Quantization of Euclidean Black Holes via the Adiabatic Invariance Method

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Physics

Eastern Mediterranean University
July 2016
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

In this thesis, we calculate the entropy of four different types of black holes and show that entropy and area of the considered black holes are equally likely quantized. The quantization of entropy/area of the black holes are employed by using the adiabatic invariance formulation of the famous Bohr-Sommerfeld theory. Moreover, we consider the exactness conditions of the first order differential equations to derive the integral solution of the adiabatic invariance. The black holes that are going to be discussed are the Schwarzschild black hole, the Kerr black hole, the Reissner-Nordström black hole and the Kerr-Newman black hole. In particular, the detailed derivations of the adiabatic invariance`s integral solutions for the latter three black holes, are given. At the end of each chapter, the quantization of the entropy/area spectrum for the considered black hole is proven via the Bekenstein's area conjecture.

Keywords: Black hole, Black hole area, Black hole entropy, Adiabatic invariance, Exact differential equations, Quantization of black hole entropy and area.

ÖZ

Bu tezde, dört farklı tür karadelik için entropi hesaplıyoruz ve ayrıca entropi ve alanın her ikisinin de benzer ve eşit şekilde kuantize olduğunu gösteriyoruz. Alan ve entropinin kuantize oldukları, Bohr-Sommerfeld'in ünlü adyabatik sabitlik formülasyonu kullanılarak gösteriliyor. Ayrıca, adyabatik sabit için integral hesaplamalarını yaparken kesin diferansiyel eşitlik çözümlerini de dikkate alıyoruz. İlgilendiğimiz karadelikler sırasıyla Schwarzschild karadeliği, Kerr karadeliği, Reissner-Nordström karadeliği ve Kerr-Newman karadeliğidir. Ayrıca, son üçünün adyabatik sabitlerinin ayrıntılı integral hesaplamaları veriliyor. Her bölümün sonunda, Bekenstein'ın alan varsayımı aracılığı ile, ele alınan kara delik için alan ve entropi tayfının kuantize oldukları doğrulanmıştır.

Hesaplanmış olan entropi ile alan kıyaslanarak orantılı ve kuantize oldukları gösterilmektedir.

Anahtar kelimeler: Karadelik, Karadelik alanı, Karadelik entropisi, Adiyabatik sabitlik, Kesin diferansiyel eşitlik, Kuantize karadelik entropisi ve alanı.

DEDICATION

This thesis is dedicated to:

- My beloved wife, Petek, who supported me during my study.
- My son, Alp.

ACKNOWLEDGMENT

I show my appreciations to my supervisor Assoc. Prof. Dr. İzzet Sakallı who serves great supervision, supportive helps and encouragements.

I also thank to the jury members of my MS defense who are Prof. Dr. Mustafa Halilsoy and Assoc. Prof. Dr. S. Habib Mazharimousavi. They helped a lot by making constructive comments/suggestions.

I also show my appreciations to my family for their endless support.

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Chapter 1

INTRODUCTION

A black hole is mostly known for its strong gravity which does not let any particles or electromagnetic radiation out. According to the general theory of relativity published by Albert Einstein in 1915 [1], if a mass is sufficiently compact, then it can deform spacetime to form a black hole. In 1972, Jacob Bekenstein suggested that black holes should have a well-defined entropy and stated out that a black hole`s entropy is proportional to its event horizon area [2-6]. This idea was also confirmed by Stephen Hawking in 1974 when he proposed that particles are emitted from black holes, which is also known as Bekenstein-Hawking radiation [7-10]. Because of the particles that are emitted from or absorbed by a black hole, Bekenstein proposed that the entropy and horizon are proportional for a black hole and they are quantized [2-6, 11-27]. Thus, the black hole must obey the laws of thermodynamics. In this thesis, my aim is to study the quantization of the basic black holes in the literature, which are Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman black holes. To this end I consider the study of Liu Cheng-Zhou [28] in which the quantization of the entropy/area of the Schwarzschild black hole is achieved by Euclideanizing the metric. Apart from [28], in my computations, the adiabatic invariance is computed with the aid of the exact differential equations [29]. For the Reissner-Nordström and the Kerr black holes I use the doublet exact differential equation solutions. However, for the Kerr-Newman black hole the triplex exact differential equation solutions [29] are used. Besides, I show that all the black holes under consideration satisfy the first law of thermodynamics, which plays great role in deriving the adiabatic invariance. In sequel, I demonstrate how the adiabatic invariant leads to the quantization of the entropy/area for each considered black hole.

The thesis is organized as follows. Chapter 2 introduces the Schwarzschild black hole. Analysis of quantization of the Reissner-Nordström black hole is given in Chapter 3. Chapter 4 is devoted to quantization of the Kerr black hole. Chapter 5 shows the similar quantization computations for the more complicated black hole: the Kerr-Newman black hole. The thesis ends with a conclusion at Chapter 6.

Chapter 2

QUANTIZATION OF SCHWARZSCHILD BLACK HOLE

2.1 Properties of Schwarzschild Black Hole

A black hole is a region in space with a very strong gravitational attraction. Around the black hole there exists a boundary called the event horizon and beyond this boundary nothing can escape, not even light.

A Schwarzschild black hole is a static black hole that has no charge and no angular momentum. It has a perfectly symmetrical spherical shape and one event horizon. It is named after the German physicist and astronomer Karl Schwarzschild who gave the first exact solution [30] to Einstein's field equations in 1916 which was recently after Einstein first introduced the general theory of relativity [1].

Schwarzschild black hole satisfies the following first law of thermodynamics:

$$dS_{BH} = \frac{dM}{T_H}. \tag{2.1}$$

2.2 Invariance and Quantization of Schwarzschild Black Hole

The Schwarzschild black hole is described by the line-element

$$ds^2 = -\left(1 - \frac{2M_t}{r}\right) dt^2 + \left(1 - \frac{2M_t}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.2}$$

When we use the transformation of t→-iτ the metric (2.2) transforms from the Minkowskian form to the Euclidean form as follows

$$ds^2 = \left(1 - \frac{2iM_\tau}{r}\right) d\tau^2 + \left(1 - \frac{2iM_\tau}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.3}$$

The adiabatic invariant quantity is given by [31]

$$I = \oint p_r \, dq_r = \oint \int_0^{p_r} dp_r{}' \, dr, \tag{2.4}$$

in which the only dynamic degree of freedom is $q_r = r$.

Since the Hamiltonian is described as the total energy, it can be considered as the mass of the black hole. Thus, we can write

$$\dot{r} = \frac{dr}{d\tau} = \frac{dH'}{dp_r{}'} = \frac{dM'}{dp_r{}'}. \tag{2.5}$$

In Eq. (2.4), instead of dr, one can use

$$dr = \frac{dM'}{dp_r{}'} d\tau . \tag{2.6}$$

Thus, we obtain

$$I = \oint \int_0^M dM' \, d\tau . \tag{2.7}$$

We use the main feature of the Euclidean time [31-34] which results in the inverse of the black hole temperature (the so-called Hawking temperature):

$$\oint d\tau = \frac{2\pi}{\kappa} = \frac{1}{T_H}, \tag{2.8}$$

where κ is the surface gravity [35] .

