Newtonian Cosmology

Hamza Saber Saeed Zebari

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Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. İzzet Sakallı
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Physics.

Asst. Prof. Dr. Ali Övgün
Supervisor

Examining Committee

1. Prof. Dr. Mustafa Halilsoy
2. Assoc. Prof. Dr. Seyedeh Fatemeh Mirekhtiary
3. Asst. Prof. Dr. Ali Övgün

ABSTRACT

In this work, we have used the Friedmann equation of general relativity with quantum correction terms to the Newtonian potential to obtain the modified Friedmann equation with quantum correction. The behavior of the solution is examined with a sign of the correction being put into consideration. With the correction, the singular and non-singular universe is clearly spelled out. The results reveal that the quantum correction effect depends on the sign in the correction. Applying cosmological constant to the modified Friedmann Equation, we found out that the scale factor of the universe is given by the first order ϵ in the expression of τ. We also compare the Friedmann Equations with Loop Quantum Gravity (LQG) to the Friedmann Equation with quantum correction and found out that, for negative γ_q the critical density is not feasible but it is feasible for positive values γ_q.

Keywords: Cosmology, quantum cosmology, Cosmological constant, Quantum bounce, Radiation universe, Dust universe.

ÖZ

Bu çalışmada, modifiye Friedmann denklemini kuantum düzeltmesi ile elde etmek için genel göreliliğin Friedmann denklemini kuantum düzeltme terimi ile Newton potansiyeline kullandık. Çözümün davranışı, dikkate alınan düzeltmenin işareti ile incelenir. Düzeltme ile, tekil ve tekil olmayan evren açıkça dile getirilir. Sonuçlar, kuantum düzeltme etkisinin düzeltme işaretine bağlı olduğunu göstermektedir. Modifiye edilmiş Friedmann Denklemine kozmolojik sabit uygulayarak, evrenin ölçek faktörünün τ ifadesinde ilk derece ϵ tarafından verildiğini öğrendik. Ayrıca, Döngü Kuantum Yerçekimi (LQG) ile Friedmann Denklemlerini kuantum düzeltmesi ile Friedmann Denklemi ile karşılaştırdık ve negatif γ_q için kritik yoğunluğun mümkün olmadığını, ancak pozitif değerler için mümkün olduğunu öğrendik γ_q.

Anahtar Kelimeler: Kozmoloji, kuantum kozmolojisi, Kozmolojik sabit, Kuantum sıçraması, Radyasyon evreni, Toz evren.

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INTRODUCTION

Our Universe is highly composed, consisting of components on larger scales which include; planets in an elliptic orbit, collection of stars into galaxies, planets orbit stars, stars collected into galaxies, galaxies are bounded together by gravitational force to form clusters and clusters are collectively held to form superclusters. An attempt to studying the whole universe as a single entity can be likened to megalomaniacs fairytale. Amazingly, with cosmology, our universe can be studied as a single entity. Cosmology is the study of the cosmos or more generally, the universe as a whole. The word Cosmology derived its origin from the Greek word "Kosmos", meaning harmony or order. That is why cosmologist tries to harmonize large and completed universe structures into a form that than be easily studied. On very small scales, there is a fluctuation in the density of the universe that ranges from subatomic quantum fluctuations to large superclusters and voids, approximately 50Mpc across characterizing the distribution of galaxies in space [1].

Even though the universe is clearly inhomogeneous and anisotropic at local scales of stars and clusters of stars, it is argued that homogeneity and isotropic only at large enough scales. However, the Friedmann-Robertson-Walker (FRW) model asserts that our Universe is exactly homogeneous and isotropic around us. By homogeneity, we mean the universe is roughly the same at all points in space and matter is evenly distributed all the space. That is to say, no part of the universe can be distinguished from the other. On the other hand, by isotropy, all directions are equal in the universe. However, there is no fundamental physical justification of homogeneity and isotropy at any epoch of time and region in space [2].

