Kantorovich Type *q*-Bernstein Polynomials

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ABSTRACT

In this thesis first of all some elementary results related with positive linear operators, properties of Bernstein operators, q-integers and some identities related with q-integers and also q-Bernstein operators and their properties are studied. Later a q-analogue of the Bernstein-Kantorovich operators, their approximation properties, local and global approximation properties and Voronovskaja type theorem for the q-Bernstein-Kantorovich operators for the case $0<q<1$ are examined.

Keywords: Kantorovich operators, q-type Kantorovich operators, q-Bernstein polynomials, local and global approximation.

ÖZ

Bu tezde ilk önce pozitif lineer operatörler ve bu operatörlerin özellikleri, bu operatörlerle ilgili sonuçlar, Bernstein operatörleri incelenmiştir. Ayrıca tamsayıların q-analoğu ve bunlarla ilgili bazı özdeşlikler verildikten sonra Bernstein operatörlerinin q-analoğu ve özellikleri çalışılmıştır. Daha sonra Bernstein-Kantorovich operatörleri ve Bernstein-Kantorovich operatörlerinin q-analoğu verilip q-Bernstein-Kantorovich operatörlerinin yakınsaklık özellikleri, lokal ve global yakınsaklık özellikleri ve $0<q<1$ için Voronovskaya tipi teorem incelenmiştir

Anahtar kelimeler: Kantorovich operatörleriö, q-tipli Kantorovich operatörleriö, q-Bernstein polinomları, lokal ve golbal yaklaşim

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NOTATIONS AND SYMBOLS

In this thesis we shall often make use of the following symbols.

:=	is the sign indicating equal definition. "$a := b$" indicates that a equantity to be defined or explained, and b provides the definition or explanation. "$b =: a$" has the same meaning,
$\mathbb{N}$	the set of natural numbers,
$\mathbb{N}_0$	the set of natural numbers including zero,
$\mathbb{R}$	the set of real numbers,
$\mathbb{R}_+$	the set of real positive numbers,
$(a.b)$	an open interval,
$[a.b]$	a closed interval,
$L^{\mathrm{p}}(X)$	the class of the $p - Lebesgue\ integrable$ functions on $X, p \geq 1$,
$\|f\|_p$	is the norm on $L^p(X)$ defined by $\|f\|_p := \left(\int_X |f(x)|dx\right)^{1/p}$, $p \geq 1$,
$C(X)$	the set of all real-valued and continuous functions defined on X,
$C[a,b]$	the set of all real-valued and continuous functions defined on the compact interval $[a,b]$,

$\Delta f(x_j)$ is the forward difference defined as

$$\Delta f(x_j) = f(x_{j+1}) - f(x_j) = f(x_j + h) - f(x_j), \text{ with stepsize } h,$$

$$\Delta^0 f(x_j) = f(x_j), \qquad \Delta^r f(x_j) = \Delta\left(\Delta^{\mathrm{r+1}} f(x_j)\right),$$

$\Delta_h^k f(x)$ is the finite difference of order $k \in \mathbb{N}$, with step size $h \in \mathbb{R}\backslash\{0\}$ and starting point $x \in X$. Its formula is given by

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh),$$

$(x-a)_q^n$ $$= \begin{cases} 1 & if\ n = 0 \\ \prod_{j=0}^{n-i} (x - q^j a) & if\ \ n \geq 1 \end{cases}$$

Chapter 1

INTRODUCTION

Positive linear operators are very important in the field of approximation theory and the theory of these operators has been an important area of research in the last few decades, especially as it affects computer-based geometric design. In the year 1885 Weierstrass proved his (fundamental) theorem on approximation by algebraic and trigonometric polynomials and this was the key moment in the development of Approximation Theory. It was a complicated and a very long proof and provoked many famous mathematicians to find simpler and more instructive proofs. Sergej N. Bernstein was one of these famous mathematicians that constructed well-known Bernstein polynomials:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1.1}$$

for any $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$. As it can seen later in this thesis, if f is continuous on the interval $[0,1]$, its sequence of Bernstein polynomials converges uniformly to f on $[0,1]$, thus giving a constructive proof of Weierstrass's Theorem.

In the last few decades interesting generalizations of Bernstein polynomials based on the $q-$integers were constructed by A. Lupas [19] and by George M. Phillips [25]. In the year 1987 Lupaş proposed the following q-analogue of Bernstein polynomials:

$$R_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1-x+qx)\dots(1-x+q^{n-1}x)}, \tag{1.2}$$

and in the year 1997, Phillips proposed the q- Bernstein polynomials $B_{n,q}(f;x)$. For each positive integer n, and $f \in C[0,1]$

$$B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1-q^s x). \tag{1.3}$$

On the other hand the classical Kantorovich operator $B_n^*, n = 1,2,\dots$ is defined by [18] as

$$B_n^*(f;x) \coloneqq (n+1)\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{k/n+1}^{k+1/n+1} f(t)\, dt$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f\left(\frac{k+t}{n+1}\right) dt,$$

$$f: [0,1] \to \mathbb{R}. \tag{1.4}$$

These operators have been widely considered in the mathematical literature. Also, some other generalizations have been introduced by different mathematicians (see, for instance [7], [6], [16]).

Here in this thesis we studied a q-type generalization of Bernstein-Kantorovich polynomial operators as follows.

$$K_{n,q}^*(f;x) := \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t, \tag{1.5}$$

Where $f \in C[0,1],\ 0 < q < 1\ and$

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k}, \quad (1-x)_q^n = \prod_{s=0}^{n-1} (1-q^s\, \mathrm{x}).$$

We evaluate the moments of $K_{n,q}^*$. We study local and global convergence properties of the q- Bernstein-Kantorovich operators and prove Voronovskaja-type asymptotic formula for the q- Bernstein-Kantorovich operators.

In chapter 2 we give some preliminary and auxiliary results related to positive linear operators. We mentioned about the norm of an operator, uniform convergence of an operator, a Hölder-type inequality for positive linear operators, the modulus of smoothness of order k. We give the definition of Bernstein Polynomials, q-integers and q-parametric Bernstein Polynomials and the theorems, lemmas, propositions related to these operators.

In chapter 3 we give the definition of classical Kantorovich operator and we give the definition of q-Bernstein-Kantorovich operator. We found a recurrence formula for q- Bernstein-Kantorovich operator and obtain explicit formulas for $K_{n,q}^*(t,x)$, $K_{n,q}^*(t^2,x)$. We found estimations for second and fourth order central moments of the q-Bernstein polynomials. Then we give local and global approximation theorems and Voronovskaja type result for q-Bernstein-Kantorovich operators.

Chapter 2

PRELIMINARY AND AUXILIARY RESULT

2.1 Positive Linear Operators

In this section we are going to give some basic definitions and some basic properties related to positive linear operators. For further information on this topic see [9].

Definition 2.1.1. Consider the mapping $L: X \to Y$ such that X and Y are linear spaces of functions. L is said to be a linear operator if

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$$

for all $f, g \in X$ and for all $\alpha, \beta \in R$. $If\ f \geq 0, f \in X$ implies that $Lf \geq 0$ then L is a positive linear operator.

Proposition 2.1.1. Assume that $L : X \to Y$ is a positive and linear operator. Then

1. L is a monotonic operator, that is , if $f, g \in X$ with $f \leq g$ then $Lf \leq Lg$.

2. for all $f \in X$ we have $|Lf| \leq L|f|$.

Definition 2.1.2. Assume that X, Y are two linear normed spaces of real functions such that $X \subseteq Y$ and let $L : X \longrightarrow Y$. Then to each linear operator L we can assign a norm $\|L\|$ defined by

$$\|L\| := sup\{\|Lf\| : f \in X, \|f\| = 1\} = sup\{\|Lf\| : f \in X\ , 0 < \|f\| \leq 1\}.$$

It can be easily verified that $\|.\|$ satisfies all the properties of a norm and so is called the operator norm.

If we select $X = Y = C[a,b]$ the following remark can be stated regarding the continuity and the operator norm.

Remark 2.1.1. Let $L : C[a,b] \to C[a,b]$ be a linear and positive operator. Then L is also continuous and $\|L\| = \|Le_0\|$ where $e_0 = t^0$.

Theorem 2.1.1. Assume that $L_n : C[a,b] \to C[a,b]$ is a sequence of positive linear operators and let $e_i = t^i$. If $lim_{n\to\infty} L_n e_i = e_i \; for \;\; i = 0,1,2$ uniformly on $[a,b]$, then

$lim_{n\to\infty} L_n f = f$ uniformly on $[a,b]$ for every $f \in C[a,b]$.

Thus from the result given above we see that the monomials $e_i = t^i$, $i = 0,1,2$, has an important role in the approximation theory of linear and positive operators on the spaces of continuous functions. In general they are called test functions.

This nice and simple result was inspirational for many researchers to extend Theorem 2.1.1 in different ways, generalizing the notion of sequence and considering different spaces. A special field of study of approximation theory arises in this way which is called the Korovkin-type approximation theory. A complete and comprehensive exposure on this topic can be found in [3].