Thus the adiabatic invariant takes the following form

$$I = 2\pi \int_0^M \frac{dM'}{\kappa'} = \int_0^M \frac{dM'}{T_H{}'} . \tag{2.9}$$

For the Schwarzschild black hole, the Hawking temperature is obtained as

$$T_H = \frac{1}{8\pi M} . \tag{2.10}$$

When we substitute this into the adiabatic invariant, it reads

$$I = \int_0^M \frac{dM'}{T_H{}'} = \int_0^M 8\pi M \, dM, \tag{2.11}$$

$$I = 4\pi M^2 = \pi r_H^2, \tag{2.12}$$

where $r_H = 2M$.

To calculate the horizon area of the Schwarzschild black hole, we set $dr = dt = 0$ in the Schwarzschild line element to obtain the "2-dimensional horizon" line element at $r = r_H$

$$d\sigma^2 = r_H{}^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{2.13}$$

which gives the metric tensor for the horizon as

$$g = \begin{pmatrix} r_H{}^2 & 0 \\ 0 & r_H{}^2 sin^2\theta \end{pmatrix}. \tag{2.14}$$

The area of the horizon is then calculated via

$$A_H = \int_0^{2\pi} d\phi \int_0^{\pi} \sqrt{detg} \, d\theta, \tag{2.15}$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} r_H^2 \sin\theta \; d\theta = r_H^2 \left(\phi \Big|_0^{2\pi} \right) \left(-\cos\theta \Big|_0^{\pi} \right), \tag{2.16}$$

$$= 4\pi r_H^2. \tag{2.17}$$

It can be seen that

$$I = \frac{A_H}{4}. \tag{2.18}$$

According to the Bohr-Sommerfeld's quantization rule [11, 36-37]

$$I = \oint pdq = nh = 2\pi n\hbar, \tag{2.19}$$

and using Eq. (2.18)

$$I = \frac{A_H}{4} = 2\pi n\hbar, \tag{2.20}$$

we get the area quantization for the Schwarzschild black hole as

$$A_H = 8\pi n\hbar. \qquad\qquad (n = 1, 2, 3, \ldots) \tag{2.21}$$

So, the minimum change in area naturally becomes

$$\Delta A_H = (A_H)_n - (A_H)_{n-1} = 8\pi\hbar\,(n - n + 1) = 8\pi\hbar. \tag{2.22}$$

The above result supports the Bekenstein's conjecture [2-6] which states that the area spectrum for a black hole is equally spaced.

Moreover, since the black hole entropy is the quarter of the black hole area:

$$S_{BH} = \frac{A_H}{4\hbar} = 2\pi n, \tag{2.23}$$

its minimum change that corresponds to the minimum information to be read on a black hole`s surface is thus given by

$$\Delta S_{BH} = \frac{\Delta A_H}{4\hbar} = 2\pi. \tag{2.24}$$

One can deduce from the above result that the entropy is also equally quantized.

Chapter 3

QUANTIZATION OF REISSNER-NORDSTRÖM BLACK HOLE

3.1 Properties of Reissner-Nordström Black Hole

Reissner-Nordström space-time is a spherically symmetric static black hole but unlike the Schwarzschild black hole it has electric charge. The metric to this charged, non-rotating and spherically symmetric black hole was discovered long time ago by Hans Reissner and Gunnar Nordström [38, 39].

The Reissner-Nordström black hole is described by the following line-element

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \tag{3.1}$$

where

$$f = \frac{(r - r_+)(r - r_-)}{r^2}, \tag{3.2}$$

And the outer (r_+) and inner (r_-) horizons are given by

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{3.3}$$

The electromagnetic four-potential, which is a relativistic vector, is given by

$$A_\mu = \left(-\frac{Q}{r}, 0,0,0\right). \tag{3.4}$$

Furthermore, the electromagnetic field tensor (Maxwell tensor) is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{3.5}$$

so that the non-zero component of the Maxwell tensor can be found as

$$F_{tr} = \partial_t A_r - \partial_r A_t = 0 - \frac{\partial}{\partial r}\left(-\frac{Q}{r}\right) = -\frac{Q}{r^2}, \tag{3.6}$$

which has its anti-symmetric partner:

$$F_{rt} = -F_{tr}. \tag{3.7}$$

It is worth noting that, one can find the Lagrangian for the electromagnetic field of the Reissner-Nordström black hole as

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.8}$$

$$L = -\frac{1}{4}(F_{tr}F^{tr} + F_{rt}F^{rt}), \tag{3.9}$$

$$L = -\frac{1}{4}\left(\frac{-Q}{r^2}\times\frac{Q}{r^2}\right) - \frac{1}{4}\left(\frac{Q}{r^2}\times\frac{-Q}{r^2}\right) = \frac{Q^2}{2r^4}. \tag{3.10}$$

To convert the Reissner-Nordström metric to the Euclidean form, we make the following transformation: $t \to -i\tau$.

Thus, the line element (3.1) becomes

$$ds^2 = f d\tau^2 + f^{-1}dr^2 + r^2 d\Omega^2. \tag{3.11}$$

We can also transform the energy vector and show that it is invariant under coordinate transformation. We first transform the time component of the electromagnetic four-potential:

$$A = A_t dt = A_t(-id\tau) = -iA_t d\tau. \tag{3.12}$$

Thus, we get

$$A = i\frac{Q}{r}d\tau, \tag{3.13}$$

and the transformed vector potential becomes

$$A_\mu = \left(i\frac{Q}{r}, 0,0,0\right), \tag{3.14}$$

from which one can read

$$A_\tau = i\frac{Q}{r}. \tag{3.15}$$

By using the identity at Eq. (3.5)

$$F_{\tau r} = \partial_\tau A_r - \partial_r A_\tau = 0 - \frac{\partial}{\partial r}\left(i\frac{Q}{r}\right) = i\frac{Q}{r^2}, \tag{3.16}$$

and recalling its anti-symmetry property

$$F_{r\tau} = -F_{\tau r} = -i\frac{Q}{r^2}, \tag{3.17}$$

we calculate the Lagrangian as

$$\begin{aligned} L &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\tau r}F^{\tau r} - \frac{1}{4}F_{r\tau}F^{r\tau}, \\ &= -\frac{1}{4}\left(i\frac{Q}{r^2} \times i\frac{Q}{r^2}\right) - \frac{1}{4}\left(-i\frac{Q}{r^2} \times -i\frac{Q}{r^2}\right), \\ &= -\frac{1}{2}\left(i\frac{Q}{r^2} \times i\frac{Q}{r^2}\right) = \frac{Q^2}{2r^4}. \end{aligned} \tag{3.18}$$

From Eq. (3.10) and Eq. (3.18), we see that the Lagrangian is invariant under the coordinate transformation.