The evolution of the universe is described by the Friedmann equation given below

$$3\left(\frac{\dot{a}^2+Kc^2}{a^2}\right) = 8\pi G\rho \quad (1.1)$$

Where k is the curvature of the universe which is flat when k=0. In a compact form, equation (1.1) can be rewritten as

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_c(t) \quad (1.2)$$

Where $H(t) = \frac{\dot{a}}{a}$ is the Hubble parameter and $\rho_c(t)$ denotes the universe's critical density at any given cosmic time. For a curve universe, its entire density is greater or less than its critical value. For larger total density than the critical density, the universe is said to be closed and for that smaller than the critical density, the universe is opened and infinite hyperbolic space. At the present time, the critical density is $\rho_{c,0}= \rho_c(t_0)$. Another parameter of interest is the density parameter which is given as

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \quad (1.3)$$

and $\Omega_{tot}(t) = \frac{\rho_{tot}(t)}{\rho_c(t)} = 1$ denotes a flat universe, if $\Omega_{tot}(t) < 1$ the universe is opened and for $\Omega_{\mathrm{tot}}(\mathrm{t}) > 1$, it is a closed universe.

Observations today reveal that the density of the universe is closer to the critical density at the recent time

$$\rho_{tot,0=}\rho_{tot}(t_0) \approx \rho_{c,0} \text{ or } \Omega_{tot,0} = \Omega_{tot}(t_0) \approx 1 \quad (1.4)$$

Measurements indicate that the present value of the universe's critical density is $\rho_{c,0} \approx 1.879 \times 10^{-29}\ gcm^{-3}$. The curvature of the universe is given in the diagram below

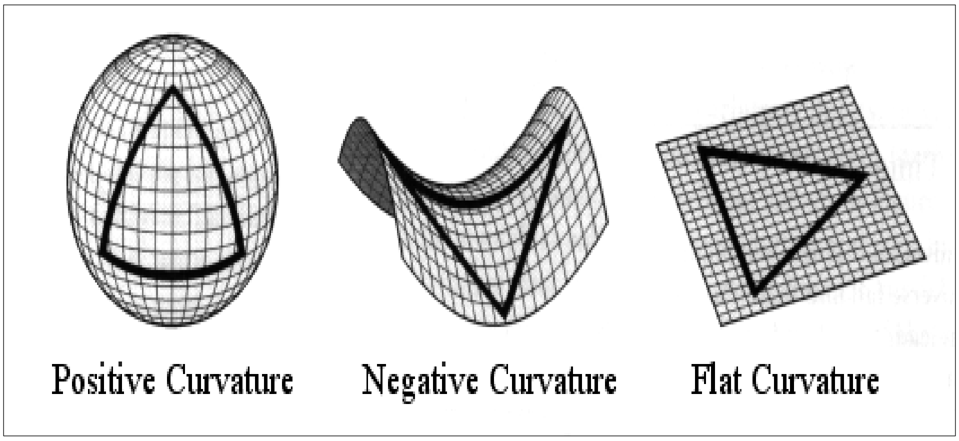

Figure 1.1: various potential outcomes of the curvature of a homogeneous and isotropic universe. [www.images.app.goo.gl/HUpB3MaEjFr1MKSD8].

Cosmic theory of inflation asserts that the universe experienced an accelerated exponential expansion in its early stage just after the big bang at time t $\sim 10^{-35}$s. The cosmic inflationary theory was introduced to solve key problems associated with the ordinary big bang theory. According to [3], basic important problems with the big bang theory without inflation are:

Flatness Problem**,** From the Friedmann equation, the critical density is $\rho_c = \frac{3H_0^2}{8\pi G}$ shows that the universe is flat. A deviation of the density from its critical value also cause changes in the curvature of the universe as

$$\Omega(t) - 1 = \frac{Kc^2}{a^2H^2} \tag{1.5}$$

The deviation increases with the time the universe began and filled it with matter or radiation. Hence, the energy density of the early universe is closer to the critical density than its present value. The theory of inflation resolves this issue by driving energy density to be closer to its critical value as the inflation ended and the universe grows rapidly from Planck scales:

$$\frac{a(t_f)}{a(t_i)} = e^{H_i(t_f - t_i)} = e^N \tag{1.6}$$

Where t_i and t_f denote the initial and final time of inflation, and $a(t_i)$ and $a(t_f)$ are scale factors at the time when inflation started and ended respectively. The theory of inflation solves problems of positive deviation of energy density for negative deviations.