In many estimates the Cauchy-Schwarz inequality is used :

$$\big(L(fg)\big)^2 \leq L(f^2)L(g^2)\,, \qquad f,g \in C[a,b].$$

Following inequality is a Hölder-type inequality for positive linear operators that reduces to the Cauchy-Schwarz inequality in the case $p = q = 2$.

Theorem 2.1.2. Let $L : C[a,b] \to C[a,b]$ be a positive linear operator, $Le_0 = e_0$. For $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in C[a,b], x \in [a,b]$ one has

$$L(|fg|; x) \leq \left(L(|f|^p; x)\right)^{\frac{1}{p}} . \left(L(|g|^q; x)\right)^{\frac{1}{q}}.$$

The following quantities play an important role for the positive linear operators $L : C[a,b] \to C[a,b]$. the moments of order n , $n \geq 0$, namely

$$L((e_1 - x)^n; x) = L((e_1 - x)^n)(x), \qquad x \in [a,b],$$

and for $n \geq 1$ also the absolute moments of odd order n, that is

$$L(|e_1 - x|^n; x) = L(|e_1 - x|^n)(x), \qquad x \in [a,b].$$

Proposition 2.1.2. Let L, p, q, f and x be given in Theorem 2.1.2 and let $0 \leq n = n_1 + n_2$ be a decomposition of the non-negative number n with $n_1, n_2 \geq 0$. Then

$$L(|e_1 - x|^n; x) \leq \left(L(|e_1 - x|^{n_1 \cdot p}; x)\right)^{1/p} \left(L(|e_1 - x|^{n_2 \cdot q}; x)\right)^{1/q}.$$

Proposition 2.1.3. Let $L : C[a,b] \to C[a,b]$ be a positive linear operator such that $Le_0 = e_0$ and $1 \leq s \leq r$. Then

$$\left(L(|e_1 - x|^s; x)\right)^{1/s} \leq \left(L(|e_1 - x|^r; x)\right)^{1/r}, x \in [a,b].$$

Proposition 2.1.4. For a linear operator L and $k \in \mathbb{N}_0$ we have

$$L((e_1 - x)^k; x) = L(e_k; x) - \sum_{l=0}^{k-1} \binom{k}{l} x^{k-l} L((e_1 - x)^l; x).$$

Remark 2.1.2. (i) Note that the equality of Proposition 2.1.4. holds without the assumption $Le_i = e_i, i \in \{0,1\}$.

(ii) The proposition means that $L((e_1 - x)^k; x)$ can be computed if we know $L(e_k; x)$ and the lower order moments $L((e_1 - x)^l; x)$, $0 \le l \le k - 1$.

Corollary 2.1.1. Let L be a linear operator with $Le_i = e_i$, $i \in \{0,1\}$. The 3rd and the 4th moments can be computed as it is given below:

$L((e_1 - x)^3; x) = L(e_3; x) - x^3 - 3xL((e_1 - x)^2; x)$,

$L((e_1 - x)^4; x) = L(e_4; x) - x^4 - \{4xL((e_1 - x)^3; x) + 6x^2L((e_1 - x)^2; x)\}$.

Definition 2.1.3. The modulus of smoothness of order k is defined by

$$\omega_k(f;\delta) = sup\{|\Delta_h^k| : 0 \le h \le \delta, x, x + kh \in [a,b]\},$$

where $k \in \mathbb{N}, \delta \in \mathbb{R}_+$ and $f \in C[a,b]$.

Proposition 2.1.5. see[9]

1) $\omega_k(f;0) = 0$.
2) $\omega_k(f;.)$ is a positive continuous and non-decreasing function on $\mathbb{R}_+$.
3) $\omega_k(f;.)$ is sub-additive, $i.e.$, $\omega_1(f;\delta_1 + \delta_2) \le \omega_1(f;\delta_1) + \omega_1(f;\delta_2)$, $\delta_i \ge 0, i = 1,2$.
4) $\forall\delta \ge 0, \omega_{k+1}(f;\delta) \le 2\omega_k(f;\delta)$.
5) $If\ f \in C^1[a,b]$ then $\omega_{k+1}(f;\delta) \le \delta. \omega_k(f';\delta), \delta \ge 0$.
6) $If\ f \in C^r[a,b]$ then $\omega_r(f;\delta) \le \delta^r sup_{\delta\in[a,b]}|f^{(r)}(\delta)|$.
7) $\forall\delta > 0$ and $n \in \mathbb{N}, \omega_k(f;n\delta) \le n^k\omega_k(f;\delta)$.
8) $\forall\delta > 0\ and\ a > 0, \omega_k(f;a\delta) \le$ $(1 + [a])^k\ \omega_k(f;\delta), where\ [a]\ is\ the\ integer\ part\ of a.$

9) $\forall \delta \geq 0$ $is\ fixed, then\ \omega_k(f;.)\ is\ a\ semi-norm\ on\ C[a,b]$.

2.2 Bernstein Polynomials

Let f be a function on $[0,1]$. For each positive integer n, we define the Bernstein polynomial

$$B_n(f\,;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\binom{n}{r} x^r(1-x)^{n-r}. \qquad (2.2.1)$$

If f is continuous on $[0,1]$, its sequence of Bernstein polynomials converges uniformly to f on $[0,1]$, which gives a constructive proof to Weierstrass's Theorem. We may ask a question as " why Bernstein created these new polynomials to prove Weierstrass's Theorem, instead of using polynomials that were already known before". For example, Taylor polynomials are not appropriate; for even setting aside questions of convergence, they are applicable only to functions that are infinitely differentiable, and not to all continuous functions. It is obvious from (2.2.1) that for all $n \geq 1$,

$$B_n(f;0) = f(0)\ \ and\ \ B_n(f;1) = f(1), \qquad (2.2.2)$$

so that a Bernstein polynomial for f interpolates f at both endpoints of the interval $[0,1]$. Moreover from the binomial expansion it follows that

$$B_n(1\,;x) = \sum_{r=0}^{n}\binom{n}{r} x^r(1-x)^{n-r} = (x+(1-x))^n = 1, \qquad (2.2.3)$$

so that the Bernstein polynomial for the constant function 1 is also 1. Also the Bernstein polynomial for the function x is x . Indeed since

$$\frac{r}{n}\binom{n}{r} = \binom{n-1}{r-1}$$

for $1 \leq r \leq n$, the Bernstein polynomial for the function t is

$$B_n(f\,;x) = \sum_{r=0}^{n} \frac{r}{n}\binom{n}{r} x^r (1-x)^{n-r}$$

$$= x\sum_{r=1}^{n}\binom{n-1}{r-1} x^{r-1}(1-x)^{n-r} = x\sum_{s=0}^{n-1}\binom{n-1}{s} x^s (1-x)^{n-1-s} \quad (2.2.4)$$

We call B_n the Bernstein operator; it maps a function f, defined on $[0,1]$, to $B_n f$, where the function $B_n f$ evaluated at x is denoted by $B_n(f;x)$. The Bernstein operator is obviously linear, since it follows from (2.2.1) that

$$B_n(\lambda f + \mu g) = \lambda B_n f + \mu B_n g, \quad (2.2.5)$$

for all functions f and g defined on $[0, 1]$, and all real λ and μ.

It can be seen from (2.2.1) that B_n is a monotone operator. It then follows from the monotonicity of B_n and (2.2.3) that

$$m \leq f(x) \leq M, x \in [0,1] \Rightarrow m \leq B_n(f;x) \leq M, x \in [0,1]. \quad (2.2.6)$$

Particularly, if we choose $m = 0$ in (2.2.6), we get

$$f(x) \geq 0, x \in [0,1] \Rightarrow B_n(f;x) \geq 0, x \in [0,1]. \quad (2.2.7)$$

It follows from (2.2.3),(2.2.4), and the linear property (2.2.5) that

$$B_n(at + b; x) = ax + b, \quad (2.2.8)$$

for all real numbers a and b. Thus we can say that the Bernstein operator reproduces linear polynomials.

Theorem 2.2.1. The Bernstein polynomial can be expressed in the following form

$$B_n(f;x) = \sum_{r=0}^{n}\binom{n}{r}\Delta^r f(0) x^r, \quad (2.2.9)$$

Where Δ is the forward difference operator, defined as

$\Delta f(x_j) = f(x_{j+1}) - f(x_j) = f(x_j + h) - f(x_j)$, with step size $h = 1/n$.

Theorem 2.2.2. The derivative of the Bernstein polynomial $B_{n+1}(f;x)$ can be expressed in the following form

$$B'_{n+1}(f;x) = (n+1)\sum_{r=0}^{n} \Delta f\left(\frac{r}{n+1}\right)\binom{n}{r} x^r(1-x)^{n-r} \tag{2.2.10}$$

for $n \geq 0$, where Δ is applied with step size $h = 1/(n+1)$. Furthermore , if f is monotonically increasing or monotonically decreasing on $[0,1]$, so are all its Bernstein polynomials.

Theorem 2.2.3. Let k be any nonnegative integer. The kth derivative of $B_{n+k}(f;x)$ can be expressed in terms of kth difference of f as

$$B^{(k)}_{n+k}(f;x) = \frac{(n+k)!}{n!}\sum_{r=0}^{n} \Delta^k f\left(\frac{r}{n+k}\right)\binom{n}{r} x^r(1-x)^{n-r} \tag{2.2.11}$$

For all $n \geq 0$, where Δ is applied with step size $h = 1/(n+k)$.

