Adomian’s Decomposition of Multi-Order Fractional Differential Equations

Ojo Gbenga Olayinka

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Prof. Dr. Cem Tanova
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Applied Mathematics and Computer Science.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Applied Mathematics and Computer Science.

Prof. Dr. Nazim Mahmudov
Supervisor

Examining Committee

1. Prof. Dr. Nazim Mahmudov
2. Asst. Prof. Dr. Mustafa Kara
3. Asst. Prof. Dr. Suzan Buranay

ABSTRACT

Adomian's Decomposition Method (ADM) was introduced about three decades ago, it has proven to be efficient, reliable and easy to compute the solutions of non-linear and linear differential equations. It can also be used to compute various types of equations such as Boundary value problems, Integral equations, Equations arising in fluid flow e.t.c.

This thesis work presents the derivation of Adomian's decomposition algorithms and the possible solution of fractional differential equations of the multi-order type in the Caputo sense. It consist of four chapters, Chapter 1 contains a brief introduction of Adomian's Decomposition Method(ADM) and definitions, while the second chapter deals with basis proofs and methodology with respect to Adomian's Decomposition Method(ADM). In Chapter 3, we applied the method of solution to multi-order fractional differential equations. We then discuss the results and make conclusion in Chapter 4.

Keywords: Adomian’s Algorithm, Caputo’s Derivative, Multi-Order Fraction Differential Equation.

ÖZ

Adomian’ın Ayrıştırma Yöntemi üç yıl önce tanımlanmıştır. Bu yöntemin lineer olmayan diferansiyel denklemlerin çözümlerini hesaplamak için verimli, güvenilir ve kolay olduğu kanıtlanmıştır. Ayrıca bu yöntem sınır değer problemleri, Rntegel denklemleri ve sıvı akışkan denklemleri gibi denklemleri hesaplamak için kullanılır.

Bu tez çalışmasında Adomian’in ayrıştırma yöntemi algoritmaları türetme ve Caputo tipli çok basamaklı fraksiyonel diferensiel denklemlerin olsaı çözümleri ifade edilmiştir. Bu tez dört bölümden oluşmaktadır. İlk bölümde Adomian’ın Ayrıştırma Yöntemi hakkında gerekli temel bilgiler ve tanımlar verilmiştir. İkinci bölümde Adomian’ın ayrıştırma yöntemi’nin metodolojisi ve bu yöntemle ilgili temel kanıtlar verilmiştir.

Üçüncü bölümde ise bu yöntemi çok basamaklı kesirli diferansiyel denklemlerin çözümünde uyguladık. Dördüncü bölümde ise bulduğumuz sonuçları tartışıp ve sonucu yazdık.

Anahtar kelimeler: Adomian'ın ayrıştırma yöntemi, Caputo Derivasyon, Çok Basamakli Kesirli diferansiyel denklemler

DEDICATION

To My Family

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Chapter 1

INTRODUCTION

This Chapter consist of Preliminary concept of Adomian's Decomposition Method (ADM) with some basic definitions. A new technique for solving Non-linear fractional differential equation was initiated by George Adomian in the 1980's, called Adomian's Decomposition Method (ADM) [6]. The procedure involves separating equations concern into Linear and Non-linear parts and treated accordingly with consideration of any given conditions. The non-linear part is decomposed into polynomial series called Adomian's polynomial and results are generated in form of a recursive series.

The issue of convergence of ADM is of great concern, several researchers investigated the convergence and concluded that the method is convergent, which produces a convergence series solution, truncating this series solution result to an approximate solution [9], [5], [7], [6].

Furthermore Charruault et al. [6] establish that the series produced by ADM is absolutely convergent as well as uniformly convergent. Since the series converges rapidly, having higher order of converge is desirable. Babolian and Bizar [5] provided a method to determine the order of convergence.

1.1 Definitions

Definition 1.1.1: Consider the following equation:

$$D_t^{\breve{\sigma}} = \frac{d^{\breve{\sigma}} y}{dt^{\breve{\sigma}}} = f\left(t, y(t), y_1^{\hat{\phi}}(t), y_2^{\hat{\phi}}(t), ..., y_n^{\hat{\phi}}(t)\right) \qquad (1)$$

subject to the incipient condition:

$$y^i(0) = \rho_{i,} \quad i = 0,1,2,....,n-1 \qquad (2)$$

where $\breve{\sigma} \geq \hat{\phi} \geq \phi_{m-1} \geq ..... \geq \hat{\phi} \geq 0, \;\; n-1 \leq \breve{\sigma} \leq n, \quad \forall n \in N.$

These equations are called multi-order fractional derivative equation, the equation (1) and (2) are examined in Caputo sense using ADM technique as the method of solution, because it makes the given equation to have a unique solution. Caputo fractional integral operator is a modification of Riemman-liouville fractional integral operator. If considered in the sense of Riemman-liouville, it is necessary to describe the incipient conditions in terms of fractional integrals and derivatives. The advantage of ADM is the possible avoidance of discretization which provides a coherent numerical solution with high accuracy and minimal calculations, making it less expensive to compute.

Definition 1.1.2: The Riemman-Liouville fractional integral operator of $\breve{\sigma} \geq 0$ is express as:

$$j^{\breve{\sigma}} f(x) = \frac{1}{\Gamma(\breve{\sigma})} \int_0^x (x-t)^{\breve{\sigma}-1} f(t) dt, \quad x > 0, \;\; \breve{\sigma} > 0, \qquad (3)$$

$$j^0 f(x) = f(x). \qquad (4)$$

The fractional derivative of $f(x)$ in Caputo's sense is express as:

$$D^{\breve{\sigma}} f(x) = j^{n-\breve{\sigma}} D^n f(x) = \frac{1}{\Gamma(n-\breve{\sigma})} \int_0^x (x-t)^{n-\breve{\sigma}-1} f^n dt \qquad (5)$$

for $x>0$, $n-1<\breve{\sigma}\le n, \quad \forall n\in m, \quad f\in C_1^n$.

They both have various properties described in literatures [11].

Properties of the $j^{\hat{\phi}}$:

$$j^{\hat{\phi}}x^{\breve{\sigma}}=\frac{\Gamma(\breve{\sigma}+1)}{\Gamma(\hat{\phi}+\breve{\sigma}+1)}x^{\hat{\phi}+\breve{\sigma}} \tag{6}$$

$$j^{\hat{\phi}}j^{\breve{\sigma}}f(x)=j^{\breve{\sigma}}j^{\hat{\phi}}f(x) \tag{7}$$

$$j^{\hat{\phi}}j^{\breve{\sigma}}=j^{\hat{\phi}+\breve{\sigma}}. \tag{8}$$

Let $m-1<\hat{\phi}<m, \quad \forall m\in N$ and $f\in C_\mu^m,\ \mu\ge -1$, so:

$$j^{\hat{\phi}}f(x)=f(x) \tag{9}$$

$$j^{\hat{\phi}}D^{\hat{\phi}}=f(x)-\sum_{n=0}^{m-1}\frac{x^n}{n!}f^{(n)}(0) \tag{10}$$

Definition 1.1.3: A function $f(x);x>0$ is contained in the space $C_\mu,\mu\in R$, $\ni k\in R$:

$$f(x_0)=x^k f_1^{'}(x) \tag{11}$$

$\forall f_1(x)\in C[0,x],\ C_\mu\subset C_\beta$ if $\breve{\sigma}<\mu$

Chapter 2

ADOMIAN'S DECOMPOSITION METHOD

In this Chapter we considered an important special function known as the Gamma Function and briefly describe the method of solution.

Consider the following equation.

2.1 The Gamma Function

The Gamma Function $\Gamma(n)$ can be defined as:

$$\Gamma(n) = \int_0^\infty e^{-s} s^{n-1} ds, \quad n \in R \tag{12}$$

It is convergent on the plane $\mathrm{Re}(n) > 0$.

Lemma 2.1.1: if $p \in C$ with $\mathrm{Re}(n) > 0$ then

$$\Gamma(p+1) = p\Gamma(p) \tag{13}$$

Proof: Using integration by part:

$$\Gamma(n+1) = \int_0^\infty e^{-s} s^n ds = -e^s s^n \Big|_0^\infty + n\int e^s s^{n-1} ds = n\Gamma(n) | \tag{14}$$

Where $\Gamma(1) = 1$ and for $n = 2, 3, \ldots$

$$\Gamma(2) = 1\Gamma(1)$$
$$\Gamma(3) = 2\Gamma(2)$$
$$\Gamma(4) = 3\Gamma(3)\ldots$$
$$\Gamma(n+1) = n\Gamma(n) = n(n-1)(n-2)\ldots = n$$

2.2 Method of Solution

Equation (1) is express in Adomian's form and pattern as follows:

$$D_t^{\breve{\sigma}} y(t)+Ly(t)+Ny(t)=\rho(t) \quad (15)$$

L is the linear operator,

N is the non-linear operator,

$D_t^{\breve{\sigma}}$ is the fractional derivative of order $\breve{\sigma}$,

$\rho(t)$ is the source term.

ADM is base on applying $j^{\breve{\sigma}}$ to equation (15). Substitution of equation (17) into equation (15) we have:

$$D_t^{\breve{\sigma}} y(t)=\rho(t)-L\sum_{k=0}^{\infty} y_k(t)-\sum_{k=0}^{\infty} A_k \quad (16)$$

Where $Ny(t)=\sum_{k=0}^{\infty} A_k$ and $Ly(t)=\sum_{k=0}^{\infty} y_k(t)$ $\quad (17)$

A_k is the Adomian's polynomial. We shall derive it by using Taylor series expansion with generalization of multi variable function

Let $f=\varphi\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)$ $\quad (18)$

Consider the power series expansion of the function $f(\lambda)$ about the point λ_0

$$f(\lambda)=A_0+A_1(\lambda_*-\lambda_0)+A_2(\lambda_*-\lambda_0)^2+A_3(\lambda_*-\lambda_0)^3+\ldots \quad (19)$$

We determine the values of $A_0, A_1, A_2, A_3, \ldots$

If $\lambda_*-\lambda_0=0 \Rightarrow \lambda_*=\lambda_0$ $\quad (20)$

$\Rightarrow f(\lambda_0=A_0)$ $\quad (21)$

The derivative of equation (19) is as follows:

$$\frac{df(\lambda)}{d\lambda}=A_1+2A_2(\lambda_*-\lambda_0)+3A_3(\lambda_*-\lambda_0)^2+4A_4(\lambda_*-\lambda_0)^3+\ldots \qquad (22)$$

Similarly, if :

$$\lambda_*=\lambda_0,\ \frac{df\ (\lambda)}{d\lambda}=A_1\Rightarrow A_1=\frac{df\ (\lambda)}{d\lambda}|_{\lambda=0}\ . \qquad (23)$$

We consider again the second and third derivative after which we generalize.

$$\frac{d^2f\ (\lambda)}{d\lambda^2}=2A_2+6A_3(\lambda_*-\lambda_0)+12A_4(\lambda_*-\lambda_0)^2+60A_5(\lambda_*-\lambda_0)^3+\ldots\ . \qquad (24)$$

Similarly, if

$$\lambda_*=\lambda_0,\ \frac{d^2f\ (\lambda)}{d\lambda^2}=2A_2\Rightarrow 2A_2=\frac{1}{2}\left(\frac{d^2f(\lambda)}{d\lambda^2}|_{\lambda=0}\right) \qquad (25)$$

$$\frac{d^3f(\lambda)}{d\lambda^3}=6A_3(\lambda_*-\lambda_0)+24A_4(\lambda_*-\lambda_0)+60A_5(\lambda_*-\lambda_0)^2+120A_6(\lambda_*-\lambda_0)^3+\ldots . \qquad (26)$$

Similarly, if :

$$\lambda_*=\lambda_0,\ \frac{d^3f\ (\lambda)}{d\lambda^3}=6A_3\Rightarrow A_3=\frac{1}{6}\left(\frac{d^3f\ (\lambda)}{d\lambda^3}|_{\lambda=0}\right). \qquad (27)$$

If we continue in same manner, we obtain:

$$A_\rho=\frac{1}{\rho!}\left[\left(\frac{d^\rho f\ (\lambda)}{d\lambda^\rho}|_{\lambda=0}\right)\right], \qquad (28)$$

where $f=\varphi\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)$ the multi- variable function, we have

$$A_\rho = \frac{1}{\rho!}\left[\left(\frac{d^\rho}{d\lambda^\rho}\varphi\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right)\right]_{\lambda=0}, \tag{29}$$

these are the Adomian Algorithms.