Horizon Problem, If the universe is isotropic and homogeneous, there should have been interactions between points with distances larger than the particle horizon in the past. This problem is resolved too by inflation theory.

Monopole Problem, Particle Theorists are of the view that the universe has magnetic monopoles, but in reality there is no observation backing this claim. This is termed a monopole problem.

Before the 1990s, cosmologists and astronomers were of the view that the expansion of the universe started by the Big Bang was decelerating that may turn into contraction in the future. Hubble however, measured the redshift from the supernova explosion and found out that the universe was expanding with an increasing rate ($\ddot{a} > 0$) instead of decreasing due to gravitational pull [4].

It is now concluded that the universe has gone into different phases of expansion after the Big Bang; from the Planck scale to inflationary astronomical scales after which radiation and matter became dominant. After inflation, the expansion slowed down, but begins to speed up and in recent times; the expansion of the universe is accelerating [5].

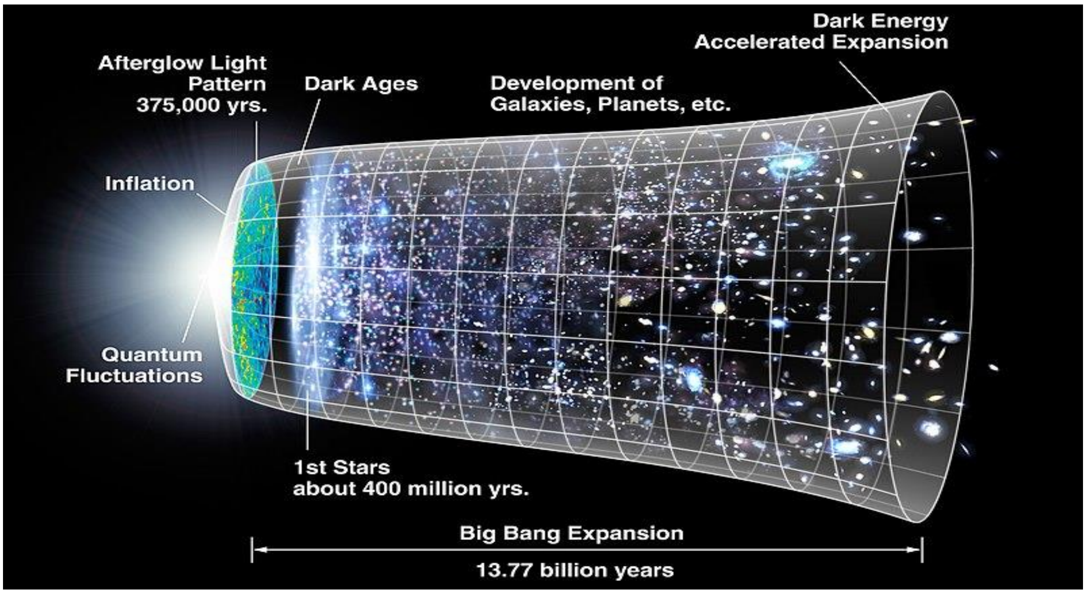

Figure 1.2: History of the universe from the Huge explosion to the present day. [cosmotography.com].

The theories of the growing universe prompted the disclosure of dark energy. In 1998, High-z Supernova Search Group and Supernova Cosmology Undertaking in 1999 published their accurately estimated information of separations of supernovas and the comparative redshifts. The accelerating expansion is widely accepted as evidence of dark energy. However, recent Cosmologist formulates that dark energy is a field energy form of gravitation which balances the gravitational attraction to maintain stability and homogeneity of the Universe [6].

The first and most prominent clarification that was offered to clarify the obvious mass errors, was to expect that there exists some type of ' dark matter'. This issue has the property that it collaborates as ordinary matter in the gravitational sense; however that doesn't cooperate with electro-magnetic radiation, making this issue imperceptible.