2.3 The *q*-Integers

Definition 2.3.1. Given a value of $q > 0$, we define $[r]$, where $n \in \mathbb{N}$ as

$$[r] = [r]_q := \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1. \end{cases} \tag{2.3.1}$$

and call $[r]$ a q-integer. It is clear that the above definition can be extended if we allow r to be any real number.

For any given value $q > 0$ let us define

$$\mathbb{N}_q = \{[n], with\ n \in \mathbb{N}\}, \tag{2.3.2}$$

and we can see from Definition 2.3.1 that

$$\mathbb{N}_q = \{0, 1, 1+q, 1+q+q^2, 1+q+q^2+q^3, \dots\} \quad . \tag{2.3.3}$$

It is clear that the set of q-integers $\mathbb{N}_q$ generalizes the set of non-negative integers $\mathbb{N}$, that we get by putting $q = 1$.

Definition 2.3.2 Let $q > 0$ be given. We define $[r]!$, where $r \in \mathbb{N}$, as

$$[r]! := \begin{cases} [\mathrm{r}][\mathrm{r}-1] \ldots [1], & r \geq 1 \\ 1, & r = 0 \end{cases} \tag{2.3.4}$$

and call $[r]!$ a q-factorial.

Definition 2.3.3. We define a q-binomial coefficient as

$$\begin{bmatrix} t \\ r \end{bmatrix} := \frac{[t][t-1] \ldots [t-r+1]}{[r]!}, \tag{2.3.5}$$

for all real t and integers $r \geq 0$, and as zero otherwise.

In this thesis we are going to deal with q-binomial coefficients for which $t = n \geq r \geq 0$, where $n \in \mathbb{N}$. Thus it is better to define them separately.

Definition 2.3.4. Let n and r be any two integers, we define

$$\begin{bmatrix} n \\ r \end{bmatrix} := \frac{[n][n-1] \ldots [n-r+1]}{[r]!} = \frac{[n]!}{[r]!\,[n-r]!}, \tag{2.3.6}$$

for $n \geq r \geq 0$, and as zero otherwise. These are called Gaussian polynomials which are named after C.F.Gauss.

The Gaussian polynomials satisfy the Pascal-type relations

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} \tag{2.3.7}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ r \end{bmatrix}. \qquad (2.3.8)$$

Definition 2.3.5. The q-analogue of $(x-a)^n$ is defined by the polynomial

$$(x-a)_q^n = \begin{cases} 1 & if\ n = 0, \\ (x-a)(x-qa)\dots(x-q^{n-1}a) & if\ n \geq 1. \end{cases}$$

Lemma 2.3.1. For a nonnegative integer n and a number a we have,

$$(x+a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} a^j x^{n-j} \qquad (2.3.9)$$

which is called the Gauss's binomial formula.

Lemma 2.3.2. For a nonnegative integer n we have,

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{j=1}^{\infty} \frac{[n][n+1]\dots[n+j-1]}{[j]!} x^j, \qquad (2.3.10)$$

which is called Heine's binomial formula.

Now we have two binomial formulas, namely Gauss's binomial formula (2.3.9)

(with x and a replaced by 1 and x respectively)

$$(1+x)_q^n = \sum_{j=0}^{n} q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix} x^j, \qquad (2.3.11)$$

and Heine's binomial formula (2.3.10)

$$\frac{1}{(1-x)_q^n} = \sum_{j=0}^{\infty} \frac{[n][n+1]\ldots[n+j-1]}{[j]!} x^j.$$

Now we may consider the question ”What happens if we let $n \to \infty$ in both formulas?”. In the ordinary calculus, i.e. when q =1, the answer is not very interesting. It depends on the value of x, it is either infinitely large or infinitely small. However, it is different in quantum calculus, because, for example, when $|q| < 1$, the infinite product

$$(1+x)_q^\infty = (1+x)(1+qx)(1+q^2x)\ldots$$

converges to some finite limit. Moreover, if we assume $|q| < 1$, we have

$$\lim_{n\to\infty} [n] = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1}{1-q} \tag{2.3.12}$$

and

$$\lim_{n\to\infty} \begin{bmatrix} n \\ j \end{bmatrix} = \lim_{n\to\infty} \frac{(1-q^n)(1-q^{n-1})\ldots\left(1-q^{n-j+1}\right)}{(1-q)(1-q^2)\ldots(1-q^j)} \ .$$

Thus

$$\lim_{n\to\infty} \begin{bmatrix} n \\ j \end{bmatrix} = \frac{1}{(1-q)(1-q^2)\ldots(1-q^j)} \ . \tag{2.3.13}$$

So, the q-analogues of integers and binomial coefficients behave differently when n is large as compared to their ordinary counterparts.

If we apply equalities (2.3.12) and (2.3.13) to Gauss's and Heine's binomial formulas, we get, as $n \to \infty$, the following two identities of formal power series in x (assuming that $|q| < 1$):

$$(1+x)_q^\infty = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{(1-q)(1-q^2)\dots(1-q^j)} \ , \tag{2.3.14}$$

$$\frac{1}{(1-x)_q^\infty} = \sum_{j=0}^{\infty} \frac{x^j}{(1-q)(1-q^2)\dots(1-q^j)}.$$

The identities given above relate the infinite products to infinite sums. They don't have classical analogues because when $q = 1$, the terms in the summations has no meaning. It is very interesting that both of the two identities were discovered by Euler, who lived before Gauss and Heine.

2.4 q-parametric Bernstein Polynomials

In this section a generalization of Bernstein polynomials based on the q-integers are discussed. These polynomials were proposed by Phillips [25] as given below;

$$B_{n,q}(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x) \tag{2.4.1}$$

where $f_r = f\left(\frac{[r]}{[n]}\right)$. Note that an empty product in (2.4.1) denotes 1. When we put $q = 1$ in (2.4.1), we obtain the classical Bernstein polynomial, defined by (2.2.1). Immediately it can be seen from (2.4.1) that

$$B_{n,q}(f;0) = f(0) \ and \ B_{n,q}(f;1) = f(1), \tag{2.4.2}$$

which gives us the interpolation at the endpoints of the interval [0,1], as we have for the classical Bernstein polynomials. It is obvious that $B_{n,q}$, which is defined by

(2.4.1), is a linear operator, and with $0 < q < 1$, it is a monotone operator that maps functions defined on $[0,1]$ *to* $\mathbb{P}_n$, the set of all polynomials of degree less than or equal to n. For a fixed $q \in (0,1)$, it is proved by II'inskii and Ostrovska that for each $f \in C[0,1]$, the sequence $\{B_{n,q}(f;x)\}$ converges to $B_{\infty,q}(f;x)$ uniformly as n approaches to infinity for $0 \le x \le 1$, where

$$B_{\infty,q}(f;x) := \begin{cases} \sum_{r=0}^{\infty} f(1-q^r) \dfrac{x^r}{(1-q)^r [r]!} \prod_{s=0}^{\infty} (1-q^s x), & 0 \le x < 1 \\ f(1), & x = 1. \end{cases}$$

The following theorem that is given below involves q-differences which yield Theorem 2.2.2 when $q = 1$.

Theorem 2.4.1. [27] The generalized Bernstein polynomial can be expressed in the following form

$$B_{n,q}(f;x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta_q^r f_0 x^r, \tag{2.4.3}$$

where

$$\Delta_q^r f_j = \Delta_q^{r-1} f_{j+1} - q^{r-1} \Delta_q^{r-1} f_j, r \ge 1,$$

with $\Delta_q^0 f_j = f_j = f([j]/[n])$.

Proof. Firstly the following identity is needed,

$$\prod_{s=0}^{n-r-1} (1-q^s x) = \sum_{s=0}^{n-r} (-1)^s q^{s(s-1)/2} \begin{bmatrix} n-r \\ s \end{bmatrix} x^s, \tag{2.4.4}$$

which reduces to a binomial expansion when we give $q = 1$. Starting with (2.4.1) and expanding the term which consists of the product of the factors $(1 - q^s)$, we get the following

$$B_{n,q}(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \sum_{s=0}^{n-r} (-1)^s q^{s(s-1)/2} \begin{bmatrix} n-r \\ s \end{bmatrix} x^s.$$

Now, let us substitute $t = r + s$. Then, since

$$\begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix} = \begin{bmatrix} n \\ t \end{bmatrix} \begin{bmatrix} t \\ r \end{bmatrix},$$

the latter double sum can be written as

$$\sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix} x^t \sum_{r=0}^{t} (-1)^{t-r} q^{(t-r)(t-r-1)/2} \begin{bmatrix} t \\ r \end{bmatrix} f_r = \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix} \Delta_q^t f_0 x^t,$$

on using the expansion for a higher-order q-difference, which is given as

$$\Delta_q^k f(x_j) = \sum_{r=1}^{k+1} (-1)^{r-1} q^{(r-1)(r-2)/2} \begin{bmatrix} k \\ r-1 \end{bmatrix} f(x_{j+k+1-r}),$$

and the proof is completed.

