le MK^{\alpha-1}\left(\frac{K}{\Gamma(\alpha+1)}+\frac{|\upsilon_1|\eta^{2\alpha-1}}{\Gamma(2\alpha)}\right) < \infty.
\end{aligned}$$

Thus, $P$ (**B**) is bounded uniformly.

(ii) $\forall k_1, k_2 \in [0,K]$, $\mathrm{w} \in \mathbf{B}$,

$$\begin{aligned}
&\Rightarrow \left|(Pw)(k_1)-(Pw)(k_2)\right| \\
&= \left|\int_0^{k_1}\frac{(k_1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi(\tau, w(\tau), (\delta w)(\tau), (\mathrm{O}w)(\tau))d\tau \right.\\
&\quad -\int_0^{k_2}\frac{(k_2-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi(\tau, w(\tau), (\delta w)(\tau), (\mathrm{O}w)(\tau))d\tau \\
&\quad \left. +\upsilon_1\left(k_1^{\alpha-1}-k_2^{\alpha-1}\right)\int_0^{\eta}\frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)}\varphi(\tau, w(\tau), (\delta w)(\tau), (\mathrm{O}w)(\tau))d\tau\right| \\
&\le M\left(\left|\frac{1}{\Gamma(\alpha)}\int_0^{k_1}\left[(k_1-\tau)^{\alpha-1}-(k_2-\tau)^{\alpha-1}\right]d\tau - \frac{1}{\Gamma(\alpha)}\int_{k_1}^{k_2}(k_2-\tau)^{\alpha-1}d\tau\right|\right.
\end{aligned}$$

$$+\left|\upsilon_1\left(k_1^{\alpha-1}-k_2^{\alpha-1}\right)\int_0^{\eta}\frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)}d\tau\right|\Bigg)\to 0 \text{ as } k_1\to k_2.$$

Hence, the operator $P$ (**B**) is an equicontinuous operator. As consequence, $P$ is a compact operator. □

The fixed point theorem as given below is essential to demonstrate the existence and uniqueness for the solution obtained by solving the problem as defined.

**Theorem 2.3.1:** Consider a Banach space $F$. Assume that operator $G:F\to F$ is completely continuous and the set:

$\mathrm{H}=\{y\in F \mid y=\mu Gy,\ 0<\mu<1\}$ is bounded.

Under these assumptions, $G$ has a unique FP in $F$.

**Theorem 2.3.2:** Assume the existence of a constant $N>0$ s.t:

$$|\varphi(k,w(k),(\delta w)(k),(\mathrm{O}w)(k))|\le N,\quad \forall k\in[0,K],\quad w\in\mathbb{R}.$$

The problem (1.1) ; (1.2)-(1.3) has at least one solution in the closed interval $[0,K]$.

***Proof :*** we consider the following set

$$\mathrm{H}=\{w\in\mathbb{R}|w=\mu Gw, 0<\mu<1\}$$

and show that the set $\mathrm{H}$ is bounded. Let $w\in\mathrm{H}$, then $w=\mu Pw$, $0<\mu<1$. For any $k\in[0,K]$, we have

$$|w(k)|\le\mu\Bigg[\int_0^k\frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))|d\tau$$

$$+|\upsilon_1|k^{\alpha-1}\int_0^{\eta}\frac{(\eta-\tau)^{\alpha-1}}{\Gamma(2\alpha-1)}|\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))|d\tau\Bigg].$$

As in part (i) of lemma (2.3.1), we have

$$\|(Pw)\| \le MK^{\alpha-1}\left(\frac{K}{\Gamma(\alpha+1)}+\frac{|\upsilon_1|\eta^{2\alpha-1}}{\Gamma(2\alpha)}\right)<\infty.$$

This lead to conclusion that H is a bounded set regardless of $\mu \in (0,1)$. From lemma (2.3.1) combined with theorem (2.3.1), it follows that the bounded operator $P$ has at least one FP. It follows that the problems defined by the equations (1.1); (1.2)-(1.3) has at least a solution. □

**Theorem 2.3.3:** Assume

$(A_1)$ the existence of a positive function $Q_1(k),\ Q_2(k),\ Q_3(k)$ such a way that

$$|\varphi(k, w(k), (\delta w)(k), (\mathrm{O}w)(k)) - \varphi(k, \upsilon(k), (\delta\upsilon)(k), (\mathrm{O}\upsilon)(k))|$$

$$\le Q_1(k)|w-\upsilon| + Q_2(k)|\delta w - \delta\upsilon| + Q_3(k)|\mathrm{O}w - \mathrm{O}\upsilon|, \quad wv \in \mathbb{R}.$$

$(A_2)$ Z$=\left(\xi_1 + |\upsilon_1| K^{\alpha-1}\xi_2\right)\left(1+\gamma_0+\delta_0\right)<1,$ with

$$\chi_0 = \sup_{k\in[0,1]}\left|\int_0^k \chi(k,\tau)\,d\tau\right|, \quad \gamma_0 = \sup_{k\in[0,1]}\left|\int_0^k \gamma(k,\tau)\,d\tau\right|,$$
$$\xi_1 = \sup_{k\in[0,K]}\left\{\left|I^{\rho}Q_1(k)\right|,\ \left|I^{\rho}Q_2(k)\right|,\ \left|I^{\rho}Q_3(k)\right|\right\},$$
$$\xi_2 = \max\left\{\left|I^{2\alpha-1}Q_1(\eta)\right|,\ \left|I^{2\alpha-1}Q_2(\eta)\right|,\ \left|I^{2\alpha-1}Q_3(\eta)\right|\right\}.$$

Hence the problems are given by the equations (1.1) ; (1.2)-(1.3) has a solution on $C[0,K]$. which is unique.

***Proof :*** Denoted by $\sup_{k\in[0,K]}|\varphi(k,0,0,0)| = M$, and consider t such that

$$\iota \geq \frac{\varepsilon M}{1-Z}.$$

Hence we can prove that $PB_\iota \subset B_\iota$, where the set $B_\iota = \{ y \in C : \|w\| \leq \iota \}$. The following relation holds

$$\begin{aligned}
&\left\|(Pw)(k)\right\| \\
&= \sup_{k\in[0,K]} \left| \int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\tau, w(\tau), (\delta w)(\tau), (\mathrm{O}w)(\tau)) d\tau \right. \\
&\qquad \left. + \upsilon_1 k^{\alpha-1} \int_0^\eta \frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)} \varphi(\tau, w(\tau), (\delta w)(\tau), (\mathrm{O}w)(\tau)) d\tau \right| \\
&\leq \sup_{k\in[0,K]} \left( \int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left( \left| \varphi(\tau, w(\tau), (\delta w)(\tau), (\mathrm{O}w)(\tau)) d\tau - \varphi(\tau,0,0,0) \right| \right. \right. \\
&\qquad \left. + \left| \varphi(\tau,0,0,0) \right| \right) d\tau \\
&\qquad + |\upsilon_1| k^{\alpha-1} \int_0^\eta \frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)} \left( \left| \varphi(\tau, w(\tau), (\delta w)(\tau), (\mathrm{O}w)(\tau)) d\tau - \varphi(\tau,0,0,0) \right| \right. \\
&\qquad \left. \left. + \left| \varphi(\tau,0,0,0) \right| \right) d\tau \right) \\
&\leq \sup_{k\in[0,K]} \left( \int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)} (Q_1(\tau)|y(\tau)| + Q_2(\tau)|(\delta y)(\tau)| + Q_3(\tau)|(\mathrm{O}y)(\tau)| + M) d\tau \right. \\
&\qquad \left. + |\upsilon_1| k^{\alpha-1} \int_0^\eta \frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)} (Q_1(\tau)|y(\tau)| + Q_2(\tau)|(\delta y)(\tau)| + Q_3(\tau)|(\mathrm{O}y)(\tau)| + M) d\tau \right) \\
&\leq \sup_{k\in[0,K]} \left( \int_0^k \frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)} (Q_1(\tau)|y(\tau)| + Q_2(\tau)|(\delta y)(\tau)| + Q_3(\tau)|(\mathrm{O}y)(\tau)| + M) d\tau \right. \\
&\qquad \left. + |\upsilon_1| k^{\alpha-1} \int_0^\eta \frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)} (Q_1(\tau)|y(\tau)| + \chi_0 Q_2(\tau)|y(\tau)| + \gamma_0 Q_3(\tau)|y(\tau)| + M) d\tau \right) \\
&\leq \sup_{k\in[0,K]} \left( \left( I^\alpha Q_1(k) + \chi_0 I^\alpha Q_2(k) + \gamma_0 I^\alpha Q_3(k) \right) \iota + \frac{M k^\rho}{\Gamma(\rho+1)} \right. \\
&\qquad \left. + |\upsilon_1| k^{\alpha-1} \left( I^{(2\alpha-1)} Q_1(\eta) + \chi_0 I^{(2\alpha-1)} Q_2(\eta) + \gamma_0 I^{(2\alpha-1)} Q_3(\eta) \right) \iota + \frac{M \eta^{2\alpha-1}}{\Gamma(2\alpha)} \right) \\
&\leq \left( \xi_1 + |\upsilon_1| K^{\alpha-1} \xi_2 \right) (1 + \chi_0 + \gamma_0) \iota + M K^{\alpha-1} \left( \frac{K}{\Gamma(\alpha+1)} + \frac{|\upsilon_1| \eta^{\alpha-1}}{\Gamma(2\alpha)} \right) \\
&= Z\iota + M\varepsilon \leq \iota.
\end{aligned}$$

Considering $(A_1)$, for every $k \in [0,K]$, we have

$$\begin{aligned}
&\left|(Pw)(k)-(P\sigma)(k)\right| \\
&\quad \le \sup_{k\in[0,K]}\left(\int_0^{k_1}\frac{(k_1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left|\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))-\varphi(\tau,\sigma(\tau),(\delta\sigma)(\tau),(\mathrm{O}\sigma)(\tau))\right|d\tau\right. \\
&\qquad \left.+|\upsilon_1|k^{\alpha-1}\int_0^{\eta}\frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)}\left|\varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))-\varphi(\tau,\sigma(\tau),(\delta\sigma)(\tau),(\mathrm{O}\sigma)(\tau))\right|d\tau\right) \\
&\quad \le \sup_{k\in[0,K]}\left(\int_0^{k_1}\frac{(k_1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(Q_1(\tau)|w-\sigma|+Q_2(\tau)|\delta w-\delta\sigma|+L_3(\tau)|\mathrm{O}w-\mathrm{O}\sigma|\right)d\tau\right. \\
&\qquad \left.+|\upsilon_1|k^{\alpha-1}\int_0^{\eta}\frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)}\left(Q_1(\tau)|w-\sigma|+Q_2(\tau)|\delta w-\delta\sigma|+Q_3(\tau)|\mathrm{O}w-\mathrm{O}\sigma|\right)d\tau\right) \\
&\quad \le \sup_{k\in[0,K]}\left(I^{\alpha}Q_1(k)+\chi_0 I^{\alpha}Q_2(k)+\gamma_0 I^{\alpha}Q_3(k)\right)\|w-\upsilon\| \\
&\qquad +|\upsilon_1|K^{\alpha-1}\left(I^{(2\alpha-1)}Q_1(\eta)+\chi_0 I^{(2\alpha-1)}Q_2(\eta)+\gamma_0 I^{(2\alpha-1)}Q_3(\eta)\right)\|w-\sigma\| \\
&\quad \le \left(\xi_1+|\upsilon_1|K^{\alpha-1}\xi_2\right)(1+\chi_0+\gamma_0)\|w-\sigma\| = \mathrm{Z}\|w-\sigma\|.
\end{aligned}$$

By assumption $(A_2)$, Z<1, this leads to the conclusion that, the operator $P$ is a contraction mapping. Therefore, by BFP theorem, we can say that $P$ consist of only one FP. This unique fixed point is also the unique solution affirmed by the problem (1.1) and (1.2)-(1.3). □

**Theorem 2.3.4:** (Krasnoselskii's fixed point theorem). Let $\mathrm{M}$ be a closed, convex and nonempty subset of a Banach space $Y$. Let $A$, $B$ be the operator such that

(i) $Ax+By \in \mathrm{M}$ whenever $x, y \in \mathrm{M}$;

(ii) $A$ is compact and continuous;

(iii) $B$ is contraction mapping.

Then there exist $z \in \mathrm{M}$ such that $z = Az + Bz$.

**Theorem 2.3.5:** Suppose the following assumption $\varphi: [0,K]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function . Under this assumption, the specific relations hold:

$(H_1)$

$$\left\|\varphi(k,w(k),(\delta w)(k),(\mathrm{O}w)(k))-\varphi(k,\upsilon(k),(\delta\upsilon)(k),(\mathrm{O}\upsilon)(k))\right|d\tau\right|$$

$$\leq Q_1(\tau)|w-\upsilon|+Q_2(\tau)|\delta w-\delta\upsilon|+Q_3(\tau)|\mathrm{O}w-\mathrm{O}\upsilon|,\ \ \forall k\in[0,K],\ \ w,\upsilon\in\mathbb{R}.$$

$(H_2)$ $|\varphi(k,w)|\leq\mu(k),\ \forall(k,w)\in[0,K]\times\mathbb{R},$ and $\mu\in C([0,K],\mathbb{R}^+).$

If

$$\frac{|\upsilon_1|K^{\alpha-1}\eta^{2\alpha-1}}{\Gamma(2\alpha)}<1,$$

the BVPs are defined by the equations (1.1) and (1.2)-(1.3) admitted at least one solution given on the interval $[0,K]$.