Substitute equation (17) into (15) we get:

$$D_t^{\breve{\sigma}} y(t) = g(t) - L\sum_{\rho=0}^{\infty} y_\rho(t) - \sum_{\rho=0}^{\infty} A_\rho, \tag{30}$$

but $j^{\breve{\sigma}} D^{\breve{\sigma}} = f(x) - \sum_{\rho=0}^{n-1} f^\rho(0^+)\frac{x^\rho}{\rho!}, x>0$. operating $j^{\breve{\sigma}}$ on equation (30) we have:

$$y(t) = \sum_{\beta=0}^{n-1} y^i(0^+)\frac{t^i}{i!} + j^{\breve{\sigma}} g(t) - j^{\breve{\sigma}} L\sum_{\rho=0}^{\infty} y_\rho(t) - j^{\breve{\sigma}}\sum_{\rho=0}^{\infty} A_\rho \tag{31}$$

$$\Rightarrow y(t) = \sum_{i=0}^{n-1} y^i(0^+)\frac{t^i}{i!} + j^{\breve{\sigma}} g(t) - j^{\breve{\sigma}} L y_\rho(t) - j^{\breve{\sigma}} N y(t), \tag{32}$$

where $y_0 = \sum_{i=0}^{n-1} y^i(0^+)\frac{t^i}{i!} + j^{\breve{\sigma}} g(t).$ $\qquad (33)$

Therefore :

$$y_1 = -j^{\breve{\sigma}} L y_0 - j^{\breve{\sigma}} A_0 \tag{34}$$

$$y_2 = -j^{\breve{\sigma}} L y_1 - j^{\breve{\sigma}} A_1 \tag{35}$$

$$y_3 = -j^{\breve{\sigma}} L y_2 - j^{\breve{\sigma}} A_2 \tag{36}$$

$$y_4 = -j^{\breve{\sigma}} L y_3 - j^{\breve{\sigma}} A_3 \ldots \ . \tag{37}$$

The result is given in series form:

$$y(t) = \sum_{\rho=0}^{\infty} y_\rho(t). \tag{38}$$

We generate the Adomian's Polynomial with the derived algorithms as follows:

$$A_\rho = \frac{1}{\rho!}\left[\frac{d^\rho}{d\eta^\rho}\varphi\left(\sum_{i=0}^{\infty}\eta^i y_i\right)\right]_{\eta=0} \tag{39}$$

when $\rho = 0$ we have:

$$A_0 = \frac{1}{0!}\left[\frac{d^0}{d\eta^0}\varphi\left(\eta^0 y_0\right)\right]_{\eta=0} = \varphi\left(y_0\right) \tag{40}$$

$\Rightarrow A_1 = \varphi\left(y_0\right)$ when $\rho = 1$ we get:

$$A_1 = \frac{1}{1!}\left[\frac{d}{d\eta}\varphi\left(\sum_{i=0}^{1}\eta^i y_i\right)\right]_{\eta=0} = 1\left[\frac{d}{d\eta}\varphi\left(\eta y + \eta^0 y_0\right)\right]_{\eta=0} \tag{41}$$

$\Rightarrow A_1 = y_1\varphi\left(y_0\right)$, when $\rho = 2$ we get:

$$A_2 = \frac{1}{2!}\left[\frac{d^2}{d\eta^2}\varphi\left(\sum_{i=0}^{2}\eta^i y_i\right)\right]_{\eta=0} = \frac{1}{2}\left[\frac{d^2}{d\eta^2}\varphi\left(\eta^2 y_2 + \eta y + \eta^0 y_0\right)\right]_{\eta=0} \tag{42}$$

$\Rightarrow A_2 = y_2\varphi^{'}\left(y_0\right) + \frac{y^2}{2!}\varphi^{''}\left(y_0\right)$, when $\rho = 3$ we get:

$$A_3 = \frac{1}{3!}\left[\frac{d^3}{d\eta^3}\varphi\left(\sum_{i=0}^{3}\eta^i y_i\right)\right]_{\eta=0} = \frac{1}{6}\left[\frac{d^3}{d\eta^3}\varphi\left(\eta^3 y_3 + \eta^2 y_2 + \eta y + \eta^0 y_0\right)\right]_{\eta=0} \tag{43}$$

$\Rightarrow A_3 = y_3\varphi^{'}\left(y_0\right) + y_1 y_2\varphi^{''}\left(y_0\right)\frac{y^3}{3!}\varphi^{'''}\left(y_0\right)$, when $\rho = 4$ we have:

$$A_4 = \frac{1}{4!}\left[\frac{d^4}{d\eta^4}\varphi\left(\sum_{i=0}^{4}\eta^i y_i\right)\right]_{\eta=0} = \frac{1}{4!}\left[\frac{d^4}{d\eta^4}\varphi\left(\eta^4 y_4 + \eta^3 y_3 + \eta^2 y_2 + \eta y + \eta^0 y_0\right)\right]_{\eta=0} \tag{44}$$

$\Rightarrow A_4 = y_4\varphi^{'}\left(y_0\right) + \left(y_1 y_3\varphi^{''}\left(y_0\right) + \frac{1}{2!}y_2^2\right) + y_1^2 y_2\varphi^{'''}\left(y_0\right) + \frac{y^4}{4!}\varphi^{iv}\left(y_0\right)$, when $\rho = 5$ we have:

$$A_4 = \frac{1}{5!}\left[\frac{d^5}{d\eta^5}\varphi\left(\sum_{i=0}^{5}\eta^i y_i\right)\right]_{\eta=0} = \frac{1}{5!}\left[\frac{d^5}{d\eta^5}\varphi\left(\eta^5 y_5 + \eta^4 y_4 + \eta^3 y_3 + \eta^2 y_2 + \eta y + \eta^0 y_0\right)\right]_{\eta=0}$$

$$\tag{45}$$

$$\Rightarrow A_5 = y_5\varphi^{'}(y_0)+\left(y_1y_4\varphi^{''}(y_0)+y_2y_3\varphi^{''}(y_0)+\frac{1}{2}y_1^2y_3\varphi^{'''}\right)+\frac{1}{2}y_1y_2^2\varphi^{'''}(y_0)$$

$$+\frac{1}{6}y_1^3y_2\varphi^{iv}(y_0)+\frac{y_1}{5!}\varphi^{v}(y_0)$$

Consider $A_i, \quad i=1,2,3,...$

$$A_0 = \varphi(y_0) \tag{46}$$

$$A_1 = y_1\varphi^{'}(y_0) \tag{47}$$

$$A_2 = y_2\varphi^{'}(y_0)+\frac{y^2}{2!}\varphi^{''}(y_0) \tag{48}$$

$$A_3 = y_3\varphi^{'}(y_0)+y_1y_2\varphi^{''}(y_0)+\frac{y^3}{3!}\varphi^{'''}(y_0) \tag{49}$$

$$A_4 = y_4\varphi^{'}(y_0)+\left(y_1y_3\varphi^{''}(y_0)+\frac{1}{2!}y_2^2\right)+y_1^2y_2\varphi^{'''}(y_0)+\frac{y^4}{4!}\varphi^{iv}(y_0) \tag{50}$$

$$A_5 = y_5\varphi^{'}(y_0)+(y_1y_4+y_2y_3)\varphi^{''}(y_0)+\left(\frac{1}{2}y_1^2y_3+\frac{1}{2}y_2^2y_1\right)\varphi^{'''}(y_0)+ \\ \frac{1}{6}y_1^3y_2\varphi^{iv}(y_0)+\frac{y_1}{5!}\varphi^{iv}(y_0) \tag{51}$$

Chapter 3

APPLICATIONS

This chapter is the heart of this thesis work where we considered the application of Adomian's Decomposition Method (ADM) to four different Fractional multi-order differential equations of linear and non-linear type. The results and other analysis were discussed in Chapter 4.

3.1 Solutions of Some Examples

The first three examples considered in this section are linear multi-order fractional differential equations, while the last example has a non-linear term.

3.1.1 Example 1

Consider the following Initial value problem:

$$\frac{d^{\hat{o}} y}{dt^{\hat{o}}} + \omega^{\hat{o}-\breve{\sigma}} \frac{d^{\breve{\sigma}} y}{dt^{\breve{\sigma}}} = 0 \qquad (52)$$

with the incipient conditions:

$$\omega = 1, y(0) = 0, y^{'}(0) = 1. \qquad (53)$$

Applying $j^{\hat{o}}$ on equation (52) we have:

$$y(t) - \sum_{i=0}^{m-1} y^{(i)}(0^{+}) \frac{t^{i}}{i!} + j^{\hat{o}} \left[\omega^{\hat{o}-\breve{\sigma}} \frac{d^{\breve{\sigma}} y}{dt^{\breve{\sigma}}} \right] = 0 \qquad (54)$$

$$y(t) - \sum_{i=0}^{m-1} y^{(i)}(0^{+}) \frac{t^{i}}{i!} + \omega^{\hat{o}-\breve{\sigma}} j^{\hat{o}-\breve{\sigma}} \sum_{i=0}^{m-1} y_{i}(t) = 0 \qquad (55)$$

$$y(t) - y(0) - y^{'}(0)t + \omega^{\hat{o}-\breve{\sigma}} j^{\hat{o}-\breve{\sigma}} y(t) + \omega^{\hat{o}-\breve{\sigma}} j^{\hat{o}-\breve{\sigma}} y(0) = 0 \qquad (56)$$

$$y(t)-y(0)-y^{'}(0)t+\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y(t)+\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y(0)=0 \tag{57}$$

$$y(t)=y(0)+y^{'}(0)t-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y(t)-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y(0) \tag{58}$$

where y_0 is defined as:

$$\sum_{i=0}^{m-1}y^{(i)}\left(0^{+}\right)\frac{t^{i}}{i!}+j^{\hat{o}}g(t) \tag{59}$$

given $y_0(t)$ as :

$$y(t)=y(0)+y^{'}(0)t-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y(0). \tag{60}$$

Using the initial conditions, we have: $y_0(t)=t$

$$y_{i+1}(t)=-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y_i(t) \tag{61}$$

when $i=0,$ it follows that

$$y_1=-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y_0(t) \tag{62}$$

$$y_1=-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}t. \tag{63}$$

Using Caputo Integral Operator, we have:

$$-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_{0}^{x}(x-t)^{\hat{o}-\breve{\sigma}-1}tdt\right] \quad \hat{o}>0, \quad x>0$$

$$y_1=-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_{0}^{x}(x-t)^{\hat{o}-\breve{\sigma}-1}tdt\right] \tag{64}$$

$$=-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_{0}^{x}\left(1-\frac{t}{x}\right)^{\hat{o}-\breve{\sigma}-1}x^{\hat{o}-\breve{\sigma}-1}tdt\right]. \tag{65}$$

