Grid Approximation of Derivatives of the Solution of Heat Conduction Equation

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ABSTRACT

In this study we propose special difference problems of the four point scheme and six point symmetric implicit scheme (Crank and Nicolson) for the approximation of first and second partial derivatives of the solution $u(x,t)$ of the first type boundary value problem for one-dimensional heat conduction equation, with constant coefficients.

A four point implicit difference problem is proposed for the approximation of $\frac{\partial u}{\partial x}$ under the assumption that the initial function belongs to the Hölder space $C^{5+\alpha}$, $0<\alpha<1$, the nonhomogeneous function given in the heat equation is from the Hölder space $C_{x,t}^{3+\alpha,\frac{3+\alpha}{2}}$, the boundary functions are from $C^{\frac{5+\alpha}{2}}$ also between the initial and boundary functions the conjugation conditions up to second order $(q=0,1,2)$ are satisfied. When the initial function belongs to $C^{7+\alpha}$, the nonhomogeneous term is from $C_{x,t}^{5+\alpha,\frac{5+\alpha}{2}}$, the boundary functions are from $C^{\frac{7+\alpha}{2}}$, also the conjugation conditions up to third order $(q=0,1,2,3)$ are satisfied, a six point implicit difference problem is given. It is proven that the solution of the constructed four and six point implicit difference problems converge to the exact value of $\frac{\partial u}{\partial x}$ on the grids of order $O(h^2+\tau)$ and $O(h^2+\tau^2)$ respectively, where, h is the step size in spatial variable x and τ is the step size in time variable t.

Furthermore, boundary value problems and implicit difference problems are given to the first derivative of the solution with respect to time variable t, $\left(\frac{\partial u}{\partial t}\right)$ and for the pure second derivative with respect to the spatial variable x. Also special implicit difference boundary value problem is proposed for the mixed second derivative of the solution, $\left(\frac{\partial^2 u}{\partial x \partial t}\right)$. When the initial function belongs to $C^{8+\alpha}$, the heat source function given in the heat equation is from $C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}$, the boundary functions are from $C^{\frac{8+\alpha}{2}}$ Hölder spaces and between the initial and boundary function the conjugation conditions of orders $q = 0,1,2,3,4$ are satisfied, it is proven that the solution of the proposed implicit difference schemes converge uniformly to the corresponding exact derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial x \partial t}$ on the grids of the order $O(h^2 + \tau)$. On the other hand, when the initial function belongs to $C^{10+\alpha}$, the heat source function is from $C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}$, the boundary functions are from $C^{\frac{10+\alpha}{2}}$ Hölder spaces and between the initial and boundary functions the conjugation conditions of orders $q = 0,1,2,3,4,5$ are satisfied, the constructed six-point symmetric (Crank-Nicolson) implicit difference boundary value problems converge with the order $O(h^2 + \tau^2)$ to the corresponding exact derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial x \partial t}$.

Finally, in order to justify the theoretical results, several numerical examples are constructed and the obtained results are presented through tables and figures.

Keywords: Finite difference method, Approximation of derivatives, Crank-Nicolson scheme, Uniform error, Heat equation.

ÖZ

Bu çalışmada sabit katsayılı tek boyutlu ısı denkleminin birinci çeşit sınır değer probleminin $u(x,t)$ çözümünün birinci ve ikinci kısmi türevlerinin yaklaşık hesaplanması için dört nokta kapali ve altı nokta simetrik kapalı fark şemalı (Crank ve Nicolson) özel fark problemleri öne sürüldü.

Başlangıç fonksiyonunun $C^{5+\alpha}$, $0<\alpha<1$, ısı denklemindeki homojen olmayan terimin $C_{x,t}^{3+\alpha,\frac{3+\alpha}{2}}$ ve sınır fonksiyonlarının $C^{\frac{5+\alpha}{2}}$ Hölder uzaylarından olduğu ayrıca başlangıç ve sınır fonksiyonları arasında ikinci dereceye kadar $(q=0,1,2)$ bağlayıcı koşulların sağladığı kabul edildiğinde $\frac{\partial u}{\partial x}$ yaklaşımı için dört nokta kapalı fark problemi öne sürüldü. Başlangıç fonksiyonunun $C^{7+\alpha}$ olduğu, homojen olmayan terimin $C_{x,t}^{5+\alpha,\frac{5+\alpha}{2}}$, sınır fornksiyonlarının ise $C^{\frac{7+\alpha}{2}}$, Hölder uzaylarından olduğu ve bağlayıcı koşulların üçüncü dereceye kadar $(q=0,1,2,3)$ sağlandığı durumda ise altı nokta kapalı fark problemi verildi. Oluşturulan dört nokta ve altı nokta kapalı fark problemlerinin düğüm noktalarında $\frac{\partial u}{\partial x}$ fonksiyonunun gerçek değerine $O(h^2+\tau)$ ve $O(h^2+\tau^2)$ mertebesinden düzgün yakınsadığı isbat edildi ki h, x değişkenindeki adım uzunluğu, τ ise zaman değişkeni t için adım uzunluğudur.

İlaveten, çözümün t değişkenine göre kısmi türevi $\left(\frac{\partial u}{\partial t}\right)$, x değişkenine göre ikinci türevi için sınır problemleri ve kapalı fark problemleri verildi. Ayrıca çözümün ikinci

dereceden karışık türevi $\left(\frac{\partial^2 u}{\partial x \partial t}\right)$ için özel bir fark sınır değer problemi önerildi.

Başlanglç değer fonksiyonun $C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}$ ısı denklemindeki ısı kaynağı fonksiyonunun $C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}$ ve sınır fonksiyonlarının $C^{\frac{8+\alpha}{2}}$, Hölder uzaylarından olduğu, ve başlangıç ile sınır forksiyonları arasında bağlama şartlarının $q = 0,1,2,3,4$ dereceden sağlandığı zaman öne sürülen kapalı fark şemalarının karşılık gelen $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ ve $\frac{\partial^2 u}{\partial x \partial t}$. türevlerine düzgün $O(h^2 + \tau)$ mertebesinden yakınsadığı gösterildi. Diğer taraftan başlangıç değer fonksiyonun $C^{10+\alpha}$, ısı kaynağı fonksiyonunun $C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}$; sınır fonksiyonlarının ise $C^{\frac{10+\alpha}{2}}$ Hölder uzaylarından ve başlangıç ile sınır fonksiyonları arasında $q = 0,1,2,3,4,5$ dereceden bağlayıcı koşulların sağlandığı durumda oluşturulan altı nokta simetrik (Crank-Nicolson) kapalı fark problemleri karşılık gelen $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ ve $\frac{\partial^2 u}{\partial x \partial t}$ türevlerine $O(h^2 + \tau^2)$ mertebesinden düzgün yakınsar.

Son olarak teoretik sonuçları desteklemesi amacı ile birçok sayısal örnekler kuruldu ve elde edilen sonuçlar tablo ve şekiller ile gösterildi.

Anahtar Kelimeler: Sonlu fark metodu, türerlerin yaklaşık hesaplanması, Crank-Nicolson şeması, düzgün hata, ısı denklemi.

DEDICATION

To my Late Father (Pa Francis Ademola Farinola (J.P))

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Chapter 1

INTRODUCTION

1.1 Motivation

Motivation is one of the key elements of learning system and it is the sole factor that has a direct impact on the success of academic. In science, especially in mathematical physics, applied thermal engineering in particular, not only the calculation of the solution of the differential equation but also the calculation of the derivatives of the solution are very important to provide information about some physical phenomena [1]. Some examples are listed below:

1. The first derivatives of the potential function defines the electrostatic field [2].
2. In heat conduction problems involving phase changes such as the problem of melting of a solid when the liquid is removed immediately on formation [3, 4, 5], the accurate calculation of the rate of heat flow into the solid $\frac{\partial u}{\partial x}$, the rate of heat absorption by melting $\frac{\partial s}{\partial t}$ where, $u(x,t)$ is the temperature and $x = s(t)$ is the distance from the initial position of heated face are considerably important.
3. The phenomena of impact of a moving foot on the transfer of heat from a constantly heated warm water into the foot immersed within a footbath [6] and the enhancement of performance by increasing the thermal efficiency of a direct absorption solar collector based on analimino-water nanofluid [7] of which the derivatives of the solution are also essential.

4. In [8], theory of the drying wood adopts the fundamental hypothesis that the rate of which transfusion takes place transversely with respect to the wood fibers $\left(\frac{\partial\theta}{\partial t}\right)$ is proportional to the slope of the moisture gradient $\left(\frac{\partial^2\theta}{\partial x^2}\right)$, where θ is the moisture content expressed as a percentage of the oven-dry weight of the wood. Therefore, accurate approximation of $\frac{\partial\theta}{\partial x}$ is very important to provide information about the moisture gradient.

1.2 Review of literature

To find highly accurate computations of the derivatives of an unknown solution of a differential equation is problematic because the differentiation operation is ill-conditioned. Also, it is well known that accuracy of the approximate derivatives depends on the accuracy of the approximate solution.

The study of approximate derivatives using finite differences was investigated in [9] where, it was proved that the high order difference derivatives uniformly converge to the corresponding derivatives of the exact solution for the two-dimensional Laplace equation in any strictly interior subdomain with the same order $O(h)$ (h is the grid step) of which the difference solution converges on the given domain.

For the Dirichlet problem of the Laplace equation on a rectangle in [10] $O(h^2)$ order of uniform convergence of the solution of the difference equation and its first and pure second difference derivatives to the solution and corresponding derivatives of the exact solution for the two-dimensional Laplace equation was proven over the whole grid domain.

Later, in [11], under the conditions that the boundary functions belong to $C^{6,\lambda}$, $0<\lambda<1$, on the sides of the rectangle and are continuous on the vertices and second, fourth order derivatives satisfy the compatibility conditions on the vertices which results from the Laplace equation, difference schemes are constructed for the first and pure second order derivatives of the solution. It is proved that the order of convergence of the solutions of these difference schemes is $O(h^4)$.

For the three dimensional Laplace equation difference schemes for obtaining the solution of the Dirichlet problem, its first derivatives and second derivatives on a cubic grid with uniform accuracy $O(h^2)$ are constructed in [12] under the agreement that the boundary functions belong to $C^{4,\lambda}$, $0<\lambda<1$, on the faces, are continuous on the edges, and their second order derivatives satisfy the compatibility condition.

In [13] difference schemes for the approximation of the first and pure second derivatives of the solution of the Dirichlet problem in a rectangular parallelepiped which converge uniformly on the cubic grid of order $O(h^4)$. are proposed when the boundary functions belong to $C^{6,\lambda}$, $0<\lambda<1$, on the faces, are continuous on the edges, and have second and fourth order derivatives satisfying the compatibility conditions.

Most recently, in [14] difference schemes on a cubic grid for obtaining the solution of the Dirichlet problem for the 3D Laplace equation on a rectangular parallelepiped, its first and pure second derivatives, difference schemes are constructed and the approximate values of the first and pure second derivatives converge with orders $O(h^6 |\text{In } h|)$ and $O(h^{5+\lambda})$, $0<\lambda<1$, respectively. It is assumed that the boundary functions on the faces have seventh derivatives satisfying the Hölder condition and on the edges their second, fourth and sixth derivatives satisfy the compatibility condition.

At the same time in [15], $O(h^{p-1})$, $p \in [4,5]$ order of approximation for the first order derivatives of the solution of the 3D Laplace equation is proven under a weaker assumption on the smoothness of the boundary functions on the faces of the parallelepiped than those used in [13].

1.3 Basic notations and first type boundary value problem

Based on Section 5, Chapter IV in [16], we give the following definitions. We denote by $A\left(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)$ the linear parabolic differential operator with real coefficients

$$A\left(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u \equiv \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} w_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} w_i(x,t)\frac{\partial u}{\partial x_i} + w(x,t)u. \quad (1.1)$$

Let Ω be a bounded domain in n-dimensional Euclidean space E_n. It is assumed that the coefficients of the operator of (1.1) are defined in a layer $D = E_n \times (0,T)$. In the cylindrical domain $Q = \Omega \times (0,T)$ with lateral surface S_T or more precisely the set of points (x,t) of E_{n+1} with $x = (x_1, x_2, ..., x_n) \in S$, $t \in [0,T]$ where S is the sufficiently smooth boundary of Ω and that $\bar{\Omega} = \Omega \bigcup S$, the first type boundary value problem is given as

$$A\left(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u = f(x,t), \quad (1.2)$$

$$u|_{t=0} = \varphi(x), \quad (1.3)$$

$$u|_{S_T} = \phi(x,t). \quad (1.4)$$

Let q be a non-negative integer. We use the notations

$$u^{(0)}(x) = \varphi(x), \qquad q = 0, \tag{1.5}$$

$$u^{(q)}(x) = \left.\frac{\partial^q u(x,t)}{\partial t^q}\right|_{t=0}, \qquad q = 1, 2, 3, \ldots, \tag{1.6}$$

and the operator

$$\hat{A}\left(x, t, \frac{\partial}{\partial x}\right)u \equiv \sum_{i,j=1}^{n} w_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{n} w_i(x,t)\frac{\partial u}{\partial x_i} - w(x,t)u. \tag{1.7}$$

From (1.2), (1.5) and (1.7), Eq. (1.6) can be rewritten as

$$u^{(1)}(x) = \hat{A}\left(x, 0, \frac{\partial}{\partial x}\right)\varphi(x) + f(x,0), \tag{1.8}$$

$$u^{(q+1)}(x) = \left.\left(\frac{\partial^q}{\partial t^q}\hat{A}\left(x, t, \frac{\partial}{\partial x}\right)u(x,t) + \frac{\partial^q}{\partial t^q} f(x,t)\right)\right|_{t=0}, \quad q = 1, 2, 3, \ldots \tag{1.9}$$

The conjugation (compatibility) conditions up to order $m \geq 0$ are

$$u^{(q)}(x)\,|_{x\in S} = \left.\frac{\partial^q \phi(x,t)}{\partial t^q}\right|_{t=0} = \phi^{(q)}(x), \quad q = 0, 1, \ldots, m. \tag{1.10}$$

Let $\bar{Q}$ and $\bar{S}_T$ be the closure of Q and S_T respectively, and $s > 0$ be a non-integer number. Let $\frac{\partial u}{\partial x} = u_x = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$ and Let D_x^j denote any derivative with respect to x of order j. Further, $C_{x,t}^{s,\frac{s}{2}}(\bar{Q})$ denotes the classical Hölder space of functions $u(x,t)$ that are continuous in $\bar{Q}$ together with all derivatives of the form $D_t^{j_0} D_x^j$ for $2j_0 + j < s$ and have finite norm defined in $C_{x,t}^{s,\frac{s}{2}}(\bar{Q})$. $C^s(\bar{\Omega})$ is the Hölder space whose elements are continuous functions $g(x)$ in $\bar{\Omega}$ having in $\bar{\Omega}$ continuous derivatives up to order [s] inclusively, and have finite norm defined in $C^s(\bar{\Omega})$ (see [16]).

Theorem 1.1: (From Theorem 5.2, Section 5, Chapter IV in [16]) suppose *s > 0* is a non-integer number, the coefficients of the operator A belongs to the class $C_{x,t}^{s,\frac{s}{2}}(\bar{Q})$, and the boundary S belongs to the class C^{s+2}. Then, for any $f \in C_{x,t}^{s,\frac{s}{2}}(\bar{Q})$, $\varphi(x) \in C^{s+2}(\bar{\Omega})$, and $\phi(x,t) \in C_{x,t}^{s+2,\frac{s}{2}+1}(\bar{S}_T)$ satisfying the compatibility conditions (1.10) up to order $\left[\frac{s}{2}\right]+1$, problem (1.2)-(1.4) has a unique solution from the class $C_{x,t}^{s+2,\frac{s}{2}+1}(\bar{Q})$.

1.4 Organization of the chapters

In this thesis, we organize the chapters as follows: In Chapter 2, we propose special difference problems of four point and six point symmetric implicit difference schemes for the derivative of the solution $u(x,t)$ of the first type boundary value problem for one dimensional heat conduction equation of constant coefficient with respect to the spatial variable x. For the construction of the four point implicit difference problem we require that:

a) the initial function belongs to $C^{5+\alpha}$, the nonhomogeneous term given in the heat equation is from $C_{x,t}^{3+\alpha,\frac{3+\alpha}{2}}$, the boundary functions are from $C^{\frac{5+\alpha}{2}}$, and the conjugation conditions of orders $q = 0,1,2$ are satisfied at the corners of the boundary. For the construction of the six point implicit difference problem it is assumed that:

b) The initial function belongs to $C^{7+\alpha}$, the nonhomogeneous term is from $C_{x,t}^{5+\alpha,\frac{5+\alpha}{2}}$, the boundary functions are from $C^{\frac{7+\alpha}{2}}$ and the conjugation conditions of orders $q = 0,1,2,3$ are satisfied.

In Chapter 3, we consider the first type boundary value problem for one dimensional heat equation of which the initial function belongs to $C^{8+\alpha}$ $0 < \alpha < 1$, the heat source function is from $C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}$, the boundary functions are from $C^{\frac{8+\alpha}{2}}$, and between the initial and the boundary functions the conjugation conditions of orders $q = 0,1,2,3,4$ are satisfied. Denoting the exact solution of this problem by $u(x,t)$, difference problems of four point implicit schemes approximating $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}$, and $\frac{\partial^2 u}{\partial x \partial t}$ are constructed. It is obtained that the solution of the constructed difference schemes converge uniformly to the exact values of $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}$, and $\frac{\partial^2 u}{\partial x \partial t}$ respectively, on the grids of order $O(h^2 + \tau)$.

In Chapter 4, we continue the extension of the method given in Chapter 2 of this research and in [17] to find the first difference derivative of $u(x,t)$ with respect to t and its second order difference derivatives with $O(h^2 + \tau^2)$ order of convergence to the corresponding exact derivatives. Here, the initial function belongs to $C^{10+\alpha}$, the heat source function is from $C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}$, the boundary functions are from $C^{\frac{10+\alpha}{2}}$, and between the initial and the boundary functions the conjugation conditions of orders

$q = 0,1,2,3,4,5$ are satisfied. The general idea of this research work is presented as an extended abstract in [18]. In Chapter 5, the concluding remarks are given.

Chapter 2

IMPLICIT METHODS FOR THE FIRST DERIVATIVE OF THE SOLUTION OF ONE-DIMENSIONAL HEAT EQUATION WITH RESPECT TO SPATIAL VARIABLE

2.1 Chapter overview

The work of this chapter is organized as follows: In Section 2, for the approximate solution of the first type boundary value problem for one dimensional heat conduction equation with constant coefficients, we use four point implicit or six point symmetric implicit schemes [19] under the assumption that the boundary value problem satisfies the conditions (a) or (b) respectively, (Chapter 1, Section 1.4). In both cases for the error function we provide a pointwise prior estimation depending on $\rho(x,t)$, which is the distance from the current grid point in the domain to the boundary. In Section 3, we consider the boundary value problem satisfying the conditions (a) and propose a special four point implicit difference problem for the approximation of $\frac{\partial u}{\partial x}$. We prove that the solution of the constructed difference scheme converges uniformly to the exact value of $\frac{\partial u}{\partial x}$ on the grids of order $O(h^2+\tau)$. In Section 4, we require that the boundary value problem satisfies the conditions (b) hence, a special six point implicit difference problem for the approximation of $\frac{\partial u}{\partial x}$ is proposed. Uniform convergence of order $O(h^2+\tau^2)$ for this scheme is shown. Section 5, justifies the theoretical results using numerical examples and the obtained results were presented via tables and figures.

2.2 Implicit difference solution of first type boundary value problem for one dimensional heat equation

Take $\Omega = (0,b)$, $\sigma_T = (0,T)$, and $\bar{\Omega}$, $\bar{\sigma}_T$ are the closure of these sets respectively, also $Q_T = \{(x,t): 0 < x < b, 0 < t \le T\}$, $\gamma_1 = \{(0,t): t \in \bar{\sigma}_T\}$, $\gamma_2 = \{(x,0): x \in \bar{\Omega}\}$, and $\gamma_3 = \{(b,t): t \in \bar{\sigma}_T\}$. Let $\gamma = \bigcup_{i=1}^{3} \gamma_i$ represent the boundary of Q_T, and $\bar{Q}_T = Q_T \cup \gamma$. We use the notations $\partial_t^k = \dfrac{\partial^k}{\partial t^k}$, $\partial_x^k = \dfrac{\partial^k}{\partial x^k}$ and $D_t^k = \dfrac{d^k}{dt^k}$, $D_x^k = \dfrac{d^k}{dx^k}$ to present the *kth* partial and ordinary derivatives respectively with respect to time variable *t*, spatial variable *x*. We consider the first type boundary value problem for a one dimensional heat equation:

$$Lu = f(x,t) \quad \text{on} \quad Q_T, \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{on} \quad \gamma_2, \tag{2.2}$$

$$u(0,t) = u_1(t) \quad \text{on} \quad \gamma_1, \ u(b,t) = u_2(t) \quad \text{on} \quad \gamma_3, \tag{2.3}$$

where $L \equiv \dfrac{\partial}{\partial t} - a\dfrac{\partial^2}{\partial x^2}$ and a is positive constant. The conjugation conditions (1.5), (1.8) and (1.9) are

$$u^{(0)}(x) = u_0(x), \tag{2.4}$$

$$u^{(1)}(x) = aD_x^2 u_0(x) + f(x,0), \tag{2.5}$$

$$u^{(q)}(x) = a\partial_x^2 u^{(q-1)}(x) + f^{(q-1)}(x), \ q = 2,3,..., \tag{2.6}$$

respectively, where $f^{(0)}(x) = f(x,0)$ and $f^{(q)}(x) = \partial_t^q f(x,t)|_{t=0}$. Also

$$u_1^{(0)}(0) = u_1(t)|_{t=0}, \quad u_2^{(0)}(0) = u_2(t)|_{t=0} \tag{2.7}$$

$$u_1^{(q)}(0) = D_t^q u_1(t)|_{t=0}, \quad u_2^{(q)}(0) = D_t^q u_2(t)|_{t=0}, \quad q = 1, 2, \ldots \tag{2.8}$$

Furthermore, the conjugation conditions up to order $m \geq 0$ in (1.10) for the one dimensional heat problem (2.1) – (2.3) are derived as

$$u^{(q)}(0) = u_1^{(q)}(0), \quad u^{(q)}(b) = u_2^{(q)}(0), \quad q = 0, 1, \ldots, m. \tag{2.9}$$

Problem 1: Let $\alpha \in (0,1)$

(i) The boundary value problem (2.1) – (2.3) satisfying the conditions

$$u_0(x) \in C^{5+\alpha}(\bar{\Omega}), \quad f(x,t) \in C_{x,t}^{3+\alpha, \frac{3+\alpha}{2}}(\bar{Q}_T) \text{ and } u_j(t) \in C^{\frac{5+\alpha}{2}}(\bar{\sigma}_T), \ j = 1, 2. \tag{2.10}$$

and the conjugation conditions (2.9) up to second order $(q = 0, 1, 2)$.

(ii) The boundary value problem (2.1) – (2.3) satisfying the conditions

$$u_0(x) \in C^{7+\alpha}(\bar{\Omega}), \quad f(x,t) \in C_{x,t}^{5+\alpha, \frac{5+\alpha}{2}}(\bar{Q}_T) \text{ and } u_j(t) \in C^{\frac{7+\alpha}{2}}(\bar{\sigma}_T), \ j = 1, 2. \tag{2.11}$$

and the conjugation conditions (2.9) up to third order $(q = 0, 1, 2, 3)$.

Theorem 2.1: [17] Problem 1(i) has a unique solution $u(x,t)$ belonging to the class $C_{x,t}^{5+\alpha, \frac{5+\alpha}{2}}(\bar{Q}_T)$. The Problem 1(ii) has a unique solution $u(x,t)$ belonging to the class $C_{x,t}^{7+\alpha, \frac{7+\alpha}{2}}(\bar{Q}_T)$.

Proof: The proof of Theorem 2.1 follows from Theorem 1.1. ■

We define

$$\bar{\omega}_h = \left\{ x_m = mh, h = \frac{b}{N}, m = 0, \ldots, N \right\}, \tag{2.12}$$

$$\omega_\tau = \left\{ t_j = j\tau, \tau = \frac{T}{M}, j = 0, \ldots, M \right\}, \tag{2.13}$$

and $\bar{\omega}_{h,\tau} = \bar{\omega}_h \times \omega_\tau$ where, the set of internal nodes are defined by

$$\omega_{h,\tau} = \omega_h \times \omega_\tau = \left\{(x_m, t_j) : m = 1, \ldots, N-1,\ j = 1, \ldots, M\right\}. \tag{2.14}$$

The set of nodes on $\gamma_i,\ i = 1, 2, 3$ are presented by

$$\omega_{0,\tau} = \left\{(0, t_j) : t_j = j\tau, \tau = \frac{T}{M}, j = 0, \ldots, M\right\}, \tag{2.15}$$

$$\bar{\omega}_{h,0} = \left\{(x_m, 0) : x_m = mh, h = \frac{b}{N}, m = 0, \ldots, N\right\}, \tag{2.16}$$

$$\omega_{b,\tau} = \left\{(b, t_j) : t_j = j\tau, \tau = \frac{T}{M}, j = 0, \ldots, M\right\}, \tag{2.17}$$

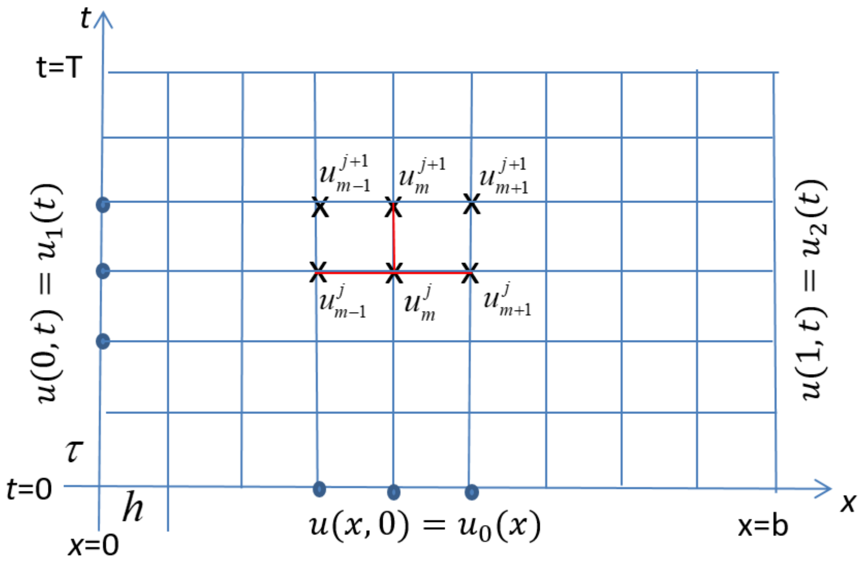

Figure 2.1: Six Point Difference Scheme

respectively. Assume that $c_1, c_2, \ldots$ are positive constants independent from h and τ; in each section, those constants are enumerated anew. For the numerical solution of the Problem 1(i), we use the four point difference problem $(\varsigma = 3)$ and for the numerical solution of the Problem 1(ii), we use the six point symmetric difference problem $(\varsigma = 6)$ [19]. We denote the solution of these difference problems by $\tilde{u}$ and use the notations $\tilde{u}_m^0 = \tilde{u}(x_m, 0)$ on $\bar{\omega}_{h,0}$, $\tilde{u}_0^j = \tilde{u}(0, t_j)$ on $\omega_{0,\tau}$, and $\tilde{u}_N^j = \tilde{u}(b, t_j)$ on $\omega_{b,\tau}$. The difference schemes are as follows:

$$\tilde{u}_{\bar{t},m}^{h,\tau} = a\Theta^{\varsigma}\tilde{u}_m^{h,\tau} + \Phi f^{h,\tau} \text{ on } \omega_{h,\tau}, \quad \varsigma = 3 \text{ or } \varsigma = 6, \tag{2.18}$$

$$\tilde{u}_m^0 = u_0(x_m) \text{ on } \bar{\omega}_{h,0}, \tag{2.19}$$

$$\tilde{u}_0^j = u_1(t_j) \text{ on } \omega_{0,\tau}, \quad \tilde{u}_N^j = u_2(t_j) \text{ on } \omega_{b,\tau}, \tag{2.20}$$

where,

$$\tilde{u}_{\bar{t},m}^{h,\tau} = \frac{\tilde{u}_m^{j+1} - \tilde{u}_m^j}{\tau}, \tag{2.21}$$

$$\Theta^3\tilde{u}_m^{h,\tau} = \frac{\tilde{u}_{m-1}^{j+1} - 2\tilde{u}_m^{j+1} + \tilde{u}_{m+1}^{j+1}}{h^2}, \tag{2.22}$$

$$\Theta^6\tilde{u}_m^{h,\tau} = \frac{1}{2}\left(\frac{\tilde{u}_{m-1}^{j+1} - 2\tilde{u}_m^{j+1} + \tilde{u}_{m+1}^{j+1}}{h^2} + \frac{\tilde{u}_{m-1}^{j} - 2\tilde{u}_m^{j} + \tilde{u}_{m+1}^{j}}{h^2}\right), \tag{2.23}$$

$$\Phi f^{h,\tau} = \begin{cases} f = f\left(x_m, t_{j+1}\right), & \text{if } \varsigma = 3, \\ \bar{f} = f\left(x_m, t_{j+\frac{1}{2}}\right), & \text{if } \varsigma = 6, \end{cases} \tag{2.24}$$

The operator $\Theta^3\tilde{u}_m^{h,\tau}$ is the central difference formula and $\Theta^6\tilde{u}_m^{h,\tau}$ is the averaging central difference formula with three points and six points respectively, for approximating $\partial_x^2 u$. Here $t_{j+\frac{1}{2}} = t_j + 0.5\tau$, $f(x,t)$ is the given function in (2.1) and $u_0(x)$ given in (2.2), $u_1(t)$, $u_2(t)$ given in (2.3) are the initial and boundary functions, respectively.