From Theorem 2.4.1 we deduce that

$$B_{n,q}(1;x) = 1. \tag{2.4.5}$$

For $f(x) = x$ we have $\Delta_q^0 f_0 = f_0 = 0$ and $\Delta_q^1 f_0 = f_1 - f_0 = 1/[n]$, and it follows from Theorem 2.4.1 that

$$B_{n,q}(t;x) = x. \tag{2.4.6}$$

For $f(x) = x^2$ we have $\Delta_q^0 f_0 = f_0 = 0$, $\Delta_q^1 f_0 = f_1 - f_0 = 1/[n]^2$, and

$$\Delta_q^2 f_0 = f_2 - (1+q)f_1 + qf_0 = \left(\frac{[2]}{[n]}\right)^2 - (1+q)\left(\frac{[1]}{[n]}\right)^2.$$

Then we find from Theorem 2.4.1 that

$$B_{n,q}(t^2;x) = x^2 + \frac{x(1-x)}{[n]}. \tag{2.4.7}$$

The above expressions for $B_{n,q}(1;x)$, $B_{n,q}(t;x)$ and $B_{n,q}(t^2;x)$ generalize their counterparts for the case q=1 and , with the help of Theorem 2.1.2, lead us to the following theorem on the convergence of the generalized Bernstein polynomials.

Theorem 2.4.2. [27] Let $\{q_n\}$ be sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then, for any $f \in C[0,1]$, $B_{n,q_n}(f;x)$ converges uniformly to $f(x)$ on [0,1].

Proof. We saw above from (2.4.5) and (2.4.6) that $B_{n,q_n}(f;x) = f(x)$ or $f(x) = 1$ and $f(x) = x$, and since $q_n \to 1$ $as\ n \to \infty$, we see from (2.4.7) that $B_{n,q_n}(f;x)$ converges uniformly to $f(x)$ for $f(x) = x^2$. Also, since $0 < q_n < 1$, it follows that B_{n,q_n} is monotone operator, and the proof is completed by applying the Bohman-Korovkin Theorem (2.1.1).

We now state the following theorems.

Theorem 2.4.3. [27] If $f(x) is\ convex\ on$ [0,1], $then$

$$B_{n,q}(f;x) \geq f(x), 0 \leq x \leq 1, \tag{2.4.8}$$

for all $n \geq 1$and for $0 < q \leq 1$.

Theorem 2.4.4. [27] If $f(x)$ is convex on [0,1],

$$B_{n-1,q}(f;x) \geq B_{n,q}(f;x), \qquad 0 \leq x \leq 1, \tag{2.4.9}$$

for all $n \geq 2, where\ B_{n-1,q}(f;x)$ and $B_{n,q}(f;x)$ are evaluated using the same value of the parameter q. The q-Bernstein polynomial are equal at $x = 0$ and $x = 1$, since they interpolate f at these points. If $f \in C[0,1]$, the inequality in (2.4.9) is strict for $0 < x < 1$ unless, for a given value of n, the function f is linear in each of the intervals $\left[\frac{[r-1]}{[n-1]}, \frac{[r]}{[n-1]}\right]$, for $1 \leq r \leq n-1$, when we have simply $B_{n-1,q}(f;x) = B_{n,q}(f;x)$.

Chapter 3

APPROXIMATION THEOREMS FOR *q*-BERNSTEIN-KANTOROVICH OPERATORS

3.1 *q*-Bernstein-Kantorovich Operators and their moments

The classical Kantorovich operator B_n^*, $n=1,2,\ldots$ is defined by [18]

$$B_n^*(f,x) := (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_{k/n+1}^{k+1/n+1} f(t)dt$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_0^1 f\left(\frac{k+t}{n+1}\right)dt, \quad f:[0,1]\to\mathbb{R}$$

Let the q-analogue of integration on the interval $[0\,,a]$ (see [17]) be defined by

$$\int_0^a f(t)d_qt := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n\ , \quad 0<q<1. \tag{3.1.1}$$

Let $0<q<1$. Based on the q-integration N. Mahmudov and P. Sabancıgil [23] proposed the Kantorovich type q-Bernstein polynomial for $f\in C[0,1]$ as follows.

$$K_{n,q}^*(f,x) = \sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 f\left(\frac{[k]+q^kt}{[n+1]}\right)d_qt\,, \qquad 0\le x\le 1, n\in\mathbb{N} \tag{3.1.2}$$

where

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k(1-x)_q^{n-k}\ , \quad (1-x)_q^n := \prod_{s=0}^{n-1}(1-q^sx). \tag{3.1.3}$$

Note that for $q \to 1^-$ the q-Bernstein-Kantorovich operator becomes the classical Bernstein-Kantorovich operator.

Lemma 3.1.1: For all $n \in \mathbb{N}, x \in [0,1]$ and $0 < q \leq 1$ we have

$$K_{n,q}^*(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i B_{n,q}\left(t^{j+i}, x\right).$$

Proof. From (3.1.2) we have

$$K_{n,q}^*(f, x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k] + q^k t}{[n+1]}\right) d_q t \ .$$

Then we get

$$K_{n,q}^*(t^m, x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k] + q^k t}{[n+1]}\right)^m d_q t \,.$$

Now from the binomial expansion

$$([k] + q^k t)^m = \sum_{j=0}^{m} \binom{m}{j} [k]^j (q^k t)^{m-j} \ ,$$

and

$$K_{n,q}^*(t^m, x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \sum_{j=0}^{m} \binom{m}{j} \frac{[k]^j q^{k(m-j)} t^{m-j}}{[n+1]^m} d_q t$$

$$= \sum_{k=0}^{n} p_{n,k}(q;x) \sum_{j=0}^{m} \binom{m}{j} \frac{[k]^j q^{k(m-j)}}{[n+1]^m} \int_0^1 t^{m-j} d_q t \ .$$

Calculating the q-integration,

$$\int_0^1 t^{m-j} d_q t = \frac{t^{m-j+1}}{[m-j+1]} |_0^1 = \frac{1}{[m-j+1]}$$

we get

$$K_{n,q}^{*}(t^m, x) = \sum_{k=0}^{n} p_{n,k}(q;x) \sum_{j=0}^{m} \binom{m}{j} \frac{[k]^j q^{k(m-j)}}{[n+1]^m [m-j+1]}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} [k]^j q^{k(m-j)} \, p_{n,k}(q;x) \ ,$$

multiplying the right hand side by $\frac{[n]^j}{[n]^j}$, we get

$$K_{n,q}^{*}(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} (q^k - 1 + 1)^{m-j} \frac{[k]^j}{[n]^j} p_{n,k}(q;x) \ ,$$

from binomial expansion

$$\left((q^k - 1) + 1\right)^{m-j} = \sum_{i=0}^{m-j} \binom{m-j}{i} (q^k - 1)^i \ ,$$

and

$$K_{n,q}^{*}(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^k - 1)^i \frac{[k]^j}{[n]^j} p_{n,k}(q;x)$$

From the Definition (2.3.1) we have that

$$[k]^i = \frac{(q^k - 1)^i}{(q-1)^i} \quad \Rightarrow (q^k - 1)^i = [k]^i (q-1)^i \ ,$$

$$[n]^i = \frac{(q^n - 1)^i}{(q-1)^i} \quad \Rightarrow (q-1)^i = \frac{(q^n-1)^i}{[n]^i}$$

and from the last two equalities we get

$$(q^k - 1)^i = (q^n - 1)^i \frac{[k]^i}{[n]^i},$$

$$K_{n,q}^*(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i \frac{[k]^{i+j}}{[n]^{i+j}} p_{n,k}(q;x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i \sum_{k=0}^{n} \frac{[k]^{i+j}}{[n]^{i+j}} p_{n,k}(q;x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i B_{n,q}(t^{i+j}, x). \blacksquare$$

Lemma 3.1.2. For all $n \in \mathbb{N}, x \in [0,1]$ and $0 < q \leq 1$ we have

$K_{n,q}^*(1,x) = 1$, $\quad K_{n,q}^*(t,x) = \frac{2q}{[2]} \frac{[n]}{[n+1]} x + \frac{1}{[2]} \frac{1}{[n+1]}$,

$K_{n,q}^*(t^2,x) = \left(\frac{4q^3+q^2+q}{[2][3]}\right) \frac{q[n][n-1]}{[n+1]^2} x^2 + \left(\frac{4q^3+5q^2+3q}{[2][3]}\right) \frac{[n]}{[n+1]^2} x + \frac{1}{[3][n+1]^2}$

Proof:

$B_{n,q}(1,x) = 1$ and we have $K_{n,q}^*(1,x) = 1$

From Lemma 3.1.1**,** equalities (2.4.5), (2.4.6) and (2.4.7) and by direct calculation we get

$$K_{n,q}^*(1;x) = \sum_{j=0}^{0} \binom{0}{0} \frac{[n]^0}{[n+1]^0 [0-0+1]} \sum_{i=0}^{0} \binom{0}{0} (q^n - 1)^0 B_{n,q}(1,x) = 1 .$$

$$K^*_{n,q}(t,x) = \binom{1}{0} \frac{[n]^0}{[n+1]^1[1-0+1]}\left(\binom{1}{0}(q^n-1)^0 B_{n,q}(t^0,x)\right.$$

$$\left. + \binom{1}{1}(q^n-1)^1 B_{n,q}(t,x)\right)$$

$$+ \binom{1}{1}\frac{[n]^1}{[n+1]^1[1-1+1]}\left(\binom{0}{0}(q^n-1)^0 B_{n,q}(t^0,x)\right)$$

$$K^*_{n,q}(t,x) = \frac{1}{[2][n+1]}\left(B_{n,q}(1,x) + (q^n-1)B_{n,q}(t,x)\right)$$

$$+ \frac{[n]}{[n+1]}B_{n,q}(t,x)$$

$$= \frac{1}{[2][n+1]} + \frac{(q^n-1)}{[2][n+1]}x + \frac{[n]}{[n+1]}x$$

$$= \left(\frac{(q^n-1)}{[2][n+1]} + \frac{[n]}{[n+1]}\right)x + \frac{1}{[2][n+1]}$$

$$= \left((q^n-1) + [2][n]\right)\frac{x}{[2][n+1]} + \frac{1}{[2][n+1]}$$

$$= \left((q^n-1) + \frac{(q+1)(q^n-1)}{(q-1)}\right)\frac{x}{[2][n+1]} + \frac{1}{[2][n+1]}$$

$$= \left(\frac{(q^n-1)(q-1) + (q+1)(q^n-1)}{(q-1)}\right)\frac{x}{[2][n+1]} + \frac{1}{[2][n+1]}$$

$$= \left(\frac{(q^n-1)(q-1+q+1)}{(q-1)}\right)\frac{x}{[2][n+1]} + \frac{1}{[2][n+1]}$$

$$= (2q[n])\frac{x}{[2][n+1]} + \frac{1}{[2][n+1]}$$

$$K_{n,q}^{*}(t,x)=\frac{2q[n]}{[2][n+1]}x+\frac{1}{[2][n+1]} \qquad \blacksquare$$

$$K_{n,q}^{*}(t^2,x)=\binom{2}{0}\frac{[n]^0}{[n+1]^2[2-0+1]}\left(\binom{2}{0}(q^n-1)^0B_{n,q}(t^0,x)\right.$$

$$\left.+\binom{2}{1}(q^n-1)^1B_{n,q}(t^1,x)+\binom{2}{2}(q^n-1)^2B_{n,q}(t^2,x)\right)$$

$$+\binom{2}{1}\frac{[n]^1}{[n+1]^2[2-1+1]}\left(\binom{1}{0}(q^n-1)^0B_{n,q}(t^1,x)\right.$$

$$\left.+\binom{1}{1}(q^n-1)^1B_{n,q}(t^2,x)\right)$$

$$+\binom{2}{2}\frac{[n]^2}{[n+1]^2[2-2+1]}\left(\binom{0}{0}(q^n-1)^0B_{n,q}(t^2,x)\right)$$

$$K_{n,q}^{*}(t^2,x)=\frac{1}{[3][n+1]^2}\left(1+2(q^n-1)x+(q^n-1)^2\left(x^2+\frac{x(1-x)}{[n]}\right)\right)$$

$$+\frac{2[n]}{[2][n+1]^2}\left(x+(q^n-1)\left(x^2+\frac{x(1-x)}{[n]}\right)\right)$$

$$+\frac{[n]^2}{[n+1]^2[1]}\left(x^2+\frac{x(1-x)}{[n]}\right)$$

$$=\frac{1}{[3][n+1]^2}+\frac{2(q^n-1)}{[3][n+1]^2}x+\frac{(q^n-1)^2}{[3][n+1]^2[n]}x$$

$$+\frac{(q^n-1)^2q[n-1]}{[3][n+1]^2[n]}x^2+\frac{2[n]}{[n+1]^2[2]}x+\frac{2[n](q^n-1)}{[n+1]^2[2][n]}x$$

$$+\frac{2[n](q^n-1)q[n-1]}{[n+1]^2[2][n]}x^2+\frac{[n]^2}{[n+1]^2[n]}x+\frac{[n]^2q[n-1]}{[n+1]^2[n]}x^2.$$

$$= \frac{1}{[3][n+1]^2} + \left(\frac{2(q^n-1)}{[3][n+1]^2} + \frac{(q^n-1)^2}{[3][n+1]^2[n]} + \frac{2[n]}{[n+1]^2[2]} + \frac{[n]}{[n+1]^2}\right.$$

$$\left. + \frac{2(q^n-1)}{[n+1]^2[2]}\right) x$$

$$+ \left(\frac{(q^n-1)^2}{[3][n+1]^2[n]} + \frac{2(q^n-1)}{[n+1]^2[2]} + \frac{[n]}{[n+1]^2}\right) q[n-1]\, x^2$$

$$= \frac{1}{[3][n+1]^2} + \big(2(q-1)[2] + (q-1)^2[2] + 2[3] + [2][3]$$

$$+ 2[3](q-1)\big) \frac{[n]x}{[n+1]^2[2][3]}$$

$$+ ((q-1)^2[2] + 2(q-1)[3] + [2][3]) \frac{q[n][n-1]}{[n+1]^2[2][3]} x^2$$

$$= \frac{1}{[n+1]^2[3]} + \big(2(q-1)(q+1) + (q^2-2q+1)(q+1) + 2(q^2+q+1)$$

$$+ (q+1)(q^2+q+1) + 2(q-1)(q^2+q+1)\big) \frac{[n]}{[n+1]^2[2][3]} x$$

$$+ \big((q+1)(q^2-2q+1) + 2(q-1)(q^2+q+1)$$

$$+ (q+1)(q^2+q+1)\big) \frac{q[n][n-1]}{[n+1]^2[2][3]} x^2$$

$$= \frac{1}{[n+1]^2[3]} + (2q^2 - 2 + q^3 - q^2 - q + 1 + 2q^2 + 2q + 2 + q^3 + 2q^2 + 1$$

$$+ 2q^3 - 2) \frac{[n]}{[n+1]^2[2][3]} x$$

$$+ (q^3 - 2q^2 + q + q^2 - 2q + 1 + 2q^3 - 2 + q^3 + 2q^2 + 2q$$

$$+ 1) \frac{q[n][n-1]}{[n+1]^2[2][3]} x^2$$

$$= \frac{1}{[n+1]^2[3]} + (4q^3 + 5q^2 + 3q) \frac{[n]}{[n+1]^2[2][3]} x$$

$$+ (4q^3 + q^2 + q) \frac{q[n][n-1]}{[n+1]^2[2][3]} x^2$$

$$K_{n,q}^{*}(t^2,x)=\left(\frac{4q^3+q^2+q}{[2][3]}\right)\frac{q[n][n-1]}{[n+1]^2}x^2+\left(\frac{4q^3+5q^2+3q}{[2][3]}\right)\frac{[n]}{[n+1]^2}x$$
$$+\frac{1}{[n+1]^2[3]}\ \blacksquare$$

Remark 3.1.1. It can be observed from the previous lemma that for the case $q=1$, we obtain the moments of the Bernstein-Kantorovich operators.

Lemma 3.1.3 For all $n\in\mathbb{N}, x\in[0,1]$ and $0<q\leq 1$ we have

$$K_{n,q}^{*}((t-x)^2,x)\leq\frac{4}{[n]}\left(x(1-x)+\frac{1}{[n]}\right),$$
$$K_{n,q}^{*}((t-x)^4,x)\leq\frac{C}{[n]^2}\left(x(1-x)+\frac{1}{[n]^2}\right),$$

where C is a positive absolute constant.

Proof. To prove this lemma we use the estimations of the 2nd and the 4th order central moments of the q-Bernstein polynomials.

$$B_{n,q}((t-x)^2,x)=\frac{1}{[n]}x(1-x),\qquad B_{n,q}((t-x)^4,x)\leq\frac{C}{[n]^2}x(1-x).$$

Indeed

$$K_{n,q}^{*}((t-x)^2,x)\ =\sum_{k=0}^{n}p_{n,k}\,(q;x)\int_0^1\left(\frac{[k]+q^kt}{[n+1]}-x\right)^2d_qt$$
$$=\sum_{k=0}^{n}p_{n,k}\,(q;x)\int_0^1\left(\frac{[k]+q^kt}{[n+1]}-\frac{[k]}{[n]}+\frac{[k]}{[n]}-x\right)^2d_qt$$
$$=\sum_{k=0}^{n}p_{n,k}\,(q;x)\int_0^1\left(\frac{q^kt}{[n+1]}-\frac{q^n[k]}{[n][n+1]}+\frac{[k]}{[n]}-x\right)^2d_qt$$

$$\leq 2\sum_{k=0}^{n} p_{n,k}\,(q;x)\int_0^1 \left(\frac{q^k t}{[n+1]} - \frac{q^n[k]}{[n][n+1]}\right)^2 d_q t$$

$$+\, 2\sum_{k=0}^{n} p_{n,k}\,(q;x)\int_0^1 \left(\frac{[k]}{[n]} - x\right)^2 d_q t$$

$$\leq\;\; 4\sum_{k=0}^{n} p_{n,k}\,(q;x)\int_0^1 \frac{q^{2k}t^2}{[n+1]^2}\, d_q t$$

$$+\, 4\sum_{k=0}^{n} p_{n,k}\,(q;x)\int_0^1 \frac{q^{2n}[k]^2}{[n]^2[n+1]^2}\, d_q t$$

$$+\, 2\sum_{k=0}^{n} p_{n,k}\,(q;x)\left(\frac{[k]}{[n]} - x\right)^2$$

$$= \frac{4}{[3][n+1]^2}\sum_{k=0}^{n} p_{n,k}\,(q;x)(q^{2k})$$

$$+\, \frac{4}{[n+1]^2}\sum_{k=0}^{n} p_{n,k}\,(q;x)\,\frac{q^{2n}[k]^2}{[n]^2[n+1]^2} + 2B_{n,q}((t-x)^2, x)$$

$$\leq \frac{4}{[3][n+1]^2} + \frac{4}{[n+1]^2} + \frac{2}{[n]}x(1-x) \leq \frac{4}{[n]}\left(x(1-x) + \frac{1}{[n]}\right) \quad \blacksquare$$

