Generalized Bezier Curves Based on Lupaş q-Analogue of the Bernstein Operator

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
January 2019
Gazimağusa, North Cyprus

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ABSTRACT

In this thesis, we study the Bezier curves and their properties. Bezier curves are one of the most important curves in Computer-Aided Geometric Design (CAGD). Bernstein functions are the basis of the Bezier curves. Therefore, we investigate the Lupaş q-analogue of Bernstein functions, their properties and corresponding Lupaş q-Bezier curves and their useful properties. Finally, we have studied the de Casteljau algorithms of Lupaş q-Bezier curve and the upgrade procedure.

Keywords: Bernstein polynomials, Bezier curve, de Casteljau algorithms, degree elevation, Lupaş q-Bezier curve, Lupaş q-Bezier surface, Lupaş q-analogue of Bernstein operator.

ÖZ

Bu tezde Bezier eğrilerilerini ve bu ozellikleri inceliyoruz. Bezier eğrileri, Bilgisayar Destekli Geometrik Tasarım'daki (CAGD) en önemli eğrilerden biridir. Bernstein fonksiyonları Bezier eğrilerinin temelidir. Bu nedenle, Bernstein fonksiyonlarının Lupaş q-analogunu, özelliklerini ve bunlara karşılık gelen Lupaş q-Bezier eğrilerini ve yararlı özelliklerini araştırıyoruz. Son olarak, Lupaş q-Bezier eğrisinin de Casteljau algoritmalarını ve yükseltme prosedürünü inceledik.

Anahtar kelimeler: Bernstein polinomları, Bezier egrileri, de Casteljau algoritması, derece yükseltme, Lupas q-Bezier egrileri, Lupaş q-Bezier yüzeyi, Lupaş q-analogue of Bernstein operatoru.

ACKNOWLEDGMENT

Firstly, I would like to appreciation my advisor Asst. Prof. Dr. Mustafa Kara, for his guidance, tolerance and unwavering encouragement. In addition I thank my father and mother Cemal Atamert and Seniz Atamert, who have provided me through moral, endless love, care, patient and emotional support. I also grateful to my other family members and friends who have supported me along the way.

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LIST OF SYMBOLS

$C[a,b]$	The set of all real-valued and continuous function defined on the compact interval $[a,b]$
N	The set of natural numbers
R	The set of real number
Z	The set of integer number
Z^{+}	The set of positive integer number
$\forall$	For all
$Б_n(h;\tau)$	A sequence of Bernstein polynomials
$Б_{\kappa}^{\eta}(\tau)$	η^{th} - degree Bernstein polynomial
$z_{\kappa}^{\eta}(\tau;q)$	η^{th} - degree Lupaş q-analogue of the Bernstein
(a,b)	An open interval
$[a,b]$	A closed interval
$[\kappa]_q$	q-analog of κ
$[\kappa]_q!$	κ-factorial
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	q-binomial coefficients

Chapter 1

INTRODUCTION

Bernstein Polynomials created by Sergei N. Bernstein [1] in 1912. In this thesis, he gave an alternative proof of the Weierstrass Approximation Theorem [13]. He introduced the following polynomials.

$$Б_n\left(h;\tau\right)=\sum_{\kappa=0}^{\eta}\binom{\eta}{\kappa}\tau^k\left(1-\tau\right)^{\eta-\kappa}h\left(\frac{\kappa}{\eta}\right) \tag{1.1}$$

$\forall\tau\in\left[0,1\right]$, $\eta\in N$ and $h\in C\left[0,1\right]$.

Bernstein polynomials are the basis of the Bézier curves [13]. These are parametric curves which are frequently used in computer graphics such as computer aided geometric design (CAGD)[18] and related fields.

Nowadays, Bézier curve is used in countless areas from modelling applications to writing type techniques. The foundations of the idea were first laid in 1959 by a French automotive engineer named Paul de Faget de Casteljau who were working at Citroen. In the same years, another French automotive engineer Pierre Bézier who carried out investigations on the parts of cylinder parts in Renault also studied a similar approach. Although these two engineers obtain the same results separately from each other, the first article published on this subject is written by Pierre Bézier in 1970. Therefore, these curves known as Bézier curves.

The rapid development of q-calculus [8] has led to the discovery of new generalizations of Bernstein polynomials involving q-integers [4,10–12,14,16-17].

In 1987, Lupas [11] introduced the first q-analogue of Bernstein operators:

$$\mathcal{L}_{\eta,q}\left(h;\tau\right)=\sum_{\kappa=0}^{\eta}\frac{h\left(\frac{[\kappa]_q}{[\eta]_q}\right)\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\frac{\kappa(\kappa-1)}{2}}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left\{\left(1-\tau\right)+q^{r-1}\tau\right\}} \tag{1.2}$$

$\forall\tau\in[0,1]$, $\eta\in N$ and $h\in C[0,1]$.

In 1996, George M. Philips [15] committed the q-analogue of the Bernstein polynomials known as Philips q-Bernstein polynomials:

$$B_{\eta,q}\left(h;\tau\right)=\sum_{\kappa=0}^{\eta}\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q h\left(\frac{[\kappa]_q}{[\eta]_q}\right)\tau^{\kappa}\prod_{j=0}^{\eta-\kappa-1}\left(1-q^{j}\tau\right) \tag{1.3}$$

where $B_{\eta,q}:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$, $\forall\eta\in N$, $\forall\tau\in[0,1]$ and arbitrary function. In 2003, Oruç and Philips [14] used the basis function of Philips q-Bernstein operator for construction of Philips q-Bezier curves and they studied the properties of the Lupas q-Bezier surfaces as well as the degree elevation, degree reduction, variation diminishing property as well as de Casteljau algorithms.

A new generalization of Bezier curves with one shape parameter which is based on the Lupas q-analogue of Bernstein operators is created by Li-Wen Han et al. in [7]. The new curves have some properties similar to classical Bezier curves. Also, they demonstrate degree elevation and de Casteljau algorithm for the generalization.

Besides, they studied the properties of the Lupas q-Bezier surfaces such as the degree elevation and de Casteljau algorithm.

This thesis is organized as follows:

In Chapter 2, the following topics are studied:

- Some useful definitions and properties associated with q-integer,
- basic and fundamental definitions and properties of q-calculus, Bernstein functions, Bézier curves, Lupas q-analogue of Bernstein operators and Lupaş q- Bézier curves.

In chapter 3, we investigate:

- definition of polynomials,
- definitions of Bernstein basis polynomials,
- the properties of Bernstein basis polynomials,
  - end-point property,
  - symmetry,
  - recursion formula,
  - non-negativity on $[0,1]$,
  - partition of unity,
  - degree raising,
  - converting form the Bernstein basis to the power basis,
  - converting form the power basis to the Bernstein basis,
  - the Bernstein polynomials as a basis,
- derivatives,
- the matrix representation of Bernstein polynomials.

- In chapter 4, we study;
- definition of the Bézier curve and Bézier polygon,

    - linear Bézier curve,
    - quadratic Bézier curve,
    - cubic Bézier curve,
- properties of Bézier curves,
    - end-points interpolation,
    - symmetry,
    - end-point tangent property,
    - variation diminishing property,
    - invariance under affine transformation,
    - convex hull property,
- derivatives,
- degree raising,
- de Casteljau algorithms,
- matrix formulation of Bézier curves.

In Chapter 5, we study

- the definition of the Lupaş q-analogue of the Bernstein function,
- properties of the Lupaş q-analogue of the Bernstein function,
    - non-negativity,
    - partition of unity,
    - end-point property,
    - q-inverse symmetry,
    - reducibility,
- degree elevation and reduction of the Lupaş q-analogue of the Bernstein functions,

- definition of Lupaş q-Bezier curve,
- properties of Lupaş q-Bezier curve,
  - geometric and affine invariant,
  - convex hull,
  - the end-point interpolation property,
  - q-inverse symmetry,
  - reducibility,
  - derivatives of the end-point property,
  - variation diminishing,
- degree elevation for Lupaş q-Bezier curve,
  - matrix representation of degree elevation of Lupaş q-Bezier curve,
- de Casteljau algorithm for Lupaş q-Bezier curve,
  - matrix representation of de Casteljau algorithm for Lupaş q-Bezier curves.

Chapter 2

PRELIMINARIES

In this chapter we define some useful properties related with Quantum Calculus, Calculus and my thesis topics.

Definition 2.1 [3] *Binomial theorem*
If r is any positive integer then,

$$(a+b)^r = \sum_{g=0}^{r} \binom{r}{g} a^{r-g} b^g$$

where

$$\binom{r}{g} = \frac{r!}{(r-g)!\,g!}, \qquad g = 0,1,2,3,\ldots,r.$$

Definition 2.2 [16] Let v denote the sequence (v_i), which may be finite or infinite.

Then we denote by $S^-(v)$ the number of strict sign changes in the sequence v.

For instance, $S^-(\text{-9,5,-6,7,-2,1}) = 5$, $S^-(\text{5,6,4,9,-7}) = 1$, and $S^-(\text{4,-4,4,-4,4,-4,...}) = \infty$

Definition 2.3 [16] Let

$$v_i = \sum_{r=0}^{s} a_{ir} u_r, \quad i = 0,1,\ldots,s,$$

where $a_{ir}, u_r, v_i \in R$. This linear transformation is said to be variation-diminishing if

$$S^-(v) \le S^-(u).$$

Theorem 2.4 [16] If $(\phi_0,\ldots,\phi_n)$ is totally positive on, then for any numbers $k_0,\ldots,k_n$

$$S^-(k_0\phi_n + \cdots + k_n\phi_n) \le S^-(k_0,\ldots,k_n).$$

Definition 2.5 [14] W is a transform on R^d is any mapping $W: R^d \to R^d$. That is, each point $x \in R^d$ is mapped to exactly one point $W(x)$ also in R^d.

Definition 2.6 [14] Let $W : R^d \to R^d$ be a transform. W is said to be linear transform iff:

(i) For all $\alpha \in R$ and all $m \in R^d$, we have $W(\alpha x) = \alpha W(m)$.

(ii) For all $m, n \in R^d$, we have $W(m+n) = W(m) + W(n)$.

Definition 2.7 [7] $(Affine\ map)$ A map $\phi : A \to B$ is called affine, if it can be represented by an $n \times m$ matrix A and a point of B such that

$$\phi x = Ax + v,$$

where v represent the image of the origin of A.

Definition 2.8 [14] An affine transform is a transform that can be written as $W(x) = T(L(x))$ where $L(.)$ is a linear transform and $T(.)$ is a translation. This can also be written as $W(x) = L(x) + t$ or $W = T_t L$.

Definition 2.9 [14] A curve is said to be affine invariant if the affine transform $\phi(.)$ applied to the points generated by the curve, i.e $\varpi(\tau) = \sum_{\kappa=0}^{\eta} \rho_\kappa Б_\kappa^\eta(\tau)$, produces precisely the same curve as transforming the control points of the curve, ρ_k, and the calculating curve, that is:

$$\phi\left(\sum_{\kappa=0}^{\eta} \rho_\kappa Б_\kappa^\eta(\tau)\right) = \sum_{\kappa=0}^{\eta} Б_\kappa^\eta(\tau) \phi(\rho_\kappa)$$

This will be satisfied if the basis functions $Б_\kappa^\eta(\tau)$ of the curve satisfy the property $\sum_{\kappa=0}^{\eta} Б_\kappa^\eta(\tau) = 1 \ \ for \ \ \tau \in [0,1]$.

Definition 2.10 [14] Let $\{y_1, y_2, \ldots, y_m\}$ be a set of points in the d-dimension Euclidean space R^d and $a_1, a_2, \ldots, a_m$ be real numbers, then:

(1) $\sum_{k=1}^{m} a_k y_k = a_1 y_1 + a_2 y_2 + \ldots + a_m y_m$ is called a linear combination of $y_1, y_2, \ldots, y_m$.

(2) If $\sum_{k=1}^{m} a_k = 1,$ then $\sum_{k=1}^{m} a_k y_k = a_1 y_1 + a_2 y_2 + \ldots + a_m y_m$ is called an affine combination of $y_1, y_2, \ldots, y_m$.

(3) If $\sum_{k=1}^{m} a_k = 1$ and $a_k \geq 0$, then $\sum_{k=1}^{m} a_k y_k = a_1 y_1 + a_2 y_2 + \ldots + a_m y_m$ is called a weighted average of $y_1, y_2, \ldots, y_m$.

Definition 2.11 [14] Let A be a set of points in R^d. The set A is convex if and only if for any two points $x, y \in A$, the line segment joining x and y is entirely in A.

Definition 2.12 [8] For all $\kappa \in Z^+$ the q-integer $[\kappa]_q$ is defined by

$$[\kappa]_q = 1 + q + q^2 + \ldots + q^{\kappa-1} := \begin{cases} \dfrac{1-q^\kappa}{1-q}, & f \;\; q \in R^+ / \{1\} \\ \kappa, & if \;\; q = 1 \end{cases}$$

Note that, $[0]_q = 0$.

Definition 2.13 [8] For each integer $\kappa \geq 0$, the q-factorial $[k]_q!$ is defined by

$$[\kappa]_q! := \begin{cases} [\kappa]_q [\kappa-1]_q \ldots [1]_q, & if \;\; \kappa = 1, 2, 3, \ldots \\ 1, & if \;\; \kappa = 0. \end{cases}$$

Definition 2.14 [8] For integers $0 \leq r \leq s$, the q-binomial coefficient is defined by

$$\begin{bmatrix} \eta \\ \kappa \end{bmatrix}_q := \frac{[\eta]_q!}{[\kappa]_q! [\eta-\kappa]_q!} := \begin{bmatrix} \eta \\ \eta-\kappa \end{bmatrix}_q$$

Definition 2.15 [8] The q-analogue of $(y-b)^n$ is a polynomial of the form

$$(y-b)_q^n := \begin{cases} 1 & if \;\; n = 0 \\ (y-b)(y-qb)(y-q^2 b)\ldots(y-q^{n-1} b) & if \;\; n \geq 1 \end{cases}$$

Proposition 2.16 [8] For any integer n,

$$D_q(x-a)_q^n = [n]_q (x-a)_q^{n-1}$$

Lemma 2.17 [8] For any integer $m > 0$ and b be a number. Gauss`s Binomial Formula defined as,

$$(y+b)_q^m = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q q^{r(r-1)/2} b^r y^{m-r}.$$

Proposition 2.18 [8] There are two q-Pascal rules, namely;

$$\begin{bmatrix} \eta \\ \kappa \end{bmatrix}_q = \begin{bmatrix} \eta-1 \\ \kappa-1 \end{bmatrix}_q + q^{\kappa} \begin{bmatrix} \eta-1 \\ \kappa \end{bmatrix}_q \tag{2.18a}$$

and

$$\begin{bmatrix} \eta \\ \kappa \end{bmatrix}_q = q^{\eta-\kappa} \begin{bmatrix} \eta-1 \\ \kappa-1 \end{bmatrix}_q + \begin{bmatrix} \eta-1 \\ \kappa \end{bmatrix}_q \tag{2.18b}$$

where $1 \le \kappa \le \eta - 1$.

Chapter 3

BERNSTEIN BASIS POLYNOMIALS

3.1 Polynomials

Polynomials are useful mathematical tools in Science, computer aided geometric designs and engineering. Therefore, firstly we need some definitions which are related to polynomials.

Definition 3.1.1 [11] A real polynomial with degree η is an expression of the form:

$$P(t) = c_n x^n + c_{n-1} x^{n-1} + ... + c_1 x + c_0 \tag{3.1}$$

where $n \in Z^+$ is an non-negative integer and $c_0, c_1, \ldots, c_n$ are the real numbers with $c_n \neq 0.$.

The highest power of x that occurs is called degree of polynomials and denoted by $deg(P)$. The numbers c_i's are called the coefficients.

In this sections we study Bernstein basis Polynomials and its useful properties.

3.2 Bernstein Basis Polynomials

Bernstein polynomials were defined by Sergei Natanovich Bernstein in 1912 as follows;

$$Б_n(h;\tau) = Б_n(\tau) = \sum_{\kappa=0}^{\eta} h\left(\frac{\kappa}{\eta}\right)\binom{\eta}{\kappa}\tau^k(1-\tau)^{\eta-\kappa}$$

where

$$h \in C[0,1] \text{ and } \tau \in [0,1].$$

Definition 3.2.1 [11] The Bernstein basis polynomials of degree η are defined as

$$B_{\kappa}^{\eta}(\tau) = \binom{\eta}{\kappa} \tau^{\kappa} (1-\tau)^{\eta-\kappa}, \quad \tau \in [0,1] \text{ and } \kappa = 0,1,\ldots,\eta \tag{3.2}$$

with binomial coefficient

$$\binom{\eta}{\kappa} = \begin{cases} \dfrac{\eta!}{\kappa!(\eta-\kappa)!} & 0 \le \kappa \le \eta, \\ 0 & otherwise \end{cases}$$

Further, if $\eta < \kappa$ or $0 > \kappa$, we set $B_{\kappa}^{\eta} = 0$.

In the following given examples contains graphs of some Bernstein basis polynomials.

Example 3.2.1:

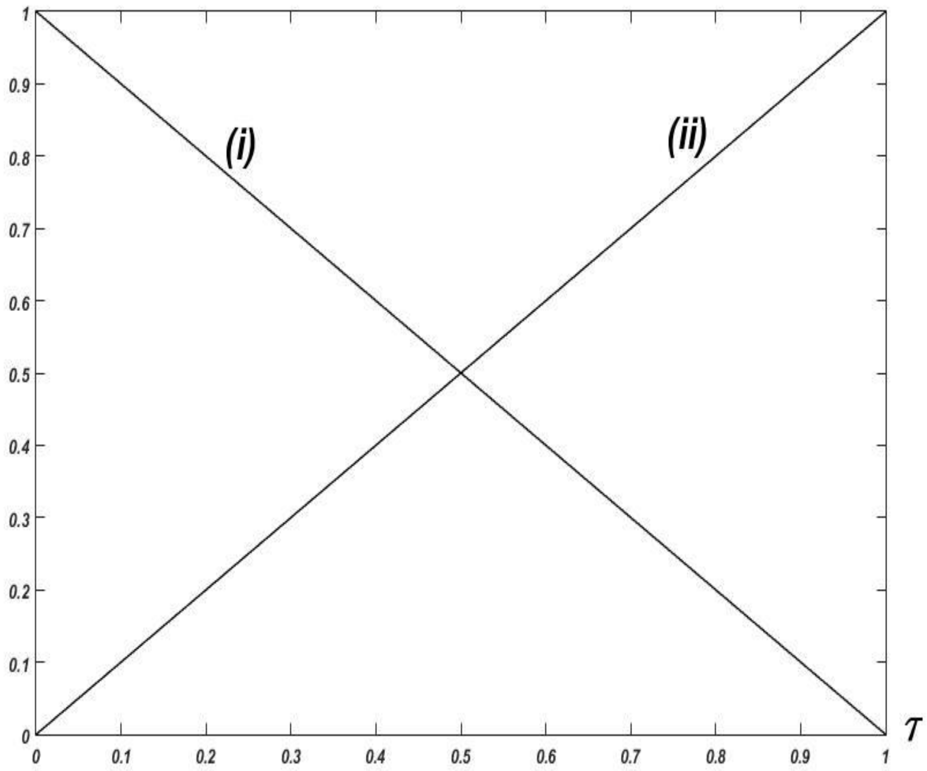


Figure 3.1: The Bernstein basis polynomials of degree 1. $(i) = B_0^1(\tau) = (1-\tau)$ and $(ii) = B_1^1(\tau) = \tau$.

Example 3.2.2:

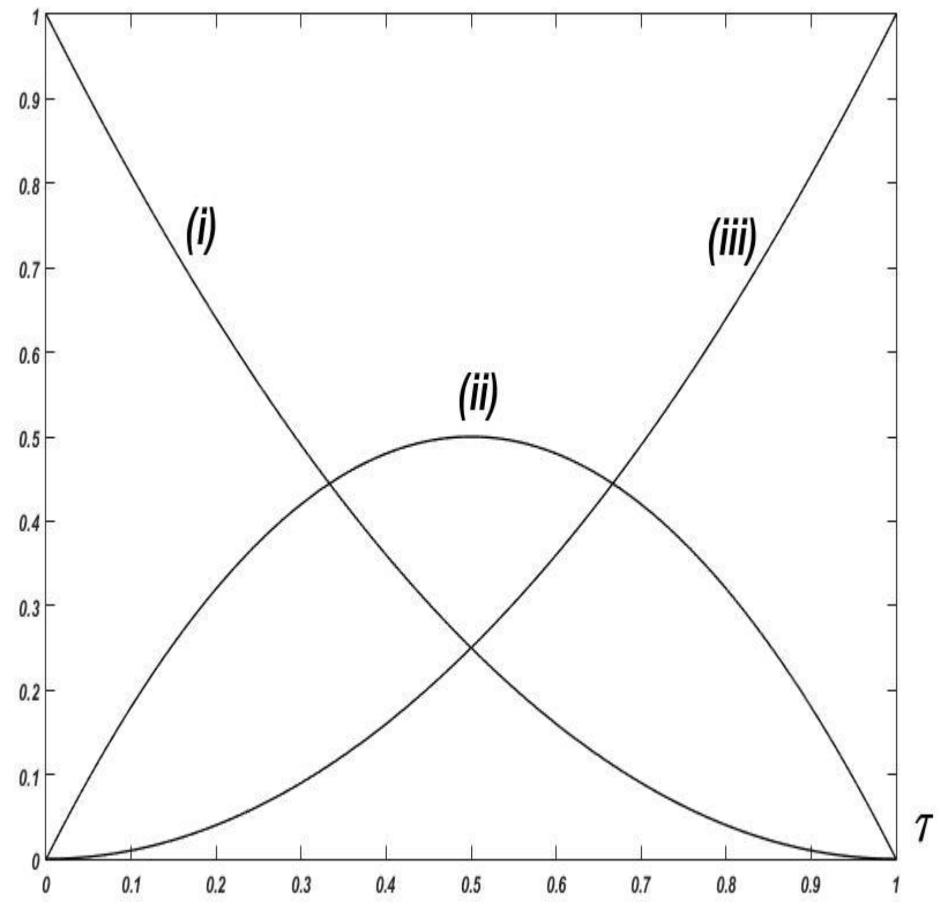


Figure 3.2: The Bernstein basis polynomials of degree 2. $(i) = B_0^2(\tau) = (1-\tau)^2$, $(ii) = B_1^2(\tau) = 2\tau(1-\tau)$ and $(iii) = B_2^2(\tau) = \tau^2$.