Consider the following systems:

$$\hat{q}_{\bar{t},m}^{h,\tau} = a\Theta^{\varsigma}\hat{q}_m^{h,\tau} + \hat{g}^{h,\tau} \text{ on } \omega_{h,\tau}, \quad \varsigma = 3 \text{ or } \varsigma = 6, \tag{2.25}$$

$$\hat{q}_m^0 = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.26}$$

$$\hat{q}_0^j = 0 \text{ on } \omega_{0,\tau}, \quad \hat{q}_N^j = 0 \text{ on } \omega_{b,\tau}, \tag{2.27}$$

$$\overline{q}_{\bar{t},m}^{h,\tau} = a\Theta^{\varsigma}\overline{q}_m^{h,\tau} + \overline{g}^{h,\tau} \quad \text{on} \quad \omega_{h,\tau}, \quad \varsigma = 3 \ \text{ or } \ \varsigma = 6, \tag{2.28}$$

$$\overline{q}_m^0 \geq 0 \quad \text{on} \quad \overline{\omega}_{h,0}, \tag{2.29}$$

$$\overline{q}_0^j \geq 0 \quad \text{on} \quad \omega_{0,\tau}, \quad \overline{q}_N^j \geq 0 \quad \text{on} \quad \omega_{b,\tau}, \tag{2.30}$$

where $\hat{g}^{h,\tau}$, $\overline{g}^{h,\tau}$ are given functions and $|\hat{g}^{h,\tau}| \leq \overline{g}^{h,\tau}$ on $\omega_{h,\tau}$ also $\hat{q}_{\bar{t},m}^{h,\tau}$, $\overline{q}_{\bar{t},m}^{h,\tau}$ are difference formulae analogous to (2.21) and $\Theta^{\varsigma}\hat{q}_m^{h,\tau}$, $\Theta^{\varsigma}\overline{q}_m^{h,\tau}$ are difference formulae analogous to (2.22) or (2.23) for $\varsigma = 3$ or $\varsigma = 6$, respectively.

Lemma 2.2: [17] The solution $\hat{q}$ of the system (2.25) – (2.27) and the solution $\overline{q}$ of the system (2.28) – (2.30) satisfy the inequality

$$|\hat{q}| \leq \overline{q} \quad \text{on } \overline{\omega}_{h,\tau} \tag{2.31}$$

for any r by the four point implicit scheme $(\varsigma = 3)$ and for $r \leq 1$, by the six point symmetric implicit scheme $(\varsigma = 6)$ where $r = \dfrac{a\tau}{h^2}$.

Proof: Taking into consideration that the canonical form of the equation $\hat{q}_{\bar{t},m}^{h,\tau} = a\Theta^3\hat{q}_m^{h,\tau} + \hat{g}^{h,\tau}$ is

$$\left(\frac{1}{\tau} + \frac{2a}{h^2}\right)\hat{q}_m^{j+1} = a\left[\frac{\hat{q}_{m-1}^{j+1} + \hat{q}_{m+1}^{j+1}}{h^2}\right] + \frac{1}{\tau}\hat{q}_m^j + \hat{g}^{h,\tau} \tag{2.32}$$

in the form $A(P)\hat{q}(P) = \sum_{Q \in Patt(P)} B(P,Q)\hat{q}(Q) + F(P)$ where $P = P(x_m, t_{j+1})$ as a node of the grid $\omega_{h,\tau}$ and *Patt(P)* consists of the nodes $Q_1 = (x_m, t_j)$, $Q_2 = (x_{m-1}, t_{j+1})$, $Q_3 = (x_{m+1}, t_{j+1}) \in \overline{\omega}_{h,\tau}$. It can be easily seen that *A(P) > 0, B(P, Q) > 0* for every $Q \in Patt(P)$ and *D(P) = 0* where $D(P) = A(P) - \sum_{Q \in Patt(P)} B(P,Q)$. Similarly the canonical form of the equation $\hat{q}_{\bar{t},m}^{h,\tau} = a\Theta^6\hat{q}_m^{h,\tau} + \hat{g}^{h,\tau}$ is

$$\left(\frac{1}{\tau}+\frac{a}{h^2}\right)\hat{q}_m^{j+1}=a\left[\frac{\hat{q}_{m-1}^{j+1}+\hat{q}_{m+1}^{j+1}}{2h^2}\right]+a\left[\frac{\hat{q}_{m-1}^{j}+\hat{q}_{m+1}^{j}}{2h^2}\right]+\left(\frac{1}{\tau}-\frac{a}{h^2}\right)\hat{q}_m^{j}+\hat{g}^{h,\tau}, \qquad (2.33)$$

where $P=P(x_m,t_{j+1})$ and *Patt(P)* consists of the nodes $Q_1=(x_m,t_j)$, $Q_2=(x_{m-1},t_{j+1})$, $Q_3=(x_{m+1},t_{j+1})$, $Q_4=(x_{m-1},t_j)$, $Q_5=(x_{m+1},t_j)$. Here *A(P) > 0, D(P) = 0* and *B(P, Q) ≥ 0* for every $Q\in Patt(P)$ if $r=\frac{a\tau}{h^2}\le 1$. The proof follows from the Comparison Theorem (see Chapter 4 in [19]) because the coefficients of the finite difference schemes (2.32) and (2.33) satisfy all conditions of the comparison theorem for any *r* and for $r\le 1$, respectively. ■

Lemma 2.3: [17] For the solution of the problem

$$\hat{q}_{\bar{t},m}^{h,\tau}=a\Theta^{\varsigma}\hat{q}_m^{h,\tau}+\beta \text{ on } \omega_{h,\tau},\quad \varsigma=3 \text{ or } \varsigma=6, \qquad (2.34)$$

$$\hat{q}_m^0=0 \text{ on } \bar{\omega}_{h,0}, \qquad (2.35)$$

$$\hat{q}_0^j=0 \text{ on } \omega_{0,\tau},\quad \hat{q}_N^j=0 \text{ on } \omega_{b,\tau}, \qquad (2.36)$$

the following inequality holds true:

$$\hat{q}\le \rho d\beta \text{ on } \bar{\omega}_{h,\tau} \qquad (2.37)$$

where

$$\beta=\beta(h,\tau)=\begin{cases} h^2+\tau & \text{for } \varsigma=3,\\ h^2+\tau^2 & \text{for } \varsigma=6.\end{cases} \qquad (2.38)$$

$$d=\max\left[\frac{b}{2a},1\right], \qquad (2.39)$$

for any *r* by the four point implicit scheme $(\varsigma=3)$ and for $r\le 1$, by the symmetric six point implicit scheme $(\varsigma=6)$. Here, $\rho=\rho(x,t)$ is the distance from the current point $(x,t)\in\bar{\omega}_{h,\tau}$ to the boundary γ of Q_T.

Proof: For the four point implicit scheme $(\varsigma = 3)$, we consider the functions

$$\bar{q}_1^3(x,t) = \frac{1}{2}(h^2+\tau)\left(\frac{bx-x^2}{a}\right) \geq 0, \qquad \bar{q}_2^3(x,t) = (h^2+\tau)t \geq 0 \quad \text{on} \quad \bar{\omega}_{h,\tau} \tag{2.40}$$

which are the solutions of $q_{\bar{t},m}^{h,\tau} = a\Theta^3 q_m^{h,\tau} + h^2 + \tau$ on $\omega_{h,\tau}$. On the basis of Lemma 2.2 we obtain

$$\hat{q} \leq \min_{i=1,2} \bar{q}_i^3(x,t) \leq \rho d(h^2+\tau) \quad \text{on} \quad \bar{\omega}_{h,\tau}\,. \tag{2.41}$$

For the six point symmetric implicit scheme $(\varsigma = 6)$, we consider the functions

$$\bar{q}_1^6(x,t) = \frac{1}{2}(h^2+\tau^2)\left(\frac{bx-x^2}{a}\right) \geq 0, \qquad \bar{q}_2^6(x,t) = (h^2+\tau^2)t \geq 0 \quad \text{on} \quad \bar{\omega}_{h,\tau}\,, \tag{2.42}$$

which are the solutions of $q_{\bar{t},m}^{h,\tau} = a\Theta^6 q_m^{h,\tau} + h^2 + \tau^2$ on $\omega_{h,\tau}$. Using Lemma 2.2 we obtain

$$\hat{q} \leq \min_{i=1,2} \bar{q}_i^6(x,t) \leq \rho d(h^2+\tau^2) \quad \text{on} \quad \bar{\omega}_{h,\tau} \tag{2.43}$$

Theorem 2.4: [17] The solution $\tilde{u}$ of the four point finite difference problem (2.18) – (2.20) $(\varsigma = 3)$ satisfies the following pointwise estimation:

$$|\tilde{u} - u| \leq c_1 \rho (h^2+\tau), \tag{2.44}$$

for any value of $r = \dfrac{a\tau}{h^2} > 0$ where *u* is the exact solution of Problem 1(i). The solution $\tilde{u}$ of the six point finite difference problem (2.18) – (2.20) $(\varsigma = 6)$ satisfies the following pointwise estimation:

$$|\tilde{u} - u| \leq c_2 \rho (h^2+\tau^2), \tag{2.45}$$

for $r \leq 1$ where *u* is the exact solution of Problem 1(ii).

Proof: On the basis of Theorem 2.1, the exact solution *u* of Problem 1(i) belongs to $C_{x,t}^{5+\alpha,\frac{5+\alpha}{2}}(\bar{Q}_T)$. Therefore, $\partial_x^4 u$ and $\partial_t^2 u$ are bounded up to the boundary. Let $\varepsilon_u^{h,\tau} = \tilde{u} - u$ on $\bar{\omega}_{h,\tau}$. Obviously the error function $\varepsilon_u^{h,\tau}$ satisfies

$$\varepsilon_{u,\bar{t},m}^{h,\tau} = a\Theta^3 \varepsilon_{u,m}^{h,\tau} + \psi_u \text{ on } \omega_{h,\tau}, \tag{2.46}$$

$$\varepsilon_{u,m}^{0} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.47}$$

$$\varepsilon_{u,0}^{j} = 0 \text{ on } \omega_{0,\tau}, \quad \varepsilon_{u,N}^{j} = 0 \text{ on } \omega_{b,\tau}, \tag{2.48}$$

where $\psi_u = a\Theta^3 u - u_{\bar{t},m} + \Phi f^{h,\tau}$. Using Taylor's formula for the function $u(x,t)$ about the node (x_m, t_{j+1}) shows that $\psi_u = O(h^2 + \tau)$ and applying Lemma 2.2 to the problem (2.34) – (2.36) for $\varsigma = 3$, (2.46) – (2.48) and on the basis of Lemma 2.3 we obtain $|\varepsilon_u^{h,\tau}| \le c_1 \rho (h^2 + \tau)$. From Theorem 2.1, the exact solution *u* of Problem 1(ii) belongs to $C_{x,t}^{7+\alpha,\frac{7+\alpha}{2}}(\bar{Q}_T)$. Hence, the derivatives $\partial_x^4 u$, $\partial_t^3 u$ are bounded up to the boundary. The error function $\varepsilon_u^{h,\tau}$ satisfies the following difference problem:

$$\varepsilon_{u,\bar{t},m}^{h,\tau} = a\Theta^6 \varepsilon_{u,m}^{h,\tau} + \psi_u \text{ on } \omega_{h,\tau}, \tag{2.49}$$

$$\varepsilon_{u,m}^{0} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.50}$$

$$\varepsilon_{u,0}^{j} = 0 \text{ on } \omega_{0,\tau}, \quad \varepsilon_{u,N}^{j} = 0 \text{ on } \omega_{b,\tau}, \tag{2.51}$$

Using Taylor's formula for the function $u(x,t)$ about the node $(x_m, t_{j+\frac{1}{2}})$ shows that $\psi_u = O(h^2 + \tau^2)$. Applying Lemma 2.2 to the six point implicit difference problem (2.34) – (2.36) for $(\varsigma = 6)$ and (2.49) – (2.51) and on the basis of Lemma 2.3 we obtain $|\varepsilon_u^{h,\tau}| \le c_2 \rho (h^2 + \tau^2)$. ■

2.3 Implicit four point difference approximation of $\partial_x u$

Problem 2:

(i) Given the Problem 1(i), we denote $p_i = \partial_x u$ on γ_i, $i = 1,2,3$ and set up the next boundary value problem for $v = \partial_x u$,

$$Lv = \partial_x f(x,t) \quad \text{on} \quad Q_T \tag{2.52}$$

$$v(x,0) = p_2 \quad \text{on} \quad \gamma_2 \tag{2.53}$$

$$v(0,t) = p_1 \quad \text{on} \quad \gamma_1, \; v(b,t) = p_3 \quad \text{on} \quad \gamma_3, \tag{2.54}$$

where $f(x,t)$ is the given function in (2.1). We take

$$p_{1h} = \frac{-3u_1(t) + 4\tilde{u}(h,t) - \tilde{u}(2h,t)}{2h} \quad \text{on} \quad \omega_{0,\tau}, \tag{2.55}$$

$$p_{2h} = \partial_x u_0(x) \quad \text{on} \quad \overline{\omega}_{h,0}, \tag{2.56}$$

$$p_{3h} = \frac{3u_2(t) - 4\tilde{u}(b-h,t) + \tilde{u}(b-2h,t)}{2h} \quad \text{on} \quad \omega_{b,\tau}, \tag{2.57}$$

and $u_0(x)$ given in (2.2), $u_1(t), u_2(t)$ given in (2.3) are the initial and boundary functions, respectively, $\tilde{u}$ is the solution of the four point difference problem (2.18) – (2.20) $(\varsigma = 3)$.

Lemma 2.5: [17] The following inequality holds:

$$| p_{ih}(\tilde{u}) - p_{ih}(u) | \le c_1(h^2 + \tau), \quad i = 1,3. \tag{2.58}$$

where u is the solution of the differential Problem 1(i) and $\tilde{u}$ is the solution of the four point difference problem (2.18) – (2.20) $(\varsigma = 3)$.

Proof: Taking into consideration Theorem 2.1, and using (2.55) and (2.57) and Theorem 2.4, we have

$$|\, p_{ih}(\tilde{u}) - p_{ih}(u)\,| \le \frac{1}{2h}\left[4(c_2 h)(h^2+\tau) + (c_2 2h)(h^2+\tau)\right] \le c_1(h^2+\tau),\ i = 1, 3. \tag{2.59}$$

Lemma 2.6: [17] The following inequality is true:

$$\max_{\omega_{0,\tau}\cup\omega_{b,\tau}} |\, p_{ih}(\tilde{u}) - p_i\,| \le c_3(h^2+\tau), \quad i = 1, 3. \tag{2.60}$$

where, $\tilde{u}$ is the solution of the four point difference problem (2.18) – (2.20) $(\varsigma = 3)$.

Proof: On the basis of Theorem 2.1, the exact solution $C_{x,t}^{5+\alpha,\frac{5+\alpha}{2}}(\bar{Q}_T)$. Then at the end points $(0,\sigma\tau)\in\omega_{0,\tau}$ and $(b,\sigma\tau)\in\omega_{b,\tau}$ of each line segment $\left[(x,t): 0\le x\le b, 0<t\le T\right]$ (2.55) and (2.57) give the second order approximation of $\partial_x u$, respectively. From the truncation error formula (see [20]) it follows that

$$\max_{\omega_{0,\tau}\cup\omega_{b,\tau}} |\, p_{ih}(u) - p_i\,| \le \frac{h^2}{3}\max_{\bar{Q}_T}|\,\partial_x^3 u\,| \le c_3 h^2, \quad i = 1, 3. \tag{2.61}$$

Using Lemma 2.5 and the estimation (2.56), (2.61) follows. ■

We construct the following difference problem for the numerical solution of Problem 2(i):

$$\tilde{v}_{\bar{t},m}^{h,\tau} = a\Theta^3\tilde{v}_m^{h,\tau} + \Phi\partial_x f^{h,\tau} \text{ on } \omega_{h,\tau}, \tag{2.62}$$

$$\tilde{v}_m^0 = p_{2h} \text{ on } \bar{\omega}_{h,0}, \tag{2.63}$$

$$\tilde{v}_0^j = p_{1h}(\tilde{u}) \text{ on } \omega_{0,\tau}, \quad \tilde{v}_N^j = p_{3h}(\tilde{u}) \text{ on } \omega_{b,\tau}, \tag{2.64}$$

where $\tilde{v}^{h,\tau}_{\bar{t},m}$ is analogue to (2.21) and $\Theta^3\tilde{v}^{h,\tau}_m$ is analogue to (2.22) using $\tilde{v}$ instead of $\tilde{u}$ and the p_{ih} are defined by (2.55) – (2.57) and $\Phi\partial_x f^{h,\tau} = \partial_x f\,|_{(x_m,t_{j+1})}$ and $\tilde{u}$ is the solution of the four point difference problem (2.18) – (2.20) $(\varsigma = 3)$.

Theorem 2.7: [17] The solution $\tilde{v}$ of the finite difference problem (2.62) – (2.64) satisfies

$$\max_{\bar{\omega}_{h,\tau}} |\tilde{v} - v| \le c_4(h^2 + \tau), \tag{2.65}$$

where $v = \partial_x u$ is the exact solution of Problem 2(i).

Proof: Let

$$\varepsilon_v^{h,\tau} = \tilde{v} - v \quad \text{on} \quad \bar{\omega}_{h,\tau}, \tag{2.66}$$

where $v = \partial_x u.$ Denote by $\left\|\varepsilon_v^{h,\tau}\right\| = \max_{\bar{\omega}_{h,\tau}}\left|\tilde{v} - v\right|.$ From (2.62) – (2.64) and (2.66) we have

$$\varepsilon^{h,\tau}_{v,\bar{t},m} = a\Theta^3\varepsilon^{h,\tau}_{v,m} + \psi_v \ \text{ on } \ \omega_{h,\tau}, \tag{2.67}$$

$$\varepsilon^0_{v,m} = 0 \ \text{ on } \ \bar{\omega}_{h,0}, \tag{2.68}$$

$$\varepsilon^j_{v,0} = p_{1h}(\tilde{u}) - v \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon^j_{v,N} = p_{3h}(\tilde{u}) - v \quad \text{on} \quad \omega_{b,\tau}, \tag{2.69}$$

where $\psi_v = a\Theta^3 v - v_{\bar{t},m} + \Phi\partial_x f^{h,\tau}.$ We take

$$\varepsilon_v^{h,\tau} = \varepsilon_v^{1,h,\tau} + \varepsilon_v^{2,h,\tau}, \tag{2.70}$$

and $\varepsilon_v^{1,h,\tau}$, $\varepsilon_v^{2,h,\tau}$ satisfy the problems

$$\varepsilon^{1,h,\tau}_{v,\bar{t},m} = a\Theta^3 \varepsilon^{1,h,\tau}_{v,m} \text{ on } \omega_{h,\tau}, \tag{2.71}$$

$$\varepsilon^{1,0}_{v,m} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.72}$$

$$\varepsilon^{1,j}_{v,0} = p_{1h}(\tilde{u}) - v \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon^{1,j}_{v,N} = p_{3h}(\tilde{u}) - v \quad \text{on} \quad \omega_{b,\tau}, \tag{2.73}$$

and

$$\varepsilon^{2,h,\tau}_{v,\bar{t},m} = a\Theta^3 \varepsilon^{2,h,\tau}_{v,m} + \psi_v \text{ on } \omega_{h,\tau}, \tag{2.74}$$

$$\varepsilon^{2,0}_{v,m} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.75}$$

$$\varepsilon^{2,j}_{v,0} = 0 \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon^{2,j}_{v,N} = 0 \quad \text{on} \quad \omega_{b,\tau}, \tag{2.76}$$

respectively. From Lemma 2.6 and by maximum principle for the solution of the problem (2.71) – (2.73) we have

$$\max_{\bar{\omega}_{h,\tau}} \left| \varepsilon^{1,h,\tau}_v \right| \le \max_{i=1,3} \max_{\bar{\omega}_{h,\tau}} \left| p_{ih}(\tilde{u}) - v \right| \le c_4 (h^2 + \tau). \tag{2.77}$$

The solution $\varepsilon^{2,h,\tau}_v$ of the problem (2.74) – (2.76) is the error of the approximate solution obtained by the finite difference method for the boundary value Problem 2(i) when the boundary values satisfy the conditions

$$p_2 \in C^{4+\alpha}(\bar{\Omega}), \quad \partial_x f(x,t) \in C^{2+\alpha, \frac{2+\alpha}{2}}_{x,t}(\bar{Q}_T), \quad p_j \in C^{\frac{4+\alpha}{2}}(\bar{\sigma}_T), \; j = 1,3. \tag{2.78}$$

$$\begin{cases} p_1^{(q)}(0) = v^{(q)}(0), \\ p_3^{(q)}(0) = v^{(q)}(b), \qquad q = 0,1,2. \end{cases} \tag{2.79}$$

Since the function $v = \partial_x u$ satisfies Eq. (2.52) with the initial function p_2 on γ_2 and boundary functions p_1, p_3 on γ_1 and γ_3, respectively, and on the basis of Theorem 1.1 and the maximum principle, we obtain

$$\max_{\bar{\omega}_{h,\tau}} \left| \varepsilon^{2,h,\tau}_v \right| \le c_5 (h^2 + \tau), \tag{2.80}$$

and using (2.70), (2.77) and (2.80) we obtain (2.65). ■

2.4 Implicit six point symmetric difference approximation of $\partial_x u$

Problem 2:

(ii) Given the Problem 1(ii), we denote $p_i = \partial_x u$ on γ_i, $i = 1, 2, 3$ and set up the boundary value problem (2.52) – (2.54) for $v = \partial_x u$.

Lemma 2.8: [17] The following inequality holds:

$$| p_{ih}(\tilde{u}) - p_{ih}(u) | \leq c_1(h^2 + \tau^2), \quad i = 1, 3. \tag{2.81}$$

where u is the solution of the differential Problem 1(ii) and $\tilde{u}$ is the solution of the symmetric six point difference problem (2.18) – (2.20) $(\varsigma = 6)$ for $r \leq 1$ and p_{ih} are defined by (2.55) – (2.57).

Proof: On the basis of Theorem 2.1, and from (2.55), (2.57) and using Theorem 2.4, we have

$$| p_{ih}(\tilde{u}) - p_{ih}(u) | \leq \frac{1}{2h}\left[4(c_2 h)(h^2 + \tau^2) + (c_2 2h)(h^2 + \tau^2)\right] \leq c_1(h^2 + \tau^2), \; i = 1, 3. \tag{2.82}$$

Lemma 2.9: [17] The following inequality is true:

$$\max_{\omega_{0,\tau} \cup \omega_{b,\tau}} | p_{ih}(\tilde{u}) - p_i | \leq c_3(h^2 + \tau^2), \quad i = 1, 3. \tag{2.83}$$

where $\tilde{u}$ is the solution of the six point difference problem (2.18) – (2.20) $(\varsigma = 6)$ for $r \leq 1$.

Proof: Using Theorem 2.1, the proof is analogous to the proof of Lemma 2.6. ■

We propose the following six point difference problem for the numerical solution of Problem 2(ii):

$$\tilde{v}^{h,\tau}_{\bar{t},m} = a\Theta^6 \tilde{v}^{h,\tau}_m + \Phi\partial_x f^{h,\tau} \text{ on } \omega_{h,\tau}, \tag{2.84}$$

$$\tilde{v}^0_m = p_{2h} \text{ on } \bar{\omega}_{h,0}, \tag{2.85}$$

$$\tilde{v}^j_0 = p_{1h}(\tilde{u}) \text{ on } \omega_{0,\tau}, \quad \tilde{v}^j_N = p_{3h}(\tilde{u}) \text{ on } \omega_{b,\tau}, \tag{2.86}$$

where p_{ih} are defined by (2.55) – (2.57) and $\Phi\partial_x f^{h,\tau} = \partial_x f\,|_{(x_m, t_{j+\frac{1}{2}})}$ and $\tilde{u}$ is the solution of the six point difference problem (2.18) – (2.20) $(\varsigma = 6)$ for $r \le 1$.

Theorem 2.10: [17] For $r \le 1$, the solution $\tilde{v}$ of the finite difference problem (2.84) – (2.86) satisfies

$$\max_{\bar{\omega}_{h,\tau}} |\tilde{v} - v| \le c_4(h^2 + \tau^2), \tag{2.87}$$

where, $v = \partial_x u$ is the exact solution of Problem 2(ii).