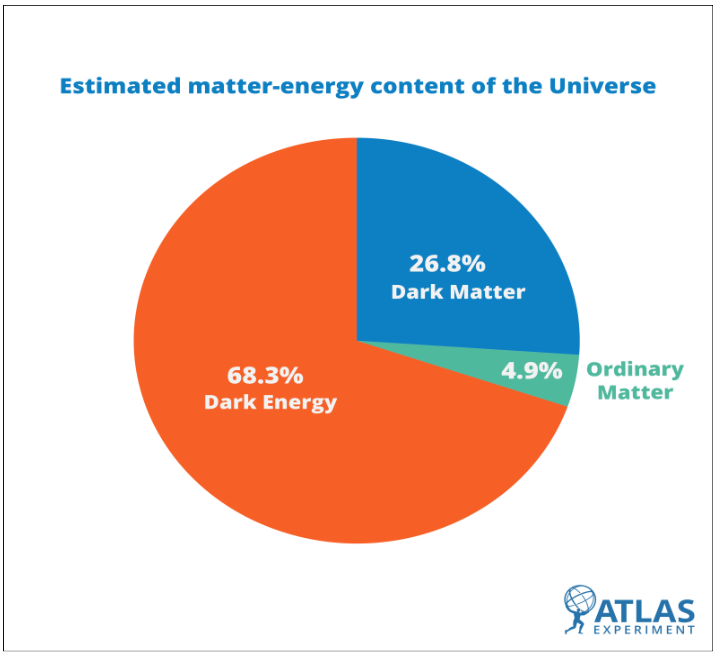

Figure 1.3: Estimated matter-energy content of the universe [6]

Notwithstanding the dark matter supposition, there is the suspicion of some type of dark energy. This energy should penetrate the whole universe and can clarify the acceleration rate at which the universe expands. According to figure 1.2, the dark energy and matter hypothesis generally 68.3% of the universe is dark energy, 26.8% is dark matter and just 4.9% of the universe is composed of obvious matter [7].

Newton's publication of his popular Philosophiae Naturalis Principia Mathematica in 1687 brought to being the main hypothesis of gravity. He used this theory to explain the empirical laws of Kepler. The early achievement of this hypothesis came up when Edmund Halley (1656-1742) effectively anticipated that the comet in 1456, 1531, 1607 and 1682 would return in 1758. Today, Newton's hypothesis of gravity still gets the job done to depict planetary and satellite movement and establishes the nonrelativistic furthest reaches of Einstein's relativistic hypothesis. Newton believed stars as suns to be similarly appropriated all through endless space in spite of the conspicuous convergence of stars in the Milky Way [8].

Quantum cosmology depends on the possibility that quantum material science ought to apply to anything in nature, including the entire universe. quantizing the entire universe is a long way from being simple, as indicated by general relativity, not just matter but as well as space and time in existence. They are dependent upon dynamical laws and have excitations (gravitational waves) that interface with one another and with the matter. Quantum cosmology is along these lines firmly identified with quantum gravity, the quantum theory of the gravitational force and space-time. Without quantum gravity, the quantum cosmology is unclear.[9]

Loop quantum gravity (LQG) expects to display the conduct of spacetime in circumstances where its atomic characteristic emerges. Among these circumstances, Loop quantum cosmology explained the universe is close to the Big Bang. The Big Bang singularity is the main point to be solved in the loop quantum cosmology since there is much probability that the general relativity initial singularity must be solved in the quantum gravity theory. (LQC) suggests that in the simple model the big bang singularity of classical general relativity is replaced by a quantum bounce.[10]

Chapter 2

RELATIVISTIC COSMOLOGY

Relativistic cosmology models are tractable when we make use of powerful simplifying assumptions known as cosmological principles. With these principles, large scale observations of our universe are studied as a whole. According to cosmological principles, at a given time and on a large scale, the universe is homogeneous and at the same time isotropic. The universe becomes inhomogeneous and anisotropic. The rotation of galaxies and galactic masses can be linked-to a somewhat non-luminous kind of matter known as dark matter. This kind of matter has nothing to do with the so-called dark energy. The Friedmann-Robertson-Walker universe is used to study relativistic cosmology.