Let $a=\frac{t}{x},\ dt=xda,\ t=ax$ when $t=0, \quad a=0$ when $t=x, \quad a=1$

We have:

$$-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1} a x^2 da\right] \qquad (66)$$

$$-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{x^{\hat{o}-\breve{\sigma}+1}}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} a da\right] \qquad (67)$$

$$-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{x^{\hat{o}-\breve{\sigma}+1}}{\Gamma(\hat{o}-\breve{\sigma})}\frac{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2)}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]. \qquad (68)$$

Therefore let $t = x$ we have y_1 given below as:

$$\frac{-\omega^{\hat{o}-\breve{\sigma}} t^{\hat{o}-\breve{\sigma}+1}}{\Gamma(\hat{o}-\breve{\sigma}+2)} \qquad (69)$$

when $i = 1$

$$y_2 = -\omega^{\hat{o}-\breve{\sigma}} j^{\hat{o}-\breve{\sigma}} y_1 t \qquad (70)$$

$$y_2 = -\omega^{\hat{o}-\breve{\sigma}} j^{\hat{o}-\breve{\sigma}}\left[\frac{-\omega^{\hat{o}-\breve{\sigma}} t^{\hat{o}-\breve{\sigma}+1}}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]. \qquad (71)$$

Using Caputo Integral Operator we have:

$$-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1}\left[\frac{-\omega^{\hat{o}-\breve{\sigma}} t^{\hat{o}-\breve{\sigma}+1}}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]dt\right] \quad \hat{o} > 0,\ x > 0$$

$$y_2 = -\omega^{\hat{o}-\breve{\sigma}}\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1}\left[\frac{-\omega^{\hat{o}-\breve{\sigma}} t^{\hat{o}-\breve{\sigma}+1}}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]dt\right] \qquad (72)$$

$$= \frac{\omega^{2(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1} t^{\hat{o}-\breve{\sigma}+1} dt \qquad (73)$$

$$= \frac{\omega^{2(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^x \left(1-\frac{t}{x}\right)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1} t^{\hat{o}-\breve{\sigma}+1} dt. \qquad (74)$$

Let $a=\frac{t}{x}$, $dt=xda$, $t=ax$ when $t=0$, $a=0$ when $t=x$, $a=1$

we have:

$$\frac{\omega^{2(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^1(1-a)^{\hat{o}-\breve{\sigma}-1}x^{\hat{o}-\breve{\sigma}-1}a^{\hat{o}-\breve{\sigma}+1}x^{\hat{o}-\breve{\sigma}+1}xda \quad (75)$$

$$\frac{\omega^{2(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^1(1-a)^{\hat{o}-\breve{\sigma}-1}x^{2(\hat{o}-\breve{\sigma})+1}a^{\hat{o}-\breve{\sigma}+1}da \quad (76)$$

$$\frac{\omega^{2(\hat{o}-\breve{\sigma})}x^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^1(1-a)^{\hat{o}-\breve{\sigma}-1}a^{\hat{o}-\breve{\sigma}+1}da \quad (77)$$

$$\frac{\omega^{2(\hat{o}-\breve{\sigma})}x^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\frac{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}{\Gamma(2(\hat{o}-\breve{\sigma})+2)}. \quad (78)$$

Therefore let $t=x$ we have y_2 given below as:

$$y_2=\frac{\omega^{2(\hat{o}-\breve{\sigma})}t^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(2(\hat{o}-\breve{\sigma})+2)} \quad (79)$$

when $i=2$:

$$y_3=-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}y_2t \quad (80)$$

$$y_3=-\omega^{\hat{o}-\breve{\sigma}}j^{\hat{o}-\breve{\sigma}}\frac{-\omega^{2(\hat{o}-\breve{\sigma})}t^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(2(\hat{o}-\breve{\sigma})+2)}. \quad (81)$$

Using Caputo Integral Operator we have:

$$-\omega^{\hat{o}-\breve{\sigma}}\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x(x-t)^{\hat{o}-\breve{\sigma}-1}\left[\frac{-\omega^{2(\hat{o}-\breve{\sigma})}t^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(2(\hat{o}-\breve{\sigma})+2)}\right]dt\right]\breve{\sigma}>0\ ,x>0$$

$$=\frac{-\omega^{3(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^x(x-t)^{\hat{o}-\breve{\sigma}-1}t^{2(\hat{o}-\breve{\sigma})+1}dt \quad (82)$$

$$= \frac{-\omega^{3(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)} \int_0^x \left(1-\frac{t}{x}\right)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1} t^{2(\hat{o}-\breve{\sigma})+1} dt \,. \quad (83)$$

Let $a = \frac{t}{x}$, $dt = xda$, $t = ax$ when $t = 0$, $a = 0$ when $t = x$, $a = 1$

we have:

$$= \frac{-\omega^{3(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)} \int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1} a x^{2(\hat{o}-\breve{\sigma})+1} xda \quad (84)$$

$$= \frac{-\omega^{3(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)} \int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} x^{3(\hat{o}-\breve{\sigma})+1} a^{2(\hat{o}-\breve{\sigma})+1} da \quad (85)$$

$$= \frac{-\omega^{3(\hat{o}-\breve{\sigma})} x^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)} \frac{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)}{\Gamma(3(\hat{o}-\breve{\sigma})+2)} \,. \quad (86)$$

Let $t = x$ we get y_3 as given below:

$$y_3 = \frac{-\omega^{3(\hat{o}-\breve{\sigma})} x^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(3(\hat{o}-\breve{\sigma})+2)} \quad (87)$$

When $i = 3$,

$$y_4 = -\omega^{\hat{o}-\breve{\sigma}} j^{\hat{o}-\breve{\sigma}} y_3 t \quad (88)$$

$$y_4 = -\omega^{\hat{o}-\breve{\sigma}} j^{\hat{o}-\breve{\sigma}} \frac{-\omega^{3(\hat{o}-\breve{\sigma})} x^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(3(\hat{o}-\breve{\sigma})+2)} \,. \quad (89)$$

Using Caputo Integral Operator, we have:

$$-\omega^{\hat{o}-\breve{\sigma}} \left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})} \int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1} \left[\frac{-\omega^{3(\hat{o}-\breve{\sigma})} t^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(3(\hat{o}-\breve{\sigma})+2)} \right] dt \right] \breve{\sigma} > 0 \ , x > 0$$

$$y_4 = \frac{-\omega^{4(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(3(\hat{o}-\breve{\sigma})+2)} \int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1} t^{3(\hat{o}-\breve{\sigma})+1} dt \quad (90)$$

$$y_4 = \frac{-\omega^{4(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(3(\hat{o}-\breve{\sigma})+2)}\int_0^x \left(1-\frac{t}{x}\right)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1} t^{3(\hat{o}-\breve{\sigma})+1} dt. \qquad (91)$$

Let $a = \frac{t}{x}$, $dt = xda$, $t = ax$ when $t = 0$, $a = 0$ when $t = x$, $a = 1$

Therefore we have:

$$y_4 = \frac{\omega^{4(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(3(\hat{o}-\breve{\sigma})+2)}\int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1} a x^{3(\hat{o}-\breve{\sigma})+1} x da \qquad (92)$$

$$y_4 = \frac{\omega^{4(\hat{o}-\breve{\sigma})}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(3(\hat{o}-\breve{\sigma})+2)}\int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} x^{4(\hat{o}-\breve{\sigma})+1} a^{3(\hat{o}-\breve{\sigma})+1} da \qquad (93)$$

$$= \frac{\omega^{4(\hat{o}-\breve{\sigma})} x^{4(\hat{o}-\breve{\sigma})+1}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(3(\hat{o}-\breve{\sigma})+2)} \frac{\Gamma(\hat{o}-\breve{\sigma})\Gamma(3(\hat{o}-\breve{\sigma})+2)}{\Gamma(4(\hat{o}-\breve{\sigma})+2)}. \qquad (94)$$

Therefore let $t = x$ we have y_4 as given below:

$$y_4 = \frac{\omega^{4(\hat{o}-\breve{\sigma})} t^{4(\hat{o}-\breve{\sigma})+1}}{\Gamma(4(\hat{o}-\breve{\sigma})+2)} \qquad (95)$$

with the recursive relativity, the terms of the decomposition series are:

$$y_1 = \frac{-\omega^{(\hat{o}-\breve{\sigma})} t^{(\hat{o}-\breve{\sigma})+1}}{\Gamma((\hat{o}-\breve{\sigma})+2)} \qquad (96)$$

$$y_2 = \frac{\omega^{2(\hat{o}-\breve{\sigma})} t^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(2(\hat{o}-\breve{\sigma})+2)} \qquad (97)$$

$$y_3 = \frac{\omega^{3(\hat{o}-\breve{\sigma})} t^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(3(\hat{o}-\breve{\sigma})+2)} \qquad (98)$$

$$y_4 = \frac{\omega^{4(\hat{o}-\breve{\sigma})} t^{4(\hat{o}-\breve{\sigma})+1}}{\Gamma(4(\hat{o}-\breve{\sigma})+2)} \qquad (99)$$

Substituting $y_0, y_1, y_2, y_3, y_4 ...$ into equation (38) we have:

$$y(t)=\frac{-\omega^{(\hat{o}-\breve{\sigma})}t^{(\hat{o}-\breve{\sigma})+1}}{\Gamma((\hat{o}-\breve{\sigma})+2)}+\frac{\omega^{2(\hat{o}-\breve{\sigma})}t^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(2(\hat{o}-\breve{\sigma})+2)}-\frac{\omega^{3(\hat{o}-\breve{\sigma})}t^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(3(\hat{o}-\breve{\sigma})+2)}+\frac{\omega^{4(\hat{o}-\breve{\sigma})}t^{4(\hat{o}-\breve{\sigma})+1}}{\Gamma(4(\hat{o}-\breve{\sigma})+2)}-\cdots \ . \tag{100}$$

The series model of the solution is given as:

$$y(t)=\sum(-1)^i\,\omega^{i(\hat{o}-\breve{\sigma})}\,\frac{t^{i(\hat{o}-\breve{\sigma})+1}}{\Gamma(i(\hat{o}-\breve{\sigma})+2)} \tag{101}$$

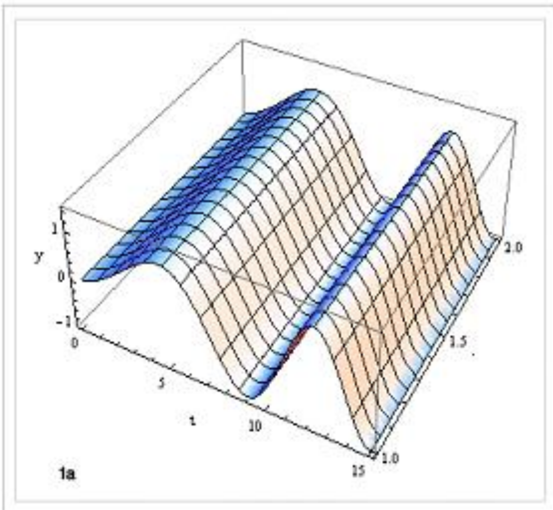
Let $\hat{o}=2,$ and $\breve{\sigma}=0$ we have:

$$y(t)=\frac{1}{\Gamma(2)}-\frac{\omega^2t^3}{\Gamma(4)}+\frac{\omega^4t^5}{\Gamma(6)}-\frac{\omega^6t^7}{\Gamma(8)}+\ldots \tag{102}$$

Which is simple harmonic oscillator's solution expressed further as:

$$y(t)=\frac{1}{\omega}\sin(\omega t) \tag{103}$$

which mean the frictional force is zero and the motion is periodic. Since it is a simple harmonic oscillator it has a constant total energy. The plane phase blue print is always closed curve (ellipse) for various values of v and x, it is periodic because it moves in clock-wise direction.