***Proof :*** By letting $\sup_{k\in[0,K]}|\mu(k)|=\|\mu\|,$ we fix

$$\bar{\iota}\geq\|\mu\|K^{\alpha-1}\left(\frac{K}{\Gamma(\alpha+1)}+\frac{|\upsilon_1|\eta^{\alpha-1}}{\Gamma(2\alpha)}\right),$$

and consider $B_{\bar{\iota}}=\{w\in C:\|w\|\leq\bar{\iota}\}$. we define the operator $P_1$ and $P_2$ on $B_{\bar{r}}$ as

$$(P_1w)(k)=\int_0^k\frac{(k-\tau)^{\alpha-1}}{\Gamma(\alpha)}\ \varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))d\tau,$$

$$(P_2w)(k)=\upsilon_1k^{\alpha-1}\int_0^\eta\frac{(\eta-\tau)^{2\alpha-2}}{\Gamma(2\alpha-1)}\ \varphi(\tau,w(\tau),(\delta w)(\tau),(\mathrm{O}w)(\tau))d\tau.$$

For $w,\upsilon\in B_{\bar{r}}$, we find that

$$\|P_1w+P_2w\|\leq\|\mu\|K^{\alpha-1}\left(\frac{K}{\Gamma(\alpha+1)}+\frac{|\upsilon_1|\eta^{\alpha-1}}{\Gamma(2\alpha)}\right)\leq\bar{\iota}.$$

Thus $P_1 w + P_2 w \in B_{\bar{\iota}}$. Consider the assumption $(H_1)$ and by equation (3.1) we conclude that $P_2$ is a contraction mapping. Since function $\varphi$ is continuous, this indicate the continuity of the operator $P_1$ .

Moreover, the operator $P_1$ is bounded uniformly on the set $B_{\bar{\iota}}$ as

$$\|P_1 w\| \le \frac{\|w\| K^{\alpha}}{\Gamma(\alpha+1)}.$$

The compactness of $P_1$ is proved as follows.

Considering the hypothesis $(H_1)$, we define:

$\sup_{(k,y,\phi y,\psi y)\in[0,K]\times B_r\times B_r\times B_r} |\varphi(k,y,\delta y,\text{O}y)| = \bar{\varphi}$, and as consequence we have the following relation:

$$\begin{aligned}
&|(P_1 w)(k_1) - (P_1 w)(k_2)| \\
&\quad = \left| \frac{1}{\Gamma(\alpha)} \int_0^{k_1} \left[ (k_1-\tau)^{\alpha-1} - (k_2-\tau)^{\alpha-1} \right] \varphi(\tau, w(\tau), (\delta w)(\tau), (\text{O}w)(\tau)) d\tau \right. \\
&\qquad \left. - \frac{1}{\Gamma(\alpha)} \int_{k_1}^{k_2} (k_2-\tau)^{\alpha-1} \varphi(\tau, w(\tau), (\delta w)(\tau), (\text{O}w)(\tau)) d\tau \right| \\
&\quad \le \frac{\bar{\varphi}}{\Gamma(\alpha+1)} \left| 2(k_2-k_1)^{\alpha} + k_1{}^{\alpha} - k_2{}^{\alpha} \right|,
\end{aligned}$$

which is not dependent on $w$ and that approaches to the zero value as $k_2 \to k_1$. That means the mapping $P_1$ is compact locally on the set $B_{\bar{\iota}}$. By A-A theorem, it follows that, operator $P_1$ is a compact on $B_{\bar{\iota}}$. which satisfies the assumptions of theorem (2.3.4). As conclusion from this theorem, it is said that the problems given by problem (1.1) ; (1.2)-(1.3) with boundaries conditions possesses at least one solution on the compact set $\left[0, K\right]$.

# Chapter 3

# A STUDY OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER WITH RIEMANN-LIOUVILLE TYPE MULTISTRIP BOUNDARY CONDITIONS

## 3.1 Introduction

In the recent decades, the fractional calculus has been widely expanded. Many researchers have been interested in a lot due to his application found in various fields of sciences ranges from chemistry, astronomy, physics, engineering mechanics etc.
In recently bibliography, one can find the modeling of dynamical system based on fractional differential equations. In all this topics, the current state and the past state of a process are always needed to well describe and forecast the processes.

Concerning the fractional differential equations, the periodicity of the equation, its asymptotic behaviors as well as numerical method of approaching the solution are sought. There exists various of FDE, one of great the interest being the nonlinear. We are interested in solving differential equation with fractional arbitrary order with the consideration of boundaries values. The R-L integral type with boundaries conditions will be considered

$$\begin{aligned} &{}^{c}D^{\rho} y(k) = \varphi(k, y(k)), \quad k \in [0, K], \\ &y(0) = 0, \quad y'(0) = 0, \ldots, \quad y^{(m-2)}(0) = 0, \qquad (1) \\ &y(K) = \sum_{j=1}^{n} \gamma_j \left[ I^{\lambda_j} y(\eta_j) - I^{\lambda_j} y(\zeta_j) \right], \end{aligned}$$

where ${}^{c}D^{\rho}$ stands for Caputo fractional derivative type with the order $\rho$, $\varphi$ a continuous function and $I^{\lambda_j}$ anti derivative of order $\lambda_j > 0,\ j = 1, 2, ..., n,$ it is called R-L fractional integral where $0 < \zeta_1 < \eta_1 < \zeta_2 < \eta_2 < ... < \zeta_n < \eta_n < K$, and $\gamma_j \in \mathbb{R}$ are constant.

Strip conditions occurs often in the modeling of some real problems. In this work, the following nonlocal strip condition is considered

$$y(1) = \sum_{j=1}^{m-2} \alpha_j \int_{\zeta_j}^{\eta_j} y(\tau) d\tau, \quad 0 < \zeta_j < \eta_j < 1, \qquad j = 1, 2, ..., (m-2). \tag{2}$$

We studied above the R-L type integral with a multiple stripe boundaries conditions. Such problem can be found a direct application in the engineering.

In this section, an alternative way to solve the problem given in the previous section is investigated. The well known fixed point theorem will be used the show the existence of some solution to the problem solved preciously.

## 3.2 First Result

Consider the following basic definitions:

**Def. 3.2.1:** Let $f(k) \in AC^n[c,d]$, the following derivative is called Caputo derivative. It is a fractional derivative of order $\rho$ ( $\rho$ a real number)

$$\begin{aligned} {}^{c}D_{a^+}^{\rho} f(k) &= \frac{1}{\Gamma(m-\rho)} \int_a^k (k-\tau)^{m-\rho-1} f^{(m)}(\tau) d\tau \\ &= I_{a^+}^{m-\rho} D^m f(y), \quad m-1 < \rho < m, \quad m = [\rho] + 1, \end{aligned} \tag{3}$$

with symbol $[\rho]$ being the truncated integer part of the given the real number $\rho$.

The symbol $AC^{n}\left[c,d\right]$ is the space of functions $f\left(k\right)$ (space of all real valued functions) which have continuous derivatives with order up to $m-1$ on the interval $\left[c,d\right]$ such a way that $f^{m-1}\left(k\right)\in AC\left[c,d\right]$.

**Def. 3.2.2:** The integral below is called the R-L fractional integral of order $\rho$:

$$I^{\rho}f\left(k\right)=\frac{1}{\Gamma\left(\rho\right)}\int_{0}^{k}\frac{f\left(\tau\right)}{\left(k-\tau\right)^{1-\rho}}d\tau,\quad \rho>0, \qquad (4)$$

with the assumption of the existence of the integral.

The following lemma is as a results of the study carried out on equation (1), it is important in the generalization of the main result.

**Lemma 3.2.1:** Considering $h\in C\left[0,K\right]$, the fractional BVP

$$\begin{aligned}
&{}^{c}D^{\rho}y\left(k\right)=h\left(k\right),\ k\in\left[0,K\right],\ \rho\in\left(n-1,n\right)\\
&y\left(0\right)=0,\ y'\left(0\right)=0,\ldots,\ y^{(m-2)}\left(0\right)=0,\\
&y\left(K\right)=\sum_{j=1}^{n}\gamma_{j}\left[I^{\lambda_j}y\left(\eta_j\right)-I^{\lambda_j}y\left(\zeta_j\right)\right],
\end{aligned} \qquad (5)$$

has a unique solution $y\left(k\right)\in AC^{n}\left[0,K\right]$ given by

$$\begin{aligned}
y\left(k\right)=&\frac{1}{\Gamma\left(\rho\right)}\int_{0}^{k}\left(k-\tau\right)^{\rho-1}h\left(\tau\right)d\tau\\
&-\frac{k^{m-1}}{\mu\Gamma\left(\rho\right)}\int_{0}^{K}\left(K-\tau\right)^{\rho-1}h\left(\tau\right)d\tau\\
&+\frac{k^{m-1}}{\lambda\Gamma\left(\rho\right)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma\left(\lambda_j\right)}\\
&\qquad\times\left[\int_{0}^{\eta_j}\int_{0}^{\tau}\left(\eta_i-\tau\right)^{\lambda_j-1}\left(\tau-w\right)^{\rho-1}h\left(w\right)dwd\tau\right.\\
&\qquad\left.-\int_{0}^{\zeta_j}\int_{0}^{\tau}\left(\zeta_j-\tau\right)^{\lambda_j-1}\left(\tau-w\right)^{\rho-1}h\left(w\right)dwd\tau\right],
\end{aligned} \qquad (6)$$

where

$$\mu=\left(K^{m-1}-\sum_{j=1}^{n}\frac{\left(\eta_j^{\lambda_j+m-1}-\zeta_j^{\lambda_j+m-1}\right)\Gamma(m)}{\Gamma\left(\lambda_j+m\right)}\right)\neq 0. \tag{7}$$

***Proof*** *:* Consider the equation (5), a general form of solution is given as follows

$$y(k)=\frac{1}{\Gamma(\rho)}\int_0^k (k-\tau)^{\rho-1}h(\tau)d\tau-c_0-c_1k-...-c_{m-1}k^{m-1}. \tag{8}$$

By using the given boundary conditions, it is found that $c_0=0,\ c_1=0,...,c_{n-2}=0.$

Now the integral given by the operator $I^{\lambda_j}$ of Riemann-Liouville on (8), leads us to

$$\begin{aligned} I^{\lambda_j}y(k) &= \frac{1}{\Gamma(\lambda_j)}\int_0^k (k-\tau)^{\lambda_j-1} \\ &\qquad\times\left(\frac{1}{\Gamma(\rho)}\int_0^\tau (\tau-w)^{\rho-1}h(w)dw-c_{m-1}\tau^{m-1}\right)d\tau \\ &= \frac{1}{\Gamma(\lambda_j)\Gamma(\rho)}\int_0^k\int_0^\tau (k-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}h(w)dwd\tau \\ &\quad -c_{m-1}\frac{1}{\Gamma(\lambda_j)}\int_0^k (k-\tau)^{\lambda_j-1}\tau^{m-1}d\tau. \end{aligned} \tag{9}$$

By using the condition $y(K)=\sum_{j=1}^{n}\gamma_j\left[I^{\lambda_j}y(\eta_j)-I^{\lambda_j}y(\zeta_j)\right],$ together with the fact that

$$\frac{1}{\Gamma(\lambda_j)}\int_0^k (k-\tau)^{\lambda_j-1}\tau^{m-1}d\tau=\frac{k^{\lambda_j+m-1}\Gamma(m)}{\Gamma\left(\lambda_j+m\right)}, \tag{10}$$

we obtain

$$\frac{1}{\Gamma(\rho)}\int_0^K (K-\tau)^{\rho-1} h(\tau)d\tau - c_{m-1}K^{m-1}$$
$$= \sum_{j=1}^{n} \frac{\gamma_j}{\Gamma(\rho)\Gamma(\lambda_j)} \times \left[ \int_0^{\eta_j}\int_0^{\tau} (\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} h(w)\,dw\,d\tau \right.$$
$$\left. - \int_0^{\zeta_j}\int_0^{\tau} (\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} h(w)\,dw\,d\tau \right]$$
$$- c_{m-1}\sum_{j=1}^{n} \gamma_j \frac{\left(\eta_j^{\lambda_j+m-1} - \zeta_j^{\lambda_j+m-1}\right)\Gamma(m)}{\Gamma(\lambda_j+m)}, \tag{11}$$

which yields

$$c_{m-1} = \frac{1}{\mu\Gamma(\rho)}\int_0^K (K-\tau)^{\rho-1} h(\tau)d\tau$$
$$- \frac{1}{\mu\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{(\lambda_j)}$$
$$\times \left[ \int_0^{\eta_j}\int_0^{\tau} (\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} h(w)\,dw\,d\tau \right.$$
$$\left. - \int_0^{\zeta_j}\int_0^{\tau} (\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} h(w)\,dw\,d\tau \right], \tag{12}$$

where $\mu$ is given by (7). By substituting the value of $c_0,\ c_1, ..., \ c_{m-2},\ c_{m-1}$ from (8), we find (6). □

## 3.3 General Results

Let $\vartheta := C([0,K],\mathbb{R})$ be a Banach space which contains all continuous function which are defined on the interval $[0,K]\times\mathbb{R}$ on which a uniform convergence topology is defined with the following norm $\|y\| = \sup_{k\in[0,K]}|y(k)|$.