Proof: The proof is analogous to the proof of Theorem 2.7. From (2.84) – (2.86) and (2.66) we have

$$\varepsilon^{h,\tau}_{v,\bar{t},m} = a\Theta^6 \varepsilon^{h,\tau}_{v,m} + \psi_v \text{ on } \omega_{h,\tau}, \tag{2.88}$$

$$\varepsilon^0_{v,m} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.89}$$

$$\varepsilon^j_{v,0} = p_{1h}(\tilde{u}) - v \text{ on } \omega_{0,\tau}, \quad \varepsilon^j_{v,N} = p_{3h}(\tilde{u}) - v \text{ on } \omega_{b,\tau}, \tag{2.90}$$

where $\psi_v = a\Theta^6 v - v_{\bar{t},m} + \Phi\partial_x f^{h,\tau}$. We take

$$\varepsilon^{h,\tau}_v = \varepsilon^{1,h,\tau}_v + \varepsilon^{2,h,\tau}_v, \tag{2.91}$$

and $\varepsilon^{1,h,\tau}_v$, $\varepsilon^{2,h,\tau}_v$ satisfy the problems

$$\varepsilon^{1,h,\tau}_{v,\bar{t},m} = a\Theta^6 \varepsilon^{1,h,\tau}_{v,m} \text{ on } \omega_{h,\tau}, \tag{2.92}$$

$$\varepsilon^{1,0}_{v,m} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.93}$$

$$\varepsilon^{1,j}_{v,0} = p_{1h}(\tilde{u}) - v \text{ on } \omega_{0,\tau}, \quad \varepsilon^{1,j}_{v,N} = p_{3h}(\tilde{u}) - v \text{ on } \omega_{b,\tau}, \tag{2.94}$$

and

$$\varepsilon^{2,h,\tau}_{v,\bar{t},m} = a\Theta^6 \varepsilon^{2,h,\tau}_{v,m} + \psi_v \text{ on } \omega_{h,\tau}, \tag{2.95}$$

$$\varepsilon^{2,0}_{v,m} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{2.96}$$

$$\varepsilon^{2,j}_{v,0} = 0 \text{ on } \omega_{0,\tau}, \quad \varepsilon^{2,j}_{v,N} = 0 \text{ on } \omega_{b,\tau}, \tag{2.97}$$

respectively. From Lemma 2.9 and by maximum principle for the solution of the system (2.92) – (2.94) we have

$$\max_{\bar{\omega}_{h,\tau}} \left| \varepsilon^{1,h,\tau}_{v} \right| \le \max_{i=1,3} \max_{\bar{\omega}_{h,\tau}} \left| p_{ih}(\tilde{u}) - v \right| \le c_4 (h^2 + \tau^2). \tag{2.98}$$

The solution $\varepsilon^{2,h,\tau}_{v}$ of the problem (2.95) – (2.97) is the error of the approximate solution obtained by the finite difference method for the boundary value Problem 2(ii) when the boundary values satisfy the conditions

$$p_2 \in C^{6+\alpha}(\bar{\Omega}), \quad \partial_x f(x,t) \in C^{4+\alpha, \frac{4+\alpha}{2}}_{x,t}(\bar{Q}_T), \quad p_i \in C^{\frac{6+\alpha}{2}}(\bar{\sigma}_T), \ i = 1,3. \tag{2.99}$$

$$\begin{cases} p_1^{(q)}(0) = v^{(q)}(0), \\ p_3^{(q)}(0) = v^{(q)}(b), \qquad q = 0,1,2,3,... \end{cases} \tag{2.100}$$

Since the function $v = \partial_x u$ satisfies Eq. (2.52) with the initial function p_2 on γ_2 and boundary functions p_1, p_3 on γ_1 and γ_3, respectively and on the basis of Theorem 1.1 and the maximum principle in Chapter 4 of [19], we obtain

$$\max_{\bar{\omega}_{h,\tau}} \left| \varepsilon^{2,h,\tau}_{v} \right| \le c_5 (h^2 + \tau^2) \tag{2.101}$$

using (2.70), (2.98) and (2.101) we obtain (2.87). ■

2.5 Numerical aspects

All the computations in this section and in the numerical aspects sections of the proceedings Chapters are carried out in double precision using the FORTRAN programming language. For all the constructed examples we take $Q_T = \{(x,t): 0 < x < 1, 0 < t \le 1\}$, $\gamma_1 = \{(0,t): 0 \le t \le 1\}$, $\gamma_2 = \{(x,0): 0 \le x \le 1\}$, $\gamma_3 = \{(1,t): 0 \le t \le 1\}$, and the constant a in the operator $L \equiv \frac{\partial}{\partial t} - a\frac{\partial^2}{\partial x^2}$ is taken as $a = 1$. In all the tables Central Processing Unit (CPU) presents the total solution time in seconds.

Example 1: [17]

$$Lu = f(x,t) \text{ on } Q_T$$
$$u(x,0) = x^{\frac{26}{5}} + \sin\left(\frac{\pi}{2}x\right) \text{ on } \gamma_2$$
$$u(0,t) = t^{\frac{13}{5}} \text{ on } \gamma_1,$$
$$u(1,t) = t^{\frac{13}{5}} + \cos(t^{\frac{13}{5}}) + 1 \text{ on } \gamma_3,$$

where, $f(x,t) = -\frac{13}{5}x^{\frac{26}{5}}t^{\frac{8}{5}}\sin(t^{\frac{13}{5}}) + \frac{13}{5}t^{\frac{8}{5}} - \frac{26}{5}\frac{21}{5}x^{\frac{16}{5}}\cos(t^{\frac{13}{5}}) + \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}x\right)$. Using the implicit four point difference scheme (2.18) – (2.20) $(\varsigma = 3)$ we obtain the following matrix form of the system of equations, for time layer $t = (j+1)\tau$, $j = 0,1,2,3,\ldots,M-1,$ as $A\tilde{u}^{j+1} = \tilde{u}^j + \tau f^{j+1}$, where $r = \frac{\tau}{h^2}$, since $a = 1$.

$$\begin{pmatrix} 1+2r & -r & 0 & 0 & 0 & 0 & 0 & 0 \\ -r & 1+2r & -r & 0 & --- & --- & --- & 0 \\ 0 & -r & 1+2r & -r & 0 & --- & --- & 0 \\ | & & \ddots & \ddots & \ddots & & & | \\ 0 & --- & --- & 0 & -r & 1+2r & -r & 0 \\ 0 & --- & --- & --- & 0 & -r & 1+2r & -r \\ 0 & 0 & 0 & 0 & 0 & 0 & -r & 1+2r \end{pmatrix} \begin{pmatrix} \tilde{u}_1^{j+1} \\ \tilde{u}_2^{j+1} \\ \tilde{u}_3^{j+1} \\ | \\ \tilde{u}_{N-3}^{j+1} \\ \tilde{u}_{N-2}^{j+1} \\ \tilde{u}_{N-1}^{j+1} \end{pmatrix} = \begin{pmatrix} \tilde{u}_1^{j} \\ \tilde{u}_2^{j} \\ \tilde{u}_3^{j} \\ | \\ \tilde{u}_{N-3}^{j} \\ \tilde{u}_{N-2}^{j} \\ \tilde{u}_{N-1}^{j} \end{pmatrix} + \tau \begin{pmatrix} f_1^{j+1} \\ f_2^{j+1} \\ f_3^{j+1} \\ | \\ f_{N-3}^{j+1} \\ f_{N-2}^{j+1} \\ f_{N-1}^{j+1} \end{pmatrix} \quad (2.102)$$

The coefficient matrix is tridiagonal band matrix. Therefore, Gauss – Thomas Method is used. The algorithm consists of three steps: decomposition and forward and back substitution [21]. In a generalized form we consider the tridiagonal system below:

$$\begin{pmatrix} \tilde{b}_1 & \tilde{c}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{a}_2 & \tilde{b}_2 & \tilde{c}_2 & 0 & --- & --- & --- & 0 \\ 0 & \tilde{a}_3 & \tilde{b}_3 & \tilde{c}_3 & 0 & --- & --- & 0 \\ | & & \ddots & \ddots & \ddots & & & | \\ 0 & --- & --- & 0 & \tilde{a}_{N-3} & \tilde{b}_{N-3} & \tilde{c}_{N-3} & 0 \\ 0 & --- & --- & --- & 0 & \tilde{a}_{N-2} & \tilde{b}_{N-2} & \tilde{c}_{N-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{a}_{N-1} & \tilde{b}_{N-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ | \\ u_{N-3} \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} \tilde{d}_1 \\ \tilde{d}_2 \\ \tilde{d}_3 \\ | \\ \tilde{d}_{N-3} \\ \tilde{d}_{N-2} \\ \tilde{d}_{N-1} \end{pmatrix} \quad (2.103)$$

and present the pseudocode to implement the Thomas algorithm as follows:

(a) *decomposition*

$DOFOR \quad j = 2 \ to \ N-1$

$\tilde{a}_j = \tilde{a}_j \ / \ \tilde{b}_{j-1}$

$\tilde{b}_j = \tilde{b}_j - \tilde{a}_j \bullet \tilde{c}_{j-1}$

$ENDDO$

(b) *forward substitution* (2.104)

$DOFOR \quad j = 2 \ to \ N-1$

$\tilde{d}_j = \tilde{d}_j - \tilde{a}_j \bullet \tilde{d}_{j-1}$

$ENDDO$

(c) *back substitution*

$u_{N-1} = \tilde{d}_{N-1} \ / \ \tilde{b}_{N-1}$

$DOFOR \quad j = N-2 \ to \ 1 \ step \ -1$

$u_j = (\tilde{d}_j - \tilde{c}_j \bullet u_{j+1}) \ / \ \tilde{b}_j$

$ENDDO$

where N is the number of intervals along spatial variable x, $N = \frac{b}{h}$. The approximate solution $\tilde{u}$ is obtained at each time level with space step size $h = 2^{-\mu}$ and time step size $\tau = 2^{-\lambda}$ where μ, λ are positive integers. Next, the boundary value problem for $v = \frac{\partial u}{\partial x}$ is constructed using the obtained approximate solution $\tilde{u}$ and the proposed Problem 2(i). The structure of the coefficient matrix A is same as in (2.102). Furthermore, the approximate solution $\tilde{v}$ of the difference problem (2.62) – (2.64) is obtained by using Gauss-Thomas Algorithm (2.104) at the same grid points. The exact solution is known as $v(x,t) = \frac{26}{5} x^{\frac{21}{5}} \cos(t^{\frac{13}{5}}) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$ and we denote the maximum errors on the grid points by $\left\|\varepsilon_v^{h,\tau}\right\| = \max_{\bar{\omega}_{h,\tau}} |\tilde{v} - v|$. Table 2.1 demonstrates the maximum errors for $r = 2^{-\omega}$, $\omega = 2,3$ and the corresponding CPU time for different step sizes with the order of convergence $\Re_{\tilde{v}}^{h,\tau}$ as;

$$\Re_{\tilde{v}}^{h,\tau} = \frac{\left\|\varepsilon_v^{2^{-\mu},2^{-\lambda}}\right\|}{\left\|\varepsilon_v^{2^{-(\mu+1)},2^{-(\lambda+2)}}\right\|} \tag{2.105}$$

of $\tilde{v}$ with respect to h and τ, for Example 1.

Table 2.1: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{v}}^{h,\tau}$, for Example 1.

$(h = 2^{-\mu}, \tau = 2^{-\lambda})$		$(h = 2^{-(\mu+1)}, \tau = 2^{-(\lambda+2)})$		$\left\|\varepsilon_v^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_v^{2^{-(\mu+1)},2^{-(\lambda+2)}}\right\|$	$\Re_v^{h,\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-5},2^{-12})$	0.344	$(2^{-6},2^{-14})$	2.109	$2.438E-02$	$6.383E-03$	3.820
$(2^{-6},2^{-14})$	2.109	$(2^{-7},2^{-16})$	15.094	$6.383E-03$	$1.633E-03$	3.910
$(2^{-5},2^{-13})$	0.656	$(2^{-6},2^{-15})$	4.250	$2.439E-02$	$6.385E-03$	3.820
$(2^{-6},2^{-15})$	4.250	$(2^{-7},2^{-17})$	30.312	$6.385E-03$	$1.633E-03$	3.910

Table 2.2 presents the maximum errors and the corresponding CPU for $h = 2^{-9}$, $r = 2^{-\lambda}$, $\lambda = 6,7,8,9,10,11$ and the order of convergence $\Re_{\tilde{v}}^{\tau}$ as;

$$\Re_{\tilde{v}}^{\tau} = \frac{\left\| \varepsilon_v^{h,2^{-\lambda}} \right\|}{\left\| \varepsilon_v^{h,2^{-(\lambda+1)}} \right\|}, \qquad (2.106)$$

of $\tilde{v}$ with respect to τ, for Example 1.

Table 2.2: Maximum errors, corresponding CPU time for different step size in time and $\Re_{\tilde{v}}^{\tau}$, for Example 1.

$(h = 2^{-\mu}, \tau = 2^{-\lambda})$		$(h = 2^{-\mu}, \tau = 2^{-(\lambda+1)})$		$\left\| \varepsilon_v^{2^{-\mu},2^{-\lambda}} \right\|$	$\left\| \varepsilon_v^{2^{-\mu},2^{-(\lambda+1)}} \right\|$	$\Re_{\tilde{v}}^{\tau}$
h, τ	CPU	h, τ	CPU			
$(2^{-9}, 2^{-6})$	0.125	$(2^{-9}, 2^{-7})$	0.125	$6.087E-02$	$3.058E-02$	1.991
$(2^{-9}, 2^{-7})$	0.125	$(2^{-9}, 2^{-8})$	0.125	$3.058E-02$	$1.531E-02$	1.997
$(2^{-9}, 2^{-8})$	0.250	$(2^{-9}, 2^{-9})$	0.250	$1.531E-02$	$7.634E-03$	2.006
$(2^{-9}, 2^{-9})$	0.438	$(2^{-9}, 2^{-10})$	0.875	$7.634E-03$	$3.791E-03$	2.014
$(2^{-9}, 2^{-10})$	0.875	$(2^{-9}, 2^{-11})$	1.750	$3.791E-03$	$1.867E-03$	2.031

According to the definition of the maximum error the third and fourth columns of Table 2.1 and Table 2.2 present the theoretical upper bound errors given in (2.65), for Example 1. Note that the $O(h^2 + \tau)$ order of convergence corresponds to $\approx 2^2$ of the quantities defined by (2.105), and $\approx 2^1$ of the quantities defined by (2.106). Figure 2.2 presents the error function $\left| \varepsilon_v^{2^{-7},2^{-17}} \right| = | \tilde{v} - v |$ for $h = 2^{-7}$, and $\tau = 2^{-17}$. The maximum errors $\left\| \varepsilon_v^{2^{-9},\tau} \right\|$ when $h = 2^{-9}$, with respect to τ, are shown in Figure 2.3 and the maximum errors $\left\| \varepsilon_v^{h,2^{-17}} \right\|$ when $\tau = 2^{-17}$, with respect to h, are demonstrated by Figure 2.4. Figure 2.5 shows the exact solution $v(x,t) = \partial_x u$, and the grid function $v^{2^{-7},2^{-17}}$

presenting the approximate solution $\tilde{v}$ of $\partial_x u$ when $h = 2^{-7}$, $\tau = 2^{-17}$ is given in Figure 2.6.

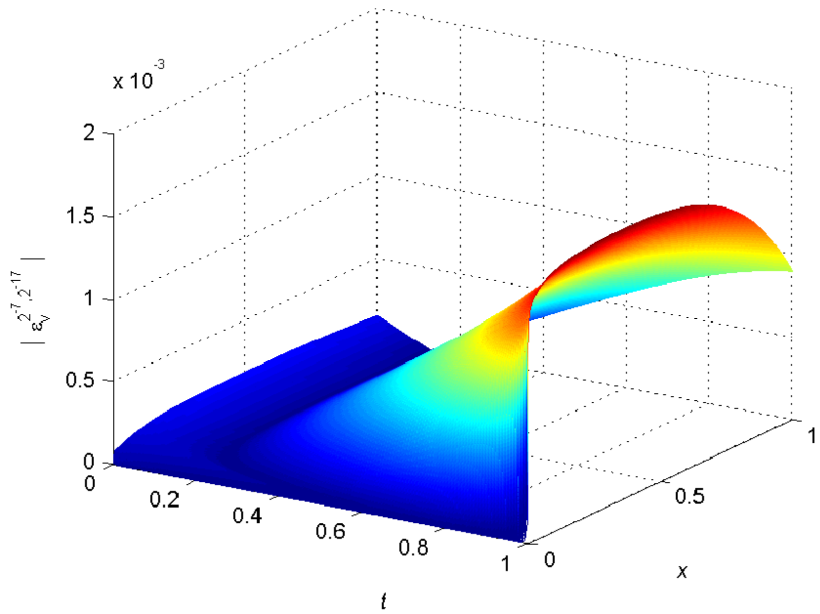

Figure 2.2: The error function $\left|\varepsilon_v^{2^{-7},2^{-17}}\right| = |\tilde{v} - v|$ when $h = 2^{-7}$, and $\tau = 2^{-17}$, for Example 1.

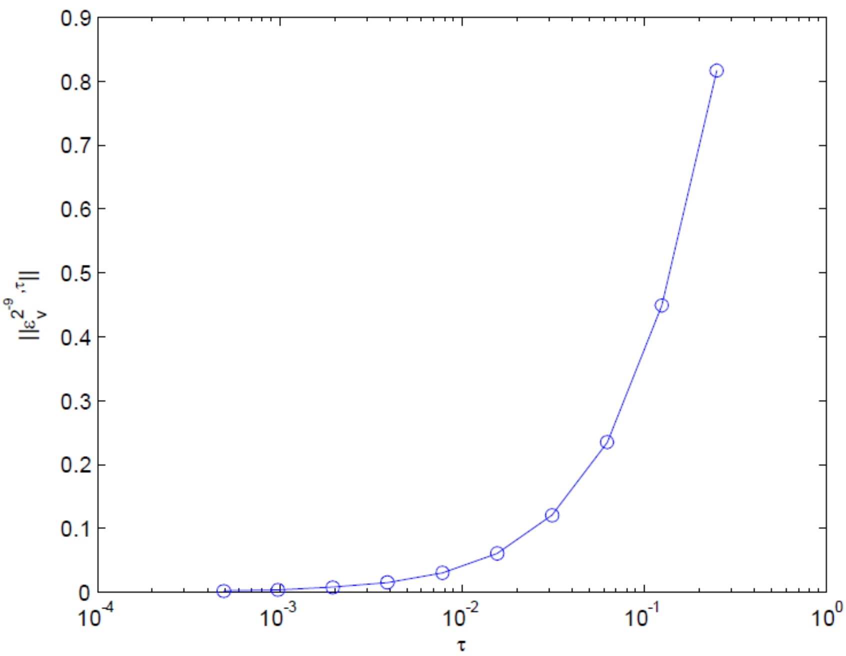

Figure 2.3: The maximum errors $\left\|\varepsilon_v^{2^{-9},\tau}\right\|$ when $h = 2^{-9}$, with respect to τ, for Example 1.

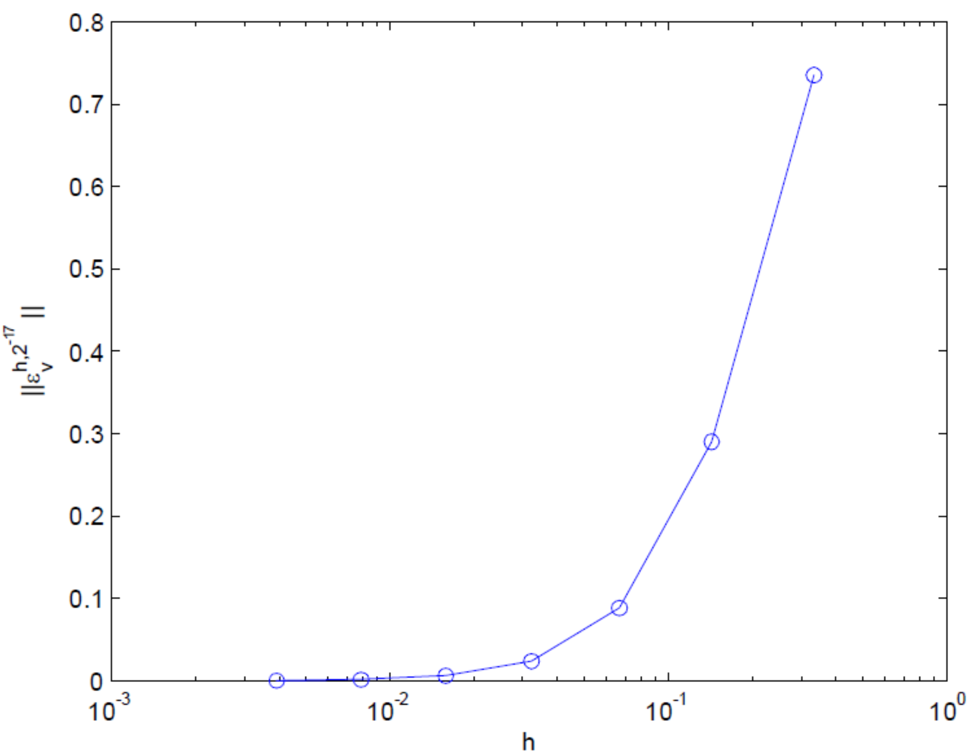

Figure 2.4: The maximum errors $\left\|\varepsilon_v^{h,2^{-17}}\right\|$ when $\tau = 2^{-17}$, with respect to h, for Example 1.

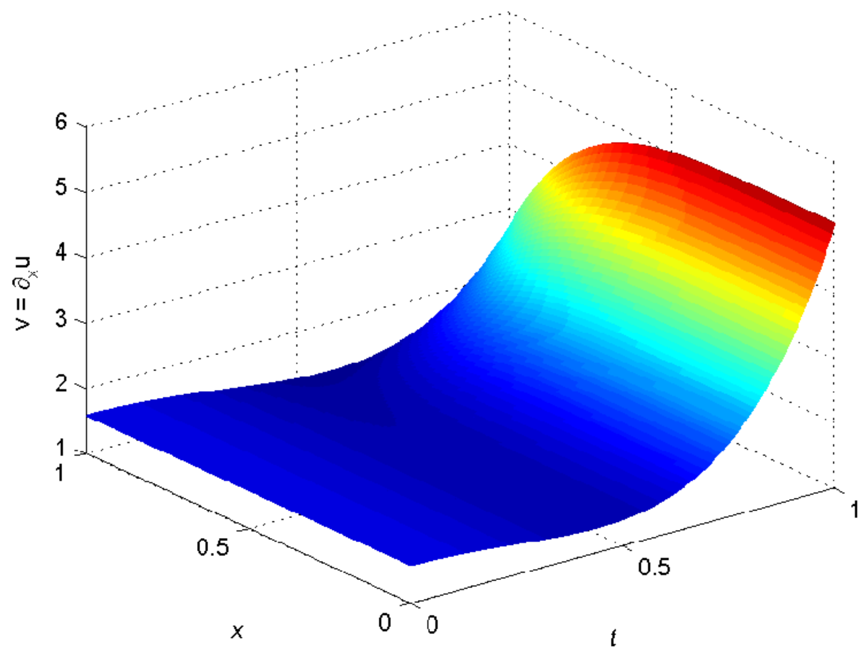

Figure 2.5: The function v presenting the exact solution $\partial_x u$, for Example 1.

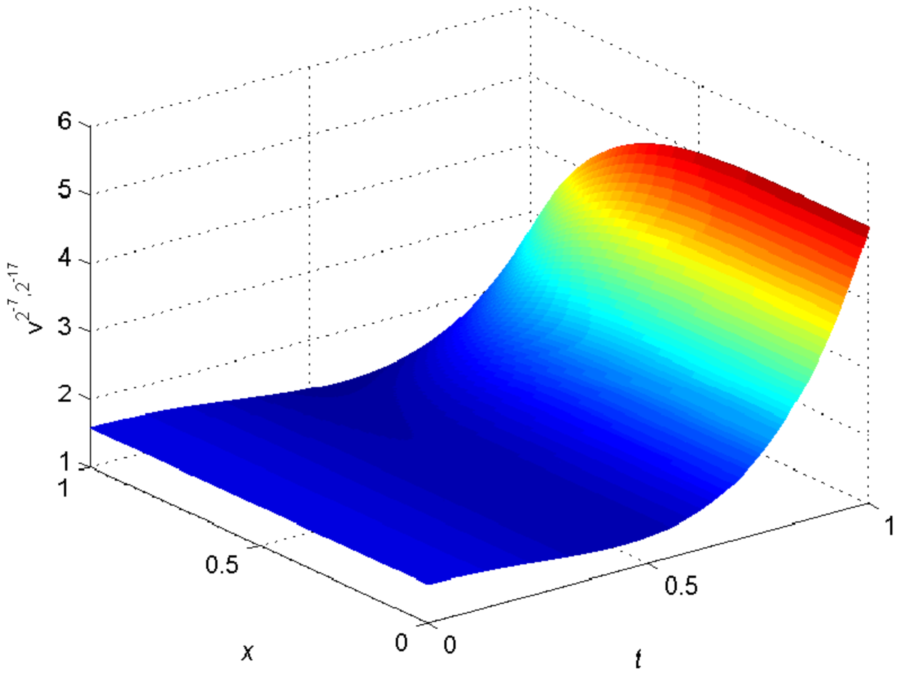

Figure 2.6: The grid function $v^{2^{-7},2^{-17}}$ presenting the approximate solution $\tilde{v}$ of $\partial_x u$ when $h = 2^{-7}$, $\tau = 2^{-17}$, for Example 1.