Example 3.2.3:

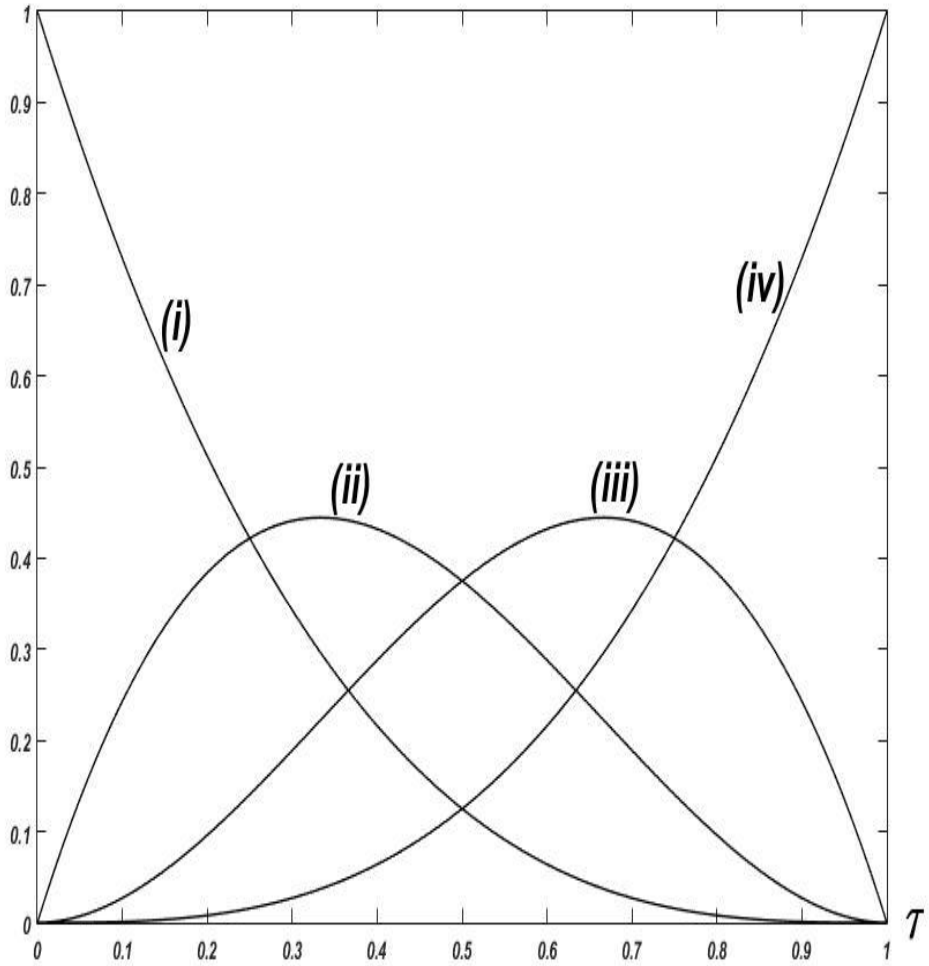


Figure 3.3: The Bernstein basis polynomials of degree 3. $(i) = B_0^3(\tau) = (1-\tau)^3$, $(ii) = B_1^3(\tau) = 3\tau(1-\tau)^2$, $(iii) = B_2^3(\tau) = 3\tau^2(1-\tau)$ and $(iv) = B_3^3(\tau) = \tau^3$.

Example 3.2.4:

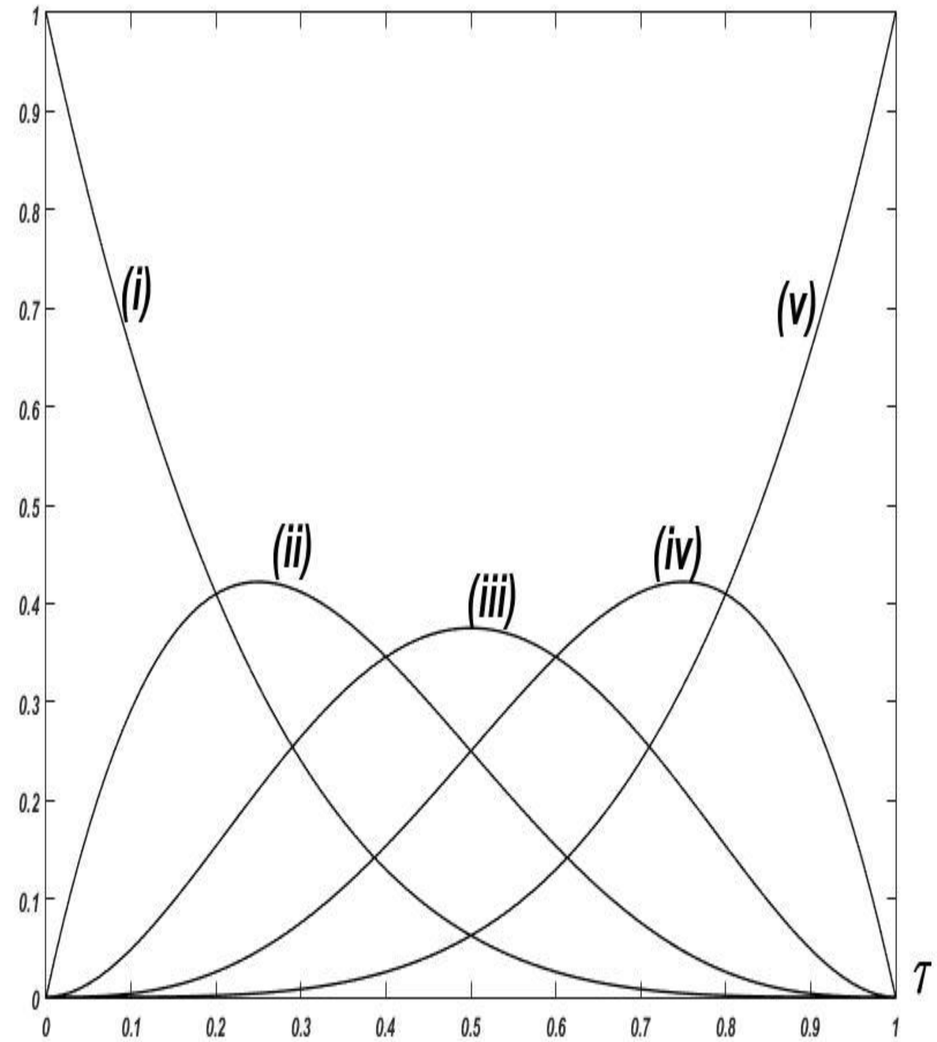


Figure 3.4: The Bernstein basis polynomials of degree 4. $(i) = Б_0^4(\tau) = (1-\tau)^4$, $(ii) = Б_1^4(\tau) = 4\tau(1-\tau)^3$, $(iii) = Б_2^4(\tau) = 6\tau^2(1-\tau)^2$, $(iv) = Б_3^4(\tau) = 4\tau^3(1-\tau)$ and $(v) = Б_4^4(\tau) = \tau^4$.

Example 3.2.5:

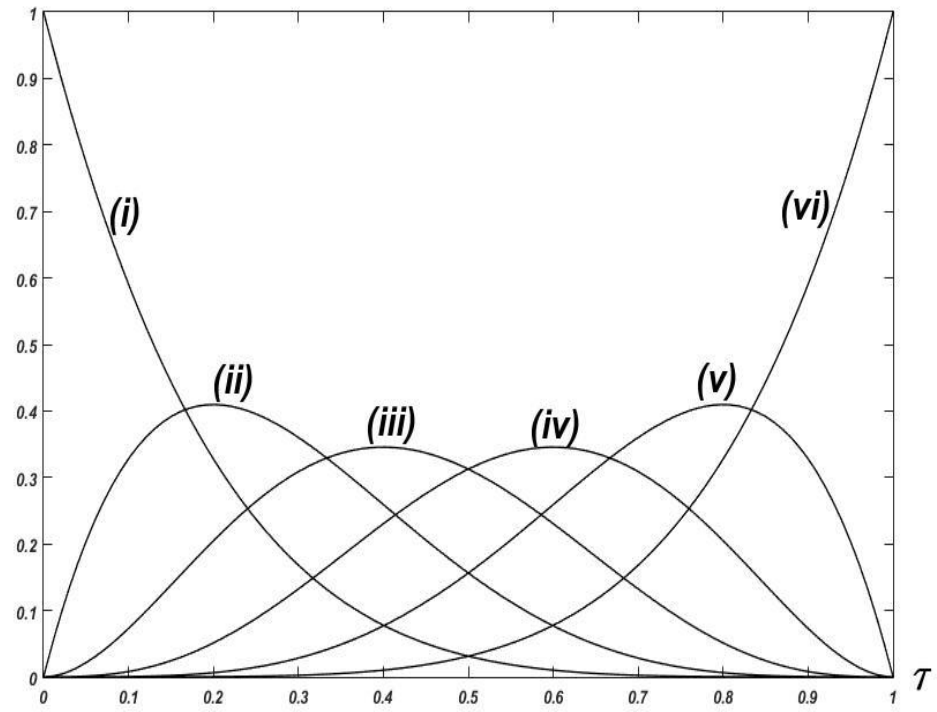


Figure 3.5: The Bernstein basis polynomials of degree 5. $(i) = Б_0^5(\tau) = (1-\tau)^5$, $(ii) = Б_1^5(\tau) = 5\tau(1-\tau)^4$, $(iii) = Б_2^5(\tau) = 10\tau^2(1-\tau)^3$, $(iv) = Б_3^5(\tau) = 10\tau^3(1-\tau)^2$, $(v) = Б_4^5(\tau) = 5\tau^4(1-\tau)$ and $(vi) = Б_5^5(\tau) = \tau^5$.

3.3 The Properties of Bernstein Basis Polynomials

Bernstein basis polynomials satisfies the following properties:

Property 3.3.1 [11] $End - point\ property$

$$Б_\kappa^\eta(0) = \begin{cases} 1 & \kappa = 0 \\ 0 & \kappa = 1, \dots, \eta \end{cases}$$

and

$$Б_\kappa^\eta(1) = \begin{cases} 0 & \kappa = 0, 1, \dots, \eta - 1 \\ 1 & \kappa = \eta \end{cases}$$

Proof. If we put $\kappa = \eta$ into (3.2), we obtain

$$Б_{\eta}^{\eta}(\tau) = \binom{\eta}{\eta} \tau^{\eta} (1-\tau)^{\eta-\eta}$$

Then, for $\tau = 1$, we have

$$Б_{\eta}^{\eta}(1) = \binom{\eta}{\eta} 1^{\eta} (0)^{\eta-\eta} = 1 .$$

Secondly, if we put $\kappa = 0$ into (3.2), we obtain

$$Б_{0}^{\eta}(\tau) = \binom{\eta}{0} \tau^{\eta} (1-\tau)^{\eta-0}$$

Then, for $\tau = 0$, we have

$$Б_{0}^{\eta}(0) = \binom{\eta}{0} 0^{\eta} (1)^{\eta} = 1 .$$

▪

Property 3.3.2 [11] *They are symmetric*

$$Б_{\kappa}^{\eta}(\tau) = Б_{\eta-\kappa}^{\eta}(1-\tau) \qquad , \qquad \kappa = 0,1,\ldots,\eta$$

Proof. From the definition of Bernstein basis polynomials (3.2), we have;

$$Б_{\kappa}^{\eta}(\tau) = \binom{\eta}{\kappa} \tau^{\kappa} (1-\tau)^{\eta-\kappa}$$

$$= \binom{\eta}{\eta-\kappa} \tau^{\eta-\kappa} (1-\tau)^{\kappa}$$

$$= Б_{\eta-\kappa}^{\eta}(1-\tau) .$$

▪

Property 3.3.3 [11] *They satisfy the recursion formula*

$$Б_{\kappa}^{\eta}(\tau) = (1-\tau) Б_{\kappa}^{\eta-1}(\tau) + \tau Б_{\kappa-1}^{\eta-1}(\tau) \tag{3.3}$$

where $Б_{-1}^{\eta} = Б_{n+1}^{\eta} = 0$.

Proof. From the definition of Bernstein basis polynomials (3.2), we have

$$Б_{\kappa}^{\eta-1}(\tau)=\binom{\eta-1}{\kappa}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa-1}$$

and

$$Б_{\kappa-1}^{\eta-1}(\tau)=\binom{\eta-1}{\kappa-1}\tau^{\kappa-1}\left(1-\tau\right)^{\eta-1-(\kappa-1)}$$

Then, from RHS of the equation of (3.3) and the binomial identity

$$(1-\tau)Б_{\kappa}^{\eta-1}(\tau)+\tau Б_{\kappa-1}^{\eta-1}(\tau)=(1-\tau)\binom{\eta-1}{\kappa}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa-1}+\tau\binom{\eta-1}{\kappa-1}\tau^{\kappa-1}\left(1-\tau\right)^{\eta-1-(\kappa-1)}$$

$$=\binom{\eta-1}{\kappa}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}+\binom{\eta-1}{\kappa-1}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}$$

$$=\left[\binom{\eta-1}{\kappa}+\binom{\eta-1}{\kappa-1}\right]\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}$$

$$=\binom{\eta}{\kappa}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}$$

$$=Б_{\kappa}^{\eta}(\tau).$$

▪

Property 3.3.4 [11] $Non-Negative\ on$ [0,1]

Bernstein basis polynomials are non-negative over the interval $[0,1]$ and are strictly positive on $(0,1)$. That is,

$$Б_{\kappa}^{\eta}(\tau)\geq 0\,,\ \tau\in[0,1] \tag{3.4}$$

and

$$Б_{\kappa}^{\eta}>0\,,\ \tau\in\left(0,1\right)$$

Proof. To show this property, we use recursive property (3.3) of Bernstein basis polynomials and mathematical induction method.

Base Case :

$$B_0^1(\tau) = 1 - \tau \text{ and } B_1^1(\tau) = \tau$$

are both non-negative over the interval $[0,1]$.

Induction hypothesis : Assume $B_\kappa^j(\tau) \geq 0$, $\forall \kappa, j < \eta$ for some η.

Then by our recursive definition :

$$B_\kappa^\eta(\tau) = (1-\tau)B_\kappa^{\eta-1}(\tau) + \tau B_{\kappa-1}^{\eta-1}(\tau)$$

RHS of the above recursive equation are all non-negative for $0 \leq \tau \leq 1$. By induction, all Bernstein bases polynomials are non-negative for $0 \leq \tau \leq 1$. Thus $B_\kappa^\eta(\tau) \geq 0$ on $[0,1]$.

If we change our hypothesis to be open interval $(0,1)$ and if we apply the same steps, we can show Bernstein bases polynomials are strictly positive on $0 < \tau < 1$.

■

Property 3.3.5 [11] *Partition of Unity*

A set of function $h_\kappa(\tau)$ is said to partition unity if they sum to 1 for all values of τ. The $\kappa+1$ Bernstein bases polynomials for a Bernstein polynomials of degree κ form a partition of unity. That is;

$$B_\kappa(\tau) = \sum_{\kappa=0}^{\eta} B_\kappa^\eta(\tau) = B_0^\eta(\tau) + B_1^\eta(\tau) + \ldots + B_\eta^\eta(\tau) = 1 \text{ , } 0 \leq \tau \leq 1. \qquad (3.5)$$

Proof. To prove this property, we need to prove following property:

$$Б_\kappa(\tau) = Б_{\kappa-1}(\tau)$$

Then,

$$\sum_{\kappa=0}^{\eta} Б_\kappa^\eta(\tau) = \sum_{\kappa=0}^{\eta-1} Б_\kappa^{\eta-1}(\tau)$$

If we use the recursive formula (3.3) of the Bernstein bases polynomials, we obtain the following results:

$$Б_\kappa(\tau) = \sum_{\kappa=0}^{\eta} Б_\kappa^\eta(\tau) = \sum_{\kappa=0}^{\eta}\left[\left(1-\tau\right)Б_\kappa^{\eta-1}(\tau) + \tau Б_{\kappa-1}^{\eta-1}(\tau)\right]$$

$$= \sum_{\kappa=0}^{\eta}\left[\left(1-\tau\right)Б_\kappa^{\eta-1}(\tau) + \tau Б_{\kappa-1}^{\eta-1}(\tau)\right]$$

$$= \left(1-\tau\right)\sum_{\kappa=0}^{\eta} Б_\kappa^{\eta-1}(\tau) + \tau \sum_{\kappa=0}^{\eta} Б_{\kappa-1}^{\eta-1}(\tau)$$

$$= \left(1-\tau\right)\left[\sum_{\kappa=0}^{\eta-1} Б_\kappa^{\eta-1}(\tau) + Б_\eta^{\eta-1}(\tau)\right] + \tau\left[\sum_{\kappa=1}^{\eta} Б_{\kappa-1}^{\eta-1}(\tau) + Б_{-1}^{\eta-1}(\tau)\right]$$

$$= \left(1-\tau\right)\sum_{\kappa=0}^{\eta-1} Б_\kappa^{\eta-1}(\tau) + \tau \sum_{\kappa=1}^{\eta} Б_{\kappa-1}^{\eta-1}(\tau)$$

$$= \left(1-\tau\right)\sum_{\kappa=0}^{\eta-1} Б_\kappa^{\eta-1}(\tau) + \tau \sum_{\kappa=0}^{\eta-1} Б_\kappa^{\eta-1}(\tau)$$

$$= \sum_{\kappa=0}^{\eta-1} Б_\kappa^{\eta-1}(\tau)$$

where we have operated $Б_\eta^{\eta-1}(\tau) = Б_{-1}^{\eta-1}(\tau) = 0$.

Once we have established this equality, it is simple to write

$$\sum_{\kappa=0}^{\eta} Б_\kappa^\eta(\tau) = \sum_{\kappa=0}^{\eta-1} Б_\kappa^{\eta-1}(\tau) = \sum_{\kappa=0}^{\eta-2} Б_\kappa^{\eta-2}(\tau) = \cdots = \sum_{\kappa=0}^{1} Б_\kappa^{1}(\tau) = \left(1-\tau\right) + \tau = 1$$

■

Property 3.3.6 [11] *Degree Raising*

Any of the lower-degree Bernstein polynomials of $\deg < \eta$ can be expressed as a linear combination of η degree Bernstein polynomials. On the other hand, any Bernstein polynomial of degree $\eta - 1$ can be written as a linear combination of Bernstein polynomials of degree η.

Proof. From the property of recursive property and formula (3.2);

$$Б_{\kappa}^{\eta}(\tau) = (1-\tau)Б_{\kappa}^{\eta-1}(\tau) + \tau Б_{\kappa-1}^{\eta-1}(\tau)$$

Then,

$$\tau Б_{\kappa}^{\eta}(\tau) = \binom{\eta}{\kappa}\tau^{\kappa+1}(1-\tau)^{\eta-\kappa} = \binom{\eta}{\kappa}\tau^{\kappa+1}(1-\tau)^{(\eta+1)-(\kappa+1)}$$

$$\tau Б_{\kappa}^{\eta}(\tau) = \frac{\binom{\eta}{\kappa}}{\binom{\eta+1}{\kappa+1}} Б_{\kappa+1}^{\eta+1}(\tau)$$

$$\tau Б_{\kappa}^{\eta}(\tau) = \frac{\kappa+1}{\eta+1} Б_{\kappa+1}^{\eta+1}(\tau) \qquad (*)$$

and

$$(1-\tau)Б_{\kappa}^{\eta}(\tau) = \binom{\eta}{\kappa}\tau^{\kappa}(1-\tau)^{\eta-\kappa+1}$$

$$(1-\tau)Б_{\kappa}^{\eta}(\tau) = \frac{\eta!}{\kappa!(\eta-\kappa)!}\frac{\kappa!(\eta+1-\kappa)!}{(\eta+1)!} Б_{\kappa}^{\eta+1}(\tau)$$

$$(1-\tau)Б_{\kappa}^{\eta}(\tau) = \frac{\eta-\kappa+1}{\eta+1} Б_{\kappa}^{\eta+1}(\tau) \qquad (**)$$

Hence,

$$(*) + (**) = Б_{\kappa}^{\eta}(\tau) = \frac{\eta-\kappa+1}{\eta+1} Б_{\kappa}^{\eta+1}(\tau) + \frac{\kappa+1}{\eta+1} Б_{\kappa+1}^{\eta+1}(\tau)$$

On the other hand,

$$\frac{1}{\binom{\eta}{\kappa}}Б_{\kappa}^{\eta}(\tau)+\frac{1}{\binom{\eta}{\kappa+1}}Б_{\kappa+1}^{\eta}(\tau)=\tau^{\kappa}(1-\tau)^{\eta-\kappa}+\tau^{\kappa+1}(1-\tau)^{\eta-(\kappa+1)}$$

$$\frac{1}{\binom{\eta}{\kappa}}Б_{\kappa}^{\eta}(\tau)+\frac{1}{\binom{\eta}{\kappa+1}}Б_{\kappa+1}^{\eta}(\tau)=\tau^{\kappa}(1-\tau)^{\eta-\kappa-1}((1-\tau)+\tau),$$

$$\frac{1}{\binom{\eta}{\kappa}}Б_{\kappa}^{\eta}(\tau)+\frac{1}{\binom{\eta}{\kappa+1}}Б_{\kappa+1}^{\eta}(\tau)=\tau^{\kappa}(1-\tau)^{\eta-\kappa-1},$$

$$\frac{1}{\binom{\eta}{\kappa}}Б_{\kappa}^{\eta}(\tau)+\frac{1}{\binom{\eta}{\kappa+1}}Б_{\kappa+1}^{\eta}(\tau)=\frac{1}{\binom{\eta-1}{\kappa}}Б_{\kappa}^{\eta-1}(\tau)$$

Then,

$$Б_{\kappa}^{\eta-1}(\tau)=\binom{\eta-1}{\kappa}\left[\frac{1}{\binom{\eta}{\kappa}}Б_{\kappa}^{\eta}(\tau)+\frac{1}{\binom{\eta}{\kappa+1}}Б_{\kappa+1}^{\eta}(\tau)\right],$$

$$Б_{\kappa}^{\eta-1}(\tau)=\frac{(\eta-1)!}{\kappa!(\eta-\kappa-1)!}\left[\frac{1}{\frac{\eta!}{\kappa!(\eta-\kappa)!}}Б_{\kappa}^{\eta}(\tau)+\frac{1}{\frac{\eta!}{(\kappa+1)!(\eta-\kappa-1)!}}Б_{\kappa+1}^{\eta}(\tau)\right]$$

$$Б_{\kappa}^{\eta-1}(\tau)=\binom{\eta-\kappa}{\eta}Б_{\kappa}^{\eta}(\tau)+\binom{\kappa+1}{\eta}Б_{\kappa+1}^{\eta}(\tau)$$

Hence, Bernstein polynomials of deg $\kappa<\eta$ can be written as a linear combination of Bernstein polynomials of degree η.