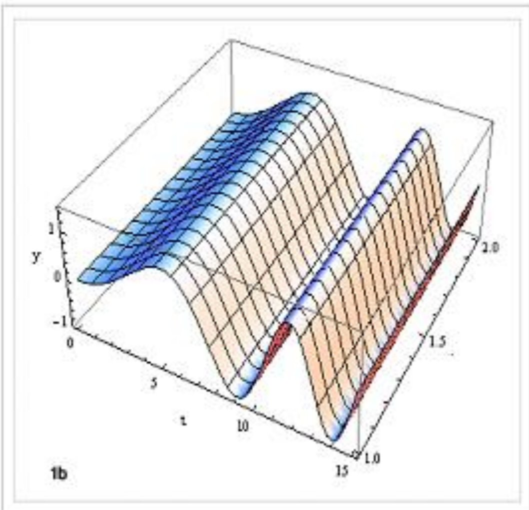


Figure 1. Solution of Example 1

Figure 1a is the solution of example 1 when $1 < \hat{o} \le 2,$ and $\breve{\sigma} = 0$ while Figure 1b is also a display of the solution of example 1 when $0 < \breve{\sigma} \le 1$and $\hat{o} = 0$ respectively.

It was observed that when we set $\breve{\sigma} = 0$ and vary $\hat{o}$ there was an increase in frequency with increase in the values of $\hat{o}$ within the interval $1 < \hat{o} \le 2$ as display in Figure 1a above. Similarly when we set $\hat{o} = 2,$ and vary $\breve{\sigma}$ we discovered that the frequency decrease with decreases in the values $\breve{\sigma}$ within the interval $0 < \breve{\sigma} \le 1$ as display in Figure 1b above. It is clear that our results for the two cases considered is in agreement with the solution obtained in other literatures using other various methods $[1]$.

3.1.2 Example 2

Consider the Initial value problem:

$$\frac{d^{\hat{o}}y}{dt^{\hat{o}}} - k\frac{d^{\breve{\sigma}}y}{dt^{\breve{\sigma}}} = 0 \qquad (104)$$

equation(104) depends on the incipient conditions:

$$k = 1, y^{'}(0) = 1, y(0) = 0 .$$

(105)

Applying $j^{\hat{o}}$ to equation(104) we have:

$$y(t) - \sum_{n=0}^{m-1} y^{(n)}(0^{+})\frac{t^{n}}{n!} - j^{\hat{o}}\left[k\frac{d^{\breve{\sigma}}y}{dt^{\breve{\sigma}}}\right] = 0 \qquad (106)$$

$$y(t) - \sum_{n=0}^{m-1} y^{(n)}(0^{+})\frac{t^{n}}{n!} - kj^{\hat{o}-\breve{\sigma}}\sum_{n=0}^{m-1} y_{n}(t) = 0 \qquad (107)$$

$$y(t) - y(0) - y^{'}(0)t - kj^{\hat{o}-\breve{\sigma}}y(t) - kj^{\hat{o}-\breve{\sigma}}y(0) = 0 \qquad (108)$$

where y_0 is defined as :

$$\sum_{n=0}^{m-1} y^{(n)}\left(0^{+}\right)\frac{t^{n}}{n!}+j^{\hat{o}}g\left(t\right). \tag{109}$$

Then,

$$y\left(t\right)=y\left(0\right)+y^{'}\left(0\right)t+kj^{\hat{o}-\breve{\sigma}}y\left(t\right)+kj^{\hat{o}-\breve{\sigma}}y\left(0\right) \tag{110}$$

where

$$y_0\left(t\right)=y\left(0\right)+y^{'}\left(0\right)t+kj^{\hat{o}-\breve{\sigma}}y\left(0\right). \tag{111}$$

Considering the equation $\left(105\right)$

$$y_0\left(t\right)=t \tag{112}$$

and

$$y_{n+1}\left(t\right)=kj^{\hat{o}-\breve{\sigma}}y_n\left(t\right) \tag{113}$$

when $n=\left(0\right)$,

$$y_1\left(t\right)=kj^{\hat{o}-\breve{\sigma}}y_0\left(t\right) \tag{114}$$

$$y_1\left(t\right)=kj^{\hat{o}-\breve{\sigma}}t\,. \tag{115}$$

Using Caputo Integral Operator we have:

$$k\left[\frac{1}{\Gamma\left(\hat{o}-\breve{\sigma}\right)}\int_0^x\left(x-t\right)^{\hat{o}-\breve{\sigma}-1}tdt\right],\hat{o}>0,x>0$$

$$y_1=k\left[\frac{1}{\Gamma\left(\hat{o}-\breve{\sigma}\right)}\int_0^x\left(x-t\right)^{\hat{o}-\breve{\sigma}-1}tdt\right],\hat{o}>0,x>0 \tag{116}$$

$$=k\left[\frac{1}{\Gamma\left(\hat{o}-\breve{\sigma}\right)}\int_0^x\left(1-\frac{t}{x}\right)^{\hat{o}-\breve{\sigma}-1}x^{\left(\hat{o}-\breve{\sigma}-1\right)}tdt\right],\hat{o}>0,x>0\,. \tag{117}$$

Let $a=\frac{t}{x},\ dt=xda,\ t=ax$ when $t=0,\quad a=0$ when $t=x,\quad a=1$

we have:

$$k\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x (1-a)^{\hat{o}-\breve{\sigma}-1}x^{(\hat{o}-\breve{\sigma}-1)}ax^2da\right] \tag{118}$$

$$k\left[\frac{x^{(\hat{o}-\breve{\sigma}+1)}}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x (1-a)^{\hat{o}-\breve{\sigma}-1}ada\right] \tag{119}$$

$$k\left[\frac{x^{(\hat{o}-\breve{\sigma}+1)}}{\Gamma(\hat{o}-\breve{\sigma})}\frac{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2)}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]. \tag{120}$$

Therefore let $t=x$ we have y_1 as given below as:

$$\left[\frac{kt^{(\hat{o}-\breve{\sigma}+1)}}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right] \tag{121}$$

when $n=1$

$$y_2(t)=kj^{\hat{o}-\breve{\sigma}}y_1(t) \tag{122}$$

$$y_2(t)=kj^{\hat{o}-\breve{\sigma}}\left[\frac{kt^{(\hat{o}-\breve{\sigma}+1)}}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]. \tag{123}$$

Using Caputo Integral Operator we have:

$$k\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1}\left[\frac{kt^{(\hat{o}-\breve{\sigma}+1)}}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]dt\right],\hat{o}>0,x>0$$

$$y_2=k\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1}\left[\frac{kt^{(\hat{o}-\breve{\sigma}+1)}}{\Gamma(\hat{o}-\breve{\sigma}+2)}\right]dt\right] \tag{124}$$

$$=\left[\frac{k^2}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1}t^{\hat{o}-\breve{\sigma}+1}dt\right] \tag{125}$$

$$=\left[\frac{k^2}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^x\left(1-\frac{t}{x}\right)^{\hat{o}-\breve{\sigma}-1}x^{\hat{o}-\breve{\sigma}-1}t^{\hat{o}-\breve{\sigma}+1}dt\right]. \qquad (126)$$

Let $a=\frac{t}{x}$, $dt=xda$, $t=ax$ when $t=0$, $a=0$ when $t=x$, $a=1$

we have:

$$\left[\frac{k^2}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^1(1-a)^{\hat{o}-\breve{\sigma}-1}x^{\hat{o}-\breve{\sigma}-1}a^{\hat{o}-\breve{\sigma}+1}x^{\hat{o}-\breve{\sigma}+1}xda\right] \qquad (127)$$

$$\left[\frac{k^2}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^1(1-a)^{\hat{o}-\breve{\sigma}-1}x^{2(\hat{o}-\breve{\sigma})+1}a^{\hat{o}-\breve{\sigma}+1}da\right] \qquad (128)$$

$$\left[\frac{k^2x^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\int_0^1(1-a)^{\hat{o}-\breve{\sigma}-1}a^{\hat{o}-\breve{\sigma}+1}da\right] \qquad (129)$$

$$\frac{k^2x^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}\frac{\Gamma(\hat{o}-\breve{\sigma})\Gamma(\hat{o}-\breve{\sigma}+2)}{\Gamma\left(2(\hat{o}-\breve{\sigma})+2\right)}. \qquad (130)$$

Therefore let $t=x$ we have y_2 as given below:

$$y_2=\frac{k^2x^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma\left(2(\hat{o}-\breve{\sigma})+2\right)} \qquad (131)$$

when $n=2$,

$$y_3(t)=kj^{\hat{o}-\breve{\sigma}}y_2(t) \qquad (132)$$

then

$$y_3(t)=kj^{\hat{o}-\breve{\sigma}}\frac{k^2x^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma\left(2(\hat{o}-\breve{\sigma})+2\right)}. \qquad (133)$$

Using Caputo Integral Operator we have:

$$k\left[\frac{1}{\Gamma(\hat{o}-\breve{\sigma})}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1}\left[\frac{k^2 x^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(2(\hat{o}-\breve{\sigma})+2)}\right]dt\right], \hat{o}>0, x>0$$

$$y_3=\left[\frac{k^3}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)}\int_0^x (x-t)^{\hat{o}-\breve{\sigma}-1} t^{2(\hat{o}-\breve{\sigma})+1}dt\right] \tag{134}$$

$$=\left[\frac{k^3}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)}\int_0^x \left(1-\frac{t}{x}\right)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1}t^{2(\hat{o}-\breve{\sigma})+1}dt\right]. \tag{135}$$

Let $a=\frac{t}{x}$, $dt=xda$, $t=ax$ when $t=0$, $a=0$ when $t=x$, $a=1$

we have:

$$\left[\frac{k^3}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)}\int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} x^{\hat{o}-\breve{\sigma}-1}ax^{2(\hat{o}-\breve{\sigma})+1}xda\right] \tag{136}$$

$$\left[\frac{k^3}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)}\int_0^1 (1-a)^{\hat{o}-\breve{\sigma}-1} x^{3(\hat{o}-\breve{\sigma})+1}a^{2(\hat{o}-\breve{\sigma})+1}da\right] \tag{137}$$

$$\left[\frac{k^3 x^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)}\frac{\Gamma(\hat{o}-\breve{\sigma})\Gamma(2(\hat{o}-\breve{\sigma})+2)}{\Gamma(3(\hat{o}-\breve{\sigma})+2)}\right] \tag{138}$$

Therefore let $t=x$ we have y_3 as given below:

$$y_3=\frac{k^3t^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(3(\hat{o}-\breve{\sigma})+2)}. \tag{139}$$