2.1 Friedmann-Robertson-Walker Universe

The FRW approach is a solution march to a space-time foliate into even homogeneity and isotropic hypersurface using the scale factor to determine its expansion law. The FRW model is the bedrock of relativistic cosmology due to its success in describing a real universe. In understanding the Newtonian cosmology, we will present the model in a way that relates to Newton's universe. In the spherical coordinates, the line element of the FRW universe is given by:

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + \mathrm{a(t)}^2 \left[\ \mathrm{d}r^2(1-\mathrm{kr}^2)^{-1} + \mathrm{r}^2\mathrm{d}\theta^2 + \mathrm{r}^2\sin^2\theta\ \mathrm{d}\emptyset^2\ \right] \qquad (2.1)$$

t, here denotes the cosmic time, the scale factor is $\mathrm{a(t)}$ that gives the expansion rate of the universe in an expansion law and k is related to the curvature of the universe.

Special Case:

Hyperspheric, If $k > 0$, the FWR metric is said to be singular at $r = k$ this type of singularity is termed coordinate singularity, it vanishes at the introduction of a new coordinate system defining $r = k^{\frac{1}{2}} sin_x$, with $0 \leq x \leq \pi$. Applying it to the equation, we obtain

$$ds^2 = -dt^2 + a(t)^2 k^{-1}[dx^2 + \sin^2 x \, d\theta^2 + \sin^2 x \sin^2\theta d\emptyset^2] \qquad (2.3)$$

which is the line element of a hypersphere that has a definite volume and the model of the universe is called closed.

Hyperbolic, For $k < 0$, the coordinate singularity does not exist and r, the radial coordinate can move from zero to infinity and the universe is said to be opened. The coordinate transform as $r = k^{\frac{1}{2}} \sinh x$ on the range $0 \leq x < 1$. With equation (2.1) we see that the hyperbolic space becomes

$$ds^2 = -dt^2 + a(t)^2 |k|^{-1}[dx^2 + \sinh^2 x \, d\theta^2 + \sinh^2 x \sin^2\theta d\emptyset^2] \qquad (2.4)$$

Flat, when $k = 0$, the metric of FRW will become

$$ds^2 = -dt^2 + [dx^2 + x^2 d\theta^2 + x^2 \sin^2\theta d\emptyset^2] \, a(t)^2 \qquad (2.5)$$

Here, x denotes the radial coordinate which spans from 0 to infinity and the spatial part represents a Euclidean space that contracts or expands with the scale factor in the (x,y,z) plane

$$ds^2 = -dt^2 + [dx^2 + dy^2 + dz^2] \, a(t)^2 \qquad (2.6)$$

To obtain the generalized form of the matric, the curvature k for simplicity can be modeled for k(-1, 0, 1) for hyperbolic, flat and hypersphere respectively. Hence, the three cases above can be rewritten as:

$$ds^2 = -dt^2 + a(t)^2[dx^2 + f_k(x)^2 d\theta^2 + f_k(x)^2 \sin^2\theta d\emptyset^2] \qquad (2.7)$$

Where

$$f_k(x) = \begin{cases} \sin_x & \text{if } k = 1 \\ x & \text{if } k = 0 \\ \sinh_x & \text{if } k = -1 \end{cases} \tag{2.8}$$

The line element (2.7) becomes meaningful when the isometries [11] of the FRW metrics are derived in the next session.

2.2 Friedmann Equations

By allowing the scale factor and the curvature of the Robertson-Walker (RW) space to vary with time, we can model the universe by taking each point in time a RW space. The generic metric is given

$$ds^2 = -dt^2 + [dx^2 + f_k(x)^2 x^2 d\omega^2]\, a(t)^2 \tag{2.9}$$

The scale factor $a(t)$ explains the expansion or contraction of the universe and is normalized such that at present time $a(t) = 1$. Putting equation (2.9) into the Einstein equations, after much algebraic work we obtain two fundamental equations known as the Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2} + \frac{\Lambda}{3} \tag{2.10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \tag{2.11}$$

The first equation is obtained from the 00-component and the second from the ii-component of the Einstein equations. Combining the two Friedmann equations we obtain an adiabatic equation

$$\frac{d}{dt}(\rho a^3 c^2) + p\frac{d}{dt}(a^3) = 0 \tag{2.12}$$

This is the first law of thermodynamics started in relativistic language;

TdS = dE+pdV = 0, which asserts that the entropy of the universe is constant.