▪

Property 3.3.7 [19] Any Bernstein polynomials of degree η can be written as a linear combination of Bernstein polynomials of $\eta + r \ (r > 0)$.

$$Б_{\kappa}^{\eta}(\tau) = \sum_{j=\kappa}^{\kappa+r} \frac{\binom{\eta}{\kappa}\binom{r}{j-\kappa}}{\binom{\eta+r}{j}} Б_{\kappa}^{\eta+r}(\tau)$$

Property 3.3.8 [11] *Converting form the Bernstein Basis to the Power Basis*

Any Bernstein polynomials of degree η can be written in terms of the power basis.

Proof. To prove this property, we need to Bernstein basis polynomials of degree η expression and binomial theorem.

$$Б_{\kappa}^{\eta}(\tau) = \binom{\eta}{\kappa}\tau^{\kappa}(1-\tau)^{\eta-\kappa}$$

$$= \binom{\eta}{\kappa}\tau^{\kappa}\sum_{j=0}^{\eta-\kappa}\binom{\eta-\kappa}{j}(-\tau)^{j}$$

$$= \binom{\eta}{\kappa}\tau^{\kappa}\sum_{j=0}^{\eta-\kappa}(-1)^{j}\binom{\eta-\kappa}{j}\tau^{j}$$

$$= \sum_{j=0}^{n-k}(-1)^{j}\binom{\eta}{\kappa}\binom{\eta-\kappa}{j}\tau^{j+\kappa}$$

$$= \sum_{j=\kappa}^{\eta}(-1)^{j-\kappa}\binom{\eta}{\kappa}\binom{\eta-\kappa}{j-\kappa}\tau^{j}$$

$$= \sum_{j=\kappa}^{\eta}(-1)^{j-\kappa}\frac{\eta!}{\kappa!(\eta-\kappa)!}\frac{(\eta-\kappa)!}{(j-\kappa)!(\eta-j)!}\tau^{j}$$

$$= \sum_{j=\kappa}^{\eta}(-1)^{j-\kappa}\frac{\eta!}{\kappa!}\frac{1}{(j-\kappa)!(\eta-j)!}\tau^{j}\frac{j!}{j!}$$

$$= \sum_{j=\kappa}^{\eta}(-1)^{j-\kappa}\binom{\eta}{j}\binom{j}{\kappa}\tau^{j}$$

■

Property 3.3.9 [11] *Converting form the Power Basis to the Bernstein basis*

Each power basis element can be written as a linear combination of Bernstein basis polynomials.

Proof. Here, we need degree elevation formula and induction hypothesis.

$$\tau^{j} = \tau(\tau^{j-1})$$

$$= \tau \sum_{\iota=j-1}^{\eta} \frac{\binom{\eta}{j-1}}{\binom{\eta}{j-1}} Б_{\kappa}^{\eta-1}(\tau) \qquad \text{(We use the induction hypothesis.)}$$

$$= \sum_{\kappa=j}^{\eta} \frac{\binom{\eta-1}{j-1}}{\binom{\eta-1}{j-1}} \tau Б_{\kappa-1}^{\eta-1}(\tau)$$

$$= \sum_{\kappa=j-1}^{\eta-1} \frac{\binom{\kappa}{j-1}}{\binom{\eta}{j-1}} \frac{\kappa}{\eta} Б_{\kappa}^{\eta}(\tau)$$

$$= \sum_{\kappa=j-1}^{\eta-1} \frac{\binom{\kappa}{j}}{\binom{\eta}{j}} Б_{\kappa}^{\eta}(\tau)$$

▪

Property 3.3.10 [11] *The Bernstein Polynomials as a Basis*

The Bernstein polynomials of degree η from a basis for the space of polynomials of degree less than equal to η.

Proof. To show this property, we need to prove two following conditions.

$K = \{Б_0^{\eta}, \ldots, Б_n^{\eta}\}$ is linearly independent.

and

$$span\left\{Б_0^{\eta},\ldots,Б_n^{\eta}\right\}=\left\{Б_0^{\eta},\ldots,Б_n^{\eta}\right\}=P^{\eta}\left(\tau\right)$$

where

$$P^{\eta}\left(\tau\right)=\{a_0+a_1\tau+a_2\tau^2+\ldots+a_{\eta}\tau^{\eta}\,/\,a_0,\ldots,a_{\eta}\in R\}$$

of degree less than or equal than η.

For the first conditions, if there exists constants $a_0,a_1,a_2,\ldots,a_{\eta}$ such that;

$$0=a_0Б_0^{\eta}\left(t\right)+a_1Б_1^{\eta}\left(t\right)+a_2Б_2^{\eta}\left(t\right)+\cdots+a_{\eta}Б_{\eta}^{\eta}\left(t\right)\ ,\forall\tau\,.$$

Then, all $a_{\kappa}`s$ must be zero. Then we write,

$$0=a_0Б_0^{\eta}\left(t\right)+a_1Б_1^{\eta}\left(t\right)+a_2Б_2^{\eta}\left(t\right)+\cdots+a_{\eta}Б_{\eta}^{\eta}\left(t\right)$$

$$0=a_0\sum_{\kappa=0}^{\eta}(-1)^{\kappa}\binom{\eta}{\kappa}\binom{\kappa}{0}\tau^{\kappa}+a_1\sum_{\kappa=0}^{\eta}(-1)^{\kappa-1}\binom{\eta}{\kappa}\binom{\kappa}{1}\tau^{\kappa}+\cdots+a_n\sum_{\kappa=0}^{\eta}(-1)^{\kappa-\eta}\binom{\eta}{\kappa}\binom{\kappa}{\eta}\tau^{\kappa}$$

$$0=a_0+\left[\sum_{\kappa=0}^{1}a_1\binom{\eta}{1}\binom{1}{1}\right]\tau^1+\left[\sum_{\kappa=0}^{2}a_2\binom{n}{2}\binom{2}{2}\right]\tau^2+\cdots+\left[\sum_{\kappa=0}^{\eta}a_{\eta}\binom{\eta}{\eta}\binom{\eta}{\eta}\right]\tau^{\eta}\qquad(*)$$

Since (*) is linearly independent;

$$a_0=0$$

$$\sum_{\kappa=0}^{1}a_1\binom{\eta}{1}\binom{1}{1}=0$$

$$\vdots$$

$$\sum_{\kappa=0}^{\eta}a_{\eta}\binom{\eta}{\eta}\binom{\eta}{\eta}=0$$

which is stand for $a_0=a_1=a_2=\cdots=a_{\eta}=0$.

Secondly, from the converting from the Bernstein basis to the Power basis property of Bernstein polynomials, we know that each power basis $\{1,\ldots,\tau^{\eta}\}$ can be written as a linear combination of Bernstein basis polynomials. That is,

$$span\ \left\{Б_0^\eta,\ldots,Б_n^\eta\right\} = P^\eta\left(\tau\right) = \left\{a_0 + a_1\tau + a_2\tau^2 + \ldots + a_\eta\tau^\eta\right\}$$

■

3.4 Derivatives

Theorem 3.3.11 [17] Derivative of η-th degree of the Bernstein basis polynomials ($Б_\kappa^\eta\left(\tau\right)$) are Bernstein basis polynomials of degree $\eta-1$.

$$\frac{d}{d\tau}Б_\kappa^\eta\left(\tau\right) = \eta\left(Б_{\kappa-1}^{\eta-1}\left(\tau\right) - Б_\kappa^{\eta-1}\left(\tau\right)\right),\quad 0 \le \kappa \le \eta \tag{3.4}$$

Proof. The derivative of Bernstein polynomials $Б_\kappa^\eta\left(\tau\right)$ is obtained as;

$$\frac{d}{d\tau}Б_\kappa^\eta\left(\tau\right) = \frac{d}{dt}\left[\binom{\eta}{\kappa}\tau^\kappa(1-\tau)^{\eta-\kappa}\right]$$

$$= \frac{\kappa\eta!}{\kappa!(\eta-\kappa)!}\tau^{\kappa-1}(1-\tau)^{\eta-\kappa} - \frac{(\eta-\kappa)\eta!}{\kappa!(\eta-\kappa)!}\tau^\kappa(1-\tau)^{\eta-\kappa-1}$$

$$= \frac{\kappa\eta\left(\eta-1\right)!}{\kappa\left(\kappa-1\right)!(\eta-\kappa)!}\tau^{\kappa-1}(1-\tau)^{\eta-\kappa} - \frac{(\eta-\kappa)\eta\left(\eta-1\right)!}{\kappa!(\eta-\kappa)\left(\eta-\kappa-1\right)!}\tau^\kappa(1-\tau)^{\eta-\kappa-1}$$

$$= \frac{\eta\left(\eta-1\right)!}{\left(\kappa-1\right)!(\eta-\kappa)!}\tau^{\kappa-1}(1-\tau)^{\eta-\kappa} - \frac{\eta\left(\eta-1\right)!}{\kappa!\left(\eta-\kappa-1\right)!}\tau^\kappa(1-\tau)^{\eta-\kappa-1}$$

$$\frac{d}{d\tau}Б_\kappa^\eta\left(\tau\right) = \eta\left(\frac{\left(\eta-1\right)!}{\left(\kappa-1\right)!(\eta-\kappa)!}\tau^{\kappa-1}(1-\tau)^{\eta-\kappa} - \frac{\left(\eta-1\right)!}{\kappa!\left(\eta-\kappa-1\right)!}\tau^\kappa(1-\tau)^{\eta-\kappa-1}\right)$$

Hence,

$$\frac{d}{d\tau}Б_\kappa^\eta\left(\tau\right) = \eta\left(Б_{\kappa-1}^{\eta-1}\left(\tau\right) - Б_\kappa^{\eta-1}\left(\tau\right)\right).$$

■

3.5 The Matrix Representation of Bernstein Polynomials

A matrix representation for the Bernstein polynomials is very useful in many applications. The main purposes of matrix representation are fast computation of matrices multiplication and generating different Bezier control polygons for the cubic curve [2].

Any $P(\tau)$ polynomials is expressed as a linear combination of $Б_{\kappa}^{\eta}(\tau)$ as the following.

$$P(\tau) = c_0 Б_0^{\eta}(\tau) + c_1 Б_1^{\eta}(\tau) + \cdots + c_{\eta} Б_{\eta}^{\eta}(\tau)$$

$$= \begin{bmatrix} Б_0^{\eta}(\tau) & Б_1^{\eta}(\tau) & \cdots & Б_{\eta}^{\eta}(\tau) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\eta} \end{bmatrix}$$

Then;

$$P(\tau) = \begin{bmatrix} 1 & \tau & \tau^2 & \cdots & \tau^{\eta} \end{bmatrix} \begin{bmatrix} g_{0,0} & 0 & 0 & \cdots & 0 \\ g_{1,0} & g_{1,1} & 0 & \cdots & 0 \\ g_{2,0} & g_{2,1} & g_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{\eta,0} & g_{\eta,1} & g_{\eta,2} & \cdots & g_{\eta,\eta} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{\eta} \end{bmatrix}$$

or

$$P(\tau) = \begin{bmatrix} \tau^{\eta} & \tau^2 & \tau & \cdots & 1 \end{bmatrix} \begin{bmatrix} g_{0,0} & g_{0,1} & g_{0,2} & \cdots & g_{0,\eta} \\ 0 & g_{1,1} & g_{1,2} & \cdots & g_{1,\eta} \\ 0 & 0 & g_{2,2} & \cdots & g_{2,\eta} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_{\eta,\eta} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{\eta} \end{bmatrix}$$

where $g_{\kappa,\eta}$\`s are the coefficients of the power basis of the Bernstein polynomials.

In the following examples, we give matrix representation of cubic and quadratic case of Bernstein polynomials.

Example 3.4.1: In a quadratic case $(\eta = 2)$,

$$
\begin{aligned}
B_0^2(\tau) &= (1-\tau)^2 = 1 - 2\tau + \tau^2 \\
B_1^2(\tau) &= 2\tau(1-\tau) = 2\tau - 2\tau^2 \\
B_2^2(\tau) &= \tau^2
\end{aligned}
$$

Then, the matrix representations;

$$
P(\tau) = \begin{bmatrix} 1 & \tau & \tau^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}
$$

or

$$
P(\tau) = \begin{bmatrix} \tau^2 & \tau & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}
$$

Example 3.4.2: If a cubic case $(\eta = 3)$

$$
\begin{aligned}
B_0^3(\tau) &= (1-\tau)^3 = 1 - 3\tau + 3\tau^2 - \tau^3 \\
B_1^3(\tau) &= 3\tau(1-\tau)^2 = 3\tau - 6\tau^2 + 3\tau^3 \\
B_2^3(\tau) &= 3\tau^2(1-\tau) = 3\tau^2 - 3\tau^3 \\
B_3^3(\tau) &= \tau^3
\end{aligned}
$$

Then, the matrix representation is;

$$
P(\tau) = \begin{bmatrix} 1 & \tau & \tau^2 & \tau^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}
$$

or

$$
P(\tau) = \begin{bmatrix} \tau^3 & \tau^2 & \tau & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -3 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}
$$

Chapter 4

BEZIER CURVES

A Bezier curve is a parametric curve used in computer graphics and related fields. Pierre Bezier introduced the Bezier curve in the 1970`s while working for Renault. Bezier curves and surface are very useful and play significant role for CAGD.

Definition [6] Bézier *Curve*

Let $\rho_0, \rho_1, ...\rho_n$ be a sequence of control points, a Bézier curve of degree η is defined by;

$$\varpi(\tau) = \sum_{\kappa=0}^{\eta} \rho_\kappa Б_\kappa^\eta(\tau) \quad , \qquad 0 \le \tau \le 1 \tag{4.1}$$

where the basis functions $Б_\kappa^\eta(\tau)$ are the Bernstein polynomials defined by;

$$Б_\kappa^\eta(\tau) = \frac{\eta!}{(\eta-\kappa)!\kappa!} \tau^\kappa (1-\tau)^{\eta-\kappa}$$

where $\begin{pmatrix} \eta \\ \kappa \end{pmatrix} = \frac{\eta!}{\kappa!(\eta-\kappa)!}$.

Definition [6] Bézier *polygo*n

Let $\rho_0, \rho_1, ...\rho_\eta$ be a set of control points of the Bezier curve, the polygon formed by connecting the Bézier points with lines, starting with ρ_0 and finishing with ρ_η, is called the Bézier polygon. The convex hull of the Bézier polygon contains the Bezier curve.

Let's investigate some specials cases of Bézier curves.

4.1 Linear Bezier Curve

Linear Bézier curve has $\eta = 1$. We know from the definition of Bezier curve, if Bézier curve of degree η have $\eta + 1$ control points. Let ρ_0 and ρ_1 are two control points, a Linear Bezier curve is simply a straight line between those two points, the curve is defined by;

$$\varpi(\tau) = \sum_{\kappa=0}^{1} \rho_\kappa B_\kappa^1(\tau) \quad , \quad \tau \in [0,1]$$

$$\varpi(\tau) = (1-\tau)\rho_0 + (\tau)\rho_1$$

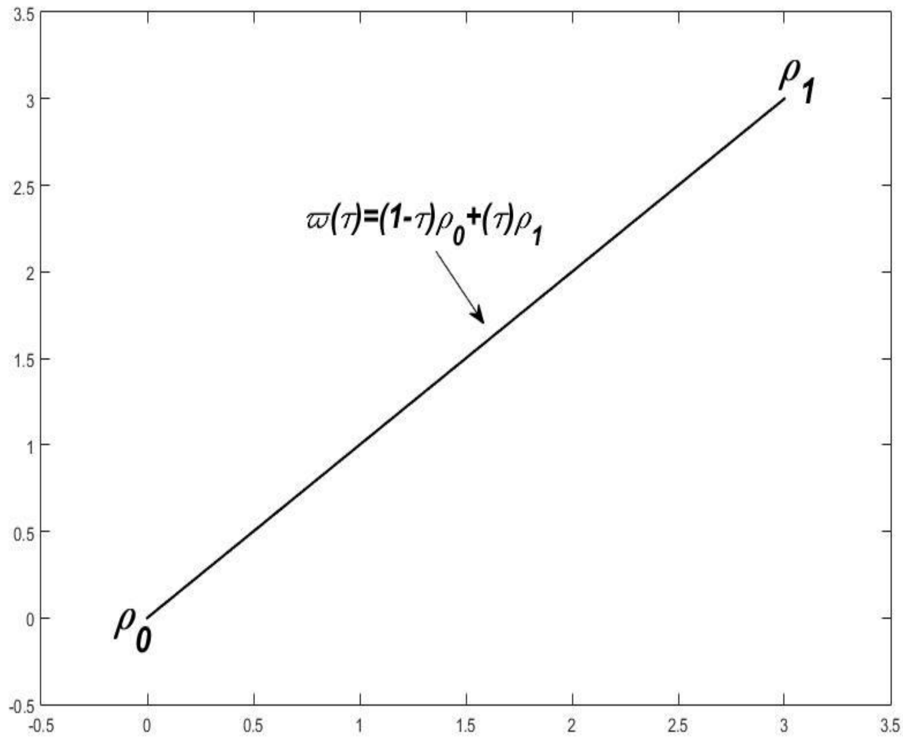


Figure 4.1: Bezier curve of degree 1.

4.2 Quadratic Bezier Curve

Let $\eta = 2$, a quadratic Bezier curve $\varpi(\tau)$ has three control points ρ_0, ρ_1 and ρ_2;

we have;

$$\varpi(\tau) = \sum_{\kappa=0}^{2} \rho_\kappa Б_\kappa^2(\tau) \quad , \qquad \tau \in [0,1]$$

$$\varpi(\tau) = (1-\tau)^2 \rho_0 + 2\tau(1-\tau)\rho_1 + \tau^2 \rho_2$$

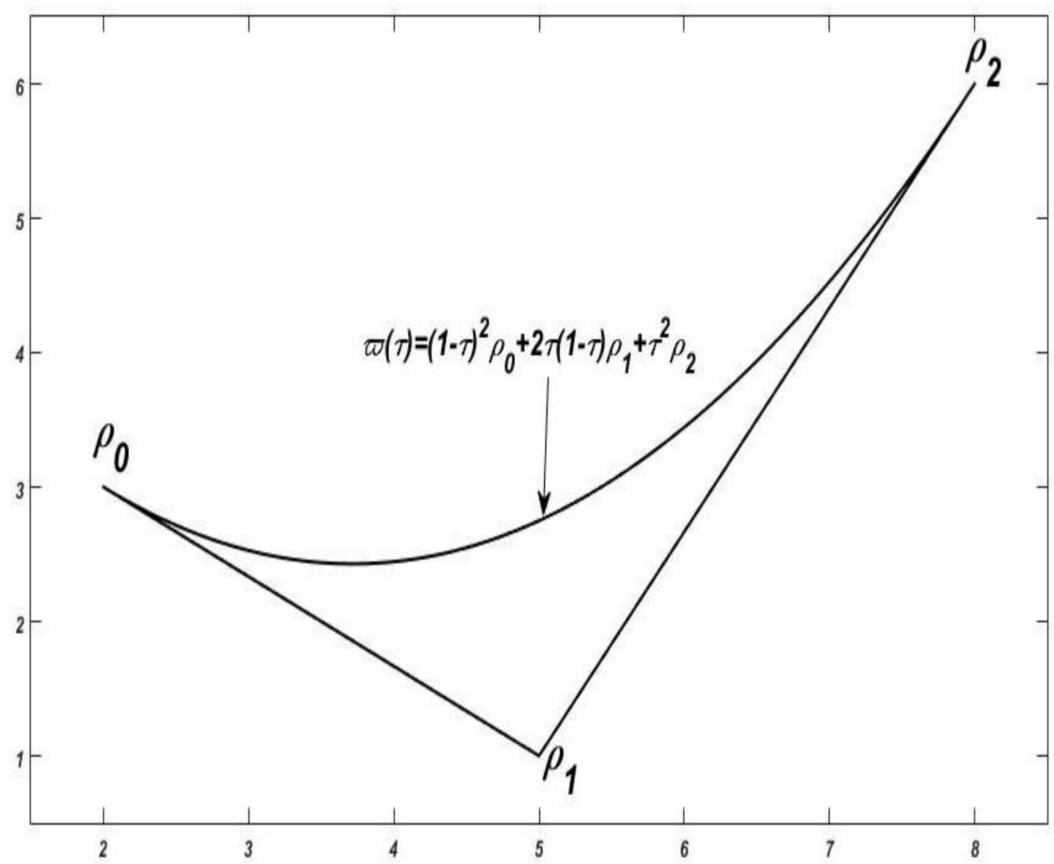


Figure 4.2: Bezier curve of degree 2.