In the Euclidean spacetime, the Killing vector is given by $l^\mu = (-i, 0,0,0)$ and the electromagnetic potential is defined by

$$\Phi_{BH} = A_\mu l^\mu \Big|_{r=r_+} = A_\tau l^\tau \Big|_{r=r_+} = i\frac{Q}{r}(-i)\Big|_{r=r_+},$$

$$\Phi_{BH} = \frac{Q}{r_+} = \frac{Q}{M + \sqrt{M^2 - Q^2}}. \tag{3.19}$$

Reissner-Nordström black hole satisfies the following first law of thermodynamics

$$dS_{BH} = \frac{dM}{T_H} - \Phi_{BH} \frac{dQ}{T_H}, \tag{3.20}$$

which can also be rewritten as

$$T_H dS_{BH} = dM - \Phi_{BH} dQ . \tag{3.21}$$

To show that the first law is satisfied, we first calculate the Hawking temperature by using

$$T_H = \left. \frac{f'}{4\pi} \right|_{r=r_+} = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{M^2 - Q^2}}{2\pi r_+^2}. \tag{3.22}$$

Then, we calculate S_{BH} by using the horizon area of the Reissner-Norström black hole at $r = r_+$ which can be calculated from the line element similar to Eq. (2.13–2.17) as $A_H = 4\pi r_+^2$,

$$S_{BH} = \frac{A_H}{4} = \pi r_+^2 = \pi \left(M + \sqrt{M^2 - Q^2}\right)^2, \tag{3.23}$$

so that we have

$$dS_{BH} = 2\pi r_+ \left(dM + \frac{MdM - QdQ}{\sqrt{M^2 - Q^2}}\right). \tag{3.24}$$

By using these findings, we can now check the 1st law of thermodynamics in the following steps. We first multiply Eq. (3.22) by Eq. (3.24) and obtain

$$T_H dS_{BH} = \frac{\sqrt{M^2 - Q^2}}{2\pi r_+^2} 2\pi r_+ \left(dM + \frac{MdM - QdQ}{\sqrt{M^2 - Q^2}}\right),$$

$$= \frac{\sqrt{M^2 - Q^2}}{r_+} \left(\frac{\left(M + \sqrt{M^2 - Q^2}\right) dM - QdQ}{\sqrt{M^2 - Q^2}} \right),$$

$$= \frac{\sqrt{M^2 - Q^2}}{r_+} \left(\frac{r_+ dM - QdQ}{\sqrt{M^2 - Q^2}} \right). \tag{3.25}$$

After some simplification one can easily get

$$T_H dS_{BH} = dM - \frac{Q}{r_+} dQ, \tag{3.26}$$

and finally obtain

$$T_H dS_{BH} = dM - \Phi_{BH} dQ, \tag{3.27}$$

which shows that the 1^{st} law of thermodynamics is satisfied.

3.2 Invariance and Quantization of Reissner-Nordström Black Hole

Theorem of Exactness for a differential equation with doublet variable:

Assume that the two functions, $H(x,y)dx \equiv H$ and $N(x,y)dy \equiv N$, are continuous and they have a continuous first-order partial differential equation

$$Hdx + Ndy = 0. \tag{3.28}$$

In this case, this differential equation is exact if

$$\frac{\partial H}{\partial y} = \frac{\partial N}{\partial x}, \tag{3.29}$$

and there exists a function $F(x,y) \equiv F$ such that

$$dF = Hdx + Ndy = 0, \tag{3.30}$$

where $H = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

We can construct the function F by using H:

$$F = \int H\, dx + g(y). \tag{3.31}$$

Then, N can be obtained by

$$N = \frac{\partial F}{\partial y} = \left(\frac{\partial}{\partial y}\int H\, dx\right) + \frac{\partial g(y)}{\partial y}. \tag{3.32}$$

By using Eq. (3.32), one can write

$$\frac{\partial g(y)}{\partial y} = N - \frac{\partial}{\partial y}\int H\, dx. \tag{3.33}$$

Because of the exactness condition, one can show that the right-hand side of Eq. (3.33) is a function of only y:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int H\, dx\right) &= \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\int H\, dx\right), \\
&= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int H\, dx\right), \\
&= \frac{\partial N}{\partial x} - \frac{\partial H}{\partial y} = 0.
\end{aligned} \tag{3.34}$$

So, one can see that $g(y)$ can be obtained by integrating the above function such that

$$g(y) = \int \left(N - \frac{\partial}{\partial y}\int H\, dx\right) dy, \tag{3.35}$$

which enables us to obtain F as

$$F = \int H\, dx + \int \left(N - \frac{\partial}{\partial y}\int H\, dx\right) dy. \tag{3.36}$$

As the entropy can be considered as an adiabatic invariant of the black hole, one can write the adiabatic invariant I of a charged static black hole by using Eq. (3.20) as

$$dI = \frac{dM}{T_H} - \Phi_{BH}\,\frac{dQ}{T_H} = 0. \tag{3.37}$$

It can be easily seen that since the differentiation of I is zero, we deduce that the parameter I is constant or invariant, which is the so-called "the adiabatic invariant". The similarities between Eq. (3.30) and Eq. (3.37) are as follows:

$$\begin{aligned} I &\to F(x,y), & M &\to x, \\ M &\to x, & J &\to y, \\ \frac{1}{T_H} &\to H(x,y), & -\frac{\Phi_{BH}}{T_H} &\to N(x,y). \end{aligned} \tag{3.38}$$

Also, one can check and verify the condition for exactness given at Eq. (3.29) by writing

$$\frac{\partial}{\partial Q}\left(\frac{1}{T_H}\right) = \frac{\partial}{\partial M}\left(-\frac{\Phi_{BH}}{T_H}\right). \tag{3.39}$$

One can see that this holds by writing T_H in the following form

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi\left(M + \sqrt{M^2 - Q^2}\right)^2}, \tag{3.40}$$

and use Φ_{BH} from Eq. (3.19) to calculate both sides of Eq. (3.43). We first calculate

$$\frac{\partial}{\partial Q}\left(\frac{1}{T_H}\right) = \frac{\partial}{\partial Q}\left[2\pi\,\frac{2M\left(M + \sqrt{M^2 - Q^2}\right) - Q^2}{\sqrt{M^2 - Q^2}}\right], \tag{3.41}$$

$$= 2\pi\frac{\partial}{\partial Q}\left[\frac{2M^2 - Q^2}{\sqrt{M^2 - Q^2}} + 2M\right], \tag{3.42}$$

$$= 2\pi\left[\frac{-2Q}{\sqrt{M^2 - Q^2}} + \frac{Q(2M^2 - Q^2)}{(M^2 - Q^2)^{3/2}}\right], \tag{3.43}$$

and find it as

$$= 2\pi\left[\frac{-2Q(M^2-Q^2)+2QM^2-Q^3}{(M^2-Q^2)^{3/2}}\right] = 2\pi\frac{Q^3}{(M^2-Q^2)^{3/2}}. \qquad (3.44)$$

We then calculate

$$\frac{\partial}{\partial M}\left(-\frac{\Phi_{BH}}{T_H}\right) = \frac{\partial}{\partial M}\left(-\frac{Q}{r_+}\times\frac{4\pi r_+^2}{r_+-r_-}\right), \qquad (3.45)$$

$$= \frac{\partial}{\partial M}\left(\frac{-4\pi Q r_+}{r_+-r_-}\right) = -2\pi\frac{\partial}{\partial M}\left(\frac{Q\left(M+\sqrt{M^2-Q^2}\right)}{\sqrt{M^2-Q^2}}\right), \qquad (3.46)$$

$$= -2\pi\left(\frac{Q}{\sqrt{M^2-Q^2}} - \frac{QM^2}{(M^2-Q^2)^{3/2}}\right), \qquad (3.47)$$

$$= -2\pi\frac{Q(M^2-Q^2)-QM^2}{(M^2-Q^2)^{3/2}}, \qquad (3.48)$$

and then we obtain

$$\frac{\partial}{\partial M}\left(-\frac{\Phi_{BH}}{T_H}\right) = 2\pi\frac{Q^3}{(M^2-Q^2)^{3/2}}. \qquad (3.49)$$