With the recursive relativity, the terms of the decomposition series are given below:

$$y(t)=\frac{kt^{(\hat{o}-\breve{\sigma})+1}}{\Gamma((\hat{o}-\breve{\sigma})+2)}+\frac{k^2t^{2(\hat{o}-\breve{\sigma})+1}}{\Gamma(2(\hat{o}-\breve{\sigma})+2)}+\frac{k^3t^{3(\hat{o}-\breve{\sigma})+1}}{\Gamma(3(\hat{o}-\breve{\sigma})+2)}+\ldots \tag{140}$$

The series model of the solution is given as:

$$y(t)=\sum_{n-0}^{\infty}\frac{k^n t^{n(\hat{o}-\breve{\sigma})+1}}{\Gamma\left(n\left(\hat{o}-\breve{\sigma}\right)+2\right)} \tag{141}$$

Let $\hat{o}=1,\ \breve{\sigma}=0$ we have:

$$y(t)=\frac{t}{\Gamma(2)}+\frac{kt^2}{\Gamma(3)}+\frac{k^2t^3}{\Gamma(4)}+\ldots \tag{142}$$

Which is the solution of exponential growth equation given by:

$$y(t)=y_o e^{(kt)} \tag{143}$$

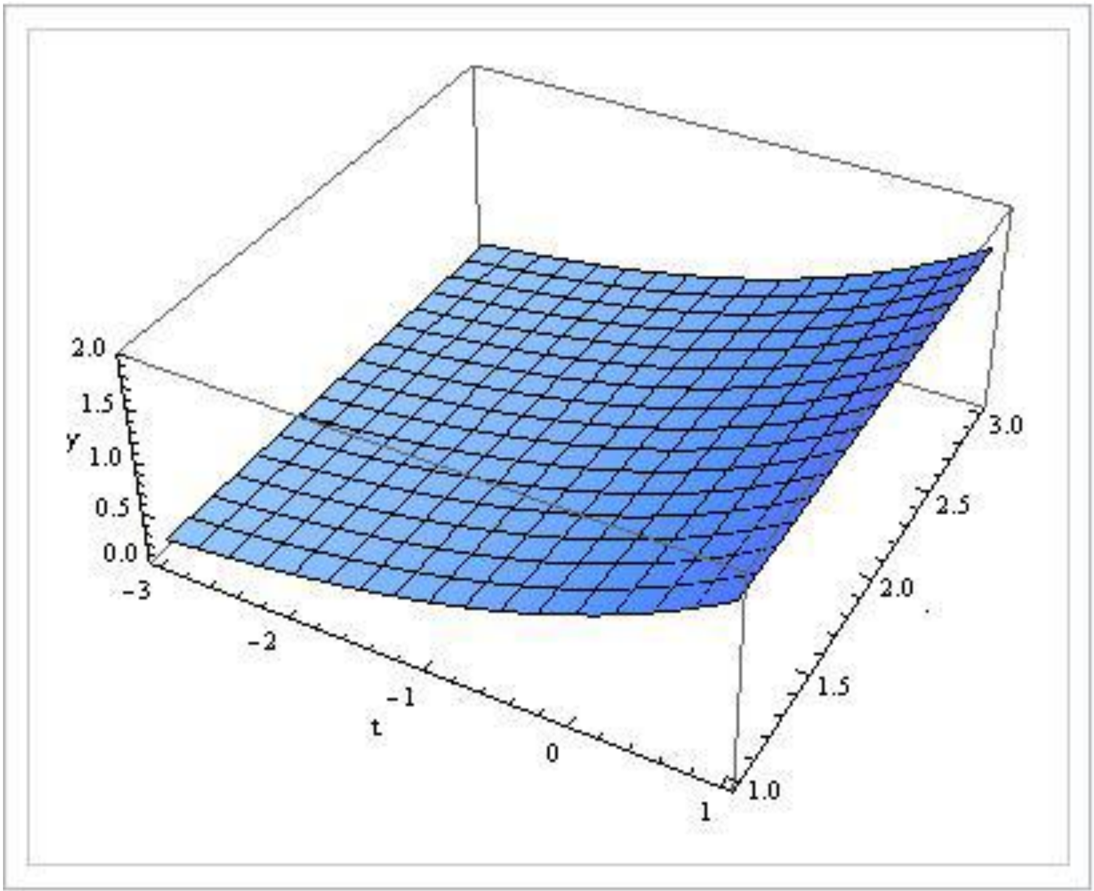


Figure 2. Solution to Example 2

If $\hat{o}=1$ and $\breve{\sigma}=0$ equation (104) becomes the equation of exponential growth.

Figure represent the solution of Example 2 when $1\le\hat{o}<2$ and $0\le\breve{\sigma}<1$.

3.1.3 Example 3

Consider the following I.V.P:

$$\frac{d^{\alpha}y}{dt^{\alpha}}-a\frac{d^{\beta_1}y}{dt^{\beta_1}}-b\frac{d^{\beta_2}y}{dt^{\beta_2}}=0 \tag{144}$$

With incipient conditions:

$$y^{(i)}(0)=\rho_i \ , \ i=0,1,2,3,.....,n-1 \tag{145}$$

Applying j^{α} to equation (144) we have:

$$y(t)-\sum_{i=0}^{n-1}y^{i}(0)\frac{t^{i}}{i!}-j^{\alpha}\left[a\frac{d^{\beta_1}y}{dt^{\beta_1}}-b\frac{d^{\beta_2}y}{dt^{\beta_2}}\right]=0 \tag{146}$$

Where:

$$\sum_{i=0}^{n-1}\frac{d^{\alpha}y}{dt^{\alpha}}=y(t)-\sum_{i=0}^{n-1}y^{i}(0)\frac{t^{i}}{i!} \tag{147}$$

The equation (144) becomes:

$$y(t)-\sum_{i=0}^{n-1}y^{i}(0)\frac{t^{i}}{i!}-aj^{\alpha-\beta_1}y(t)-a\sum_{i=0}^{n-1}y^{(i)}(0)\,j^{\alpha-\beta_1}\frac{t^{i}}{i!}-bj^{\alpha-\beta_2}y(t)$$
$$-b\sum_{i=0}^{n-1}y^{(i)}(0)\,j^{\alpha-\beta_2}\frac{t^{i}}{i!}=0 \tag{148}$$

Recall equation (45), we have:

$$y(t)-\sum_{i=0}^{n-1}\rho_i\frac{t^i}{i!}-aj^{\alpha-\beta_1}y(t)-a\sum_{i=0}^{l-1}\rho_i j^{\alpha-\beta_1}\frac{t^i}{i!}-bj^{\alpha-\beta_2}y(t)-\sum_{i=0}^{r-1}\rho_i j^{\alpha-\beta_2}\frac{t^i}{i!}=0 \qquad (149)$$

Where in the Caputo sense we have:

$$\sum_{i=0}^{l-1}\rho_i j^{\alpha-\beta_1}=\sum_{i=0}^{l-1}\rho_i\frac{t^{\alpha-\beta_i+i}}{\Gamma(\alpha-\beta_i+i+1)} \qquad (150)$$

that is:

$$j^{\alpha-\beta_1}\frac{t^i}{i!}=\frac{1}{\Gamma(\alpha-\beta_1)}\int_0^x (x-t)^{\alpha-\beta_1-1}\frac{t^i}{i!}dt \qquad (151)$$

$$j^{\alpha-\beta_1}\frac{t^i}{i!}=\frac{1}{\Gamma(\alpha-\beta_1)}\int_0^x \left(1-\frac{t}{x}\right)^{\alpha-\beta_1-1}x^{\alpha-\beta_1-1}\frac{t^i}{i!}dt \qquad (152)$$

Let $u=\frac{t}{x},\; dt=xdu,\; t=ux$ when $t=0,\quad u=0$ when $t=x,\quad u=1$, we have:

$$j^{\alpha-\beta_1}\frac{t^i}{i!}=\frac{1}{\Gamma(\alpha-\beta_1)}\int_0^1 (1-u)^{\alpha-\beta_1-1}x^{\alpha-\beta_1-1}\frac{u^i x^i}{i!}xdu \qquad (153)$$

$$j^{\alpha-\beta_1}\frac{t^i}{i!}=\frac{1}{\Gamma(\alpha-\beta_1)i!}\int_0^1 (1-u)^{\alpha-\beta_1-1}x^{\alpha-\beta_1+i}u^i x^i du \qquad (154)$$

$$j^{\alpha-\beta_1}\frac{t^i}{i!}=\frac{x^{\alpha-\beta_1+i}}{\Gamma(\alpha-\beta_1)i!}\int_0^1 (1-u)^{\alpha-\beta_1-1}u^i du \qquad (155)$$

$$=\frac{x^{\alpha-\beta_1+i}}{\Gamma(\alpha-\beta_1)i!}\frac{\Gamma(\alpha-\beta_1)\Gamma(i+1)}{\Gamma(\alpha-\beta_1+1+i)} \qquad (156)$$

$$=\frac{x^{\alpha-\beta_1+i}}{\Gamma(\alpha-\beta_1)i!}\frac{\Gamma(\alpha-\beta_1)!}{\Gamma(\alpha-\beta_1+1+i)} \qquad (157)$$

$$=\frac{x^{\alpha-\beta_1+i}}{\Gamma(\alpha-\beta_1+1+i)i!} \qquad (158)$$

therefore:

$$\sum_{i=0}^{l-1}\rho_i j^{\alpha-\beta_1}=\sum_{i=0}^{l-1}\rho_i\frac{t^{\alpha-\beta_i+i}}{\Gamma(\alpha-\beta_1+i+1)} \qquad (159)$$

Similarly:

$$\sum_{i=0}^{r-1} \rho_i j^{\alpha-\beta_2} = \sum_{i=0}^{r-1} \rho_i \frac{x^{\alpha-\beta_2+i}}{\Gamma(\alpha-\beta_2+i+1)} \tag{160}$$

Then the equation (146) becomes:

$$\begin{aligned} & y(t) - \sum_{i=0}^{n-1} \rho_i \frac{t^i}{i!} - aj^{\alpha-\beta_1} y(t) - a\sum_{i=0}^{l-1} \rho_i \frac{x^{\alpha-\beta_1+i}}{\Gamma(\alpha-\beta_1+i+1)} - bj^{\alpha-\beta_2} y(t) \\ & -b\sum_{i=0}^{l-1} \rho_i \frac{x^{\alpha-\beta_2+i}}{\Gamma(\alpha-\beta_2+i+1)} = 0 \end{aligned} \tag{161}$$

Let

$$\psi_1(t) = \sum_{i=0}^{n-1} \rho_i \frac{t^i}{i!} \tag{162}$$

$$\psi_2(t) = \sum_{i=0}^{l-1} \rho_i \frac{t^{\alpha-\beta_1+i}}{\Gamma(\alpha-\beta_i+i+1)} \tag{163}$$

$$\psi_3(t) = \sum_{i=0}^{s-1} \rho_i \frac{t^{\alpha-\beta_2+i}}{\Gamma(\alpha-\beta_2+i+1)} \tag{164}$$

Then:

$$y(t) - \psi_1(t) - aj^{\alpha-\beta} y(t) - a\psi_2(t) - bj^{\alpha-\beta_2} y(t) - b\psi_3(t) = 0 \tag{165}$$

$$y(t) = \psi_1(t) + aj^{\alpha-\beta} y(t) + a\psi_2(t) + bj^{\alpha-\beta_2} y(t) + b\psi_3(t) \perp \tag{166}$$