4.3 Cubic Bezier Curve

Four control points $\rho_0, \rho_1, \rho_2, \rho_3$ in the plane or in three-dimensional space define a cubic Bezier curve, the curve starts at ρ_0 going toward ρ_1 and arrives at ρ_3 coming from the direction of ρ_2, usually it will not pass through ρ_1 or ρ_2, these points are only there to provide directional information, the parametric form of the curve is[6];

$$\varpi(\tau) = \sum_{\kappa=0}^{3} \rho_\kappa Б_\kappa^3(\tau) \quad , \qquad \tau \in [0,1]$$

$$\varpi(\tau) = (1-\tau)^3 \rho_0 + 3\tau(1-\tau)^2 \rho_1 + 3\tau^2(1-\tau)\rho_2 + \tau^3 \rho_3$$

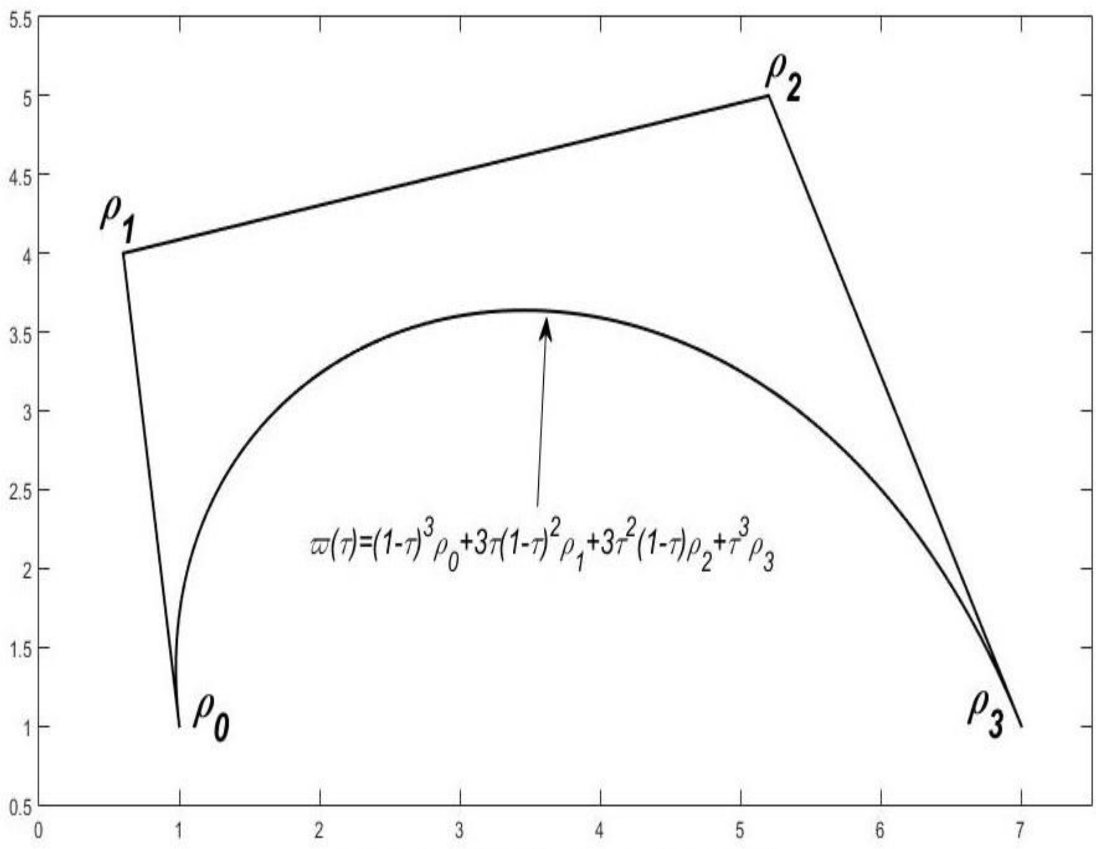


Figure 4.3: Bezier curve of degree 3.

To next section we give some properties of Bezier curves.

4.4 Properties of Bezier Curves

Property 4.4.1 [6] $End - points\ Interpolation$

Bezier curves $\varpi(\tau)$ always passes through the first and last control points of ρ_0 and ρ_η. That is;

$$\varpi(0) = \rho_0 \quad \text{and } \varpi(1) = \rho_\eta$$

Proof. If we put $\tau = 0$ into (4.1), we obtain

$$\varpi(0) = \sum_{\kappa=0}^{\eta} \rho_\kappa Б_\kappa^\eta(0)$$

From the end-point property of Bernstein basis polynomials (property 3.3.1), we have $Б_0^\eta(0) = 1$. Therefore;

$$\varpi(0) = \rho_0 .$$

In addition, when we put $\tau = 1$ into (4.1);

$$\varpi(1) = \sum_{\kappa=0}^{\eta} \rho_\kappa Б_\kappa^\eta(1)$$

From the end-point property of Bernstein basis polynomials (property 3.3.1), we have $Б_\eta^\eta(1) = 1$. Therefore;

$$\varpi(1) = \rho_\eta .$$

■

Property 4.4.2 [6] $Symmetry$

Reversing the order of the control points produce the same curve.

Proof. Let

$$\rho_\kappa^* = \rho_{\eta-\kappa} \qquad (\kappa = 0,1,\ldots,\eta)$$

Then, we have

$$\varpi^*(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa^* Б_\kappa^\eta(\tau)=\sum_{\kappa=0}^{\eta}\rho_{\eta-\kappa} Б_\kappa^\eta(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa Б_{\eta-\kappa}^\eta(\tau)$$

Due to the symmetry property of Bernstein polynomials (Property3.3.2), $Б_{\eta-\kappa}^\eta(\tau)=Б_\kappa^\eta(1-\tau)$.

$$\varpi^*(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa Б_\kappa^\eta(1-\tau)=\varpi(1-\tau)$$

▪

Property 4.4.3 [6] $End - point\ tangent\ property$

The end-point tangent vector are parallel to $\rho_1-\rho_0$ and $\rho_\eta-\rho_{\eta-1}$.

$$\varpi'(0)=\eta(\rho_1-\rho_0)$$

and

$$\varpi'(1)=\eta(\rho_\eta-\rho_{\eta-1})$$

Proof. From the definition of Bezier curve (4.1), we have;

$$\varpi(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa Б_\kappa^\eta(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa\binom{\eta}{\kappa}\tau^\kappa(1-\tau)^{\eta-\kappa}$$

Then, we take derivatives of Bezier curve with respect to τ, we obtain;

$$\varpi'(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa\binom{\eta}{\kappa}\left[\kappa\tau^{\kappa-1}(1-\tau)^{\eta-\kappa}-\left((\eta-\kappa)\tau^\kappa(1-\tau)^{\eta-\kappa-1}\right)\right]$$

for $\tau=0$,

$$\varpi'(0)=\eta(\rho_1-\rho_0)$$

for $\tau=1$,

$$\varpi'(1)=\eta(\rho_\eta-\rho_{\eta-1})$$

Property 4.4.4 [6] *Variation Diminishing Property*

Let planar Bezier curves are variation dimension, this means that the number of intersections of a straight line is no greater than the number of intersections of a line with the control polygon.

Proof. In this proof $Z_{\tau\in I\subseteq(0,+\infty)}\left[g(\tau)\right]$ denote the number of positive roots of any polynomials $g(\tau)$ on the interval I. That is.

$$Z_{0<\tau<1}\left[a_0+a_1\tau+\cdots+a_\eta\tau^\eta\right]=S^-\left(a_0+a_1\tau+\cdots+a_\eta\tau^\eta\right)\le S^-\left(a_0,a_1,\ldots,a_\eta\right). \qquad (4.2)$$

Let C denote a planar Bezier curve, M is any straight line, and let $I(C,M)$ the number of times C crosses M. Establish the rectangular coordinate system whose abscissa axis is M. Due to Bezier curves are geometric invariant, we can denote (x_i,y_i) $(i=0,1,\ldots,\eta)$ the new coordinates of the control points. Let P denote the control polygon and $I(P,M)$ the number of times P crosses M. Then we will prove that $I(C,M)\ \le\ I(P,M)$.

We make a parameter transformation. Let $u=\dfrac{\tau}{1-\tau}$, $\tau\in(0,1)$, so that $u\in(0,+\infty)$.

Then

$$I(C,M)=Z_{0<\tau<1}\left[\sum_{\kappa=0}^{\eta}y_\kappa Б_\kappa^\eta(\tau)\right]=Z_{0<\tau<1}\left[y_\kappa\binom{\eta}{\kappa}\tau^\kappa(1-\tau)^{\eta-\kappa}\right]$$

$$=Z_{0<u<+\infty}\left[\sum_{\kappa=0}^{\eta}y_\kappa\binom{\eta}{\kappa}\tau^\kappa(1-\tau)^{\eta-\kappa}\right]=Z_{0<u<+\infty}\left[\sum_{\kappa=0}^{\eta}y_\kappa\binom{\eta}{\kappa}u^\kappa\right]$$

$$\le S^-\left(\binom{\eta}{0}\mathrm{y}_0,\binom{\eta}{1}y_1,\ldots,\binom{\eta}{\eta}y_\eta\right)=S^-\left(y_0,y_1,\ldots,y_\eta\right)=I\left(P,\mathrm{M}\right)$$

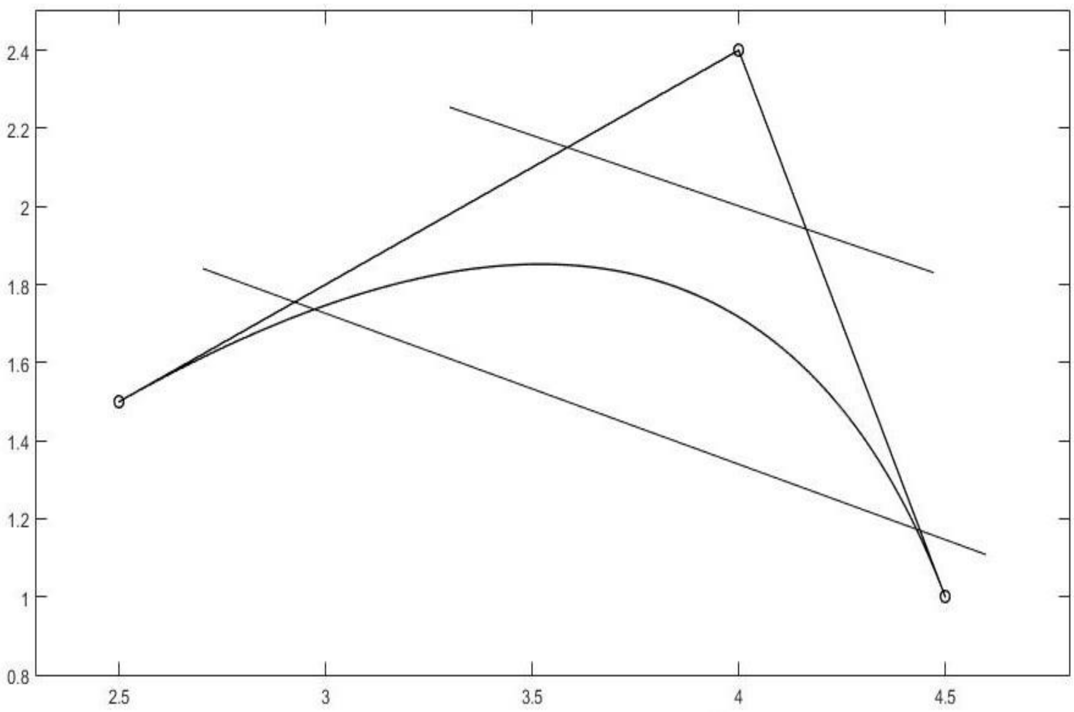


Figure 4.4: Variation diminishing property of quadratic Bezier curves.

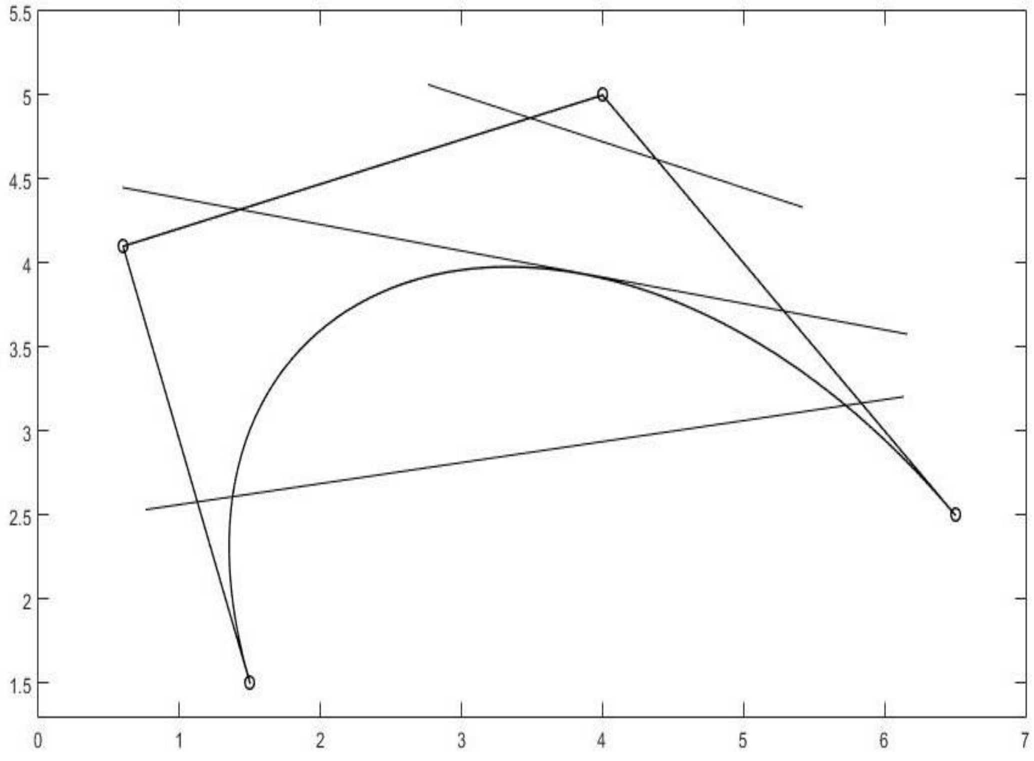


Figure 4.5: Variation diminishing property of cubic Bezier Curves.

Property 4.4.5 [6] $Invariance\ Under\ Affine\ Transformations$

If ϕ is an affine transform, then

$$\phi\left(\sum_{\kappa=0}^{\eta}\rho_{\kappa}Б_{\kappa}^{\eta}(\tau)\right)=\sum_{\kappa=0}^{\eta}Б_{\kappa}^{\eta}(\tau)\phi(\rho_{\kappa})$$

Proof. Let $\varpi(\tau)=\sum_{\kappa=0}^{\eta}\rho_{\kappa}Б_{\kappa}^{\eta}(\tau)$ where $\tau\in[0,1]$. Since, partition of unity property of $Б_{\kappa}^{\eta}(\tau)$, every point $\varpi(\tau)$ is an affine combination of control points $\rho_0,\ldots\rho_n$. From that, $\varpi(\tau)$is affine invariant. If we assume ϕ is an affine transform in $\mathbb{R}^d$, then

$$\phi(\varpi(\tau))=\phi\left(\sum_{\kappa=0}^{\eta}\rho_{\kappa}Б_{\kappa}^{\eta}(\tau)\right)=A\sum_{\kappa=0}^{\eta}\rho_{\kappa}Б_{\kappa}^{\eta}(\tau)+v$$

$$=\sum_{\kappa=0}^{\eta}\rho_{\kappa}AБ_{\kappa}^{\eta}(\tau)+\sum_{\kappa=0}^{\eta}Б_{\kappa}^{\eta}(\tau)v$$

$$=\sum_{\kappa=0}^{\eta}(A\rho_{\kappa}+v)Б_{\kappa}^{\eta}(\tau)$$

$$=\sum_{\kappa=0}^{\eta}\phi_{\kappa}(\rho_{\kappa})Б_{\kappa}^{\eta}(\tau)$$

Property 4.4.6 [6] $Convex\ hull\ property$

A Bezier curve lies in the convex hull of the control points, that is

$$\varpi(\tau)\in\varpi H(\rho_0,\ldots,\rho_\eta)\text{ for all }\tau\in[0,1].$$

Proof. Every point in the $\varpi(\tau)$ has the form;

$$a_0\rho_0+a_1\rho_1+\ldots+a_\eta\rho_\eta\quad\text{with}\quad a_{\kappa}=Б_{\kappa}^{\eta}(\tau).$$

Since non-negativity (3.4) and partition of unity property (3.5) of Bernstein polynomials, we have;

$$Б_\kappa^\eta(\tau) \geq 0 \quad \text{and} \quad \sum_{\kappa=0}^{\eta} Б_\kappa^\eta(\tau) = 1$$

Hence; $\varpi(\tau) \in \varpi H(\rho_0, \ldots, \rho_\eta)$.

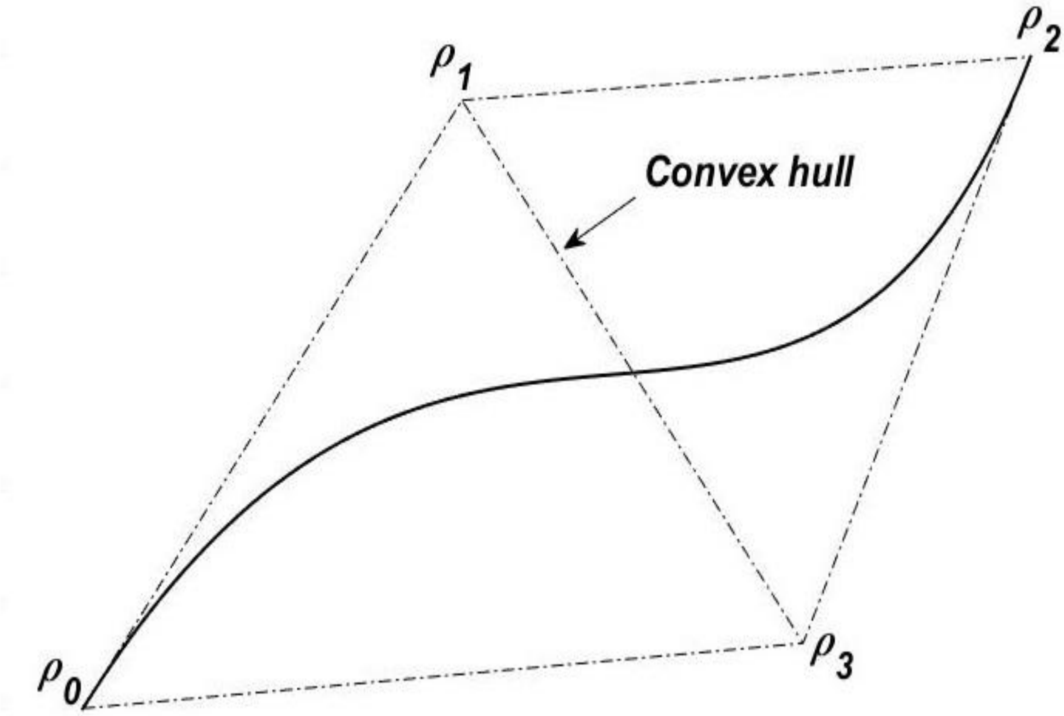


Figure 4.6: The curve lies in the convex hull of the control points.

4.5 The Derivative of a Bezier Curve

Theorem 4.5.1 [6] The derivative of $\varpi(\tau)$ of order η is;

$$\frac{d}{d\tau}\varpi(\tau) = \eta \sum_{\kappa=0}^{\eta-1} (\rho_{\kappa+1} + \rho_\kappa) Б_\kappa^{\eta-1}(\tau)$$

Proof. From the derivative of $Б_\kappa^\eta(\tau)$ (3.4);

$$\frac{d}{d\tau}\varpi(\tau) = \frac{d}{d\tau}\left(\sum_{\kappa=0}^{\eta} \rho_\kappa Б_\kappa^\eta(\tau)\right) = \sum_{\kappa=0}^{\eta} \rho_\kappa \frac{d}{d\tau}\left(Б_\kappa^\eta(\tau)\right)$$

$$= \eta \sum_{\kappa=0}^{\eta} \rho_\kappa \left(Б_{\kappa-1}^{\eta-1}(\tau) - Б_{\kappa}^{\eta-1}(\tau) \right)$$

Then;

$$\frac{d}{d\tau} \varpi(\tau) = \eta \sum_{\kappa=0}^{\eta} Б_{\kappa-1}^{\eta-1}(\tau) \rho_\kappa - \eta \sum_{\kappa=0}^{\eta} Б_{\kappa}^{\eta-1}(\tau) \rho_\kappa$$

from $Б_{-1}^{\eta-1}(\tau) = 0$ and $Б_{\eta}^{\eta-1}(\tau) = 0$,

$$\frac{d}{d\tau} \varpi(\tau) = \eta \sum_{\kappa=1}^{\eta} Б_{\kappa-1}^{\eta-1}(\tau) \rho_\kappa - \eta \sum_{\kappa=0}^{\eta-1} Б_{\kappa}^{\eta-1}(\tau) \rho_\kappa$$

and

$$\frac{d}{d\tau} \varpi(\tau) = \eta \sum_{\kappa=0}^{\eta-1} Б_{\kappa}^{\eta-1}(\tau) \rho_{\kappa+1} - \eta \sum_{\kappa=0}^{\eta-1} Б_{\kappa}^{\eta-1}(\tau) \rho_\kappa$$

Hence;

$$\frac{d}{d\tau} \varpi(\tau) = \eta \sum_{\kappa=0}^{\eta-1} \left(\rho_{\kappa+1} + \rho_\kappa \right) Б_{\kappa}^{\eta-1}(\tau)$$

4.6 Degree Raising

Definition 4.6.1 [18] Any Bezier curve of degree η (with control points ρ_κ) can be expressed in terms of a new basis of degree $\eta+1$.The new control point ρ_κ^* are given by

$$\rho_\kappa^* = \frac{\kappa}{\eta+1} \rho_{\kappa-1} + \left(1 - \frac{\kappa}{\eta+1} \right) \rho_\kappa \qquad \kappa = 0, 1, \ldots, \eta+1$$

where $\rho_{-1} = \rho_{\eta+1} = 0$.