Example 2: [17]

$$
\begin{aligned}
&Lu = f(x,t) \quad \text{on} \quad Q_T \\
&u(x,0) = \frac{5}{36}\frac{5}{18}x^{\frac{36}{5}} + \sin\left(\frac{\pi}{2}x\right) \quad \text{on} \quad \gamma_2 \\
&u(0,t) = \frac{5}{18}t^{\frac{18}{5}} \quad \text{on} \quad \gamma_1, \\
&u(1,t) = \frac{5}{36}t^{\frac{18}{5}} + \frac{5}{36}\frac{5}{18}\cos(t^{\frac{18}{5}}) + 1 \quad \text{on} \quad \gamma_3,
\end{aligned}
$$

where $f(x,t) = -\frac{5}{36}x^{\frac{36}{5}}t^{\frac{13}{5}}\sin(t^{\frac{18}{5}}) + t^{\frac{13}{5}} - \frac{31}{18}x^{\frac{26}{5}}\cos(t^{\frac{18}{5}}) + \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}x\right)$. Using the implicit six point difference scheme (2.18) – (2.20) $(\varsigma = 6)$ we obtain the following system of equations in matrix form at each time level, as $A\tilde{u}^{j+1} = B\tilde{u}^{j} + \tau f^{j+\frac{1}{2}}$, where $r = \frac{\tau}{h^2}$ for $a = 1$.

$$
\begin{pmatrix}
1+r & -0.5r & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5r & 1+r & -0.5r & 0 & --- & --- & --- & 0 \\
0 & -0.5r & 1+r & -0.5r & 0 & --- & --- & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \vdots \\
0 & --- & --- & 0 & -0.5r & 1+r & -0.5r & 0 \\
0 & --- & --- & --- & 0 & -0.5r & 1+r & -0.5r \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5r & 1+r
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_1^{j+1} \\ \tilde{u}_2^{j+1} \\ \tilde{u}_3^{j+1} \\ \vdots \\ \tilde{u}_{N-3}^{j+1} \\ \tilde{u}_{N-2}^{j+1} \\ \tilde{u}_{N-1}^{j+1}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
1-r & 0.5r & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5r & 1-r & 0.5r & 0 & --- & --- & --- & 0 \\
0 & 0.5r & 1-r & 0.5r & 0 & --- & --- & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \vdots \\
0 & --- & --- & 0 & 0.5r & 1-r & 0.5r & 0 \\
0 & --- & --- & --- & 0 & 0.5r & 1-r & 0.5r \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5r & 1-r
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_1^{j} \\ \tilde{u}_2^{j} \\ \tilde{u}_3^{j} \\ \vdots \\ \tilde{u}_{N-3}^{j} \\ \tilde{u}_{N-2}^{j} \\ \tilde{u}_{N-1}^{j}
\end{pmatrix}
+\tau
\begin{pmatrix}
f_1^{j+0.5} \\ f_2^{j+0.5} \\ f_3^{j+0.5} \\ \vdots \\ f_{N-3}^{j+0.5} \\ f_{N-2}^{j+0.5} \\ f_{N-1}^{j+0.5}
\end{pmatrix}
\qquad (2.107)
$$

The approximate solution $\tilde{u}$ is obtained by applying Gauss – Thomas algorithm (2.104) for solving the algebraic system of equations (2.107) at each time level for $\tau = 2^{-\lambda}$ where λ is nonnegative integer. Next the boundary value problem for $v = \frac{\partial u}{\partial x}$ is constructed from the proposed Problem 2(ii) using the obtained approximate solution $\tilde{u}$. The structure of the obtained algebraic linear system is analogues to (2.107). Furthermore, the approximate solution $\tilde{v}$ for $\frac{\partial u}{\partial x}$ is obtained at the same grid points by solving the problem (2.84) – (2.86) using Gauss-Thomas algorithm (2.104), and compared on the grids with the known exact solution $v(x,t) = \frac{5}{18} x^{\frac{31}{5}} \cos(t^{\frac{18}{5}}) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$. We use

$$\tilde{\mathfrak{R}}_{\tilde{v}}^{h,\tau} = \frac{\left\| \varepsilon_v^{2^{-\mu},2^{-\lambda}} \right\|}{\left\| \varepsilon_v^{2^{-(\mu+1)},2^{-(\lambda+1)}} \right\|} \tag{2.108}$$

to present the order of convergence of $\tilde{v}$ with respect to h and τ. Note that the $O(h^2+\tau^2)$ order of convergence corresponds to $\approx 2^2$ of the quantity by (2.108). Table 2.3 shows the maximum errors for $h=2^{-\mu}$, $\mu=4,5,6,7,8$ and $\tau=2^{-\lambda}$, $\lambda=13,14,15,16,17,$ respectively, and the corresponding CPU time for each step sizes and the orders $\tilde{\mathfrak{R}}_{\tilde{v}}^{h,\tau}$. The third and fourth columns of this table presents the theoretical upper bound errors given in (2.87). Figure 2.7 present the error function $\left|\varepsilon_v^{2^{-7},2^{-17}}\right|$ for $h=2^{-7}$, and $\tau=2^{-17}$. The maximum errors $\left\|\varepsilon_v^{h,2^{-17}}\right\|$ when $\tau=2^{-17}$, with respect to h, is demonstrated by Figure 2.8. Figure 2.9 shows the exact solution $v(x,t)=\partial_x u$, and the grid function $v^{2^{-7},2^{-17}}$ presenting the approximate solution $\tilde{v}$ of $\partial_x u$ when $h=2^{-7}$, $\tau=2^{-17}$.

Table 2.3: Maximum errors, corresponding CPU time for different step sizes in space and time and $\tilde{\mathfrak{R}}_{\tilde{v}}^{h,\tau}$, for Example 2.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-(\mu+1)},\tau=2^{-(\lambda+1)})$		$\left\|\varepsilon_v^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_v^{2^{-(\mu+1)},2^{-(\lambda+1)}}\right\|$	$\tilde{\mathfrak{R}}_{\tilde{v}}^{h,\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-4},2^{-13})$	0.453	$(2^{-5},2^{-14})$	1.312	$1.0567E-02$	$3.1449E-03$	3.360
$(2^{-5},2^{-14})$	1.312	$(2^{-6},2^{-15})$	4.312	$3.1449E-03$	$8.5353E-04$	3.687
$(2^{-6},2^{-15})$	4.312	$(2^{-7},2^{-16})$	15.328	$8.5353E-04$	$2.2209E-04$	3.842
$(2^{-7},2^{-16})$	15.328	$(2^{-8},2^{-17})$	58.609	$2.2209E-04$	$5.6628E-05$	3.921

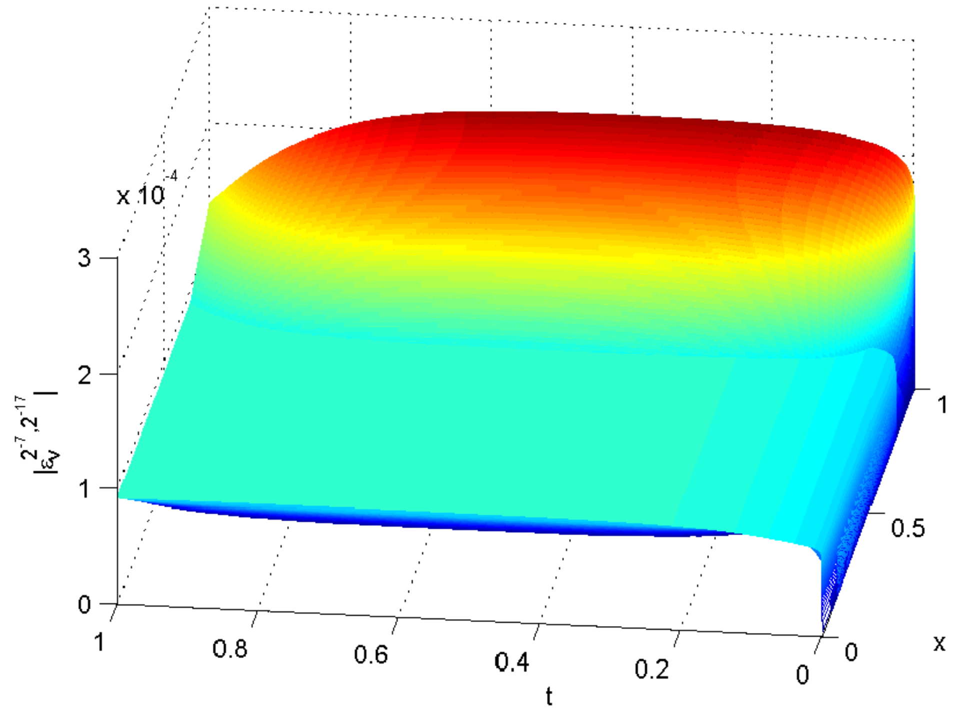

Figure 2.7: The error function $\left|\varepsilon_v^{2^{-7},2^{-17}}\right| = \left|\tilde{v} - v\right|$ when $h = 2^{-7}$, and $\tau = 2^{-17}$, for Example 2.

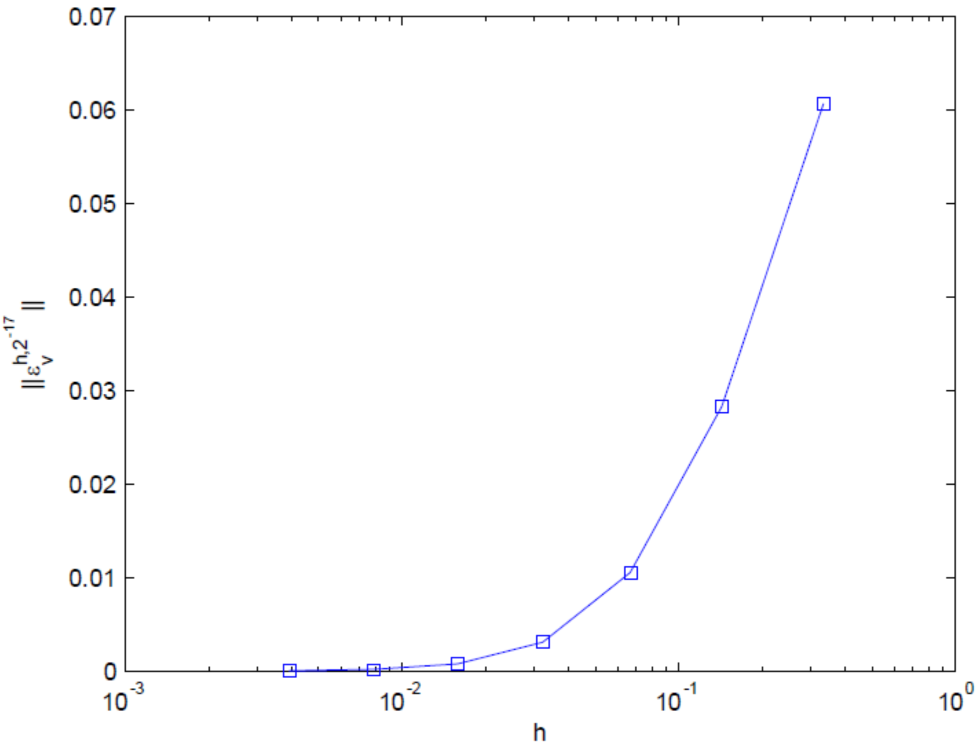

Figure 2.8: The maximum errors $\left\|\varepsilon_v^{h,2^{-17}}\right\|$ for $\tau = 2^{-17}$ with respect to h, of Example 2.

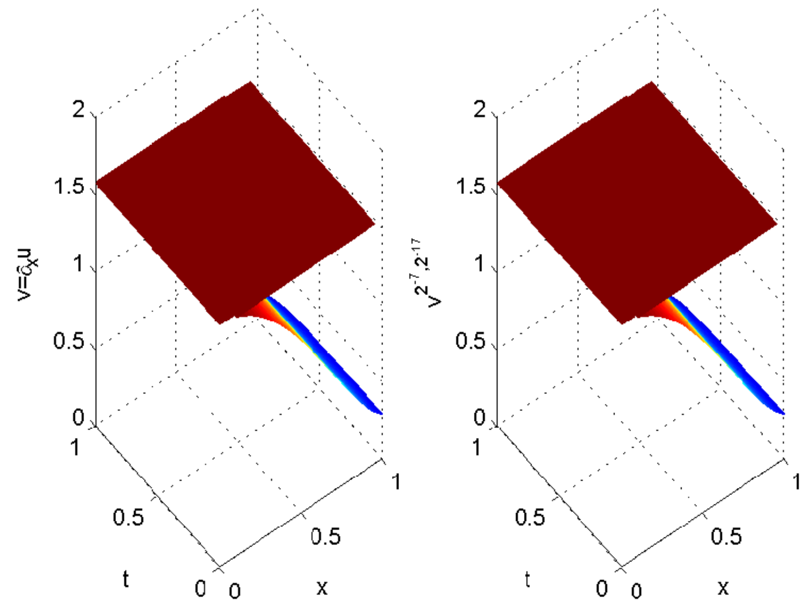

Figure 2.9: The exact solution $\partial_x u$, and the grid function $v^{2^{-7},2^{-17}}$ when $h = 2^{-7}$, $\tau = 2^{-17}$, for Example 2.

Example 3: [17]

$$\begin{aligned} &Lu = f(x,t) \quad \text{on} \quad Q_T, \\ &u(x,0) = e^{-x} \quad \text{on} \quad \gamma_2, \\ &u(0,t) = 1 + 0.001t^{\frac{25}{7}} \quad \text{on} \quad \gamma_1, \\ &u(1,t) = 0.0001\sin(t^{\frac{25}{7}}) + 0.001t^{\frac{25}{7}} + e^{-1} \quad \text{on} \quad \gamma_3, \end{aligned}$$

Where,

$$f(x,t) = 0.0001\frac{25}{7}x^{\frac{50}{7}}t^{\frac{18}{7}}\cos(t^{\frac{25}{7}}) + 0.001\frac{25}{7}t^{\frac{18}{7}} - 0.0001\frac{50}{7}\frac{43}{7}x^{\frac{36}{5}}\sin(t^{\frac{25}{7}}) - e^{-x}.$$

The initial function, the boundary functions and the nonhomogeneous term $f(x,t)$ satisfy the conditions (2.11) of Problem 1(ii). Using the proposed implicit six point difference problem (2.18) – (2.20) $(\varsigma = 6)$ we obtain the approximate solution $\tilde{u}$ at each time level. Next the boundary value problem for $v = \frac{\partial u}{\partial x}$ is constructed from the proposed Problem 2(ii) using the obtained approximate solution $\tilde{u}$; then the approximate solution $\tilde{v}$ of $v = \partial_x u$ is obtained at the same grid points by solving the system of equations resulting from (2.84) – (2.86). Let $v^{2^{-\mu},2^{-\lambda}}(x,t)$ be the approximate solution $\tilde{v}$ at (x,t) when $h = 2^{-\mu}$ and $\tau = 2^{-\lambda}$. The exact solution v is not given. To verify the order of convergence of the computed solution $\tilde{v}$ to the exact solution v we compute the solution at grid points with successively reduced step sizes h and τ by a factor of two and the ratio of the absolute successive errors (see Chapter 2 of [22]). Table 2.4 presents $v^{2^{-\mu},2^{-\lambda}}(x,t)$ at the grid points (0.125, 1), (0.25, 1), (0.375, 1), (0.5, 1), (0.625, 1), (0.75, 1) and (0.875, 1) for the pairs (μ,λ) = (5, 13), (6,14), (7, 15), (8, 16) which means that the step sizes h in x and τ in t are halved successively.

Table 2.4: The approximate solution $\tilde{v}$ on $t=1$, for Example 3.

x	$v^{2^{-5},2^{-13}}(x,1)$	$v^{2^{-6},2^{-14}}(x,1)$	$v^{2^{-7},2^{-15}}(x,1)$	$v^{2^{-8},2^{-16}}(x,1)$
0.125	−0.88218321	−0.88241771	−0.88247701	−0.88249191
0.25	−0.77852029	−0.77872997	−0.77878291	−0.77879622
0.375	−0.68703982	−0.68722539	−0.68727216	−0.68728390
0.5	−0.60630576	−0.60646779	−0.60650853	−0.60651874
0.625	−0.53504253	−0.53518148	−0.53521631	−0.53522502
0.75	−0.47210904	−0.47222525	−0.47225424	−0.47226147
0.875	−0.41647280	−0.41656648	−0.41658968	−0.41659545

Table 2.5 demonstrates the absolute error ratios

$$r_1 = \left| \frac{v^{2^{-5},2^{-13}}(x,1) - v^{2^{-6},2^{-14}}(x,1)}{v^{2^{-6},2^{-14}}(x,1) - v^{2^{-7},2^{-15}}(x,1)} \right|,$$

$$r_2 = \left| \frac{v^{2^{-6},2^{-14}}(x,1) - v^{2^{-7},2^{-15}}(x,1)}{v^{2^{-7},2^{-15}}(x,1) - v^{2^{-8},2^{-16}}(x,1)} \right|,$$

and the corresponding orders

$$p_1 = \log_2 \left| \frac{v^{2^{-5},2^{-13}}(x,1) - v^{2^{-6},2^{-14}}(x,1)}{v^{2^{-6},2^{-14}}(x,1) - v^{2^{-7},2^{-15}}(x,1)} \right|,$$

$$p_2 = \log_2 \left| \frac{v^{2^{-6},2^{-14}}(x,1) - v^{2^{-7},2^{-15}}(x,1)}{v^{2^{-7},2^{-15}}(x,1) - v^{2^{-8},2^{-16}}(x,1)} \right|,$$

for the considered points at $t=1$. By analyzing the values of p_1 and p_2 in the third and fifth columns of Table 2.5, respectively, we conclude that the order of convergence is quadratic in the two variables x and t on $t=1$. Figure 2.10 illustrates the grid function $v^{2^{-8},2^{-16}}$ presenting the approximate solution $\tilde{v}$ of $v = \partial_x u$ when $h = 2^{-8}$, $\tau = 2^{-16}$.

Table 2.5: The absolute error ratios at some grid points on $t=1$ and the orders p_1, p_2, for Example 3.

x	r_1	p_1	r_2	p_2
0.125	3.9544688	19835	3.9798658	1.9927
0.25	3.9607102	1.9858	3.9774606	1.9919
0.375	3.9675005	1.9882	3.9838160	1.9942
0.5	3.9777172	1.9917	3.9902057	1.9965
0.625	3.9893770	1.9962	3.9988519	1.9996
0.75	4.0086237	2.0031	4.0096819	2.0035
0.875	4.0379310	2.0136	4.0207972	2.0075

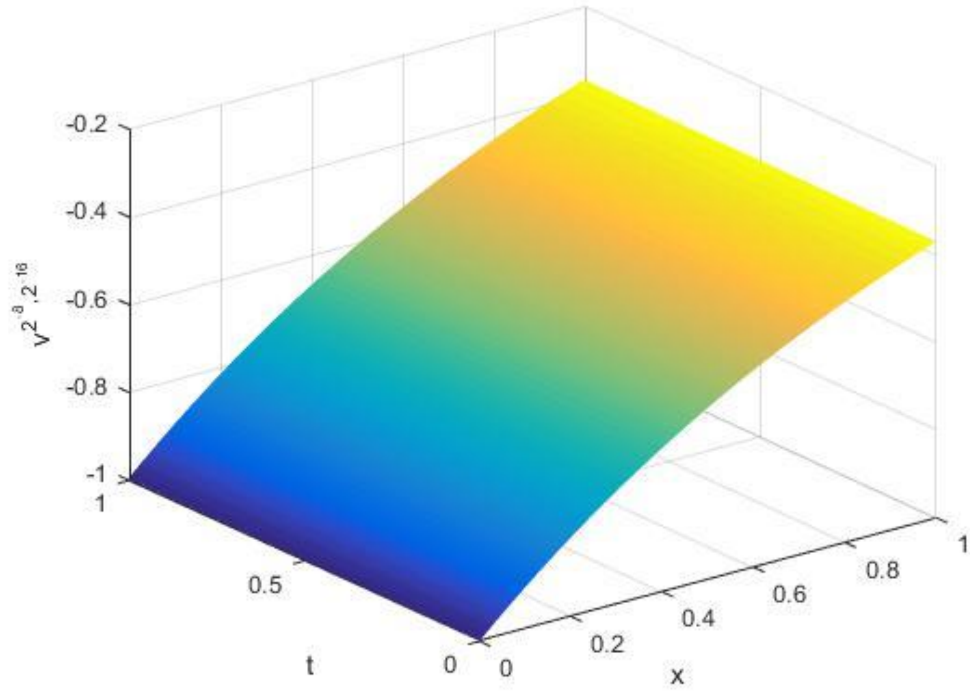

Figure 2.10: The grid function $v^{2^{-8},2^{-16}}$ presenting the approximate solution $\tilde{v}$ of $v = \partial_x u$ when $h = 2^{-8}$, $\tau = 2^{-16}$, for Example 3.

Chapter 3

FOUR POINT IMPLICIT METHODS FOR THE APPROXIMATION OF SECOND DERIVATIVES TO HEAT EQUATION WITH CONSTANT COEFFICIENTS

3.1 Chapter overview

This chapter extends the methods given in Chapter 2 of this dissertation and in [17] for finding the first difference derivative of $u(x,t)$ with respect to t and its second order difference derivatives with $O(h^2+\tau)$ order of convergence to the corresponding exact derivatives. The general idea of this research work is presented as an extended abstract in [23].

Here, we consider the first type boundary value problem for one dimensional heat equation of which the initial function belongs to $C^{8+\alpha}$ the heat source function is from $C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}$, the boundary functions are from $C^{\frac{8+\alpha}{2}}$, and between the initial and the boundary functions the conjugation conditions of orders $q=0,1,2,3,4$ are satisfied. Difference problems of four point implicit schemes approximating $\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial t^2}$, and $\frac{\partial^2 u}{\partial x \partial t}$ are constructed, which converge uniformly to the exact values of $\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial t^2}$, and $\frac{\partial^2 u}{\partial x \partial t}$ respectively, on the grids of order $O(h^2+\tau)$.

Under the above assumption, we organized this chapter as follows: In Section 2 boundary value problems for $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ are given and difference boundary value problems of implicit schemes approximating $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ are constructed. Moreover, for the error function we provide a pointwise prior estimation depending on $\rho(x,t)$ which is the distance from the current grid point in the domain to the boundary. In Section 3, we propose a special implicit difference problem for the approximation of $\frac{\partial^2 u}{\partial t^2}$ and prove that the solution of the constructed difference scheme converge uniformly to the exact value $\frac{\partial^2 u}{\partial t^2}$ on the grids of order $O(h^2+\tau)$. In Section 4, a special implicit difference problem for the approximation of $\frac{\partial^2 u}{\partial x \partial t}$ is given, of which the solution converge uniformly to the exact value of $\frac{\partial^2 u}{\partial x \partial t}$ on the grids of order $O(h^2+\tau)$. To justify the theoretical results, a numerical example is constructed and obtained results are presented through tables and figures in Section 5.

3.2 Implicit schemes for the approximation of $\partial_t u$ and $\partial_x^2 u$

Let the following problem be given:

Problem 1:

(iii) The boundary value problem (2.1) – (2.3) satisfying the conditions

$$u_0(x) \in C^{8+\alpha}(\bar{\Omega}), \quad f(x,t) \in C_{x,t}^{6+\alpha, \frac{6+\alpha}{2}}(\bar{Q}_T) \text{ and } u_i(t) \in C^{\frac{8+\alpha}{2}}(\bar{\sigma}_T),\ i=1,2. \qquad (3.1)$$

and the conjugation conditions (2.9) of order $0,1,2,3,4$.

Also, let $w=\partial_t u$ and $\phi=\partial_x^2 u$ then w and ϕ satisfy the following boundary value problems respectively, see also [18, 23].

Problem 3:

$$Lw=\partial_t f(x,t)=F(x,t) \quad \text{on} \quad Q_T, \tag{3.2}$$

$$w(x,0)=u^{(1)}(x)=w_0(x) \quad \text{on} \quad \gamma_2, \tag{3.3}$$

$$w(0,t)=D_t u_1(t)=w_1(t) \quad \text{on} \quad \gamma_1, \tag{3.4}$$

$$w(b,t)=D_t u_2(t)=w_2(t) \quad \text{on} \quad \gamma_3. \tag{3.5}$$

Problem 4:

$$L\phi=\partial_x^2 f(x,t)=G(x,t) \quad \text{on} \quad Q_T, \tag{3.6}$$

$$\phi(x,0)=\partial_x^2 u_0(x)=\phi_0(x) \quad \text{on} \quad \gamma_2, \tag{3.7}$$

$$\phi(0,t)=\frac{1}{a}\left[D_t u_1(t)-f(0,t)\right]=\phi_1(t) \quad \text{on} \quad \gamma_1, \tag{3.8}$$

$$\phi(b,t)=\frac{1}{a}\left[D_t u_2(t)-f(b,t)\right]=\phi_2(t) \quad \text{on} \quad \gamma_3. \tag{3.9}$$

where, $f(x,t)$ is the heat source function given in (2.1), $u_0(x)$ and $u_1(t),\ u_2(t)$ are the initial and boundary functions given in (2.2), (2.3) respectively, also $u^{(1)}(x)$ is as defined in (2.5). Furthermore,

$$F(x,t)\in C_{x,t}^{4+\alpha,\frac{4+\alpha}{2}}(\bar{Q}_T), \quad w_0(x)\in C^{6+\alpha}(\bar{\Omega}), \quad w_i(t)\in C^{\frac{6+\alpha}{2}}(\bar{\sigma}_T), \quad i=1,2, \tag{3.10}$$

$$G(x,t)\in C_{x,t}^{4+\alpha,\frac{4+\alpha}{2}}(\bar{Q}_T), \quad \phi_0(x)\in C^{6+\alpha}(\bar{\Omega}), \quad \phi_i(t)\in C^{\frac{6+\alpha}{2}}(\bar{\sigma}_T), \quad i=1,2, \tag{3.11}$$

both satisfying the conjugation conditions (2.9) of order 0, 1, 2, 3.

Lemma 3.1: The Problem 1(iii) has unique solution $u \in C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}(\bar{Q}_T)$ and the constructed Problem 3 with (3.10) and Problem 4 with (3.11) have unique solution w and ϕ respectively, belonging to the space $C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}(\bar{Q}_T)$.

Proof: From Theorem 1.1, Problem 3 has unique solution $u \in C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}(\bar{Q}_T)$. Taking into account that Problem 3 with (3.10) and Problem 4 with (3.11) are also first type boundary value problems analogous to the problem (2.1) – (2.3) on the basis of Theorem 1.1 the proof follows.

To realize the numerical solution of the Problem 4 with (3.10), we propose the following implicit difference problem, of which the solution is $\tilde{w}$,

$$\tilde{w}_{\bar{t},m}^{h,\tau} = a\Theta^3 \tilde{w}_m^{h,\tau} + \varphi_{\tilde{w},m} \text{ on } \omega_{h,\tau}, \tag{3.12}$$

$$\tilde{w}_m^0 = w_0(x_m) \text{ on } \bar{\omega}_{h,0}, \tag{3.13}$$

$$\tilde{w}_0^j = w_1(t_j) \text{ on } \omega_{0,\tau}, \quad \tilde{w}_N^j = w_2(t_j) \text{ on } \omega_{b,\tau}, \tag{3.14}$$

where, $\tilde{w}_m^0 = \tilde{w}(x_m,0)$, $\tilde{w}_0^j = \tilde{w}(0,t_j)$, $\tilde{w}_N^j = \tilde{w}(b,t_j)$ and

$$\tilde{w}_{\bar{t},m}^{h,\tau} = \frac{\tilde{w}_m^{j+1} - \tilde{w}_m^j}{\tau}, \tag{3.15}$$

$$\Theta^3 \tilde{w}_m^{h,\tau} = \frac{\tilde{w}_{m-1}^{j+1} - 2\tilde{w}_m^{j+1} + \tilde{w}_{m+1}^{j+1}}{h^2}, \tag{3.16}$$

$$\varphi_{\tilde{w},m} = F\left(x_m, t_{j+1}\right) = \partial_t f\left(x_m, t_{j+1}\right). \tag{3.17}$$

For the numerical solution of Problem 4 with (3.11), we propose

$$\tilde{\phi}_{\bar{t},m}^{h,\tau} = a\Theta^3 \tilde{\phi}_m^{h,\tau} + \varphi_{\tilde{\phi},m} \text{ on } \omega_{h,\tau}, \tag{3.18}$$

$$\tilde{\phi}_m^0 = \phi_0(x_m) \text{ on } \bar{\omega}_{h,0}, \tag{3.19}$$

$$\tilde{\phi}_0^j = \phi_1(t_j) \text{ on } \omega_{0,\tau}, \quad \tilde{\phi}_N^j = \phi_2(t_j) \text{ on } \omega_{b,\tau}, \tag{3.20}$$

and denote the solution of this difference system by $\tilde{\phi}$, where, $\tilde{\phi}^{h,\tau}_{\bar{t},m}$, and $\Theta^3\tilde{\phi}^{h,\tau}_m$ are formulae analogous to (3.15) and (3.16) respectively, and $\tilde{\phi}^0_m = \tilde{\phi}(x_m, 0)$, $\tilde{\phi}^j_0 = \tilde{\phi}(0, t_j)$, $\tilde{\phi}^j_N = \tilde{\phi}(b, t_j)$ also $\varphi_{\tilde{\phi},m} = G\left(x_m, t_{j+1}\right) = \partial^2_x f\big|_{\left(x_m, t_{j+1}\right)}$. By the maximum principle the difference problems (3.12) – (3.14) and (3.18) – (3.20) have unique solution.

