# Convergent Sequences and Statistical Limit Points

**Erdem Baytunç**

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
September 2018
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Assoc. Prof. Dr. Ali Hakan Ulusoy
Acting Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Hüseyin Aktuğlu
Supervisor

Examining Committee

1. Prof. Dr. Hüseyin Aktuğlu
2. Prof. Dr. Mehmet Ali Özarslan
3. Asst. Prof. Dr. Halil Gezer

# ABSTRACT

In the present thesis, we prepare a summary of the existing theory of statistical, lacunary statistical, $\lambda$-statistical, $A$-statistical limit points and some related topics for sequences of real numbers by using different research papers.

In Chapter 1, you can find a summary of the existing theory of convergent sequences. The real number sequences and some of their important properties are all given in this chapter.

In Chapter 2, we give the definitions and some important properties of statistical convergence, lacunary, $\lambda$ and $A$-statistical convergence. In this chapter we also discuss implication and inclusion relations between these new type convergences. All implications and inclusions are illustrated by examples.

Chapter 3, is devoted to the main work of this thesis. This chapter starts with the definitions of statistical limit point and statistical cluster point and continue with the discussion of similarities and differences between statistical and ordinary limit points of sequences of real numbers. Later the same study is repeated for lacunary statistical, $\lambda$-statistical and $A$-statistical limit points for sequences of real numbers.

**Keywords:** Statistical convergence, lacunary statistical convergence, $\lambda$-statistical convergence, $A$-statistical convergence, statistical limit points, lacunary statistical limit points, $\lambda$-statistical limit points, $A$-statistical limit points.

# ÖZ


Biz bu tezde mevcut teoride bilinen, istatistiksel limit noktaları, lacunary istatistiksel limit noktaları, $\lambda$-istatistiksel limit noktaları ve $A$-istatistiksel limit noktaları ve bunlarla ilgili konuların bir derlemesini yaptık.

Birinci bölümde, yakınsak dizilerle ilgili mevcut teorinin bir özeti ile yakınsak reel değerli diziler ve bunların önemli özelliklerinin içerildiği kısa bir özet bulabilirsiniz.

İkinci bölümde istatistiksel yakınsaklık, lacunary istatistiksel yakınsaklık, $\lambda$-istatistiksel yakınsaklık ve $A$-istatistiksel yakınsaklık tanımlarını ve bazı önemli özelliklerini verdik. Bu bölümde ayrıca bu kavramlarla ilgili içerilme ve kapsanma özellikleri tartışılmıştır ve bu özellikler örneklendirilmiştir.

Üçüncü bölüm, tezin esas konusuna ayrılmıştır. Bu bölüm istatistiksel limit noktası ile istatistiksel değme noktalarının tanımları ile başlar ve istatistiksel limit noktaları ve bilinen anlamda limit noktalarının benzerlik ve farklılıklarının tartışılması ile devam eder. Bu bölümün devamında benzer tartışma lacunary istatistiksel limit noktaları, $\lambda$-istatistiksel limit noktaları ve $A$-istatistiksel limit noktaları içinde tekrarlanmıştır.



**Anahtar Kelimeler:** İstatistiksel yakınsaklık, lacunary istatistiksel yakınsaklık, $\lambda$-istatistiksel yakınsaklık, $A$-istatistiksel yakınsaklık, istatistiksel limit noktaları, lacunary istatistiksel limit noktaları, $\lambda$-istatistiksel limit noktaları, $A$-istatistiksel limit noktaları.

# ACKNOWLEDGEMENT

Firstly, I would like to thank my supervisor Prof. Dr. Hüseyin Aktuğlu for his continuous support and supervision. During this research, he provided great advices and feedbacks. Then, I would like to thank Prof. Dr. Mehmet Ali Özarslan and Assist. Prof. Dr. Halil Gezer for their helpful suggestions.

I would also like to give my special thanks to my family and friends for their spiritual and financial supports.

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# Chapter 1

## INTRODUCTION

### 1.1 Sequences in Real Numbers

In this thesis we mainly focus on limit points of sequences of real numbers in statistical, lacunary statistical, $\lambda$-statistical and A-statistical sense. Therefore the present chapter is devoted to a short summary of concepts of limit points in ordinary sense, infinite matrices, matrix transformations and density functions.

**Definition 1.1.1** *( [27] ) A sequence $x(k)$ is a function whose domain is $\mathbb{N}$. In general, the sequence is represented by $(x_k)$ or $\{x_k\}_{k=1}^{\infty}$. Furthermore, it is worthwhile to note that in this notation, k stands for the index of the sequence, and $x_k$ is called the $k^{th}$ term of $(x_k)$.*

$\varpi$ represents the set of all real valued sequences.

**Definition 1.1.2** *( [27] ) A sequence $(x_k)$ is called bounded, if $\exists B \in \mathbb{N}$ , with $|x_k| \leq B$ $\forall k \in \mathbb{N}$.*

**Definition 1.1.3** *( [23] ) A sequence $(x_k)$ is called monotone increasing (or monotone decreasing), if for every k,*

$$x_k \leq x_{k+1} (or\ x_{k+1} \leq x_k),$$

*holds true.*

*Furthermore, a sequence $(x_k)$ is called strictly increasing (or strictly decreasing), if* *for every k,*

$$x_k < x_{k+1} (or\, x_{k+1} < x_k),$$

*holds true.*

**Example 1.1.1** *A sequence $(x_k)$, which is defined as*

$$(x_k) = \left(\frac{1}{3}\right)^k$$

*is decreasing.*

**Example 1.1.2** *A sequence $(y_k)$, which is defined as*

$$(y_k) = 3^k$$

*is increasing.*

**Definition 1.1.4** *( [27] ) A sequence $(x_k)$ is convergent to $\eta$, provided that $\forall \varepsilon > 0$, $\exists H(\varepsilon)$, such that for every $k \geq H(\varepsilon)$,*

$$|x_k - \eta| < \varepsilon.$$

*This convergency is denoted by $x_k \longrightarrow \eta$ or $\lim_k x_k = \eta$.*

The set of all convergent sequences is represented by $C$.

**Remark 1.1.1** *If $x \to \eta$, then the set $\{k : |x_k - \eta| \geq \varepsilon\}$ is a finite set for all $\varepsilon > 0$.*

**Definition 1.1.5** *( [27] ) If a sequence is not convergent, then it is called divergent.*

**Theorem 1.1.1** *( [27] ) Limit of $(x_k)$ is unique, if $(x_k) \in C$.*

**Theorem 1.1.2** *( [27] ) If $(x_k) \in C$, then the sequence $(x_k)$ is bounded, but not the vice versa.*

**Example 1.1.3** *The sequence*

$$(x_k) = \big(0,1,0,1,0,1,\cdots\big)$$

*is bounded but not convergent.*

**Definition 1.1.6** *( [18] ) For a sequence $(x_k)$, the real number $b = \sup x_k$, the supremum (or least upper bound) of $(x_k)$, is a number that satisfies the following items;*

*1) For every k, $x_k \leq b$*

*2) There exists $x_N$ provides, $x_N > b - \varepsilon$, for every $\varepsilon > 0$.*

**Definition 1.1.7** *( [18] ) For a sequence $(x_k)$, the real number $a = \inf x_k$, the infimum (or greatest lower bound) of $(x_k)$, is a number that satisfies the following items ;*

*1) For every k, $a \leq x_k$*

*2) There exists $x_N$ provides, $x_N < a + \varepsilon$, for every $\varepsilon > 0$.*

**Example 1.1.4** *Consider the sequence $(x_k) = \frac{1}{k}$, then $\inf x_k = 0$ and $\sup x_k = 1$.*

**Proposition 1.1.1** *( [18] ) If a real valued sequence $(x_k)$ is bounded, then*

$\inf x_k \leq \sup x_k$.

**Theorem 1.1.3** *( [27] )If a sequence $(x_k)$ is monotone and bounded, then $(x_k)$ is convergent. Moreover, if a sequence $(x_k)$ is monotone increasing and bounded, then it converges to $\sup x_k$, and if a sequence $(x_k)$ is monotone decreasing and bounded, then it converges to $\inf x_k$.*

**Proof.** Assume that $(x_k) \in \varpi$ is bounded and increasing and let $t = \sup x_k$. Then, from the definition of supremum, $\forall \varepsilon > 0$, $\exists x_N$ so that

$$t \geq x_N > t - \varepsilon.$$

But $(x_k)$ is increasing, therefore $x_k \geq x_N$ for every $k \geq N$.

Thus, $x_k > t - \varepsilon$ for every $k \geq N$.

It indicates that,

$$t - \varepsilon < x_k \leq t < t + \varepsilon,$$

or

$$|x_k - t| < \varepsilon,$$

which is the definition of $\lim_k x_k = t$. ■

**Theorem 1.1.4** *( [27] ) If a sequence $(x_k)$ is monotone increasing (decreasing) and not bounded above(bounded below), then $x_k \to \infty$ ($x_k \to -\infty$), as $k \to \infty$.*

**Theorem 1.1.5** *( [27] ) Consider the convergent sequences $(x_k)$ and $(y_k)$, and a real number $c$. Then,*

*(i)* $\lim_k (x_k + y_k) = \lim_k x_k + \lim_k y_k$,

*(ii)* $\lim_k (cx_k) = c \lim_k x_k$,

*(iii)* $\lim_k (x_k y_k) = \lim_k x_k . \lim_k y_k$,

*(iv) If* $\lim_k y_k \neq 0$, *then*

$$\lim_k \frac{x_k}{y_k} = \frac{\lim_k x_k}{\lim_k y_k}.$$

**Definition 1.1.8** *Consider the sequence $(x_k)$, let's accept that $(k_n)$ is a strictly increasing sequence of $\mathbb{N}$. A sequence $(x_{k_n})$, whose $n^{th}$ term is $x_{k_n}$, is called a subsequence of $(x_k)$.*

**Example 1.1.5** *Consider the sequence* $(x_k) := (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, *then the sequence*

$$(x_{4k}) := \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \cdots\right)$$

*is one of the subsequence of* $(x_k)$.

**Theorem 1.1.6** *( [27] ) Let* $(x_{k_n})$ *be a subsequence of* $(x_k)$*. If*

$$(x_k) \longrightarrow \eta,$$

*then for every* $\{k_n\}$,

$$(x_{k_n}) \longrightarrow \eta.$$

*In other words, if* $x$ *is convergent to* $\eta$*, every subsequence is convergent to* $\eta$.

**Proposition 1.1.2** *( [18] ) Let* $(x_{k_n})$ *be a subsequence of* $(x_k)$*,*

1) *If* $(x_k)$ *is bounded above, then* $\sup x_{k_n} \leq \sup x_k$,

2) *If* $(x_k)$ *is bounded below, then* $\inf x_k \leq \inf x_{k_n}$.

**Definition 1.1.9** *A real number* $\rho$ *is called limit(accumulation) point of* $(x_k) \in \varpi$*, if* $(x_k)$ *has at least one point different than* $\rho$ *which is in the interval* $|x_k - \rho| < \varepsilon$ *for all* $\varepsilon > 0$.

The above mentioned theorem shows that every bounded real valued sequence, which has infinitely many terms, possesses at least one limit point.

**Definition 1.1.10** *( [26] ) A real number* $\sigma$ *is called a cluster point of* $(x_k) \in \varpi$*, if* $\forall \varepsilon > 0$*, infinitely many terms of* $(x_k)$ *implies* $|x_k - \sigma| < \varepsilon$.

A sequence having a cluster point does not specifically imply that it must have a limit.

**Example 1.1.6** *A real valued sequence* $(x_k) = (-1)^k$ *is not convergent. However, the numbers* $1$ *and* $-1$ *are the cluster points of the sequence* $x$.

Some sequences may not have a cluster point.

**Example 1.1.7** *A real valued sequence $(x_k) = e^k$ has no cluster point.*

**Theorem 1.1.7** *( [26] ) If $x \in \varpi$ is bounded and a number $\sigma$ is the only cluster point of $x$,then $x \to \sigma$.*

Boundedness condition can not be removed.

**Example 1.1.8** *Let $x \in \varpi$ and defined by $x_k = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, ...\}$, then $0$ is the only cluster point of $x$. However, $x$ is not convergent.*

**Theorem 1.1.8** *( [27] ) Suppose that $(x_k)$, $(y_k)$ and $(w_k)$ are real valued sequences. If*

$$x_k \longrightarrow x,$$

*and*

$$y_k \longrightarrow x,$$

*as $k \to \infty$, and, if $\exists H \in \mathbb{N}$ such that*

$$x_k \leq w_k \leq y_k, \text{ for every } k > H$$

*then,*

$$w_k \longrightarrow x.$$

**Theorem 1.1.9** *( [27] ) Every real valued and bounded sequence has a convergent subsequence.*

**Theorem 1.1.10** *( [27] ) If $(x_k) \longrightarrow x$ and $(y_k) \longrightarrow y$ and $\exists H \in \mathbb{N}$ such that, for every $k > H$ implies $x_k < y_k$. Then, $x \leq y$.*

**Definition 1.1.11** *A sequence $(x_n)$ is called Cauchy, provided that $\forall \varepsilon > 0$, $\exists H(\varepsilon)$ such that $\forall n, m > H(\varepsilon)$, implies $|x_n - x_m| < \varepsilon$.*

**Theorem 1.1.11** *( [27] ) If $(x_k)$ is a Cauchy sequence, then it is bounded.*

**Theorem 1.1.12** *( [27] ) $(x_k) \in C$ iff it is Cauchy.*

Note that : if and only if is abreviated as iff.

**Definition 1.1.12** *( [18] ) The set $\mathbb{R} \cup \{\pm\infty\}$ is called extended real numbers.*

**Definition 1.1.13** *( [18] ) Suppose that $(x_k)$ represents a sequence of extended real numbers.Then,*

*(1) $x_k \longrightarrow \infty$, if for every real number $P$, $\exists H \in \mathbb{N}$ so that for every $k \geq H$, $x_k > P$,*

*(2) $x_k \longrightarrow -\infty$, if for every real number $P$,$\exists H \in \mathbb{N}$ so that for every $k \geq H$, $x_k < P$.*

**Definition 1.1.14** *( [27] ) Suppose that $(x_k) \in \varpi$*

*(1) The limit superior of $(x_k)$ is denoted by $\lim\limits_{k\to\infty}{}^{*} x_k$ and defined as*

$$\lim_{k}{}^{*} x_k = \lim_{k\to\infty} [sup\{x_n | n \geq k\}].$$

*(2) The limit inferior of $(x_k)$ is denoted by $\lim\limits_{k\to\infty}{}_{*} x_k$ and defined as*

$$\lim_{k}{}_{*} x_k = \lim_{k\to\infty} [inf\{x_n | n \geq k\}].$$

**Definition 1.1.15** *( [26] ) For the real valued and bounded sequence $x$ and $M_x$ represents the set of all cluster points of $x$. Then, limit inferior of $x$ equals to the smallest point of the set $M_x$ and limit superior of $x$ equals to the greatest point of the set $M_x$.*

**Example 1.1.9** *Let* $(x_k) = (-1)^k$*, then* $M_x = \{-1, 1\}$*. So,*

$$\lim_{k}{}_{*} x_k = -1,$$

*and*

$$\lim_{k}{}^{*} x_k = 1.$$

**Theorem 1.1.13** *( [18] ) A sequence* $(x_k) \in C$ *iff*

$$\lim_{k}{}_{*} x_k = \lim_{k}{}^{*} x_k$$

**Proof.** If $x_k \to \eta$, then $\eta$ is the only cluster point of $x$. Under this assumption, the smallest and greatest point of the set of cluster point equals to $\eta$.