From lemma (3.2.1), the operator $P:\vartheta \to \vartheta$ is defined as

$$
\begin{aligned}
&(Py)(k) \\
&= \frac{1}{\Gamma(\rho)}\int_0^k (k-\tau)^{\rho-1} f\left(\tau, y(\tau)\right)d\tau \\
&\quad - \frac{k^{n-1}}{\mu\Gamma(\rho)}\int_0^K (K-\tau)^{\rho-1} f\left(\tau, y(\tau)\right)d\tau \\
&\quad + \frac{k^{n-1}}{\mu\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)} \\
&\qquad \times\left[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} f\left(w, y(w)\right)dw\,d\tau\right. \\
&\qquad\qquad \left. - \int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} f\left(w, y(w)\right)dw\,d\tau\right], \\
&\qquad\qquad\qquad k\in[0,K].
\end{aligned}
\tag{13}
$$

It is clear that the problem (1) may have a solution only if the following associated fixed point equation $Py = y$ possesses a solution; that means admitted a fixed point. Previously, the Banach's contraction mapping was used to show existence and the uniqueness of solution to problem (1).

Let us consider the following notation for convenience

$$
\begin{aligned}
Z &= \frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{\rho+m-1}}{|\mu|\Gamma(\rho+1)} \\
&\quad + \frac{K^{m-1}}{|\mu|}\sum_{j=1}^{n}\gamma_j\,\frac{\left(\eta_j^{\rho+\lambda_j}-\zeta_j^{\rho+\lambda_j}\right)\Gamma(m)}{\Gamma\left(\rho+\lambda_j+1\right)}.
\end{aligned}
\tag{14}
$$

**Theorem 3.3.1:** Assume the continuous real valued function $\varphi:[0,K]\times\mathbb{R}\to\mathbb{R}$ satisfies the following assumption:

$(A_3)$
$$\left|\varphi(k,y)-\varphi(k,x)\right|\le L\left|y-x\right|,$$

$$\forall k\in[0,1],\ L>0,\quad y,x\in\mathbb{R}. \qquad (15)$$

The boundary value problem (1) is defined above may have a unique solution under the condition

$$L<\frac{1}{\mathrm{Z}}, \qquad (16)$$

with $Z$ defined by (14).

***Proof :*** consider $\iota\ge MZ(1-LZ)$, we define $B_\iota=\{y\in\vartheta:\|y\|\le\iota\}$, where $M=\sup_{k\in[0,K]}<\infty$ and $Z$ is given as defined by (14). Then we prove that $PB_\iota\subset B_\iota$. For $y\in B_\iota$, by means of the inequality $\left|\varphi(\tau,y(\tau))\right|\le\left|\varphi(\tau,y(\tau))-\varphi(\tau,0)\right|+\left|\varphi(\tau,0)\right|\le L\|y\|+M\le L\iota+M$, it can easily be proved that

$$\|Py\|=(L\iota+M)\mathrm{Z}\le\iota. \qquad (17)$$

Now, for $y,x\in\vartheta$ and for each $k\in[0,K]$, we obtain

$$\begin{aligned}
&\left\|(Py)-(Px)\right\|\\
&\le\sup_{k\in[0,K]}\left\{\frac{1}{\Gamma(\rho)}\int_0^k(k-\tau)^{\rho-1}\left|\varphi(\tau,y(\tau))-\varphi(\tau,x(\tau))\right|d\tau\right.\\
&\qquad+\frac{k^{m-1}}{\mu\Gamma(\rho)}\int_0^K(K-\tau)^{\rho-1}\left|\varphi(\tau,y(\tau))-\varphi(\tau,x(\tau))\right|d\tau\\
&\qquad+\frac{k^{m-1}}{\mu\Gamma(\rho)}\\
&\qquad\quad\times\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)}
\end{aligned}$$

$$\times\left[\int_0^{\eta_j}\int_0^{\tau}\left(\eta_j-\tau\right)^{\lambda_j-1}\left(\tau-w\right)^{\rho-1}\left|\varphi\left(w,y(w)\right)-\varphi\left(w,x(w)\right)\right|dw\,d\tau\right.$$

$$\left.\left.-\int_0^{\zeta_j}\int_0^{\tau}\left(\zeta_j-\tau\right)^{\lambda_j-1}\left(\tau-w\right)^{\rho-1}dw\left|\varphi\left(w,y(w)\right)-\varphi\left(w,x(w)\right)\right|d\tau\right]\right\}$$

$$\leq LZ\|y-x\|. \tag{18}$$

The value $Z$ is a function of the parameter of the problem. Since $LZ<1$, the application $P$ is a contraction mapping. Therefore, by the Banach's contraction mapping fixed point theorem, the problem (1) has a unique solution on the interval $[0,K]$. □

**Example 3.3.1:** Examine the boundary four-strip nonlocal valued problem:

$$\begin{aligned}
&{}^{c}D^{9/2}y(k)=\varphi\left(k,y(k)\right),\quad \mathrm{k}\in[0,2],\\
&y(0)=0,\quad y'(0)=0,\quad y''(0)=0,\quad y'''(0)=0,\\
&y(K)=\sum_{j=1}^{4}\gamma_j\left[I^{\lambda_j}y(\eta_j)-I^{\lambda_j}y(\zeta_j)\right],
\end{aligned} \tag{19}$$

where

$\rho=9/2, m=5, \zeta_1=1/4, \eta_1=1/2, \zeta_2=2/3, \eta_2=1, \zeta_3=5/4, \eta_3=4/3, \zeta_4=3/2,$
$\eta_4=7/4, \gamma_1=5, \gamma_2=10, \gamma_3=15, \gamma_4=25, \lambda_1=5/4, \lambda_2=7/4, \lambda_3=9/4, \lambda_4=11/4.$

Consider the numerical value of the parameters as given above, it follows that

$$\mu=\left(K^{m-1}-\sum_{j=1}^{n}\frac{\left(\eta_j^{\lambda_i+m-1}-\zeta_j^{\lambda_j+m-1}\right)\Gamma(m)}{\Gamma\left(\lambda_j+m\right)}\right)$$

$$\simeq 9.334784,$$

$$Z=\frac{K^{\rho}}{\Gamma(\rho+1)}-\frac{K^{\rho+m-1}}{|\mu|\Gamma(\rho+1)}+\frac{K^{m-1}}{|\mu|}\sum_{j=1}^{4}\gamma_j\frac{\left(\eta_j^{\rho+\lambda_j}-\zeta_j^{\rho+\lambda_j}\right)\Gamma(m)}{\Gamma\left(\rho+\lambda_j+1\right)}$$

$$\simeq 1.406972. \tag{20}$$

Let us chose

$$\varphi(k, y(k)) = \frac{1}{\sqrt[3]{(k+8)}}\left(\tan^{-1} y\right) + \sqrt{4+3\sin 2k}. \tag{21}$$

Obviously, $L = 1/2$ as $|\varphi(k,y) - \varphi(k,x)| \le (1/2)|y-x|$ and $L < 1/Z$, where $Z \simeq 1.406972$. Theorem (3.3.1) is satisfied, therefore problem (19) has an unique solution where $\varphi(k, y(k))$ is defined by (21).

Consider the unbounded nonlinear equation:

$$\varphi(k, y(k)) = \frac{y}{7} + \frac{1}{\sqrt[3]{(k+8)}}\left(\tan^{-1} y\right) + \sqrt{4+3\sin 2k}, \tag{22}$$

we have $L = 9/14$ and $L < 1/Z$ $(Z \simeq 1.406972)$. Previously, the problem (19) with $\varphi(k, x(k))$ defined by (22) has a unique solution.

The Leray-Schauder alternative is used for the purpose in what the follows:

**Theorem 3.3.2:** Assume the existence $L_1 > 0$ s.t: $|\varphi(k,y)| \le L_1$, for $k \in [0, K]$, $y \in \mathbb{R}$. Then equation (1) has at least one solution.

***Proof :*** We prove here that operator $P$ is completely continuous. You might also observe the continuity of operator $P$ through the continuity of $f$. Consider $\vartheta \subset \varpi$ a bounded set. Assume that $|\varphi(k,y)| \le L_1$, for $y \in \vartheta$, lead us to

$$\begin{aligned} &|(Py)(k)| \\ &\quad \le \frac{1}{\Gamma(\rho)}\int_0^k (k-\tau)^{\rho-1}|\varphi(\tau, y(\tau))|\,d\tau \\ &\quad\quad + \frac{k^{m-1}}{|\mu|\Gamma(\rho)}\int_0^K (K-\tau)^{\rho-1}|\varphi(\tau, y(\tau))|\,d\tau \end{aligned}$$

$$
\begin{aligned}
&+\frac{k^{m-1}}{|\mu|\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)}\\
&\quad\times\Bigg[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\\
&\qquad\times|\varphi(w,y(w))|\,dwd\tau\\
&\quad-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\\
&\qquad\times|\varphi(w,y(w))|\,dwd\tau\Bigg]\\
&\le L_1\Bigg[\frac{1}{\Gamma(\rho)}\int_0^{k}(k-\tau)^{\rho-1}d\tau+\frac{k^{m-1}}{|\mu|\Gamma(\rho)}\int_0^{K}(K-\tau)^{\rho-1}d\tau\\
&\quad+\frac{k^{m-1}}{\mu\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)}\\
&\qquad\times\Bigg[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}\\
&\qquad\times(\tau-w)^{\rho-1}dwd\tau\\
&\qquad-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}\\
&\qquad\times(\tau-w)^{\rho-1}\mathrm{dwd}\tau\Bigg]\Bigg]\\
&\le L_1\Bigg\{\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{\rho+m-1}}{|\lambda|\Gamma(\rho+1)}\\
&\quad+\frac{K^{m-1}}{|\mu|}\sum_{j=1}^{n}\gamma_j\frac{\left(\eta_j^{\rho+\lambda_j}-\zeta_j^{\rho+\lambda_j}\right)}{\Gamma(\rho+\lambda_j+1)}\Bigg\}=L_2,
\end{aligned}
\tag{24}
$$

that implies that $\|(Py)\|\le L_2$. Furthermore, we find the

$$
\begin{aligned}
&|(\rho y)'(k)|\\
&\quad\le\frac{1}{\Gamma(\rho-1)}\int_0^{k}(k-\tau)^{\rho-2}|\varphi(\tau,y(\tau))|\,d\tau\\
&\quad+\frac{(m-1)k^{m-2}}{|\mu|\Gamma(\rho)}\int_0^{K}(K-\tau)^{\rho-1}|\varphi(\tau,y(\tau))|\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(m-1)k^{m-2}}{|\mu|\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)}\\
&\quad\times\left[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\right.\\
&\quad\quad\times|\varphi(w,y(w))|\,dw\,d\tau\\
&\quad-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\\
&\quad\quad\left.\times|\varphi(w,y(w))|\,dw\,d\tau\right]\\
&\le L_1\left[\frac{1}{\Gamma(\rho-1)}\int_0^{k}(k-\tau)^{\rho-2}d\tau+\frac{(m-1)k^{m-2}}{|\mu|\Gamma(\rho)}\int_0^{K}(K-\tau)^{\rho-1}d\tau\right.\\
&\quad+\frac{(m-1)k^{m-2}}{\mu\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)}\\
&\quad\times\left(\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\right.\\
&\quad\quad\times|\varphi(w,y(w))|\,dw\,d\tau\\
&\quad-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\\
&\quad\quad\left.\left.\times|\varphi(w,y(w))|\,dw\,d\tau\right)\right]\\
&\le\left\{\frac{K^{\rho-1}}{\Gamma(\rho)}+\frac{K^{\rho+m-2}}{|\lambda|\Gamma(\rho+1)}\right.\\
&\quad\left.+\frac{(m-1)K^{m-1}}{|\mu|}\sum_{j=1}^{n}\gamma_j\frac{\left(\eta_j^{\rho+\lambda_j}-\zeta_j^{\rho+\lambda_j}\right)}{\Gamma(\rho+\lambda_j+1)}\right\}=L_3.
\end{aligned}
\tag{25}
$$

Finally, for $k_1,k_2\in[0,K]$, we have the following inequality:

$$
\left|(Py)(k_2)-(Py)(k_1)\right|\le\int_{k_1}^{k_2}\left|(Py)'(\tau)\right|\le L_3(k_2-k_1). \tag{26}
$$

That proved equicontinuity of the operator $P$ on the Interval $[0,K]$. By A-A theorem, $P:\vartheta\to\vartheta$ is completely continuous operator.

Let us consider the set

$$\upsilon=\{y\in\vartheta \mid y=\gamma Py,\ 0<\gamma<1\}, \tag{27}$$

and prove that the set $\upsilon$ is bounded let $y\in\upsilon$, then $y=\gamma Py$, $0<\gamma<1$. $\forall k\in[0,K]$,

$$\begin{aligned}\Rightarrow |y(k)| &= \gamma |(Py)(k)| \\ &\le \frac{1}{\Gamma(\rho)}\int_0^k (k-\tau)^{\rho-1}|\varphi(\tau,y(\tau))|d\tau \\ &\quad+\frac{k^{m-1}}{|\mu|\Gamma(\rho)}\int_0^K (K-\tau)^{\rho-1}|\varphi(\tau,y(\tau))|d\tau \\ &\quad+\frac{k^{m-1}}{|\mu|\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)} \\ &\qquad\times\Bigg[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} \\ &\qquad\qquad\times|\varphi(w,y(w))|dwd\tau \\ &\qquad-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} \\ &\qquad\qquad\times|\varphi(w,y(w))|dwd\tau\Bigg] \\ &\le L_1\Bigg\{\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{\rho+m-1}}{|\mu|\Gamma(\rho+1)} \\ &\qquad+\frac{K^{m-1}}{|\mu|}\sum_{j=1}^{n}\gamma_j\frac{\left(\eta_j^{\rho+\lambda_j}-\zeta_j^{q+\beta_j}\right)}{\Gamma(\rho+\lambda_j+1)}\Bigg\}=M_1.\end{aligned} \tag{28}$$

Thus, $\|y\|\le M_1$ for any $k\in[0,K]$. So, the set $\upsilon$ is bounded. Therefore, by Theorem (3.3.2), the operator $P$ admits at least one fixed point. Which mean equation (1) has at least a solution. □

**Example 3.3.2:** Let us consider now the BVP as given previously in example 1, as well as the function defined by

$$\varphi(k, y(k)) = \frac{3e^{\sqrt{(2-|y(k)|)^3}}\left[\cos 4k + 2\ln\left(1+4\sin^2 y(k)\right)\right]}{\sqrt{(10+\cos y(k))}}. \qquad (29)$$

One can easily see, that $|\varphi(k, y)| \le L_1$ with $L_1 = e^{(2\sqrt{2})}(1+\ln 25)$. Therefore the conditions of theorem (3.3.3) holds. So by theorem (3.3.3), equation (19) with $\varphi(k, y(k))$ is defined. Equation (29) indicate that it has at least one real value which is solution.