Theorem 3.2: Let w be the solution of the Problem 3 with (3.10) and $\tilde{w}$ be the solution of the difference problem (3.12) – (3.14). The following pointwise estimation holds true

$$|\tilde{w} - w| \le c_1 \rho (h^2 + \tau) \tag{3.21}$$

Proof: On the basis of Lemma 3.1 the exact solution $w \in C^{6+\alpha, \frac{6+\alpha}{2}}_{x,t}(\bar{Q}_T)$. Let $\varepsilon^{h,\tau}_w = \tilde{w} - w$ then the error function $\varepsilon^{h,\tau}_w$ satisfies the following difference problem

$$\varepsilon^{h,\tau}_{w,\bar{t},m} = a\Theta^3 \varepsilon^{h,\tau}_{w,m} + \psi_w \text{ on } \omega_{h,\tau}, \tag{3.22}$$
$$\varepsilon^0_{w,m} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{3.23}$$
$$\varepsilon^j_{w,0} = 0 \text{ on } \omega_{0,\tau}, \quad \varepsilon^j_{w,N} = 0 \text{ on } \omega_{b,\tau}, \tag{3.24}$$

where $\psi_w = a\Theta^3 w - w_{\bar{t},m} + \varphi_{\tilde{w},m}$ and $\varphi_{\tilde{w},m}$ is as given in (3.17). Using Taylor's formula for the function $w(x,t)$ about the node (x_m, t_{j+1}) gives that $\psi_w = O(h^2 + \tau)$. Applying Lemma 2.2 to the problem (2.34) – (2.36) for $\varsigma = 3$ and (3.22) – (3.24) and on the basis of Lemma 2.3 we obtain $|\varepsilon^{h,\tau}_w| \le c_1 \rho (h^2 + \tau)$. ■

Theorem 3.3: The following inequality holds

$$|\tilde{\phi} - \phi| \le c_2 \rho (h^2 + \tau) \tag{3.25}$$

where, ϕ is the solution of the Problem 4 with (3.11) and $\tilde{\phi}$ is the solution of the difference problem (3.18) – (3.20).

Proof: The proof is analogous to the proof of Theorem 3.2.

3.3 Implicit difference problem for the approximation of $\partial_t^2 u$

When the Problem 1(iii) is given we set up the Problem 3 with (3.10) for $w = \partial_t u$ and use the difference system (3.12) – (3.14) for obtaining the approximate solution $\tilde{w}$. We denote $q_i = \partial_t^2 u$ on γ_i, $i = 1, 2, 3$ and construct the following boundary value problem for $z = \partial_t^2 u$, see also [23].

Problem 5:

$$Lz = \partial_t^2 f(x,t) \quad \text{on} \quad Q_T, \tag{3.26}$$

$$z(x,0) = q_2 \quad \text{on} \quad \gamma_2, \tag{3.27}$$

$$z(0,t) = q_1 \quad \text{on} \quad \gamma_1, \; z(b,t) = q_3 \quad \text{on} \quad \gamma_3, \tag{3.28}$$

where, $f(x,t)$ is the heat source function in (2.1).

We take

$$q_{1\tau} = D_t^2 u_1(t) = D_t w_1(t) \text{ on } \omega_{0,\tau}, \tag{3.29}$$

$$q_{2\tau}(\tilde{w}) = \frac{1}{\tau}\left[\tilde{w}(x,\tau) - w_0(x)\right] \text{ on } \bar{\omega}_{h,0}, \tag{3.30}$$

$$q_{3\tau} = D_t^2 u_2(t) = D_t w_2(t) \quad \text{on} \quad \omega_{b,\tau}, \tag{3.31}$$

where, $w_0(x) = u^{(1)}(x)$ is as defined in (2.5) and $u_1(t)$, $u_2(t)$ are given boundary functions in (2.3).

Lemma 3.4: The following inequality holds:

$$|q_{2\tau}(\tilde{w}) - q_{2\tau}(w)| \le c_1(h^2 + \tau) \quad (2.32)$$

where, w is the solution of Problem 3 with (3.10) and $\tilde{w}$ is the solution of the difference problem (3.12) – (3.14).

Proof: From Theorem 3.2, we have

$$|q_{2\tau}(\tilde{w}) - q_{2\tau}(w)| \le \frac{1}{\tau}\left[(c_1\tau)(h^2+\tau)\right] = c_1(h^2+\tau) \quad (3.33)$$

Lemma 3.5: The following inequality is true:

$$\max_{\bar{\omega}_{h,0}} |q_{2\tau}(\tilde{w}) - q_2| \le c_2(h^2+\tau) \quad (3.34)$$

where, $\tilde{w}$ is the solution of the difference problem (3.12) – (3.14).

Proof: On the basis of Lemma 3.1, $w \in C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}(\bar{Q}_T)$ and at the end points $(mh,0) \in \bar{\omega}_{h,0}$ of each line segment $\left[(x,t): 0 \le x \le b, 0 \le t \le T\right]$ the equation (3.30) gives the first order approximation of $\partial_t w$. From the truncation error formula (see [20]) it follows that

$$\max_{\bar{\omega}_{h,0}} |q_{2\tau}(w) - q_2| \le \frac{\tau}{2}\max_{\bar{Q}_T} |\partial_t^2 w| \le c_2\tau, \quad (3.35)$$

Using Lemma 3.4 and the estimation (3.32), (3.35) follows (3.34) ■

We construct the following difference problem for the numerical solution of Problem 5

$$\tilde{z}_{\bar{t},m}^{h,\tau} = a\Theta^3\tilde{z}_m^{h,\tau} + \varphi_{\tilde{z},m} \text{ on } \omega_{h,\tau}, \quad (3.36)$$

$$\tilde{z}_m^0 = q_{2\tau}(\tilde{w}) \text{ on } \bar{\omega}_{h,0}, \quad (3.37)$$

$$\tilde{z}_0^j = q_{1\tau} \text{ on } \omega_{0,\tau}, \quad \tilde{z}_z^j = q_{3\tau} \text{ on } \omega_{b,\tau}, \quad (3.38)$$

where, $q_{i\tau}$, $i=1,2,3$ are defined by (3.29) – (3.31) and $\varphi_{\tilde{z},m}=\partial_t^2 f\,|_{(x_m,t_{j+1})}$.

Theorem 3.6: The solution $\tilde{z}$ of the finite difference problem (3.36) – (3.38) satisfies

$$\max_{\bar{\omega}_{h,\tau}} |\tilde{z}-z| \le c_3(h^2+\tau), \tag{3.39}$$

where $z=\partial_t^2 u$ is the exact solution of Problem 5.

Proof: Let

$$\varepsilon_z^{h,\tau}=\tilde{z}-z \quad \text{on} \quad \bar{\omega}_{h,\tau}, \tag{3.40}$$

where, $z=\partial_t^2 u.$ Denote by $\left\|\varepsilon_z^{h,\tau}\right\|=\max_{\bar{\omega}_{h,\tau}}\left|\tilde{z}-z\right|.$ From (3.36) – (3.38) and (3.40) we have

$$\varepsilon_{z,\bar{t},m}^{h,\tau}=a\Theta^3\varepsilon_{z,m}^{h,\tau}+\Psi_z \text{ on } \omega_{h,\tau}, \tag{3.41}$$

$$\varepsilon_{z,m}^{0}=q_{2\tau}(\tilde{w})-z \text{ on } \bar{\omega}_{h,0}, \tag{3.42}$$

$$\varepsilon_{z,0}^{j}=0 \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon_{z,N}^{j}=0 \text{ on } \omega_{b,\tau}, \tag{3.43}$$

where $\Psi_z=a\Theta^3 z-z_{\bar{t},m}+\varphi_{\tilde{z},m}.$ We take

$$\varepsilon_z^{h,\tau}=\varepsilon_z^{1,h,\tau}+\varepsilon_z^{2,h,\tau} \tag{3.44}$$

and $\varepsilon_z^{1,h,\tau}$, $\varepsilon_z^{2,h,\tau}$ satisfy the problems

$$\varepsilon_{z,\bar{t},m}^{1,h,\tau}=a\Theta^3\varepsilon_{z,m}^{1,h,\tau} \text{ on } \omega_{h,\tau}, \tag{3.45}$$

$$\varepsilon_{z,m}^{1,0}=q_{2\tau}(\tilde{w})-z \text{ on } \bar{\omega}_{h,0}, \tag{3.46}$$

$$\varepsilon_{z,0}^{1,j}=0 \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon_{z,N}^{1,j}=0 \text{ on } \omega_{b,\tau}, \tag{3.47}$$

and

$$\varepsilon_{z,\bar{t},m}^{2,h,\tau}=a\Theta^3\varepsilon_{z,m}^{2,h,\tau}+\Psi_z \text{ on } \omega_{h,\tau}, \tag{3.48}$$

$$\varepsilon_{z,m}^{2,0}=0 \text{ on } \bar{\omega}_{h,0}, \tag{3.49}$$

$$\varepsilon_{z,0}^{2,j}=0 \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon_{z,N}^{2,j}=0 \text{ on } \omega_{b,\tau}. \tag{3.50}$$

respectively. From Lemma 3.5 and by maximum principle for the solution of the problem (3.45) – (3.47) we have

$$\max_{\bar{\omega}_{h,\tau}} \left|\varepsilon_z^{1,h,\tau}\right| \le \max_{\bar{\omega}_{h,\tau}} \left|q_{2\tau}(\tilde{w}) - z\right| \le c_4(h^2 + \tau). \quad (3.51)$$

The solution $\varepsilon_z^{2,h,\tau}$ of the problem (3.48) – (3.50) is the error of the approximate solution obtained by the finite difference method for the boundary value Problem 5 when

$$q_2 \in C^{4+\alpha}(\bar{\Omega}), \quad \partial_t^2 f(x,t) \in C_{x,t}^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q}_T), \quad q_i \in C^{\frac{4+\alpha}{2}}(\bar{\sigma}_T), \; i = 1,3. \quad (3.52)$$

and

$$\begin{cases} q_1^{(q)}(0) = z^{(q)}(0), \\ q_3^{(q)}(0) = z^{(q)}(b), \qquad q = 0,1,2. \end{cases} \quad (3.53)$$

Since the function $z = \partial_t^2 u$ satisfies the equation (3.26) with the initial function q_2 on γ_2 and boundary functions q_1, q_3 on γ_1 and γ_3 respectively, using (3.52), (3.53) and on the basis of Theorem 1.1 and the maximum principle in Chapter 4 of [19] we obtain

$$\max_{\bar{\omega}_{h,\tau}} \left|\varepsilon_z^{2,h,\tau}\right| \le c_5(h^2 + \tau). \quad (3.54)$$

Using (3.44), (3.51) and (3.54) we obtain (3.39). ■

3.4 Implicit difference problem for the approximation of $\partial_x \partial_t u$

Given the Problem 1(iii), we setup the Problem 3 with (3.10) for $w = \partial_t u$ and use the difference system (3.12) – (3.14) for obtaining the approximate solution $\tilde{w}$. We denote $p_i = \partial_x \partial_t u$ on γ_i, $i = 1,2,3$ respectively, and give the following boundary value problem for $y = \partial_x \partial_t u$ (see also [23]).

Problem 6:

$$Ly = \partial_x \partial_t f(x,t) \quad \text{on} \quad Q_T \tag{3.55}$$

$$y(x,0) = p_2 \quad \text{on} \quad \gamma_2 \tag{3.56}$$

$$y(0,t) = p_1 \quad \text{on} \quad \gamma_1, \quad y(b,t) = p_3 \quad \text{on} \quad \gamma_3, \tag{3.57}$$

where, $f(x,t)$ is the given function in (2.1). We take

$$p_{1h}(\tilde{w}) = \frac{-3w_1(t) + 4\tilde{w}(h,t) - \tilde{w}(2h,t)}{2h} \quad \text{on} \quad \omega_{0,\tau}, \tag{3.58}$$

$$p_{2h} = D_x w_0(x) \quad \text{on} \quad \bar{\omega}_{h,0}, \tag{3.59}$$

$$p_{3h}(\tilde{w}) = \frac{3w_2(t) - 4\tilde{w}(b-h,t) + \tilde{w}(b-2h,t)}{2h} \quad \text{on} \quad \omega_{b,\tau}, \tag{3.60}$$

where, $w_0(x) = u^{(1)}(x)$ and $w_1 = D_t u_1(t),\ w_2 = D_t u_2(t).$

We construct the following difference problem for the numerical solution of Problem 6 and denote this solution by $\tilde{y}$

$$\tilde{y}^{h,\tau}_{\bar{t},m} = a\Theta^3 \tilde{y}^{h,\tau}_m + \varphi_{\tilde{y},m} \quad \text{on} \quad \omega_{h,\tau}, \tag{3.61}$$

$$\tilde{y}^0_m = p_{2h} \quad \text{on} \quad \bar{\omega}_{h,0}, \tag{3.62}$$

$$\tilde{y}^j_0 = p_{1h}(\tilde{w}) \quad \text{on} \quad \omega_{0,\tau}, \quad \tilde{y}^j_N = p_{3h}(\tilde{w}) \quad \text{on} \quad \omega_{b,\tau}. \tag{3.63}$$

Here, p_{ih} are defined by (3.58) – (3.60) $\tilde{y}^{h,\tau}_{\bar{t},m}$, $\Theta^3 \tilde{y}^{h,\tau}_m$ are formulae analogous to (3.15) and (3.16) respectively, and $\tilde{y}^0_m = \tilde{y}(x_m, 0)$, $\tilde{y}^j_0 = \tilde{y}(0, t_j)$, $\tilde{y}^j_N = \tilde{y}(b, t_j)$ and $\varphi_{\tilde{y},m} = \partial_x \partial_t f \big|_{(x_m, t_{j+1})}$.

Lemma 3.7: The following inequality holds

$$|\, p_{ih}(\tilde{w}) - p_{ih}(w) \,| \le c_1(h^2 + \tau), \quad i = 1, 3. \tag{3.64}$$

where w is the solution of the Problem 3 with (3.10) and $\tilde{w}$ is the solution of the difference problem (3.12) – (3.14).

Proof: From (3.58), (3.60) and Theorem 3.2, we have

$$|p_{ih}(\tilde{w})-p_{ih}(w)| \le \frac{1}{2h}\left[4(ch)(h^2+\tau)+(c2h)(h^2+\tau)\right] \le c_1(h^2+\tau), \quad i=1,3. \tag{3.65}$$

Lemma 3.8: The following inequality is true

$$\max_{\omega_{0,\tau}\cup\omega_{b,\tau}} |p_{ih}(\tilde{w})-p_i| \le c_2(h^2+\tau), \quad i=1,3. \tag{3.66}$$

where, $\tilde{w}$ is the solution of the difference problem (3.12) – (3.14).

Proof: On the basis of Lemma 3.1, the exact solution of Problem 3 belongs to $C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}(\bar{Q}_T)$. Then at the end points $(0,\sigma\tau)\in\omega_{0,\tau}$ and $(b,\sigma\tau)\in\omega_{b,\tau}$ of each line segment $\left[(x,t): 0\le x\le b, 0\le t\le T\right]$ the equations (3.58) and (3.60) give the second order approximation of $\partial_x w$ respectively. From the truncation error formula (see [20]) it follows that

$$\max_{\omega_{0,\tau}\cup\omega_{b,\tau}} |p_{ih}(w)-p_i| \le \frac{h^2}{3}\max_{\bar{Q}_T}|\partial_x^3 w| \le c_3 h^2, \quad i=1,3. \tag{3.67}$$

On the basis of Lemma 3.7 using the estimation (3.64) and (3.67) follows (3.66). ■

Theorem 3.9: The solution $\tilde{y}$ of the finite difference problem (3.61) – (3.63) satisfies

$$\max_{\bar{\omega}_{h,\tau}} |\tilde{y}-y| \le c_4(h^2+\tau), \tag{3.68}$$

where $y=\partial_x\partial_t u$ is the exact solution of the Problem 6.

Proof: Let

$$\varepsilon_y^{h,\tau} = \tilde{y} - y \quad \text{on} \quad \bar{\omega}_{h,\tau}, \tag{3.69}$$

where $y = \partial_x \partial_t u$. Denote by $\left\| \varepsilon_y^{h,\tau} \right\| = \max_{\bar{\omega}_{h,\tau}} \left| \tilde{y} - y \right|.$ From (3.61) – (3.63) and (3.69) we have

$$\varepsilon_{y,\bar{t},m}^{h,\tau} = a\Theta^3 \varepsilon_{y,m}^{h,\tau} + \psi_y \ \text{ on } \ \omega_{h,\tau}, \tag{3.70}$$

$$\varepsilon_{y,m}^{0} = 0 \ \text{ on } \ \bar{\omega}_{h,0}, \tag{371}$$

$$\varepsilon_{y,0}^{j} = p_{1h}(\tilde{w}) - y \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon_{y,N}^{j} = p_{3h}(\tilde{w}) - y \ \text{ on } \ \omega_{b,\tau}, \tag{3.72}$$

where $\psi_y = a\Theta^3 y - y_{\bar{t},m} + \varphi_{\tilde{y},m}.$ We take

$$\varepsilon_y^{h,\tau} = \varepsilon_y^{1,h,\tau} + \varepsilon_y^{2,h,\tau}, \tag{3.73}$$

and $\varepsilon_y^{1,h,\tau}$, $\varepsilon_y^{2,h,\tau}$ satisfy the difference problems

$$\varepsilon_{y,\bar{t},m}^{1,h,\tau} = a\Theta^3 \varepsilon_{y,m}^{1,h,\tau} \ \text{ on } \ \omega_{h,\tau}, \tag{3.74}$$

$$\varepsilon_{y,m}^{1,0} = 0 \ \text{ on } \ \bar{\omega}_{h,0}, \tag{3.75}$$

$$\varepsilon_{y,0}^{1,j} = p_{1h}(\tilde{w}) - y \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon_{y,N}^{1,j} = p_{3h}(\tilde{w}) - y \ \text{ on } \ \omega_{b,\tau}, \tag{3.76}$$

and

$$\varepsilon_{y,\bar{t},m}^{2,h,\tau} = a\Theta^3 \varepsilon_{y,m}^{2,h,\tau} + \psi_y \ \text{ on } \ \omega_{h,\tau}, \tag{3.77}$$

$$\varepsilon_{y,m}^{2,0} = 0 \ \text{ on } \ \bar{\omega}_{h,0}, \tag{3.78}$$

$$\varepsilon_{y,0}^{2,j} = 0 \quad \text{on} \quad \omega_{0,\tau}, \quad \varepsilon_{y,N}^{2,j} = 0 \ \text{ on } \ \omega_{b,\tau}. \tag{3.79}$$

respectively. From Lemma 3.8 and by maximum principle for the solution of the system (3.74) – (3.76) we have

$$\max_{\bar{\omega}_{h,\tau}} \left| \varepsilon_y^{1,h,\tau} \right| \le \max_{i=1,3} \max_{\bar{\omega}_{h,\tau}} \left| p_{ih}(\tilde{w}) - y \right| \le c_4(h^2 + \tau). \tag{3.80}$$

The solution $\varepsilon_y^{2,h,\tau}$ of the problem (3.77) – (3.79) is the error of the approximate solution obtained by the finite difference method for the Problem 6 when the boundary values satisfy the conditions

$$p_2 \in C^{5+\alpha}(\bar{\Omega}), \quad \partial_x\partial_t f(x,t) \in C_{x,t}^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}_T), \quad p_i \in C^{\frac{5+\alpha}{2}}(\bar{\sigma}_T),\ i=1,3. \tag{3.81}$$

$$\begin{cases} p_1^{(q)}(0)=y^{(q)}(0), \\ p_3^{(q)}(0)=y^{(q)}(b), \qquad q=0,1,2. \end{cases} \tag{3.82}$$

Since the function $y=\partial_x\partial_t u$ satisfies the equation (3.55) with the initial function p_2 on γ_2 and boundary functions p_1, p_3 on γ_1 and γ_3, respectively using (3.81) and (3.82) and on the basis of Theorem 1.1 and the maximum principle in Chapter 4 of [19] we obtain

$$\max_{\bar{\omega}_{h,\tau}}\left|\varepsilon_y^{2,h,\tau}\right| \le c_5(h^2+\tau). \tag{3.83}$$

using (3.73), (3.80) and (3.83) we obtain (3.68). ■

3.5 Numerical aspects

Example 4: we consider the following boundary value problem

$$\begin{aligned} Lu &= f(x,t) \quad \text{on} \quad Q_T, \\ u(x,0) &= 1-x^{\frac{41}{5}} \quad \text{on} \quad \gamma_2, \\ u(0,t) &= \cos(\pi t) \quad \text{on} \quad \gamma_1, \\ u(1,t) &= t^{\frac{41}{10}}\sin(1)+\cos(\pi t)-1 \quad \text{on} \quad \gamma_3, \end{aligned}$$

where

$$f(x,t)=\sin(x^{\frac{41}{5}})\left[\frac{41}{10}t^{\frac{31}{10}}+\left(\frac{41}{5}\right)^2 x^{\frac{72}{5}}t^{\frac{41}{10}}\right]-\pi\sin(\pi t)+\frac{41}{5}\frac{36}{5}x^{\frac{31}{5}}\left[1-t^{\frac{41}{10}}\cos(x^{\frac{41}{5}})\right].$$

3.5.1 Numerical results for $\partial_t u$ and $\partial_x^2 u$

Using the proposed Problem 3 with (3.10) and Problem 4 with (3.11) we construct the boundary value problems for $w = \partial_t u$ and $\phi = \partial_x^2 u$, respectively. Then for the approximate solution of the Problem 3 with (3.10) the difference system (3.12) – (3.14) and for the Problem 4 with (3.11) the difference system (3.18) – (3.20) are solved directly by applying Gauss-Thomas Method. Using the exact solutions,

$$w(x,t) = \frac{41}{10} t^{\frac{31}{10}} \sin(x^{\frac{41}{5}}) - \pi \sin(\pi t),$$

$$\phi(x,t) = \frac{41}{5}\frac{36}{5} x^{\frac{31}{5}} \left[t^{\frac{41}{10}} \cos(x^{\frac{41}{5}}) - 1 \right] - \left(\frac{41}{5}\right)^2 x^{\frac{72}{5}} t^{\frac{41}{10}} \sin(x^{\frac{41}{5}}),$$

we denote the maximum error on the grid points by $\left\| \varepsilon_w^{h,\tau} \right\| = \max_{\bar{\omega}_{h,\tau}} | \tilde{w} - w |$ and by $\left\| \varepsilon_\phi^{h,\tau} \right\| = \max_{\bar{\omega}_{h,\tau}} | \tilde{\phi} - \phi |$. Table 3.1 demonstrates the maximum errors for $r = \frac{\tau}{h^2} = 2^{-\omega}$, $\omega = 2, 3$ and the corresponding process time for different step sizes and the order of convergence $\Re_{\tilde{w}}^{h,\tau}$

$$\Re_{\tilde{w}}^{h,\tau} = \frac{\left\| \varepsilon_w^{2^{-\mu}, 2^{-\lambda}} \right\|}{\left\| \varepsilon_w^{2^{-(\mu+1)}, 2^{-(\lambda+2)}} \right\|} \tag{3.84}$$

of $\tilde{w}$ to the exact solution $w = \partial_t u$ with respect to h and τ, for Example 4.

Table 3.1: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{w}}^{h,\tau}$, for Example 4.

$(h = 2^{-\mu}, \tau = 2^{-\lambda})$		$(h = 2^{-(\mu+1)}, \tau = 2^{-(\lambda+2)})$		$\left\| \varepsilon_w^{2^{-\mu}, 2^{-\lambda}} \right\|$	$\left\| \varepsilon_w^{2^{-(\mu+1)}, 2^{-(\lambda+2)}} \right\|$	$\Re_{\tilde{w}}^{h,\tau}$
h, τ	CPU	h, τ	CPU			
$(2^{-5}, 2^{-12})$	0.469	$(2^{-6}, 2^{-14})$	2.938	$1.3101E-02$	$3.3328E-03$	3.9309
$(2^{-6}, 2^{-14})$	2.938	$(2^{-7}, 2^{-16})$	22.422	$3.3328E-03$	$8.3892E-04$	3.9727
$(2^{-5}, 2^{-13})$	0.812	$(2^{-6}, 2^{-15})$	5.812	$1.3335E-02$	$3.3906E-03$	3.9329
$(2^{-6}, 2^{-15})$	5.812	$(2^{-7}, 2^{-17})$	44.542	$3.3906E-03$	$8.5244E-04$	3.9775

Table 3.2 presents the maximum errors and the corresponding central processing unit time for $h=2^{-9}$, $r=2^{-\lambda}$, $\lambda=7,8,9,10,11$ and the order of convergence $\Re_{\tilde{w}}^{\tau}$

$$\Re_{\tilde{w}}^{\tau}=\frac{\left\|\varepsilon_{w}^{h,2^{-\lambda}}\right\|}{\left\|\varepsilon_{w}^{h,2^{-(\lambda+1)}}\right\|}, \tag{3.85}$$

of $\tilde{w}$ in time variable t (with respect to τ).

Table 3.2: Maximum errors, corresponding CPU time for different step size in time and $\Re_{\tilde{w}}^{\tau}$, for Example 4.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-\mu},\tau=2^{-(\lambda+1)})$		$\left\|\varepsilon_{w}^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_{w}^{2^{-\mu},2^{-(\lambda+1)}}\right\|$	$\Re_{\tilde{w}}^{\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-9},2^{-7})$	0.234	$(2^{-9},2^{-8})$	0.359	$2.2916E-02$	$1.1459E-02$	1.9998
$(2^{-9},2^{-8})$	0.359	$(2^{-9},2^{-9})$	0.672	$1.1459E-02$	$5.7182E-03$	2.0040
$(2^{-9},2^{-9})$	0.672	$(2^{-9},2^{-10})$	1.312	$5.7182E-03$	$2.8450E-03$	2.0099
$(2^{-9},2^{-10})$	1.312	$(2^{-9},2^{-11})$	2.609	$2.8450E-03$	$1.4078E-03$	2.0209

Note that the $O(h^2+\tau)$ order of convergence corresponds to $\approx 2^2$ of the quantities defined by (3.84), and $\approx 2^1$ of the quantities defined by (3.85) respectively. Figure 3.1 shows the exact solution $w=\partial_t u$ and the grid functions $w^{2^{-11},2^{-14}}$ denoting the approximate solution $\tilde{w}$ for $h=2^{-11}$ and $\tau=2^{-14}$ for Example 4. The error function $\left|\varepsilon_{w}^{2^{-11},2^{-14}}\right|=|\tilde{w}-w|$ for $h=2^{-11}$ and $\tau=2^{-14}$ is given in Figure 3.2. The exact solution $\phi=\partial_x^2 u$ and the grid functions $\phi^{2^{-11},2^{-14}}$ denoting the approximate solution $\tilde{\phi}$ for

$h=2^{-11}$ and $\tau=2^{-14}$ are presented in Figure 3.3. Also the error function $\left|\varepsilon_{\phi}^{2^{-11},2^{-14}}\right|=|\tilde{\phi}-\phi|$ for $h=2^{-11}$ and $\tau=2^{-14}$ is shown in Figure 3.4.