So, $\lim_{k}{}_{*} x_k = \eta = \lim_{k}{}^{*} x_k$.

On the other hand, if $\lim_{k}{}_{*} x_k = \eta = \lim_{k}{}^{*} x_k$, then $x$ is bounded and $x$ has only one cluster point and it is $\eta$. From theorem 1.1.7, $x$ is convergent to $\eta$. ∎

## 1.2 Matrix Representation

In this section, we briefly discuss infinite, conservative and regular matrices and matrix transformations.

**Definition 1.2.1** *Suppose that* $C = (c_{nk})$ *and* $D = (d_{nk})$ *are two infinite matrices. Then, the sum and scalar product of infinite matrices are defined as follows;*

*1)* $C + D = (c_{nk} + d_{nk})$ *(matrices addition)*

*2)* $\lambda C = (\lambda c_{nk})$ *(scalar multiplication)*

*hold true, noting that* $\lambda$ *represents a constant number.*

**Definition 1.2.2** *A non-negative, infinite matrix is defined as an infinite matrix with non-negative components.*

**Definition 1.2.3** *Let $D := (d_{nk})$ stand for an infinite matrix, the D-transform of a sequence $(x_k)$ is represented by $Dx := (Dx)_n$, and defined as*

$$(Dx)_n = \sum_{k=1}^{\infty} d_{nk} x_k$$

*if the series converges for all n.*

**Definition 1.2.4** *An infinite matrix D is called conservative if $Dx \in C$ for each $x \in C$.*

There exist conditions to understand whether any infinite matrix is conservative or not.

**Theorem 1.2.1** *(Kojima-Shurer) Suppose that $D = (d_{nk})$ is an infinite matrix. Then, $D = (d_{nk})$ is conservative iff*

*(i)* $\sup_n \sum_{k=1}^{\infty} |d_{nk}| < \infty$,

*(ii)* $\lim_n d_{nk} = \mu_k$ *for every k*,

*(iii)* $\lim_n \sum_{k=1}^{\infty} d_{nk} = \mu$.

For example, the following matrix is conservative,

$$H = (h_{nk}) = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 1-\frac{1}{n} & 0 & \frac{1}{n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

**Definition 1.2.5** *An infinite matrix D is called regular matrix iff for each sequence $(x_k) \in C$, with $x \to \eta$, implies that*

$$\lim_{k \to \infty} (Dx)_k = \eta.$$

The necessary and sufficient conditions for regularity of an infinite matrix is known as the Silverman-Toeplitz Theorem.

**Definition 1.2.6** *(Silverman-Toeplitz, [22]) Consider an infinite matrix $D = (d_{nk})$, then the matrix $D$ is regular iff*

*(R-1)* $\sup_n \sum_{k=1}^{\infty} |d_{nk}| < \infty$,

*(R-2)* $\lim_n d_{nk} = 0$ *for every* $k$,

*(R-3)* $\lim_n \sum_{k=1}^{\infty} d_{nk} = 1$

*hold.*

The set of all non-negative regular matrices is denoted by $(C,C;\eta)$.

**Example 1.2.1** *([8]) Let $C_1 = (c_{nk}) \in (C,C;\eta)$, where*

$$c_{nk} = \begin{cases} \frac{1}{n}, & if\ 1 \leq k \leq n \\ 0, & otherwise \end{cases}$$

*or equivalently,*

$$c_{nk} = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & \dots & 0 & \dots \\ \vdots & & & & \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & \dots \\ \vdots & \vdots & & \ddots & \end{pmatrix},$$

*is a regular matrix which is known as the Cesaro matrix of order one(or shortly, $C_1$).*

**Example 1.2.2** *([6]) Let $(t_k) \in \varpi$, and $R = (r_{nk})$ be a nonnegative matrix, regular matrix with*

$$r_{nk} = \begin{cases} \frac{t_k}{t_n}, & if\ 1 \leq k \leq n \\ 0, & otherwise \end{cases}$$

*where* $t_n = \sum_{k=1}^{n} t_k$.

*The matrix R is a regular matrix and known as Riesz matrix.*

**Definition 1.2.7** *([24]) A sequence of numbers* $\{f_n\}_{n=1}^{\infty}$ *is called Fibonacci numbers if*

$$f_n = f_{n-1} + f_{n-2}; \;\; n \geq 2,$$

*and* $f_0 = 0$, $f_1 = 1$ *are hold.*

**Example 1.2.3** *([24]) The Fibonacci matrix* $F = (f_{nk})$ *is a nonnegative infinite matrix, which is defined as*

$$f_{nk} = \begin{cases} \frac{f_k}{f_{n+2}-1}, & if\ 1 \leq k \leq n \\ 0, & otherwise \end{cases}$$

*or equivalently,*

$$f_{nk} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0 & 0 & \dots \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0 & 0 & \dots \\ \frac{1}{12} & \frac{1}{12} & \frac{2}{12} & \frac{3}{12} & \frac{5}{12} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*is a regular matrix.*

## 1.3 Density

The concept of statistical convergence and related topics are based on density functions. Therefore, all readers needs to know the idea and at least basic properties of density functions. For this reason, in the present section we introduced the definition and some properties of density functions.

**Definition 1.3.1** *([8]) Let* $S, R \subseteq \mathbb{N}$, *then the symmetric difference of S and R denoted* *by $S\Delta R$ and defined as*

$$S\Delta R = (S\setminus R)\cup(R\setminus S).$$

*Moreover, if the symmetric difference of $S$ and $R$ is finite then $S$ is called asymptotically equal to $R$ and denoted by $S\sim R$.*

**Definition 1.3.2** *([8]) A function $\delta$ from the space of all subset of natural numbers to the closed interval $[0,1]$ is called an asymptotic density function (or density function), if the following four axioms hold :*

*(D-1)* *If $S\sim R$, then $\delta(S)=\delta(R)$;*

*(D-2)* *If $S\cap R=\emptyset$, then $\delta(S)+\delta(R)\leq\delta(S\cup R)$;*

*(D-3)* *For every $S,R$; $\delta(S)+\delta(R)\leq 1+\delta(S\cap R)$;*

*(D-4)* $\delta(\mathbb{N})=1$.

*where $S$ and $R$ are subsets of natural numbers.*

**Definition 1.3.3** *([8])If the density of any subset $S\subseteq\mathbb{N}$ is represented by $\delta(S)$, then $\bar{\delta}(S)$, the upper density associated with $\delta(S)$, can be defined by*

$$\bar{\delta}(S)=1-\delta(\mathbb{N}\backslash S).$$

**Proposition 1.3.1** *([8]) For sets $S$ and $R$ of natural numbers, consider $\delta$ as a lower asymptotic density, which has $\bar{\delta}$ as an associated upper density. Then, the following propositions hold:*

*1)* $S\subseteq R\Rightarrow\delta(S)\leq\delta(R)$,

*2)* $S\subseteq R\Rightarrow\bar{\delta}(S)\leq\bar{\delta}(R)$, *3)* $\bar{\delta}(S \cup R) \leq \bar{\delta}(S) + \bar{\delta}(R)$ *for any* $S, R \subseteq \mathbb{N}$*,*

*4)* $\delta(\emptyset) = \bar{\delta}(\emptyset) = 0$*,*

*5)* $S \sim R \Rightarrow \bar{\delta}(S) = \bar{\delta}(R)$*,*

*6)* $\bar{\delta}(\mathbb{N}) = 1$*,*

*7)* $\delta(S) \leq \bar{\delta}(S)$.

**Proof.** 1) Since $S \cap (R \setminus S) = \emptyset$, then using (D-2) we have,

$$\delta(S) + \delta(R \setminus S) \leq \delta(S \cup (R \setminus S)).$$

From the assumption $S \subseteq R$,

$$\delta(S \cup (R \setminus S)) = \delta(R).$$

And, from the definition of density $\delta(R \setminus S) \geq 0$, so that

$$\delta(S) \leq \delta(S) + \delta(R \setminus S).$$

Thus,

$$\delta(S) \leq \delta(R).$$

2) Assume that $S \subseteq R$, then $(\mathbb{N} \setminus S) \supset (\mathbb{N} \setminus R)$. From (1)

$(\mathbb{N} \setminus S) \supset (\mathbb{N} \setminus R)$ provides

$$\delta(\mathbb{N} \setminus S) \geq \delta(\mathbb{N} \setminus R).$$

Multiply both sides by -1 and add 1. We get,

$$1 - \delta(\mathbb{N} \setminus S) \leq 1 - \delta(\mathbb{N} \setminus R).$$

So, we conclude that

$$\bar{\delta}(S) \leq \bar{\delta}(R).$$

3) From the definition of upper density,

$$\bar{\delta}(S) = 1 - \delta(\mathbb{N} \setminus S),$$

and

$$\bar{\delta}(R) = 1 - \delta(\mathbb{N} \setminus R).$$

We get,

$$\bar{\delta}(S) + \bar{\delta}(R) = 2 - \delta(S) - \delta(R)$$

$$= 2 - (\delta(\mathbb{N} \setminus S) + \delta(\mathbb{N} \setminus R))$$

$$= 2 - (1 + \delta((\mathbb{N} \setminus S) \cap (\mathbb{N} \setminus R))).$$

Let use $(\mathbb{N} \setminus S) \cap (\mathbb{N} \setminus R)) = \mathbb{N} \setminus (S \cup R)$.

We conclude that,

$$\bar{\delta}(S) + \bar{\delta}(R) = 1 - \delta(\mathbb{N} \setminus (S \cup R))$$

$$= \bar{\delta}(S \cup R).$$

4) Let use the property (D-2), we have

$$\delta(S) + \delta(R) \leq \delta(S \cup R)$$

if $S \cap R = \emptyset$.

Assume that $S = \emptyset$, then we attain;

$$\emptyset \cup R = R,$$

and

$$\emptyset \cap R = \emptyset.$$

So, we can use this conclusions in the property (D-2),

$$\delta(\emptyset) + \delta(R) \leq \delta(\emptyset \cup R)$$

$$= \delta(R).$$

We conclude that, $\delta(\emptyset) \leq 0$ and from the definition of density $\delta(\emptyset) \geq 0$.

As a result,

$$\delta(\emptyset) = 0.$$

It is similar to prove $\bar{\delta}(\emptyset) = 0$, by using definition of upper density.

5) From the definition of $S \sim R$, we get

$$\begin{aligned} S\Delta R &= (R \setminus S) \cup (S \setminus R) \\ &= ((\mathbb{N} \setminus S) \setminus (\mathbb{N} \setminus R)) \cup ((\mathbb{N} \setminus R) \setminus (\mathbb{N} \setminus S)) \\ &= (\mathbb{N} \setminus S)\Delta(\mathbb{N} \setminus R), \end{aligned}$$

which provides that,

$$\delta(\mathbb{N} \setminus S) = \delta(\mathbb{N} \setminus R).$$

Thus, we get

$$\bar{\delta}(S) = \bar{\delta}(R).$$

6) From the definiton of upper density;

$$\begin{aligned} \bar{\delta}(\mathbb{N}) &= 1 - \delta(\mathbb{N} \setminus \mathbb{N}) \\ &= 1 - \delta(\emptyset) \\ &= 1. \end{aligned}$$

7) By the property (D-2), we have

$$\delta(S) + \delta(R) \leq \delta(S \cup R),$$

when $S \cap R = \emptyset$.

Choose $(\mathbb{N} \setminus S)$ and $S$ instead of $S$ and $R$, respectively.

We get,

$$\begin{aligned} \delta(\mathbb{N} \setminus S) + \delta(S) &\leq \delta((\mathbb{N} \setminus S) \cup S) \\ &= \delta(\mathbb{N}) \\ &= 1. \end{aligned}$$

Thus,

$$\begin{aligned} \delta(S) &\leq 1 - \delta(\mathbb{N} \setminus S) \\ &= \bar{\delta}(S). \end{aligned}$$

■

**Definition 1.3.4** *([8]) A subset $K \subset \mathbb{N}$ is called to have natural density with respect to $\delta$, if*

$$\delta(K) = \bar{\delta}(K).$$

**Example 1.3.1** *Consider the asymptotic density function*

$$\delta(K) = \lim_{n\to\infty} {}_* \frac{|K(n)|}{n},$$

*where $|K(n)|$ represents the number of elements in $\mathbb{N} \cap K$, then*

$$\delta(K) = \bar{\delta}(K)$$

*iff*

$$\delta(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}.$$

*This density function is known as natural density.*

Furthermore, recall that the characteristic sequence of K is represented by $\chi_K$ and it is a sequence of 0's and 1's and the Cesàro matrix, which is denoted by $C_1$, is defined as;

$$C_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & \dots & 0 & \dots \\ \vdots & & & & \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & \dots \\ \vdots & \vdots & & \ddots & \end{pmatrix}.$$

Then the $n^{th}$ term of the sequence $C_1\chi_K$ is equal to $\frac{|K(n)|}{n}$. Therefore,

$$\delta(K) = \lim_{n\to\infty} (C_1.\chi_K)_n,$$

and axioms (D-1)-(D-4) are satisfied for this function. In other words, the natural density $\delta(K)$ can be defined by using the Cesàro matrix $C_1$.

**Example 1.3.2** *Consider the set $M = \{m \in \mathbb{N} : m = k^2\}$ where k is natural number.*

*Clearly, $M(n) \leq \sqrt{n}$ in which $M(n)$ represents the number of elements belonging to set $M$ in the first n natural numbers. Then,*

$$\begin{aligned}\delta(M) &= \lim_n \frac{|M(n)|}{n} \\ &\leq \lim_n \frac{\sqrt{n}}{n} \\ &= 0.\end{aligned}$$

**Example 1.3.3** *If $T = \{n \in \mathbb{N} : n = 5k\}$, so that $k \in \mathbb{N}$. Then*

$$\begin{aligned}\delta(T) &= \lim_n \frac{|T(n)|}{n} \\ &= \frac{1}{5}.\end{aligned}$$

**Lemma 1.3.1** *If $K = \{n \in \mathbb{N} : n = ak + b\}$ with $k \in \mathbb{N}$. So*

$$\delta(K) = \lim_n \frac{1}{a}.$$

**Example 1.3.4** *If $K$ is a finite subset of $\mathbb{N}$, then obviously $\delta(K) = 0$.*

The example which is presented in Example 1.3.1 implies that one may create a density with the aid of the summability method.