By rearranging:

$$y(t) = \psi_1(t) + a\psi_2(t) + b\psi_3(t) + aj^{\alpha-\beta} y(t) + bj^{\alpha-\beta_2} y(t) \tag{167}$$

Therefore the terms of $y(t)$ is determined by:

$$y_0 = \psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{168}$$

$$y_{k+1} = \left(aj^{\alpha-\beta} y(t) + bj^{\alpha-\beta_2} y(t)\right) y_k(t), k \geq 1 \tag{169}$$

With the recursive relativity, the term of the decomposition series are given below:

$$y_1 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right) y_1(t) \tag{170}$$

$\forall k = 0$,

$$\Rightarrow y_1 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right) \psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{171}$$

Also when $k = 1$ we have:

$$y_2 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right) y_1(t) \tag{172}$$

$\forall k = 1$

$$\Rightarrow y_2 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)\left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right) \psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{173}$$

$$\Rightarrow y_2 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)^2 \psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{174}$$

Also when $k = 2$ we have:

$$\Rightarrow y_2 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)\left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)\left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right) \psi_1(t) + a\psi_2(t) + b\psi_3(t)$$

$$\tag{176}$$

$$y_3 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)^3 \psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{177}$$

Also when $k = 3$ we have:

$$y_4 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right) y_3(t) \tag{178}$$

$\forall k = 3$,

$$y_4 = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)^4 \psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{179}$$

$$y_k = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)^k y_{k-1}(t), \forall k \tag{180}$$

$$y_k = \left(aj^{\alpha-\beta} + bj^{\alpha-\beta_2}\right)^k \psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{181}$$

Expanding the operator using binomial formula a series solution is obtained:

$$y(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \left(\binom{k}{j} a^j b^{k-j} j^{k\alpha - j\beta_1 - k\beta_2 + j^\alpha}\right)\psi_1(t) + a\psi_2(t) + b\psi_3(t) \tag{182}$$

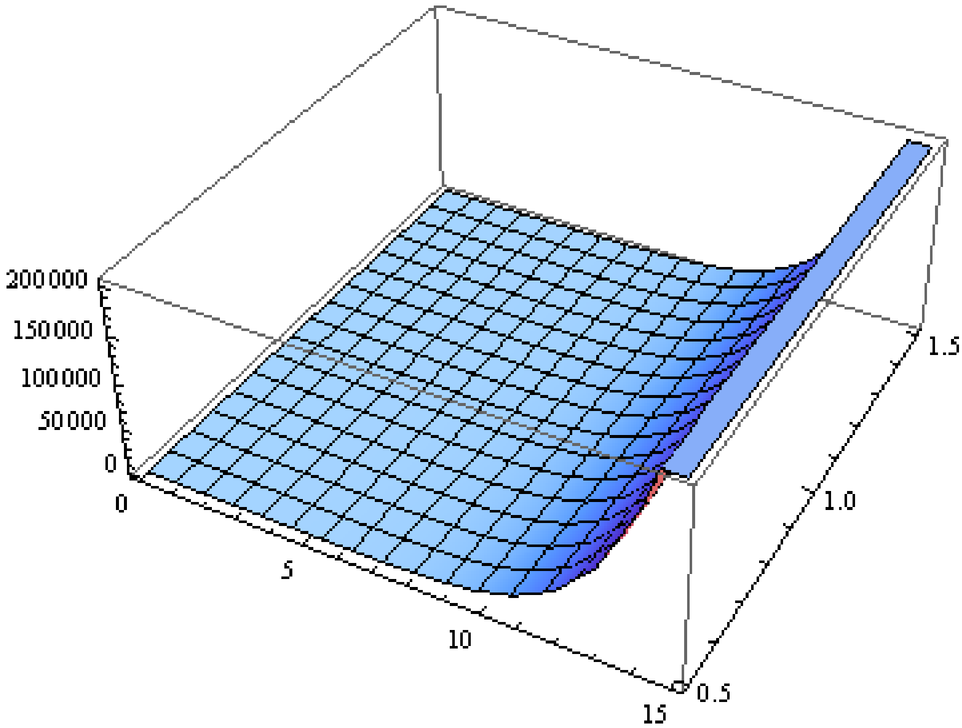


Figure 3. Solution to Example 3

The solution of Example 3 is display above for the values of $0.5 \le \beta_1 \le 1.5$ and $\beta_2 = 0$ when $\alpha = 2$ $\beta_1 = \frac{3}{2}$ and $\beta_2 = 0$ in equation (144) we obtained the solution of Bagley Torvik equation. From Figure 3 above, we discover that the amplitude increases with increase in β_1 within the interval $0.5 \le \beta_1 \le 1.5$. The examples considered so far shows the efficiency of the method of solutions for three different multi-order fractional differential equations.

3.1.4 Example 4

Now consider the non-linear case:

$$\frac{d^{\hat{\phi}}y}{dt^{\hat{\phi}}} - y(t)\frac{d^{\breve{\sigma}}y}{dt^{\breve{\sigma}}} - 1 = 0 \quad 1 < \hat{\phi} \le 2, \quad 0 \le \breve{\sigma} < \hat{\phi} \tag{183}$$

with the incipient condition $y^{i}(0) = 0 \quad i = 0,1,\ldots,m-1$

Applying $j^{\hat{\phi}}$ we have:

$$j^{\hat{\phi}}\left[\frac{d^{\hat{\phi}}y}{dt^{\hat{\phi}}} - y(t)\frac{d^{\breve{\sigma}}y}{dt^{\breve{\sigma}}} - 1\right] = 0 \tag{184}$$

From equation (17):

$$y(t)\frac{d^{\breve{\sigma}}y}{dt^{\breve{\sigma}}} = Ny(t)\ ,\ Ny(t) = \sum_{\rho=0}^{\infty} A_{\rho} \text{ and } j^{\hat{\phi}}\left(\sum_{\rho=0}^{\infty} A_{\rho}\right) = j^{\hat{\phi}} A_{\rho} \tag{185}$$

We have:

$$j^{\hat{\phi}}\left[\frac{d^{\hat{\phi}}y}{dt^{\hat{\phi}}} - y(t)\frac{d^{\breve{\sigma}}y}{dt^{\breve{\sigma}}} - 1\right] = 0$$

$$j^{\hat{\phi}}\frac{d^{\breve{\sigma}}y}{dt^{\breve{\sigma}}} - j^{\hat{\phi}}\sum_{k=0}^{\infty} A_{\rho} - j^{\hat{\phi}}(1) = 0 \tag{187}$$

$$y(t) - \sum_{i=0}^{n-1} y^{i}(0)\frac{t^{i}}{i!} - j^{\hat{\phi}} A_{\rho} - j^{\hat{\phi}}(1) = 0 \tag{188}$$

Applying initial condition, we have:

$$y(t) - j^{\hat{\phi}} A_{\rho} - j^{\hat{\phi}}(1) = 0 \tag{189}$$

Therefore:

$$y(t) = j^{\hat{\phi}} A_{\rho} + j^{\hat{\phi}}(1) = 0 \tag{190}$$

The terms of $y(t)$ are determined by:

$$y_0(t) = j^{\hat{\phi}}(1) \tag{191}$$

$$y_{\rho+1}(t) = j^{\hat{\phi}} A_{\rho} \tag{192}$$

The general formula for the Adomian's Polynomial will be used to calculate the non-linear function.

$$A_{\rho} = \frac{1}{\rho!} \frac{d^{\rho}}{d\lambda^{\rho}} \left[\phi \left(\sum_{j=0}^{\infty} \lambda^{j} y_{j} \right) \right]_{\lambda=0} \tag{193}$$

Taking the derivative using chain rule we have:

$$= \frac{1}{\rho!} \frac{d^{\rho}}{d\lambda^{\rho}} \left[\phi \left(\sum_{j=0}^{\infty} \lambda^{j} y_{j} \right) \left(\sum_{j=0}^{\infty} \lambda^{j} D_{t}^{\breve{\sigma}} y_{j} \right) \right]_{\lambda=0} \tag{194}$$

$$= \frac{1}{\rho!} \sum_{j=0}^{\rho} \binom{\rho}{j} j! y_{j} (\rho - j)! D_{t}^{\breve{\sigma}} y_{\rho-j} \tag{195}$$

$$\frac{1}{\rho!} \sum_{j=0}^{\rho} \frac{\rho!}{j!(\rho-j)!} j! y_{j} (\rho - j)! D_{t}^{\breve{\sigma}} y_{\rho-j} \tag{196}$$

therefore:

$$A_{\rho} = \frac{1}{\rho!} \sum_{j=0}^{\rho} y_{j} D_{t}^{\breve{\sigma}} y_{\rho-j} \tag{197}$$

Thus

$$A_0 = y_0 D_t^{\breve{\sigma}} y_0 \tag{198}$$

$$A_1 = y_1 D_t^{\breve{\sigma}} y_0 + y_0 D_t^{\breve{\sigma}} y_1 \tag{199}$$

$$A_2 = y_2 D_t^{\bar{\sigma}} y_0 + y_1 D_t^{\bar{\sigma}} y_1 + y_0 D_t^{\bar{\sigma}} y_2 \quad (200)$$

$$A_3 = y_3 D_t^{\bar{\sigma}} y_0 + y_2 D_t^{\bar{\sigma}} y_1 + y_1 D_t^{\bar{\sigma}} y_2 + y_0 D_t^{\bar{\sigma}} y_3 \quad (201)$$

Using the recursive relation, we have:

$$y_0 = j^{\hat{\phi}}(1) \quad (202)$$

$$= \frac{1}{\Gamma(\hat{\phi})} \int_0^x (x-t)^{\hat{\phi}-1} dt \quad (203)$$

$$= \frac{1}{\Gamma(\hat{\phi})} \int_0^x \left(1 - \frac{t}{x}\right)^{\hat{\phi}-1} x^{\hat{\phi}-1} dt \quad (204)$$

Let $b = \frac{t}{x}$, $dt = xdb$ $t = bx$ when $t = 0,$ $b = 0$ when $t = x,$ $b = 1$, we have:

$$\frac{1}{\Gamma(\hat{\phi})} \int_0^1 (1-b)^{\hat{\phi}-1} x^{\hat{\phi}-1} x db \quad (205)$$

$$\frac{x^{\hat{\phi}}}{\Gamma(\hat{\phi})} \int_0^1 (1-b)^{\hat{\phi}-1} db \quad (206)$$

$$\frac{x^{\hat{\phi}}}{\Gamma(\hat{\phi})} \frac{(1-b)^{\hat{\phi}}}{\hat{\phi}} \Big|_0^1 \quad (207)$$

$$\frac{x^{\hat{\phi}}}{\Gamma(\hat{\phi})} \frac{(-1)}{\hat{\phi}} \quad (208)$$

$$\frac{-x^{\hat{\phi}}}{\Gamma(\hat{\phi}+1)} \quad (209)$$

Where $\hat{\phi}\Gamma(\hat{\phi}) = \Gamma(\hat{\phi}+1)$ and let $t = x$, we have:

$$\frac{-t^{\hat{\phi}}}{\Gamma\left(\hat{\phi}+1\right)} \tag{210}$$

So let $\frac{1}{\Gamma\left(\hat{\phi}+1\right)}=a_0$ therefore

$$y_0=-a_0t^{\hat{\phi}} \tag{211}$$

When $\rho=0$:

$$y_1(t)=j^{\hat{\phi}}(A_0)\ ,\ A_0=y_0D_t^{\breve{\sigma}}y_0 \tag{212}$$

therefore:

$$y_1(t)=j^{\hat{\phi}}\left[y_0D_t^{\breve{\sigma}}y_0\right] \tag{213}$$

Using the identity $D_t^{\breve{\sigma}}k^{\breve{\sigma}}=\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\breve{\sigma}-\hat{\phi}+1\right)}k^{\breve{\sigma}-\hat{\phi}}$

$$\Rightarrow D_t^{\breve{\sigma}}y_0=-D_t^{\breve{\sigma}}a_0t^{\hat{\phi}}=-a_0\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\breve{\sigma}-\hat{\phi}+1\right)}t^{\left(\breve{\sigma}-\hat{\phi}\right)} \tag{214}$$