2.3 The Newtonian derivation of the Friedmann equations

The Friedmann equations can also be derived from the principles of Newtonian gravity. For a sphere of mass $\mathrm{M} = \frac{4\pi r^3 \rho}{3}$ and radius r, the total potential energy is

$$E = PE + KE = -\frac{GMm}{r} + \frac{1}{2} m\dot{r}^2 \tag{2.13}$$

If $r = ar_0$ and dividing by m yields

$$\varepsilon = \frac{E}{m} = -\frac{4\pi G\rho {r_0}^2 a^2}{3} + \frac{1}{2} {r_0}^2 \dot{a}^2 \tag{2.14}$$

After rearrangement we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{1}{2} {r_0}^2 \dot{a}^2 \tag{2.15}$$

Or

$$H(t)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \tag{2.16}$$

Where $\mathrm{k} = -2\varepsilon/{r_0}^2 c^2$ and $H(t)$ is known as Hubble's parameter which tells us the expansion rate of the universe. Equation (2.16) is the Friedman Equation which can be solved to obtain $a(t)$ in terms of other parameters.

2.4 Friedmann equations with a cosmological constant

Adding cosmological constant as proposed by Einstein, the acceleration equation takes the form

$$\frac{\ddot{a}}{a} + \frac{4\pi G\rho}{3} - \frac{\Lambda c^2}{3} = 0 \tag{2.17}$$

And also, the Friedmann equation

$$H(t)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \tag{2.18}$$

Where c is the speed of light taken to be 1 in a natural unit. It is worth noting that, even if Λ is assumed to be constant, normalized cosmological $\Omega_\Lambda = \Lambda/3H^2$ is time-dependent. The figure below shows how the density parameter varies with cosmic age for the open, flat and closed universe.

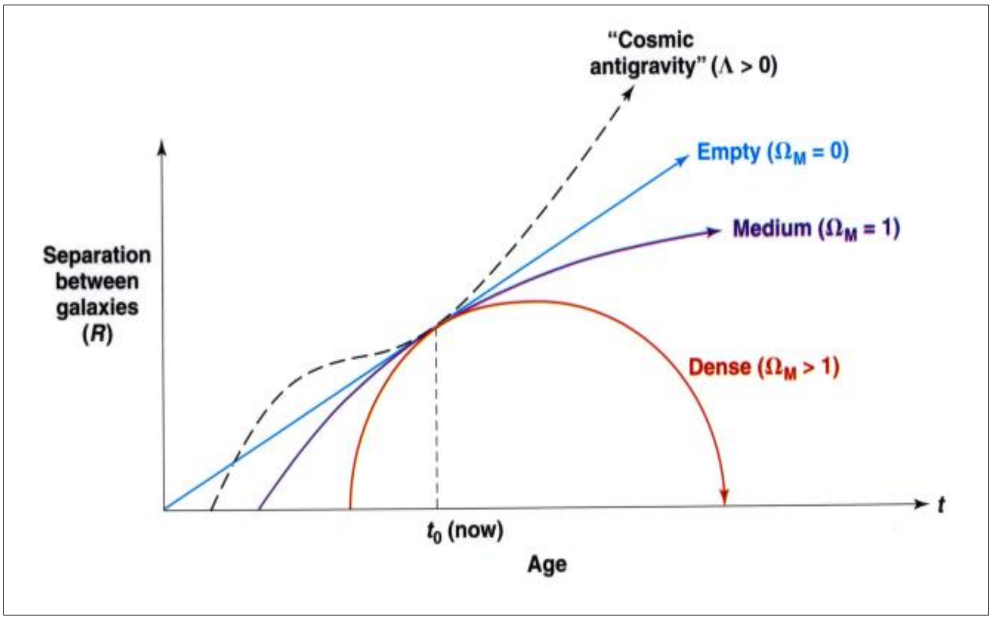

Figure 2.1: the density parameters with cosmic age for open, flat and closed universe.