In what follows, we show another existence of the result for problem (1), based on the following well known result.

**Theorem 3.3.3:** Let $Y$ be Banach space, and $\Omega$ is bounded, open subset of $Y$ with $\theta \in \Omega$, let $K : \bar{\Omega} \to Y$ be completely continuous operator s.t:

$$\|Kw\| \le \|w\|, \quad \forall w \in \partial\Omega. \qquad (30)$$

$\Leftrightarrow$ $K$ has a fixed point in $\bar{\Omega}$.

**Theorem 3.3.4:** Let us assume the existence of a small number $\tau > 0$ s.t: $|\varphi(k, y)| \le \upsilon |y|$ for $0 < |y| < \tau$, with $0 < \upsilon < 1/Z$, with $Z$ is defined by (14). As a consequence $\exists$ at least one solution to the equation (1).

***Proof :*** We define $P_\eta = \{y \in \vartheta \,|\, \|y\| < \eta\}$ and take $y \in \vartheta$ such that $\|y\| = \eta$, that is, $y \in \partial P_\eta$. Previously, the completely continuity of the operator $P$ is easy to show.

$$\begin{aligned}
&\|Py\| \\
&\leq \sup_{k\in[0,K]}\left\{\frac{1}{\Gamma(\rho)}\int_0^k (k-\tau)^{\rho-1}\left|\varphi(\tau, y(\tau))\right| d\tau\right. \\
&\quad + \frac{k^{m-1}}{|\mu|\Gamma(\rho)}\int_0^K (K-\tau)^{\rho-1}\left|\varphi(\tau, y(\tau))\right| d\tau \\
&\quad + \frac{k^{m-1}}{|\mu|\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)} \\
&\qquad\qquad \times\left[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\right. \\
&\qquad\qquad\qquad \times\left|\varphi(w, y(w))\right| dw\, d\tau \\
&\qquad\qquad\qquad -\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} \\
&\qquad\qquad\qquad\quad \left.\left.\times\left|\varphi(w, y(w))\right| dw\, d\tau\right]\right\} \\
&\leq Z\upsilon\|y\|,
\end{aligned} \tag{31}$$

by regarding the condition $(\upsilon Z \leq 1)$, it follows that $\|Py\| \leq \|y\|$, $y \in \partial P_r$. Therefore, by theorem (3.3.4), the operator $P$ is admitted at least one FP, which means the problem (1) admitted at least one solution.

**Example 3.3.3:** Consider the following problem

$$\begin{aligned}
\varphi(k, y(k)) &= y\left(b^5 + y^4(k)\right)^{(1/5)} + 2\left(1+\cos\left(k^4+3\right)\right)^5 \\
&\quad \times\left(1-\cos y(k)\right), \quad y \neq 0, \ \text{b>0}.
\end{aligned} \tag{32}$$

If $y$ is small enough and if all its power are neglected then

$$\begin{aligned}
&\left| y\left(b^5 + y^4(k)\right)^{1/5} + 2\left(1+\cos\left(k^4+3\right)\right)^5\left(1-\cos y(k)\right)\right| \\
&\quad \leq b\,|y|.
\end{aligned} \tag{33}$$

Pick $b \le 1/Z$, the assumptions of theorem (3.3.5) is verified. Thus, from theorem (3.3.5) the problem (19) with $\varphi(k, y(k))$ is defined by (32) has at least a solution.

**Lemma 3.3.1:** (Nonlinear alternative for single valued maps)

Let Banach space $H$ be closed, convex subset $D$ of $H$, and $W$ an open subset of $D$ with $0 \in W$. Assume $F : \bar{W} \to D$ is continuous and compact mapping (i.e., $F(\bar{W})$ is a relatively compact subset of $D$) map, Then:

(i) $H$ has a FP in $\bar{W}$, otherwise

(ii) $\exists w \in \partial W$, $W \in D$ and $\gamma \in (0,1)$ s.t: $w = \gamma F(w)$.

**Theorem 3.3.5:** Consider the following assumptions

$(A_1)$ $\exists\ \sigma \in C([0,1], \mathbb{R})$ function and a nonlinear decreasing function $\mathrm{X}: \mathbb{R}^+ \to \mathbb{R}^+$ s.t:

$|\varphi(k, y)| \le \sigma(k)\mathrm{X}(\|y\|)$, for all $(k, y) \in [0, K] \times \mathbb{R}$;

$(A_2)$ $\exists$ a constant $M > 0$ s.t:

$$\frac{M}{\mathrm{X}(M)\mathrm{Z}\|\sigma\|} > 1. \tag{34}$$

Then problem (1) defined above has a solution with boundary value conditions on the interval $[0, K]$.

***Proof :*** Let us define the operator $P : \vartheta \to \vartheta$ as given in (13). Let us prove that the operator $P$ maps any bounded sets into another bounded sets in $C([0, K], \mathbb{R})$.

Consider $\iota > 0$, and $B_\iota = \{y \in C([0, K], \mathbb{R}) : \|y\| \le \iota\}$ a bounded set in $C([0, K], \mathbb{R})$.

$$\Rightarrow \|(Py)\|$$

$$\leq \sup_{k\in[0,K]}\left\{\frac{1}{\Gamma(\rho)}\int_0^k (k-\tau)^{\rho-1}\left|\varphi(\tau,y(\tau))\right|d\tau\right.$$

$$+\frac{k^{m-1}}{|\mu|\Gamma(\rho)}\int_0^K (K-\tau)^{\rho-1}\left|\varphi(\tau,y(\tau))\right|d\tau$$

$$+\frac{k^{m-1}}{|\mu|\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)}$$

$$\times\left[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\right.$$

$$\times\left|\varphi(w,y(w))\right|dw\,d\tau$$

$$-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}$$

$$\left.\times\left|\varphi(w,y(w))\right|dw\,d\tau\right]$$

$$\leq \mathrm{X}(\iota)\left\{\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{\rho+m-1}}{|\lambda|\Gamma(\rho+1)}\right.$$

$$\left.+\frac{K^{m-1}}{|\mu|}\sum_{j=1}^{n}\gamma_j\frac{\left(\eta_j^{\rho+\lambda_j}-\zeta_j^{\rho+\lambda_j}\right)}{\Gamma(\rho+\lambda_j+1)}\right\}\|\sigma\|. \tag{35}$$

We demonstrated that $F$ maps all bounded sets into equicontinuous sets $C([0,1],\mathbb{R})$ Let $k',k''\in[0,1]$ where $k'<k''$ and $y\in B_\iota$, with $B_\iota$ a bounded set which comes from $C([0,1],\mathbb{R})$:

$$\left|(Py)(k'')-(Py)(k')\right|$$

$$=\left|\frac{1}{\Gamma(\rho)}\int_0^{k''}(k''-\tau)^{\rho-1}\varphi(\tau,y(\tau))d\tau\right.$$

$$-\frac{1}{\Gamma(\rho)}\int_0^{k'}(k'-\tau)^{\rho-1}\varphi(\tau,y(\tau))d\tau$$

$$-\frac{\left[(k'')^{n-1}-(k')^{n-1}\right]}{\mu\Gamma(\rho)}\int_0^{T}(K-\tau)^{q-1}\left|\varphi(\tau,y(\tau))\right|d\tau$$

$$
\begin{aligned}
&+\frac{\left[(k'')^{n-1}-(k')^{n-1}\right]}{\mu\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)} \\
&\qquad\times\left[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\right. \\
&\qquad\qquad\times\left|\varphi(w,y(w))\right|dw\,d\tau \\
&\qquad-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1} \\
&\qquad\qquad\left.\times\left|\varphi(w,y(w))\right|dw\,d\tau\right]\Bigg| \\
&\le\frac{1}{\Gamma(\rho)}\int_0^{k'}\left|(k''-\tau)^{q-1}-(k'-\tau)^{q-1}\right|\mathrm{X}(r)\sigma(\tau)d\tau \\
&\quad+\frac{1}{\Gamma(\rho)}\int_{k'}^{k''}\left|k''-\tau\right|^{\rho-1}\mathrm{X}(r)\sigma(\tau)d\tau \\
&\quad+\frac{\left|(k'')^{n-1}-(k')^{n-1}\right|}{|\mu|\Gamma(\rho)}\int_0^{K}(K-\tau)^{\rho-1}\mathrm{X}(r)\sigma(\tau)d\tau \\
&\quad+\frac{\left|(k'')^{n-1}-(k')^{n-1}\right|}{|\mu|\Gamma(\rho)}\sum_{j=1}^{n}\frac{\gamma_j}{\Gamma(\lambda_j)} \\
&\qquad\times\left[\int_0^{\eta_j}\int_0^{\tau}(\eta_j-\tau)^{\lambda_j-1}(\tau-w)^{q-1}\mathrm{X}(r)\sigma(\tau)dw\,d\tau\right. \\
&\qquad\left.-\int_0^{\zeta_j}\int_0^{\tau}(\zeta_j-\tau)^{\lambda_j-1}(\tau-w)^{\rho-1}\mathrm{X}(r)\sigma(\tau)dw\,d\tau\right]\Bigg].
\end{aligned}
\tag{36}
$$

It is clear to see that the RHS of the inequality given above approaches zero independently of the variable $y\in B_\iota$ as $k''-k'\to 0$. As $P: C([0,K],\mathbb{R})\to C([0,K],\mathbb{R})$ is satisfied the assumptions above. It's follows from A-A theorem $P$ is completely continuous.

Assume $y$ a solution of the equation. Consider any real number $k\in[0,K]$, and perform similar computation as above, leads to

$$\begin{aligned}|y(k)| &= |\gamma(Py)(k)| \\ &\leq \mathrm{X}(r)\left\{\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{\rho+m-1}}{|\mu|\Gamma(\rho+1)}\right. \\ &\qquad \left.+\frac{K^{m-1}}{|\mu|}\sum_{j=1}^{n}\gamma_j\frac{\left(\eta_j^{\rho+\lambda_j}-\zeta_j^{\rho+\lambda_j}\right)}{\Gamma\left(\rho+\lambda_j+1\right)}\right\}\|\sigma\|.\end{aligned} \tag{37}$$

In consequence, we have

$$\frac{\|y\|}{\mathrm{X}(\|y\|)\mathrm{Z}\|\sigma\|}\leq 1. \tag{38}$$

Thus by $(A_2)$, there exists a constant $M$ such that $\|y\|\neq M$. Let us defined the set

$$\upsilon=\{y\in C([0,K],\mathbb{R}):\|y\|<\mathrm{M}+1\}. \tag{39}$$

The defined operator $P:\overline{U}\to C([0,K],\mathbb{R})$ is continuous moreover it is completely continuous operator. Based on $\upsilon$, choice, there does not exists $y\in\partial\upsilon|\ y=\gamma P(y)$ for some value of $\mu\in(0,1)$ Finally, the nonlinear alternative of Leray-Schauder type (Lemma 3.3.1), we conclude that the operator $P$ has a fixed point $y\in\overline{U}$ that fixed point is the solution of equation (1) as stated above. □

**Example 3.3.4:** Recall the Example (3.3.1) with its boundary conditions

$$\varphi(k,y(k))=\frac{1}{\sqrt{k+4}}\left(1+\frac{|y|}{1+|y|}\right)\leq\sigma(k)\mathrm{X}(\|y\|). \tag{40}$$

Then $\sigma(k)=1/\sqrt{k+4}$ and $X(\|y\|)=2$. By using $\|\sigma\|=1/2$, $Z \simeq 1.406972$, the condition $(A_2)$ leads us previously to $M>Z$. Thus all the assumptions provided on the Theorem (3.3.6) are satisfied. As conclusion, based on the theorem (3.3.6), the problem which is given by (19) with $\varphi(k,y(k))$ and given by (40) has a solution.

If the unbounded nonlinearity is chose by:

$$\varphi(k,y(k))=\frac{1}{\sqrt{k+4}}\left(1+\frac{|y|}{1+|y|}+\frac{|y|}{2}\right). \tag{41}$$

Then $\varphi(k,y(k))\le\sigma(t)X(\|y\|)$ with $\sigma(k)=1/\sqrt{k+4}$ and $X(\|y\|)=2+\|y\|/2$. By using the earlier arguments, with $\|\sigma\|=1/2$, $Z \simeq 1.406972$, we find that $M>M_1$, $M_1 \simeq 2.170392$. Hence, the problem (19) with $\varphi(k,y(k))$ which is given by (41) has at least one solution.