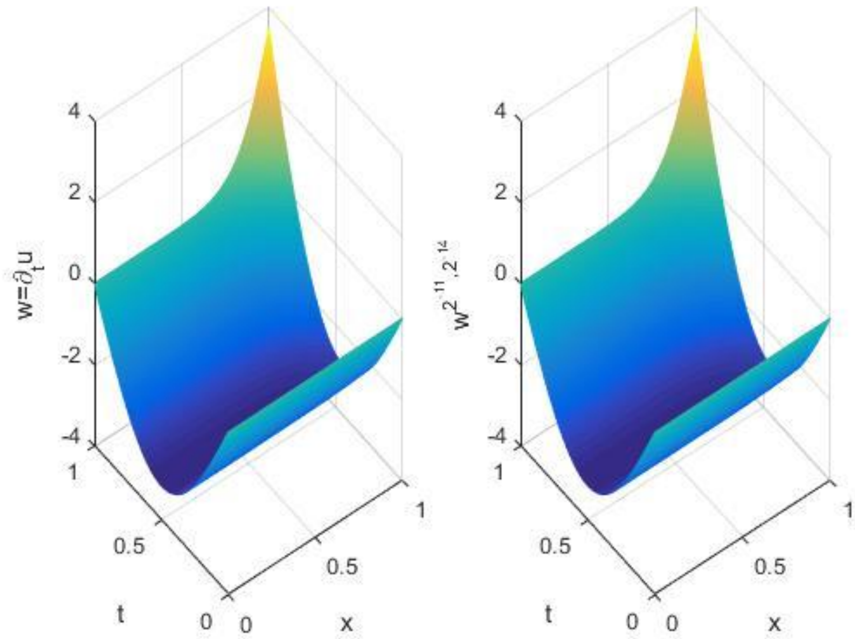

Figure 3.1: The exact solution $w=\partial_t u$ and the grid functions $w^{2^{-11},2^{-14}}$ presenting the approximate solution $\tilde{w}$ when $h=2^{-11}$ and $\tau=2^{-14}$, for Example 4.

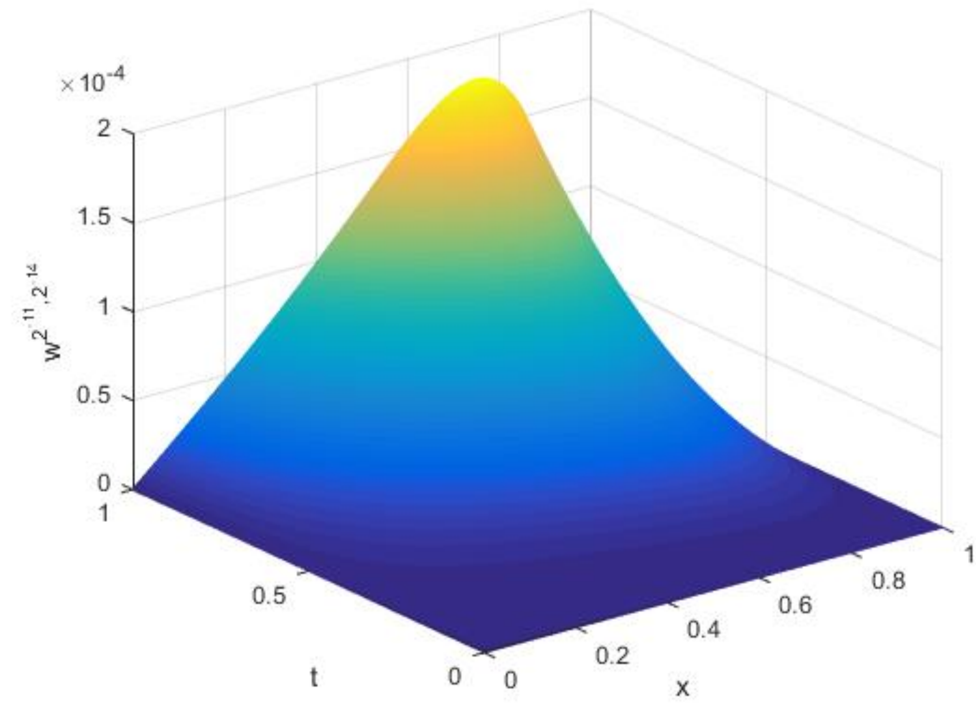

Figure 3.2: The error function $\left|\varepsilon_w^{2^{-11},2^{-14}}\right|$ representing $|\tilde{w}-w|$ when $h=2^{-11}$ and $\tau=2^{-14}$, for Example 4.

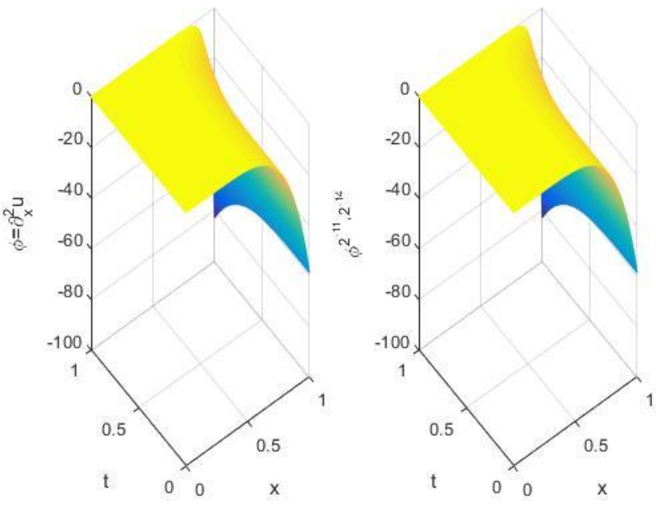

Figure 3.3: The exact solution $\phi=\partial_x^2 u$ and the grid functions $\phi^{2^{-11},2^{-14}}$ presenting the approximate solution $\tilde{\phi}$ when $h=2^{-11}$ and $\tau=2^{-14}$, of Example 4.

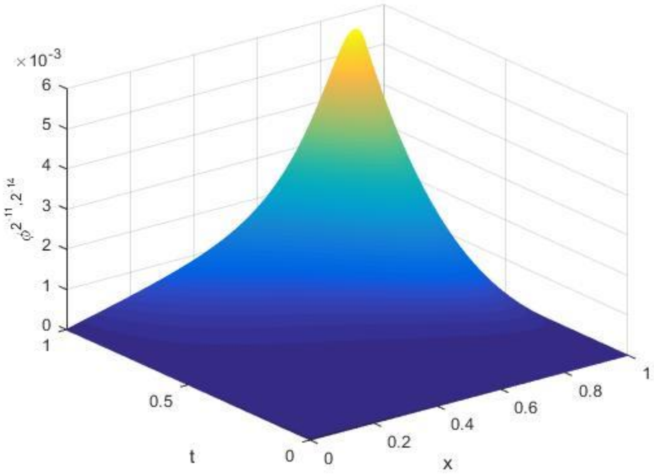

Figure 3.4: The error function $\left|\varepsilon_{\phi}^{2^{-11},2^{-14}}\right|$ presenting $|\tilde{\phi}-\phi|$ for $h=2^{-11}$ and $\tau=2^{-14}$, of Example 4.

3.5.2 Numerical results for $\partial_t^2 u$

Here, the boundary value problem for $z=\partial_t^2 u$ is constructed using the proposed Problem 5 and the approximate solution $\tilde{w}$. Further, the approximate solution $\tilde{z}$ of the difference problem (3.36) – (3.38) is obtained at the same grid points. Using the exact solution

$$z(x,t)=\frac{41}{10}\frac{31}{10}t^{\frac{21}{10}}\sin(x^{\frac{41}{5}})-\pi^2\cos(\pi t)$$

we present the maximum errors by $\left\|\varepsilon_z^{h,\tau}\right\|=\max_{\bar{\omega}_{h,\tau}}|\tilde{z}-z|$. The order of convergence of the approximate solution $\tilde{z}$ to the exact solution $z=\partial_t^2 u$ in spatial variable x and in time variable t is denoted by $\Re_{\tilde{z}}^{h,\tau}$ analogous to the formula (3.84). The order of convergence of $\tilde{z}$ to the exact solution z in time variable t is represented by $\Re_{\tilde{z}}^{\tau}$ analogous to the formula (3.85). Table 3.3 shows the maximum errors and the corresponding CPU time for $r=2^{-\omega}$, $\omega=2,3$ and the order of convergence $\Re_{\tilde{z}}^{h,\tau}$, while Table 3.4, shows the maximum errors for $h=2^{-9}$ and $\tau=2^{-\lambda}$, $\lambda=7,8,9,10,11$, and the corresponding elapsed time with the order of convergence $\Re_{\tilde{z}}^{\tau}$, for Example 4. Figure 3.5 demonstrates the exact solution $z=\partial_t^2 u$ and the grid functions $z^{2^{-11},2^{-14}}$ denoting the approximate solution $\tilde{z}$ for $h=2^{-11}$ and $\tau=2^{-14}$, for Example 4. The Figure 3.6 illustrates the error function $\left|\varepsilon_z^{2^{-11},2^{-14}}\right|=|\tilde{z}-z|$ for $h=2^{-11}$ and $\tau=2^{-14}$, for the same example.

Table 3.3: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{z}}^{h,\tau}$, for Example 4.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-(\mu+1)},\tau=2^{-(\lambda+2)})$		$\left\|\varepsilon_z^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_z^{2^{-(\mu+1)},2^{-(\lambda+2)}}\right\|$	$\Re_{\tilde{z}}^{h,\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-5},2^{-12})$	0.453	$(2^{-6},2^{-14})$	3.344	$4.1795E-02$	$1.0631E-02$	3.9314
$(2^{-6},2^{-14})$	3.344	$(2^{-7},2^{-16})$	26.000	$1.0631E-02$	$2.6706E-03$	3.9808
$(2^{-5},2^{-13})$	0.922	$(2^{-6},2^{-15})$	6.828	$4.2309E-02$	$1.0758E-02$	3.9328
$(2^{-6},2^{-15})$	6.828	$(2^{-7},2^{-17})$	52.375	$1.0758E-02$	$2.7008E-03$	3.9833

Table 3.4: Maximum errors, corresponding CPU time for different step size in time and $\Re_{\tilde{z}}^{\tau}$, for Example 4.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-\mu},\tau=2^{-(\lambda+1)})$		$\left\|\varepsilon_z^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_z^{2^{-\mu},2^{-(\lambda+1)}}\right\|$	$\Re_{\tilde{z}}^{\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-9},2^{-7})$	0.266	$(2^{-9},2^{-8})$	0.406	$5.1406E-02$	$2.5678E-02$	2.0019
$(2^{-9},2^{-8})$	0.406	$(2^{-9},2^{-9})$	0.781	$2.5678E-02$	$1.2708E-03$	2.0206
$(2^{-9},2^{-9})$	0.781	$(2^{-9},2^{-10})$	1.531	$1.2708E-02$	$6.3529E-03$	2.0004
$(2^{-9},2^{-10})$	1.531	$(2^{-9},2^{-11})$	3.078	$6.3529E-03$	$3.1301E-03$	2.0296

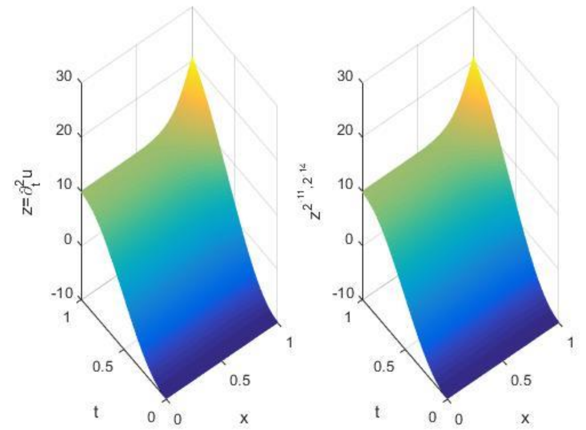

Figure 3.5: The exact solution $z=\partial_t^2 u$ and the grid functions $z^{2^{-11},2^{-14}}$ presenting the approximate solution $\tilde{z}$ for $z=\partial_t^2 u$ when $h=2^{-11}$ and $\tau=2^{-14}$, of Example 4.

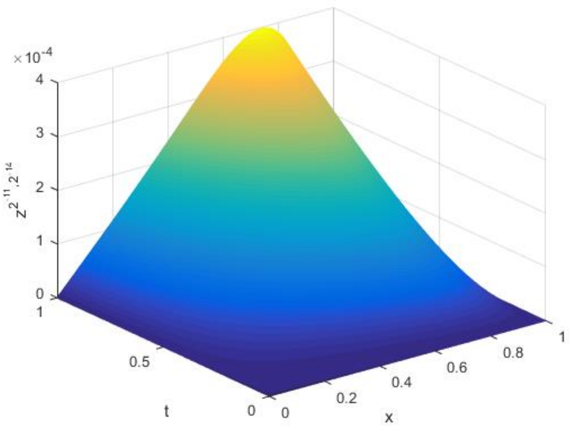

Figure 3.6: The error function $\left|\varepsilon_z^{2^{-11},2^{-14}}\right|$ presenting $|\tilde{z}-z|$ for $h=2^{-11}$ and $\tau=2^{-14}$, of Example 4.

3.5.3 Numerical results for $\partial_x\partial_t u$

We setup the boundary value problems for $y=\partial_x\partial_t u$ from the proposed Problem 6 using the approximate solution $\tilde{w}$. The approximate solution $\tilde{y}$ of the difference system (3.61) – (3.63) is obtained at the same grid points. By virtue of the exact solution

$$y(x,t)=\frac{41}{10}\frac{41}{5}x^{\frac{36}{5}}t^{\frac{31}{10}}\cos(x^{\frac{41}{5}})$$

We denote the maximum error by $\left\|\varepsilon_y^{h,\tau}\right\| = \max_{\bar{\omega}_{h,\tau}} |\tilde{y} - y|$. The maximum errors, corresponding CPU time for different step sizes in space and

$$\Re_{\tilde{y}}^{h} = \frac{\left\|\varepsilon_y^{2^{-\mu},2^{-\lambda}}\right\|}{\left\|\varepsilon_y^{2^{-(\mu+1)},2^{-\lambda}}\right\|}, \tag{3.86}$$

which is the order of convergence of $\tilde{y}$ to the exact solution $y = \partial_x \partial_t u$ in the spatial variable x (with respect to h) are given in Table 3.5. Table 3.6 presents the processing unit time for different step sizes and the corresponding maximum errors with the order of convergence $\Re_{\tilde{y}}^{\tau}$ of $\tilde{y}$ analogous to the formula (3.85) to the exact solution $y = \partial_x \partial_t u$ in time variable t, for Example 4. Figure 3.7 demonstrates the exact solution $y = \partial_x \partial_t u$ and the grid function $y^{2^{-11},2^{-14}}$ denoting the approximate solution $\tilde{y}$ for $h = 2^{-11}$ and $\tau = 2^{-14}$. The error function $\left|\varepsilon_y^{2^{-11},2^{-14}}\right| = |\tilde{y} - y|$ for $h = 2^{-11}$ and $\tau = 2^{-14}$ is shown in Figure 3.8. The maximum errors $\left\|\varepsilon_y^{h,2^{-14}}\right\|$ when $\tau = 2^{-14}$, with respect to h, and the maximum errors $\left\|\varepsilon_y^{2^{-9},\tau}\right\|$ when $h = 2^{-9}$, with respect to τ are illustrated by Figure 3.9 and Figure 3.10, respectively.

Table 3.5: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{y}}^{\tau}$, for Example 4.

$(h = 2^{-\mu}, \tau = 2^{-\lambda})$		$(h = 2^{-(\mu+1)}, \tau = 2^{-\lambda})$		$\left\|\varepsilon_y^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_y^{2^{-(\mu+1)},2^{-\lambda}}\right\|$	$\Re_{\tilde{y}}^{h,\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-7}, 2^{-14})$	7.203	$(2^{-8}, 2^{-14})$	13.859	0.1170	$3.0533E-02$	3.832
$(2^{-8}, 2^{-14})$	13.859	$(2^{-9}, 2^{-14})$	27.453	$3.0533E-02$	$6.9746E-03$	4.378
$(2^{-9}, 2^{-14})$	27.453	$(2^{-10}, 2^{-14})$	55.016	$6.9746E-03$	$1.0375E-03$	6.723
$(2^{-10}, 2^{-14})$	55.016	$(2^{-11}, 2^{-14})$	73.091	$1.0375E-03$	$1.9800E-04$	5.240

Table 3.6: Maximum errors, corresponding CPU time for different step size in time and $\Re_{\tilde{y}}^{\tau}$, for Example 4.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-\mu},\tau=2^{-(\lambda+1)})$		$\left\|\varepsilon_y^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_y^{2^{-\mu},2^{-(\lambda+1)}}\right\|$	$\Re_{\tilde{y}}^{\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-9},2^{-7})$	0.375	$(2^{-9},2^{-8})$	0.438	0.1661	$7.9536E-02$	2.088
$(2^{-9},2^{-8})$	0.438	$(2^{-9},2^{-9})$	0.859	$7.9536E-02$	$3.6201E-02$	2.197
$(2^{-9},2^{-9})$	0.859	$(2^{-9},2^{-10})$	1.672	$3.6201E-02$	$1.4559E-02$	2.487
$(2^{-9},2^{-10})$	1.672	$(2^{-9},2^{-11})$	3.422	$1.4559E-02$	$4.9560E-03$	2.938

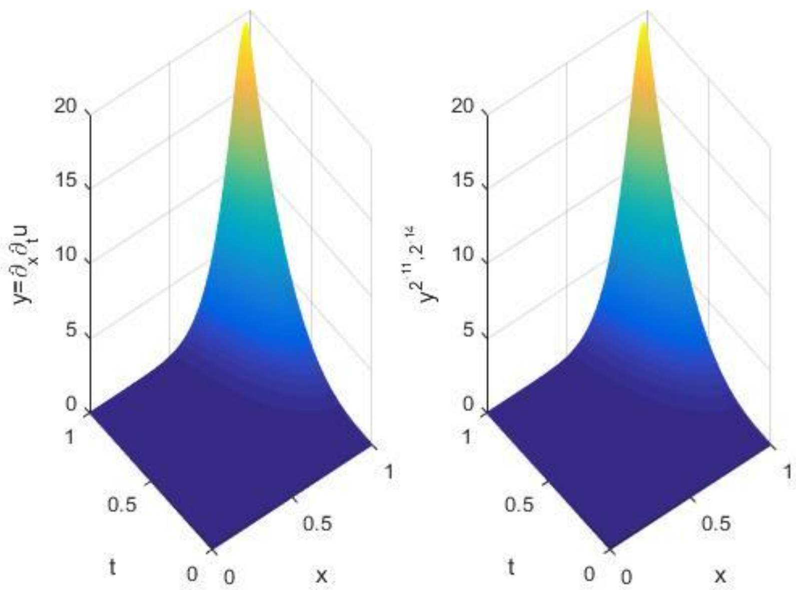

Figure 3.7: The exact solution $y=\partial_x\partial_t u$ and the grid function $y^{2^{-11},2^{-14}}$ presenting the approximate solution $\tilde{y}$ for $h=2^{-11}$ and $\tau=2^{-14}$, of Example 4.

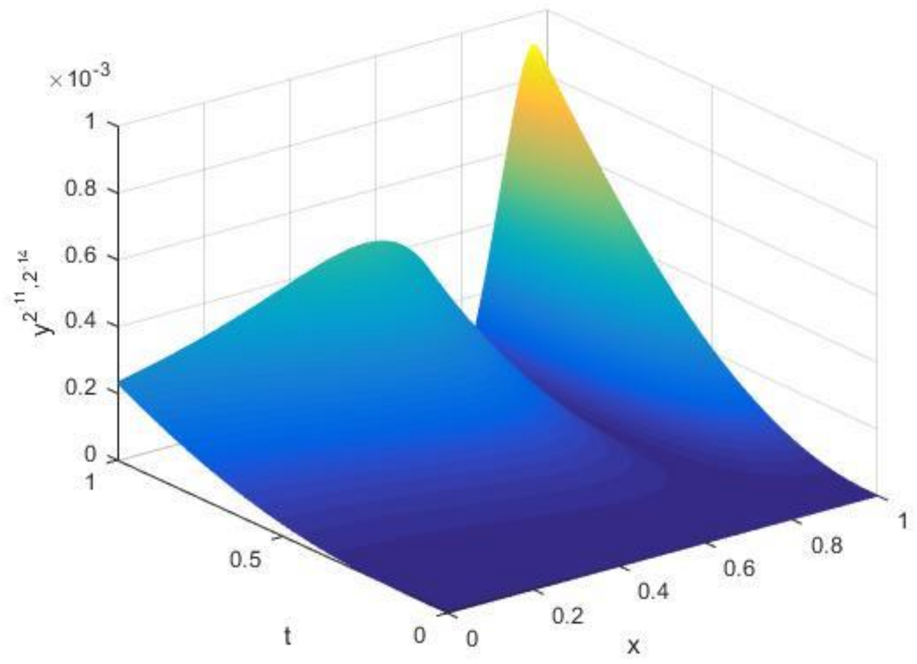

Figure 3.8: The error function $\left|\varepsilon_y^{2^{-11},2^{-14}}\right|$ presenting $|\tilde{y}-y|$ for $h=2^{-11}$ and $\tau=2^{-14}$, of Example 4.

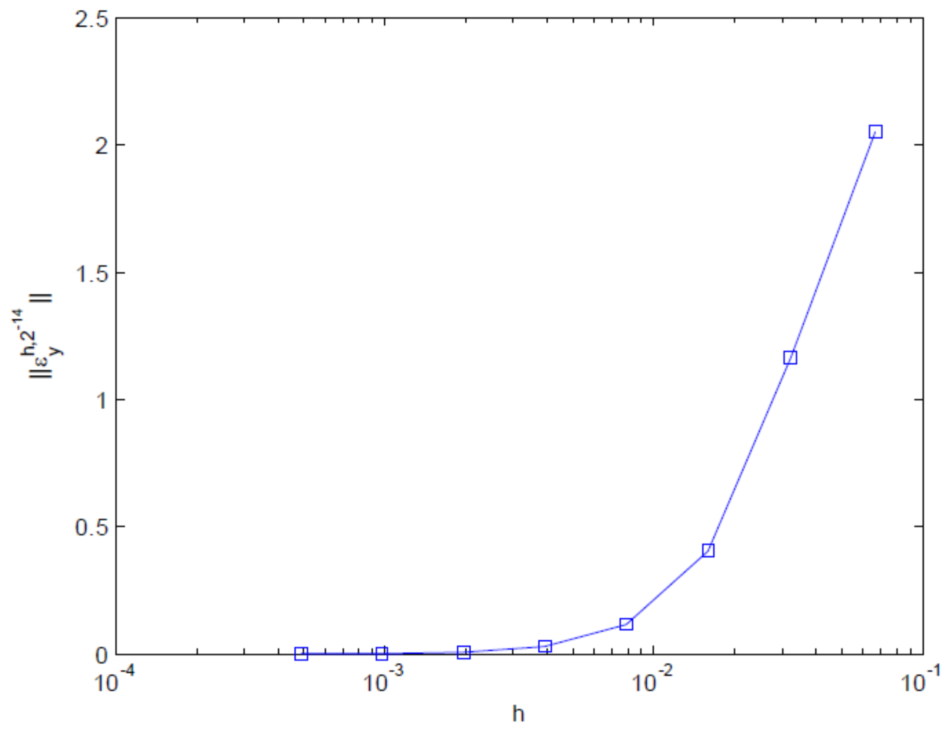

Figure 3.9: The maximum errors $\left\|\varepsilon_y^{h,2^{-14}}\right\|$ when $\tau=2^{-14}$, with respect to h, for Example 4.

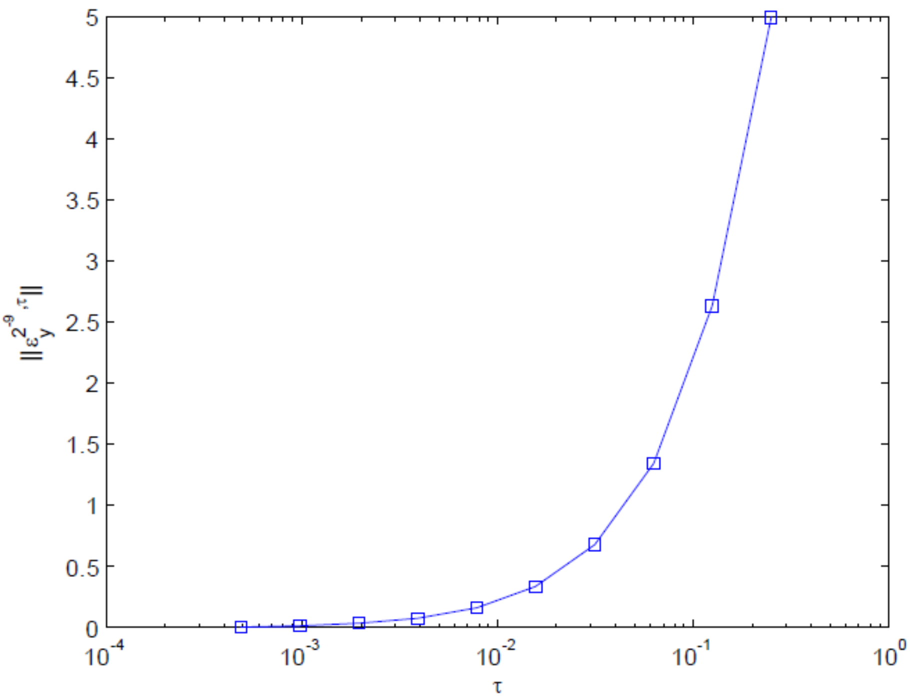

Figure 3.10: The maximum errors $\left\|\varepsilon_y^{2^{-9},\tau}\right\|$ when $h = 2^{-9}$, with respect to τ , for Example 4.

Chapter 4

SIX POINT IMPLICIT METHODS FOR THE APPROXIMATION OF SECOND DERIVATIVES TO HEAT EQUATION WITH CONSTANT COEFFICIENTS

4.1 Chapter overview

The work in this chapter is organized as follows: In Section 2, the first type boundary value problem for one dimensional heat equation is considered requiring that the initial function belongs to $C^{10+\alpha}$ the heat source function is from $C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}$, the boundary functions are from $C^{\frac{10+\alpha}{2}}$, and between the initial and the boundary functions the conjugation conditions of orders $q = 0,1,2,3,4,5$ are satisfied. We give the boundary value problems for $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ based on the assumptions, and difference problems of symmetric six point implicit schemes approximating $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ are constructed. For the error function we provide a pointwise prior estimation depending on $\rho(x,t)$ which is the distance from the current grid point in the domain to the boundary. In Section 3, and Section 4, special six point implicit difference problem for the approximation of $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x \partial t}$ respectively are proposed and it is proved that the solution of the constructed difference schemes converge uniformly to the exact value of the respective derivatives on the grids of order $O(h^2 + \tau^2)$. In Section 5, we constructed a numerical example to justify the theoretical results and the obtained results are presented through tables and figures.