As Eq. (3.44) and Eq. (3.49) are equal, Eq. (3.39) holds. So, we can use exact differential equation solution given at (3.36) to calculate the adiabatic invariant quantity I. One can easily use the similarities at Eq. (3.38) to re-write Eq. (3.36) in this form:

$$I = \int\frac{1}{T_H}dM + \int\left(-\frac{\Phi_{BH}}{T_H} - \frac{\partial}{\partial Q}\int\frac{1}{T_H}dM\right)dQ. \qquad (3.50)$$

One can easily check that the first integral in the above equation yields

$$\int \frac{1}{T_H} dM = 2\pi M \left(M + \sqrt{M^2 - Q^2}\right). \tag{3.51}$$

Then, we find

$$\frac{\partial}{\partial Q} \int \frac{1}{T_H} dM = -\frac{2\pi M Q}{\sqrt{M^2 - Q^2}}, \tag{3.52}$$

and calculate the second integral in Eq. (3.50) as

$$\begin{aligned} &\int \left(-\frac{\Phi_{BH}}{T_H} - \frac{\partial}{\partial Q} \int \frac{1}{T_H} dM\right) dQ, \\ &= \int \left(-2\pi \frac{Q\left(M + \sqrt{M^2 - Q^2}\right)}{\sqrt{M^2 - Q^2}} + \frac{2\pi M Q}{\sqrt{M^2 - Q^2}}\right) dQ, \\ &= -2\pi \int Q\, dQ = -\pi Q^2 . \end{aligned} \tag{3.53}$$

Thus, we finally obtain the adiabatic invariant as follows

$$\begin{aligned} I &= 2\pi M \left(M + \sqrt{M^2 - Q^2}\right) - \pi Q^2 , \\ &= \pi(2Mr_+ - Q^2) = \pi r_+^2. \end{aligned} \tag{3.54}$$

As the horizon area is

$$A_H = 4\pi r_+^2 , \tag{3.55}$$

we see that the adiabatic invariant is nothing but the quarter of the horizon area.

According to the Bohr-Sommerfeld quantization rule [11, 35-36]

$$I = nh = 2\pi n\hbar , \tag{3.56}$$

and according to Eq. (3.55) we have

$$A_H = 8\pi n\hbar. \qquad (n = 1, 2, 3, \dots) \tag{3.57}$$

We can also get the change in entropy as

$$\Delta S_{BH} = \frac{\Delta A_H}{4\hbar} = 2\pi. \tag{3.58}$$

The above results are fully in agreement with the previous results in the literature.

Chapter 4

QUANTIZATION OF KERR BLACK HOLE

4.1 Properties of Kerr Black Hole

It is the first rotating black hole solution to the Einstein's field equations. This black hole solution was derived in 1963 [40] by the mathematician Roy Patrick Kerr from New Zealand while he was working at the University of Texas. In astrophysics, it is believed that the Kerr black hole is formed in the gravitational collapse of a spinning massive star. In the metric given below, if the angular momentum vanishes, the metric reduces to the Schwarzschild black hole. On the other hand, unlike the Schwarzschild black hole it does not have spherical symmetry since it is oblate.

The Kerr black hole is described by the following line-element:

$$ds^2 = -\frac{\rho^2}{\Sigma}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2}sin^2\theta(d\phi - \omega dt)^2, \tag{4.1}$$

with the following identities given as

$$\omega = \frac{2Mar}{\Sigma}, \tag{4.2}$$

$$\Sigma = (r^2 + a^2)^2 - \Delta a^2 sin^2\theta, \tag{4.3}$$

$$\rho^2 = r^2 + a^2 cos^2\theta, \tag{4.4}$$

$$\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-), \tag{4.5}$$

where the outer and inner radii are given by

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{4.6}$$

The angular momentum of the Kerr black hole, expressed in terms of mass (M) and the rotation parameter (a), is given by [35]

$$J = Ma. \tag{4.7}$$

Some of the properties of the Kerr metric can be deduced from its line element. It is not static as it is not invariant under time reversal. It is stationary since it does not depend explicitly on time. As it does not depend explicitly on ϕ, we understand that it is axisymmetric. Finally, since it is invariant under the inversion of t and ϕ simultaneously, the Kerr black hole rotates in the opposite direction when the time is reversed.

Kerr black hole satisfies the following first law of thermodynamics:

$$T_H dS_{BH} = dM - \Omega_{BH} dJ. \tag{4.8}$$

The angular velocity of the Kerr black hole is

$$\Omega_{BH} = -\left.\frac{g_{t\phi}}{g_{\phi\phi}}\right|_{r=r_+}. \tag{4.9}$$

The line-element can be re-organized in the following way to determine the coefficients of $g_{t\phi}$ and $g_{\phi\phi}$:

$$ds^2 = -\frac{\rho^2}{\Sigma}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$+\frac{\Sigma}{\rho^2} sin^2\theta(d\phi^2 + \omega^2 dt^2 - 2\omega dt d\phi). \tag{4.10}$$

From Eq. (4.10) one can easily see that

$$g_{t\phi} = -\frac{\Sigma}{\rho^2}\omega sin^2\theta, \tag{4.11}$$

and

$$g_{\phi\phi} = \frac{\Sigma}{\rho^2}sin^2\theta. \tag{4.12}$$

So, the angular velocity can be calculated as

$$\Omega_{BH} = -\left.\frac{-\frac{\Sigma}{\rho^2}\omega sin^2\theta}{\frac{\Sigma}{\rho^2}sin^2\theta}\right|_{r=r_+} = \omega|_{r=r_+},$$

$$= \left.\frac{2Mar}{\Sigma}\right|_{r=r_+},$$

$$= \frac{2Mar_+}{(r_+^2 + a^2)^2}, \tag{4.13}$$

and can further be simplified by using $r_+^2 = 2Mr_+ - a^2$ to obtain

$$\Omega_{BH} = \frac{2Mar_+}{(2Mr_+)^2} = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}. \tag{4.14}$$

Also, Ω_{BH} can be obtained in terms of M and J by using Eq. (4.7)

$$\Omega_{BH} = \frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}. \tag{4.15}$$

We can obtain the Hawking temperature from

$$T_H = \frac{r_+ - r_-}{4\pi(r_+^2 + r_+r_-)}. \tag{4.16}$$

Using the identity $r_+r_- = a^2$, we can re-arrange Eq. (4.16) as

$$T_H = \frac{2\sqrt{M^2 - a^2}}{4\pi\ 2Mr_+} = \frac{\sqrt{M^2 - a^2}}{4\pi M\left(M + \sqrt{M^2 - a^2}\right)}. \tag{4.17}$$

At this point, we can again use the identity at Eq. (4.7) and obtain

$$T_H = \frac{\sqrt{M^4 - J^2}}{4\pi M\left(M^2 + \sqrt{M^4 - J^2}\right)}. \tag{4.18}$$