**Proposition 1.3.2** *([8]) If $A \in (C,C;\eta)$, then $\delta_A(K)$ which is defined by*

$$\delta_A(K) = \lim_{n\to\infty} (A.\chi_K)_n$$

*is called A-density of K.*

# Chapter 2

## NEW TYPE CONVERGENCES

Statistical convergence has been initiated by H. Fast and H. Steinhaus independently in 1951. After that, statistical convergence is used by many researchers in different directions([7, 21, 16, 10]). Moreover some non-trivial extensions like lacunary statistical convergence, $\lambda$-statistical convergence, A-statistical convergence and $\alpha\beta$-statistical convergence are introduced and discussed by different researchers. This chapter is devoted to these new type convergences([12, 19, 1]).

### 2.1 Statistical Convergence

As it is mentioned before, if the sequence $\chi_K$ is the characteristic sequence of the set $K$ and the matrix $C_1 = (c_{nk})$ is defined by

$$c_{nk} := \begin{cases} \frac{1}{n}, & if\ 1 \leq k \leq n \\ 0, & otherwise. \end{cases}$$

Then,

$$\delta(K) = \lim_n (C_{nk}\chi_K)_n$$

is called the natural density of $K$.

**Definition 2.1.1** *([7]) Let $x = (x_k) \in \varpi$. If $(x_k)$ satisfies the condition,*

$$\delta(\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}) = \lim_{n\to\infty} \frac{|\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}|}{n} = 0$$

*for all $\varepsilon > 0$, then $x$ is called statistically convergent to $\eta$ and denoted by $C_{st} - \lim x = \eta$ or $x_k \to \eta(C_{st})$. A sequence which is not statistically convergent called statistically divergent.*

The set of all statistical convergent sequences is denoted by $C_{st}$.

**Theorem 2.1.1** *([10]) "Ordinary convergence implies statistical convergence."*

**Proof.** Assume that $x_k \to \eta$, then this assumption implies that the set $\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}$ is a finite set. Since the density of finite set is zero(Example 1.3.4), $\delta(\{k : |x_k - \eta| \geq \varepsilon\}) = 0$, $\forall \varepsilon > 0$. Therefore, $x$ statistically convergent to $\eta$. ■

The most significant difference between ordinary and statistical convergence is given by the next remark.

**Remark 2.1.1** *For the ordinary convergence, if $x$ is convergent to $\eta$, then at most finitely many terms of the sequence are allowed to be outside the all $\varepsilon$ - neighborhoods of the limit $\eta$. But in statistical sense, there can be infinitely many terms of the sequence $x = (x_k)$ outside of each $\varepsilon$-neighborhoods under the condition that their natural density is zero.*

**Example 2.1.1** *Consider the sequence $x = (x_k)$, where*

$$x_k = \begin{cases} 3, & \text{if} \quad k = m^3 \\ 0, & \text{if} \quad k \neq m^3 \end{cases}$$

*since $\delta(\{k^3 : k \in \mathbb{N}\}) = 0$, we have $C_{st} - \lim x = 0$ . However, since infinitely many terms of $(x_k)$ are out of each $\varepsilon$ - neighborhoods, then $x$ does not converge to 0 or 3 in ordinary sense.*

Another important difference between ordinary and statistical convergence is the boundedness condition. In the ordinary case, every convergent sequence has to be bounded.

Whereas, in the statistical case, $x \in C_{st}$ need not to be bounded. The following example demonstrates this difference.

**Example 2.1.2** *For* $x = (x_k) \in \varpi$, *where*

$$x_k = \begin{cases} k^2, & \text{if} \quad k = m^2 \\ 9, & \text{if} \quad k \neq m^2 \end{cases}$$

*Obviously,* $C_{st} - \lim x = 9$ *but* $x$ *is not bounded, and this implies that* $x$ *is not ordinary convergent to 9.*

**Lemma 2.1.2** *If* $C_{st} - \lim x = \eta_1$ *and* $C_{st} - \lim y = \eta_2$, *then*

*(i)* $C_{st} - \lim(x + y) = \eta_1 + \eta_2$,

*(ii)* $C_{st} - \lim(x.y) = \eta_1.\eta_2$,

*(iii)* $C_{st} - \lim(c.x) = c.\eta_1$, *for any* $c \in \mathbb{R}$.

**Proof.**

(i) $\forall \varepsilon > 0$, the next inclusion holds,

$$\{k : |(x_k + y_k) - (\eta_1 + \eta_2)| \geq \varepsilon\} \subset \{k : |x_k - \eta_1| \geq \frac{\varepsilon}{2}\} \cup \{k : |y_k - \eta_2| \geq \frac{\varepsilon}{2}\}.$$

Clearly, as a consequence of above inclusion we have,

$$C_{st} - \lim(x + y) = \eta_1 + \eta_2.$$

(ii) Since $C_{st} - \lim x = \eta_1$, define a set $F$ such that,

$$\delta(F) = \delta(\{k : |x_k - \eta_1| < 1\}) = 1.$$

It is obvious that,

$$|x_k y_k - \eta_1 \eta_2| \leq |x_k||y_k - \eta_2| + |\eta_2||x_k - \eta_1|.$$

For every $k \in F$, we have

$$|x_k| < |\eta_1| + 1.$$

Therefore,

$$|x_k y_k - \eta_1 \eta_2| \leq (|\eta_1| + 1)|y_k - \eta_2| + |\eta_2||x_k - \eta_1|. \tag{2.1.1}$$

Given $\varepsilon > 0$, pick $\mu$ such that,

$$0 < 2\mu < \frac{\varepsilon}{|\eta_1| + |\eta_2| + 1}. \tag{2.1.2}$$

Now define,

$$G_1 = \{k : |x_k - \eta_1| < \mu\},$$

and

$$G_2 = \{k : |y_k - \eta_2| < \mu\}.$$

It is obvious that

$$\delta(G_1) = \delta(G_2) = 1,$$

because of

$$C_{st} - \lim x = \eta_1$$

and

$$C_{st} - \lim y = \eta_2.$$

Therefore, by using (D-2) of Definition 1.3.2, we get

$$\delta(F \cap G_1 \cap G_2) = 1,$$

or equivalently,

$$\delta(k : |x_k y_k - \eta_1 \eta_2| \geq \varepsilon) = 0,$$

which completes the proof of (ii).

(iii) In case of $c = 0$, this condition is satisfied. Assuming $c \neq 0$, and defining $y_k = c$ for every $k \in \mathbb{N}$, the conclusion follows by $(ii)$.

■

**Example 2.1.3** *Consider $x = (x_k) \in \varpi$ and $y = (y_k) \in \varpi$ which are defined as*

$$x_k := \begin{cases} 1, & k = m^2 \\ 0, & k = m^2 + 1 \\ 2, & otherwise \end{cases}$$

*and*

$$y_k := \begin{cases} \frac{1}{k} + 1, & otherwise \\ 0, & k = m^2 \end{cases}$$

*respectively. Then, the sequences $(x_k)$ and $(y_k)$ are not convergent in the ordinary sense. But*

$$C_{st} - \lim x = 2,$$

*and*

$$C_{st} - \lim y = 1.$$

*From Lemma (2.1.2) we have,*

$$C_{st} - \lim(x + y) = 3,$$

$$C_{st} - \lim(x.y) = 2,$$

*and*

$$C_{st} - \lim(3x) = 6.$$

**Definition 2.1.2** *([10]) If a sequence $x = (x_k)$ provides property $P$ for every $k \notin K$ with $\delta(K) = 0$, then it is said that $(x_k)$ satisfies $P$ "almost all k", and it is abreviated by "a.a.k.".*

The next lemmas can be given as a result of this definition.

**Lemma 2.1.3** *([10]) For a sequence $(x_k)$, $C_{st} - \lim x = \eta$ iff $\forall \varepsilon > 0$,*

$$|x_k - \eta| < \varepsilon \;\; a.a.k.$$

**Theorem 2.1.4** *$(x_k) \to \eta(C_{st})$ iff $\exists (k_n)$ so that $\delta(\{k_n : n \in \mathbb{N}\}) = 1$ and $\lim_{k\to\infty} x_{k_n} = \eta$.*

**Definition 2.1.3** *([25]) $(x_k) \in \varpi$ is called statistically divergent to $\infty$ if $\forall T \in \mathbb{R}$,*

$$\delta(\{k \in \mathbb{N} : x_k > T\}) = 1.$$

**Example 2.1.4** *Consider $(x_k) \in \varpi$ where*

$$x_k = \begin{cases} \sqrt{k}, & \text{otherwise} \\ 2, & k = m^3 \end{cases},$$

*then $(x_k)$ is statistically divergent to $\infty$.*

**Definition 2.1.4** *([25]) $(x_k) \in \varpi$ is called statistically divergent to $-\infty$ if $\forall M \in \mathbb{R}$,*

$$\delta(\{k \in \mathbb{N} : x_k < M\}) = 1.$$

**Example 2.1.5** *Consider $(x_k) \in \varpi$ where*

$$x_k = \begin{cases} -k, & \text{otherwise} \\ \sqrt{k}, & k = m^3 \end{cases}$$

*then $(x_k)$ is statistically divergent to $-\infty$.*

**Theorem 2.1.5** *Any statistical divergent sequence to $\infty$ (or to $-\infty$) is a divergent sequence.*

**Proof.** Let $(x_k)$ be a statistically divergent to $\infty$ (or $-\infty$). Then for all real number $T$,

$$\delta(\{k \in \mathbb{N} : |x_k| > T\}) = 1 \text{ (or } \delta(\{k \in \mathbb{N} : |x_k| < T\}) = 1).$$

It is obvious that, the sequence $(x_k)$ is not bounded. So, the sequence $(x_k)$ can not be convergent because every convergent sequence is bounded. Then, $(x_k)$ is a divergent sequence. ■

**Remark 2.1.2** *A divergent sequence need not to be statistically divergent.*

**Example 2.1.6** *Consider $x := (x_k) \in \varpi$ where*

$$x_k = \begin{cases} k, & \text{if } k = n^2 \\ \\ 6, & \text{otherwise}. \end{cases}$$

*Then, the sequence $x$ is divergent but not statistically divergent.*

**Definition 2.1.5** *([10]) $x := (x_k) \in \varpi$ is called statistically Cauchy sequence if $\forall \varepsilon > 0$, $\exists H = H(\varepsilon)$ such that,*

$$\lim_n \frac{1}{n} |k \leq n : |x_k - x_H| \geq \varepsilon| = 0.$$

**Theorem 2.1.6** *([10]) $(x_k) \in \varpi$ is statistically Cauchy iff $(x_k) \in C_{st}$.*

## 2.2 Lacunary Statistical Convergence

In this section we shall discuss lacunar statistically convergent sequences. We will also discuss inclusion relations with statistical convergence.

**Definition 2.2.1** *([12]) A lacunary sequence $\theta = \{k_r\}$ is an increasing sequence of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Furthermore, $I_r := (k_{r-1}, k_r]$ and $q_r := \frac{k_r}{k_{r-1}}$.*

The set of all lacunary sequences is represented by $\Theta$.

**Example 2.2.1** *A sequence $\theta = \{r! - 1\}$ is a lacunary sequence with $I_r = ((r-1)! - 1, r! - 1]$ and $q_r = \frac{r!-1}{(r-1)!-1}$.*

**Definition 2.2.2** *([5, 12]), Let the sequence $\chi_K$ be the characteristic sequence of the* *set K and a matrix* $C_\theta = (c_{nk})_{n,k=1}^{\infty}$ *be defined by*

$$c_{nk} := \begin{cases} \frac{1}{h_r}, & if \quad k \in I_r \\ \\ 0, & if \quad k \notin I_r \end{cases}$$

*Then,*

$$\delta_\theta(K) = \lim_n (C_\theta \chi_K)_n$$

*is called the lacunary-density of K.*

*Furthermore,*

$$\delta_\theta(K) = \lim_{r\to\infty} \frac{|I_r \cap K|}{h_r}.$$

**Definition 2.2.3** *([12]) For a lacunary sequence* $\theta = \{k_r\}$*, a number sequence* $x := (x_k)$ *is called lacunary statistical convergent to* $\eta$ *if* $\forall \varepsilon > 0$

$$\delta_\theta(K(\varepsilon)) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - \eta| \geq \varepsilon\}|$$

$$= 0,$$

*where* $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}$*. Lacunary convergence of x to* $\eta$ *is denoted by* $\theta_{st} - \lim x = \eta$ *or* $x_k \to \eta(\theta_{st})$.

The set of all lacunary statistical convergent sequences is represented by $\theta_{st}$.

**Example 2.2.2** *Consider* $x = (x_k) \in \varpi$*, where*

$$x_k = \begin{cases} 1, & if \quad k = 2^r \\ \\ 0, & if \quad k \neq 2^r \end{cases}$$

*and* $\theta = \{k_r\}$ *is a lacunary sequence and defined as* $\{k_r\} = 2^r - 1$*, where r is a natural number.*

*We should check the limit*

$$\lim_{r\to\infty} \frac{|I_r \cap K(\varepsilon)|}{h_r} = \lim_{r\to\infty} \frac{|\{k \in I_r : |x_k - \eta| \geq \varepsilon\}|}{h_r},$$

*for lacunary statistical convergence.*

*Clearly,*

$$h_r = 2^{r-1}.$$

*Choose* $\eta = 1$.

*Then, the set* $K(\varepsilon) \cap I_r$ *has* $2^{r-1} - 1$ *elements.*

*In other words,*

$$|K(\varepsilon) \cap I_r| = |\{k \in I_r : |x_k - 1| \geq \varepsilon\}|$$

$$= 2^{r-1} - 1.$$

*So,*

$$\lim_{r \to \infty} \frac{|\{k \in I_r : |x_k - 1| \geq \varepsilon\}|}{h_r} = 1.$$

*Thus,* $\delta_\theta(K(\varepsilon)) = 1$ *implies that* $(x_k)$ *is not lacunary statistical convergent to* $1$.

*Choose* $\eta = 0$.

*Then, the set* $K(\varepsilon) \cap I_r$ *has only one element for each r.*

*In other words for every r,* $|K(\varepsilon) \cap I_r| = |\{k \in I_r : |x_k - 0| \geq \varepsilon\}| = 1$.

*So,*

$$\lim_{r \to \infty} \frac{1}{2^{r-1}} = 0.$$

*Thus,* $\delta_\theta(K(\varepsilon)) = 0$ *implies that* $(x_k)$ *is lacunary statistical convergent to* $0$.

**Example 2.2.3** *Consider* $\theta = \{2^r - 1\} \in \Theta$, *and* $x \in \varpi$ *which is defined as*

$$x_k = \begin{cases} 2, & if \quad k \text{ is even} \\ 3, & if \quad k \text{ is odd}. \end{cases}$$

*It is obvious that* $h_r = 2^{r-1}$.