So therefore:

$$y_1(t)=j^{\hat{\phi}}(A_0)=j^{\hat{\phi}}\left[-y_0a_0\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\breve{\sigma}-\hat{\phi}+1\right)}t^{\left(\breve{\sigma}-\hat{\phi}\right)}\right] \tag{215}$$

$$\frac{1}{\Gamma\left(\hat{\phi}\right)}\int_0^x(x-t)^{\hat{\phi}-1}\left(-\left(-a_0(t)^{\hat{\phi}}\right)\right)a_0\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\breve{\sigma}-\hat{\phi}+1\right)}t^{\left(\hat{\phi}-\breve{\sigma}\right)}dt \tag{216}$$

$$\frac{1}{\Gamma\left(\hat{\phi}\right)}\int_0^x(x-t)^{\hat{\phi}-1}a_0(t)^{\hat{\phi}}\frac{1}{\Gamma\left(\hat{\phi}+1\right)}\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\breve{\sigma}-\hat{\phi}+1\right)}t^{\left(\hat{\phi}-\breve{\sigma}\right)}dt \tag{217}$$

$$\frac{a_0}{\Gamma(\hat{\phi})\Gamma(\breve{\sigma}-\hat{\phi}+1)}\int_0^x (x-t)^{\hat{\phi}-1} t^{(2\hat{\phi}-\breve{\sigma})} dt \tag{218}$$

$$\frac{a_0}{\Gamma(\hat{\phi})\Gamma(\breve{\sigma}-\hat{\phi}+1)}\int_0^x \left(1-\frac{x}{t}\right)^{\hat{\phi}-1} x^{\hat{\phi}-1} t^{(2\hat{\phi}-\breve{\sigma})} dt \tag{219}$$

Let $b=\frac{t}{x}$, $dt = xdb$ $t = bx$ when $t=0$, $b=0$ when $t=x$, $b=1$, we have:

$$\frac{a_0}{\Gamma(\hat{\phi})\Gamma(\hat{\phi}-\breve{\sigma}+1)}\int_0^1 (1-b)^{\hat{\phi}-1} x^{2\hat{\phi}-\breve{\sigma}} x^{\hat{\phi}-1} b^{(2\hat{\phi}-\breve{\sigma})} dt \tag{220}$$

$$\frac{a_0}{\Gamma(\hat{\phi})\Gamma(\hat{\phi}-\breve{\sigma}+1)}\int_0^1 (1-b)^{\hat{\phi}-1} x^{3\hat{\phi}-\breve{\sigma}} b^{(2\hat{\phi}-\breve{\sigma})} db \tag{221}$$

$$\frac{a_0 x^{3\hat{\phi}-\breve{\sigma}}}{\Gamma(\hat{\phi})\Gamma(\hat{\phi}-\breve{\sigma}+1)}\int_0^1 (1-b)^{\hat{\phi}-1} b^{(2\hat{\phi}-\breve{\sigma})} db \tag{222}$$

$$\frac{a_0 x^{3\hat{\phi}-\breve{\sigma}}}{\Gamma(\hat{\phi})\Gamma(\hat{\phi}-\breve{\sigma}+1)} \frac{\Gamma(\hat{\phi})\Gamma(2\hat{\phi}-\breve{\sigma}+1)}{\Gamma(\hat{\phi}+2\hat{\phi}-\breve{\sigma}+1)} \tag{223}$$

$$\frac{a_0 x^{3\hat{\phi}-\breve{\sigma}}}{\Gamma(\hat{\phi}-\breve{\sigma}+1)} \frac{\Gamma(2\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-\breve{\sigma}+1)} \tag{224}$$

Let $x=t$, we have :

$$\frac{a_0 t^{3\hat{\phi}-\breve{\sigma}}}{\Gamma(\hat{\phi}-\breve{\sigma}+1)} \frac{\Gamma(2\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-\breve{\sigma}+1)} \tag{225}$$

So let $\frac{a_0 t^{3\hat{\phi}-\breve{\sigma}}}{\Gamma(\hat{\phi}-\breve{\sigma}+1)} \frac{\Gamma(2\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-\breve{\sigma}+1)} = a_1$ we have:

$$y_1(t) = a_1 t^{3\hat{\phi}-\breve{\sigma}} \tag{226}$$

When $\rho = 1$:

$$y_2(t) = j^{\hat{\phi}} A_1, \quad A_1 = y_1 D_t^{\breve{\sigma}} y_0 + y_0 D_t^{\breve{\sigma}} y_1 \tag{227}$$

$$y_2(t) = j^{\hat{\phi}} \left[y_1 D_t^{\breve{\sigma}} y_0 + y_0 D_t^{\breve{\sigma}} y_1 \right], \tag{228}$$

Similarly,

$$D_t^{\breve{\sigma}} y_0 = -D_t^{\breve{\sigma}} a_0 t^{\hat{\phi}} = -a_0 \frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\breve{\sigma}-\hat{\phi}+1\right)} \tag{229}$$

$$D_t^{\breve{\sigma}} y_1 = D_t^{\breve{\sigma}} a_1 t^{3\hat{\phi}-\breve{\sigma}} = a_1 \frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} t^{3\hat{\phi}-2\breve{\sigma}} \tag{230}$$

$$\Rightarrow y_2(t) = j^{\hat{\phi}} \left[-a_1 t^{3\hat{\phi}-\breve{\sigma}} \frac{a_0 \Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} t^{\hat{\phi}-\breve{\sigma}} - a_0 t^{\hat{\phi}} \frac{a_1 \Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} t^{3\hat{\phi}-2\breve{\sigma}} \right] \tag{231}$$

$$\Rightarrow y_2(t) = \frac{1}{\Gamma\left(\hat{\phi}\right)} \int_0^x (x-t)^{\hat{\phi}-1} \left[-a_1 t^{4\hat{\phi}-2\breve{\sigma}} \frac{a_0 \Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} - a_0 t^{4\hat{\phi}-2\breve{\sigma}} \frac{a_1 \Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} \right] dt \tag{232}$$

$$\Rightarrow y_2(t) = \frac{1}{\Gamma\left(\hat{\phi}\right)} \int_0^x (x-t)^{\hat{\phi}-1} - a_1 a_0 t^{4\hat{\phi}-2\breve{\sigma}} \left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} + \frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} \right] dt \tag{233}$$

$$y_2(t) = \left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} + \frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} \right] \frac{-a_1 a_0}{\Gamma\left(\hat{\phi}\right)} \int_0^x (x-t)^{\hat{\phi}-1} t^{4\hat{\phi}-2\breve{\sigma}} dt \tag{234}$$

Let $b = \dfrac{t}{x}$, $dt = xdb$ $t = bx$ when $t = 0$, $b = 0$ when $t = x$, $b = 1$, we have:

$$y_2(t) = \left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} + \frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} \right] \frac{-a_1 a_0}{\Gamma\left(\hat{\phi}\right)} \int_0^1 \left(1-\frac{t}{x}\right)^{\hat{\phi}-1} x^{\hat{\phi}-1} t^{4\hat{\phi}-2\breve{\sigma}} dt \tag{235}$$

$$y_2(t)=\left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}\right]\frac{-a_1a_0}{\Gamma\left(\hat{\phi}\right)}\int_0^1(1-b)^{\hat{\phi}-1}x^{\hat{\phi}-1}x^{4\hat{\phi}-2\breve{\sigma}}b^{4\hat{\phi}-2\breve{\sigma}}xdb \tag{236}$$

$$y_2(t)=\left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}\right]\frac{-a_1a_0}{\Gamma\left(\hat{\phi}\right)}\int_0^1(1-b)^{\hat{\phi}-1}x^{5\hat{\phi}-2\breve{\sigma}}b^{4\hat{\phi}-2\breve{\sigma}}xdb \tag{237}$$

$$y_2(t)=\left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}\right]\frac{-a_1a_0x^{5\hat{\phi}-2\breve{\sigma}}}{\Gamma\left(\hat{\phi}\right)}\int_0^1(1-b)^{\hat{\phi}-1}b^{4\hat{\phi}-2\breve{\sigma}}xdb \tag{238}$$

$$y_2(t)=\left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}\right]\frac{-a_1a_0x^{5\hat{\phi}-2\breve{\sigma}}}{\Gamma\left(\hat{\phi}\right)}\frac{\Gamma\left(\hat{\phi}\right)\Gamma\left(4\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)} \tag{239}$$

$$y_2(t)=\left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}\right]\frac{-a_1a_0x^{5\hat{\phi}-2\breve{\sigma}}\Gamma\left(4\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)} \tag{240}$$

Let $x=t$ we have:

$$y_2(t)=\left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}\right]\frac{-a_1a_0t^{5\hat{\phi}-2\breve{\sigma}}\Gamma\left(4\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)} \tag{241}$$

Let

$$\left[\frac{\Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}\right]\frac{\Gamma\left(4\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}=a_2 \tag{242}$$

Therefore:

$$y_2 = -a_2 t^{5\hat{\phi}-2\breve{\sigma}} \tag{243}$$

When $\rho = 2$ we have:

$$y_3(t) = j^{\hat{\phi}} A_2 = j^{\hat{\phi}}\left[y_2 D_t^{\breve{\sigma}} y_0 + y_1 D_t^{\breve{\sigma}} y_1 + y_0 D_t^{\breve{\sigma}} y_2 \right] \tag{244}$$