By using a similar calculation we have :

$$K^*_{n,q}((t-x)^4, x) = \sum_{k=0}^{n} p_{n,k}\,(q;x)\int_0^1 \left(\frac{[k] + q^k t}{[n+1]} - x\right)^4 d_q t$$

$$= \sum_{k=0}^{n} p_{n,k}\,(q;x)\int_0^1 \left(\frac{[k] + q^k t}{[n+1]} - \frac{[k]}{[n]} + \frac{[k]}{[n]} - x\right)^4 d_q t$$

$$= \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{q^k t}{[n+1]} - \frac{q^n [k]}{[n][n+1]} + \frac{[k]}{[n]} - x \right)^4 d_q t$$

$$\leq 4 \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{q^k t}{[n+1]} - \frac{q^n [k]}{[n][n+1]} \right)^4 d_q t$$

$$+ 4 \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]}{[n]} - x \right)^4 d_q t$$

$$\leq \ 32 \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \frac{q^{4k} t^4}{[n+1]^4} d_q t$$

$$+ 32 \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \frac{q^{4n} [k]^4}{[n]^4 [n+1]^4} d_q t$$

$$+ 4 \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]}{[n]} - x \right)^4 d_q t$$

$$= \frac{32}{[5][n+1]^4} \sum_{k=0}^{n} p_{n,k}(q;x)(q^{4k}) + \frac{32}{[n+1]^4} \sum_{k=0}^{n} p_{n,k}(q;x) \frac{q^{4n}[k]^4}{[n]^4}$$

$$+ 4B_{n,q}((t-x)^4, x) \leq \frac{32}{[5][n+1]^4} + \frac{32}{[n+1]^4} + \frac{4C}{[n]^2} x(1-x)$$

$$\leq \frac{32}{[n]^4} + \frac{32}{[n]^4} + \frac{4C}{[n]^2} x(1-x)$$

$$\leq \frac{C}{[n]^2} \left(x(1-x) + \frac{1}{[n]^2} \right) \quad \blacksquare$$

Lemma 3.1.4 Assume that $0 < q_n < 1$, $q_n \longrightarrow 1$ and $q_n^n \longrightarrow a$ as $n \longrightarrow \infty$. Then we have

$$\lim_{n \longrightarrow \infty} [n]_{q_n} K_{n,q_n}^*(t-x;x) = -\frac{1+a}{2} x + \frac{1}{2},$$

$$\lim_{n \longrightarrow \infty} [n]_{q_n} K_{n,q_n}^*((t-x)^2;x) = -\frac{1}{3} x^2 - \frac{2}{3} a x^2 + x .$$

Proof. To give the proof of this lemma we are going to use formulas for $K^*_{n,q_n}(t;x)$ and $K^*_{n,q_n}(t^2;x)$ which was given before in lemma 3.2.

$$[n]_{q_n}K^*_{n,q_n}(t-x;x) = [n]_{q_n}\left\{K^*_{n,q_n}(t;x)-x\right\}$$

Taking the limit,

$$\lim_{n\to\infty}[n]_{q_n}K^*_{n,q_n}(t-x;x) = \lim_{n\to\infty}[n]_{q_n}\left\{\frac{2q_n}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}}x+\frac{1}{[2]_{q_n}}\frac{1}{[n+1]_{q_n}}-x\right\}$$

$$= \lim_{n\to\infty}\left\{[n]_{q_n}\left(\frac{2q_n}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}}-1\right)x+\frac{1}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}}\right\}$$

$$= \lim_{n\to\infty}\left\{[n]_{q_n}\left(\frac{-(1+q_n^{n+1})}{[2]_{q_n}[n+1]_{q_n}}\right)x+\frac{1}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}}\right\}$$

$$= \lim_{n\to\infty}\left\{\left(\frac{-[n]_{q_n}}{[n+1]_{q_n}}\frac{(1+q_n^{n+1})}{[2]_{q_n}}\right)x+\frac{1}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}}\right\}$$

$$= -\frac{1+a}{2}x+\frac{1}{2} \quad \blacksquare$$

$$\lim_{n\to\infty}[n]_{q_n}K^*_{n,q_n}((t-x)^2;x) = \lim_{n\to\infty}[n]_{q_n}K^*_{n,q_n}\left((t^2-2xt+x^2);x\right)$$

$$= \lim_{n\to\infty}[n]_{q_n}K^*_{n,q_n}\left(\left(t^2-x^2-2x(t-x)\right);x\right)$$

$$= \lim_{n\to\infty}[n]_{q_n}\left(K^*_{n,q_n}(t^2;x)-x^2-2xK^*_{n,q_n}\left((t-x);x\right)\right)$$

$$= \lim_{n\to\infty}[n]_{q_n}\left(\left(\frac{4{q_n}^3+{q_n}^2+q_n}{[2]_{q_n}[3]_{q_n}}\right)\frac{q_n[n]_{q_n}[n-1]_{q_n}}{[n+1]^2_{q_n}}x^2\right.$$

$$+\left(\frac{4{q_n}^3+5{q_n}^2+3q_n}{[2]_{q_n}[3]_{q_n}}\right)\frac{[n]_{q_n}}{[n+1]^2_{q_n}}x+\frac{1}{[n+1]^2_{q_n}[3]_{q_n}}-x^2$$

$$\left.-2xK^*_{n,q_n}\left((t-x);x\right)\right)$$

$$= \lim_{n\to\infty}[n]_{q_n}\left(\frac{q_n(q_n+2)}{[3]_{q_n}}\frac{[n]_{q_n}^2-[n]_{q_n}}{[n+1]_{q_n}^2}-1\right)x^2$$

$$-\lim_{n\to\infty}[n]_{q_n}\left(\frac{4q_n{}^3+5q_n{}^2+3q_n}{[2]_{q_n}[3]_{q_n}}\right)\frac{[n]_{q_n}}{[n+1]_{q_n}^2}x$$

$$+\lim_{n\to\infty}[n]_{q_n}\frac{1}{[n+1]_{q_n}^2[3]_{q_n}}-\lim_{n\to\infty}[n]_{q_n}\,2xK_{n,q_n}^*\big((t-x);x\big)$$

$$= \lim_{n\to\infty}q_n(1-q_n^n)(2q_n+q_n^2+2)\,x^2-\lim_{n\to\infty}(4q_n+3q_n^2+2q_n^3)\,x^2$$

$$+\lim_{n\to\infty}\frac{4q_n{}^3+5q_n{}^2+3q_n}{[2]_{q_n}[3]_{q_n}}x-2xK_{n,q_n}^*\big((t-x);x\big)$$

$$= 5(1-a)x^2-9x^2+2x-(1-a)x+x$$

3.2 Local and Global Approximation

First we consider the following K-functional:

$$K_2(f,\delta^2) := inf\{\|f-g\|+\delta^2\|g''\|: g\in C^2[0,1]\},\qquad \delta\geq 0,$$

where

$$C^2[0,1] := \{g: g,g',g''\ \in C[0,1]\}.$$

Then from the known result [10], there exists an absolute constant $C_0\geq 0$ such that

$$K_2(f,\delta^2)\leq C_0\omega_2(f,\delta) \tag{3.2.1}$$

where

$$\omega(f,t) := \sup_{|x-y|\leq t}|f(x)-f(y)|$$

and

$$\omega_2(f,\delta) := \sup_{0<h\leq\delta}\sup_{x\pm h\in[0,1]}|f(x-h)-2f(x)+f(x+h)|$$

is the second modulus of smoothness of $f\in C[0,1]$. Our first main result is stated below.