*Choose* $\eta = 2$.

*For every interval* $I_r$*, a set* $K(\varepsilon) \cap I_r = \{k \in I_r : |x_k - 2| \geq \varepsilon\}$ *has* $\frac{2^{r-1}-1}{2}$ *number of elements.*

*So,*

$$\lim_{r \to \infty} \frac{\frac{2^{r-1}-1}{2}}{2^{r-1}} = \frac{1}{2}.$$

*Therefore,* $\delta_\theta(K(\varepsilon)) = \frac{1}{2}$. *Then, the sequence* $(x_k)$ *is not lacunary statistically convergent to* $2$.

*Similarly, if we choose* $\eta = 3$, *then* $\delta_\theta(K(\varepsilon)) = \frac{1}{2}$ *such that* $(x_k)$ *is not lacunary statistically convergent to* $3$.

**Lemma 2.2.1** *Suppose that* $\theta_{st} - \lim x = \eta_1$ *and* $\theta_{st} - \lim y = \eta_2$. *Then,*

*(i)* $\theta_{st} - \lim(x+y) = \eta_1 + \eta_2$,

*(ii)* $\theta_{st} - \lim(x.y) = \eta_1.\eta_2$,

*(iii)* $\theta_{st} - \lim(c.x) = c.\eta_1$ *for any* $c \in \mathbb{R}$.

**Definition 2.2.4** $(x_k) \in \varpi$ *is called lacunary statistical divergent to* $\infty$ *if for every real number K,*

$$\delta_\theta(\{k \in \mathbb{N} : x_k > K\}) = 1.$$

**Example 2.2.4** *Consider* $\theta = \{k_r\} \in \Theta$, *where* $\{k_r\} = \{2^r - 1\}$. *Assume that* $x \in \varpi$ *is defined as*

$$x_k = \begin{cases} 0, & if\ k = r^2 \\ k, & otherwise. \end{cases}$$

*So,*

$$\delta_\theta(\{k \in \mathbb{N} : x_k > K\}) = 1$$

*for every real number* $K$. *Consequently, the sequence* $x$ *is lacunary statistical divergent to* $\infty$.

**Definition 2.2.5** $(x_k) \in \varpi$ *is called lacunary statistical divergent to* $-\infty$ *if for every real number M,*

$$\delta_\theta(\{k \in \mathbb{N} : x_k < M\}) = 1.$$

**Example 2.2.5** *Let $\theta = \{k_r\} \in \Theta$, where $\{k_r\} = \{r! - 1\}$. Assume that $x := (x_k) \in \varpi$ is defined as*

$$x_k = \begin{cases} 0, & if\ k = r! \\ -2k, & otherwise. \end{cases}$$

*So,*

$$\delta_\theta(\{k \in \mathbb{N} : x_k < M\}) = 1$$

*for every real number $M$. Therefore, the sequence $x$ is lacunary statistical divergent to $-\infty$.*

**Lemma 2.2.2** *([12]) For $\theta \in \Theta$, $C_{st} - \lim x = \eta$ provides $\theta_{st} - \lim x = \eta$ iff*

$$\lim_r{}_* q_r > 1.$$

**Example 2.2.6** *Consider $\theta = \{3^r - 1\} \in \Theta$ and $x = (x_k) \in \varpi$, where*

$$x_k = \begin{cases} 3, & if\ k = r^2 \\ 2, & otherwise. \end{cases}$$

*Let us check whether $x \in \theta_{st}$ or not. First of all, we have that $\lim_r{}_* q_r = 3$. So, it is enough to check whether $x \in C_{st}$ or not. Due to*

$$\delta(\{k \in \mathbb{N} : |x_k - 2| > \varepsilon\}) = 0,$$

*$x \to 2(C_{st})$. Therefore, $x \to 2(\theta_{st})$ from Lemma 2.2.2.*

**Lemma 2.2.3** *([12]) For $\theta \in \Theta$, $\theta_{st} - \lim x = \eta$ provides $C_{st} - \lim x = \eta$ iff*

$$\lim_r{}^* q_r < \infty.$$

**Example 2.2.7** *Consider the lacunary sequence $\theta = \{r^{r+1}\}$ and define $x$ where*

$$x_k = \begin{cases} 1, & if\ k_{r-1} < k \leq 2k_{r-1} \\ 0, & otherwise \end{cases}$$

*Since*

$$\lim_{r\to\infty} \frac{|\{k \in I_r : |x_k - 0| \geq \varepsilon\}|}{h_r} \leq \lim_r \frac{k_{r-1}}{h_r} = 0,$$

*the sequence x is lacunary statistical convergent to* 0.

*On the other hand, the sequence x is not statitistical convergent.*

**Theorem 2.2.4** *([12]) Let $\theta \in \Theta$,then*

$$C_{st} - \lim x = \theta_{st} - \lim x = \eta$$

*iff*

$$1 < \lim_r{}_* q_r \leq \lim_r{}^* q_r < \infty.$$

**Theorem 2.2.5** *([12]) If $x \in C_{st}$ and $x \in \theta_{st}$, then $C_{st} - \lim x = \theta_{st} - \lim x$.*

## 2.3 $\lambda$-Statistical Convergence

The concept of $\lambda-$statistical convergence has been introduced by M. Mursaleen in ([19]). Later, the concept of $\lambda-$statistical convergence has been studied by different authors in different ways. In this section we shall discuss briefly, the concept of $\lambda-$statistical convergence and its properties.

"Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that $\lambda_{n+1} \leq \lambda_n + 1$, with $\lambda_1 = 1$, and $I_n = [n - \lambda_n + 1, n]$."

By using $(\lambda_n)$, the $\lambda-$ density can be defined in as follows;

**Definition 2.3.1** *([19]) Let $K \subseteq \mathbb{N}$. Then, $\lambda$-density of K is denoted by $\delta_\lambda(K)$, and defined as,*

$$\delta_\lambda(K) = \lim_{n\to\infty} \frac{|\{k \in I_n : k \in K\}|}{\lambda_n}.$$

**Remark 2.3.1** *([19])A sequence $\chi_K$ is a characteristic sequence of the set K and a matrix*

*$\lambda = (\lambda_{nk})_{n,k=1}^{\infty}$ is defined by*

$$\lambda_{nk} := \begin{cases} \frac{1}{\lambda_n}, & if \quad k \in I_n \\ 0, & if \quad k \notin I_n. \end{cases}$$

*Then,*

$$\delta_\lambda(K) = \lim_n (\lambda_{nk}\chi_K)_n$$

*is called the $\lambda$-density of $K$.*

It shows that, $\lambda$-density is a special condition of $A$- density.

**Example 2.3.1** *Let $(\lambda_n)$ be a nonnegative real valued sequence defined as $\lambda_n = \sqrt{n}$, then the interval $I_n = [n - \sqrt{n} + 1, n]$. Now, Consider the sequence $(x_k)$ which is*

$$x_k = \begin{cases} 1, & if\, k = 3n \\ 2, & if\, k = 3n+1 \\ 3, & if\, k = 3n+2. \end{cases}$$

*If*

*$K = \{3n : n \in \mathbb{N}\}$, $N = \{3n+1 : n \in \mathbb{N}\}$ and $M = \{3n+2 : n \in \mathbb{N}\}$, then*

$$\delta_\lambda(K) = \frac{1}{3},$$
$$\delta_\lambda(N) = \frac{1}{3},$$

*and*

$$\delta_\lambda(M) = \frac{1}{3}.$$

**Definition 2.3.2** *([19]) A sequence $x = (x_k)$ is called $\lambda$-statistically convergent to $\eta$ provided that $\forall \varepsilon > 0$ the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}$ has $\lambda$-density zero.*

*In other words,*

$$\delta_\lambda(K(\varepsilon)) = \lim_{n\to\infty} \frac{|\{k \in I_n : |x_k - \eta| \geq \varepsilon\}|}{\lambda_n} = 0.$$

*In this case, this convergence is represented by $\lambda_{st} - \lim x = \eta$ or $x \to \eta(\lambda_{st})$.*

The set of all $\lambda$-statistically convergent sequences is represented by $\lambda_{st}$.

**Example 2.3.2** *Let $\lambda_n$ be a nonnegative real valued sequence defined as $\lambda_n = \sqrt{n}$, then $I_n = [n - \sqrt{n} + 1, n]$. Consider the sequences $(x_k)$ and $(y_k)$ where*

$$x_k = \begin{cases} 1, & if\ k = \sqrt{n}+1 \\ 2, & otherwise \end{cases}$$

*and*

$$y_k = \begin{cases} 4, & if\ k = 2n \\ 5, & if\ k = 2n+1 \end{cases}.$$

$$\lim_n \frac{|\{k \in I_n : |x_k - 2| \geq \varepsilon\}|}{\lambda_n} = 0,$$

$$\lim_n \frac{|\{k \in I_n : |x_k - 1| \geq \varepsilon\}|}{\lambda_n} = 1,$$

$$\lim_n \frac{|\{k \in I_n : |y_k - 4| \geq \varepsilon\}|}{\lambda_n} = \frac{1}{2},$$

$$\lim_n \frac{|\{k \in I_n : |y_k - 5| \geq \varepsilon\}|}{\lambda_n} = \frac{1}{2}.$$

*So, $x_k \to 2(\lambda_{st})$ and $(y_k)$ is not $\lambda$-statistically convergent.*

**Remark 2.3.2** *([19]) If $\lambda_n = n$, then $\delta_\lambda(K)$ is reduced to $\delta(K)$ and $\lambda$-statistical convergence reduces to statistical convergence.*

**Example 2.3.3** *Choose $\lambda_n = n$. Then, $I_n = [n - \lambda_n + 1, n]$ becomes $I_n = [1, n]$ and the* *matrix $\lambda = (\lambda_{nk})$ becomes,*

$$\lambda_{nk} = \begin{cases} \frac{1}{n}, & if\ 1 \leq k \leq n \\ 0, & otherwise \end{cases}$$

*which is equal to $C_1$. It indicates that,*

$$\delta_\lambda(K) = \lim_n (\lambda_{nk}\chi_K)_n$$

$$= \delta(K)$$

*for any subset K, where $\chi_K$ is the characteristic sequence of K.*

**Theorem 2.3.1** *Ordinary convergence implies $\lambda$-statistically convergence.*

**Proof.** Let $x \to \eta$, then the set $\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}$ is finite. Therefore,

$$\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\} \supseteq \{k \in I_n : |x_k - \eta| \geq \varepsilon\}$$

holds true. Thus,

$$\frac{1}{\lambda_n}|\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}| \geq \frac{1}{\lambda_n}|\{k \in I_n : |x_k - \eta| \geq \varepsilon\}|,$$

take limit on both sides as n $\to \infty$, completes the proof. ■

**Definition 2.3.3** *$(x_k) \in \varpi$ is called $\lambda$-statistical divergent to $\infty$ if $\forall K \in \mathbb{R}$,*

$$\delta_\lambda(\{k \in \mathbb{N} : x_k > K\}) = 1.$$

**Definition 2.3.4** *$(x_k) \in \varpi$ is called $\lambda$-statistical divergent to $-\infty$ if $\forall M \in \mathbb{R}$,*

$$\delta_\lambda(\{k \in \mathbb{N} : x_k < M\}) = 1.$$

**Theorem 2.3.2** *([19]) $\lim_{n}{}_* \frac{\lambda_n}{n} > 0$ if and only if $C_{st} \subseteq \lambda_{st}$.*

**Remark 2.3.3** *([19]) Under the condition $\lim_{n}{}_* \frac{\lambda_n}{n} = 0$, above theorem does not hold.*

**Theorem 2.3.3** *([15]) For a real valued sequence x, if $\lambda_n$ implies $\lim_n \frac{\lambda_n}{n} = 1$, then $\lambda_{st} \subseteq C_{st}$.*

## 2.4 A-Statistical Convergence

In 1981, Freedman and Sember generalized the natural density function $\delta$ which is based on $C_1$([8]). They replace $C_1$ by any non-negative, regular matrix $A$.

**Definition 2.4.1** *([8, 16])Let $K = \{k_i\}$ be an index set and let $\chi_K$ be the characteristic sequence of $K$. In this case, the $A-$density of $K$ is introduced as follows;*

$$\delta_A(K) = \lim_{n\to\infty} (A\chi_K)_n$$

*in which A represents a non-negative regular matrix.*

*Furthermore,*

$$\delta_A(K) = \lim_{n\to\infty} \sum_{k\in K} a_{nk}$$

$$= \lim_{n\to\infty} \sum_{i} a_{n,k_i}.$$

**Lemma 2.4.1** *If $A_{st} - \lim x = \eta_1$ and $A_{st} - \lim y = \eta_2$. Then,*

*(i)* $A_{st} - \lim(x+y) = \eta_1 + \eta_2$,

*(ii)* $A_{st} - \lim(x.y) = \eta_1.\eta_2$,

*(iii)* $A_{st} - \lim(c.x) = c.\eta_1$ *for any* $c \in \mathbb{R}$.

If $\delta_A(K)$ is known, then $\delta_A(\mathbb{N}\backslash K)$ can be found by

$$\delta_A(\mathbb{N}\backslash K) = 1 - \delta_A(K).$$

**Example 2.4.1** *Consider the matrix $A = (a_{nk})$, where*

$$a_{nk} = \begin{cases} 1 \quad , \quad k = n^4 \\ 0 \quad , \quad k \neq n^4 \end{cases}.$$

*Let choose,*

$$K_1 = \{k \in \mathbb{N} : k = n^4\},$$

*and*

$$K_2 = \{k \in \mathbb{N} : k \neq n^4\}.$$

*Therefore,* $\delta_A(K_1) = 1$ *and* $\delta_A(K_2) = 0$.

**Definition 2.4.2** *([16])A sequence x is called A-statistical convergent to* $\eta$*, if* $\forall \varepsilon > 0$,

$$\delta_A(K(\varepsilon)) = \delta_A(\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\})$$

$$= 0.$$

*In that case, this convergency can be written as* $A_{st} - \lim x = \eta$.

The set $A_{st}$ represents all $A$-statistical convergent sequences.

**Remark 2.4.1** *If a matrix* $A \in (C, C; \eta)$ *is equals to* $C_1$ *which is Cesaro matrix, then A-statistical density is reduced to natural density. Furthermore, A-statistical convergence is reduced to statistical convergence([16]).*

**Example 2.4.2** *Consider* $C_1 = (c_{nk}) \in (C, C; \eta)$ *where*

$$c_{nk} = \begin{cases} \frac{1}{n}, & k \leq n \\ 0, & otherwise, \end{cases}$$

*which is known as Cesaro matrix.*

*Let a sequence x is defined as*

$$x_k = \begin{cases} 2, & if \quad k \text{ is odd} \\ 3, & if \quad k \text{ is even} \end{cases}$$

*Let the set* $K(\varepsilon)$ *is defined as*

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - 2| \geq \varepsilon\},$$

*and the set* $M(\varepsilon)$ *is defined as*

$$M(\varepsilon) = \{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\}.$$

*Then,* $\forall \varepsilon > 0$ *we get,*

$$\delta_{C_1}(K(\varepsilon)) = \delta_{C_1}(\{k \in \mathbb{N} : |x_k - 2| \geq \varepsilon\})$$

$$= \lim_{n\to\infty} (C_1\chi_K)_n$$

$$= \frac{1}{2},$$

*where* $\chi_K$ *is characteristic sequence of* $K(\varepsilon)$.