To calculate the horizon area of the Kerr black hole, we first obtain the line element of 2-dimensional horizon of the Kerr black hole, again by setting $dt = dr = 0$ at $r = r_+$:

$$d\sigma^2 = \rho_+^2 d\theta^2 + \frac{(r_+^2 + a^2)^2}{\rho_+^2} sin^2\theta d\phi^2, \tag{4.19}$$

$$= \rho_+^2 d\theta^2 + \frac{(2Mr_+)^2}{\rho_+^2} sin^2\theta d\phi^2, \tag{4.20}$$

where

$$\rho_+^2 = r_+^2 + a^2 cos^2\theta. \tag{4.21}$$

From Eq. (4.20) one can read the metric tensor for the Kerr horizon, that is

$$g = \begin{bmatrix} \rho_+^2 & 0 \\ 0 & \left(\frac{2Mr_+}{\rho_+}\right)^2 sin^2\theta \end{bmatrix}. \tag{4.22}$$

By using Eq. (2.15) we calculate the horizon area of the Kerr black hole as

$$A_H = 2Mr_+ \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\ d\theta, \tag{4.23}$$

$$= 4\pi Mr_+ \left(-\cos\theta \Big|_0^{\pi}\right), \tag{4.24}$$

$$= 8\pi Mr_+, \tag{4.25}$$

which can also be re-organized in this way:

$$
\begin{aligned}
A_H &= 8\pi M\left(M+\sqrt{M^2-a^2}\right),\\
&= 8\pi\left(M^2+\sqrt{M^4-J^2}\right).
\end{aligned}
\tag{4.26}
$$

Now, one can obtain the entropy as

$$
S_{BH} = \frac{A_H}{4} = 2\pi\left(M^2+\sqrt{M^4-J^2}\right). \tag{4.27}
$$

To show that Eq. (4.8) holds, we calculate $dS_{BH} = \frac{\partial S_{BH}}{\partial M}dM + \frac{\partial S_{BH}}{\partial J}dJ$:

$$
\begin{aligned}
dS_{BH} &= 2\pi\left(2MdM + \frac{4M^3dM-2JdJ}{2\sqrt{M^4-J^2}}\right)\\
&= \frac{4\pi M\left(M^2+\sqrt{M^4-J^2}\right)}{\sqrt{M^4-J^2}}dM - \frac{2\pi J}{\sqrt{M^4-J^2}}dJ.
\end{aligned}
\tag{4.28}
$$

We obtain $T_H dS_{BH}$ as

$$
\begin{aligned}
T_H dS_{BH} &= \left(\frac{\sqrt{M^4-J^2}}{4\pi M\left(M^2+\sqrt{M^4-J^2}\right)}\right)\\
&\quad\times\left(\frac{4\pi M\left(M^2+\sqrt{M^4-J^2}\right)}{\sqrt{M^4-J^2}}dM - \frac{2\pi J}{\sqrt{M^4-J^2}}dJ\right),
\end{aligned}
\tag{4.29}
$$

which can further be simplified as

$$
\begin{aligned}
T_H dS_{BH} &= dM - \frac{J}{2M\left(M^2+\sqrt{M^4-J^2}\right)}dJ,\\
&= dM - \Omega_{BH}dJ.
\end{aligned}
\tag{4.30}
$$

As Eq. (4.8) holds we verify that the Kerr black hole metric satisfies the 1st law of thermodynamics.

4.2 Invariance and Quantization of Kerr Black Hole

For a rotating uncharged black hole the adiabatic invariant is calculated via

$$dI = \frac{1}{T_H} dM - \frac{\Omega_{BH}}{T_H} dJ = 0. \tag{4.31}$$

By using the similarities seen in Eq. (3.30), one can prove that Eq. (4.31) is an exact differential equation. Namely, we have the condition of

$$\frac{\partial}{\partial J}\left(\frac{1}{T_H}\right) = \frac{\partial}{\partial M}\left(-\frac{\Omega_{BH}}{T_H}\right). \tag{4.32}$$

Left hand side of the above equation can be computed by using

$$\frac{1}{T_H} = \left(\frac{\sqrt{M^4 - J^2}}{4\pi M\left(M^2 + \sqrt{M^4 - J^2}\right)}\right)^{-1},$$

$$= 4\pi M\left(\frac{M^2}{\sqrt{M^4 - J^2}} + 1\right), \tag{4.33}$$

and then differentiating it with respect to J, we have

$$\frac{\partial}{\partial J}\left(\frac{1}{T_H}\right) = \frac{4\pi M^3 J}{(M^4 - J^2)^{3/2}}. \tag{4.34}$$

Then we use Eq. (4.15) together with Eq. (4.33) to obtain

$$-\frac{\Omega_{BH}}{T_H} = -\frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)} 4\pi M \frac{\left(M^2 + \sqrt{M^4 - J^2}\right)}{\sqrt{M^4 - J^2}},$$

$$= -2\pi \frac{J}{\sqrt{M^4 - J^2}}. \tag{4.35}$$

One can easily calculate the right-hand side of Eq. (4.32) as

$$\frac{\partial}{\partial M}\left(-\frac{\Omega_{BH}}{T_H}\right) = \frac{4\pi M^3 J}{(M^4 - J^2)^{3/2}}. \tag{4.36}$$

One can immediately observe that Eq. (4.34) and Eq. (4.36) are equal: the proof of the exactness condition (4.32).

As the exactness condition is satisfied, one can amalgamate Eq. (3.36) with Eq. (4.31) to obtain the adiabatic invariant for the Kerr black hole as the following

$$I = \int \frac{1}{T_H} dM + \int \left(-\frac{\Omega_{BH}}{T_H} - \frac{\partial}{\partial J}\int \frac{1}{T_H} dM\right) dJ. \tag{4.37}$$

Using Eq. (4.33), the first integral of Eq. (4.37) can be easily evaluated:

$$\int \frac{1}{T_H} dM = \int 4\pi M \left(\frac{M^2}{\sqrt{M^4 - J^2}} + 1\right) dM$$

$$= 4\pi \left[\int \left(\frac{M^3}{\sqrt{M^4 - J^2}}\right) dM + \int M dM\right]. \tag{4.38}$$

Let $u = M^4 - J^2$, then $du = 4M^3 dM$. By using this substitution, this integral can be easily solved:

$$\int \frac{1}{T_H} dM = 4\pi \left(\frac{\sqrt{M^4 - J^2}}{2} + \frac{M^2}{2}\right) = 2\pi \left(M^2 + \sqrt{M^4 - J^2}\right). \tag{4.39}$$

Then, we find

$$\frac{\partial}{\partial J}\int \frac{1}{T_H}dM = -2\pi \frac{J}{\sqrt{M^4 - J^2}}, \tag{4.40}$$

and by using Eq. (4.35) one can see that the second integral yields

$$-\frac{\Omega_{BH}}{T_H} - \frac{\partial}{\partial J}\int \frac{1}{T_H}dM = 0. \tag{4.41}$$

Then the adiabatic invariant becomes a simple integral:

$$I = \int \frac{1}{T_H}dM = 2\pi\left(M^2 + \sqrt{M^4 - J^2}\right). \tag{4.42}$$

We can write this in the following way

$$I = 2\pi M\left(M + \sqrt{M^2 - a^2}\right) = 2\pi M r_+. \tag{4.43}$$

Recalling the area, $A = 8\pi M r_+$, we deduce from Eq. (4.43) that $I = \frac{A}{4} = 2\pi n\hbar$. Therefore, the area (and hence the entropy) is equally likely quantized, $A = 8\pi n\hbar$: Bekenstein`s conjecture is also valid for the Kerr black hole.