Where:

$$D_t^{\breve{\sigma}} y_0 = -D_t^{\breve{\sigma}} a_0 t^{\hat{\phi}} = \frac{-a_0 \Gamma\left(\hat{\phi}+1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} t^{\hat{\phi}-\breve{\sigma}} \tag{245}$$

$$D_t^{\breve{\sigma}} y_1 = D_t^{\breve{\sigma}} a_1 t^{3\hat{\phi}-\breve{\sigma}} = \frac{a_1 \Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)} t^{3\hat{\phi}-2\breve{\sigma}} \tag{246}$$

$$D_t^{\breve{\sigma}} y_2 = -D_t^{\breve{\sigma}} a_1 t^{3\hat{\phi}-\breve{\sigma}} = \frac{-a_2 \Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-3\breve{\sigma}+1\right)} t^{5\hat{\phi}-2\breve{\sigma}} \tag{247}$$

$$y_3(t) = j^{\hat{\phi}} \begin{bmatrix} a_2 t^{5\hat{\phi}-2\breve{\sigma}} \dfrac{a_0 \Gamma\left(\hat{\phi}-1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} t^{\hat{\phi}-\breve{\sigma}} + a_1 t^{3\hat{\phi}-\breve{\sigma}} \dfrac{a_1 \Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} t^{3\hat{\phi}-2\breve{\sigma}} \\ + a_0 t^{\hat{\phi}} \dfrac{a_2 \Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-3\breve{\sigma}+1\right)} t^{5\hat{\phi}-2\breve{\sigma}} \end{bmatrix} \tag{248}$$

$$y_3(t) = j^{\hat{\phi}} \begin{bmatrix} a_2 t^{6\hat{\phi}-3\breve{\sigma}} \dfrac{a_0 \Gamma\left(\hat{\phi}-1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)} + a_1^2 t^{6\hat{\phi}-3\breve{\sigma}} \dfrac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)} t^{6\hat{\phi}-3\breve{\sigma}} \\ + a_0 \dfrac{a_2 \Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-3\breve{\sigma}+1\right)} t^{6\hat{\phi}-2\breve{\sigma}} \end{bmatrix} \tag{249}$$

$$y_3(t)=\frac{1}{\Gamma(\hat{\phi})}\int_0^x (x-t)^{\hat{\phi}-1} t^{6\hat{\phi}-3\breve{\sigma}} \times\left[a_2\frac{a_0\Gamma(\hat{\phi}-1)}{\Gamma(\hat{\phi}-\breve{\sigma}+1)}+a_1^2\frac{\Gamma(3\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-2\breve{\sigma}+1)}+a_0\frac{a_2\Gamma(5\hat{\phi}-2\breve{\sigma}+1)}{\Gamma(5\hat{\phi}-3\breve{\sigma}+1)}\right]dt \tag{250}$$

$$y_3(t)=\left[a_2\frac{a_0\Gamma(\hat{\phi}-1)}{\Gamma(\hat{\phi}-\breve{\sigma}+1)}+a_1^2\frac{\Gamma(3\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-2\breve{\sigma}+1)}+a_0\frac{a_2\Gamma(5\hat{\phi}-2\breve{\sigma}+1)}{\Gamma(5\hat{\phi}-3\breve{\sigma}+1)}\right] \times\frac{1}{\Gamma(\hat{\phi})}\int_0^x (x-t)^{\hat{\phi}-1} t^{6\hat{\phi}-3\breve{\sigma}}dt \tag{251}$$

$$y_3(t)=\left[a_2\frac{a_0\Gamma(\hat{\phi}-1)}{\Gamma(\hat{\phi}-\breve{\sigma}+1)}+a_1^2\frac{\Gamma(3\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-2\breve{\sigma}+1)}+a_0\frac{a_2\Gamma(5\hat{\phi}-2\breve{\sigma}+1)}{\Gamma(5\hat{\phi}-3\breve{\sigma}+1)}\right] \times\frac{1}{\Gamma(\hat{\phi})}\int_0^x \left(1-\frac{t}{x}\right)^{\hat{\phi}-1} x^{\hat{\phi}-1} t^{6\hat{\phi}-3\breve{\sigma}}dt \tag{252}$$

Let $b=\frac{t}{x}$, $dt=xdb$ $t=bx$ when $t=0$, $b=0$ when $t=x$, $b=1$, we have:

$$y_3(t)=\left[a_2\frac{a_0\Gamma(\hat{\phi}-1)}{\Gamma(\hat{\phi}-\breve{\sigma}+1)}+a_1^2\frac{\Gamma(3\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-2\breve{\sigma}+1)}+a_0\frac{a_2\Gamma(5\hat{\phi}-2\breve{\sigma}+1)}{\Gamma(5\hat{\phi}-3\breve{\sigma}+1)}\right] \times\frac{1}{\Gamma(\hat{\phi})}\int_0^1 (1-b)^{\hat{\phi}-1} x^{7\hat{\phi}-3\breve{\sigma}} b^{6\hat{\phi}-3\breve{\sigma}}db \tag{253}$$

$$y_3(t)=\left[a_2\frac{a_0\Gamma(\hat{\phi}-1)}{\Gamma(\hat{\phi}-\breve{\sigma}+1)}+a_1^2\frac{\Gamma(3\hat{\phi}-\breve{\sigma}+1)}{\Gamma(3\hat{\phi}-2\breve{\sigma}+1)}+a_0\frac{a_2\Gamma(5\hat{\phi}-2\breve{\sigma}+1)}{\Gamma(5\hat{\phi}-3\breve{\sigma}+1)}\right] \times\frac{x^{7\hat{\phi}-3\breve{\sigma}}}{\Gamma(\hat{\phi})}\int_0^1 (1-b)^{\hat{\phi}-1} b^{6\hat{\phi}-3\breve{\sigma}}db \tag{254}$$

$$y_3(t)=\left[a_2\frac{a_0\Gamma\left(\hat{\phi}-1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+a_1^2\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}+a_0\frac{a_2\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-3\breve{\sigma}+1\right)}\right]\times\frac{x^{7\hat{\phi}-3\breve{\sigma}}}{\Gamma\left(\hat{\phi}\right)}\frac{\Gamma\left(\hat{\phi}\right)\Gamma\left(6\hat{\phi}-3\breve{\sigma}+1\right)}{\Gamma\left(7\hat{\phi}-3\breve{\sigma}+1\right)} \quad (255)$$

$$y_3(t)=\left[a_2\frac{a_0\Gamma\left(\hat{\phi}-1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+a_1^2\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}+a_0\frac{a_2\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-3\breve{\sigma}+1\right)}\right]\times\frac{x^{7\hat{\phi}-3\breve{\sigma}}\Gamma\left(6\hat{\phi}-3\breve{\sigma}+1\right)}{\Gamma\left(7\hat{\phi}-3\breve{\sigma}+1\right)} \quad (256)$$

Let $x=t$ we have:

$$y_3(t)=\left[a_2\frac{a_0\Gamma\left(\hat{\phi}-1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+a_1^2\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}+a_0\frac{a_2\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-3\breve{\sigma}+1\right)}\right]\frac{t^{7\hat{\phi}-3\breve{\sigma}}\Gamma\left(6\hat{\phi}-3\breve{\sigma}+1\right)}{\Gamma\left(7\hat{\phi}-3\breve{\sigma}+1\right)} \quad (257)$$

Let:

$$\left[a_2\frac{a_0\Gamma\left(\hat{\phi}-1\right)}{\Gamma\left(\hat{\phi}-\breve{\sigma}+1\right)}+a_1^2\frac{\Gamma\left(3\hat{\phi}-\breve{\sigma}+1\right)}{\Gamma\left(3\hat{\phi}-2\breve{\sigma}+1\right)}+a_0\frac{a_2\Gamma\left(5\hat{\phi}-2\breve{\sigma}+1\right)}{\Gamma\left(5\hat{\phi}-3\breve{\sigma}+1\right)}\right]=a_3 \quad (258)$$

Therefore

$$y_3=a_3t^{7\hat{\phi}-3\breve{\sigma}} \quad (259)$$

The solution is in series form express below as:

$$y(t)=-a_0t^{\hat{\phi}}+a_1t^{3\hat{\phi}-\breve{\sigma}}-a_1t^{5\hat{\phi}-2\breve{\sigma}}+a_1t^{7\hat{\phi}-3\breve{\sigma}}-\ldots \quad (260)$$

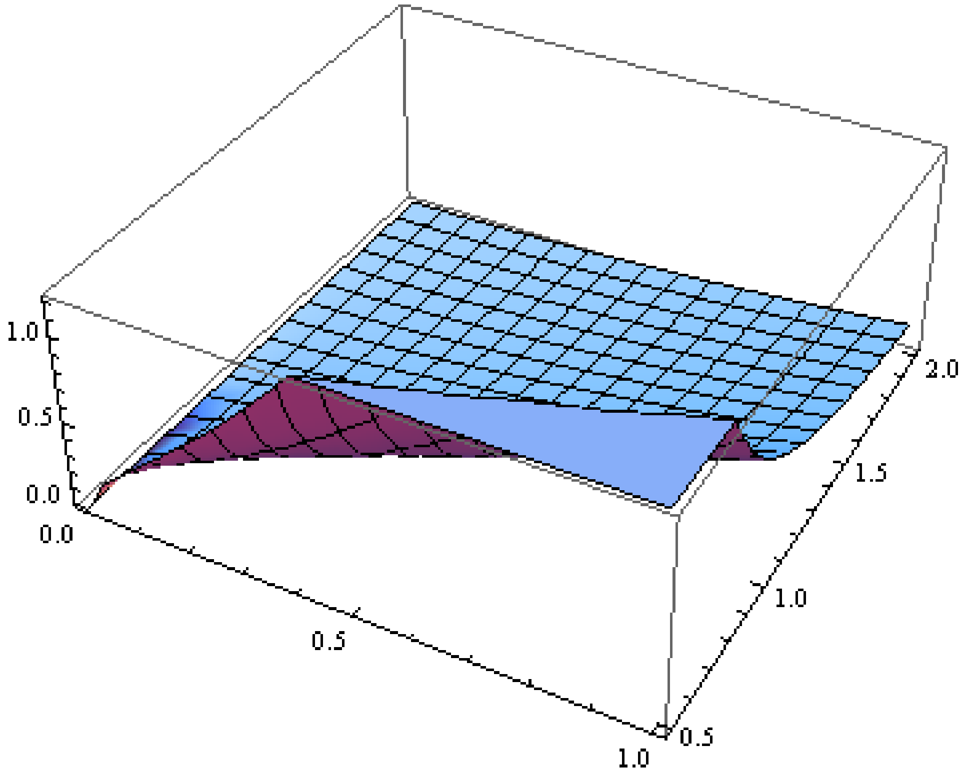


Figure 4. Solution of Example 4

The solution obtained is in conformity with the solution obtained by Shawagfeh [16] when $\breve{\sigma} = 0$, but the solution above is obtained by varying the value of $\hat{\phi}$ within the interval $0.5 \le \hat{\phi} \le 2$ and $\breve{\sigma} = 0$. This solution is only considered for four terms, more terms can be consider to improve the solution.

Chapter 4

DISCUSSION OF RESULTS

4.1 Conclusion

The effectiveness of ADM technique on multi-order fractional differential equations of both linear and non linear type was explored. Four different examples were considered, the results obtained were compared to those existing in literature for evaluation purpose.

Several of researchers have applied ADM technique to solve several problems existing in real life situation as written in Chapter 1, which contains the preamble. Since the introduction of the technique about three decade ago, sources confirm the effectiveness, efficient and the accuracy of ADM technique.

However, this thesis work only highlighted it advantages without comparing it pros and cons.

In the second Chapter, rudimentary analysis of ADM technique with the consideration of a very important special function known as the gamma function was reviewed. In Chapter 3 we considered four examples and the estimation of these examples were easily carried out and computed without the aid of computer. The solution obtained in the study is in the form of convergence series which exhibit some recursive relationship.

It is evidently clear that there exist a relationship of some sorts between the simple harmonic motion and the sine wave as shown in this thesis work, therefore this study is applicable in the music industries for fine tuning and production of various musical instruments such as microphones, loudspeaker, acoustic instruments like guitars, pianos, violins e.t.c. Also in the auto-mobile industries, the technology behind the shock absorbers of our vehicles depends greatly on this study or similar studies.

Electrical power generation is also an area of application due to electromagnetic wave. The operation of gravimeters for the detection of differences in the value of gravity at that location, known as gravity surveys is extensively useful in the oil, gas and mining industry to locate crude oil and precious metal deposit. The fourth examined equation displays an important advantage of ADM technique, inspite of the easy computation there is no need for linearization or assumption of any sort

In general, the Adomian's Decomposition Method (ADM) is a convenient tool, it can be applied directly to problems. It also demonstrates computational ease which can be computed manually.

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