*Therefore, x is not* $C_1$*-statistically convergent to 2.*

*And,*

$$\delta_{C_1}(M(\varepsilon)) = \delta_{C_1}(\{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\})$$

$$= \lim_{n\to\infty} (C_1\chi_M)_n$$

$$= \frac{1}{2},$$

*where* $\chi_M$ *is characteristic sequence of* $M(\varepsilon)$.

*Similarly, x is not* $C_1$*-statistically convergent to 3.*

From the above example, $x$ is not $C_1$-statistical convergent does not mean that it is not $A$-statistical convergent for other non-negative regular matrix.

**Example 2.4.3** *Let a matrix* $A = (a_{nk}) \in (C, C; \eta)$ *be defined as*

$$a_{nk} = \begin{cases} 1, & if \quad k = 2n \\ 0, & if \quad otherwise \end{cases}$$

*And, let a sequence x is defined from above theorem.*

*Likewise,* $K(\varepsilon)$ *and* $M(\varepsilon)$ *is defined from above theorem.*

*Then,*

$$\delta_A(K(\varepsilon)) = \delta_A(\{k \in \mathbb{N} : |x_k - 2| \geq \varepsilon\})$$

$$= \lim_{n\to\infty} (A\chi_K)_n$$

$$= 1.$$

*And,*

$$
\begin{aligned}
\delta_A(M(\varepsilon)) &= \delta_A(\{k\in\mathbb{N} : |x_k-3|\geq\varepsilon\}) \\
&= \lim_{n\to\infty}(A\chi_M)_n \\
&= 0.
\end{aligned}
$$

*This equality implies that* $x\to 3(A_{st})$.

**Remark 2.4.2** *If a matrix* $A\in(C,C;\eta)$ *is equal to* $C_\theta$*, which is defined in Definition 2.2.2, then A-density becomes lacunary density. Furthermore, A-statistical convergence becomes lacunary statistical convergence([5]).*

**Example 2.4.4** *Considering matrix* $C_\theta=(c_{nk})$*, where*

$$
c_{nk}=\begin{cases} \frac{1}{h_r}, & if\ k\in I_r \\ 0, & if\ otherwise, \end{cases}
$$

*and* $\theta=\{3^r-1\}\in\Theta$*. Let* $x\in\varpi$*, which is defined as*

$$
x_k=\begin{cases} 1, & k=3^r \\ 0, & k\neq 3^r, \end{cases}
$$

*and the sets* $M(\varepsilon)$ *and* $N(\varepsilon)$ *are defined as*

$$
M(\varepsilon)=\{k\in\mathbb{N} : |x_k-1|\geq\varepsilon\},
$$

*and*

$$
N(\varepsilon)=\{k\in\mathbb{N} : |x_k-0|\geq\varepsilon\}.
$$

*Then, we get for every* $\varepsilon>0$,

$$
\begin{aligned}
\delta_{C_\theta}(M(\varepsilon)) &= \delta_{C_\theta}(\{k\in I_r : |x_k-1|\geq\varepsilon\}) \\
&= \lim_n(C_\theta\chi_M)_n \\
&= 1,
\end{aligned}
$$

*where* $\chi_M$ *represents the characteristic sequence of* $M(\varepsilon)$.

*In the same way,*

$$\begin{aligned}\delta_{C_\theta}(N(\varepsilon)) &= \delta_{C_\theta}(\{k \in I_r : |x_k| \geq \varepsilon\})\\ &= \lim_n (C_\theta \chi_N)_n\\ &= 0,\end{aligned}$$

*where $\chi_N$ represents the characteristic sequence of $N(\varepsilon)$.*

*Therefore, x is $C_\theta$-statistically(lacunary statistically) convergent to* $0$.*(Or, $x \to 0(\theta_{st})$.)*

**Remark 2.4.3** *If $A \in (C, C; \eta)$ is equal to $\lambda = (\lambda_{nk})$, which is defined in Remark 2.3.1, then A-density becomes $\lambda$-density. Moreover, A-statistical convergence becomes $\lambda$-statistical convergence([19]).*

**Example 2.4.5** *Considering matrix $\lambda = (\lambda_{nk})$ in which*

$$\lambda_{nk} = \begin{cases} \frac{1}{\lambda_n}, & if\ k \in I_n \\ 0, & otherwise, \end{cases}$$

*and suppose that $\lambda = (\lambda_n)$ is defined as $\lambda_n = \sqrt[3]{n}$.*

*Let a sequence x be defined as*

$$x_k = \begin{cases} 3, & if\ k = \sqrt{n} + 2 \\ 4, & otherwise, \end{cases}$$

*and the sets $K(\varepsilon)$ and $L(\varepsilon)$ are defined as*

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\},$$

*and*

$$L(\varepsilon) = \{k \in \mathbb{N} : |x_k - 4| \geq \varepsilon\}.$$

*Thus, we get for all $\varepsilon > 0$,*

$$\begin{aligned}\delta_\lambda(K(\varepsilon)) &= \lim_n (\lambda \chi_K)_n\\ &= 1.\end{aligned}$$

*where $\chi_K$ represents the characteristic sequence of $K(\varepsilon)$.*

*So, x is not $C_\lambda$-statistically($\lambda$-statistically) convergent to* $3$. *Similarly, we get for all*

$\varepsilon > 0$,

$$\delta_\lambda(L(\varepsilon)) = \lim_n (\lambda \chi_L)_n$$

$$= 0.$$

*where $\chi_L$ represents the characteristic sequence of $L(\varepsilon)$.*

*Therefore, x is $C_\lambda$-statistically($\lambda$-statistically) convergent to 3.(Or, $x \to 3(\lambda_{st})$.)*

**Remark 2.4.4** *If a matrix $A \in (C,C;\eta)$ is equal to I, which is identity matrix, then A-statistical convergence becomes ordinary convergence.*

**Example 2.4.6** *Let $I = (I_{nk})$ be a identity matrix, which is defined as*

$$I_{nk} = \begin{cases} 1, & k = n \\ 0, & otherwise. \end{cases}$$

*Obviously, $I \in (C,C;\eta)$.*

*Assume that $x \in \varpi$ is defined as*

$$x_k = \begin{cases} 3, & k = \sqrt{n} \\ 4, & otherwise \end{cases}$$

*Define the sets $M(\varepsilon)$ and $N(\varepsilon)$ as*

$$M(\varepsilon) = \{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\},$$

*and*

$$N(\varepsilon) = \{k \in \mathbb{N} : |x_k - 4| \geq \varepsilon\}.$$

*So,*

$$\delta_I(M(\varepsilon)) = \delta_I(\{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\})$$

$$= \lim_n (I\chi_M)_n$$

*does not exists.*

*Similarly,*

$$\delta_I(N(\varepsilon)) = \delta_I(\{k \in \mathbb{N} : |x_k - 4| \geq \varepsilon\})$$

$$= \lim_n (I\chi_N)_n$$

*does not exists.*

*As a result, x is not I-statistically convergent(ordinary convergent) to any number.*

**Example 2.4.7** *Consider a matrix $A = (a_{nk}) \in (C,C;\eta)$ is defined as*

$$a_{nk} = \begin{cases} 1, & if \quad k = n^2 \\ 0, & if \quad otherwise \end{cases}$$

*And, let a sequence x is defined as*

$$x_k = \begin{cases} 3, & if \quad k = n^2 \\ 4, & if \quad otherwise \end{cases}$$

*For a set $M(\varepsilon) = \{k \in \mathbb{N} : |x_k - 4| \geq \varepsilon\}$,*

$$\begin{aligned} \delta_A(M(\varepsilon)) &= \delta_A(\{k \in \mathbb{N} : |x_k - 4| \geq \varepsilon\}) \\ &= \lim_{n\to\infty}(A\chi_M)_n \\ &= 1. \end{aligned}$$

*Similarly, for a set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\}$,*

$$\begin{aligned} \delta_A(K(\varepsilon)) &= \delta_A(\{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\}) \\ &= \lim_{n\to\infty}(A\chi_K)_n \\ &= 0. \end{aligned}$$

*Therefore, x is A-statistically convergent to* 3.

For different nonnegative regular matrices a sequence $x$ can converge to different points.

**Example 2.4.8** *If we replace the nonnegative regular matrix $A = (a_{nk})$, in the above example by the matrix where;*

$$a_{nk} = \begin{cases} 1, & if\ k = n^2 + 1 \\ 0, & if\ otherwise. \end{cases}$$

*Then, we have,*

$$\begin{aligned}\delta_A(K(\varepsilon)) &= \delta_A(\{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\}) \\ &= \lim_n (A\chi_K)_n \\ &= 1.\end{aligned}$$

*Moreover,*

$$\begin{aligned}\delta_A(M(\varepsilon)) &= \delta_A(\{k \in \mathbb{N} : |x_k - 4| \geq \varepsilon\}) \\ &= \lim_n (A\chi_M)_n \\ &= 0.\end{aligned}$$

*Therefore,* $x \to 4(A_{st})$.

**Remark 2.4.5** *According to one nonnegative regular matrix A, x is A-statistically convergent does not mean that x is A-statistically convergent for every nonnegative regular matrix* $A$.

**Example 2.4.9** *Let us to change the nonnegative regular matrix* $A = (a_{nk})$ *in Example 2.4.7 by*

$$a_{nk} = \begin{cases} \frac{1}{2}, & k = n^2 \\ \frac{1}{2}, & k = n^2 + 1 \\ 0, & otherwise. \end{cases}$$

*For the sequence x in Example 2.4.7, we have*

$$\begin{aligned}\delta_A(K(\varepsilon)) &= \delta_A(\{k \in \mathbb{N} : |x_k - 3| \geq \varepsilon\}) \\ &= \lim_n (A\chi_K)_n \\ &= \frac{1}{2}.\end{aligned}$$

*In a similar way,*

$$\begin{aligned}\delta_A(M(\varepsilon)) &= \delta_A(\{k \in \mathbb{N} : |x_k - 4| \geq \varepsilon\}) \\ &= \lim_n (A\chi_M)_n \\ &= \frac{1}{2}.\end{aligned}$$

*Hence, $x \notin A_{st}$.*

**Definition 2.4.3** *$(x_k) \in \varpi$ is called A-statistically divergent to $\infty$ if for all $P \in \mathbb{R}$,*

$$\delta_A(\{k \in \mathbb{N} : x_n > P\}) = 1.$$

**Example 2.4.10** *Consider a matrix $A = (a_{nk}) \in (C, C; \eta)$, where*

$$a_{nk} = \begin{cases} 1, & if\ k = 2n \\ 0, & if\ otherwise. \end{cases}$$

*Define a sequence x as,*

$$x_k = \begin{cases} k^2, & if\ k = 2n \\ 3, & otherwise. \end{cases}$$

*Then,*

$$\delta_A(\{k \in \mathbb{N} : x_k > P\}) = 1$$

*for all real number P. Therefore, x is A-statistically divergent to $\infty$.*

**Definition 2.4.4** *$(x_k) \in \varpi$ is called A-statistically divergent to $-\infty$ if for all real number T,*

$$\delta_A(\{k \in \mathbb{N} : x_k < T\}) = 1.$$

# Chapter 3

## LIMIT POINTS IN STATISTICAL SENSE

### 3.1 Statistical Limit Points

Consider a sequence $x$, the range of $x$ is represented by $\{x_k : k \in \mathbb{N}\}$.

For $K = \{k(j) : j \in \mathbb{N}\}$ the sequence $\{x_{k(j)}\}$ is called a subsequence of $x$, and it is denoted by $\{x\}_K$.

**Definition 3.1.1** *([11]) The subsequence $\{x\}_K$ is said to be a thin subsequence (or subsequence of density zero) if $\delta(K) = 0$. Otherwise, the subsequence $\{x\}_K$ is said to be a nonthin subsequence of $x$.*

*Note that: $K$ is a nonthin subset of $\mathbb{N}$, if $\delta(K) > 0$ or $\delta(K)$ is undefined.*

**Definition 3.1.2** *([11]) The real number $\lambda$ is called a statistical limit point of $x \in \varpi$, if there exists a nonthin subsequence of $x$, which converges to $\lambda$.*

*For $x \in \varpi$, $\Lambda_x$ and $L_x$ represents the set of all statistical limit points and the set of all ordinary limit points of $x$, respectively.*

**Example 3.1.1** *Define $x \in \varpi$ by*

$$x_k = \begin{cases} 2, & if \quad k = n^2 \\ 1, & if \quad otherwise. \end{cases}$$

*So, $L_x = \{1, 2\}$ but $\Lambda_x = \{1\}$.*

For any sequence $x$, it is clear that $\Lambda_x \subset L_x$, but in general converse implication does not hold. Moreover, for some cases the sets $\Lambda_x$ and $L_x$ can be very different. The next example shows such a difference.

**Example 3.1.2** *([11]) Consider that $x \in \varpi$ is defined as*

$$x_k = \begin{cases} t_n, & if\ k = n^2 \\ k, & otherwise \end{cases}$$

*where $\{t_n\}_{k=1}^{\infty}$ is a sequence whose range is the set of all rational numbers.*

*It follows that $L_x = \mathbb{R}$, because the set $\{t_k : k \in \mathbb{N}\}$ is dense in $\mathbb{R}$. However, $\Lambda_x = \emptyset$ because the set of squares has density zero and on the set of nonsquares of $x$ is not convergent.*

**Definition 3.1.3** *([11]) A real number $\gamma$ is called a statistical cluster point of $x \in \varpi$ if $\forall \varepsilon > 0$,*

$$\delta(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0.$$

For any $x \in \varpi$, the set $\Gamma_x$ represents the set of all statistical cluster points of the sequence $x$.

Clearly, $\Gamma_x \subset L_x$,for all $x \in \varpi$.

**Proposition 3.1.1** *([11]) For any sequence $x$, $\Lambda_x \subseteq \Gamma_x$.*

**Proof.** Assume that $\lambda \in \Lambda_x$. It means that there exists a nonthin subsequence $\{x_{k(j)}\}$ of $x$ such that $\lim\limits_j x_{k(j)} = \lambda$, and

$$\lim_n{}^* \frac{|\{j : |x_{k(j)} - \lambda| < \varepsilon\}|}{n} = c > 0$$

Furthermore, the set

$$\{j : |x_{k(j)} - \lambda| \geq \varepsilon\}$$

is finite because of $\lim_{j\to\infty} x_{k(j)} = \lambda$. So,

$$\{k(j) : j \in \mathbb{N}\} \setminus \{finite\ set\} \subseteq \{k \in \mathbb{N} : |x_k - \lambda| < \varepsilon\}.$$

Then,

$$\begin{aligned}\frac{|\{k \le n : |x_k - \lambda| < \varepsilon\}|}{n} &\ge \frac{|\{k(j) \le n : j \in \mathbb{N}\}|}{n} - \frac{O(1)}{n} \\ &\ge \frac{c}{2} \\ &\ne 0\end{aligned}$$

for infinitely many n.