Chapter 3

NEWTONIAN COSMOLOGY BY LAGRANGIAN FORMALISM

The theory of general relativity that formulates geometry is used to describe modern cosmology, so the cosmological effects are linked with space-time geometry. Differential manifolds; objects used to describe curved space-time are used in this description. On the other hand, mathematical tools like algebraic tensors or continuous group are also involved in the description.

Milne and McCrea [12,13] used principles based on Newtonian Theory that is associated with gravitational phenomena to describe space-time curvature. Using this classical physics approach with the least mathematical difficulty the universe is studied just as with the Einstein Cosmology. The same results are obtained provided that the cosmological principle of homogeneity and isotropy is not violated. In this approach, the Universe's expansion is not dynamically inborn to the universe itself and it is known as a static universe. For this to be complete, Hubble's observations pertaining to the expanding universe should be incorporated. [12-13] approach aimed towards admitting that the observed expansion of the universe is related to the motion of celestial bodies or galaxies in the universe. Hence, cosmic expansion was thought to be an effect of particle motion and therefore no need to introducing curved spacetime of general relativity. Initially, the formulation of the Newtonian Cosmology was done without considering pressure but was included after some decade [14-15]. Also, the term containing cosmological constant can be included in the Newtonian approach.

Cosmological equations in Newtonian cosmology are derived from the equation of motion of particles under the effect of gravitational interaction [16-17]. Hence, the formalism of Lagrange and Hamilton provides the necessary classical equation and integral of motion respectively.

From the generalization of the Copernican principle, the principle of modern cosmology gets its simplification. It asserts that in each cosmological epoch, the universe is even in every point, aside by local irregularities. Thus for determined Newton time, $t =$ constant, the universe is isotropic and homogenous.

3.1 The Equation of cosmology

Equations of Cosmology in cosmic layers comprised a finite large volume of an expanding gas cloud. The galaxies formed the particles that formed this gas. We Expect that the pressure in the cloud is given by $p = \rho(t)$. For null pressure, we assumed that the cosmic gas is dust or matter. The expansion of the universe is governed by two basic equations. Like statements of energy, the total energy (potential and gravitational energy) for expanding galaxies is constant due to their relativistic motion. Hubble's law describes the receding velocity of galaxies to behave a direct proportionality to a distant observer by

$$\frac{da}{dt} \equiv \dot{a} = Ha \qquad (3.1)$$

Where a is the distance of the receding galaxies from an observer and $H(t)$ is the Hubble parameter. This theory is expected to be observed in anisotropic and uniformly expanding Universe.

According to Newton, gravitational force attracts galaxies together and slows down the rate at which the universe expands. Due to homogeneity, distances a can be measured from a galaxy placed at any location; the gravitational force on a galaxy separated by distance a from a mass of the homogeneous mass of the universe embedded in a sphere is equivalent to that at the center of the sphere. Expressed in fixed rectangular coordinates, the kinetic energy is a function of velocity $\dot{a}$, is the motion of the galaxies is in a conservation field, the potential energy varies only with position a. This can be expressed thus:

$$T = T(\dot{a}) \tag{3.2}$$

$$U = U(a) \tag{3.3}$$

Using the Lagrangian which is expressed by the sum of $T(\dot{a})$ and $U(a)$ for a system, the Lagrangian L, is a function of position and velocity L = L(q, $\dot{q}$). The expansion of a moving galaxy of mass m is

$$L(a, \dot{a}) = T - U = \frac{1}{2}m\dot{a}^2 + \frac{GMm}{a} \tag{3.4}$$

Here, we are using the usually generalized coordinates $q_i = a$ and $\dot{q}_i = \dot{a}$ that the total mass of the universe is denoted by M which is taken to be evenly spread around a sphere of radius r. Assuming that there exists a cosmic force on the galaxy whose magnitude is

$$F_\Lambda = \frac{1}{3}\Lambda \mathrm{mR} \tag{3.5}$$

Where Λ represents cosmological constant and there is an extra potential energy term given by

$$U_\Lambda = -\int_0^a F_\Lambda da' = -\frac{1}{6}\Lambda \mathrm{m}a^2 \tag{3.6}$$