# Chapter 4

# NONLOCAL HADAMARD FRATIONAL INTEGRAL CONDITIONS FOR NONLINEAR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

## 4.1 Introduction

Existence and the uniqueness of the solution of nonlinear R-L Fractional type of differential equation was considered a nonlocal Hadamard Fractional integral boundary conditions which is defined as follows:

$$ {}_{RL}D^{\rho}\xi(k) = \mathrm{O}\big(k, \xi(k)\big), \quad k \in [0, K], \tag{1.1} $$

$$ \xi(0) = 0, \qquad \xi(K) = \sum_{j=1}^{m} \alpha_j \, {}_{H}I^{p_j} \xi(\eta_j), \tag{1.2} $$

whenever $1 < \rho \le 2$, ${}_{RL}D^{\rho}$ is recognized as standard R-L Fractional derivative of order $\rho$, ${}_{H}I^{p_j}$ recognized as Hadamard Fractional integral of order $p_j$, $p_j > 0$, $\eta_j \in (0, K)$, $\mathrm{O}: [0, K] \times \mathbb{R} \to \mathbb{R}$, and $\alpha_j \in \mathbb{R}$, $\mathrm{j} = 1, 2, \ldots, m$ are real constant such that

$$ \sum_{j=1}^{m} \frac{\alpha_j \eta_j^{\rho-1}}{(\rho-1)^{p_j}} \ne K^{\rho-1}. $$

There are several important and interesting results about the existence and the uniqueness of the solutions as well as the stabilities of the solutions. The analytic and numerical methods have both been used in recent research to investigate the solution of differential equation of fractional type.

Naturally, fractional-order operators are nonlocal with consideration of properties arising from many processing or phenomena. Also fractional calculus is a powerful tool for modeling a lot of real world situation or problems. However, it is obvious that the majority of work in the fields involve fractional derivatives of either RLFD or fractional derivatives of Caputo type. Beyond these two derivatives type, the HFD is another type of fractional derivative which was introduced by Hadamard year 1892 . In this chapter, the solution to the same problem as defined from the beginning is going to be approach based on Hadamard fractional derivative. The difference between these methods studied previously is going to be highlighted in the course of the chapter.

## 4.2 First Results

In this section, important results and definitions are introduced as well as some result that will be proved in the course of the work. The definitions are basically from the fractional calculus.

**Def 4.2.1:** The equation below is called the R-L Fractional derivative of real order $\rho > 0$. The derivative is defined on a continuous function $\text{O}: [0, \infty] \to \mathbb{R}$ as follows:

$$ {}_{RL}D^{\rho}\text{O}(k) = \frac{1}{\Gamma(m-\rho)} \left( \frac{d}{dk} \right)^{m} \int_{0}^{k} (k-\tau)^{m-\rho-1} \text{O}(\tau)\, d\tau, \quad \text{m-1} < \rho < m, $$

where $m = [\rho]+1$, $[\rho]$ (the integer part) and $\Gamma$ is recognized as the gamma function, it is defined as:

$$ \Gamma(\rho) = \int_{0}^{\infty} e^{-\tau} \tau^{\rho-1} d\tau. $$

**Def 4.2.2:** The R-L Fractional integral of order $\rho > 0$ is a continuous function $\text{O}: [0, \infty] \to \mathbb{R}$ is describe as

$$ {}_{RL}I^{\rho}\mathrm{O}(k)=\frac{1}{\Gamma(\rho)}\int_{0}^{k}(k-\tau)^{\rho-1}\mathrm{O}(\tau)\,d\tau. $$

**Def 4.2.3:** The Hadamard derivative of fractional order $\rho$ for a function $\mathrm{O}:[0,\infty]\to\mathbb{R}$ is express as:

$$ {}_{H}D^{\rho}\mathrm{O}(k)=\frac{1}{\Gamma(m-\rho)}\left(k\,\frac{d}{dk}\right)^{m}\int_{0}^{k}\left(\log\frac{k}{\tau}\right)^{m-\rho-1}\frac{\mathrm{O}(\tau)}{\tau}\,d\tau,\ \ m\text{ -}1<\rho<m,\ \ m=[\rho]+1, $$

where $\log(\cdot)=\log_{e}(\cdot)$, provided the integral exists.

**Def 4.2.4:** The Hadamard fractional integral of order $\rho\in\mathbb{R}^{+}$ for a function $\mathrm{O}(k)$, $\forall k>0$, is define as:

$$ {}_{H}I^{\rho}\mathrm{O}(k)=\frac{1}{\Gamma(\rho)}\int_{0}^{k}\left(\log\frac{k}{\tau}\right)^{\rho-1}\mathrm{O}(\tau)\,\frac{d\tau}{\tau}, $$

Provides the integral exists.

**Lemma 4.2.1:** Consider two positive real number $\rho>0$ ; $m>0$. The formula below hold:

$$ \left({}_{H}I^{\rho}\tau^{m}\right)(k)=m^{-\rho}k^{m}\text{ and }\left({}_{H}D^{\rho}\tau^{m}\right)(k)=m^{\rho}k^{m}. $$

**Lemma 4.2.2:** Consider a positive real number $\rho>0$ and $\xi\in C(0,K)\cap L(0,K)$. The following FDE

$$ {}_{RL}D^{\rho}\xi(k)=0 $$

has a unique solution

$$ \xi(k)=c_{1}k^{\rho-1}+c_{2}k^{\rho-2}+\cdots+c_{m}k^{\rho-m}, $$

where $c_j \in \mathbb{R},\ j=1,2,\dots,m,$ and $m-1<\rho<m.$

**Lemma 4.2.3:** Let $\rho>0.$ Then for $\xi \in C(0,K)\cap L(0,K)$ we have

$$_{RL}I^{\rho}\ _{RL}D^{\rho}\xi(k)=c_1k^{\rho-1}+c_2k^{\rho-2}+\cdots+c_mk^{\rho-m},$$

where $c_j \in \mathbb{R},\ j=1,2,\dots,m,$ and $m-1<\rho<m.$

**Lemma 4.2.4:** Let

$\sum_{j=1}^{m}\left(\left(\alpha_j\eta_j^{\rho-1}\right)/(\rho-1)^{p_j}\right)\neq K^{\rho-1},\ 1<\rho\leq 2,\ p_j>0,\ \alpha_j \in \mathbb{R},\ \eta_j\in(0,K),\ j=1,2,\dots$ $,m,$ and $\varphi \in ([0,K],\mathbb{R})$. It follows that the nonlocal HFI problem for the nonlinear R-L fractional differential equation

$$_{RL}D^{\rho}\xi(k)=\varphi(k),\quad 0<k<K, \tag{2.1}$$

with the boundary conditions

$$\xi(0)=0,\qquad \xi(K)=\sum_{j=1}^{m}\alpha_j\ _{H}I^{p_j}\xi(\eta_j), \tag{2.2}$$

has a unique solution given by

$$\xi(k)=\ _{RL}I^{\rho}\varphi(k)-\frac{k^{\rho-1}}{\mu}\left(_{RL}I^{\rho}\varphi(K)-\sum_{j=1}^{m}\alpha_j\left(_{H}I^{p_j}\ _{RL}I^{\rho}\varphi\right)(\eta_j)\right), \tag{2.3}$$

where

$$\mu:=K^{\rho-1}-\sum_{j=1}^{m}\frac{\alpha_j\eta_j^{\rho-1}}{(\rho-1)^{p_j}}\neq 0. \tag{2.4}$$

***Proof :*** By using lemmas (2.2)-(2.3), the equation (2.1) is equivalent to the integral equation below

$$\xi(k)=\ _{RL}I^{\rho}\varphi(k)-c_1k^{\rho-1}+c_2k^{\rho-2}, \tag{2.5}$$

for $c_1, c_2 \in \mathbb{R}$. The first condition of (2.2) implies that $c_2 = 0$. We considering the HFI type of order $p_j > 0$ for (2.5) it follows from the properties of the HFI which given by:

$$\left( {}_H I^{p_j} \tau^{\rho-1} \right)(k) = (\rho-1)^{-p_j} k^{\rho-1},$$

that

$${}_H I^{p_j} \xi(k) = \left( {}_H I^{P_j} {}_{RL} I^{\rho} \varphi \right)(k) - c_1 \left( {}_H I^{p_j} \tau^{\rho-1} \right)(k) = \left( {}_H I^{P_j} {}_{RL} I^{\rho} \varphi \right)(k) - c_1 \frac{k^{\rho-1}}{(\rho-1)^{p_j}}.$$

The second condition of (2.2) implies that

$${}_{RL} I^{\rho} \varphi(K) - c_1 K^{\rho-1} = \sum_{j=1}^{m} \alpha_j \left( {}_H I^{p_j} {}_{RL} I^{\rho} \varphi \right)(\eta_j) - c_1 \sum_{j=1}^{m} \frac{\alpha_j \eta_j^{\rho-1}}{(\rho-1)^{p_j}}.$$

Thus,

$$c_1 = \frac{1}{\mu} \left( {}_{RL} I^{\rho} \varphi(K) - \sum_{j=1}^{m} \alpha_j \left( {}_H I^{p_j} {}_{RL} I^{\rho} \varphi \right)(\eta_j) \right).$$

Substitute the values of $c_1$ and $c_2$ lead to the solution of (2.3). □

## 4.3 General Results

In order to ensure convenience, throughout this chapter the following expressions are used:

$${}_{RL} I^{\alpha} \mathrm{O}(\tau, \xi(\tau))(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-\tau)^{\alpha-1} \mathrm{O}(\tau, \xi(\tau)) d\tau, \quad z \in \{k, K\},$$

for $k \in [0, K]$ and

$$ {}_{H}I^{p_j}\, {}_{RL}I^{\alpha}\mathrm{O}(\tau,\xi(\tau))(\eta_j) = \frac{1}{\Gamma(p_j)\Gamma(\alpha)}\int_0^{\eta_j}\int_0^{\iota}\left(\log\frac{\eta_j}{\iota}\right)^{p_j-1} \times(\iota-\tau)^{\alpha-1}\frac{\mathrm{O}(\tau,\xi(\tau))}{\iota}d\tau d\iota, $$

where $\eta_j \in (0,K)$ for $j = 1,2,...,m$.

Let set $\mathrm{C} = C([0,K],\mathbb{R})$ be a Banach space (space of all continuous functions) define from the interval $[0,T]$ which has the norm given by $\|\xi\| = \sup_{k\in[0,K]}|\xi(k)|$. Similarly to the Lemma 2.4, the operator is define $\mathrm{A}: C \to C$ by

$$ (\mathrm{A}\xi)(k) = {}_{RL}I^{\rho}\mathrm{O}(\tau,\xi(\tau))(k) - \frac{k^{\rho-1}}{\mu}\left({}_{RL}I^{\rho}\mathrm{O}(\tau,\xi(\tau))(k) - \sum_{j=1}^{m}\alpha_j\left({}_{H}I^{p_j}\, {}_{RL}I^{\rho}\mathrm{O}(\tau,\xi(\tau))\right)(\eta_j)\right). \tag{3.1} $$

It is obviously that problem (1.1)-(1.2) possesses a solution provided that the operator $\mathrm{A}$ has a fixed point.

In what follow, for convenience, we defined a constant

$$ \Omega = \frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)} + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}|\alpha_j|\,\rho^{-p_j}\eta_j^{q}. \tag{3.2} $$

In the next subsection we show the existence, as well as the uniqueness results, for the BVP (1.1)-(1.2) based on a new formulation of the fixed point theorems.

### 4.3.1 Existence and Uniqueness Result Through BFP Theorem

**Theorem 4.3.1:** Let us assume that:

Assume $(H_1)$ the existence of a constant $L > 0$ s.t:

$$\left|\mathrm{O}(k,\xi) - \mathrm{O}(k,x)\right| \le L\left|\xi - x\right|, \text{ for each } k \in [0,K] \text{ and } \xi, x \in \mathbb{R}.$$

If

$$L\Omega < 1, \tag{3.3}$$

with $\Omega$ is expressed by (3.2), therefore, BVP (1.1)-(1.2) has a unique solution on $[0,K]$.

***Proof :*** The BVP (1.1)-(1.2) is written as a FP problem as follow $\xi = \mathrm{A}\xi$, with the operator $\mathrm{A}$ given by (3.1). The fixed point of operator $\mathrm{A}$, is the solution to the equation (1.1)-(1.2). From the Banach contraction mapping theorem, we will prove that the operator $\mathrm{A}$ has a unique fixed point.

Let $\sup_{k\in[0,K]}\left|\mathrm{O}(k,0)\right| = Q < \infty$, and we select the real value

$$\iota > \frac{Q\Omega}{1 - L\Omega}. \tag{3.4}$$

Now, we shall show that $\mathrm{A}B_\iota \subset B_\iota$, where $B_\iota = \{\xi \in C : \|\xi\| \le \iota\}$. For any $\xi \in B_\iota$, it follow that

$$\begin{aligned}
\left|(\mathrm{A}\xi)(k)\right| \le \sup_{k\in[0,K]} \Bigg\{ & {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))\right|(k) + \frac{k^{\rho-1}}{|\mu|}\, {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))\right|(K) \\
& + \frac{k^{\rho-1}}{|\mu|}\sum_{j=1}^{m}\left|\alpha_j\right| {}_{H}I^{p_j}\, {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))\right|(\eta_j)\Bigg\} \\
\le {}& {}_{RL}I^{\rho}\left(\left|\mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,0)\right| + \left|\mathrm{O}(\tau,0)\right|\right)(K) \\
& + \frac{K^{\rho-1}}{|\mu|}\, {}_{RL}I^{\rho}\left(\left|\mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,0)\right| + \left|\mathrm{O}(\tau,0)\right|\right)(K) \\
& + \frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}\left|\alpha_j\right| {}_{H}I^{p_j}\, {}_{RL}I^{\rho}\left(\left|\mathrm{O}(\tau,y(\tau)) - \mathrm{O}(\tau,0)\right| + \left|\mathrm{O}(\tau,0)\right|\right)(\eta_j)
\end{aligned}$$

$$\leq (L\|\xi\|+Q)\, {}_{RL}I^{\rho}(1)(K)+(L\|\xi\|+Q)\frac{K^{\rho-1}}{|\mu|}\, {}_{RL}I^{\rho}(1)(K)$$

$$+(L\|\xi\|+Q)\frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\, {}_{H}I^{p_j}\, {}_{RL}I^{\rho}(1)(\eta_j)$$

$$=(L\iota+Q)\left(\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)}+\frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}|\alpha_j|\rho^{-p_j}\eta_j^{\rho}\right)$$

$$=(L\iota+Q)\Omega\leq\iota,$$

that implies $\mathrm{A}B_{\iota}\in B_{\iota}$.