Theorem 3.2.1. There exists an absolute constant $C > 0$ such that

$$\left|K_{n,q}^{*}(f;x) - f(x)\right| \le C\omega_2\left(f, \sqrt{\frac{\delta_n(x)}{[n]}}\right) + \omega\left(f, \left|\frac{(1+q^{n+1})x-1}{[2][n+1]}\right|\right),$$

where

$$f \in C[0,1], \delta_n(x) = \varphi^2(x) + \frac{1}{[n]}, \varphi^2(x) = x(1-x), 0 \le x \le 1 \text{and } 0 < q < 1.$$

Proof. Let

$$\widetilde{K_{n,q}^{*}}(f;x) = K_{n,q}^{*}(f;x) + f(x) - f(a_n x + b_n),$$

where $f \in C[0,1], a_n = \frac{2q}{1+q}\frac{[n]}{[n+1]}$ and $b_n = \frac{1}{1+q}\frac{1}{[n+1]}$. Using the Taylor formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)ds, \quad g \in C^2[0,1],$$

we have

$$\widetilde{K_{n,q}^{*}}(g;x) = K_{n,q}^{*}(g;x) + g(x) - g(a_n x + b_n),$$

since

$$K_{n,q}^{*}(g;x) = g(x) + g'(x)K_{n,q}^{*}(t;x) - xg'(x) + K_{n,q}^{*}\left(\int_x^t (t-s)g''(s)ds\ ;x\right)$$

$$= g(x) + g'(x)(a_n x + b_n) - xg'(x) + K_{n,q}^{*}\left(\int_x^t (t-s)g''(s)ds\ ;x\right)$$

$$g(a_n x + b_n) = g(x) + g'(x)(a_n x + b_n - x) + \int_x^{a_n x + b_n} (a_n x + b_n - s)g''(s)ds$$

we get

$$\widetilde{K_{n,q}^*}(g;x) = g(x) + g'(x)(a_nx+b_n) - xg'(x) + K_{n,q}^*\left(\int_x^t (t-s)g''(s)ds\ ;x\right)$$

$$+ g(x) - g(x) - g'(x)(a_nx+b_n-x)$$

$$-\int_x^{a_nx+b_n}(a_nx+b_n-s)g''(s)ds$$

$$= g(x) + K_{n,q}^*\left(\int_x^t (t-s)g''(s)ds\ ;x\right)$$

$$-\int_x^{a_nx+b_n}(a_nx+b_n-s)g''(s)ds$$

$$\widetilde{K_{n,q}^*}(g;x) - g(x) = K_{n,q}^*\left(\int_x^t (t-s)g''(s)ds\ ;x\right)$$

$$-\int_x^{a_nx+b_n}(a_nx+b_n-s)g''(s)ds\ , \qquad g \in C^2[0,1].$$

hence

$$\left|\widetilde{K_{n,q}^*}(g;x) - g(x)\right|$$

$$\le K_{n,q}^*\left(\int_x^t |t-s||g''(s)|ds\ ;x\right)$$

$$+\left|\int_x^{a_nx+b_n}|a_nx+b_n-s||g''(s)|ds\right|$$

$$\le \|g''\|K_{n,q}^*((t-x)^2;x) + \|g''\|(a_nx+b_n-x)^2$$

$$\le \|g''\|\left\{\frac{4}{[n]}\left(x(1-x)+\frac{1}{[n]}\right)+\frac{4}{[n]^2}x^2+\frac{2}{[n]^2}\right\}$$

$$\le \|g''\|\left\{\frac{10}{[n]}\left(x(1-x)+\frac{1}{[n]}\right)\right\}$$

$$= \frac{10}{[n]}\delta_n(x)\|g''\| \qquad (3.2.2)$$

Using (3.2.2) and the uniform boundedness of $\widetilde{K_{n,q}^*}$ we get

$$K_{n,q}^{*}(f;x) = \widetilde{K_{n,q}^{*}}(f;x) - f(x) + f(a_n x + b_n),$$

$$\begin{aligned} K_{n,q}^{*}(f;x) - f(x) \\ &= \widetilde{K_{n,q}^{*}}(f;x) - f(x) + f(a_n x + b_n) - f(x) + g(x) - g(x) \\ &+ \widetilde{K_{n,q}^{*}}(g;x) - \widetilde{K_{n,q}^{*}}(g;x) \\ &= \widetilde{K_{n,q}^{*}}(f-g;x) + \left(\widetilde{K_{n,q}^{*}}(g;x) - g(x)\right) + \left(g(x) - f(x)\right) \\ &+ \left(f(a_n x + b_n) - f(x)\right) \end{aligned}$$

so

$$\begin{aligned} \left|K_{n,q}^{*}(f;x) - f(x)\right| \\ &\leq \left|\widetilde{K_{n,q}^{*}}(f-g;x)\right| + \left|\widetilde{K_{n,q}^{*}}(g;x) - g(x)\right| + |f(x) - g(x)| \\ &+ |f(a_n x + b_n) - f(x)|. \end{aligned}$$

On the other hand

$$\begin{aligned} \widetilde{K_{n,q}^{*}}(f-g;x) &= K_{n,q}^{*}(f-g;x) + (f-g)(x) - (f-g)(a_n x + b_n) \\ &\leq K_{n,q}^{*}(f-g;x) + (f-g)(x) + (f-g)(x) \\ \left|\widetilde{K_{n,q}^{*}}(f-g;x)\right| &\leq \left|K_{n,q}^{*}(f-g;x)\right| + |f(x) - g(x)| + |f(x) - g(x)| \\ &\leq \|f(x) - g(x)\| + \|f(x) - g(x)\| + \|f(x) - g(x)\| \\ &= 3\|f(x) - g(x)\| \end{aligned}$$

and we have

$$|x - y| \leq \omega|x - y|$$

$$|a_n x + b_n - x| \leq \omega|a_n x + b_n - x| = \omega|(a_n - 1)x + b_n|$$

$$\left|K_{n,q}^{*}(f;x)-f(x)\right| \leq 4\|f-g\| + \frac{10}{[n]}\delta_n(x)\|g''\| + \omega(f|(a_n-1)x+b_n|)$$

$$\leq 10\left(\|f-g\| + \frac{\delta_n(x)}{[n]}\|g''\|\right) + \omega(f|(a_n-1)x+b_n|)$$

Taking the infimum on the right hand side over all $g \in C^2[0,1]$, we get

$$\left|K_{n,q}^{*}(f;x)-f(x)\right| \leq 10K_2\left(f,\frac{\delta_n(x)}{[n]}\right) + \omega(f|(a_n-1)x+b_n|)$$

We know that

$$a_n - 1 = \frac{-(1+q^{n+1})}{[2][n+1]}, \qquad \text{and } b_n = \frac{1}{[2][n+1]}$$

then

$$|(a_n-1)x+b_n| = \left|\frac{(1+q^{n+1})-1}{[2][n+1]}\right|$$

By using (3.2.2) we obtain

$$\left|K_{n,q}^{*}(f;x)-f(x)\right| \leq C\omega_2\left(f,\sqrt{\frac{\delta_n(x)}{[n]}}\right) + \omega\left(f\left|\frac{(1+q^{n+1})-1}{[2][n+1]}\right|\right) \ \blacksquare$$

Corollary 3.2.1. Let q_n be a sequence such that $q_n \in (0,1), q_n \to 1\ as\ n \to \infty$. For any $f \in C^2[0,1]$ we have

$$\lim_{n\to\infty}\left\|K_{n,q}^{*}(f)-f\right\| = 0.$$

Next we present the direct global approximation theorem for the operators $K_{n,q}^{*}$. In order to state the theorem we need the weighted K-functional of second order for $f \in C[0,1]$ defined by

$$K_{2,\phi}(f,\delta^2) := \inf\{\|f-g\| + \delta^2\|\phi^2 g''\| : g \in W^2(\varphi)\}, \delta \geq 0, \varphi^2(x) = x(1-x)$$

where

$$W^2(\varphi) := \{g \in C[0,1] : g' \in AC[0,1], \qquad \varphi^2 g'' \in C[0,1]\},$$

and $g' \in AC[0,1]$ means that g is differentiable and g' is absolutely continuous in $[0,1]$. Moreover, the Ditzian-Totik modulus of second order is given by

$$\omega_2^{\varphi}(f,\delta) := \underset{0<h\leq\delta}{sup} \ \underset{x\pm h\varphi(x)\in[0,1]}{sup} |f(x-\varphi(x)h) - 2f(x) + f(x+\varphi(x)h)|$$

It is well known that the K-functional $K_{2,\phi}(f,\delta^2)$ and the Ditzian-Totik modulus $\omega_2^{\varphi}(f,\delta)$ are equivalent (see [10])

Theorem 3.2.2. There exists an absolute constant $C > 0$ such that

$$\left\|K_{n,q}^*(f) - f\right\| \leq C\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{[n]}}\right) + \overrightarrow{\omega_{\psi}}\left(f, \frac{1}{[n]}\right),$$

where $\in C[0,1], 0 < q < 1, \varphi^2(x) = x(1-x), \psi(x) = 2x + 1$.