4.2 Crank-Nicolson schemes for the approximation of $\partial_t u$ and $\partial_x^2 u$

Let the following problem be given:

Problem 1:

(iv) The boundary value problem (2.1) – (2.3) with the assumption

$$u_0(x) \in C^{10+\alpha}(\bar{\Omega}), \quad f(x,t) \in C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}(\bar{Q}_T) \text{ and } u_i(t) \in C^{\frac{10+\alpha}{2}}(\bar{\sigma}_T), \ i=1,2. \tag{4.1}$$

and satisfying the conjugation conditions (2.9) of order $0,1,2,3,4,5.$

Let $w = \partial_t u$ and $\phi = \partial_x^2 u$ and further w satisfies Problem 3 and ϕ satisfies Problem 4, where,

$$F(x,t) \in C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}(\bar{Q}_T), \quad w_0(x) \in C^{8+\alpha}(\bar{\Omega}), \quad w_i(t) \in C^{\frac{8+\alpha}{2}}(\bar{\sigma}_T), \quad i=1,2, \tag{4.2}$$

$$G(x,t) \in C_{x,t}^{6+\alpha,\frac{6+\alpha}{2}}(\bar{Q}_T), \quad \phi_0(x) \in C^{8+\alpha}(\bar{\Omega}), \quad \phi_i(t) \in C^{\frac{8+\alpha}{2}}(\bar{\sigma}_T), \quad i=1,2, \tag{4.3}$$

both satisfying the conjugation conditions (2.9) of order 0, 1, 2, 3, 4.

Lemma 4.1: The Problem 1(iv) has unique solution $u \in C_{x,t}^{10+\alpha,\frac{10+\alpha}{2}}(\bar{Q}_T)$ and the constructed boundary value Problem 3 with (4.2) and Problem 4 with (4.3) have unique solution w and ϕ respectively, belonging to the space $C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}(\bar{Q}_T)$.

Proof: From Theorem 1.1, Problem 1(iv) has unique solution $u \in C_{x,t}^{10+\alpha,\frac{10+\alpha}{2}}(\bar{Q}_T)$. Taking into account that Problem 3 and Problem 4 with (4.2) and (4.3) respectively are also first type boundary value problems analogous to the problem (2.1) – (2.3) on the basis of Theorem 1.1 the proof follows.

To realize the numerical solution of the Problem 3 satisfying (4.2) and Problem 4 with (4.3), we propose the following implicit six point difference problems, of which the solution of (4.4) – (4.6) is denoted by $\tilde{w}$, and the solution of (4.10) – (4.12) is presented by $\tilde{\phi}$ see also [18].

$$\tilde{w}^{h,\tau}_{\bar{t},m} = a\Theta^6 \tilde{w}^{h,\tau}_m + \varphi_{\tilde{w},m} \text{ on } \omega_{h,\tau}, \tag{4.4}$$

$$\tilde{w}^0_m = w_0(x_m) \text{ on } \bar{\omega}_{h,0}, \tag{4.5}$$

$$\tilde{w}^j_0 = w_1(t_j) \text{ on } \omega_{0,\tau}, \quad \tilde{w}^j_N = w_2(t_j) \text{ on } \omega_{b,\tau}, \tag{4.6}$$

where, $\tilde{w}^0_m = \tilde{w}(x_m, 0)$, $\tilde{w}^j_0 = \tilde{w}(0, t_j)$, $\tilde{w}^j_N = \tilde{w}(b, t_j)$ and

$$\tilde{w}^{h,\tau}_{\bar{t},m} = \frac{\tilde{w}^{j+1}_m - \tilde{w}^j_m}{\tau}, \tag{4.7}$$

$$\Theta^6 \tilde{w}^{h,\tau}_m = \frac{1}{2}\left[\frac{\tilde{w}^{j+1}_{m-1} - 2\tilde{w}^{j+1}_m + \tilde{w}^{j+1}_{m+1}}{h^2} + \frac{\tilde{w}^j_{m-1} - 2\tilde{w}^j_m + \tilde{w}^j_{m+1}}{h^2}\right], \tag{4.8}$$

$$\varphi_{\tilde{w},m} = F\left(x_m, t_{j+\frac{1}{2}}\right) = \partial_t f\left(x_m, t_{j+\frac{1}{2}}\right). \tag{4.9}$$

$$\tilde{\phi}^{h,\tau}_{\bar{t},m} = a\Theta^6 \tilde{\phi}^{h,\tau}_m + \varphi_{\tilde{\phi},m} \text{ on } \omega_{h,\tau}, \tag{4.10}$$

$$\tilde{\phi}^0_m = \phi_0(x_m) \text{ on } \bar{\omega}_{h,0}, \tag{4.11}$$

$$\tilde{\phi}^j_0 = \phi_1(t_j) \text{ on } \omega_{0,\tau}, \quad \tilde{\phi}^j_N = \phi_2(t_j) \text{ on } \omega_{b,\tau}, \tag{4.12}$$

where $\tilde{\phi}^{h,\tau}_{\bar{t},m}$, $\Theta^6 \tilde{\phi}^{h,\tau}_m$ are formulae analogous to (4.7) and (4.8) respectively, and $\tilde{\phi}^0_m = \tilde{\phi}(x_m, 0)$, $\tilde{\phi}^j_0 = \tilde{\phi}(0, t_j)$, $\tilde{\phi}^j_N = \tilde{\phi}(b, t_j)$ also $\varphi_{\tilde{\phi},m} = G\left(x_m, t_{j+\frac{1}{2}}\right) = \partial^2_x f\Big|_{\left(x_m, t_{j+\frac{1}{2}}\right)}$. Here $t_{j+\frac{1}{2}} = t_j + 0.5\tau$, $f(x,t)$ is the given function in (2.1) and $u_0(x)$ given in (2.2),

$u_1(t),\ u_2(t)$ given in (2.3) are the initial and boundary functions respectively. Using maximum principle the difference problems (4.4) – (4.6) and (4.10) – (4.12) have unique solution.

Theorem 4.2: Let w be the solution of the differential Problem 3 with (4.2) and $\tilde{w}$ be the solution of the difference problem (4.4) – (4.6). The following pointwise estimation holds true

$$|\tilde{w}-w| \leq c_1\rho(h^2+\tau^2) \tag{4.13}$$

For $r \leq 1$.

Proof: On the basis of Theorem 1.1 the exact solution $w \in C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}(\bar{Q}_T)$. Let $\varepsilon_w^{h,\tau} = \tilde{w} - w$ then the error function $\varepsilon_w^{h,\tau}$ satisfies the following difference problem

$$\varepsilon_{w,\bar{t},m}^{h,\tau} = a\Theta^6\varepsilon_{w,m}^{h,\tau} + \psi_w \text{ on } \omega_{h,\tau}, \tag{4.14}$$

$$\varepsilon_{w,m}^{0} = 0 \text{ on } \bar{\omega}_{h,0}, \tag{4.15}$$

$$\varepsilon_{w,0}^{j} = 0 \text{ on } \omega_{0,\tau},\quad \varepsilon_{w,N}^{j} = 0 \text{ on } \omega_{b,\tau}, \tag{4.16}$$

where $\psi_w = a\Theta^6 w - w_{\bar{t},m} + \varphi_{\tilde{w},m}$ and $\varphi_{\tilde{w},m}$ is as given in (4.9). Using Taylor's formula for the function $w(x,t)$ about the node $\left(x_m, t_{j+\frac{1}{2}}\right)$ gives that $\psi_w = O(h^2+\tau^2)$.

Applying Lemma 2.2 to the problem (4.10) – (4.12) and (4.14) – (4.16) and on the basis of Lemma 2.3 we obtain $|\varepsilon_w^{h,\tau}| \leq c_1\rho(h^2+\tau^2)$. ■

Theorem 4.3: The following inequality holds

$$|\tilde{\phi}-\phi| \leq c_2\rho(h^2+\tau^2) \tag{4.17}$$

For $r \leq 1$ where, ϕ is the solution of the differential Problem 4 with (4.3) and $\tilde{\phi}$ is the solution of the difference problem (4.10) – (4.12).

Proof: The proof is analogous to the proof of Theorem 4.2.

4.3 Six point implicit scheme for the approximation of $\partial_t^2 u$

When Problem 1(iv) is given we setup the boundary value Problem 3 with (4.2) for $w = \partial_t u$ and use the difference system (4.4) – (4.6) for obtaining the approximate solution $\tilde{w}$. We denote $q_i = \partial_t^2 u$ on γ_i, $i = 1, 2, 3$ and construct Problem 5, given in Chapter 3, Section 3 for $z = \partial_t^2 u$, see also [18].

We take $q_{1\tau}$ as same in (3.29),

$$q_{2\tau}(\tilde{w}) = \frac{1}{2\tau}\left[-3w_0(x) + 4\tilde{w}(x,\tau) - \tilde{w}(x,2\tau)\right] \text{ on } \bar{\omega}_{h,0}, \qquad (4.18)$$

and $q_{3\tau}$ as in (3.31), where, $w_0(x) = u^{(1)}(x)$ as defined in (2.5).

Lemma 4.4: The following inequality holds:

$$|q_{2\tau}(\tilde{w}) - q_{2\tau}(w)| \le c_1(h^2 + \tau^2) \qquad (4.19)$$

For $r \le 1$ where, w is the solution of the differential Problem 3 with (4.2) and $\tilde{w}$ is the solution of the difference problem (4.4) – (4.6).

Proof: From Theorem 4.2, we have

$$|q_{2\tau}(\tilde{w}) - q_{2\tau}(w)| \le \frac{1}{2\tau}\left[4(c\tau)(h^2+\tau^2) + (c2\tau)(h^2+\tau^2)\right] \le c_1(h^2+\tau^2) \qquad (4.20)$$

Lemma 4.5: The following inequality is true:

$$\max_{\bar{\omega}_{h,0}} |q_{2\tau}(\tilde{w}) - q_2| \le c_2(h^2 + \tau^2) \qquad (4.21)$$

for $r \le 1$ where, $\tilde{w}$ is the solution of the difference problem (4.4) – (4.6).

Proof: On the basis of Lemma 4.1, $w \in C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}(\bar{Q}_T)$ and at the end points $(mh,0) \in \bar{\omega}_{h,0}$ of each line segment $\left[(x,t): 0 \le x \le b, 0 \le t \le T\right]$ the equation (4.18) gives the second order approximation of $\partial_t w$. From the truncation error formula (see [20]) it follows that

$$\max_{\bar{\omega}_{h,0}} | q_{2\tau}(w) - q_2 | \le \frac{\tau^2}{3} \max_{\bar{Q}_T} | \partial_t^3 w | \le c_2 \tau^2, \qquad (4.22)$$

Using Lemma 4.4 and the estimation (4.19), (4.22) follows (4.21). ■

We construct the following difference problem for the second order accurate in space and in time for numerical solution of Problem 5.

$$\tilde{z}_{\bar{t},m}^{h,\tau} = a\Theta^6 \tilde{z}_m^{h,\tau} + \varphi_{\tilde{z},m} \text{ on } \omega_{h,\tau}, \qquad (4.23)$$

$$\tilde{z}_m^0 = q_{2\tau}(\tilde{w}) \text{ on } \bar{\omega}_{h,0}, \qquad (4.24)$$

$$\tilde{z}_0^j = q_{1\tau} \text{ on } \omega_{0,\tau}, \quad \tilde{z}_z^j = q_{3\tau} \text{ on } \omega_{b,\tau}, \qquad (4.25)$$

where, $q_{i\tau}$, $i = 1,2,3$ are defined by (3.29) , (4.18) and (3.31) respectively and $\varphi_{\tilde{z},m} = \partial_t^2 f \,|_{\left(x_m, t_{j+\frac{1}{2}}\right)}$.

Theorem 4.6: The solution $\tilde{z}$ of the finite difference problem (4.23) – (4.25) satisfies

$$\max_{\bar{\omega}_{h,\tau}} | \tilde{z} - z | \le c_3(h^2 + \tau^2), \qquad (4.26)$$

for $r \le 1$ where, $z = \partial_t^2 u$ is the exact solution of Problem 5.

Proof: Let

$$\varepsilon_z^{h,\tau} = \tilde{z} - z \quad \text{on} \quad \bar{\omega}_{h,\tau}, \qquad (4.27)$$

where $z=\partial_t^2 u$. Denote by $\left\|\varepsilon_z^{h,\tau}\right\|=\max_{\bar{\omega}_{h,\tau}}\left|\tilde{z}-z\right|$. From (4.23) – (4.25) and (4.27) we have

$$\varepsilon_{z,\bar{t},m}^{h,\tau}=a\Theta^6\varepsilon_{z,m}^{h,\tau}+\Psi_z \text{ on } \omega_{h,\tau}, \tag{4.28}$$

$$\varepsilon_{z,m}^{0}=q_{2\tau}(\tilde{w})-z \text{ on } \bar{\omega}_{h,0}, \tag{4.29}$$

$$\varepsilon_{z,0}^{j}=0 \text{ on } \omega_{0,\tau}, \quad \varepsilon_{z,N}^{j}=0 \text{ on } \omega_{b,\tau}, \tag{4.30}$$

where $\Psi_z=a\Theta^6 z-z_{\bar{t},m}+\varphi_{\tilde{z},m}$. We take

$$\varepsilon_z^{h,\tau}=\varepsilon_z^{1,h,\tau}+\varepsilon_z^{2,h,\tau} \tag{4.31}$$

and $\varepsilon_z^{1,h,\tau}$, $\varepsilon_z^{2,h,\tau}$ satisfy the problems

$$\varepsilon_{z,\bar{t},m}^{1,h,\tau}=a\Theta^6\varepsilon_{z,m}^{1,h,\tau} \text{ on } \omega_{h,\tau}, \tag{4.32}$$

$$\varepsilon_{z,m}^{1,0}=q_{2\tau}(\tilde{w})-z \text{ on } \bar{\omega}_{h,0}, \tag{4.33}$$

$$\varepsilon_{z,0}^{1,j}=0 \text{ on } \omega_{0,\tau}, \quad \varepsilon_{z,N}^{1,j}=0 \text{ on } \omega_{b,\tau}, \tag{4.34}$$

and

$$\varepsilon_{z,\bar{t},m}^{2,h,\tau}=a\Theta^6\varepsilon_{z,m}^{2,h,\tau}+\Psi_z \text{ on } \omega_{h,\tau}, \tag{4.35}$$

$$\varepsilon_{z,m}^{2,0}=0 \text{ on } \bar{\omega}_{h,0}, \tag{4.36}$$

$$\varepsilon_{z,0}^{2,j}=0 \text{ on } \omega_{0,\tau}, \quad \varepsilon_{z,N}^{2,j}=0 \text{ on } \omega_{b,\tau}. \tag{4.37}$$

respectively. From Lemma 4.5 and by maximum principle for the solution of the system (4.32) – (4.34) we have

$$\max_{\bar{\omega}_{h,\tau}}\left|\varepsilon_z^{1,h,\tau}\right| \le \max_{\bar{\omega}_{h,\tau}}\left|q_{2\tau}(\tilde{w})-z\right| \le c_4(h^2+\tau^2). \tag{4.38}$$

The solution $\varepsilon_z^{2,h,\tau}$ of problem (4.35) – (4.37) is the error of the approximate solution obtained by the finite difference method for the Problem 5 when

$$q_2\in C^{6+\alpha}(\bar{\Omega}), \quad \partial_t^2 f(x,t)\in C_{x,t}^{4+\alpha,\frac{4+\alpha}{2}}(\bar{Q}_T), \quad q_i\in C^{\frac{6+\alpha}{2}}(\bar{\sigma}_T),\ i=1,3. \tag{4.39}$$

and

$$\begin{cases} q_1^{(q)}(0)=z^{(q)}(0), \\ q_3^{(q)}(0)=z^{(q)}(b), \qquad q=0,1,2,3 \end{cases} \tag{4.40}$$

Since the function $z=\partial_t^2 u$ satisfies the equation (3.26) with the initial function q_2 on γ_2 and boundary functions q_1, q_3 on γ_1 and γ_3 respectively, using (4.39), (4.40) and on the basis of maximum principle in Chapter 4 of [19] we obtain

$$\max_{\bar{\omega}_{h,\tau}} \left|\varepsilon_z^{2,h,\tau}\right| \le c_5(h^2+\tau^2). \tag{4.41}$$

Using (4.31), (4.38) and (4.41) we obtain (4.26). ■

4.4 Six point implicit scheme for the approximation of $\partial_x\partial_t u$

Given the Problem 1(iv), we setup the Boundary Value Problem 3 with (4.2) for $w=\partial_t u$ and use the difference system (4.4) – (4.6) for obtaining the approximate solution $\tilde{w}$. We denote $p_i=\partial_x\partial_t u$ on γ_i, $i=1,2,3$ respectively and construct Problem 6 for $y=\partial_x\partial_t u$.

We construct the following difference problem for the numerical solution of Problem 6 and denote this solution by $\tilde{y}$

$$\tilde{y}_{\bar{t},m}^{h,\tau} = a\Theta^6\tilde{y}_m^{h,\tau} + \varphi_{\tilde{y},m} \text{ on } \omega_{h,\tau}, \tag{4.42}$$

$$\tilde{y}_m^0 = p_{2h} \text{ on } \bar{\omega}_{h,0}, \tag{4.43}$$

$$\tilde{y}_0^j = p_{1h}(\tilde{w}) \text{ on } \omega_{0,\tau}, \quad \tilde{y}_N^j = p_{3h}(\tilde{w}) \text{ on } \omega_{b,\tau}. \tag{4.44}$$

Here, p_{ih} are defined by (3.58) – (3.60) $\tilde{y}_{\bar{t},m}^{h,\tau}$, $\Theta^6\tilde{y}_m^{h,\tau}$ are formulae analogous to (4.7) and (4.8) respectively, and $\tilde{y}_m^0=\tilde{y}(x_m,0)$, $\tilde{y}_0^j=\tilde{y}(0,t_j)$, $\tilde{y}_N^j=\tilde{y}(b,t_j)$ and $\varphi_{\tilde{y},m}=\partial_x\partial_t f\Big|_{\left(x_m,t_{j+\frac{1}{2}}\right)}$.

Lemma 4.7: The following inequality holds

$$| p_{ih}(\tilde{w}) - p_{ih}(w) | \le c_1(h^2 + \tau^2), \quad i = 1,3. \tag{4.45}$$

for $r \le 1$ where *w* is the solution of the differential Problem 3 with (4.2) and $\tilde{w}$ is the solution of the difference problem (4.4) – (4.6).

Proof: From (3.58), (3.60) and Theorem 4.2, we have

$$| p_{ih}(\tilde{w}) - p_{ih}(w) | \le \frac{1}{2h}\left[4(ch)(h^2+\tau^2) + (c2h)(h^2+\tau^2)\right] \le c_1(h^2+\tau^2), \quad i = 1,3. \tag{4.46}$$

Lemma 4.8: The following inequality is true

$$\max_{\omega_{0,\tau} \cup \omega_{b,\tau}} | p_{ih}(\tilde{w}) - p_i | \le c_2(h^2 + \tau^2), \quad i = 1,3. \tag{4.47}$$

for $r \le 1$ where, $\tilde{w}$ is the solution of the difference problem (4.4) – (4.6).

Proof: On the basis of Lemma 4.1, the exact solution of Problem 3 with (4.2) belongs to $C_{x,t}^{8+\alpha,\frac{8+\alpha}{2}}(\bar{Q}_T)$. Then at the end points $(0,\sigma\tau) \in \omega_{0,\tau}$ and $(b,\sigma\tau) \in \omega_{b,\tau}$ of each line segment $\left[(x,t): 0 \le x \le b, 0 \le t \le T\right]$ the equations (3.58) and (3.60) give the second order approximation of $\partial_x w$ respectively. From the truncation error formula (see [20]) it follows that

$$\max_{\omega_{0,\tau} \cup \omega_{b,\tau}} | p_{ih}(w) - p_i | \le \frac{h^2}{3} \max_{\bar{Q}_T} | \partial_x^3 w | \le c_3 h^2, \quad i = 1,3. \tag{4.48}$$

On the basis of Lemma 4.7 using the estimation (4.45) and (4.48) follows (4.47). ■

Theorem 4.9: The solution $\tilde{y}$ of the finite difference problem (4.42) – (4.44) satisfies

$$\max_{\bar{\omega}_{h,\tau}} | \tilde{y} - y | \le c_4(h^2 + \tau^2), \tag{4.49}$$

for $r \le 1$ where, $y = \partial_x \partial_t u$ is the exact solution of Problem 6.

Proof: On the basis of maximum principle in Chapter 4 of [19], the proof follows from Lemma 4.8 and is analogous to the proof of Theorem 3.9.

4.5 Numerical aspects

Example 5: [18] We consider the following boundary value problem

$$Lu = f(x,t) \quad \text{on} \quad Q_T,$$

$$u(x,0) = 0.005 + \sin(2\pi x) \quad \text{on} \quad \gamma_2,$$

$$u(0,t) = 0.005\cos(t^{\frac{51}{10}}) \quad \text{on} \quad \gamma_1,$$

$$u(1,t) = 0.0005\sin(t^{\frac{51}{10}}) + 0.005\cos(t^{\frac{51}{10}}) + \sin(2\pi) \quad \text{on} \quad \gamma_3,$$

where,

$$f(x,t) = \sin(x^{\frac{51}{10}})\left[-0.0005\frac{51}{5}\frac{46}{5}x^{\frac{41}{5}} - 0.005\frac{51}{10}t^{\frac{41}{10}}\right] + 0.0005\frac{51}{10}x^{\frac{51}{5}}t^{\frac{41}{10}}\cos(x^{\frac{41}{10}}) + 4\pi^2\sin(2\pi x) \tag{4.50}$$

4.5.1 Numerical results for $\partial_t u$

The boundary value problems for $w = \partial_t u$ is constructed using the proposed Problem 3 with (4.2). Further, the approximate solution $\tilde{w}$ of the difference system (4.4) – (4.6) is obtained by using Gauss-Thomas Algorithm (2.104) since the obtained algebraic system of equations at each time level has a structure analogues to (2.107). By the known exact solution

$$w(x,t) = 0.0005\frac{51}{10}x^{\frac{51}{5}}t^{\frac{41}{10}}\cos(t^{\frac{51}{10}}) - 0.005\frac{51}{10}t^{\frac{41}{10}}\sin(t^{\frac{51}{10}}) \tag{4.51}$$

we denote the maximum error by $\left\| \varepsilon_w^{h,\tau} \right\| = \max_{\bar{\omega}_{h,\tau}} |\tilde{w} - w|$. Table 4.1 presents the maximum errors and the corresponding CPU time for different step sizes in space and the order of convergence of the approximate solution $\tilde{w}$ to the exact solution $w = \partial_t u$ in spatial variable x (with respect to h).

$$\Re_{\tilde{w}}^{h} = \frac{\left\| \varepsilon_w^{2^{-\mu}, 2^{-\lambda}} \right\|}{\left\| \varepsilon_w^{2^{-(\mu+1)}, 2^{-\lambda}} \right\|}. \tag{4.52}$$

Table 4.2, shows CPU for different step sizes of the maximum errors and the order of convergence $\Re_{\tilde{w}}^{h,\tau}$ analogues to (2.108) with respect to x and t of $\tilde{w}$ to the exact solution w in x and t.

Table 4.1: Maximum errors, corresponding CPU time for different step size in space and $\Re_{\tilde{w}}^{h}$ for Example 5.

$(h = 2^{-\mu}, \tau = 2^{-\lambda})$		$(h = 2^{-(\mu+1)}, \tau = 2^{-\lambda})$		$\left\| \varepsilon_w^{2^{-\mu}, 2^{-\lambda}} \right\|$	$\left\| \varepsilon_w^{2^{-(\mu+1)}, 2^{-\lambda}} \right\|$	$\Re_{\tilde{w}}^{h}$
h, τ	CPU	h, τ	CPU			
$(2^{-4}, 2^{-16})$	4.375	$(2^{-5}, 2^{-16})$	6.969	$2.6585E-05$	$6.6997E-06$	3.9681
$(2^{-5}, 2^{-16})$	6.969	$(2^{-6}, 2^{-16})$	12.578	$6.6997E-06$	$1.6794E-06$	3.9893
$(2^{-6}, 2^{-16})$	12.578	$(2^{-7}, 2^{-16})$	23.672	$1.6794E-06$	$4.2013E-07$	3.9973
$(2^{-7}, 2^{-16})$	23.672	$(2^{-8}, 2^{-16})$	46.375	$4.2013E-07$	$1.0505E-07$	3.9993

Table 4.2: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{w}}^{h,\tau}$ for Example 5.

$(h = 2^{-\mu}, \tau = 2^{-\lambda})$		$(h = 2^{-(\mu+1)}, \tau = 2^{-(\lambda+1)})$		$\left\| \varepsilon_w^{2^{-\mu}, 2^{-\lambda}} \right\|$	$\left\| \varepsilon_w^{2^{-(\mu+1)}, 2^{-(\lambda+1)}} \right\|$	$\Re_{\tilde{w}}^{h,\tau}$
h, τ	CPU	h, τ	CPU			
$(2^{-4}, 2^{-13})$	0.625	$(2^{-5}, 2^{-14})$	1.719	$2.6585E-05$	$6.6996E-06$	3.9682
$(2^{-5}, 2^{-14})$	1.719	$(2^{-6}, 2^{-15})$	6.219	$6.6996E-06$	$1.6794E-06$	3.9893
$(2^{-6}, 2^{-15})$	6.219	$(2^{-7}, 2^{-16})$	23.672	$1.6794E-06$	$4.2013E-07$	3.9973
$(2^{-7}, 2^{-16})$	23.672	$(2^{-8}, 2^{-17})$	92.891	$4.2013E-07$	$1.0505E-07$	3.9993

Note that the $O(h^2+\tau^2)$ order of convergence corresponds to $\approx 2^2$ of the quantity defined by (4.52). The exact solution $w=\partial_t u$ and the grid functions $w^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{w}$ when $h=2^{-7}$ and $\tau=2^{-16}$ are shown in Figure 4.1 for Example 5, while the error function $\left|\varepsilon_w^{2^{-7},2^{-16}}\right|=|\tilde{w}-w|$ for $h=2^{-7}$, and $\tau=2^{-16}$ is given in Figure 4.2.

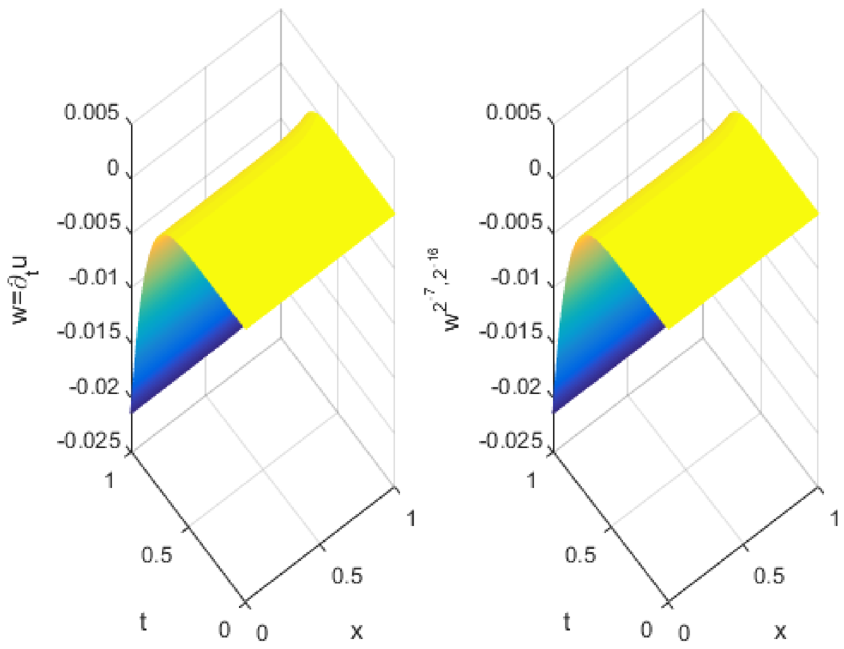

Figure 4.1: The exact solution $w=\partial_t u$ and the grid functions $w^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{w}$ for $h=2^{-7}$ and $\tau=2^{-16}$, of Example 5.