Next, we let $\xi, x\in C$. Then for $k\in[0,K]$, we have

$$|(\mathrm{A}\xi)(k)-(\mathrm{A}x)(k)|$$

$$\leq {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))-\mathrm{O}(\tau,x(\tau))\right|(k)$$

$$+\frac{K^{\rho-1}}{|\lambda|}\, {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))-\mathrm{O}(\tau,x(\tau))\right|(K)$$

$$+\frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\, {}_{H}I^{p_j}\, {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))-\mathrm{O}(\tau,x(\tau))\right|(\eta_j)$$

$$\leq L\left(\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)}+\frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}|\alpha_j|\rho^{-p_j}\eta_j^{\rho}\right)\|\xi-x\|$$

$$=L\Omega\|\xi-x\|,$$

which implies that $\|\mathrm{A}\xi-\mathrm{A}x\|\leq L\Omega\|\xi-x\|$. As $L\Omega<1$, A is a contraction. The B'sFP theorem is used to conclude the operator A possesses one and only one fixed point. This FP is exactly the solution of the BVP given by (1.1)-(1.2). □

**4.3.2 Existence, Uniqueness of Fixed Point Through Banach's Fixed Point and Holder Inequality**

**Theorem 4.3.2:** Suppose $\mathrm{O}:[0,K]\times\mathbb{R}\to\mathbb{R}$ a continuous function which satisfies the following condition:

$(H_2)$ $|\mathrm{O}(k,\xi)-\mathrm{O}(k,x)|\leq\gamma(k)|\xi-x|$, for $k\in[0,K]$, $\xi, x\in\mathbb{R}$ and $\gamma\in L^{\frac{1}{\sigma}}([0,K],$

$\mathbb{R}), \sigma\in(0,1)$.

Denote $\|\gamma\| = \left( \int_0^K |\gamma(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma}$.

If

$$\|\gamma\| \left\{ \frac{K^{\rho-\sigma}}{\Gamma(\rho)} \left( \frac{1-\sigma}{\rho-\sigma} \right)^{1-\sigma} + \frac{K^{2\rho-\sigma-1}}{|\mu|\Gamma(\rho)} \left( \frac{1-\sigma}{\rho-\sigma} \right)^{1-\sigma} + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho)} \left( \frac{1-\sigma}{\rho-\sigma} \right)^{1-\sigma} \sum_{j=1}^{m} |\alpha_j| \rho^{-p_j} \eta_j^{\rho-\sigma} \right\} < 1,$$

thus the BVP (1.1)-(1.2) has a unique solution.

***Proof :*** $\forall\ \xi, x \in C([0,K],\mathbb{R})$ and $\forall k \in [0,K]$, by Holder's inequality we have:

$$\begin{aligned}
&\left|(\mathrm{A}\xi)(k) - (\mathrm{A}x)(k)\right| \\
&\leq {}_{RL}I^{\rho} \left| \mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,x(\tau)) \right|(k) \\
&\quad + \frac{K^{\rho-1}}{|\mu|} {}_{RL}I^{\rho} \left| \mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,x(\tau)) \right|(K) \\
&\quad + \frac{K^{\rho-1}}{|\mu|} \sum_{j=1}^{m} |\alpha_j| {}_{H}I^{p_j} {}_{RL}I^{\rho} \left| \mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,x(\tau)) \right|(\eta_j) \\
&= \frac{1}{\Gamma(\rho)} \int_0^k (k-\tau)^{\rho-1} \delta(\tau) |\xi(\tau) - x(\tau)| d\tau \\
&\quad + \frac{K^{\rho-1}}{|\mu|} \int_0^K (K-\tau)^{\rho-1} \delta(\tau) |\xi(\tau) - x(\tau)| d\tau \\
&\quad + \frac{K^{\rho-1}}{|\mu|} \sum_{j=1}^{m} \frac{|\alpha_j|}{\Gamma(p_j)\Gamma(\alpha)} \int_{0_+}^{\eta_j} \int_{0_+}^{\tau} \left( \log \frac{\eta_j}{\tau} \right)^{p_j-1} (\tau-\iota)^{\rho-1} \gamma(\iota) |\xi(\iota) - x(\iota)| d\iota \frac{d\tau}{\tau} \\
&\leq \frac{1}{\Gamma(\rho)} \left( \int_0^k \left( (k-\tau)^{\rho-1} \right)^{\frac{1}{1-\sigma}} d\tau \right)^{1-\sigma} \left( \int_0^k (\gamma(\tau))^{\frac{1}{\sigma}} d\tau \right)^{\sigma} \|\xi - x\| \\
&\quad + \frac{K^{\rho-1}}{|\mu|} \left( \int_0^K \left( (K-\tau)^{\rho-1} \right)^{\frac{1}{1-\sigma}} d\tau \right)^{1-\sigma} \left( \int_0^K (\gamma(\tau))^{\frac{1}{\sigma}} d\tau \right)^{\sigma} \|\xi - x\| \\
&\quad + \frac{K^{\rho-1}}{|\mu|} \sum_{j=1}^{m} \frac{|\alpha_j|}{\Gamma(\rho)\Gamma(p_j)} \int_0^{\eta_j} \left( \log \frac{\eta_j}{\tau} \right)^{p_j-1} \left( \int_0^{\tau} \left( (\tau-\iota)^{\rho-1} \right)^{\frac{1}{1-\sigma}} d\iota \right)^{1-\sigma} \\
&\qquad \times \left( \int_0^{\tau} (\gamma(\iota))^{\frac{1}{\sigma}} d\iota \right)^{\sigma} \frac{d\tau}{\tau} \|\xi - x\|
\end{aligned}$$

$$\le \|\gamma\| \frac{K^{\rho-\sigma}}{\Gamma(\rho)}\left(\frac{1-\sigma}{\rho-\sigma}\right)^{1-\sigma} \|\xi - x\| + \|\gamma\| \frac{K^{2\rho-\sigma-1}}{|\mu|\Gamma(\rho)}\left(\frac{1-\sigma}{\rho-\sigma}\right)^{1-\sigma} \|\xi - x\|$$

$$+ \|\gamma\| \frac{K^{\rho-1}}{|\mu|\Gamma(\rho)}\left(\frac{1-\sigma}{\rho-\sigma}\right)^{1-\sigma} \sum_{j=1}^{m} \frac{|\alpha_j|}{\Gamma(p_j)} \int_0^{\eta_j} \left(\log \frac{\eta_j}{\tau}\right)^{p_i-1} \tau^{\rho-\sigma} \frac{d\tau}{\tau} \|\xi - x\|$$

$$\le \|\gamma\| \left\{ \frac{K^{\rho-\sigma}}{\Gamma(\rho)}\left(\frac{1-\sigma}{\rho-\sigma}\right)^{1-\sigma} + \frac{K^{2\rho-\sigma-1}}{|\mu|\Gamma(\rho)}\left(\frac{1-\sigma}{\rho-\sigma}\right)^{1-\sigma} \right.$$

$$\left. + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho)}\left(\frac{1-\sigma}{\rho-\sigma}\right)^{1-\sigma} \sum_{j=1}^{m} |\alpha_j| (\rho-\sigma)^{p_j} \eta_j^{\rho-\sigma} \right\} \|\xi - x\|.$$

The BFP theorem is used to conclude that the operator A possesses one and only one fixed point. The solution of the BVP described by (1.1)-(1.2) is the fixed point. □

**4.3.3 Existence and Uniqueness of the Solution Through Nonlinear Contractions**

**Def 4.3.1:** Let $E$ be a Banach space and $\mathrm{A}: E \to E$ be a mapping. $\mathrm{A}$ is nonlinear contraction mapping if there is a continuous and nondecreasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ s.t: $\Psi(0) = 0$ and $\Psi(\varepsilon) = \varepsilon,\ \varepsilon > 0$ with the property:

$$\|\mathrm{A}\xi - \mathrm{A}x\| \le \Psi(\|\xi - x\|), \quad \forall \xi, x \in \mathrm{E}.$$

**Lemma 4.3.1:** Consider the space $E$ a Banach space and $\mathrm{A}: E \to E$ a nonlinear contraction then the mapping $\mathrm{A}$ has a unique FP in the Banach space $E$.

**Theorem 4.3.3:** Let $\mathrm{O}: [0, K] \times \mathbb{R} \to \mathbb{R}$ be a continuous mapping that satisfied the assumption:

$(H_3)$ $\quad |\mathrm{O}(k,\xi) - \mathrm{O}(k,x)| \le \varphi(k) \dfrac{|\xi - x|}{H^* + |\xi - x|}$, for $k \in [0, K]$, $\xi, x \ge 0$, where $\varphi: [0, K] \to \mathbb{R}^+$ is continuous with the constant $H^*$ defined as:

$$H^{*} := {}_{RL}I^{\rho}\varphi(K) + \frac{K^{\rho-1}}{|\mu|}\,{}_{RL}I^{\rho}\varphi(K) + \frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\,{}_{RL}I^{\rho}\varphi(\eta_j).$$

It follows that the BVP (1.1)-(1.2) has a unique solution on the said interval above.

***Proof :*** Define the operator $\mathrm{A}: C \to C$ as in (3.1) and a continuous nondecreasing function $\Psi: \mathbb{R}^{+} \to \mathbb{R}^{+}$ by

$$\Psi(\varepsilon) = \frac{H^{*}\varepsilon}{H^{*}+\varepsilon}, \quad \forall \varepsilon \geq 0.$$

It is observed that the function $\Psi$ satisfies $\Psi(0)=0$ and $\Psi(\varepsilon) < \varepsilon,\ \forall \varepsilon > 0.$

$\forall\ \xi, x \in C$ and for each $k \in [0, K]$,

$$\begin{aligned}
\Rightarrow &\left|(\mathrm{A}\xi)(k) - (\mathrm{A}x)(k)\right| \\
&\leq {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,x(\tau))\right|(k) \\
&\quad + \frac{K^{\rho-1}}{|\mu|}\,{}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,x(\tau))\right|(K) \\
&\quad + \frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\,{}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau)) - \mathrm{O}(\tau,x(\tau))\right|(\eta_j) \\
&\leq {}_{RL}I^{\rho}\left(\varphi(\tau)\frac{|\xi - x|}{H^{*}+|\xi - x|}\right)(K) + \frac{K^{\rho-1}}{|\mu|}\,{}_{RL}I^{\rho}\left(\varphi(\tau)\frac{|\xi - x|}{H^{*}+|\xi - x|}\right)(K) \\
&\quad + \frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\,{}_{RL}I^{\rho}\left(\varphi(\tau)\frac{|\xi - x|}{H^{*}+|\xi - x|}\right)(\eta_j) \\
&\leq \frac{\Psi(\|\xi - x\|)}{H^{*}}\left({}_{RL}I^{\rho}\varphi(K) + \frac{K^{\rho-1}}{|\mu|}\,{}_{RL}I^{\rho}\varphi(K) + \frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\,{}_{RL}I^{\rho}\varphi(\eta_j)\right) \\
&= \Psi(\|\xi - x\|).
\end{aligned}$$

It follows that $\|\mathrm{A}\xi - \mathrm{A}x\| \leq \Psi(\|\xi - x\|)$. Therefore, operator A is nonlinear contraction. The lemma (4.3.1) is then used to conclude that the operator A has one and only one FP. The solution of the BVP describe by (1.1)-(1.2) is the fixed point.□

**4.3.4 Existence of Solution Through Krasnoselskii's Fixed Point Theorem (KFP)**

**Theorem 4.3.4:** Consider the function $O:[0,K]\times\mathbb{R}\to\mathbb{R}$ to be continuous and satisfying the hypothesis $(H_1)$. To that, let us add:

$(H_4)$ $|O(k,\xi)|\le\omega(k)$, $\forall(k,\xi)\in\mathbb{R}$, and $\omega\in C([0,K]\times\mathbb{R}^+)$.

Thus the BVP (1.1)-(1.2) possesses at least on solution on the interval $[0,K]$ is given by

$$L\left(\frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)}+\frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}|\alpha_j|\rho^{-p_j}\eta_j^{\rho}\right)<1. \tag{3.5}$$

***Proof :*** We set $\sup_{k\in[0,K]}\omega(k)=\|\omega\|$ and choosing

$$\delta\ge\|\omega\|\Omega, \tag{3.6}$$

(with $\Omega$ given as in (3.2)), let us consider the set $B_\delta=\{\xi\in C([0,K],\mathbb{R}):\|\xi\|\le\delta\}$.