Chapter 5

QUANTIZATION OF KERR-NEWMAN BLACK HOLE

5.1 Properties of Kerr-Newman Black Hole

The Kerr-Newman black hole is characterized by three physical parameters which are mass, angular momentum and electric charge. Its metric is the most generic stationary black hole solution [41-42] to the Einstein-Maxwell equations. When both the angular momentum and the electric charge are taken to be zero, one gets back the Schwarzschild metric. When only the electric charge vanishes, the metric corresponds to the metric of a spinning black hole, which is nothing but the Kerr metric. Similarly, in a case that the angular momentum vanishes but there exists an electric charge, the metric reduces to the Reissner-Nordström metric.

The Kerr-Newman black hole is described by the following line-element

$$ds^2 = \left(1 - \frac{2Mr - Q^2}{\rho^2}\right)dt^2 + 2dt\,dr + 2\frac{a\sin^2\theta}{\rho^2}(2Mr - Q^2)dt\,d\phi$$

$$-2a\sin^2\theta\,dr\,d\phi - \rho^2 d\theta^2$$

$$+\frac{\sin^2\theta}{\rho^2}[\Delta a^2\sin^2\theta - (a^2 + r^2)^2]d\phi^2, \tag{5.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{5.2}$$

$$\Delta = r^2 + a^2 - 2Mr + Q^2, \tag{5.3}$$

$$= (r - r_+)(r - r_-), \tag{5.4}$$

in which

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{5.5}$$

The Hawking temperature, angular momentum, electromagnetic potential, and angular velocity, respectively, are given as

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}, \tag{5.6}$$

$$J = Ma, \tag{5.7}$$

$$\Phi_{BH} = \frac{Qr_+}{r_+^2 + a^2}, \tag{5.8}$$

$$\Omega_{BH} = \frac{a}{r_+^2 + a^2}. \tag{5.9}$$

To obtain the line element of 2-dimensional horizon of the Kerr-Newman black hole, we again set $dt = dr = 0$ at $r = r_+$:

$$d\sigma^2 = -\rho_+^2 d\theta^2 - \frac{sin^2\theta}{\rho_+^2}(r_+^2 + a^2)^2 d\phi^2, \tag{5.10}$$

where

$$\rho_+^2 = r_+^2 + a^2 cos^2\theta. \tag{5.11}$$

From Eq. (5.10) one can read the metric tensor for the Kerr-Newman horizon, that is

$$g = \begin{bmatrix} -\rho_+^2 & 0 \\ 0 & -\frac{sin^2\theta}{\rho_+^2}(r_+^2 + a^2)^2 \end{bmatrix}. \tag{5.12}$$

By using Eq. (2.15) we calculate the horizon area of the Kerr-Newman black hole as

$$A_H = (r_+^2 + a^2) \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \ d\theta, \tag{5.13}$$

$$= 2\pi(r_+^2 + a^2)\left(-\cos\theta \Big|_0^\pi\right), \tag{5.14}$$

$$= 4\pi(r_+^2 + a^2), \tag{5.15}$$

$$= 4\pi(2Mr_+ - Q^2). \tag{5.16}$$

From this, the black hole entropy reads

$$S_{BH} = 2\pi M r_+ - \pi Q^2$$

$$= 2\pi M\left(M + \sqrt{M^2 - a^2 - Q^2}\right) - \pi Q^2. \tag{5.17}$$

The Kerr black hole satisfies the following first law of thermodynamics

$$T_H dS_{BH} = dM - \Omega_{BH} dJ - \Phi_{BH} dQ. \tag{5.18}$$

To prove this, we start by writing

$$dS_{BH} = \frac{\partial S_{BH}}{\partial M} dM + \frac{\partial S_{BH}}{\partial J} dJ + \frac{\partial S_{BH}}{\partial Q} dQ. \tag{5.19}$$

It is obvious that

$$T_H \frac{\partial S_{BH}}{\partial M} = 1, \tag{5.20}$$

$$T_H \frac{\partial S_{BH}}{\partial J} = -\Omega_{BH}, \tag{5.21}$$

$$T_H \frac{\partial S_{BH}}{\partial Q} = -\Phi_{BH}. \tag{5.22}$$

In the following part, we calculate them one by one in order to check the validity of the 1st law of thermodynamics.

In the first part we substitute the identity $J = Ma$ into Eq. (5.17) to get

$$S_{BH} = 2\pi M\left(M + \sqrt{M^2 - \frac{J^2}{M^2} - Q^2}\right) - \pi Q^2, \tag{5.23}$$

and then calculate

$$\frac{\partial S_{BH}}{\partial M} = \left[2\pi\left(M + \sqrt{M^2 - \frac{J^2}{M^2} - Q^2}\right) + 2\pi M\left(1 + \frac{2M + 2\frac{J^2}{M^3}}{2\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}}\right)\right],$$

$$= 2\pi\left[\left(M + \sqrt{M^2 - \frac{J^2}{M^2} - Q^2}\right) + \left(\frac{M\sqrt{M^2 - \frac{J^2}{M^2} - Q^2} + M^2 + \frac{J^2}{M^2}}{\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}}\right)\right],$$

$$= 2\pi\,\frac{\left(M\sqrt{M^2 - \frac{J^2}{M^2} - Q^2} + M^2 - \frac{J^2}{M^2} - Q^2 + M\sqrt{M^2 - \frac{J^2}{M^2} - Q^2} + M^2 + \frac{J^2}{M^2}\right)}{\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}},$$

$$= 2\pi\,\frac{\left(2M\sqrt{M^2 - \frac{J^2}{M^2} - Q^2} + 2M^2 - Q^2\right)}{\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}},$$

$$= 2\pi\frac{2Mr_+ - Q^2}{\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}}, \tag{5.24}$$

which corresponds to

$$\frac{\partial S_{BH}}{\partial M} = 2\pi\frac{r_+^2 + a^2}{\sqrt{M^2 - a^2 - Q^2}}, \tag{5.25}$$

and multiply it with Eq. (5.6) to verify Eq. (5.20)

$$T_H\frac{\partial S_{BH}}{\partial M} = \left(\frac{1}{2\pi}\frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}\right)\left(2\pi\frac{r_+^2 + a^2}{\sqrt{M^2 - a^2 - Q^2}}\right) = 1. \tag{5.26}$$

We use Eq. (5.23) to calculate

$$\frac{\partial S_{BH}}{\partial J} = 2\pi M \frac{1}{2\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}} \left(-\frac{2J}{M^2}\right), \tag{5.27}$$

$$= -2\pi \frac{J}{M\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}}. \tag{5.28}$$