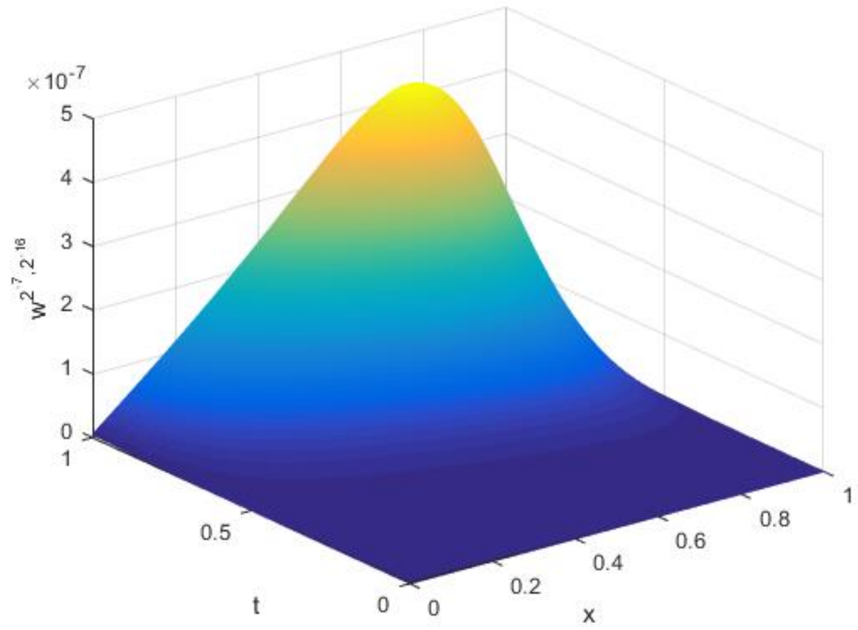

Figure 4.2: The error function $\left|\varepsilon_w^{2^{-7},2^{-16}}\right| = |\tilde{w} - w|$ when $h = 2^{-7}$ and $\tau = 2^{-16}$, for Example 5.

4.5.2 Numerical results for $\partial_x^2 u$

We present the numerical results for the approximate solution of pure second derivative of the solution $u(x,t)$ with respect to x. The boundary value problems for $\phi = \partial_x^2 u$ is constructed using the proposed Problem 4 satisfying (4.3). Then, the approximate solution $\tilde{\phi}$ of the difference problems (4.10) – (4.12) is obtained at the same grid points using the Algorithm (2.104). By the known exact solution.

$$\phi(x,t) = 0.0005\frac{51}{5}\frac{46}{5}x^{\frac{41}{5}}\sin(t^{\frac{51}{10}}) - 4\pi^2\sin(2\pi x), \tag{4.53}$$

we denote the maximum errors by $\left\|\varepsilon_\phi^{h,\tau}\right\| = \max_{\bar{\omega}_{h,\tau}} |\tilde{\phi} - \phi|$. Table 4.3 shows the maximum errors and the corresponding processing time for different step sizes in space and the order of convergence $\Re_{\tilde{\phi}}^h$ of the approximate solution $\tilde{\phi}$ to the exact solution $\phi = \partial_x^2 u$

for Example 5, analogous to the formula (4.52) in spatial variable x (with respect to h). Table 4.4, shows the maximum errors, the elapsed time and the order of convergence $\Re_{\tilde{\phi}}^{h,\tau}$ of $\tilde{\phi}$ to the exact solution ϕ in time variable t analogous to the formula (2.108). As expected, $O(h^2+\tau^2)$ order of convergence corresponds to $\approx 2^2$ of the quantities was achieved. Figure 4.3 demonstrates the exact solution $\phi=\partial_x^2 u$ and the grid functions $\phi^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{\phi}$ when $h=2^{-7}$ and $\tau=2^{-16}$ for Example 5. The error function $\left|\varepsilon_{\phi}^{2^{-7},2^{-16}}\right|=|\tilde{\phi}-\phi|$ for $h=2^{-7}$, and $\tau=2^{-16}$ is given in Figure 4.4.

Table 4.3: Maximum errors, corresponding CPU time for different step size in space and $\Re_{\tilde{\phi}}^{h}$, for Example 5.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-(\mu+1)},\tau=2^{-\lambda})$		$\left\|\varepsilon_{\phi}^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_{\phi}^{2^{-(\mu+1)},2^{-\lambda}}\right\|$	$\Re_{\tilde{\phi}}^{h}$
h,τ	CPU	h,τ	CPU			
$(2^{-4},2^{-16})$	3.344	$(2^{-5},2^{-16})$	4.797	0.5116	0.1272	4.0220
$(2^{-5},2^{-16})$	4.797	$(2^{-6},2^{-16})$	7.484	0.1272	$3.1745E-02$	4.0069
$(2^{-6},2^{-16})$	7.484	$(2^{-7},2^{-16})$	12.703	$3.1745E-02$	$7.9334E-03$	4.0014
$(2^{-7},2^{-16})$	12.703	$(2^{-8},2^{-16})$	23.250	$7.9334E-03$	$1.9832E-03$	4.0003

Table 4.4: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{\phi}}^{h,\tau}$, for Example 5.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-(\mu+1)},\tau=2^{-(\lambda+1)})$		$\left\|\varepsilon_{\phi}^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_{\phi}^{2^{-(\mu+1)},2^{-(\lambda+1)}}\right\|$	$\Re_{\tilde{\phi}}^{h,\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-4},2^{-13})$	0.516	$(2^{-5},2^{-14})$	1.141	0.5116	0.1272	4.0220
$(2^{-5},2^{-14})$	1.141	$(2^{-6},2^{-15})$	3.672	0.1272	$3.1745E-02$	4.0069
$(2^{-6},2^{-15})$	3.672	$(2^{-7},2^{-16})$	12.703	$3.1745E-02$	$7.9334E-03$	4.0014
$(2^{-7},2^{-16})$	12.703	$(2^{-8},2^{-17})$	46.344	$7.9334E-03$	$1.9832E-03$	4.0003

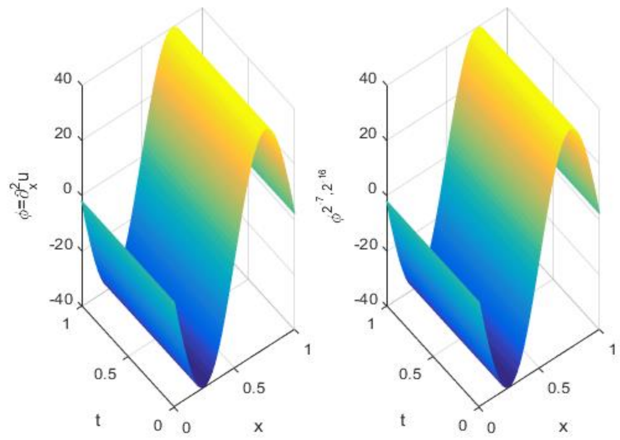

Figure 4.3: The exact solution $\phi = \partial_x^2 u$ and the grid functions $\phi^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{\phi}$ when $h = 2^{-7}$ and $\tau = 2^{-16}$, for Example 5.

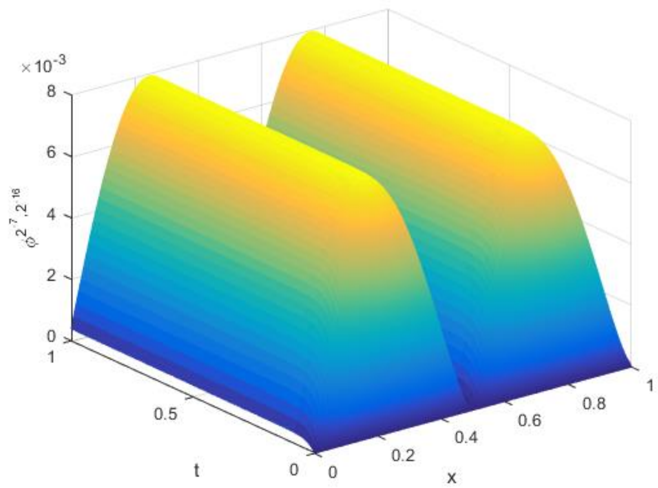

Figure 4.4: The error function $\left|\varepsilon_{\phi}^{2^{-7},2^{-16}}\right| = |\tilde{\phi} - \phi|$ when $h = 2^{-7}$, and $\tau = 2^{-16}$, for Example 5.

4.5.3 Numerical results for $\partial_t^2 u$

Now we present the numerical results for the approximate solution of pure second derivative of the solution $u(x,t)$ with respect to t. First we construct the boundary value Problem 3 with (4.2) and the approximate solution $\tilde{w}$ is obtained by solving the system (4.4) – (4.6). Then the boundary value problem for $z = \partial_t^2 u$ is constructed using the proposed Problem 5. Further, the approximate solution $\tilde{z}$ of the difference problems (4.23) – (4.25) is obtained at the same grid points. Using the exact solution

$$z(x,t)=\sin(t^{\frac{51}{10}})\left[-0.0005\left(\frac{51}{10}\right)^2 x^{\frac{51}{5}} t^{\frac{41}{5}} -0.005\frac{51}{10}\frac{41}{10}t^{\frac{31}{10}}\right]$$

$$+\cos(t^{\frac{51}{10}})\left[-0.005\left(\frac{51}{10}\right)^2 t^{\frac{41}{5}} +0.0005\frac{51}{10}\frac{41}{10}x^{\frac{51}{5}}t^{\frac{31}{10}}\right] \tag{4.54}$$

we denote the maximum errors by $\left\|\varepsilon_z^{h,\tau}\right\|=\max_{\bar{\omega}_{h,\tau}}|\tilde{z}-z|$. Table 4.5 present the maximum errors, CPU time and the order of convergence $\Re_{\tilde{z}}^{h}$ of the approximate solution to the exact solution $z=\partial_t^2 u$ in spatial variable x analogous to the formula (4.52). Table 4.6, shows the maximum errors, process time and the order of convergence $\Re_{\tilde{z}}^{h,\tau}$ of $\tilde{z}$ to the exact solution z with respect to h and τ analogous to the formula (2.108). The exact solution $z=\partial_t^2 u$ and the grid functions $z^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{z}$ when $h=2^{-7}$, $\tau=2^{-16}$ are shown in Figure 4.5 while the error function $\left|\varepsilon_z^{2^{-7},2^{-16}}\right|=|\tilde{z}-z|$ for $h=2^{-7}$, and $\tau=2^{-16}$ is presented in Figure 4.6.

Table 4.5: Maximum errors, corresponding CPU time for different step size in space and $\Re_{\tilde{z}}^{h}$, for Example 5.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-(\mu+1)},\tau=2^{-\lambda})$		$\left\|\varepsilon_z^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_z^{2^{-(\mu+1)},2^{-\lambda}}\right\|$	$\Re_{\tilde{z}}^{h}$
h,τ	CPU	h,τ	CPU			
$(2^{-4},2^{-16})$	6.234	$(2^{-5},2^{-16})$	10.562	$7.8539E-05$	$1.9792E-05$	3.9682
$(2^{-5},2^{-16})$	10.562	$(2^{-6},2^{-16})$	19.922	$1.9792E-05$	$4.9611E-06$	3.9894
$(2^{-6},2^{-16})$	19.922	$(2^{-7},2^{-16})$	38.891	$4.9611E-06$	$1.2411E-06$	3.9973
$(2^{-7},2^{-16})$	38.891	$(2^{-8},2^{-16})$	73.516	$1.2411E-06$	$3.1033E-07$	3.9993

Table 4.6: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{z}}^{h,\tau}$, for Example 5.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-(\mu+1)},\tau=2^{-(\lambda+1)})$		$\left\|\varepsilon_z^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_z^{2^{-(\mu+1)},2^{-(\lambda+1)}}\right\|$	$\Re_{\tilde{z}}^{h,\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-4},2^{-13})$	0.844	$(2^{-5},2^{-14})$	2.641	$7.8539E-05$	$1.9972E-05$	3.9682
$(2^{-5},2^{-14})$	2.641	$(2^{-6},2^{-15})$	9.953	$1.9972E-05$	$4.9611E-06$	3.9894
$(2^{-6},2^{-15})$	9.953	$(2^{-7},2^{-16})$	38.891	$4.9611E-06$	$1.2411E-06$	3.9973
$(2^{-7},2^{-16})$	38.891	$(2^{-8},2^{-17})$	81.016	$1.2411E-06$	$3.1034E-07$	3.9992

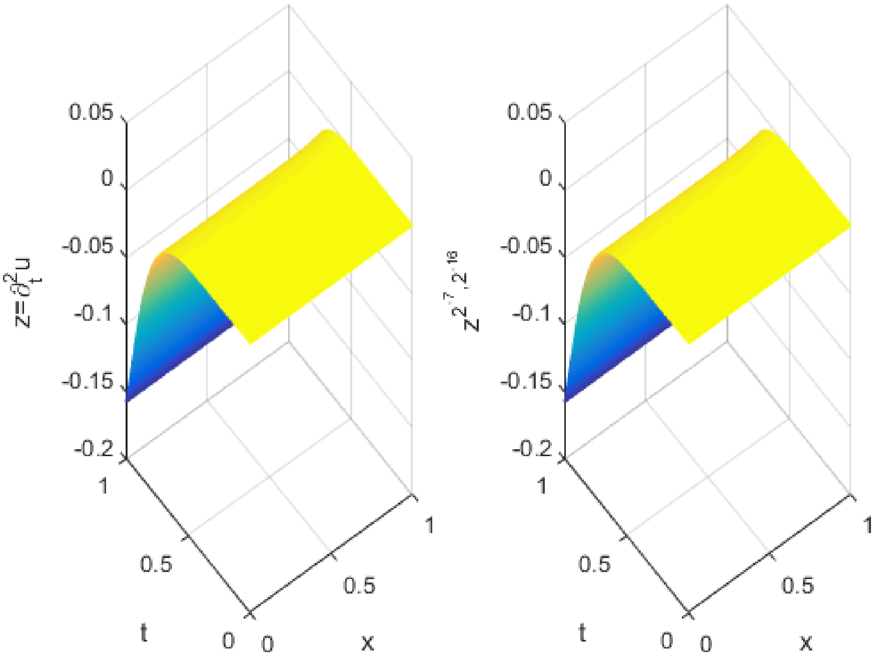

Figure 4.5: The exact solution $z=\partial_t^2 u$ and the grid functions $z^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{z}$ when $h=2^{-7}$, $\tau=2^{-16}$, for Example 5.

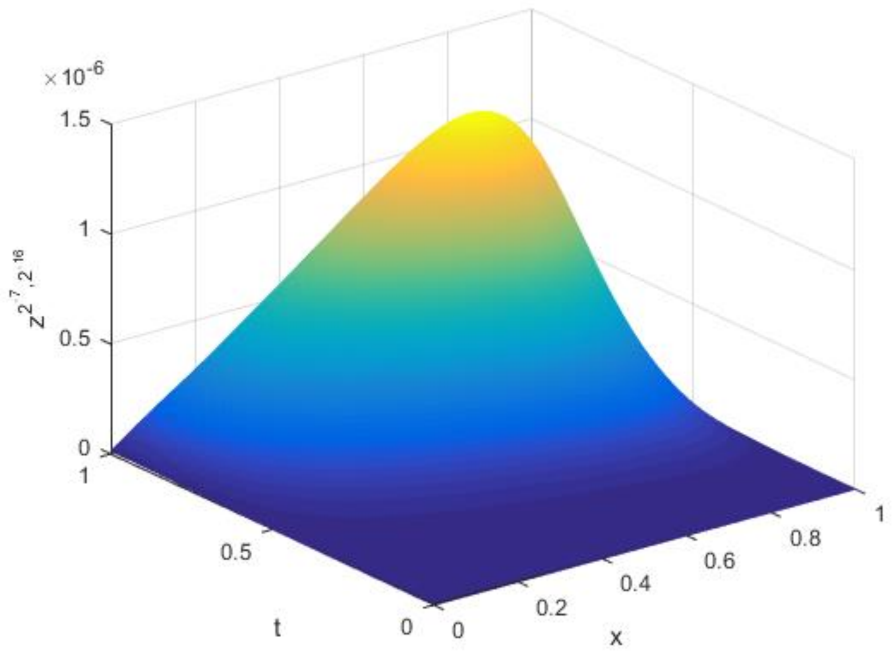

Figure 4.6: The error function $\left|\varepsilon_z^{2^{-7},2^{-16}}\right| = |\tilde{z} - z|$ for $h = 2^{-7}$, $\tau = 2^{-16}$, of Example 5.

4.5.4 Numerical results for $\partial_x \partial_t u$

Finally, we present the numerical results for the approximate solution of mixed second derivative of the solution $u(x,t)$. First we construct the boundary value Problem 3 with (4.2) and the approximate solution $\tilde{w}$ is obtained by solving the problem (4.4) – (4.6). Then, we construct the Problem 6 for $y = \partial_x \partial_t u$ approximate solution $\tilde{w}$. We them, obtain the approximate solution $\tilde{y}$ of the difference problem (4.42) – (4.44) at the same grid points. Using the exact solution

$$y(x,t) = 0.0005 \frac{51}{5} \frac{51}{10} x^{\frac{46}{5}} t^{\frac{41}{10}} \cos(x^{\frac{51}{10}}) \tag{4.55}$$

we denote the maximum error by $\left\|\varepsilon_y^{h,\tau}\right\| = \max_{\bar{\omega}_{h,\tau}} |\tilde{y} - y|$. Table 4.7 presents the CPU time, maximum errors and the order of convergence $\Re_{\tilde{y}}^h$ of the approximate solution $\tilde{y}$ to the exact solution $y = \partial_x \partial_t u$ in spatial variable *x* analogous to the formula

(4.52). Table 4.8, shows the maximum errors and its processing unit time and the order of convergence $\Re_{\tilde{y}}^{h,\tau}$ of $\tilde{y}$ to the exact solution y in time variable t analogous to the formula (2.108). The exact solution $y = \partial_x \partial_t u$ and the grid function $y^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{y}$ when $h = 2^{-7}$, $\tau = 2^{-16}$ are given in Figure 4.7 and the error function $\left|\varepsilon_y^{2^{-7},2^{-16}}\right| = |\tilde{y} - y|$ for $h = 2^{-7}$, $\tau = 2^{-16}$ is given in Figure 4.8 for Example 5. Figure 4.9 shows the maximum errors $\left\|\varepsilon_y^{h,2^{-16}}\right\|$ when $\tau = 2^{-16}$, with respect to h for y.

Table 4.7: Maximum errors, corresponding CPU time for different step size in space and $\Re_{\tilde{y}}^{h}$, for Example 5.

$(h=2^{-\mu},\tau=2^{-\lambda})$ h,τ	CPU	$(h=2^{-(\mu+1)},\tau=2^{-\lambda})$ h,τ	CPU	$\left\|\varepsilon_y^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_y^{2^{-(\mu+1)},2^{-\lambda}}\right\|$	$\Re_{\tilde{y}}^{h}$
$(2^{-4},2^{-16})$	4.906	$(2^{-5},2^{-16})$	7.719	$7.8539E-05$	$1.9792E-05$	3.9682
$(2^{-5},2^{-16})$	7.719	$(2^{-6},2^{-16})$	14.422	$1.9792E-05$	$4.9611E-06$	3.9894
$(2^{-6},2^{-16})$	14.422	$(2^{-7},2^{-16})$	27.547	$4.9611E-06$	$1.2411E-06$	3.9973
$(2^{-7},2^{-16})$	27.547	$(2^{-8},2^{-16})$	54.328	$1.2411E-06$	$3.1033E-07$	3.9993

Table 4.8: Maximum errors, corresponding CPU time for different step sizes in space and time and $\Re_{\tilde{y}}^{h,\tau}$, for Example 5.

$(h=2^{-\mu},\tau=2^{-\lambda})$		$(h=2^{-(\mu+1)},\tau=2^{-(\lambda+1)})$		$\left\|\varepsilon_y^{2^{-\mu},2^{-\lambda}}\right\|$	$\left\|\varepsilon_y^{2^{-(\mu+1)},2^{-(\lambda+1)}}\right\|$	$\Re_{\tilde{y}}^{h,\tau}$
h,τ	CPU	h,τ	CPU			
$(2^{-4},2^{-13})$	0.688	$(2^{-5},2^{-14})$	2.016	$7.8539E-05$	$1.9972E-05$	3.9682
$(2^{-5},2^{-14})$	2.016	$(2^{-6},2^{-15})$	7.156	$1.9972E-05$	$4.9611E-06$	3.9894
$(2^{-6},2^{-15})$	7.156	$(2^{-7},2^{-16})$	27.547	$4.9611E-06$	$1.2411E-06$	3.9973
$(2^{-7},2^{-16})$	27.547	$(2^{-8},2^{-17})$	69.009	$1.2411E-06$	$3.1034E-07$	3.9992

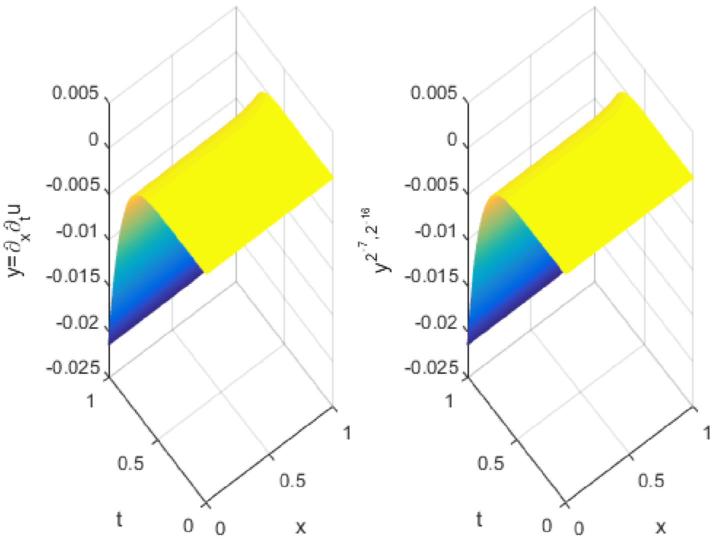

Figure 4.7: The exact solution $y = \partial_x \partial_t u$ and the grid function $y^{2^{-7},2^{-16}}$ presenting the approximate solution $\tilde{y}$ when $h = 2^{-7}$, $\tau = 2^{-16}$, for Example 5.

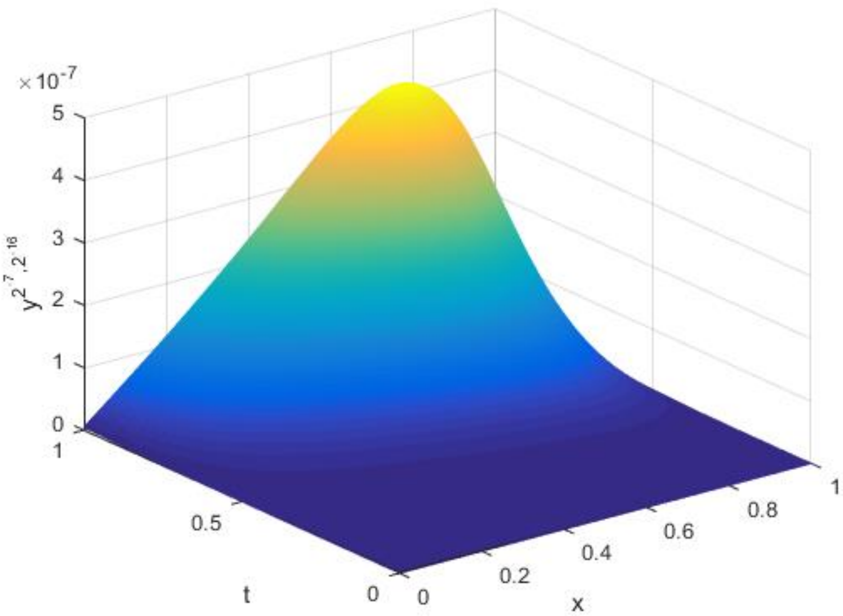

Figure 4.8: The error function $\left|\varepsilon_y^{2^{-7},2^{-16}}\right| = |\tilde{y} - y|$ for $h = 2^{-7}$, $\tau = 2^{-16}$, of Example 5.

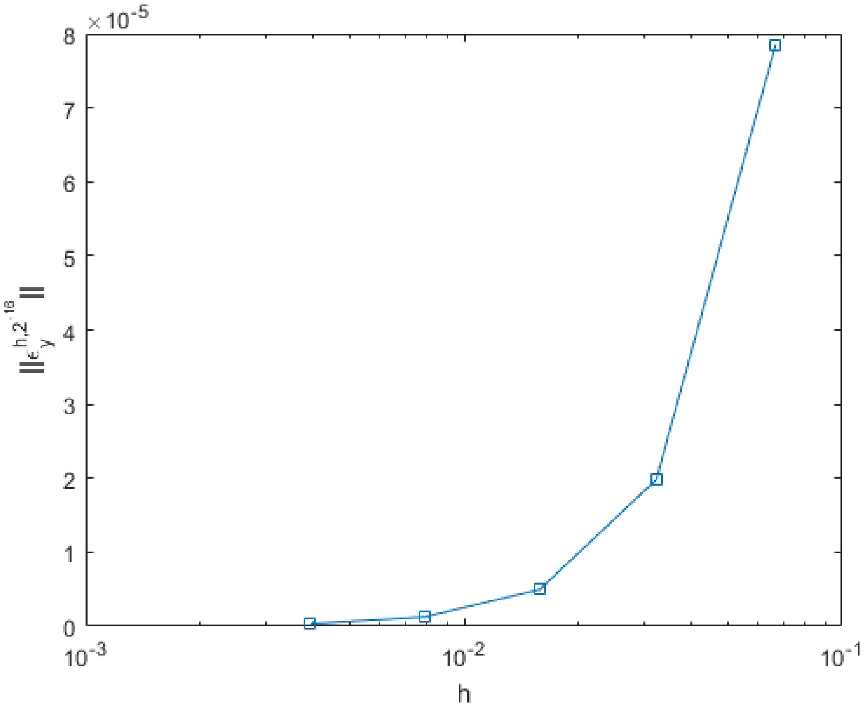

Figure 4.9: The maximum errors $\left\|\varepsilon_y^{h,2^{-16}}\right\|$ for $\tau = 2^{-16}$, with respect to h, of Example 5.

Chapter 5

CONCLUSION AND FINAL REMARKS

We study the finite difference approximation of $\partial_x u, \partial_t u, \partial_x^2 u, \partial_t^2 u$ and $\partial_x \partial_t u$ of which, $u(x,t)$ is the solution of the first type boundary value problem for one dimensional heat equation with constant coefficients. Difference boundary value problems of four point and six point implicit schemes are constructed. It is assumed that the initial function, boundary functions and the nonhomogeneous term in the heat equation possess a number of derivatives in the variables x and t necessary in this connection for performing current and subsequent manipulation in approximating the considered derivatives. We prove that the solution of the proposed four point and six point difference schemes converge uniformly to the exact value of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial t^2}$, and $\frac{\partial^2 u}{\partial x \partial t}$ on the grids of order $O(h^2+\tau)$ and $O(h^2+\tau^2)$ respectively, where, h is the step size in x and τ is the step size in time

Remark: These results can be used in some domain decomposition methods allowing parallel computation [24, 25] and also the methodology may be extended to two-dimensional heat equation.

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