The following two operators $A_1$ and $A_2$ are defined on $B_\delta$ by

$$A_1\xi(k)={}_{RL}I^{\rho}O(\tau,\xi(\tau))(k),\ k\in[0,K],$$
$$A_2\xi(k)=-\frac{k^{\rho-1}}{\mu}\left({}_{RL}I^{\rho}O(\tau,\xi(\tau))(K)-\sum_{j=1}^{m}\alpha_j\left({}_{H}I^{p_j}\ {}_{RL}I^{\rho}O(\tau,\xi(\tau))\right)(\eta_j)\right),$$
$$k\in[0,K].$$

For any $\xi,x\in B_\delta$, we have

$$|A_1\xi(k)+A_2x(k)|$$
$$\le\sup_{k\in[0,K]}\left\{{}_{RL}I^{\rho}|O(\tau,\xi(\tau))|(k)+\frac{k^{\rho-1}}{|\mu|}{}_{RL}I^{\rho}|O(\tau,x(\tau))|(K)\right.$$
$$\left.+\frac{k^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\ {}_{RL}I^{\rho}|O(\tau,x(\tau))|(\eta_j)\right\}$$

$$\leq \|\omega\| \left( \frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)} + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)} \sum_{j=1}^{m} |\alpha_j| \rho^{-p_j} \eta_j^{\rho} \right)$$
$$= \|\omega\| \Omega \leq \delta.$$

It shows that $A_1\xi + A_2 x \in B_\delta$. It is obvious to see by using (3.5) that $A_2$ is a contraction mapping.

$A_1$ is continuous because $O$ is continuous. On the other hand, $A_1$ is uniformly bounded on $B_\delta$ because

$$\|A_1\xi\| \leq \frac{K^{\rho}}{\Gamma(\rho+1)} \|\omega\|.$$

Let us now prove that the operator $A_1$ is compact.

Consider $\sup_{(k,y)\in[0,K]\times B_\rho} |O(k,\xi)| = \bar{O} < \infty,$ it follows that

$$\left|(A_1\xi)(k_2) - (A_1\xi)(k_1)\right| = \frac{1}{\Gamma(\rho)} \left| \int_0^{k_1} (k_2-\tau)^{\rho-1} (k_1-\tau)^{\rho-1} O(\tau,\xi(\tau)) d\tau + \int_{k_1}^{k_2} (k_2-\tau)^{\rho-1} O(\tau,\xi(\tau)) d\tau \right|$$
$$\leq \frac{\bar{O}}{\Gamma(\rho+1)} \left| k_1^{\rho} - k_2^{\rho} \right|,$$

which holds independently of $y \to 0$ as $k_2 \to k_1$. Thus, $A_1$ is equicontinuous.

This means that the operator $A_1$ is relatively compact on the interval $B_\delta$, by A-A theorem, the operator $A_1$ is compact on the interval $B_\delta$. These hold with the conditions of the lemma (4.3.2). It follows by the lemma that the BVP (1.1)-(1.2) has at least one solution on the interval $[0, K]$.

### 4.3.5 Existence of the Solution Through Leray-Schauder's Nonlinear Alternative

**Theorem 4.3.5:** Assume that

$(H_5)$ $\exists$ a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a function $\mathrm{T} \in C([0,K] \times \mathbb{R}^+)$ such that

$$\left|\mathrm{O}(k,w)\right| \le \mathrm{T}(k)\psi(\|\xi\|), \text{ for each } (k,\xi) \in [0,K] \times \mathbb{R};$$

$(H_6)$ $\exists$ a constant $Q > 0$ s.t:

$$\frac{Q}{\psi(Q)\|\mathrm{T}\|\Omega} > 1,$$

where $\Omega$ given by (3.2).

The BVP (1.1)-(1.2) has at least one solution on the interval $[0, K]$.

***Proof :*** Consider the mapping $\mathrm{A}$ to be defined as in (3.1). Let us prove first that the operator $\mathrm{A}$ is a mapping of bounded sets into bounded sets in $C([0,K],\mathbb{R})$. $\forall \iota > 0$, let $B_\iota = \{\xi \in C([0,K],\mathbb{R}) : \|\xi\| \le \iota\}$ be a bounded ball in $C([0,K],\mathbb{R})$. Then for $k \in [0, K]$ we have

$$\left|(\mathrm{A}\xi)(k)\right| \le \sup_{k\in[0,K]} \Bigg\{ {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))\right|(k) + \frac{k^{\rho-1}}{|\mu|}\, {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))\right|(K)$$

$$+\frac{k^{\rho-1}}{|\mu|}\sum_{j=1}^{m}\left|\alpha_j\right| {}_{H}I^{p_j}\, {}_{RL}I^{\rho}\left|\mathrm{O}(\tau,\xi(\tau))\right|(\eta_j)\Bigg\}$$

$$\le \psi(\|\xi\|)\, {}_{RL}I^{\rho}\mathrm{T}(\tau)(K) + \psi(\|\xi\|)\frac{K^{\rho-1}}{|\mu|}\, {}_{RL}I^{\rho}\mathrm{O}(\tau,\xi(\tau))(K)$$

$$+\psi(\|\xi\|)\frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}\left|\alpha_j\right| {}_{H}I^{p_j}\, {}_{RL}I^{\rho}\mathrm{T}(\tau)(\eta_j)$$

$$\leq \psi\left(\|\xi\|\right)\|T\|\left(\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)}+\frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}|\alpha_j|\rho^{-p_j}\eta_j^{\rho}\right),$$

and consequently,

$$\|A\xi\| \leq \psi(\iota)\|T\|\Omega.$$

Consequently, we show that operator A maps every bounded sets into equicontinuous set of $C([0,K],\mathbb{R})$. Let $\nu_1,\nu_2 \in [0,K]$ with $\nu_1<\nu_2$ and $\xi \in B_\iota$. It follows that

$$\begin{aligned}
&\left|(A\xi)(\nu_2)-(A\xi)(\nu_1)\right| \\
&\leq \frac{1}{\Gamma(\rho)}\left|\int_0^{\nu_1}\left[(\nu_2-\tau)^{\rho-1}-(\nu_1-\tau)^{\rho-1}\right]O(\tau,\xi(\tau))d\tau+\int_{\nu_1}^{\nu_2}(\nu_2-\tau)^{\rho-1}O(\tau,\xi(\tau))d\tau\right| \\
&+\frac{\left(\nu_1^{\rho-1}-\nu_2^{\rho-1}\right)}{|\mu|}\,{}_{RL}I^{\rho}\left|O(\tau,\xi(\tau))\right|(K) \\
&+\frac{\left(\nu_1^{\rho-1}-\nu_2^{\rho-1}\right)}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\,{}_{RL}I^{\rho}\left|O(\tau,\xi(\tau))\right|(\eta_j) \\
&\leq \frac{1}{\Gamma(\rho)}\left|\int_0^{\nu_1}\left[(\nu_2-\tau)^{\rho-1}-(\nu_1-\tau)^{\rho-1}\right]T(\tau)d\tau+\int_{\nu_1}^{\nu_2}(\nu_2-\tau)^{\rho-1}(\tau)d\tau\right| \\
&+\frac{\left(\nu_1^{\rho-1}-\nu_2^{\rho-1}\right)\psi(\iota)}{|\mu|}\,{}_{RL}I^{\rho}\left|T(\tau)\right|(K)+\frac{\left(\nu_1^{\rho-1}-\nu_2^{\rho-1}\right)\psi(\iota)}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\,{}_{RL}I^{\rho}\left|T(\tau)\right|(\eta_j).
\end{aligned}$$

As $\nu_2-\nu_1 \to 0$, the RHS of the inequality above approaches zero not depending on the parameter $\xi \in B_\iota$. It follows from A-A theorem that operator A: $C([0,K],\mathbb{R}) \to C([0,K],\mathbb{R})$ is completely continuous.

Consider $\xi$ to be a solution. $\forall k \in [0,K]$, similar to the previous computations,

$$|\xi(k)| \leq \psi\left(\|\xi\|\right)\|T\|\Omega,$$

which leads to

$$\frac{\|\xi\|}{\psi(\|\xi\|)\|T\|\Omega} \leq 1.$$

Based on $(H_6)$, $\exists$ a real value $M$ such that $\|\xi\| \neq M$. Assume that

$$W = \{\xi \in C([0,K],\mathbb{R}) : \|\xi\| \leq M\}.$$

It is clear to observe that $A: \overline{W} \to C([0,K],\mathbb{R})$ is continuous , moreover, it is completely continuous. A suitable choice of the set $W$, implies that there is no $\xi \in \partial W$ such that $\xi = \mu A\xi$. By choosing $\mu \in (0,1)$, the nonlinear alternative of Leray-Schauder type is enough to conclude that the operator A has a fixed point $\xi \in \overline{U}$ . The FP of the unique solution to the BVP (1.1)-(1.2). □

#### 4.3.6 Existence Result Through Leray-Schauder's Degree Theory

**Theorem 4.3.6:** Let $O: [0,K] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that

$(H_5)$ $\exists$ a constants $\kappa$, $0 < \kappa < \Omega^{-1}$ and $Q > 0$ s.t:

$$|O(k,\xi)| \leq \kappa|\xi| + Q \text{ for all } (k,\xi) \in [0,K] \times \mathbb{R},$$

$\Omega$ is as defined by (3.2).

Thus BVP (1.1)-(1.2) has at least one solution on $[0,K]$.

***Proof :*** Define an operator $A: C \to C$ as in (3.1), in view of the FP problem

$$\xi = A\xi. \tag{3.7}$$

Let demonstrate the existence of at least one solution $\xi \in C[0,K]$ that satisfied (3.7). Consider $B_1 \subset C[0,K]$, as:

$$B_{\mathrm{I}} = \left\{ \xi \in C : \max_{k \in [0,K]} |\xi| < \mathrm{I} \right\},$$

with $\mathrm{I} > 0$. Let us prove that $\mathrm{A} : \bar{B}_{\mathrm{I}} \to C[0,K]$ satisfies the conditions

$$\xi \neq \theta \mathrm{A} \xi, \quad \forall \xi \in \partial \mathrm{B}_{\mathrm{I}}, \ \forall \theta \in [0,1]. \tag{3.8}$$

We set

$$H(\theta,\xi) = \theta \mathrm{A}\xi, \ \ \xi \in \mathrm{C}, \ \ \theta \in [0,1].$$

As proved in the Theorem 3.6 , operator A is continuous, equicontinuous, and uniformly bounded. Therefore, by A-A theorem, a continuous map $\varphi_\theta$ defined by $\varphi_\theta(\xi) = \xi - H(\theta,\xi) = \xi - \theta \mathrm{A}\xi$ is completely continuous. When the equation (3.8) holds, it follows from the Leray-Schauder degree are well defined, and by the homotopy invariance of topological degree we have the relation

$$\begin{aligned}\deg(\varphi_\theta, B_R, 0) &= \deg(I - \theta \mathrm{A}, B_{\mathrm{I}}, 0) \\ &= \deg(\varphi_0, B_{\mathrm{I}}, 0) = \deg(I, B_{\mathrm{I}}, 0) = 1 \neq 0, \ \ 0 \in B_{\mathrm{I}},\end{aligned}$$

where $I$ is unit operator. The nonzero property of the Leray-schauder degree leads us to $\varphi_1(\xi) = \xi - A\xi = 0$ that holds for at least one $\xi \in B_{\mathrm{I}}$. Assume that $\xi = \theta \mathrm{A}\xi$ for a real value $\theta \in [0,1]$ and $\forall k \in [0,K]$ s.t:

$$\begin{aligned}|\xi(k)| &= |\theta(\mathrm{A}\xi)(k)| \\ &\leq {}_{RL}I^{\rho} |\mathrm{O}(\tau,\xi(\tau))|(k) + \frac{k^{\rho-1}}{|\mu|} {}_{RL}I^{\rho} |\mathrm{O}(\tau,\xi(\tau))|(K) \\ &\quad + \frac{k^{\rho-1}}{|\mu|} \sum_{j=1}^{m} |\alpha_j| \, {}_{H}I^{p_j} \, {}_{RL}I^{\rho} |\mathrm{O}(\tau,\xi(\tau))|(\eta_j) \\ &\leq (\kappa|\xi| + Q) \, {}_{RL}I^{\rho}(1)(K) + (\kappa|\xi| + Q) \frac{K^{\rho-1}}{|\mu|} \, {}_{RL}I^{\rho}(1)(K)\end{aligned}$$

$$+\left(\kappa|\xi|+Q\right)\frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}\left|\alpha_j\right|{}_{H}I^{p_j}{}_{RL}I^{\rho}\mathrm{T}(\tau)(\eta_j)$$

$$\leq\left(\kappa|\xi|+Q\right)\left(\frac{K^{\rho}}{\Gamma(\rho+1)}+\frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)}+\frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}\left|\alpha_j\right|\rho^{-p_j}\eta_j^{\rho}\right)$$

$$\leq\left(\kappa|\xi|+Q\right)\Omega,$$

which, on taking the norm $\sup_{k\in[0,K]}\left|\xi(k)\right|=\|\xi\|$ and solving the equation for $\|\xi\|$, yields

$$\|\xi\|\leq\frac{\Omega Q}{1-\Omega\kappa}.$$

If $\mathrm{I}=\frac{\Omega Q}{1-\Omega\kappa}+1,$ the inequality given by (3.8) holds. □

## 4.4 Examples

The illustrations of our results is given via some examples in this section

**Example 4.4.1:** Consider a nonlocal Hadamard FICs for nonlinear R-L Fractional differential equation:

$$\begin{cases}{}_{RL}D^{\frac{3}{2}}\xi(k)=\dfrac{\sin^2(\pi k)}{\left(e^k+3\right)^2}\cdot\dfrac{\left|\xi(k)\right|}{\left|\xi(k)\right|+1}+\dfrac{\sqrt{3}}{2},\quad k\in[0,3],\\ \xi(0)=0,\quad \xi(3)+\sqrt{5}\,{}_{H}I^{1/2}\xi\left(\dfrac{9}{4}\right)=\dfrac{4}{5}\,{}_{H}I^{\sqrt{2}}\xi\left(\dfrac{3}{4}\right)+\dfrac{\sqrt{3}}{2}\,{}_{H}I^{\pi}\xi\left(\dfrac{3}{2}\right).\end{cases}\qquad(4.1)$$

Here $\rho=3/2,\ \mathrm{m}=3,\ \mathrm{K}=3, \alpha_1=4/5,\ \alpha_2=\sqrt{3}/2,\ \alpha_3=-\sqrt{5},\ p_1=\sqrt{2},\ p_2=\pi,\ p_3=1/2, \eta_1=3/4, \eta_2=3/2, \eta_3=9/4$, and $\mathrm{O}(k,\xi)=\left(\sin^2(\pi k)\big/\left(e^k+3\right)^2\right)\left(|\xi|\big/\left(1+|\xi|\right)\right).$

Since $\left|\mathrm{O}(k,\xi)-\mathrm{O}(k,x)\right|\leq(1/16)\left|\xi-x\right|$, $(H_1)$ is satisfied with $L=1/16.$ By using a Maple program, we can find that

$$\Omega := \frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)} + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)} \sum_{j=1}^{m} |\alpha_j| \rho^{-p_j} \eta_j^{\rho} \approx 7.2399010270.$$

Therefore, $L\Omega \approx 0.4524938142 < 1.$ Consequently, the BVP (4.1) has unique solution on the closed interval $[0,3]$ according to theorem (4.3.1).