Therefore,

$$\delta(\{k \in \mathbb{N} : |x_k - \lambda| \le \varepsilon\}) \ne 0.$$

It indicates that, $\lambda \in \Gamma_x$. ■

**Remark 3.1.1** *([11]) For some real valued sequence x , $\Gamma_x$ may not be a subset of $\Lambda_x$.*

**Example 3.1.3** *Choose a uniformly distributed sequence in $[0,1]$, which is defined as $\{0,0,0,1,0,\frac{1}{2},1,0,\frac{1}{3},\frac{2}{3},1,0,...\}$. Then density of $x_k$ in any subinterval with length c, is equal to c.*

*Therefore, for any real number $\lambda$ in any subinterval of $[0,1]$,*

$$\delta(\{k \in \mathbb{N} : |x_k - \lambda| \le \varepsilon\}) > \varepsilon > 0.$$

*So, $\Gamma_x = [0,1]$. However, select a real number $\gamma \in [0,1]$ and a subsequence $\{x\}_M$ which is ordinarily convergent to $\gamma$. $\forall \varepsilon > 0$ and infinitely many n, $M_n$ can be written as*

$$M_n \subseteq \{m \in M_n : |x_m - \gamma| < \varepsilon\} \cup \{m \in M_n : |x_m - \gamma| \ge \varepsilon\},$$

*where $\{m \in M_n : |x_m - \gamma| \ge \varepsilon\}$ is a finite set, because of $\{x\}_M$ is ordinarily convergent to $\gamma$.*

*If density is taken for both sides,*

$$\begin{aligned}\frac{|M_n|}{n} + 0 &\le \frac{|\{m \in M_n : |x_m - \lambda| < \varepsilon\}|}{n} + \frac{|\{m \in M_n : |x_m - \lambda| \ge \varepsilon\}|}{n} \\ &\le 2\varepsilon + O(1).\end{aligned}$$

*As a result $\delta(M) \leq 2\varepsilon$, and because of $\varepsilon$ is arbitrary number, $\delta(M) = 0$.*

*So, $\Lambda_x = \emptyset$.*

**Theorem 3.1.1** *Assume that $x \to \eta(C_{st})$, then $\Lambda_x = \{\eta\}$ and $\Gamma_x = \{\eta\}$.*

**Proof.** Assume that $C_{st} - \lim x = \eta$, then

$$\delta(K(\varepsilon)) = \lim_n \frac{|\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}|}{n}$$

$$= 0.$$

It causes,

$$\delta(\mathbb{N} \backslash K(\varepsilon)) = \lim_n \frac{|\{k \in \mathbb{N} : |x_k - \eta| < \varepsilon\}|}{n}$$

$$= 1.$$

So, $\eta \in \Gamma_x$.

For the uniqueness of the statistical cluster point, assume that $\eta_1$ is an other statistical cluster point then, by using the fact that $C_{st} - \lim x = \eta$ , we get $\eta = \eta_1$ so $\Gamma_x = \{\eta\}$. Similarly in a parallel way one can show that $\Lambda_x = \{\eta\}$ .

■

**Remark 3.1.2** *$\Lambda_x = \{\eta\}$ and $\Gamma_x = \{\eta\}$ does not means that $x \to \eta(C_{st})$.*

**Example 3.1.4** *Say that $x \in \varpi$ is defined as $x_k = [1 + (-1)^k]k$. Then, $\Lambda_x = \Gamma_x = \{0\}$. But, $x$ is not statistically convergent.*

**Proposition 3.1.2** *([11]) The set $\Gamma_x$ is a closed point set.*

**Proof.** This theorem will be proved using the property, which is $\bar{\Gamma}_x = \Gamma_x \cup \Gamma'_x$. Say that $c$ is any accumulation point of $\Gamma_x$. From the definition of accumulation point, $\exists p \in (c - \varepsilon, c + \varepsilon) \cap \Gamma_x$ for every $\varepsilon > 0$.

Pick, $\varepsilon' > 0$, so that $(p-\varepsilon', p+\varepsilon') \subseteq (c-\varepsilon, c+\varepsilon)$.

Because of $p \in \Gamma_x$,

$$\delta(\{k \in \mathbb{N} : x_k \in (p-\varepsilon', p+\varepsilon')\}) \neq 0,$$

and this implies that,

$$\delta(\{k \in \mathbb{N} : x_k \in (c-\varepsilon, c+\varepsilon)\}) \neq 0.$$

Hence $c \in \Gamma_x$, then $\Gamma'_x \subseteq \Gamma_x = \bar{\Gamma}_x$ where $\bar{\Gamma}_x$ is the closure of $\Gamma_x$ and $\Gamma'_x$ is the set of all accumulation points of the set $\Gamma_x$. ■

**Theorem 3.1.2** *([11]) If $x, y \in \varpi$ implies that $x_k = y_k$ for a.a.k., then $\Lambda_x = \Lambda_y$ and $\Gamma_x = \Gamma_y$.*

**Proof.** Considering that $\delta(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$, and let $\alpha \in \Lambda_x$ be an arbitrary element. It provides that there exists a nonthin subsequence $\{x\}_M$ of $x$ which is ordinarily convergent to $\alpha$.

Therefore;

$$M \cap \{k \in \mathbb{N} : x_k \neq y_k\} \subseteq \{k \in \mathbb{N} : x_k \neq y_k\}$$

so that,

$$\delta(M \cap \{k \in \mathbb{N} : x_k \neq y_k\}) \leq \delta(\{k \in \mathbb{N} : x_k \neq y_k\}).$$

Hence, $M' = M \cup \{k \in \mathbb{N} : x_k = y_k\}$ does not have density zero.

Hence, $\{y\}_{M'}$ is a nonthin subsequence of $\{y\}_M$ which is ordinarily convergent to $\alpha$.

So, $\alpha \in \Lambda_y$ and $\Lambda_x \subseteq \Lambda_y$.

Similarly one can show that $\Lambda_y \subseteq \Lambda_x$. The assertion that $\Gamma_x = \Gamma_y$ can be proved in a parallel way. ■

A sufficient connection between $\Lambda_x$ and $\Gamma_x$ is given by next theorem.

**Theorem 3.1.3** *([11]) For a sequence $x$, there exists a sequence $y$ implying $L_y = \Gamma_x$*

*and $y_k = x_k$ for almost all k. Furthermore, the range of y is a subset of x.*

**Remark 3.1.3** *([11]) If $\Gamma_x$ is replaced by $\Lambda_x$, the above theorem may not be true because $L_y$ is closed set but $\Lambda_x$ need not to be closed.*

**Example 3.1.5** *For the sequence x, which is defined as*

$$x_k = \frac{1}{r} \text{ where } k = 2^{r-1}(2t+1),$$

*where $r-1$ is the number of factors of 2 in the prime factorization of k.*

*Clearly we have,*

$$\delta(\{k : x_k = \frac{1}{r}\}) = \frac{1}{2^r} > 0.$$

*Thus, $\Lambda_x = \left\{\frac{1}{r}\right\}_{r=1}^{\infty}$*

*Furthermore,*

$$\delta(\{k : 0 < x_k < \frac{1}{r}\}) = \frac{1}{2^r}.$$

*It follows that $0 \in \Gamma_x$. Therefore $\Gamma_x = \{0\} \cup \left\{\frac{1}{r}\right\}_{r=1}^{\infty}$.*

*If we can show that $0 \notin \Lambda_x$, then we prove that $\Lambda_x$ is not closed set. If $\{x\}_M$ is a subsequence of x, that has limit zero, then we can demonstrate that $\delta(M) = 0$.*

*For every r,*

$$\begin{aligned} |M_n| &= |\{k \in M_n : x_k \geq \frac{1}{r}\}| + |\{k \in M_n : x_k < \frac{1}{r}\}| \\ &\leq O(1) + |\{k \in \mathbb{N} : x_k < \frac{1}{r}\}| \\ &\leq O(1) + \frac{n}{2^r}. \end{aligned}$$

*Therefore, $\delta(M) \leq \frac{1}{2^r}$, and for arbitrary number r, $\delta(M) = 0$.*

**Definition 3.1.4** *([2]) A sequence $(x_k)$ is called statistically monotonic increasing (decreasing) iff*

1. *There exists $K \subseteq \mathbb{N}$, such that*

$$K = \{k_1 < k_2 < k_3 < \ldots\} \quad \text{with } \delta(K) = 1,$$

2. $(x_{k_n})$ *is monotonically increasing (decreasing) sequence.*

**Example 3.1.6** *Assume that $x_k \in \varpi$ is defined by*

$$x_k = \begin{cases} 5, & k = n^2 \\ 2k, & k \neq n^2 \end{cases}$$

*and $K = \{k \in \mathbb{N} : k \neq n^2\}$. It is clear that $\delta(K) = 1$. Furthermore, $(x_{k_n}) = \{4, 6, 10, ...\}$ is monotonically increasing. Therefore, the sequence $x$ is statistically monotonic increasing.*

**Example 3.1.7** *Consider $(x_k) \in \varpi$, where*

$$x_k = \begin{cases} 2k, & k = n^2 \\ 5, & k \neq n^2 \end{cases}$$

*then there is no any $K \subseteq \mathbb{N}$ such that $\{x\}_K$ is monotonically increasing with $\delta(K) = 1$.*

**Proposition 3.1.3** *([11]) Assume that $x \in \varpi$ and $K := \{k \in \mathbb{N} : x_k \leq x_{k+1}\}$ with $\delta(K) = 1$. If $x$ is bounded sequence on $K$, then $x \in C_{st}$.*

**Theorem 3.1.4** *([11]) For $x \in \varpi$, if $x$ has a bounded nonthin subsequence, then $x$ has a statistical cluster point.*

**Proof.** For a sequence $x$, Theorem 3.1.3 guarrantees that there exists a real valued sequence $y$, which implies $L_y = \Gamma_x$ with $\delta(\{k \in \mathbb{N} : y_k \neq x_k\}) = 0$. Thus, the sequence $y$ must have a bounded nonthin subsequence, because of the Bolzano-Weierstrass Theorem $L_y \neq \emptyset$. Therefore, $\Gamma_x \neq \emptyset$. ■

**Definition 3.1.5** *([13]) A sequence $x$ is called statistically bounded provided that $\exists M \in \mathbb{R}$ so that*

$$\delta(\{k \in \mathbb{N} : |x_k| > M\}) = 0.$$

**Theorem 3.1.5** *"Every bounded sequence is statistically bounded."*

**Proof.** Say that the seqeuence $x$ is bounded. From the definiton of boundedness condition, $\exists M \in \mathbb{R}$ such that $|x_k| < M \ \forall k \in \mathbb{N}$.

It shows that,

$$\delta(\{k \in \mathbb{N} : |x_k| < M\}) = 1.$$

Then,

$$\delta(\{k \in \mathbb{N} : |x_k| \geq M\}) = 0.$$

Therefore, $x$ is statistically bounded. ■

**Remark 3.1.4** *Statistically boundedness does not satisfies boundedness in the ordinary sense.*

**Example 3.1.8** *Consider $x := (x_k) \in \varpi$ which is defined by*

$$x_k = \begin{cases} k^2, & if \quad k = n^2 \\ 3, & if \quad k \neq n^2. \end{cases}$$

*Then,*

$$\delta(\{k \in \mathbb{N} : |x_k| > 3\}) = 0.$$

*So, $x$ is statistically bounded but it is not bounded in the ordinary sense.*

**Theorem 3.1.6** *"Every statistical convergent sequence is statistically bounded."*

**Proof.** Assume that $C_{st} - \lim x = \eta$. This implies that, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}|}{n} = 0.$$

Moreover,

$$\{k \in \mathbb{N} : |x_k| \geq |\eta| + \varepsilon\} \subset \{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}.$$

From the subset property of density,

$$\delta(\{k \in \mathbb{N} : |x_k| \geq |\eta| + \varepsilon\}) = 0$$

because of

$$\delta(\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}) = 0.$$

As a result, $x$ is statistically bounded. ■

Let choose the sequence $x$ as $(x_k) = (-2)^k$. In this case, $x$ is statistically bounded but not statistically convergent. So statistically boundedness condition does not imply statistical convergence.

**Theorem 3.1.7** *For $x = (x_k) \in \varpi$, the sequence $x$ is statistically bounded iff there exists a statistically bounded sequence $y = (y_k)$ such that,*

$$x_k = y_k \ a.a.k.$$

**Remark 3.1.5** *In ordinary case, all subsequence of a bounded sequence is bounded. On the other hand, in statistical case, every subsequence of a statistically bounded sequence need not to be statistically bounded.*

**Example 3.1.9** *Consider $x \in \varpi$ where*

$$x_k = \begin{cases} 2k, & if \quad k = n^2 \\ \\ 4, & if \quad k \neq n^2 \end{cases}$$

*Then, $x$ is statistically bounded. However, $(y_k) = (x_{k^2})$, which is defined as $(y_k) = \{2, 8, 18, 32, ...\}$, is a subsequence of $x$, but it is not statistically bounded.*

**Definition 3.1.6** *([13]) Considering the sets,*

$$E_x = \{e \in \mathbb{R} : \delta(\{k : x_k > e\}) \neq 0\},$$

*and*

$$F_x = \{f \in \mathbb{R} : \delta(\{k : x_k < f\}) \neq 0\},$$

*for* $x \in \varpi$.

*Then, statistical limit superior and statistical limit inferior of x is defined as;*

$$C_{st} - \lim^{*} x = \begin{cases} \sup E_x, & if \quad E_x \neq \emptyset, \\ -\infty, & if \quad E_x = \emptyset, \end{cases}$$

*and*

$$C_{st} - \lim_{*} x = \begin{cases} \inf F_x, & if \quad F_x \neq \emptyset, \\ +\infty, & if \quad F_x = \emptyset, \end{cases}$$

*respectively.*

**Example 3.1.10** *Suppose that* $x \in \varpi$ *is defined as*

$$x_k = \begin{cases} k^2, & \text{if } k \text{ is an even square} \\ 2, & \text{if } k \text{ is an even nonsquare} \\ -k^3, & \text{if } k \text{ is an odd square} \\ 4, & \text{if } k \text{ is an odd nonsquare} \end{cases}$$

*Thus* $\Gamma_x = \{2, 4\} = \Lambda_x$.