Proof. From the property of degree raising of $Б_{\kappa}^{\eta}(\tau)$(3.3.6), we have;

$$(1-\tau) Б_{\kappa}^{\eta}(\tau) = \frac{\eta - \kappa + 1}{\eta + 1} Б_{\kappa}^{\eta-1}(\tau)$$

and

$$\tau Б_{\kappa}^{\eta}\left(\tau\right)=\frac{\kappa+1}{n+1}B_{i+1}^{n+1}(t)$$

Degree raising, obtained by simply multiplying the equation of the degree n of Bezier curve by $\left[\left(1-\tau\right)+\tau\right]=1$:

$$\varpi\left(\tau\right)=\left[\left(1-\tau\right)+\tau\right]\varpi\left(\tau\right)$$

$$=\left[\left(1-\tau\right)+\tau\right]\sum_{\kappa=0}^{\eta}\rho_{\kappa}Б_{\kappa}^{\eta}\left(\tau\right)$$

$$=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\left[\left(1-\tau\right)Б_{\kappa}^{\eta}\left(\tau\right)+\tau Б_{\kappa}^{\eta}\left(\tau\right)\right]$$

$$=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\left[\left(1-\tau\right)\binom{\eta}{\kappa}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}+\tau\binom{\eta}{\kappa}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}\right]$$

$$=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\left[\binom{\eta}{\kappa}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa+1}+\binom{\eta}{\kappa}\tau^{\kappa+1}\left(1-\tau\right)^{\eta-\kappa}\right]$$

$$=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\left[\frac{\binom{\eta}{\kappa}}{\binom{\eta+1}{\kappa}}Б_{\kappa}^{\eta+1}\left(\tau\right)+\frac{\binom{\eta}{\kappa}}{\binom{\eta+1}{\kappa+1}}Б_{\kappa+1}^{\eta+1}\left(\tau\right)\right]$$

$$=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\left[\frac{\eta!}{(\eta-\kappa)!\kappa!}\frac{\kappa!(\eta+1-\kappa)!}{(\eta+1)!}Б_{\kappa}^{\eta+1}\left(\tau\right)+\frac{\eta!}{\kappa!(\eta-\kappa)!}\frac{(\kappa+1)!(\eta-\kappa)!}{(\eta+1)!}Б_{\kappa+1}^{\eta+1}\left(\tau\right)\right]$$

$$=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\left[\frac{\eta-\kappa+1}{\eta+1}Б_{\kappa}^{\eta+1}\left(\tau\right)+\frac{\kappa+1}{\eta+1}Б_{\kappa+1}^{\eta+1}\left(\tau\right)\right]$$

$$=\sum_{\kappa=0}^{\eta}\frac{\eta-\kappa+1}{\eta+1}\rho_{\kappa}Б_{\kappa}^{\eta+1}\left(\tau\right)+\sum_{\kappa=0}^{\eta}\frac{\kappa+1}{\eta+1}\rho_{\kappa}Б_{\kappa+1}^{\eta+1}\left(\tau\right)$$

$$=\sum_{\kappa=0}^{\eta}\frac{\eta-\kappa+1}{\eta+1}\rho_{\kappa}Б_{\kappa}^{\eta+1}\left(\tau\right)+\sum_{\kappa=0}^{\eta+1}\frac{\kappa}{\eta+1}\rho_{\kappa-1}Б_{\kappa}^{\eta+1}\left(\tau\right)$$

$$=\sum_{\kappa=0}^{\eta+1}\frac{\eta-\kappa+1}{\eta+1}\rho_{\kappa}Б_{\kappa}^{\eta+1}\left(\tau\right)+\sum_{\kappa=0}^{\eta+1}\frac{\kappa}{\eta+1}\rho_{\kappa-1}Б_{\kappa}^{\eta+1}\left(\tau\right)$$

$$= \sum_{\kappa=0}^{\eta+1} \left[\frac{(\eta+1-\kappa)\rho_\kappa + \kappa\rho_{\kappa-1}}{\eta+1} \right] Б_\kappa^{\eta+1}(\tau)$$

$$\varpi(\tau) = \sum_{\kappa=0}^{\eta+1} \rho_\kappa^* Б_\kappa^{\eta+1}(\tau)$$

where;

$$\rho_\kappa^* = \alpha_\kappa \rho_{\kappa-1} + (1-\alpha_\kappa)\rho_\kappa \ , \quad \alpha_\kappa = \frac{\kappa}{\eta+1}$$

In the next example we give degree raising of a cubic Bezier curve.

Example 4.6.1: The degree raising of cubic Bezier curve for $\eta = 3$, the new control points ρ_κ^* are:

$$\rho_0^* = \rho_o$$

$$\rho_1^* = \frac{1}{4}\rho_o + \frac{3}{4}\rho_1$$

$$\rho_2^* = \frac{2}{4}\rho_1 + \frac{2}{4}\rho_2$$

$$\rho_3^* = \frac{3}{4}\rho_2 + \frac{1}{4}\rho_3$$

$$\rho_4^* = \rho_3$$

We illustrate graphically in next figure.

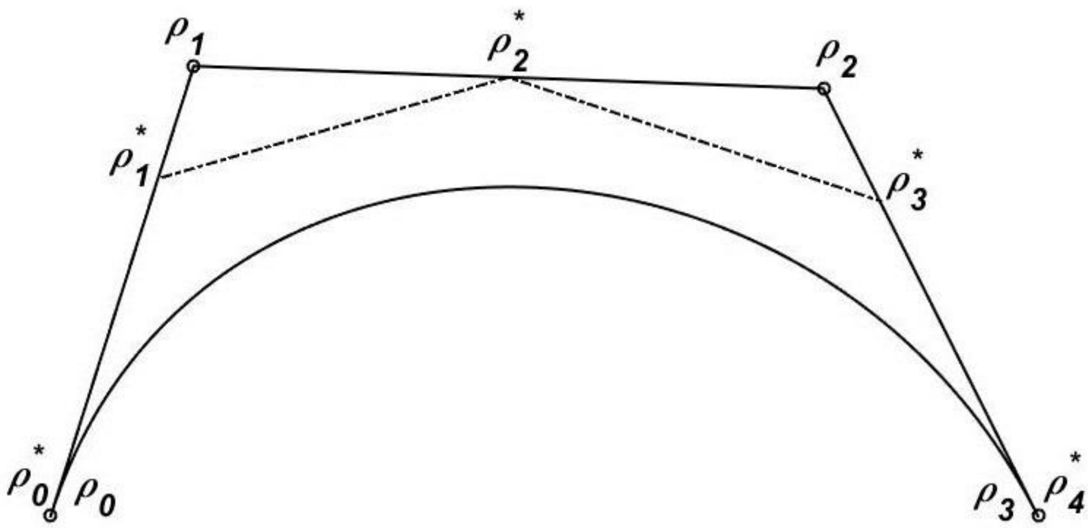
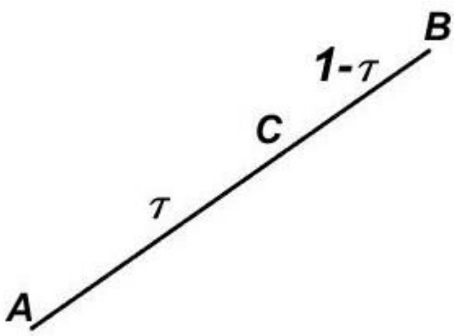


Figure 4.7: Degree elevation of a cubic Bezier curve.

4.7 The de Casteljau Algorithm

The principal concept of de Casteljau`s algorithm is to choose a point C on a line segment AB such that C divides the line segment AB in the ratio of $\tau : 1-\tau$.



Definition 4.7.1 [6] A curve

$$\varpi(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa^0 Б_\kappa^\eta(\tau)=\sum_{\kappa=0}^{\eta-1}\rho_\kappa^1 Б_\kappa^{\eta-1}(\tau)=\ldots=\sum_{\kappa=0}^{0}\rho_\kappa^\eta Б_\kappa^0(\tau)=\rho_0$$

where

$$\rho_\kappa^r(\tau)=(1-\tau)\rho_\kappa^{r-1}(\tau)+\tau\rho_{\kappa+1}^{r-1}(\tau) \qquad \begin{cases} r=1,\ldots\ldots,\eta \\ \kappa=0,\ldots,\eta-r \end{cases}$$

$$\rho_\kappa^0(\tau)=\rho_\kappa\,.$$

Then $\rho_0^\eta(\tau)$ is the point with parameter value τ on the Bezier curve ρ_η.

Proof. From the recursive formula of Bernstein polynomials (3.3), we obtain;

$$=\sum_{\kappa=0}^{\eta}\rho_\kappa Б_\kappa^\eta(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa\left[(1-\tau)Б_\kappa^{\eta-1}(\tau)+\tau Б_{\kappa-1}^{\eta-1}(\tau)\right]$$

$$=\sum_{\kappa=0}^{\eta-1}\rho_\kappa(1-\tau)Б_\kappa^{\eta-1}(\tau)+\sum_{i=1}^{n}\rho_\kappa\,\tau Б_{\kappa-1}^{\eta-1}(\tau)$$

$$=\sum_{\kappa=0}^{\eta-1}\rho_\kappa(1-\tau)Б_\kappa^{\eta-1}(\tau)+\sum_{i=0}^{n-1}\rho_{\kappa+1}\,\tau Б_\kappa^{\eta-1}(\tau) \quad =\sum_{\kappa=0}^{\eta-1}\left[\rho_\kappa(1-\tau)+\rho_{\kappa+1}(\tau)\right]Б_\kappa^{\eta-1}(\tau)$$

$$\varpi(\tau)=\sum_{\kappa=0}^{\eta-1}\rho_\kappa^1 Б_\kappa^{\eta-1}(\tau)$$

where

$$\rho_\kappa^1=\rho_\kappa(1-\tau)+\rho_{\kappa+1}(\tau)=\rho_\kappa^0(1-\tau)+\rho_{\kappa+1}^0(\tau) \qquad \text{for } \kappa=0,\ldots,\eta-1\,.$$

if we apply the same argument to the above Bezier curve;

$$\varpi(\tau)=\sum_{\kappa=0}^{\eta-1}\rho_\kappa^1 Б_\kappa^{\eta-1}(\tau)$$

yields

$$\varpi(\tau)=\sum_{\kappa=0}^{\eta-2}\rho_\kappa^2 Б_\kappa^{\eta-2}(\tau)$$

where $\rho_\kappa^2 = \rho_\kappa^1 (1-\tau) + \rho_{\kappa+1}^1 \tau$ for $\kappa = 0,\ldots,\eta-2.$

In general,

$$\varpi(\tau) = \sum_{\kappa=0}^{\eta-r} \rho_\kappa^r Б_\kappa^{\eta-r}(\tau)$$

where $\rho_\kappa^r = \rho_\kappa^{r-1}(1-\tau) + \rho_{\kappa+1}^{r-1}\tau$ for $\kappa = 0,\ldots,\eta-r$. Taking $r=\eta$ yields

$$\varpi(\tau) = \sum_{\kappa=0}^{0} \rho_\kappa^r Б_\kappa^{\eta-n}(\tau) = \rho_0^\eta .$$

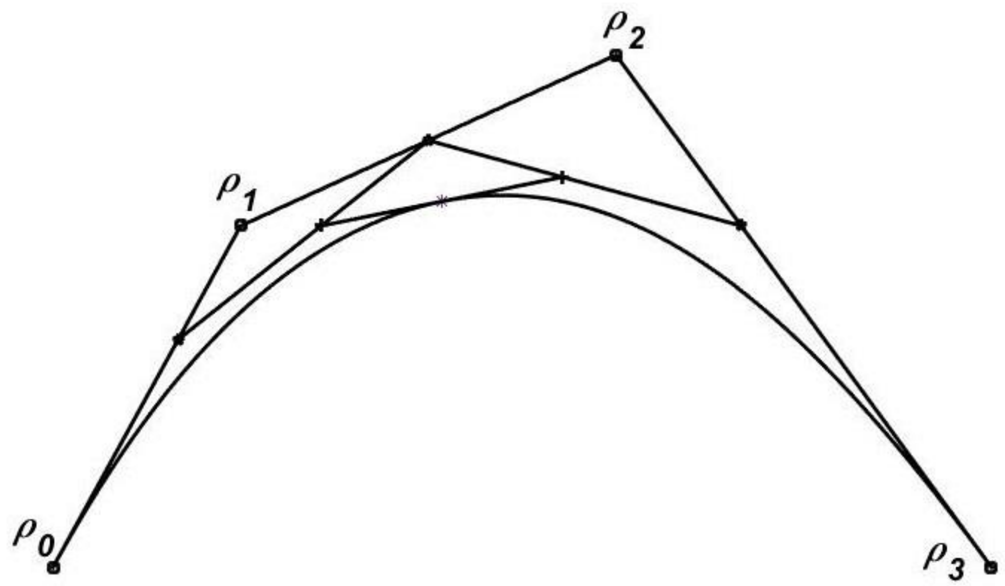


Figure 4.8: De Casteljau algorithms of cubic Bezier curves.

Next section we give matrix formulation of Bezier curve.

4.8 Matrix Formulation of Bezier Curve

A curve of the form $\varpi(\tau)=\sum_{\kappa=0}^{\eta}\rho_\kappa Б_\kappa^\eta(\tau)$ can be interpreted as a dot product;

$$\varpi(\tau)=\begin{bmatrix}\rho_0 & \cdots & \rho_\eta\end{bmatrix}\begin{bmatrix}Б_0^\eta(\tau)\\ \vdots \\ Б_\eta^\eta(\tau)\end{bmatrix}.$$

In addition,

$$\begin{bmatrix}Б_0^\eta(\tau)\\ \vdots \\ Б_\eta^\eta(\tau)\end{bmatrix}=\begin{bmatrix}m_{0,0} & \cdots & m_{0,\eta}\\ \vdots & \ddots & \vdots \\ m_{\eta.0} & \cdots & m_{\eta,\eta}\end{bmatrix}\begin{bmatrix}\tau^0\\ \vdots \\ \tau^\eta\end{bmatrix}$$

where,

$$m_{\kappa,j}=(-1)^{j-\kappa}\binom{\eta}{j}\binom{j}{\kappa}\qquad \begin{matrix}\kappa=0,\ldots,\eta\\ j=0,\ldots,\eta\end{matrix}$$

In the next examples, we give the matrix representation of Quadratic Bezier curve and Cubic Bezier curve.

Example 4.8.1: *Quadratic Bezier curve*

Let $\eta=2$;

$$\varpi(\tau)=\rho_0Б_0^2(\tau)+\rho_1Б_1^2(\tau)+\rho_2Б_2^2(\tau)$$

Then, the matrix representation of quadratic Bezier curve is;

$$\varpi(\tau)=\begin{bmatrix}\rho_0 & \rho_1 & \rho_2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -2 & 2 & 0\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}(1-\tau)^2\\ 2\tau(1-\tau)\\ \tau^2\end{bmatrix}$$

Example 4.8.2: *Cubic Bezier curve*

Let $\eta=3$;

$$\varpi(\tau)=\rho_0Б_0^3(\tau)+\rho_1Б_1^3(\tau)+\rho_2Б_2^3(\tau)+\rho_3Б_3^3(\tau)$$

$$\varpi(\tau)=(1-\tau)^3\rho_0+3\tau(1-\tau)^2\rho_1+3\tau^2(1-\tau)\rho_2+\tau^3\rho_3$$

Then, the matrix representation of cubic Bezier curve is;

$$\varpi(\tau) = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} (1-\tau)^3 \\ 3\tau(1-\tau)^2 \\ 3\tau^2(1-\tau) \\ \tau^3 \end{bmatrix}$$

Chapter 5

GENERALIZED BEZIER CURVES BASED ON LUPAŞ q-ANALOGUES OF THE BERNSTEIN OPERATOR

5.1 Lupaş q-Analogues of the Bernstein Function

The Lupaş q-analogue of the Bernstein operator introduced by George M. Philips [8] in 2010.

Definition 5.1.1 [7] Let $h \in C[0,1]$. The linear operator $L_{\eta,q} : C[0,1] \to C[0,1]$ is defined by

$$\mathcal{L}_{\eta,q}(h;\tau) = \sum_{k=0}^{\eta} \frac{\begin{bmatrix}\eta \\ \kappa\end{bmatrix}_q q^{\frac{\kappa(\kappa-1)}{2}} \tau^{\kappa} (1-\tau)^{\eta-\kappa}}{\prod_{r=1}^{\eta} \left\{(1-\tau) + q^{r-1}\tau\right\}} h\left(\frac{[\kappa]_q}{[\eta]_q}\right) \tag{5.1}$$

$\mathcal{L}_{n,q}$ is called the Lupaş q-analoque of the Bernstein operator.

Definition 5.1.2 [7] Given a real number $q > 0$, the Lupaş q-analogues of the Bernstein functions of degree η defined by;

$$z_{\kappa}^{\eta}(\tau;q) = \frac{\begin{bmatrix}\eta \\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2} \tau^{\kappa} (1-\tau)^{\eta-\kappa}}{\prod_{r=1}^{\eta} \left(1-\tau + q^{r-1}\tau\right)}, \quad \kappa = 0,1,\ldots,\eta\,,\ \tau \in [0,1] \tag{5.2}$$

Example. The Lupas q-analogues of the Bernstein funtions of degree $\eta = 3$;

$$z_0^3(\tau;q) = \frac{(1-\tau)^3}{((1-\tau)+q\tau)((1-\tau)+q^2\tau)}$$

$$z_1^3(\tau;q) = \frac{(1+q+q^2)\tau(1-\tau)^2}{((1-\tau)+q\tau)((1-\tau)+q^2\tau)}$$

$$z_2^3(\tau;q) = \frac{(1+q+q^2)\tau^2(1-\tau)}{((1-\tau)+q\tau)((1-\tau)+q^2\tau)}$$

$$z_3^3(\tau;q) = \frac{\tau^3}{((1-\tau)+q\tau)((1-\tau)+q^2\tau)}$$

We illustrate graphically in next figure.

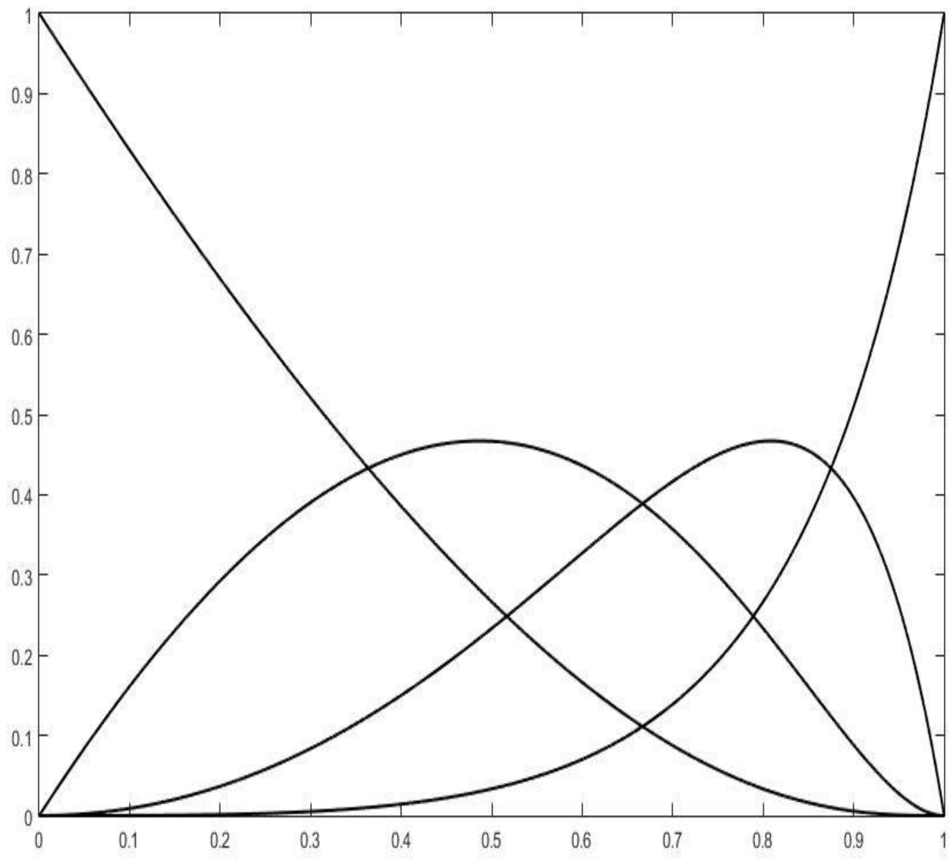


Figure 5.1: Lupas q-analogues of the Bernstein functions of degree 3 with q=0.5.