**Example 4.4.2:** Consider the nonlocal Hadamard FICs for nonlinear R-L Fractional differential equation:

$$\begin{cases} {}_{RL}D^{\frac{4}{3}}\xi(k) = \dfrac{e^k}{e^k+8} \cdot \dfrac{|\xi(k)|}{|\xi(k)|+2} + 1, \quad k \in \left[0, \dfrac{3}{2}\right], \\ \xi(0) = 0, \\ \xi\left(\dfrac{3}{2}\right) + \dfrac{2}{3} {}_{H}I^{\sqrt{2}/2}\xi\left(\dfrac{3}{5}\right) + \pi \, {}_{H}I^{\sqrt{3}}\xi\left(\dfrac{6}{5}\right) = \dfrac{1}{5} {}_{H}I^{1/4}\xi\left(\dfrac{3}{10}\right) + \dfrac{1}{\sqrt{3}} {}_{H}I^{6/5}\xi\left(\dfrac{9}{10}\right). \end{cases} \tag{4.2}$$

Here

$\rho = 4/3,\ \mathrm{m} = 4,\ \mathrm{K} = 3/2,\ \alpha_1 = 1/5,\ \alpha_2 = -2/3,\ \alpha_3 = 1/\sqrt{3},\ \alpha_4 = -\pi/2,\ p_1 = 1/4$

$p_2 = \sqrt{2}/2,\ p_3 = 6/5,\ p_4 = \sqrt{3},\ \eta_1 = 3/10,\ \eta_2 = 3/5,\ \eta_3 = 9/10,$ and $\eta_4 = 6/5.$

Since $|\mathrm{O}(k,\xi) - \mathrm{O}(k,x)| = \left(2e^k/(e^k+8)\right)|\xi - x|,$ then $(H_2)$ satisfies with $\gamma(k)$ $= 2e^k/(e^k+8)$ and $\sigma = 1/2,$ with the help of Maple program, we show that:

$$\begin{aligned} &\|\gamma\| \left\{ \frac{K^{\rho-\sigma}}{\Gamma(\rho)} \left( \frac{1-\sigma}{\rho-\sigma} \right)^{1-\sigma} + \frac{K^{2\rho-\sigma-1}}{|\mu|\Gamma(\rho)} \left( \frac{1-\sigma}{\rho-\sigma} \right)^{1-\sigma} \right. \\ &\left. + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho)} \left( \frac{1-\sigma}{\rho-\sigma} \right)^{1-\sigma} \sum_{j=1}^{m} |\alpha_j| \rho^{-p_j} \eta_j^{\rho-\sigma} \right\} \|\xi - x\| \\ &\approx 0.9380422264 < 1. \end{aligned}$$

It follows directly from the theorem (4.3.2), the BVP (4.2) has a unique solution on the interval $[0,3/2]$.

**Example 4.4.3:** Consider the nonlocal Hadamard FICs for nonlinear R-L Fractional differential equation:

$$\begin{cases} {}_{RL}D^{\frac{7}{6}}\xi(k)=\dfrac{k}{(k+2)^2}\cdot\dfrac{|\xi(k)|}{|\xi(k)|+1}+3k+\dfrac{4}{5}, \quad k\in[0,2], \\ \xi(0)=0, \quad \xi(2)=2\,{}_{H}I^{\sqrt{\pi}}\xi\left(\dfrac{2}{5}\right)+\dfrac{2}{3}\,{}_{H}I^{5/4}\xi\left(\dfrac{4}{3}\right)+\sqrt{3}\,{}_{H}I^{3/7}\xi\left(\dfrac{3}{2}\right). \end{cases} \tag{4.3}$$

Here

$\rho=7/6,\ \mathrm{m}=3,\ K=2,\ \alpha_1=2,\ \alpha_2=2/3,\ \alpha_3=\sqrt{3},\ p_1=\sqrt{\pi},\ p_2=5/4,\ p_3=3/7,$
$\eta_1=2/5,\ \eta_2=4/3,\ \eta_3=3/2,$ and $\mathrm{O}(k,\xi)=\left(k^2|\xi|\big/(k+2)^2\right)(|\xi|+1)+3k+(4/5),$

pick $\varphi(k)=k^2/4$ and

$$\begin{aligned} H^* &:= {}_{RL}I^{\rho}\varphi(K)+\frac{K^{\rho-1}}{|\mu|}\,{}_{RL}I^{\rho}\varphi(K)+\frac{K^{\rho-1}}{|\mu|}\sum_{j=1}^{m}|\alpha_j|\,{}_{H}I^{p_j}\,{}_{RL}I^{\rho}\varphi(\eta_j) \\ &\approx 0.6432886158. \end{aligned}$$

Clearly,

$$\begin{aligned} |\mathrm{O}(k,\xi)-\mathrm{O}(k,x)| &= \frac{k^2}{(k+2)^2}\left|\frac{|\xi|-|x|}{1+|\xi|+|x|+|\xi||x|}\right| \\ &\le \frac{k^2}{4}\left(\frac{|\xi-x|}{0.6432886158+|\xi-x|}\right). \end{aligned}$$

$\Leftrightarrow$ theorem (4.3.3), that the BVP (4.3) has a unique solution on the closed interval $[0,\ 2]$.

**Example 4.4.4:** Consider nonlocal Hadamard FICs for nonlinear R-L Fractional differential equation:

$$\begin{cases} {}_{RL}D^{\frac{6}{5}}\xi(k) = \dfrac{e^{-k}\sin^2(2k)}{(k+3)^2}\cdot\dfrac{|\xi(k)|}{|\xi(k)|+1}+\dfrac{k-1}{k+1}, \quad k\in[0,2\pi], \\ \xi(0)=0, \\ \xi(2\pi)+\sqrt{3}\,{}_HI^{1/2}\xi\left(\dfrac{\pi}{3}\right)+\dfrac{3}{4}\,{}_HI^{3/4}\xi\left(\dfrac{2\pi}{3}\right) \\ = {}_HI^{4/5}\xi(\pi)+\dfrac{1}{9}\,{}_HI^{4/3}\xi\left(\dfrac{4\pi}{3}\right)+2\,{}_HI^{2/3}\xi\left(\dfrac{5\pi}{3}\right). \end{cases} \tag{4.4}$$

Here

$\rho = 5/4$, m $=5$, $K = 2\pi$, $\alpha_1 = \sqrt{3}$, $\alpha_2 = -3/4$, $\alpha_3 = 1$, $\alpha_4 = 1/9$, $\alpha_5 = 2$, $p_1 = 1/2$

$p_2 = 3/4$, $p_3 = 4/5$, $p_4 = 4/3$, $p_5 = 2/3$, $\eta_1 = \pi/3$, $\eta_2 = 2\pi/3$, $\eta_3 = \pi$, $\eta_4 = 4\pi/3$, $\eta_5 = 5\pi/3$, and $f(k,y) = \left(e^{-k^2}\sin^2(2k)|y|\right)\Big/\left(\left((k+3)^2\right)(|y|+1)\right)+(k-1)/(k+1)$.

Since $|f(k,y)-f(k,x)| \le P(1/16)|y-x|$, $(H_1)$ satisfied with $L = 1/36$. By using a Maple program, we show that

$$L\left(\frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)}+\frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}|\alpha_j|\rho^{-p_j}\eta_j^{\rho}\right)\approx 0.9518560542<1.$$

Clearly,

$$\mathrm{O}(k,\xi) = \left|\frac{e^{-k}\sin^2(2k)}{(k+3)^2}\cdot\frac{|\xi(k)|}{|\xi(k)|+1}+\frac{k-1}{k+1}\right| \le \frac{e^{-k^2}}{9}+\frac{k-1}{k+1}.$$

From Theorem (4.3.4), it follows that the BVP (4.4) has at least one solution on the interval $[0,2\pi]$.

**Example 4.4.5:** Consider nonlocal Hadamard FICs for nonlinear R-L Fractional differential equation:

$$\begin{cases} {}_{RL}D^{\frac{6}{5}}\xi(k) = \dfrac{1}{64}\left(1+k^2\right)\left(\dfrac{\xi^2}{|\xi(k)|+1} + \dfrac{\sqrt{|\xi|}}{2\left(1+\sqrt{|\xi|}\right)} + \dfrac{1}{2}\right), \ k \in [0,e], \\ \xi(0) = 0, \quad \xi(e) = \dfrac{1}{2}\,{}_H I^{\sqrt{2}}\xi\left(\dfrac{1}{2}\right) - 5\,{}_H I^{\sqrt{3}}\xi\left(\dfrac{2}{3}\right) + \sqrt{3}\,{}_H I^{\sqrt{5}}\xi(1). \end{cases} \tag{4.5}$$

Here

$\rho = 6/5,\ \mathrm{m} = 3,\ K = e,\ \alpha_1 = 1/2,\ \alpha_2 = -5,\ \alpha_3 = \sqrt{3},\ p_1 = \sqrt{2},\ p_2 = \sqrt{3},\ p_1 = \sqrt{5}$

$\eta_1 = 1/2,\ \eta_2 = 2/3,\ \eta_3 = 1,$ and $\mathrm{O}(k,\xi) = (1/64)\left(1+k^2\right)\left(\left(\xi^2/(|\xi|+1)\right) + \left(\sqrt{\xi}\right)(2(1 + \sqrt{\xi}))\right) + (1/2).$ It is easy to verify that

$$\Omega := \frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)} + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)} \sum_{j=1}^{m} |\alpha_j|\,\rho^{-p_j}\eta_j^{\rho} \approx 3.905177250.$$

Clearly,

$$|\mathrm{O}(k,y)| = \left| \frac{1}{64}\left(1+k^2\right)\left(\frac{\xi^2}{|\xi(k)|+1} + \frac{\sqrt{|\xi|}}{2\left(1+\sqrt{|\xi|}\right)} + \frac{1}{2}\right) \right| \le \frac{1}{64}\left(1+k^2\right)\left(|\xi|+1\right).$$

By choosing $\mathrm{T}(k) = (1/64)\left(1+k^2\right)$ and $\psi(|\xi|) = |\xi|+1,$ we can show that

$$\frac{Q}{\psi(Q)\|\mathrm{T}\|\Omega} > 1,$$

we implying that $Q > 1.048704821.$ It follows from Theorem (4.3.6), that the BVP (4.5) has at least one solution on the interval $[0,e]$.

**Example 4.4.6:** Consider nonlocal Hadamard FICs for nonlinear R-L Fractional differential equation as follows:

$$\begin{cases} {}_{RL}D^{\frac{7}{4}}\xi(k) = \dfrac{1}{2\pi}\sin\left(\dfrac{\pi}{2}\xi\right).\dfrac{|\xi|}{|\xi|+1}, \quad k \in [0,1], \\ \xi(0)=0, \quad \xi(1) = 3\,{}_H I^{1/2}\xi\left(\dfrac{1}{2}\right) - 2\,{}_H I^{3/2}\xi\left(\dfrac{3}{4}\right). \end{cases} \tag{4.6}$$

Here

$\rho = 7/4,\ \mathrm{m}=2,\ \mathrm{K}=1,\ \alpha_1 = 3,\ \alpha_2 = -2,\ p_1 = 1/2,\ p_2 = 3/2,\ \eta_1 = 1/2,\ \eta_2 = 3/4$

and $\mathrm{O}(k,\xi) = (1/2\pi)(\sin(\pi\xi/2))(|\xi|/(|\xi|+1))+1.$ We can show that

$$\Omega := \frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{2\rho-1}}{|\mu|\Gamma(\rho+1)} + \frac{K^{\rho-1}}{|\mu|\Gamma(\rho+1)}\sum_{j=1}^{m}|\alpha_j|\,\rho^{-p_j}\eta_j^{\rho} \approx 1.582207843.$$

Since

$$|\mathrm{O}(k,\xi)| = \left|\frac{1}{2\pi}\sin\left(\frac{\pi}{2}\xi\right).\frac{|\xi|}{|\xi|+1}\right| \le \frac{1}{4}|\xi|+1,$$

$(H_5)$ is satisfied with $\kappa = 1/4$ and $Q = 1$ such that

$$\kappa = \frac{1}{4} < \frac{1}{\Omega} \approx 0.6320282158.$$

From the Theorem (4.3.7), it follows that the BVP (4.6) has at least one solution on the interval $[0,1]$.

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