*Because of*

$$\delta(\{k \in \mathbb{N} : |x_k| > 4\}) = 0,$$

*x is statistically bounded.*

*Furthermore,* $E_x = (-\infty, 4)$ *satisfies that*

$$C_{st} - \lim^{*} x = 4,$$

*and* $F_x = (2, \infty)$ *satisfies that*

$$C_{st} - \lim_{*} x = 2.$$

**Remark 3.1.6** *For a sequence x, if* $C_{st} - \lim^{*} x$ *and* $C_{st} - \lim_{*} x$ *are finite numbers, then the* $C_{st} - \lim^{*} x$ *is the greatest value of* $\Gamma_x$*, and similarly* $C_{st} - \lim_{*} x$ *is the lowest value* *of* $\Gamma_x$.

**Theorem 3.1.8** *([13]) Suppose that $C_{st} - \lim^* x = \beta$ is a finite number, then for each $\varepsilon > 0$,*

$$\delta(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0, \tag{3.1.1}$$

*and*

$$\delta(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0. \tag{3.1.2}$$

*On the other hand, if (3.1.1) and (3.1.2) holds for all $\varepsilon > 0$, then $C_{st} - \lim^* x = \beta$.*

**Theorem 3.1.9** *([13]) Suppose that $C_{st} - \lim_* x = \alpha$ is a finite number, then for each $\varepsilon > 0$,*

$$\delta(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}) \neq 0, \tag{3.1.3}$$

*and*

$$\delta(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}) = 0. \tag{3.1.4}$$

*On the other hand, if (3.1.3) and (3.1.4) holds for all $\varepsilon > 0$, then $C_{st} - \lim_* x = \alpha$.*

**Theorem 3.1.10** *([13]) For every $x \in \varpi$,*

$$C_{st} - \lim_* x \leq C_{st} - \lim^* x.$$

**Proof.** Case 1 : Suppose that $C_{st} - \lim^* x = \beta$ is a finite number, and say

$\alpha := C_{st} - \lim_* x$. Given $\varepsilon > 0$, we demonstrate that $\beta + \alpha \in F_x$, so that $\alpha \leq \beta + \varepsilon$.

We know that

$$\delta(\{k \in \mathbb{N} : x_k > \beta + \frac{\varepsilon}{2}\}) = 0,$$

because of $C_{st} - \lim^* x = \beta$ and using Theorem 3.1.8.

It follows that,

$$\delta(\{k \in \mathbb{N} : x_k \leq \beta + \frac{\varepsilon}{2}\}) = 1$$

which provides that,

$$\delta(\{k \in \mathbb{N} : x_k < \beta + \frac{\varepsilon}{2}\}) = 1.$$

Thus, $\beta + \alpha \in F_x$.

From the definition $\alpha = \inf F_x$, then we conclude that $\alpha \leq \beta + \varepsilon$. Therefore, for arbitrary $\varepsilon > 0$, gives us $\alpha \leq \beta$.

Case 2 : Suppose that $C_{st} - \lim^* x = -\infty$, then that provides $E_x = \emptyset$. It indicates that,

$$\delta(\{k \in \mathbb{N} : x_k > e\}) = 0$$

$\forall e \in \mathbb{R}$.

This satisfies,

$$\delta(\{k \in \mathbb{N} : x_k \leq e\}) = 1.$$

So, $\forall f \in \mathbb{R}$,

$$\delta(\{k \in \mathbb{N} : x_k < f\}) \neq 0.$$

Thus, $C_{st} - \lim_* x = -\infty$.

Case 3 : The case $C_{st} - \lim^* x = \infty$ is clear. ■

**Remark 3.1.7** *([13]) Statistical boundedness condition provides that $C_{st} - \lim^* x$ and $C_{st} - \lim_* x$ are finite, for $x \in \varpi$.*

**Theorem 3.1.11** *([13]) If $x \in C_{st}$ is statistically bounded, then*

$$C_{st} - \lim_* x = C_{st} - \lim^* x.$$

**Proof.** Let $\alpha := C_{st} - \lim_* x$ and $\beta := C_{st} - \lim^* x$.

Necessity. Assume that $C_{st} - \lim x = \eta$, then $\forall \varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - \eta| \geq \varepsilon\}) = 0.$$

So;

$$\delta(\{k \in \mathbb{N} : x_k \geq \eta + \varepsilon\}) = 0,$$

and

$$\delta(\{k \in \mathbb{N} : x_k \leq \eta - \varepsilon\}) = 0.$$

These two sets, each having density zero, provides that $\beta \leq \eta$ and $\eta \leq \alpha$. Thus, $\beta \leq \alpha$. From theorem 3.1.10, $\alpha \leq \beta$. So, $\alpha = \beta$.

Sufficiency. Let $\alpha = \beta$. Define $\eta := \alpha$. Then, for $\varepsilon > 0$, and from Theorem 3.1.8 and 3.1.9 provides that $\delta(\{k \in \mathbb{N} : x_k > \eta + \frac{\varepsilon}{2}\}) = 0$ and $\delta(\{k \in \mathbb{N} : x_k < \eta - \frac{\varepsilon}{2}\}) = 0$ respectively. Thus $C_{st} - \lim x = \eta$. ■

## 3.2 Lacunary Statistical Limit Points

**Definition 3.2.1** *([12])Let $\{x\}_K$ be a subsequence of $x$, where $K = \{k(j) : j \in \mathbb{N}\}$, and $\theta$ be a lacunary sequence. The subsequence $\{x\}_K$ is called lacunary thin subsequence if*

$$\delta_\theta(K) = \lim_{r\to\infty} \frac{|\{k_{r-1} < k(j) \leq k_r : j \in \mathbb{N}\}|}{h_r} = 0.$$

*Otherwise, the subsequence $\{x\}_K$ is called lacunary nonthin subsequence of $x$.*

**Example 3.2.1** *Let $\theta \in \Theta$, which is defined as $\theta = \{2^r - 1\}$ where $r$ is a natural number.*

*Then, $I_r = (2^{r-1} - 1, 2^r - 1]$ and $h_r = 2^{r-1}$.*

*Let the real valued sequence $x := (x_k)$ be defined as*

$$x_k = \begin{cases} 2, & if \quad k = 3n \\ 3, & if \quad k = 3n+1 \\ k, & if \quad k = 3n+2. \end{cases}$$

*and assume that $K = \{3k : k \in \mathbb{N}\}$, $M = \{3k+1 : k \in \mathbb{N}\}$ and $N = \{3k+2 : k \in \mathbb{N}\}$.*

*Then*

$$\delta_\theta(K) = \frac{1}{3} \neq 0.$$

*So, we can say that $\{x\}_K$ is lacunary nonthin subsequnce of $x$.*

*In a similar way, we have $\delta_\theta(M) = \frac{1}{3}$ and $\delta_\theta(N) = \frac{1}{3}$, then $\{x\}_M$ and $\{x\}_N$ are also lacunary nonthin subsequences of $x$.*

**Definition 3.2.2** *([5]) A number $\lambda$ is called a lacunary statistical limit point of $x$ if there exists a lacunary nonthin subsequence of $x$, which is ordinarily convergent to $\lambda$.*

The set $\Lambda_x^\theta$ represents the set of all lacunary statistical limit points of $x$.

**Example 3.2.2** *Let $\theta \in \Theta$ and $x \in \varpi$ which are defined in the previous example. We picked and found 3 lacunary nonthin subsequence of $x$. We should check if these subsequence of $x$ are ordinarily convergent or not.*

*Then,*

$$\lim_k x_K = 2,$$

$$\lim_k x_M = 3$$

*and*

$$\lim_k x_N = \infty.$$

*As a result, $\Lambda_x^\theta = \{2,3\}$.*

**Definition 3.2.3** *([5])Consider the sequence $x$, $\gamma$ is called a lacunary statistical cluster point of $x$ if*

$$\delta_\theta(K_\varepsilon) = \lim_r \frac{|\{k \in I_r : |x_k - \gamma| < \varepsilon\}|}{h_r}$$

$$\neq 0$$

*in which $K_\varepsilon = \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$.*

A set $\Gamma_x^\theta$ represents all lacunary cluster points of $x$.

**Example 3.2.3** *Consider* $\theta = \{r! - 1\} \in \Theta$*, then* $I_r = ((r-1)! - 1, r! - 1]$ *and* $h_r = (r-1)!(r-1)$.

*Assume that* $x \in \varpi$ *is defined by*

$$x_k = \begin{cases} 2 & if\ k = r! + 1 \\ 3 & otherwise \end{cases}$$

*Say that,* $M_\varepsilon = \{k \in \mathbb{N} : |x_k - 2| < \varepsilon\}$ *and* $N_\varepsilon = \{k \in \mathbb{N} : |x_k - 3| < \varepsilon\}$*. Clearly,*

$$\delta_\theta(M_\varepsilon) = 0.$$

*So,* $2 \notin \Gamma_x^\theta$.

*In a similar way, we can check*

$$\delta_\theta(N_\varepsilon) = 1.$$

*Consequently,* $3 \in \Gamma_x^\theta$*. Thus,* $\Gamma_x^\theta = \{3\}$.

**Example 3.2.4** *Let* $\theta = \{2^r - 1\}$ *where* $r \in \mathbb{N}$*. In this case,* $I_r = (2^{r-1} - 1, 2^r - 1]$*. Define* $x = (x_k)$*, where*

$$x_k = \begin{cases} 2, & if\ \ k \in \left(2^{r-1} - 1,\ 2^{r-1} - 1 + \frac{(2^r - 2^{r-1})}{2}\right) \\ 4, & if\ \ k \in \left(2^{r-1} - 1 + \frac{(2^r - 2^{r-1})}{2},\ 2^{r-1}\right) \\ 0, & otherwise. \end{cases}$$

*Obviously,* $\Gamma_x^\theta = \Lambda_x^\theta = \{2, 4\}$*. On the other hand,* $L_x = \{0, 2, 4\}$.

**Proposition 3.2.1** *For any sequence x,* $\Lambda_x^\theta \subseteq \Gamma_x^\theta$.

**Definition 3.2.4** *([5])* $x \in \varpi$ *is called lacunary statistically bounded provided that there exists* $M \in \mathbb{R}$ *and a subsequence* $\{x\}_K$ *of x implies that*

$$\delta_\theta(\{k \in K : |x_k| < M\}) = 1.$$

*In other words, if* $\exists M \in \mathbb{R}$ *such that* $\delta_\theta(\{k \in \mathbb{N} : |x_k| \geq M\}) = 0$*, then x is called lacunary statistically bounded.*

**Example 3.2.5** *Consider the lacunary seqeuence $\theta$, which is defined as $\{k_r\} = 2^r - 1$, and let a sequence x be,*

$$x_k = \begin{cases} k^2, & if\ k = r^2 \\ 3, & if\ k \neq r^2 \end{cases}$$

*We can pick a subsequence $x_M$ of x, which is defined as $M = \{k \in \mathbb{N} : k \neq r^2\}$ where r is a natural number.*

*Then for $T > 3$,*

$$\delta_\theta(\{k \in M : |x_k| < T\}) = \lim_{r\to\infty} \frac{|\{k \in M : k \in I_r\}|}{2^{r-1}}$$
$$= 1.$$

*Thus, x is lacunary statistical bounded.*

**Lemma 3.2.1** *([5]) Consider the lacunary sequence $\theta$,*

1. $\Lambda_\theta(x) \subset \Lambda(x)$, *in case of* $\lim_r{}_* q_r > 1$,

2. $\Lambda(x) \subset \Lambda_\theta(x)$, *in case of* $\lim_r{}^* q_r < \infty$.

**Lemma 3.2.2** *([5]) Consider the lacunary sequence $\theta$,*

1. $\Gamma_\theta(x) \subset \Gamma(x)$, *in case of* $\lim_r{}_* q_r > 1$,

2. $\Gamma(x) \subset \Gamma_\theta(x)$, *in case of* $\lim_r{}^* q_r < \infty$.

**Theorem 3.2.3** *([5]) Consider the lacunary sequence $\theta$,*

1. $\Lambda_\theta(x) = \Lambda(x)$, *iff,* $1 < \lim_r{}_* q_r \leq \lim_r{}^* q_r < \infty$,

2. $\Gamma(x) = \Gamma_\theta(x)$, *iff,* $1 < \lim_r{}_* q_r \leq \lim_r{}^* q_r < \infty$.

**Definition 3.2.5** *([5]) Considering*

$$E_x^\theta = \{e \in \mathbb{R} : \delta_\theta(\{k \in I_r : x_k > e\}) \neq 0\}$$

*and*

$$F_x^\theta = \{f \in \mathbb{R} : \delta_\theta(\{k \in I_r : x_k < f\}) \neq 0\}$$

*for* $x \in \varpi$.

*Then, lacunary statistical limit superior and lacunary statistical limit inferior of x is defined by;*

$$\theta_{st} - \lim{}^* x := \begin{cases} \sup E_x^\theta, & if \quad E_x^\theta \neq \emptyset, \\ -\infty, & if \quad E_x^\theta = \emptyset, \end{cases}$$

*and*

$$\theta_{st} - \lim{}_* x := \begin{cases} \inf F_x^\theta, & if \quad F_x^\theta \neq \emptyset, \\ +\infty, & if \quad F_x^\theta = \emptyset. \end{cases}$$

A real number $\theta_{st} - \lim_* x$ is the least value of $\Gamma_x^\theta$ and $\theta_{st} - \lim^* x$ is the greatest value of $\Gamma_x^\theta$ ([5]).