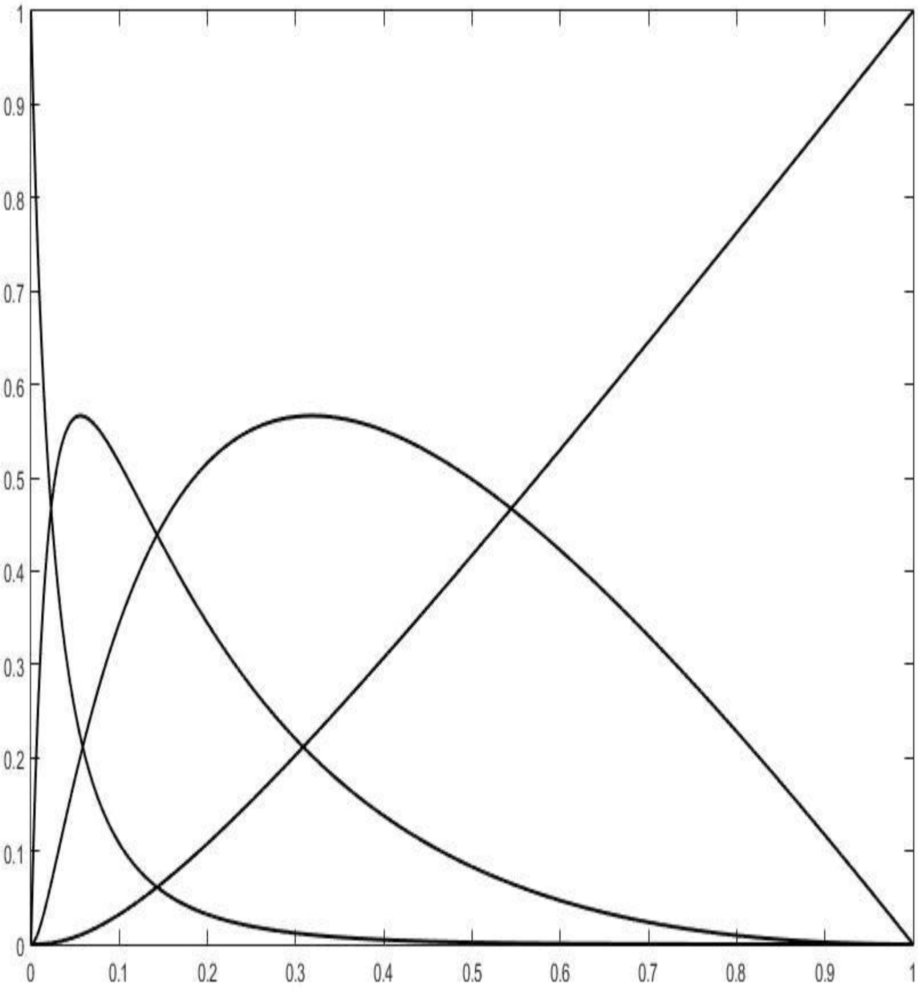


Figure 5.2: Lupas q-analogues of the Bernstein functions of degree 3 with q=6.

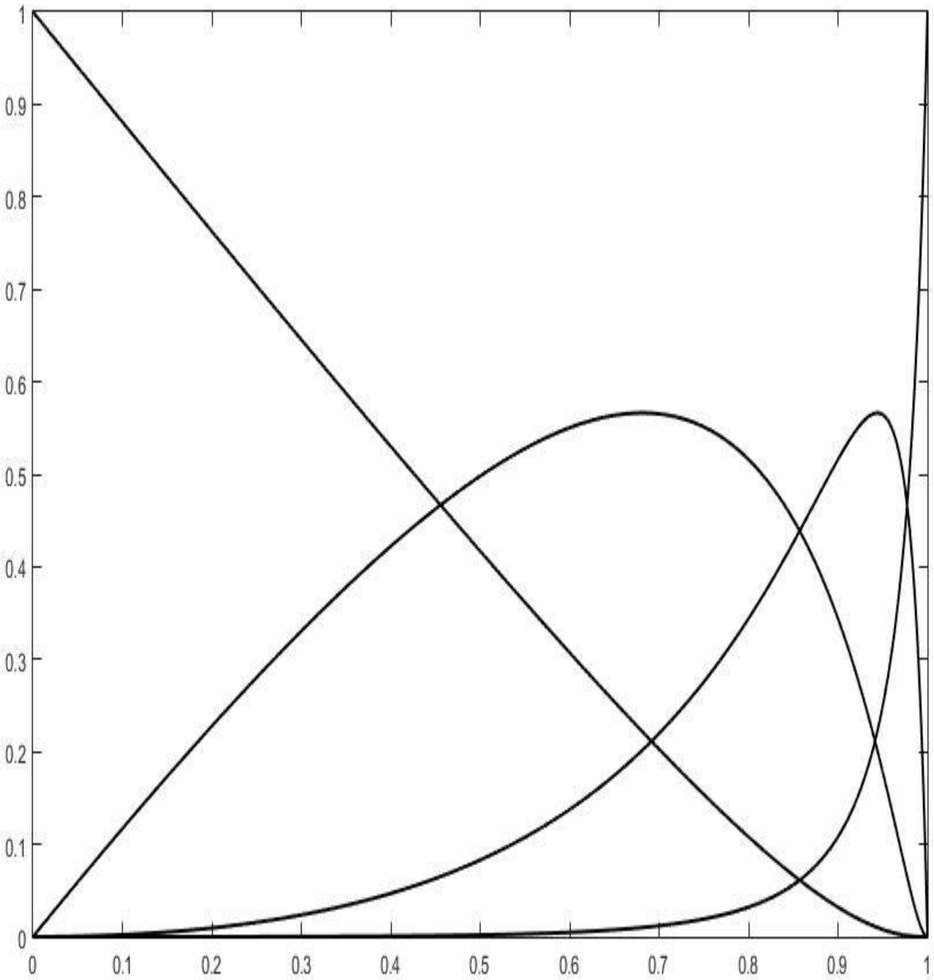


Figure 5.3: Lupas q-analogues of the Bernstein functions of degree 3 with q=1/6.

In the next section, we give some general properties of the $z_{\kappa}^{\eta}(\tau;q)$.

5.2 Properties of the Lupaş q-Analogues of the Bernstein Functions

Theorem 5.2.1. [7] The Lupaş q-analogues of the Bernstein functions possess the following properties:

1.*Non-negativity* [7]**:**

$$z_{\kappa}^{\eta}(\tau;q) \geq 0, \ \kappa = 0,1,\ldots,\eta, \ \tau \in [0,1].$$

2. Partition of unit [7]**:**

$$\sum_{\kappa=0}^{\eta} z_{\kappa}^{\eta}\left(\tau;q\right)=1,\ \tau\in[0,1].$$

Proof. From the q-analogue of Newton binomial theorem we have;

$$\left(1+\tau\right)\left(1+q\tau\right)\cdots\left(1+q^{\eta-1}\tau\right)=\sum_{\kappa=0}^{\eta}\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}$$

$$\sum_{\kappa=0}^{\eta}\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}=\sum_{\kappa=0}^{\eta}\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\left(1-\tau\right)^{\eta}\left(\frac{\tau}{1-\tau}\right)^{\kappa}$$

$$=\left(1-\tau\right)^{\eta}\left(1+\frac{\tau}{1-\tau}\right)\left(1+q\frac{\tau}{1-\tau}\right)\cdots\left(1+q^{\eta-1}\frac{\tau}{1-\tau}\right)$$

$$=\frac{\left(1-\tau\right)^{\eta}}{\left(1-\tau\right)^{\eta}}\left(1-\tau+q\tau\right)\cdots\left(1-\tau+q^{\eta-1}\tau\right)$$

$$=\left(1-\tau+q\tau\right)\cdots\left(1-\tau+q^{\eta-1}\tau\right)=\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)$$

so

$$\sum_{\kappa=0}^{\eta}\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}=\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)$$

and

$$\frac{\sum_{\kappa=0}^{\eta}\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}=1$$

3. End-point property [7]**:**

$$z_{\kappa}^{\eta}\left(0;q\right)=\begin{cases}1, & \kappa=0,\\ 0, & \kappa\neq 0,\end{cases}\quad\text{and}\quad z_{\kappa}^{\eta}\left(1;q\right)=\begin{cases}1, & \kappa=\eta,\\ 0, & \kappa\neq\eta,\end{cases}$$

Proof. From the definition of the Lupaş q-analogues of the Bernstein functions possess the following properties:

$$z_{\kappa}^{\eta}(0;q)=\frac{\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}0^{\kappa}(1)^{\eta-\kappa}}{\prod_{r=1}^{\eta}(1)}\quad,\qquad \kappa=0,1,\ldots,\eta,$$

If $\kappa=0 \Rightarrow \quad z_{\kappa}^{\eta}(0;q)=1$.

and if $\kappa\neq 0 \Rightarrow z_{\kappa}^{\eta}(0;q)=0$.

Secondly from the definition of Lupas q-analogues of Bernstein function;

$$z_{\kappa}^{\eta}(1;q)=\frac{\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}1^{\kappa}(0)^{\eta-\kappa}}{\prod_{r=1}^{\eta}q^{r-1}}\quad,\quad \kappa=0,1,\ldots,\eta,$$

If $\kappa=\eta \Rightarrow \quad z_{\kappa}^{\eta}(1;q)=1$.

and if $\kappa\neq\eta \Rightarrow z_{\kappa}^{\eta}(1;q)=0$.

4. q-inverse symmetry [7]**:**

$$z_{\eta-\kappa}^{\eta}(\tau;q)=z_{\kappa}^{\eta}(1-\tau;1/q),\ \kappa=0,1,\ldots,\eta\ .$$

Proof. From the definition of Lupas q-analogues of Bernstein function:

$$z_{\eta-\kappa}^{\eta}(\tau;q)=\frac{\begin{bmatrix}\eta\\ \eta-\kappa\end{bmatrix}_q q^{(\eta-\kappa)(\eta-\kappa-1)/2}\tau^{\eta-\kappa}(1-\tau)^{\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

$$=\frac{\begin{bmatrix}\eta\\ \eta-\kappa\end{bmatrix}_q q^{(\eta-\kappa)(\eta-\kappa-1)/2}\tau^{\eta-\kappa}(1-\tau)^{\kappa}}{q^{\eta(\eta-1)/2}\tau^{\eta}\prod_{r=1}^{\eta}\left(1+\frac{1-\tau}{q^{r-1}\tau}\right)}$$

$$= \frac{\begin{bmatrix} \eta \\ \kappa \end{bmatrix}_{1/q} (1/q)^{\kappa(\kappa-1)/2} \tau^{\eta-\kappa} (1-\tau)^{\kappa}}{\prod_{r=1}^{\eta} \left(1 + \frac{1-\tau}{q^{r-1}\tau}\right)}$$

hence

$$z_{\eta-\kappa}^{\eta}(\tau;q) = z_{\kappa}^{\eta}(1-\tau;1/q)\ .$$

5. Reducibility [7]:

When $q = 1$, Lupas q-analogues of Bernstein function reduces to the classical Bernstein bases.

Proof. If $q = 1$ into the formula (5.2), we obtain

$$z_{\kappa}^{\eta}(\tau;1) = \frac{\begin{bmatrix} \eta \\ \kappa \end{bmatrix}_{q} 1^{\kappa(\kappa-1)/2} \tau^{\kappa} (1-\tau)^{\eta-\kappa}}{\prod_{r=1}^{\eta} \left(1-\tau+1^{r-1}\tau\right)} = z_{\kappa}^{\eta}(\tau)$$

5.3 Degree Elevation and Reduction for the Lupaş q-Analogues of the Bernstein Functions

Degree elevation technique for increase the flexibility of a curve. For degree elevation and reduction of Lupas q-analogues of the Bernstein functions, the following identities are very useful:

$$\frac{q^{\eta}\tau}{(1-\tau)+q^{\eta}\tau} z_{\kappa}^{\eta}(\tau;q) = \left(1 - \frac{[\eta-\kappa]_q}{[\eta+1]_q}\right) z_{\kappa+1}^{\eta+1}(\tau;q) \tag{5.3}$$

$$\frac{(1-\tau)}{(1-\tau)+q^{\eta}\tau} z_{\kappa}^{\eta}(\tau;q) = \left(\frac{[\eta+1-\kappa]_q}{[\eta+1]_q}\right) z_{\kappa}^{\eta+1}(\tau;q) \tag{5.4}$$

Theorem 5.3.1[7] *Degree raising*

Each Lupaş q-analogue of the corresponding Bernstein function of degree η is a linear combination of two Lupaş q-analogues of the Bernstein functions of degree $\eta+1$.

$$z_{\kappa}^{\eta}\left(\tau;q\right)=\frac{\left[\eta+1-\kappa\right]_q}{\left[\eta+1\right]_q}z_{\kappa}^{\eta+1}(\tau;q)+\left(1-\frac{\left[\eta-\kappa\right]_q}{\left[\eta+1\right]_q}\right)z_{\kappa+1}^{\eta+1}(\tau;q) \tag{5.5}$$

Proof.

$$z_{\kappa}^{\eta}\left(\tau;q\right)=z_{\kappa}^{\eta}\left(\tau;q\right)\left(1-\frac{q^{\eta}\tau}{1-t+q^{\eta}\tau}+\frac{q^{\eta}\tau}{1-\tau+q^{\eta}\tau}\right)$$

$$=z_{\kappa}^{\eta}\left(\tau;q\right)\left(\frac{1-\tau+q^{\eta}\tau}{1-\tau+q^{\eta}\tau}-\frac{q^{\eta}\tau}{1-\tau+q^{\eta}\tau}+\frac{q^{\eta}\tau}{1-\tau+q^{\eta}\tau}\right)$$

$$=\frac{1-\tau}{1-\tau+q^{\eta}\tau}\frac{\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}+\frac{q^{\eta}\tau}{1-\tau+q^{\eta}\tau}\frac{\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

Using formula (5.3) and (5.4), we obtain;

$$z_{\kappa}^{\eta}(\tau;q)=\frac{\left[\eta+1-\kappa\right]_q}{\left[\eta+1\right]_q}z_{\kappa}^{\eta+1}(\tau;q)+\left(1-\frac{\left[\eta-\kappa\right]_q}{\left[\eta+1\right]_q}\right)z_{\kappa+1}^{\eta+1}(\tau;q).$$

Theorem 5.3.2 [7] *Degree reduction*

Each Lupaş q-analogue of the Bernstein function of degree η is a linear combination of two Lupaş q-analogues of the Bernstein functions of degree $\eta-1$.

$$(i)\quad z_{\kappa}^{\eta}(\tau;q)=\frac{q^{\kappa-1}\tau}{1-\tau+q^{\eta-1}\tau}z_{\kappa-1}^{\eta-1}(\tau;q)+\frac{q^{\kappa-1}\left(1-\tau\right)}{1-\tau+q^{\eta-1}\tau}z_{\kappa}^{\eta-1}(\tau;q),\quad \kappa=0,1,\ldots,\eta\ , \tag{5.6}$$

$$(ii)\quad z_{\kappa}^{\eta}(\tau;q)=\frac{q^{\eta-1}\tau}{1-\tau+q^{\eta-1}\tau}z_{\kappa-1}^{\eta-1}(\tau;q)+\frac{1-\tau}{1-\tau+q^{\eta-1}\tau}z_{\kappa}^{\eta-1}(\tau;q)\ ,\quad \kappa=0,1,\ldots,\eta\,, \tag{5.7}$$

Proof.

(i) If we use the definition of q-binomial coefficients of the Pascal-type relations (2.18) and formula (2.18a), we obtain the following equality:

$$z_{\kappa}^{\eta}(\tau;q)=\frac{\left(\begin{bmatrix}\eta-1\\ \kappa-1\end{bmatrix}_q+q^{\kappa}\begin{bmatrix}\eta-1\\ \kappa\end{bmatrix}_q\right)q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

$$=\frac{\begin{bmatrix}\eta-1\\ \kappa-1\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}+\frac{q^{\kappa}\begin{bmatrix}\eta-1\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

Hence

$$z_{\kappa}^{\eta}(\tau;q)=\frac{q^{\kappa-1}t}{1-\tau+q^{\eta-1}\tau}\,z_{\kappa-1}^{\eta-1}(\tau;q)+\frac{q^{\kappa}\left(1-\tau\right)}{1-\tau+q^{\eta-1}\tau}\,b_{\kappa}^{\eta-1}(\tau;q)\,.$$

(ii) If we use the definition of q-binomial coefficients of the Pascal-type relations (2.18) and formula (2.18b), we obtain the following equality:

$$z_{\kappa}^{\eta}(\tau;q)=\frac{\left(q^{\eta-\kappa}\begin{bmatrix}\eta-1\\ \kappa-1\end{bmatrix}_q+\begin{bmatrix}\eta-1\\ \kappa\end{bmatrix}_q\right)q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

$$=\frac{q^{\eta-\kappa}\begin{bmatrix}\eta-1\\ \kappa-1\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}+\frac{\begin{bmatrix}\eta-1\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}\left(1-\tau\right)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

Therefore

$$z_{\kappa}^{\eta}(\tau;q)=\frac{q^{\eta-1}\tau}{1-\tau+q^{\eta-1}\tau}\,z_{\kappa-1}^{\eta-1}(\tau;q)+\frac{1-\tau}{1-\tau+q^{\eta-1}\tau}\,z_{\kappa}^{\eta-1}(\tau;q)$$

5.4 Lupaş q-Bezier Curves

Definition [7] *Lupaş q − Bezier curves*

Given a set of control points $\{\rho_0, \ldots, \rho_\eta\}$ where $\rho_\kappa \in R^d$ $(\kappa = 0,1,\ldots,\eta)$ and $q > 0$, the Lupaş q-Bezier curves of degree η is:

$$\rho(\tau;q) = \sum_{\kappa=0}^{\eta} \rho_\kappa z_\kappa^\eta(\tau;q) \tag{5.8}$$

where

$$z_\kappa^\eta(\tau;q) = \frac{\begin{bmatrix} \eta \\ \kappa \end{bmatrix}_q q^{\kappa(\kappa-1)/2} \tau^\kappa (1-\tau)^{\eta-\kappa}}{\prod_{r=1}^{\eta} \left(1-\tau+q^{r-1}\tau\right)} \quad \text{and} \quad \tau \in [0,1].$$

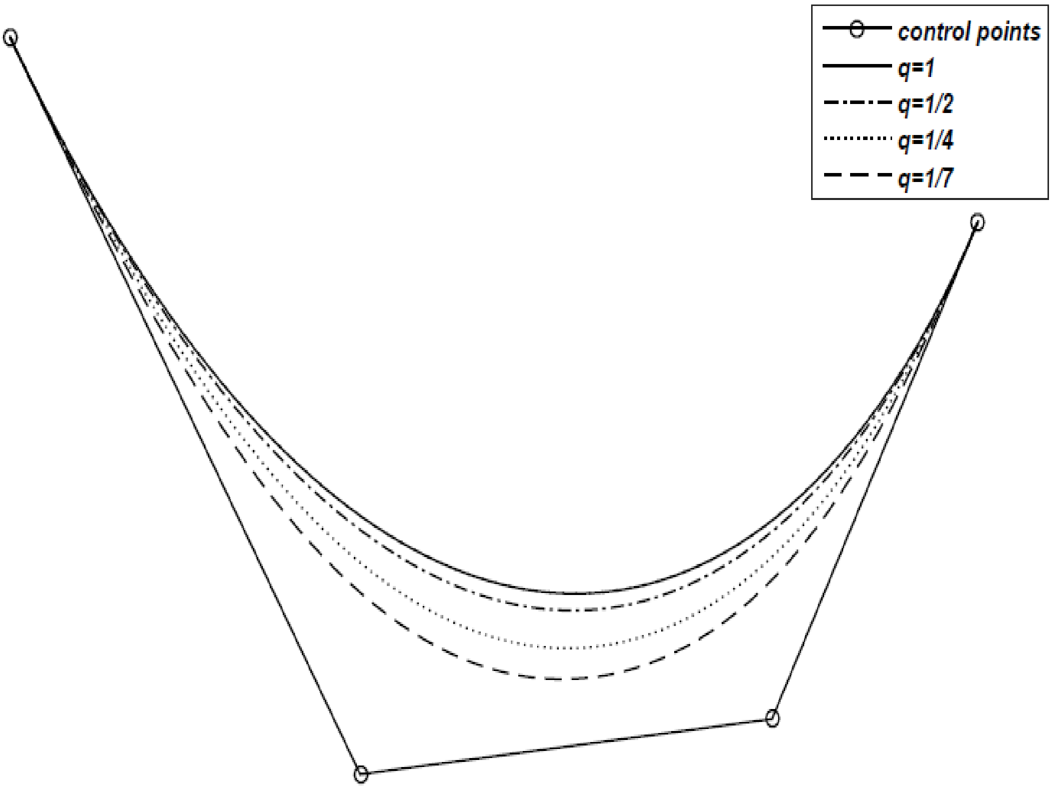


Figure 5.4: The effect of the shape of cubic q-Bezier by $0 < q < 1$.

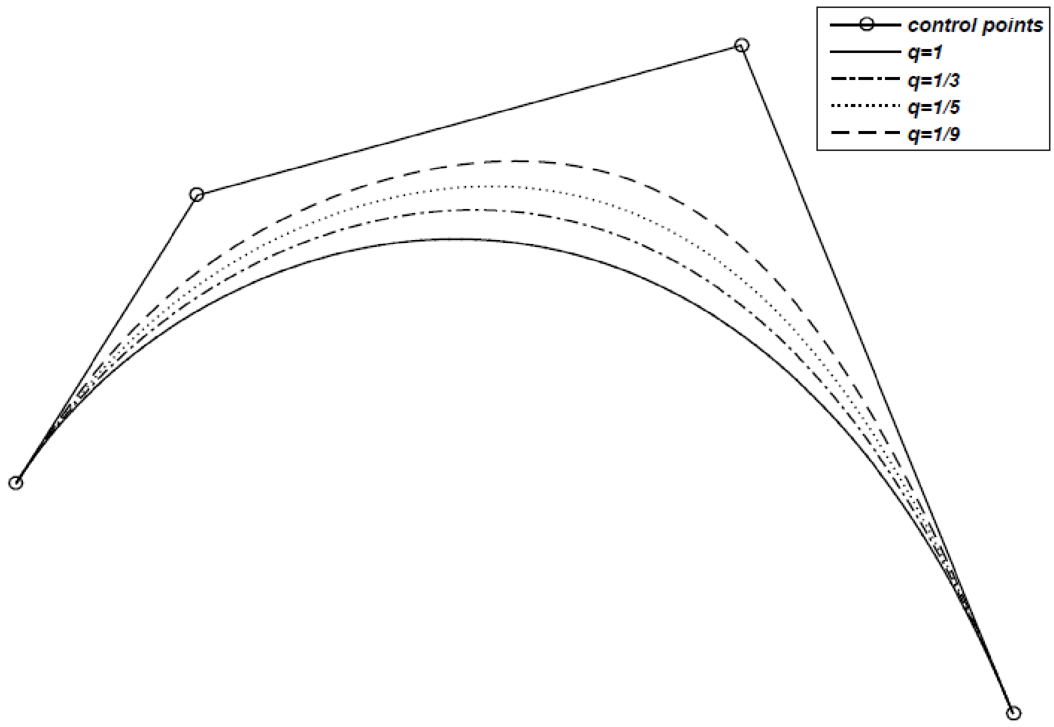


Figure 5.5: The effect of the shape of cubic q-Bezier by $0 < q < 1$.