Then, we calculate and simplify

$$T_H \frac{\partial S_{BH}}{\partial J} = \left(\frac{1}{2\pi} \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}\right)\left(-2\pi \frac{J}{M\sqrt{M^2 - \frac{J^2}{M^2} - Q^2}}\right),$$

$$= -\frac{J}{M(r_+^2 + a^2)}. \tag{5.29}$$

By substituting $J = Ma$ into Eq. (5.29) one can easily verify Eq. (5.21) as

$$T_H \frac{\partial S_{BH}}{\partial J} = -\frac{a}{r_+^2 + a^2} = -\Omega_{BH} . \tag{5.30}$$

Since this is verified, we move on to the third part and therefore the last condition of the exactness (5.22):

$$\frac{\partial S_{BH}}{dQ} = 2\pi M \frac{-2Q}{2\sqrt{M^2 - a^2 - Q^2}} - 2\pi Q ,$$

$$= -2\pi Q \frac{M + \sqrt{M^2 - a^2 - Q^2}}{\sqrt{M^2 - a^2 - Q^2}} = -2\pi Q \frac{r_+}{\sqrt{M^2 - a^2 - Q^2}}. \tag{5.31}$$

By using Eq. (5.6) and Eq. (5.31), we calculate

$$T_H \frac{\partial S_{BH}}{\partial Q} = \left(\frac{1}{2\pi} \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}\right)\left(-2\pi Q \frac{r_+}{\sqrt{M^2 - a^2 - Q^2}}\right). \tag{5.32}$$

Simplifying the above equation, we verify Eq. (5.22):

$$T_H \frac{\partial S_{BH}}{\partial Q} = -\frac{Q r_+}{r_+^2 + a^2} = -\Phi_{BH}. \tag{5.33}$$

As all the three conditions, Eq. (5.20 - 5.22), are satisfied, one can conclude that the 1st law of thermodynamics for the Kerr-Newman black hole is satisfied.

5.2 Invariance and Quantization of Kerr-Newman Black Hole

One can easily obtain the adiabatic invariant by writing Eq. (5.18) in the following form

$$dI = \frac{1}{T_H} dM - \frac{\Omega_{BH}}{T_H} dJ - \frac{\Phi_{BH}}{T_H} dQ = 0. \tag{5.34}$$

As we have triplet first order differential equation, the following three exactness conditions [29] must be satisfied simultaneously

$$\frac{\partial}{\partial J}\left(\frac{1}{T_H}\right) = \frac{\partial}{\partial M}\left(-\frac{\Omega_{BH}}{T_H}\right), \tag{5.35}$$

$$\frac{\partial}{\partial Q}\left(\frac{1}{T_H}\right) = \frac{\partial}{\partial M}\left(-\frac{\Phi_{BH}}{T_H}\right), \tag{5.36}$$

$$\frac{\partial}{\partial Q}\left(-\frac{\Omega_{BH}}{T_H}\right) = \frac{\partial}{\partial J}\left(-\frac{\Phi_{BH}}{T_H}\right). \tag{5.37}$$

To see whether the above conditions hold or not, we first start to compute

$$\frac{1}{T_H} = \frac{2\pi M\left(2M^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} - Q^2\right)}{\sqrt{M^4 - J^2 - Q^2 M^2}}, \tag{5.38}$$

$$-\frac{\Omega_{BH}}{T_H} = \left(-\frac{a}{r_+^2 + a^2}\right) \times \left(\frac{1}{2\pi}\frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}\right)^{-1}$$

$$= \frac{-2\pi J}{\sqrt{M^4 - J^2 - Q^2M^2}}, \tag{5.39}$$

and

$$-\frac{\Phi_{BH}}{T_H} = -\frac{2\pi Q\left(M^2 + \sqrt{M^4 - J^2 - Q^2M^2}\right)}{\sqrt{M^4 - J^2 - Q^2M^2}}. \tag{5.40}$$

The left-hand side of Eq. (5.35) is

$$\frac{\partial}{\partial J}\left(\frac{1}{T_H}\right) = \frac{\partial}{\partial J}\left(\frac{2\pi M(2M^2 - Q^2) + 4\pi M\sqrt{M^4 - J^2 - Q^2M^2}}{\sqrt{M^4 - J^2 - Q^2M^2}}\right),$$

$$= \frac{\partial}{\partial J}\left(\frac{(4\pi M^3 - 2\pi Q^2 M)}{\sqrt{M^4 - J^2 - Q^2M^2}} + 4\pi M\right),$$

$$= \frac{(4M^3 - 2Q^2M)\pi J}{(M^4 - J^2 - Q^2M^2)^{3/2}}. \tag{5.41}$$

The right-hand side of Eq. (5.35) is

$$\frac{\partial}{\partial M}\left(-\frac{\Omega_{BH}}{T_H}\right) = \frac{(4M^3 - 2Q^2M)\pi J}{(M^4 - J^2 - Q^2M^2)^{3/2}}. \tag{5.42}$$

So, one can see that Eq. (5.35) is verified.

We now calculate the second exactness condition given by equation Eq. (5.36). The left-hand side is calculated as

$$\frac{\partial}{\partial Q}\left(\frac{1}{T_H}\right) = \frac{\partial}{\partial Q}\left[\frac{2\pi M\left(2M^2 + 2\sqrt{M^4 - J^2 - Q^2M^2} - Q^2\right)}{\sqrt{M^4 - J^2 - Q^2M^2}}\right], \tag{5.43}$$

$$= \frac{\partial}{\partial Q}\left(\frac{4\pi M^3 - 2\pi Q^2 M}{\sqrt{M^4 - J^2 - Q^2M^2}} + 4\pi M\right), \tag{5.44}$$

$$= \frac{-4\pi MQ}{\sqrt{M^4 - J^2 - Q^2M^2}} - \frac{(4\pi M^3 - 2\pi Q^2 M) \times (-2QM^2)}{2(M^4 - J^2 - Q^2M^2)^{3/2}}, \tag{5.45}$$

$$= \frac{-4\pi MQ(M^4 - J^2 - Q^2M^2) + 4\pi QM^5 - 2\pi Q^3M^3}{(M^4 - J^2 - Q^2M^2)^{3/2}} \tag{5.46}$$

$$\frac{\partial}{\partial Q}\left(\frac{1}{T_H}\right) = \pi Q \frac{4MJ^2 + 2Q^2M^3}{(M^4 - J^2 - Q^2M^2)^{3/2}}. \tag{5.47}$$

The right-hand side of Eq. (5.36) becomes

$$\frac{\partial}{\partial M}\left(-\frac{\Phi_{BH}}{T_H}\right) = \frac{\partial}{\partial M}\left(-\frac{2\pi QM^2}{\sqrt{M^4 - J^2 - Q^2M^2}} - 2\pi Q\right), \tag{5.48}$$

$$= \frac{-4\pi QM}{\sqrt{M^4 - J^2 - Q^2M^2}} + \frac{\pi QM^2(4M^3 - 2Q^2M)}{(M^4 - J^2 - Q^2M^2)^{3/2}}, \tag{5.49}$$

$$\frac{\partial}{\partial M}\left(-\frac{\Phi_{BH}}{T_H}\right) = \pi Q \frac{4MJ^2 + 2Q^2M^3}{(M^4 - J^2 - Q^2M^2)^{3/2}}. \tag{5.50}$$

Eq. (5.47) and Eq. (5.50) are equal, thus Eq. (5.36) is verified.