**Example 3.2.6** *For* $\theta = \{2^r - 1\} \in \Theta$,

*let,*

$$x_k = \begin{cases} 2, & if\ k = 4r \\ 4, & if\ k = 4r+1 \\ 6, & if\ k = 4r+2 \\ 8, & if\ k = 4r+3. \end{cases}$$

*Thus,* $\Lambda_x^\theta = \{2,4,6,8\} = \Gamma_x^\theta$. *Also, the sequence x is lacunary statistical bounded.*

$E_x^\theta = (-\infty, 8)$, *so that*

$$\theta_{st} - \lim{}^* x = 8,$$

*and* $F_x^\theta = (2, \infty)$, *so that*

$$\theta_{st} - \lim{}_* x = 2.$$

**Example 3.2.7** *For $\theta = \{2^r - 1\} \in \Theta$, consider x which is defined by*

$$x_k = \begin{cases} 3, & if \;\; k \in \left(2^{r-1} - 1,\, 2^{r-1} - 1 + \frac{(2^r - 2^{r-1})}{3}\right] \\ 4, & if \;\; k \in \left(2^{r-1} - 1 + \frac{(2^r - 2^{r-1})}{3},\, 2^{r-1} - 1 + \frac{2(2^r - 2^{r-1})}{3}\right] \\ 5, & if \;\; k \in \left(2^{r-1} - 1 + \frac{2(2^r - 2^{r-1})}{3},\, 2^r - 1\right]. \end{cases}$$

*So, $\Lambda_x^\theta = \Gamma_x^\theta = \{3,4,5\}$. It indicates that, $E_x^\theta = (-\infty, 5)$ and $F_x^\theta = (3, \infty)$.*

*Therefore,$\theta_{st} - \lim^* x = 5$, and $\theta_{st} - \lim_* x = 3$.*

**Theorem 3.2.4** *([5]) If a sequence x is lacunary statistically bounded, then $\theta_{st} - \lim_* x$ and $\theta_{st} - \lim^* x$ are finite numbers.*

**Theorem 3.2.5** *([5])Lacunary statistically bounded sequence x is lacunary statistically convergent to a real number $\eta$ iff $\theta_{st} - \lim_* x = \theta_{st} - \lim^* x = \eta$.*

## 3.3 $\lambda$-Statistical Limit Points

**Definition 3.3.1** *([19]) Let $\{x\}_K$ be a subsequence of x, where $K = \{k(j) : j \in \mathbb{N}\}$, and consider $I_n = [n - \lambda_n + 1, n]$, $\lambda = (\lambda_n)$ which is defined in Section 2.3. The subsequence $x_K$ is said to be $\lambda$-thin subsequence if*

$$\delta_\lambda(K) = \lim_{n\to\infty} \frac{|\{k(j) \in I_n : j \in \mathbb{N}\}|}{\lambda_n}$$
$$= 0.$$

*Conversely, $x_K$ is called $\lambda$-nonthin subsequence of x if $\delta_\lambda(K) \neq 0$.*

**Example 3.3.1** *For $(\lambda_n) = \sqrt{n}$ and $x \in \varpi$, suppose that $\{x\}_K$ is a subsequence of x, in which $K := \{2k : k \in \mathbb{N}\}$. Thus,*

$$\delta_\lambda(K) = \lim_{n\to\infty} \frac{|\{k \in K : k \in I_n\}|}{\lambda_n}$$
$$= \frac{1}{2}$$

*such that $x_K$ is a $\lambda$-nonthin subsequence of x.*

**Definition 3.3.2** *A real number $\lambda$ is called a $\lambda$-statistical limit point of $x$, if there exists a $\lambda$-nonthin subsequence of $x$ which convergent to $\lambda$ in ordinary sense.*

The set of all $\lambda$-statistical limit points of $x$ is represented by $\Lambda_x^\lambda$.

**Example 3.3.2** *Let $\lambda = (\lambda_n)$ be a nonnegative real valued sequence and defined as $\lambda_n = \sqrt{n}$. Assume that a sequence $x$ is defined as*

$$x_k = \begin{cases} 1, & if\ k = 2n \\ -1, & if\ k = 2n+1, \end{cases}$$

*and, consider the sets $K = \{2k : k \in \mathbb{N}\}$ and $L = \{2k+1 : k \in \mathbb{N}\}$.*

*Then,*

$$\delta_\lambda(K) = \lim_n (\lambda \chi_K)_n$$
$$= \frac{1}{2}$$

*where $\chi_K$ represents the characteristic sequence of $K$. Also, the subsequence $\{x\}_K$ convergent to* $1$.

*And,*

$$\delta_\lambda(L) = \lim_n (\lambda_{nk} \chi_L)_n$$
$$= \frac{1}{2}$$

*where $\chi_L$ represents the characteristic sequence of $K$. Also, the subsequence $\{x_L\}$ is ordinarily convergent to $-1$. Therefore, $\Lambda_x^\lambda = \{-1, 1\}$.*

**Definition 3.3.3** *Consider the sequence $x$, a real number $\gamma$ is called $\lambda$-statistical cluster point of $x$ provided that $\delta_\lambda(K_\varepsilon) \neq 0$ in which $K_\varepsilon = \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$.*

The set of all $\lambda$-statistical cluster points of $x$ is represented by $\Gamma_x^\lambda$.

**Example 3.3.3** *Consider $(\lambda_n) \in \varpi$, where $\lambda_n = \sqrt{n}$. And, $x \in \varpi$ where*

$$x_k = \begin{cases} 2, & if\ k = n^4 \\ 3, & otherwise. \end{cases}$$

*Let, $K_\varepsilon = \{k \in \mathbb{N} : |x_k - 2| < \varepsilon\}$ and $L_\varepsilon = \{k \in \mathbb{N} : |x_k - 3| < \varepsilon\}$.*

*Then,*

$$\begin{aligned} \delta_\lambda(K_\varepsilon) &= \delta_\lambda(\{k \in \mathbb{N} : |x_k - 2| < \varepsilon\}) \\ &= \lim_n (\lambda \chi_K)_n \\ &= 0, \end{aligned}$$

*where $\chi_K$ is the characteristic function of $K$.*

*And,*

$$\begin{aligned} \delta_\lambda(L_\varepsilon) &= \delta_\lambda(\{k \in \mathbb{N} : |x_k - 3| < \varepsilon\}) \\ &= \lim_n (\lambda \chi_L)_n \\ &= 1, \end{aligned}$$

*where $\chi_L$ is the characteristic function of $L$. Therefore, $\Gamma_x^\lambda = \{3\}$.*

**Theorem 3.3.1** *If $x \in \varpi$,then $\Lambda_x^\lambda \subset \Gamma_x^\lambda$.*

**Theorem 3.3.2** *If $x \in \varpi$,then $\Gamma_x^\lambda \subset L_x$.*

**Theorem 3.3.3** *If $(x_k),(y_k) \in \varpi$ implies that $\lim_n \frac{|\{k \in I_n : x_k \neq y_k\}|}{\lambda_n} = 0$, then $\Lambda_x^\lambda = \Lambda_y^\lambda$ and $\Gamma_x^\lambda = \Gamma_y^\lambda$.*

**Definition 3.3.4** *If any $M \in \mathbb{R}$ implies that $\delta_\lambda(\{k \in \mathbb{N} : |x_k| \geq M\}) = 0$, then $x = (x_k) \in \varpi$ is called $\lambda$-statistically bounded.*

**Example 3.3.4** *Let $\lambda = (\lambda_n)$ be defined as $\lambda_n = \sqrt[3]{n}$. Consider a sequence $x$, where*

$$x_k = \begin{cases} k, & if\ k = \sqrt[3]{n} + 1 \\ 1, & otherwise. \end{cases}$$

*Then,*

$$\delta_\lambda(\{k \in \mathbb{N} : |x_k| > 1\}) = 0.$$

*Thus, the sequence x is $\lambda$-statistically bounded.*

**Definition 3.3.5** *Considering*

$$E_x^\lambda = \{e \in \mathbb{R} : \delta_\lambda(\{k \in I_n : x_k > e\}) \neq 0\}$$

*and*

$$F_x^\lambda = \{f \in \mathbb{R} : \delta_\lambda(\{k \in I_n : x_k < f\}) \neq 0\}$$

*for $x \in \varpi$.*

*Then, $\lambda$-statistical limit superior and $\lambda$-statistical limit inferior of x is defined as;*

$$\lambda_{st} - \lim{}^* x = \begin{cases} \sup E_x^\lambda, & if \quad E_x^\lambda \neq \emptyset, \\ -\infty, & if \quad E_x^\lambda = \emptyset, \end{cases}$$

*and*

$$\lambda_{st} - \lim{}_* x = \begin{cases} \inf F_x^\lambda, & if \quad F_x^\lambda \neq \emptyset, \\ +\infty, & if \quad F_x^\lambda = \emptyset, \end{cases}.$$

*respectively.*

A real number $\lambda_{st} - \lim_* x$ is the least value of $\Gamma_x^\lambda$, and $\lambda_{st} - \lim^* x$ is the greatest value of $\Gamma_x^\lambda$.

## 3.4 A-Statistical Limit Points

**Definition 3.4.1** *([4]) For a matrix $A \in (C, C; \eta)$, consider $\{x\}_K$ is a subsequence of x, the subsequence $\{x\}_K$ is called A-thin subsequence (or subsequence of A-density zero) in case of $\delta_A(K) = 0$. Otherwise, the subsequence $\{x\}_K$ is called A-nonthin subsequence of x.*

**Example 3.4.1** *Consider $A = (a_{nk}) \in (C, C; \eta)$, where*

$$a_{nk} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*and $x \in \varpi$, where*

$$x_k = \begin{cases} 7, & if\ k = 2n \\ 8, & otherwise. \end{cases}$$

*Define, $S = \{2k : k \in \mathbb{N}\}$, and $R = \{2k+1 : k \in \mathbb{N}\}$. So,*

$$\delta_A(S) = \lim_n (A\chi_S)_n = 0.$$

*Similarly,*

$$\delta_A(R) = \lim_n (A\chi_R)_n = 1.$$

*It indicates that, $\{x\}_S$ is A-thin and $\{x\}_R$ is A-nonthin subsequence $x$.*

**Definition 3.4.2** *([3]) A number $\lambda$ is called A-statistically limit point of $x$ if there exists a A-nonthin subsequence of $x$, which is ordinarily convergent to $\lambda$.*

The set $\Lambda_x^A$ represents all A-statistical limit points of $x$.

**Example 3.4.2** *Let a nonnegative regular matrix $A = (a_{nk})$ be defined as*

$$a_{nk} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \dots \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \dots \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{5} & \dots \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*and a sequence $x$ is defined as*

$$x_k = \begin{cases} 2, & if \quad k \text{ is odd} \\ -2, & if \quad k \text{ is even} \end{cases}$$

*For the subset $K = \{k \in \mathbb{N} : |x_k - 2| \geq \varepsilon\}$,*

$$\delta_A(K) = \lim_{n\to\infty} (A\chi_K)_n$$
$$= 0$$

*where $\chi_K$ is characteristic function of $K$.*

*A subsequence $\{x\}_K$ is not A-nonthin subsequence, then $2 \notin \Lambda_x^A$.*

*However, for the subset $M = \{k \in \mathbb{N} : |x_k + 2| \geq \varepsilon\}$,*

$$\delta_A(M) = \lim_{n\to\infty} (A\chi_M)_n$$
$$= 1.$$

*A subsequence $\{x\}_M$ is A-nonthin subsequence of $x$, because of $\delta_A(M) = 1$. Moreover, the subsequence $\{x\}_M$ is ordinarily convergent to $-2$.*

*Finally, the number $-2$ is A-limit point of $x$. As a result, $\Lambda_x^A = \{-2\}$.*

**Definition 3.4.3** *([3]) For a sequence $x$, a real number $\gamma$ is called A-statistical cluster point of $x$ if $\delta_A(K_\varepsilon) \neq 0$ in which $K_\varepsilon = \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$.*

It must be noted that $\delta_A(K_\varepsilon) \neq 0$ means not only positive number but also does not have A-density.

A set $\Gamma_x^A$ represents all A-statistically cluster points of $x$.

**Example 3.4.3** *For a matrix $A \in (C, C; \eta)$, where*

$$a_{nk} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*and, let a sequence x which is defined as*

$$x_k = \begin{cases} 3 & , if \quad k \text{ is even} \\ 5 & , if \quad k \text{ is odd} \end{cases}$$

*For a set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - 3| < \varepsilon\}$, $\delta_A(K_\varepsilon)$ does not exist.*

*And, for a set $M_\varepsilon = \{k \in \mathbb{N} : |x_k - 5| < \varepsilon\}$, $\delta_A(M_\varepsilon)$ does not exist. Therefore, the real numbers* 3 *and* 5 *are A-statistical cluster points of x, then $\Gamma_x^A = \{3, 5\}$.*

**Proposition 3.4.1** *For any sequence x, $\Lambda_x^A \subseteq \Gamma_x^A$.*

**Definition 3.4.4** *([3]) A sequence x is called A-statistically bounded provided that there exists a subsequence $\{x_K\}$ of x implies $\delta_A(K) = 1$.*

*In other words, if any $M \in \mathbb{R}$ implies that $\delta_A(\{k \in \mathbb{N} : |x_k| \geq M\}) = 0$, then x is called A-statistically bounded.*

**Example 3.4.4** *Consider the nonnegative regular matrix $A = (a_{nk})$ which is defined as*

$$a_{nk} = \begin{cases} 1, & if\ k = n^2 \\ 0, & otherwise \end{cases}$$

*and for a sequence x which is defined as*

$$x_k = \begin{cases} 5, & if\ k = n^2 \\ k^2, & otherwise \end{cases}$$

*Choose a set $K = \{k^2 : k \in \mathbb{N}\}$, then a subsequence $x_K$ is bounded with*

$$\delta_A(K) = 1.$$

*So, a sequence x is A-statistically bounded.*

**Definition 3.4.5** *([3]) Considering the sets,*

$$E_x^A = \{e \in \mathbb{R} : \delta_A(\{k : x_k > e\}) \neq 0\},$$

*and*

$$F_x^A = \{f \in \mathbb{R} : \delta_A(\{k : x_k < f\}) \neq 0\},$$

*for $x \in \varpi$.*

*Then, A-statistical limit superior and A-statistical limit inferior of x is defined as;*

$$A_{st} - \lim{}^* x = \begin{cases} \sup E_x^A, & if \quad E_x^A \neq \emptyset, \\ -\infty, & if \quad E_x^A = \emptyset, \end{cases}$$

*and*

$$A_{st} - \lim{}_* x = \begin{cases} \inf F_x^A, & if \quad F_x^A \neq \emptyset, \\ +\infty, & if \quad F_x^A = \emptyset. \end{cases}$$

A real number $A_{st} - \lim_* x$ is the least valuef of $\Gamma_x^A$ and $A_{st} - \lim^* x$ is the greatest value of $\Gamma_x^A$.

**Example 3.4.5** *Consider a nonnegative regular matrix $A = (a_{nk})$ which is defined as*

$$a_{nk} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \dots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*Let x be a sequence, where*

$$x_k = \begin{cases} 1, & if\ k = 4n \\ 2, & if\ k = 4n+1 \\ 3, & if\ k = 4n+2 \\ 4, & if\ k = 4n+3. \end{cases}$$

*Thus,* $\Gamma_x^A = \{1,2,3,4\} = \Lambda_x^A$. *Also,* $E_x^A = (-\infty,4)$ *and* $F_x^A = (1,\infty)$. *It follows that,*

$$A_{st} - \lim{}^{*} x = 4,$$

*and*

$$A_{st} - \lim{}_{*} x = 1.$$

**Theorem 3.4.1** *([3]) If a sequence x is A-statistically bounded, then* $A_{st} - \lim_{*} x$ *and* $A_{st} - \lim^{*} x$ *are finite numbers.*

**Theorem 3.4.2** *([3]) A-statistically bounded sequence x is A-statistically convergent to a real number* $\eta$ *iff*

$$A_{st} - \lim{}_{*} x = A_{st} - \lim{}^{*} x = \eta.$$

**Example 3.4.6** *Consider* $A \in (C,C;\eta)$ *and* $x \in \varpi$*, which are defined in Example 3.4.4. Clearly,* $\Gamma_x^A = \{5\}$. *So,*

$$A_{st} - \lim^* x = A_{st} - \lim_* = 5.$$

*From Theorem 3.4.2,* $x_k \to 5(A_{st})$.

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