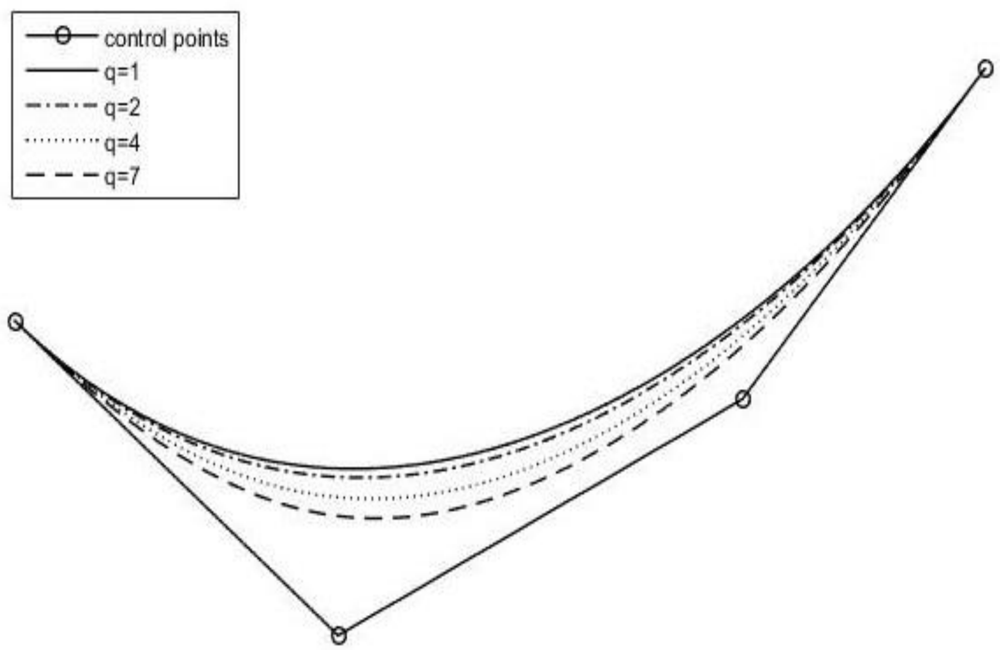


Figure 5.6: The effect of the shape of cubic q-Bezier $q > 1$.

In the next section we will discuss some basic properties of Lupaş q-Bezier curves.

5.4.1 Properties of Lupaş q-Bezier Curves

Property 5.4.1.1 [7] Lupaş q-Bezier have geometric and affine invariance.

Proof. Since, $z_\kappa^\eta(\tau;q)$ are partition of unity, $\rho(\tau;q)$ is affine invariant. Let ϕ is an affine transform in R^d, then

$$\phi(\rho(\tau;q)) = \phi\left(\sum_{\kappa=0}^{\eta} \rho_\kappa z_\kappa^\eta(\tau;q)\right) = A\sum_{\kappa=0}^{\eta} \rho_\kappa z_\kappa^\eta(\tau;q) + v$$

$$= \sum_{\kappa=0}^{\eta} \rho_\kappa A z_\kappa^\eta(\tau;q) + \sum_{\kappa=0}^{\eta} z_\kappa^\eta(\tau;q) v$$

$$= \sum_{\kappa=0}^{\eta} (A\rho_\kappa + v) z_\kappa^\eta(\tau;q)$$

$$= \sum_{\kappa=0}^{\eta} \phi_\kappa(\rho_\kappa) z_\kappa^\eta(\tau;q)$$

Property 5.4.1.2 [7] Lupaş q-Bezier curves lie inside the convex hull of its control polygon.

Proof. Every point in the $\rho(\tau;q)$ has the following terms;

$$x_0\rho_0 + x_1\rho_1 + \ldots + x_\eta\rho_\eta \quad \text{with} \quad x_\kappa = z_\kappa^\eta(\tau;q).$$

Furthermore, from the property 3.3.4 (non-negative) and property 3.3.5 (partition of unity), $\rho(\tau;q)$ is convex combination of the $\rho_0, \rho_1, \ldots, \rho_\eta$.

Property 5.4.1.3 [7] $The\ end - point\ interpolation\ property$

$$\rho(0,q) = \rho_0$$

$$\rho(1;q) = \rho_\eta$$

Proof. From formula (5.8);

$$\rho(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_\kappa z_\kappa^\eta(\tau;q)=\rho_0 z_0^\eta(\tau;q)+\rho_1 z_1^\eta(\tau;q)+\ldots+\rho_\eta z_\eta^\eta(\tau;q) \qquad (5.9)$$

For $\tau=0$ into (5.9), yields

$$\rho(0;q)=\rho_0\, z_0^\eta(0;q)$$

Due to the end-point property of Lupas q-analogues of the Bernstein functions, we obtain the following result;

$$\rho(0;q)=\rho_0$$

Similarly, if we substitute $\tau=1$ into (5.8), we obtain;

$$\rho(1;q)=\sum_{\kappa=0}^{\eta}\rho_\eta z_\eta^\eta(1;q)$$

From the end-point property of Lupas q-analogues of the Bernstein functions. Hence,

$$\rho(1;q)=\rho_\eta$$

Property 5.4.1.4 [7] $q-inverse\ symmetry$

The Lupaş q-Bezier curves obtained by reversing the order of the control points are the same as the Lupaş q-Bezier curves with q replaced by $1/q$.

Proof.

Let $\rho_\kappa^*=\rho_{\eta-\kappa}$, $\kappa=0,1,\ldots,\eta$, then

$$\rho^*(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_\kappa^* z_\kappa^\eta(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_{\eta-\kappa} z_\kappa^\eta(\tau;q)$$

$$\rho^*(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_{\eta-\kappa}\frac{\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^\kappa(1-\tau)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

then

$$\rho^*(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_\kappa \frac{\begin{bmatrix}\eta\\ \eta-\kappa\end{bmatrix}_q q^{\eta-\kappa(\eta-\kappa-1)/2}\tau^{\eta-\kappa}(1-\tau)^\kappa}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}$$

From the definition of the Lupaş q-analogues of the Bernstein function, we obtain;

$$\rho^*(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_\kappa z_\kappa^\eta(1-\tau;1/q)=\rho(1-\tau;1/q).$$

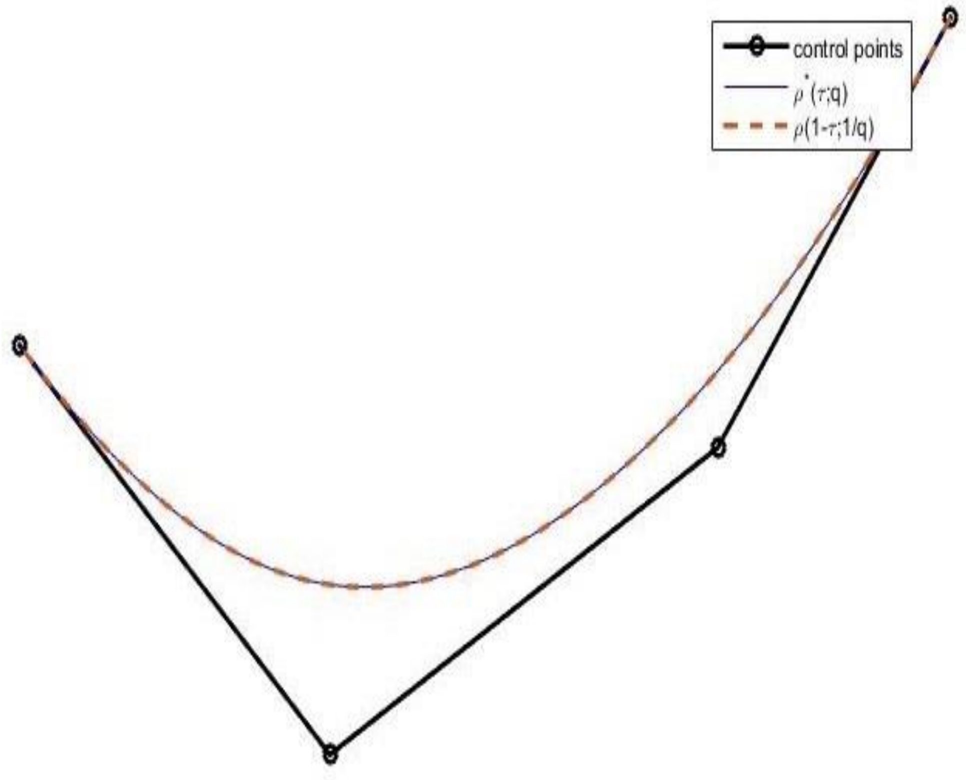


Figure 5.7: The effect of the shape of cubic q-inverse symmetry of Lupas q-Bezier curve for q=1/5 by $0<q<1$.

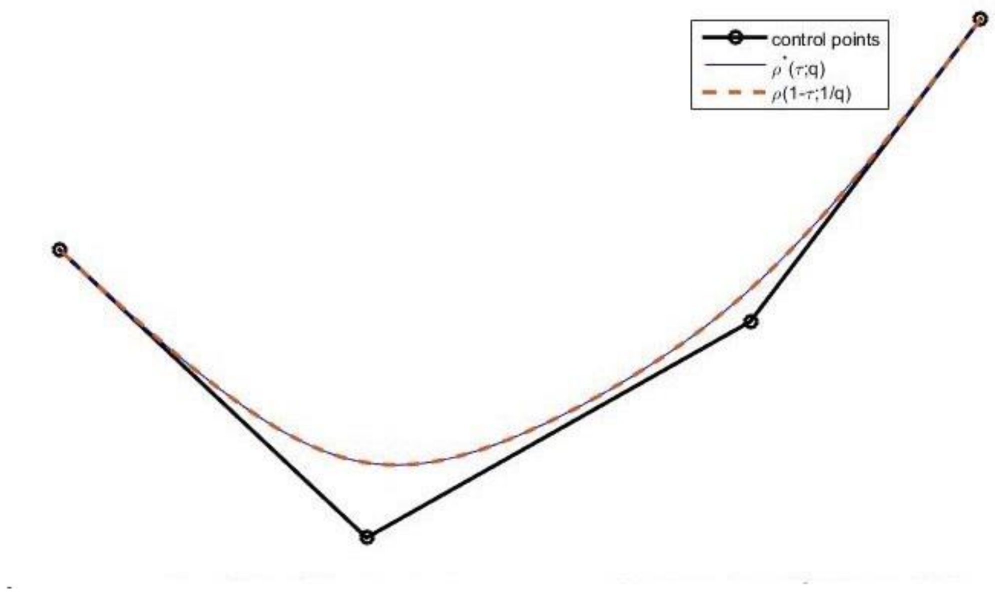


Figure 5.8: The effect of the shape of cubic q-inverse symmetry of Lupas q-Bezier curve for q=1/23 by $0 < q < 1$.

Property 5.4.1.5 [7] *Reducibility*

It is easily seen that when $q = 1$, the Lupas q-Bezier curve (5.8) reduces the classical Bezier curves (4.1).

Theorem 5.4.1.6 [7] *The end − point property of derivative*

$$\rho`(0;q) = [\eta]_q (\rho_1 - \rho_0), \qquad \rho`(1;q) = \frac{[\eta]_q}{q^{\eta-1}} (\rho_\eta - \rho_{\eta-1}),$$

i.e. Lupaş q-Bezier curves are tangent to fore-and-aft edges of its control polygon at end points.

Proof. From the definition of Lupas q-Bezier curve (5.8):

$$\rho(\tau;q) = \sum_{\kappa=0}^{\eta} \rho_\kappa z_\kappa^\eta(\tau;q) = \frac{\sum_{\kappa=0}^{\eta} \rho_\kappa \begin{bmatrix} \eta \\ \kappa \end{bmatrix}_q q^{\kappa(\kappa-1)/2} \tau^\kappa (1-\tau)^{\eta-\kappa}}{\prod_{r=1}^{\eta} \left(1 - \tau + q^{r-1}\tau\right)} \triangleq \frac{S(\tau;q)}{L(\tau;q)}$$

Then

$$\rho(\tau;q) L(\tau;q) = S(\tau;q).$$

If we take derivatives of both sides with respect to τ , we obtain the following result;

$$\rho^{'}(\tau;q)L(\tau;q)+\rho(\tau;q)L^{'}(\tau;q)=S^{'}(\tau;q). \qquad (5.10)$$

Let $d_{\kappa}^{\eta}(\tau;q)=\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}(1-\tau)^{\eta-\kappa}$, then

$$S(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^{\kappa}(1-\tau)^{\eta-\kappa}=\sum_{\kappa=0}^{\eta}\rho_{\kappa}\, d_{\kappa}^{\eta}(\tau;q)$$

From the extension of Newton`s binomial formula, we obtain;

$$L(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_{\kappa}d_{\kappa}^{\eta}(\tau;q).$$

Due to

$$\left[d_{\kappa}^{\eta}(\tau;q)\right]^{'}=\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\,\kappa\tau^{\kappa-1}(1-\tau)^{\eta-\kappa}-\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\,\tau^{\kappa}(\eta-\kappa)(1-\tau)^{\eta-\kappa-1} \qquad (5.11)$$

Besides that, we know the following equalities;

$$\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q=\frac{[\eta]_q}{[\kappa]_q}\begin{bmatrix}\eta-1\\ \kappa-1\end{bmatrix}_q$$

$$\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q=\frac{[\eta]_q}{[\eta-\kappa]_q}\begin{bmatrix}\eta-1\\ \kappa\end{bmatrix}_q$$

So (5.11) becomes;

$$\left[d_{\kappa}^{\eta}(\tau;q)\right]^{'}=\frac{[\eta]_q}{[\kappa]_q}\begin{bmatrix}\eta-1\\ \kappa-1\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\,\kappa\tau^{\kappa-1}(1-\tau)^{\eta-\kappa}-\frac{[\eta]_q}{[\eta-\kappa]_q}\begin{bmatrix}\eta-1\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\,\tau^{\kappa}(\eta-\kappa)(1-\tau)^{\eta-\kappa-1}$$

$$\triangleq e_{\kappa}^{\eta}d_{\kappa-1}^{\eta-1}(\tau;q)-g_{\eta-\kappa}^{\eta}\,d_{\kappa}^{\eta-1}(\tau;q)$$

where $e_{\kappa}^{\eta}=\frac{[\eta]_q}{[\kappa]_q}q^{\kappa-1}\kappa,\quad g_{\kappa}^{\eta}=\frac{[\eta]_q}{[\kappa]_q}\kappa$.

Then;

$$S(0;q)=\rho_0, \qquad L(0;q)=1$$

$$S^{'}\left(0;q\right)=e_1^{\eta}\rho_1-g_{\eta}^{\eta}\rho_0, \qquad L^{'}\left(0;q\right)=e_1^{\eta}-g_{\eta}^{\eta}$$

Therefore (5.10) becomes;

$$\rho^{'}\left(0;q\right)L\left(0;q\right)+\rho\left(0;q\right)L^{'}\left(0;q\right)=S^{'}\left(0;q\right)$$

$$\rho^{'}\left(0;q\right)+\rho_0\left(e_1^{\eta}-g_{\eta}^{\eta}\right)=e_1^{\eta}\rho_1-g_{\eta}^{\eta}\rho_0$$

$$\rho^{'}\left(0;q\right)=e_1^{\eta}\rho_1-g_{\eta}^{\eta}\rho_0-e_1^{\eta}\rho_0+g_{\eta}^{\eta}\rho_0$$

$$\rho^{'}\left(0;q\right)=e_1^{\eta}\left(\rho_1-\rho_0\right)$$

where

$$e_1^{\eta}=\left[\eta\right]_q$$

Secondly we calculation same steps for $\tau=1$;

$$S\left(1;q\right)=\rho_{\eta}\,q^{\eta(\eta-1)/2}, \qquad L\left(1;q\right)=q^{\eta(\eta-1)/2}$$

$$S^{'}\left(1;q\right)=\left(e_{\eta}^{\eta}\rho_{\eta}-g_1^{\eta}\rho_{\eta-1}\right)q^{(\eta-1)(\eta-2)/2}, \quad L^{'}\left(1;q\right)=\left(e_{\eta}^{\eta}-g_1^{\eta}\right)q^{(\eta-1)(\eta-2)/2}.$$

Finally again (5.10) becomes;

$$\rho^{'}\left(1;q\right)L\left(1;q\right)+\rho\left(1;q\right)L^{'}\left(1;q\right)=S^{'}\left(1;q\right)$$

$$\rho^{'}\left(1;q\right)q^{\eta(\eta-1)/2}+\rho_{\eta}\left(e_{\eta}^{\eta}-g_1^{\eta}\right)q^{(\eta-1)(\eta-2)/2}=\left(e_{\eta}^{\eta}\rho_{\eta}-g_1^{\eta}\rho_{\eta-1}\right)q^{(\eta-1)(\eta-2)/2}$$

$$\rho^{'}\left(1;q\right)=\frac{\left(e_{\eta}^{\eta}\rho_{\eta}-g_1^{\eta}\rho_{\eta-1}-e_{\eta}^{\eta}\rho_{\eta}+g_1^{\eta}\rho_{\eta}\right)q^{(\eta-1)(\eta-2)/2}}{q^{\eta(\eta-1)/2}}$$

$$\rho^{'}\left(1;q\right)=\frac{g_1^{\eta}}{q^{\eta-1}}\left(\rho_{\eta}-\rho_{\eta-1}\right).$$

where

$$g_1^{\eta}=\left[\eta\right]_q.$$

Theorem 5.4.1.7 [7] **"**Planar Lupas q-Bezier are variation diminishing, which the number of intersection points of any straight line with a Lupas q-Bezier is at most the number of intersection points of same straight line with control polygon."

Proof. In this proof $Z_{\tau\in L\subseteq(0,+\infty)}\left[g(\tau)\right]$ denote the number of roots of any polynomials $g(\tau)$ on the interval L. For vector $V=(v_0,v_1,\ldots,v_\eta)$ and $S^-(v_0,v_1,\ldots,v_\eta)$ to demonstrate the exact sign changes number in the or V. Due to $(1,\tau,\ldots,\tau^m)$ is totally positive on $[0,1]$, then for any sequence of real numbers $b_0,b_1,\ldots,b_m$,

$$Z_{0<\tau<1}\left[b_0+b_1\tau+\cdots+b_\eta\tau^\eta\right]=S^-\left(b_0+b_1\tau+\cdots+b_\eta\tau^\eta\right)\leq S^-\left(b_0,b_1,\ldots,b_\eta\right).$$

Let ρ denote a planar $\rho(\tau;q)$, Y is any straight line, and let $L(\rho,Y)$ the number of times ρ crosses Y. Establish the rectangular coordinate system whose abscissa axis is Y. Because curves are geometric invariant, we can denote (r_k,s_k) $(k=0,1,\ldots,m)$ the new coordinates of the control points. Let Z denote the control polygon and $L(Z,Y)$ the number of times Z crosses Y. Then, we will prove that $L(\rho,Y)\leq L(Z,Y)$.

We make a parameter transformation. Let $\mu=\dfrac{\tau}{1-\tau}$, $\tau\in(0,1)$, so that $\mu\in(0,+\infty)$.

Then

$$L(\rho,Y)=Z_{0<\tau<1}\left[\sum_{\kappa=0}^{\eta}y_\eta\, z_\kappa^\eta(\tau;q)\right]=Z_{0<\tau<1}\left[\sum_{\kappa=0}^{\eta}\frac{y_\kappa\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\tau^\kappa(1-\tau)^{\eta-\kappa}}{\prod_{r=1}^{\eta}\left(1-\tau+q^{r-1}\tau\right)}\right]$$

$$L(\rho,Y)=Z_{0<\mu<+\infty}\left[\sum_{\kappa=0}^{\eta}\frac{y_\kappa\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q q^{\kappa(\kappa-1)/2}\mu^\kappa}{\prod_{r=1}^{\eta}\left(1+q^{r-1}\mu\right)}\right]=Z_{0<\mu<+\infty}\left[\sum_{\kappa=0}^{\eta}y_\kappa\begin{bmatrix}\eta\\ \kappa\end{bmatrix}_q\mu^\kappa\right]$$

$$L(\rho,Y) \le S^{-}\left(\begin{bmatrix} m \\ 0 \end{bmatrix}_q y_0, \begin{bmatrix} m \\ 1 \end{bmatrix}_q y_1, \ldots, \begin{bmatrix} m \\ m \end{bmatrix}_q y_m\right) = S^{-}(y_0, y_1, \ldots, y_m) = L(Z,Y)$$

5.4.2 Degree Elevation for Lupaş q-Bezier Curves

Definition 5.4.2.1: [7] $Degree\ elevation$

Any Lupas q-Bezier curves of degree η with control points ρ_κ can be expressed of a new basis of degree $\eta+1$. The new control point ρ_κ^* are given by

$$\rho_\kappa^* = \left(1 - \frac{[\eta+1-\kappa]_q}{[\eta+1]_q}\right)\rho_{\kappa-1} + \frac{[\eta+1-\kappa]_q}{[\eta+1]_q}\rho_\kappa, \quad \kappa = 0,1,\ldots,\eta+1, \tag{5.12}$$

Note that $\rho_{-1} = \rho_{\eta+1} = 0$.