We can now move on to the third and final condition for exactness which is given at Eq. (5.37). The left-hand side gives

$$\frac{\partial}{\partial Q}\left(-\frac{\Omega_{BH}}{T_H}\right) = \frac{\partial}{\partial Q}\left(\frac{-2\pi J}{\sqrt{M^4 - J^2 - Q^2 M^2}}\right),$$

$$= -\frac{2\pi J Q M^2}{(M^4 - J^2 - Q^2 M^2)^{3/2}}, \tag{5.51}$$

and the right-hand side is calculated as

$$\frac{\partial}{\partial J}\left(-\frac{\Phi_{BH}}{T_H}\right) = \frac{\partial}{\partial J}\left(-\frac{2\pi Q M^2}{\sqrt{M^4 - J^2 - Q^2 M^2}} - 2\pi Q\right),$$

$$= -2\pi Q M^2 \left(-\frac{1}{2}\right)\frac{2J}{(M^4 - J^2 - Q^2 M^2)^{3/2}}, \tag{5.52}$$

and finally we apply simplification to Eq. (5.52) to get

$$\frac{\partial}{\partial J}\left(-\frac{\Phi_{BH}}{T_H}\right) = -\frac{2\pi J Q M^2}{(M^4 - J^2 - Q^2 M^2)^{3/2}}. \tag{5.53}$$

From above, one can verify that Eq. (5.37) also holds.

The adiabatic invariant quantity for a rotating and charged black hole is given by

$$dI = \frac{\partial I}{\partial M} dM + \frac{\partial I}{\partial J} dJ + \frac{\partial I}{\partial Q} dQ = 0. \tag{5.54}$$

As it can be seen from above, the structure of the differential equation is in the form of triplet exact differential equation. Since the exactness conditions (Eq. 5.35 – Eq. 5.37) are all verified, we can now solve the adiabatic invariant differential equation Eq. (5.34) as

$$I = \int \frac{\partial I}{\partial M} dM + \psi(J, Q) = \int \frac{dM}{T_H} + \psi(J, Q). \tag{5.55}$$

Also,

$$\frac{\partial I}{\partial J}=-\frac{\Omega_{BH}}{T_H}=\frac{\partial}{\partial J}\int\frac{dM}{T_H}+\frac{\partial\psi}{\partial J}, \tag{5.56}$$

from which one can write

$$\frac{\partial\psi}{\partial J}=-\frac{\Omega_{BH}}{T_H}-\frac{\partial}{\partial J}\int\frac{dM}{T_H}, \tag{5.57}$$

and by taking the integral, ψ can be obtained as

$$\psi=-\int\left(\frac{\Omega_{BH}}{T_H}+\frac{\partial}{\partial J}\int\frac{dM}{T_H}\right)dJ+g(Q). \tag{5.58}$$

So, we obtain the adiabatic invariant as

$$I=\int\frac{dM}{T_H}-\int\left(\frac{\Omega_{BH}}{T_H}+\frac{\partial}{\partial J}\int\frac{dM}{T_H}\right)dJ+g(Q). \tag{5.59}$$

Let

$$\int\frac{dM}{T_H}-\int\left(\frac{\Omega_{BH}}{T_H}+\frac{\partial}{\partial J}\int\frac{dM}{T_H}\right)dJ=X_1, \tag{5.60}$$

then, we can write

$$\frac{\partial I}{\partial Q}=-\frac{\Phi_{BH}}{T_H}=\frac{\partial}{\partial Q}X_1+\frac{\partial g}{\partial Q}, \tag{5.61}$$

and obtain $\frac{\partial g}{\partial Q}$ as

$$\frac{\partial g}{\partial Q}=-\frac{\Phi_{BH}}{T_H}-\frac{\partial X_1}{\partial Q}. \tag{5.62}$$

From this, we can obtain g in the following way

$$g=-\int\left(\frac{\Phi_{BH}}{T_H}+\frac{\partial X_1}{\partial Q}\right)dQ, \tag{5.63}$$

and by using Eq. (5.60) we insert g into Eq. (5.59) to get

$$I = X_1 - \int \left(\frac{\Phi_{BH}}{T_H} + \frac{\partial X_1}{\partial Q}\right) dQ. \tag{5.64}$$

This is the main equation that we use for calculating the adiabatic invariant. We begin the calculation with the following integration

$$\int \frac{dM}{T_H} = 2\pi \left(M^2 + \sqrt{M^4 - J^2 - Q^2 M^2}\right), \tag{5.65}$$

and taking its derivative with respect to J

$$\frac{\partial}{\partial J} \int \frac{dM}{T_H} = -\frac{2\pi J}{\sqrt{M^4 - J^2 - Q^2 M^2}}. \tag{5.66}$$

In sequel, we see that

$$\int \left(\frac{\Omega_{BH}}{T_H} + \frac{\partial}{\partial J} \int \frac{dM}{T_H}\right) dJ = 0. \tag{5.67}$$

When we substitute this into Eq. (5.60) we get X_1 as

$$X_1 = 2\pi \left(M^2 + \sqrt{M^4 - J^2 - Q^2 M^2}\right). \tag{5.68}$$

Then we proceed with a differentiation:

$$\frac{\partial X_1}{\partial Q} = \frac{2\pi Q M^2}{\sqrt{M^4 - J^2 - Q^2 M^2}}, \tag{5.69}$$

and use it in Eq. (5.64) to get the adiabatic invariant as

$$\begin{aligned} I = X_1 - \int \left(\frac{\Phi_{BH}}{T_H} + \frac{\partial X_1}{\partial Q}\right) dQ &= 2\pi M^2 + 2\pi\sqrt{M^4 - J^2 - Q^2 M^2} - \pi Q^2, \\ &= \pi \left(2M^2 + 2M\sqrt{M^2 - a^2 - Q^2} - Q^2\right), \\ &= \pi(r_+^2 + a^2). \end{aligned} \tag{5.70}$$

Since

$$A_H = 4\pi(r_+^2 + a^2), \tag{5.71}$$

Eq. (5.70) yields

$$I = \frac{A_H}{4} = 2\pi n\hbar \quad \rightarrow \quad A_H = 8\pi n\hbar. \tag{5.72}$$

We see that the area of the Kerr-Newman black hole is also equidistant. Namely, our computations support the Bekenstein's conjecture.

Chapter 6

CONCLUSION

I have thoroughly studied the entropy and area of four different types of asymptotically flat black holes. After showing that each black hole verifies the first law of thermodynamics, I have derived the integral solution for the adiabatic invariance. In particular, I have used the exact differential equation solutions with their appropriate exactness conditions to obtain the adiabatic invariants of the considered black holes. Furthermore, by employing the Bohr-Sommerfeld's quantization rule for the adiabatic invariance obtained, I have showed that the area and correspondingly the entropy (since it is equal to the quarter of the black hole area) of the black holes are equally likely quantized. My results are in full agreement with the original Bekenstein's conjecture.

I plan to extend my studies to the other type of black hole solutions like the higher dimensional black holes, asymptotically non-flat black holes, black holes belonging to the other gravity models (massive gravity, $f(R)$, Chern-Simons…) etc.

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