Proof. Let

$$\widetilde{K_{n,q}^*}(f;x) = K_{n,q}^*(f;x) + f(x) - f(a_n x + b_n),$$

where $f \in C[0,1], a_n = \frac{2q}{1+q}\frac{[n]}{[n+1]}$ and $b_n = \frac{1}{1+q}\frac{1}{[n+1]}$. Using the Taylor formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)ds, \quad g \in W^2(\varphi)\,,$$

we have

$$\widetilde{K_{n,q}^*}(g;x) = g(x) + K_{n,q}^*\left(\int_x^t (t-x)g''(s)ds \ ;x\right)$$

$$-\int_x^{a_nx+b_n} (a_nx+b_n-s)g''(s)ds \quad g \in W^2(\varphi)\,,$$

Hence

$$\left|\widetilde{K_{n,q}^*}(g;x) - \ g(x)\right|$$

$$\leq K_{n,q}^*\left(\int_x^t |t-s||g''(s)|ds\,;x\right)$$

$$+\left|\int_x^{a_nx+b_n} |a_nx+b_n-s||g''(s)|ds\right|. \tag{3.2.3}$$

Since the function δ_n^2 is concave on $[0,1]$, we have for $u = t+\tau(x-t), \tau \in [0,1]$, the following estimate

$$\frac{|t-s|}{\delta_n^2(s)} = \frac{\tau|x-t|}{\delta_n^2(s)} \leq \frac{\tau|x-t|}{\delta_n^2(t)+\tau\big(\delta_n^2(x)-\delta_n^2(t)\big)} \leq \frac{|x-t|}{\delta_n^2(x)}.$$

Hence, by (3.2.3), we find

$$\left|\widetilde{K_{n,q}^*}(g;x) - \ g(x)\right|$$

$$\leq \|\delta_n^2 g''\| K_{n,q}^*\left(\left|\int_x^t \frac{|t-s|}{\delta_n^2(s)}ds\right|;x\right)$$

$$+ \|\delta_n^2 g''\|\left|\int_x^{a_nx+b_n} \frac{|a_nx+b_n-s|}{\delta_n^2(s)}ds\right|$$

$$\leq \frac{\|\delta_n^2 g''\|}{\delta_n^2(x)}\big(K_{n,q}^*((t-x)^2;x) + (a_nx+b_n-x)^2\big)$$

$$\leq \frac{\|\delta_n^2 g''\|}{\delta_n^2(x)}\left\{\frac{4}{[n]}\left(x(1-x)+\frac{1}{[n]}\right)+\frac{4}{[n]^2}x^2+\frac{2}{[n]^2}\right\}$$

$$\leq \frac{\|\delta_n^2 g''\|}{\delta_n^2(x)}\left\{\frac{10}{[n]}\left(x(1-x)+\frac{1}{[n]}\right)\right\}=\frac{10}{[n]}\|\delta_n^2 g''\|.$$

Since

$$\|\delta_n^2 g''\| \leq \|\varphi^2 g''\| + \frac{1}{[n+1]}\|g''\|$$

we have

$$\left|\widetilde{K_{n,q}^*}(g;x)-\ g(x)\right| \leq \frac{10}{[n]}\left(\|\varphi^2 g''\|+\frac{1}{[n]}\|g''\|\right). \tag{3.2.4}$$

Using (3.2.4) and the uniform boundedness of $\widetilde{K_{n,q}^*}$we get

$$\left|K_{n,q}^*(f;x)-f(x)\right|$$

$$\leq \left|\widetilde{K_{n,q}^*}(f-g;x)\right|+\left|\widetilde{K_{n,q}^*}(g;x)-g(x)\right|+|f(x)-g(x)|$$

$$+\ |f(a_n x+b_n)-f(x)|$$

$$\leq 4\|f-g\|+\frac{10}{[n]}\left(\|\varphi^2 g''\|+\frac{1}{[n]}\|g''\|\right)+|f(a_n x+b_n)-f(x)|.$$

Taking the infimum on the right hand side over all $g \in W^2(\varphi)$, we obtain

$$\left|K_{n,q}^*(f;x)-f(x)\right| \leq 10K_{2,\varphi}\left(f,\frac{1}{[n]}\right)+|f(a_n x+b_n)-f(x)|. \tag{3.2.5}$$

On the other hand

$$|f(a_n x+b_n)-f(x)| = \left|f\left(x+\psi(x)\big((a_n-1)x+b_n\big)\right)-f(x)\right|$$

$$\leq \sup\left|f\left(x+\psi(t)\left(-\frac{1+q^{n+1}}{\psi(x)[2][n+1]}x+\frac{1}{\psi(x)[2][n+1]}\right)\right)-f(x)\right|$$

$$\leq \overrightarrow{\omega_\psi}\left(f;\left|-\frac{1+q^{n+1}}{\psi(x)[2][n+1]}x+\frac{1}{\psi(x)[2][n+1]}\right|\right)$$

$$\leq \overrightarrow{\omega_\psi}\left(f;\frac{\left|K_{n,q}^*(t;x)-x\right|}{\psi(x)}\right)\leq \overrightarrow{\omega_\psi}\left(f;\frac{2x+1}{\psi(x)[2][n]}\right). \qquad (3.2.6)$$

Hence, by (3.2.5) and (3.2.6), using the equivalence of $K_{2,\varphi}\left(f,\frac{1}{[n]}\right)$ and the Ditzian-Totik modulus $\omega_2^\varphi\left(f,\sqrt{\frac{1}{[n]}}\right)$ we get the desired estimate.

Next we give the proof of Voronovskaja type result for q-Bernstein-Kantorovich operators.

Theorem 3.2.3. Assume that $q_n\in(0,1), q_n\to 1$ and $q_n^n\to a$ as $n\to\infty$. For any $f\in C^2[0,1]$ the following equality holds

$$\lim_{n\to\infty}[n]_{q_n}\left(K_{n,q_n}^*(f;x)-f(x)\right)$$

$$=f'(x)\left(-\frac{1+a}{2}x+\frac{1}{2}\right)+\frac{1}{2}f''(x)\left(-\frac{1}{3}x^2-\frac{2}{3}ax^2+x\right),$$

uniformly on $[0,1]$.

Proof. Let $f\in C^2[0,1]$ and $x\in[0,1]$ be fixed. By the Taylor formula we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2, \qquad (3.2.7)$$

where $r(t;x)$ is the Peano form of the remainder, $r(.;x)\in C[0,1]$ and $\lim_{t\to x}r(t;x)=0$. Applying K_{n,q_n}^* to (3.2.7) we get

$$K_{n,q_n}^*(f;x)=f(x)+f'(x)K_{n,q_n}^*\left((t-x);x\right)+\frac{1}{2}f''(x)K_{n,q_n}^*((t-x)^2;x)$$

$$+K_{n,q_n}^*(r(t;x)(t-x)^2;x)$$

$$K_{n,q_n}^*(f;x) - f(x)$$

$$= f'(x)K_{n,q_n}^*(t-x;x) + \frac{1}{2}f''(x)K_{n,q_n}^*((t-x)^2;x)$$

$$+ K_{n,q_n}^*(r(t;x)(t-x)^2;x)$$

multiplying both side by $[n]_{q_n}$ we get

$$[n]_{q_n}\left(K_{n,q_n}^*(f;x) - f(x)\right)$$

$$= f'(x)[n]_{q_n}K_{n,q_n}^*(t-x;x) + \frac{1}{2}f''(x)[n]_{q_n}K_{n,q_n}^*((t-x)^2;x)$$

$$+ [n]_{q_n}K_{n,q_n}^*(r(t;x)(t-x)^2;x)$$

By the Cauchy-Schwartz inequality, we have

$$K_{n,q_n}^*(r(t;x)(t-x)^2;x) \leq \sqrt{K_{n,q_n}^*(r^2(t;x);x)}\sqrt{K_{n,q_n}^*((t-x)^4;x)}\,. \qquad (3.2.8)$$

Observe that $r^2(x;x) = 0$ and $r^2(.\,;x) \in C[0,1]$. Then it follows from Corollary 3.2.1 that

$$\lim_{n\to\infty} K_{n,q_n}^*(r^2(t;x);x) = r^2(x;x) = 0 \qquad (3.2.9)$$

uniformly with respect to $x \in [0,1]$. Then from (3.2.8) and (3.2.9) we get immediately

$$\lim_{n\to\infty}[n]_{q_n} K_{n,q_n}^*(r(t;x)(t-x)^2;x) = 0. \qquad (3.2.10)$$

Now from (3.2.10) and Lemma 3.1.4 we get

$$\lim_{n\to\infty}[n]_{q_n}\left(K_{n,q_n}^*(f;x) - f(x)\right)$$

$$= f'(x)\left(-\frac{1+a}{2}x + \frac{1}{2}\right) + \frac{1}{2}f''(x)\left(-\frac{1}{3}x^2 - \frac{2}{3}ax^2 + x\right) \blacksquare$$

Chapter 4

CONCLUSION

As a result here in this thesis we studied a q-generalization of Bernstein-Kantorovich polynomial operators

$$K_{n,q}^*(f;x) := \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t,$$

where $f \in C[0,1],\ 0 < q < 1\ \ and$

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k}, \quad (1-x)_q^n = \prod_{s=0}^{n-1} (1-q^s\,\mathrm{x})\,.$$

We calculated the moments of $K_{n,q}^*$. We studied local and global convergence properties of the q- Bernstein-Kantorovich operators and proved Voronovskaja-type asymptotic formula for these operators. We found a recurrence formula for q-Bernstein-Kantorovich operator and obtain explicit formulas for $K_{n,q}^*(t,x)$, $K_{n,q}^*(t^2,x)$. We found estimations for second and fourth order central moments of the q-Bernstein Kantorovich polynomials.

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