Proof. From the definition of Lupas q-Bezier curve (5.8), we have;

$$\rho(\tau;q) = \sum_{\kappa=0}^{\eta} \rho_\kappa z_\kappa^\eta(\tau;q)$$

From the degree elevation of the Lupas q-analogues of the Bernstein functions(5.5), we obtain;

$$\rho(\tau;q) = \sum_{\kappa=0}^{\eta} \rho_\kappa \left[\frac{[\eta-\kappa+1]_q}{[\eta+1]_q} z_\kappa^{\eta+1}(\tau;q) + \left(1 - \frac{[\eta-\kappa]_q}{[\eta+1]_q}\right) z_{\kappa+1}^{\eta+1}(\tau;q)\right]$$

$$= \sum_{\kappa=0}^{\eta} \frac{[\eta-\kappa+1]_q}{[\eta+1]_q} \rho_\kappa z_\kappa^{\eta+1}(\tau;q) + \sum_{\kappa=1}^{\eta+1}\left(1 - \frac{[\eta+1-\kappa]_q}{[\eta+1]_q}\right)\rho_{\kappa-1} z_\kappa^{\eta+1}(\tau;q)$$

$$= \sum_{\kappa=0}^{\eta+1}\left[\frac{[\eta+1-\kappa]_q \rho_\kappa + \left(1 - [\eta+1-\kappa]_q\right)\rho_{\kappa-1}}{[\eta+1]_q}\right] z_\kappa^{\eta+1}(\tau;q)$$

$$= \sum_{\kappa=0}^{\eta+1} \rho_\kappa^* z_\kappa^{\eta+1}(\tau;q)$$

Where;

$$\rho_{\kappa}^{*}=\left(1-\frac{[\eta+1-\kappa]_q}{[\eta+1]_q}\right)\rho_{\kappa-1}+\frac{[\eta+1-\kappa]_q}{[\eta+1]_q}\rho_{\kappa} \qquad , \quad \kappa=0,1,\dots,\eta+1$$

Next section, we give the matrix representation of degree elevation of $\rho(\tau;q)$.

5.4.2.1 Matrix Representation of Degree Elevation of Lupaş *q*-Bezier Curves

Let $\rho=\left(\rho_0,\rho_1,\dots,\rho_\eta\right)^T$ indicate the vector of control points of the initial Lupas q-Bezier curve of degree η, and $\rho^{(1)}=\left(\rho_0^*,\rho_1^*,\dots,\rho_{\eta+1}^*\right)$ shows that the control points of the degree elevated Lupaş q-Bezier curve of degree $\eta+1$.

Firstly, we apply the degree elevation algorithm of Lupas q-Bezier curves (formula (5.12), we obtain following results:

$$\rho_0^*=\frac{[\eta+1]_q}{[\eta+1]_q}\rho_0$$

$$\rho_1^*=\left(\frac{[\eta+1]_q-[\eta]_q}{[\eta+1]_q}\right)\rho_0+\frac{[\eta]_q}{[\eta+1]_q}\rho_1$$

$$\rho_2^*=\left(\frac{[\eta+1]_q-[\eta-1]_q}{[\eta+1]_q}\right)\rho_1+\left(\frac{[\eta-1]_q}{[\eta+1]_q}\right)\rho_2$$

$$\rho_3^*=\left(\frac{[\eta+1]_q-[\eta-2]_q}{[\eta+1]_q}\right)\rho_2+\left(\frac{[\eta-2]_q}{[\eta+1]_q}\right)\rho_3$$

$\vdots$

$\vdots$

$$\rho_{\eta-2}^*=\left(\frac{[\eta+1]_q-[3]_q}{[\eta+1]_q}\right)\rho_{\eta-3}+\left(\frac{[3]_q}{[\eta+1]_q}\right)\rho_{\eta-2}$$

$$\rho^{*}_{\eta-1} = \left(\frac{[\eta+1]_q - [2]_q}{[\eta+1]_q} \right) \rho_{\eta-2} + \left(\frac{[2]_q}{[\eta+1]_q} \right) \rho_{\eta-1}$$

$$\rho^{*}_{\eta} = \left(\frac{[\eta+1]_q - [1]_q}{[\eta+1]_q} \right) \rho_{\eta-1} + \left(\frac{[1]_q}{[\eta+1]_q} \right) \rho_{\eta}$$

$$\rho^{*}_{\eta+1} = \left(\frac{[\eta+1]_q - [0]_q}{[\eta+1]_q} \right) \rho_{\eta} + \left(\frac{[0]_q}{[\eta+1]_q} \right) \rho_{\eta+1}$$

Then, the degree elevation procedure of for Lupas q-Bezier curves of degree $\eta+1$ can be represented as the following:

$$\rho^{(1)} = T_{\eta+1}\rho$$

where

$$\begin{matrix} \rho_0(\tau;q) & \rho_1(\tau;q) & \cdots & \rho_\eta(\tau;q) & \rho_{\eta+1}(\tau;q) \end{matrix}$$

$$T_{\eta+1} = \frac{1}{[\eta+1]_q} \begin{pmatrix} [\eta+1]_q & 0 & \cdots & 0 & 0 \\ [\eta+1]_q - [\eta]_q & [\eta]_q & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & [2]_q & 0 \\ 0 & 0 & \cdots & [\eta+1]_q - [1]_q & [1]_q \\ 0 & 0 & \cdots & 0 & [\eta+1]_q \end{pmatrix}_{(\eta+2)x(\eta+1)}$$

Generally; $\forall r \in N$, the degree elevated of Lupas q-Bezier curve of control points of degree $\eta+1$ is:

$$\rho^{(r)} = T_{\eta+1} \cdots T_{\eta+2} T_{\eta+1} P \,.$$

As $r \to \infty$, the control polygon $\rho^{(r)}$ converges to a Lupas q-Bezier curve.

5.4.3 De Casteljau Algorithm for Lupaş q-Bezier Curves

Lupaş q-Bezier curves of degree η can be written as two kinds of linear combination of two Lupaş q-Bezier curves of degree $\eta-1$.

Definition 5.4.3.1 [7] $De\ Casteljau\ algorithms$

A curve

$$\rho(\tau;q)=\sum_{\kappa=0}^{\eta-1}\rho_\kappa^1(\tau;q)=\cdots=\sum\rho_\kappa^r(\tau;q)z_\kappa^{\eta-r}(\tau;q)=\cdots=\rho_0^\eta(\tau;q).$$

Where

$$\begin{cases}\rho_\kappa^0(\tau;q)\equiv\rho_\kappa^0\equiv\rho_\kappa, \quad \kappa=0,1,\ldots,\eta\\ \rho_\kappa^r(\tau;q)=\dfrac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau}\rho_{\kappa+1}^{r-1}(\tau;q)+\dfrac{1-\tau}{1-\tau+q^{\eta-r}\tau}\rho_\kappa^{r-1}(\tau;q),\\ r=1,2,\ldots,\eta, \quad \kappa=0,1,\ldots,\eta-r\end{cases} \tag{5.13}$$

or

$$\begin{cases}\rho_\kappa^0(\tau;q)\equiv\rho_\kappa^0\equiv\rho_\kappa, \quad \kappa=0,1,\ldots,\eta\\ \rho_\kappa^r(\tau;q)=\dfrac{q^\kappa\tau}{1-\tau+q^{\eta-r}\tau}\rho_{\kappa+1}^{r-1}(\tau;q)+\dfrac{q^\kappa(1-\tau)}{1-\tau+q^{\eta-r}\tau}\rho_\kappa^{r-1}(\tau;q),\\ r=1,2,\ldots,\eta, \quad \kappa=0,1,\ldots,\eta-r\end{cases} \tag{5.14}$$

Proof. From the degree reduction of $z_\kappa^\eta(\tau;q)$ formula (5.7), we have;

$$\rho(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_\kappa z_\kappa^\eta(\tau;q)=\sum_{\kappa=0}^{\eta}\rho_\kappa\left[(1-\tau)z_\kappa^{\eta-1}(\tau;q)+(\tau)z_{\kappa-1}^{\eta-1}(\tau;q)\right]$$

$$=\sum_{\kappa=0}^{\eta-1}\rho_\kappa(1-\tau)z_\kappa^{\eta-1}(\tau;q)+\sum_{\kappa=1}^{\eta}\rho_\kappa(\tau)z_{\kappa-1}^{\eta-1}(\tau;q)$$

$$=\sum_{\kappa=0}^{\eta-1}\rho_\kappa(1-\tau)z_\kappa^{\eta-1}(\tau;q)+\sum_{\kappa=0}^{\eta-1}\rho_{\kappa+1}(\tau)z_\kappa^{\eta-1}(\tau;q)$$

$$=\sum_{\kappa=0}^{\eta-1}\left[\rho_\kappa(1-\tau)+\rho_{\kappa+1}\tau\right]z_\kappa^{\eta-1}(\tau;q)$$

$$=\sum_{\kappa=0}^{\eta-1}\rho_\kappa^1 z_\kappa^{\eta-1}(\tau;q)$$

where

$$\rho_\kappa^1 = \rho_\kappa\left(1-\tau\right)+\rho_{\kappa+1}\tau = \rho_\kappa^0\left(1-\tau\right)+\rho_{\kappa+1}^0\tau \quad \text{for} \quad k=0,1,\ldots,\mathrm{m}-1.$$

If we apply same argument to the above to $\rho\left(\tau;q\right)$;

$$\rho\left(\tau;q\right)=\sum_{\kappa=0}^{\eta-1}\rho_\kappa^1 z_\kappa^{\eta-1}\left(\tau;q\right)$$

then

$$\rho\left(\tau;q\right)=\sum_{\kappa=0}^{\eta-2}\rho_\kappa^2 z_\kappa^{\eta-2}\left(\tau;q\right)$$

where

$$\rho_\kappa^2=\rho_\kappa^1\left(1-\tau\right)+\left(\tau\right)\rho_{\kappa+1}^1 \qquad , \qquad \kappa=0,1,\ldots,\eta-2$$

Generally

$$\rho\left(\tau;q\right)=\sum_{\kappa=0}^{\eta-r}\rho_\kappa^r\left(\tau;q\right)z_\kappa^{\eta-r}\left(\tau;q\right)$$

where

$$\rho_\kappa^r\left(\tau;q\right)=\frac{1-\tau}{1-\tau+q^{\eta-r}\tau}\rho_\kappa^{r-1}\left(\tau;q\right)+\frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau}P_{\kappa+1}^{r-1}\left(\tau;q\right) \text{ for } \kappa=0,\ldots,\eta-r \text{ .}$$

Taking $r=\eta$. Yields;

$$\rho\left(\tau;q\right)=\sum_{\kappa=0}^{0}\rho_\kappa^\eta\left(\tau;q\right)z_\kappa^0\left(\tau;q\right)=\rho_0^\eta\left(\tau;q\right)$$

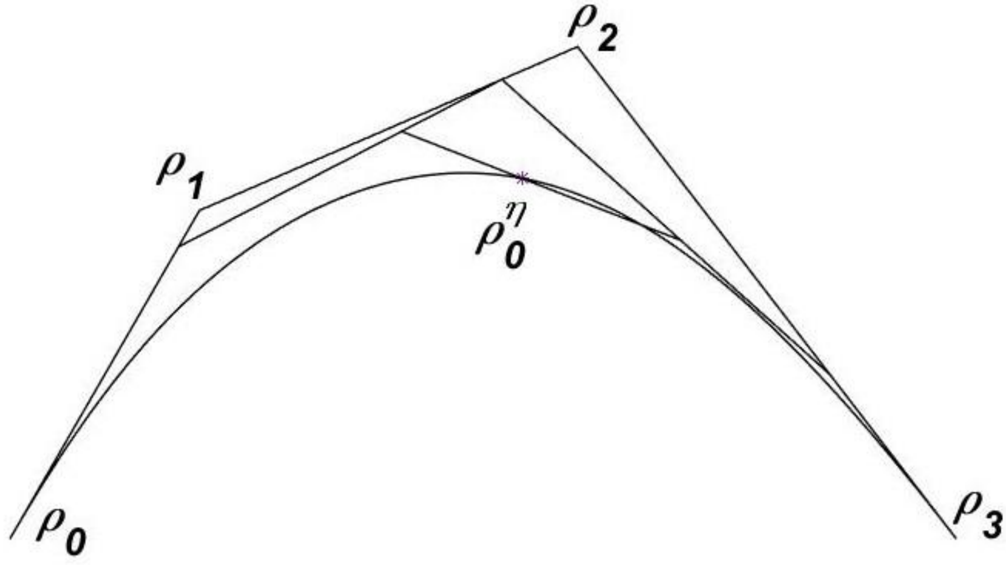


Figure 5.9: The de Casteljau algorithms of cubic Lupas q-Bezier curves for q=2.

To next section, we investigate the matrix representation of de Casteljau algorithm for Lupas q-Bezier curves.

5.4.3.1 Matrix Representation of De Casteljau Algorithm for Lupaş q-Bezier Curve

Let $\rho^0 = \left(\rho_0, \rho_1, \ldots, \rho_\eta\right)^T$, $\rho^r = \left(\rho_0^r, \rho_1^r, \ldots, \rho_{\eta-r}^r\right)^T$, if we apply the de Casteljau algorithm of Lupaş q-Bezier curve(5.13), we get the following result;

$$\rho_0^r(\tau;q) = \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau}\rho_1^{r-1}(\tau;q) + \frac{1-\tau}{1-\tau+q^{\eta-r}\tau}\rho_0^{r-1}(\tau;q)$$

$$\rho_1^r(\tau;q) = \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau}\rho_2^{r-1}(\tau;q) + \frac{1-\tau}{1-\tau+q^{\eta-r}\tau}\rho_1^{r-1}(\tau;q)$$

$$\rho_2^r(\tau;q) = \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau}\rho_3^{r-1}(\tau;q) + \frac{1-\tau}{1-\tau+q^{\eta-r}\tau}\rho_2^{r-1}(\tau;q)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\rho_{\eta-1}^{r}\left(\tau;q\right)=\frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau}\rho_{\eta}^{r-1}\left(\tau;q\right)+\frac{1-\tau}{1-\tau+q^{\eta-r}\tau}\rho_{\eta-1}^{r-1}\left(\tau;q\right)$$

$$\rho_{\eta}^{r}\left(\tau;q\right)=\frac{q^{\eta-r}\tau}{1-t+q^{\eta-r}\tau}\rho_{\eta+1}^{r-1}\left(\tau;q\right)+\frac{1-\tau}{1-\tau+q^{\eta-r}\tau}\rho_{\eta}^{r-1}\left(\tau;q\right)$$

Secondly, if we apply the de Casteljau algorithm of Lupaş q-Bezier curve(5.14), we get the following result;

$$\rho_{0}^{r}\left(\tau;q\right)=\frac{\tau}{1-\tau+q^{\eta-r}\tau}\rho_{1}^{r-1}\left(\tau;q\right)+\frac{\left(1-\tau\right)}{1-\tau+q^{\eta-r}\tau}\rho_{0}^{r-1}\left(\tau;q\right)$$

$$\rho_{1}^{r}\left(\tau;q\right)=\frac{q\tau}{1-\tau+q^{\eta-r}\tau}\rho_{2}^{r-1}\left(\tau;q\right)+\frac{q\left(1-\tau\right)}{1-\tau+q^{\eta-r}\tau}\rho_{1}^{r-1}\left(\tau;q\right)$$

$$\rho_{2}^{r}\left(\tau;q\right)=\frac{q^{2}\tau}{1-\tau+q^{\eta-r}\tau}\rho_{3}^{r-1}\left(\tau;q\right)+\frac{q^{2}\left(1-\tau\right)}{1-\tau+q^{\eta-r}\tau}\rho_{2}^{r-1}\left(\tau;q\right)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\rho_{\eta-r-1}^{r}\left(\tau;q\right)=\frac{q^{\eta-r-1}\tau}{1-\tau+q^{\eta-r}t}\tau\rho_{\eta-r}^{r-1}\left(\tau;q\right)+\frac{q^{\eta-r-1}\left(1-\tau\right)}{1-\tau+q^{\eta-r}\tau}\rho_{\eta-r-1}^{r-1}\left(\tau;q\right)$$

$$\rho_{\eta-r}^{r}\left(\tau;q\right)=\frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau}\rho_{\eta-r+1}^{r-1}\left(\tau;q\right)+\frac{q^{\eta-r}\left(1-\tau\right)}{1-\tau+q^{\eta-r}\tau}\rho_{\eta-r}^{r-1}\left(\tau;q\right)$$

then the De Casteljau algorithm procedure of Lupas q-Bezier curves can be expressed as;

$$\rho^{r}\left(\tau;q\right)=M_{r}\left(\tau;q\right)\cdots M_{2}\left(\tau;q\right)M_{1}\left(\tau;q\right)\rho^{0}$$

where $M_{r}\left(\tau;q\right)$ is a $\left(\eta-r+1\right)x\left(\eta-r+2\right)$ matrix. Then;

$$\begin{array}{cccccc} \rho_0^{r-1}(\tau;q) & \rho_1^{r-1}(\tau;q) & \rho_2^{r-1}(\tau;q) & \cdots & \rho_\eta^{r-1}(\tau;q) & \rho_{\eta+1}^{r-1}(\tau;q) \end{array}$$

$$M_r(\tau;q)=\begin{pmatrix} \frac{1-\tau}{1-\tau+q^{\eta-r}\tau} & \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1-\tau}{1-\tau+q^{\eta-r}\tau} & \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1-\tau}{1-\tau+q^{\eta-r}\tau} & \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1-\tau}{1-\tau+q^{\eta-r}\tau} & \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau} \end{pmatrix}$$

or

$$\begin{array}{cccccc} \rho_0^{r-1}(\tau;q) & \rho_1^{r-1}(\tau;q) & \rho_2^{r-1}(\tau;q) & \cdots & \rho_{\eta-r}^{r-1}(\tau;q) & \rho_{\eta-r+1}^{r-1}(\tau;q) \end{array}$$

$$M_r(\tau;q)=\begin{pmatrix} \frac{1-\tau}{1-\tau+q^{\eta-r}\tau} & \frac{\tau}{1-\tau+q^{\eta-r}\tau} & 0 & \cdots & 0 & 0 \\ 0 & \frac{q(1-\tau)}{1-\tau+q^{\eta-r}\tau} & \frac{q\tau}{1-\tau+q^{\eta-r}\tau} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{q^{\eta-r-1}(1-\tau)}{1-\tau+q^{\eta-r}\tau} & \frac{q^{\eta-r-1}\tau}{1-\tau+q^{\eta-r}\tau} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \frac{q^{\eta-r}(1-\tau)}{1-\tau+q^{\eta-r}\tau} & \frac{q^{\eta-r}\tau}{1-\tau+q^{\eta-r}\tau} \end{pmatrix}$$

REFERENCES

[1] Bernstein, S. N. (1912). Constructive proof of Weierstrass approximation theorem. *Comm. Kharkov Math. Soc*.

[2] Bézier, P. (1970). Numerical control: mathematics and applications.

[3] Cross, E. M., & Chaffin, W. W. (1982). Use of the binomial theorem in interpreting results of multiple tests of significance. *Educational and Psychological Measurement*, *42*(1), 25-34.

[4] Dişibüyük, Ç., & Oruc, H. (2008). Tensor product q-Bernstein polynomials. *BIT Numerical Mathematics*, *48*(4), 689.

[5] Elhabian, S. Y., Rara, H. M., & Farag, A. A. (2011, August). Towards Efficient and Compact Phenomenological Representation of Arbitrary Bidirectional Surface Reflectance. In *BMVC* (pp. 1-11).

[6] Farin, G. (2014). *Curves and surfaces for computer-aided geometric design: a practical guide*. Elsevier.

[7] Han, L. W., Chu, Y., & Qiu, Z. Y. (2014). Generalized Bézier curves and surfaces based on Lupaş q-analogue of Bernstein operator. *Journal of Computational and Applied Mathematics*, *261*, 352-363.

[8] Kac, V., & Cheung, P. (2001). *Quantum calculus*. Springer Science & Business Media.

[9] MacCracken, R., & Joy, K. I. (1996, August). Free-form deformations with lattices of arbitrary topology. In *Proceedings of the 23rd annual conference on Computer graphics and interactive techniques* (pp. 181-188). ACM.

[10] Mahmudov, N., & Sabancigil, P. Some approximation properties of Lupaş q-analogue of Bernstein operators,(2010). *arXiv preprint arXiv:1012.4245*.

[11] Mursaleen, M., & Khan, A. (2013). Generalized-Bernstein-Schurer operators and some approximation theorems. *Journal of Function Spaces and Applications*, *2013*.

[12] Mursaleen, M., Ansari, K. J., & Khan, A. (2015). On (p, q)-analogue of Bernstein operators. *Applied Mathematics and Computation*, *266*, 874-882.

[13] Operators, L. (1960). Approximation Theory, Hindustan Publ. *Corp., Delhi, India*.

[14] Oruç, H., & Phillips, G. M. (2003). q-Bernstein polynomials and Bézier curves. *Journal of Computational and Applied Mathematics*, *151*(1), 1-12.

[15] Phillips, G. M. (1996). Bernstein polynomials based on the q-integers. *Annals of numerical Mathematics*, *4*, 511-518.

[16] Phillips, G. M. (1997). A de Casteljau algorithm for generalized Bernstein polynomials. *BIT Numerical Mathematics*, *37*(1), 232-236.

[17] Phillips, G. M. (2003). *Interpolation and approximation by polynomials* (Vol. 14). Springer Science & Business Media.

[18] Sederberg, T. W. Computer aided geometric design course notes, 2011.

[19] Zhang, J. W. C. (1999). Bézier curves and surfaces. *Graphical Models and Image Processing*, *61*(1), 2-15.