From equation (3.4), the Lagrangian transforms to

$$L(a, \dot{a}) = T - U_{eff} = T - U - U_\Lambda = \frac{1}{2}m\dot{a}^2 + \frac{GMm}{a} + \frac{1}{6}\Lambda \mathrm{m}a^2 \tag{3.7}$$

From the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{3.8}$$

We can deduce that

$$\frac{\partial L}{\partial a} = -\frac{GMm}{a^2} + \frac{1}{3}\Lambda \mathrm{ma} \tag{3.9}$$

$$\frac{\partial L}{\partial \dot{a}} = m\dot{a} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) = m\ddot{a} \tag{3.10}$$

Putting equations (3.9) and (3.10) into equation (3.8), the equation of motion for the galaxy becomes

$$\ddot{a} = -\frac{GM}{a^2} + \frac{1}{3}\Lambda \mathrm{a} \tag{3.11}$$

If we consider a spherical mass of density ρ given by

$$M = \frac{4}{3}\pi a^3 \rho \tag{3.12}$$

Then the equation of motion becomes

$$\ddot{a} = -\frac{4}{3}\pi G\rho a + \frac{1}{3}\Lambda \mathrm{a} \tag{3.13}$$

Equation (3.13) is the Newtonian Cosmological form and a is the scale factor relating to the expansion of the Universe. This equation is similar to the Einstein equation derived in general relativity [18].

3.2 Cosmic Differential Equation in Newtonian Form

From the cosmological principle, that time is homogenous, in the inertial frame of reference, the Lagrangian can be described in a closed system and do not explicitly depend on time. In this context, the Lagrangian is also time-independent since the system is influenced by some uniform force field. Therefore, the Hamiltonian H is the only constant of motion defined by

$$H = \left(\frac{\partial L}{\partial \dot{q}}\right)\partial \dot{q} - L \tag{3.14}$$

From equation (3.7), we get

$$E = (m\dot{a})\dot{a} - \frac{1}{2}m\dot{a}^2 - \frac{GMm}{a} - \frac{1}{6}\Lambda \mathrm{m} a^2 \tag{3.15}$$

We have renamed the constant H by the total energy E, which is also a constant of motion. Equation (3.15) can be rewritten as

$$\dot{a}^2 = \frac{C}{a} + \frac{1}{3}\Lambda a^2 - k \tag{3.16}$$

Where $C = 8\pi G\rho a^3/3$ and $k = -2E/m$ are constants. This is the cosmic differential equation that governs the expansion of the universe and a represents the scale factor. Equation (3.16) is the Friedmann equation in the Newtonian form [19]

Chapter 4

MODIFIED NEWTONIAN DYNAMICS

In 1983, the mass discrepancy was detected in stellar systems whenever gravitational acceleration decreases some threshold value [20]. A theory of the MOND was proposed by M. Milgrom to supplement non-baryonic dark matter [21]. According to this theory, for a body accelerating less than $a_0 = 1.2 \times 10^{-8}\ \frac{cm}{s^2}$, gravitation does not agree with the predictions of the Newtonian dynamic [22]. Below this threshold, gravity changes and behave asymptotically as $g = \sqrt{g_N g_0}$, and g_N is Newton's acceleration. As the strong field of the sun becomes dominant in all dynamic processes and accelerations are unnoticeable within the solar system, there is a transition from the Newtonian to the Modified Newtonian Dynamic. In other words, the Newtonian Dynamics becomes invalid below acceleration regimes typically, that of the galaxies.

Aside from [21], there are few literatures showing that though revolutionary, the approach is simple and can explain most galactic features without needing non-baryonic dark matter [22-23]. The dynamics of galactic groups and clusters can also be effectively described by the MOND [24], including gravitational lensings and some approximations [25-26].

4.1 Scales of the MOND Acceleration

Distances between physical bodies to show by a universal dimensionless scale factor that fluctuates just with cosmological time are permitted by the cosmic principles of homogeneity and isotropy. Newton's equation of motion is a tool for deriving Friedmann Equation that gives the evolution of the universe scale factor. If we consider a uniform sphere of radius r, then

$$\ddot{r} = -\frac{GM}{r^2} \tag{4.1}$$

Where M denotes gravitational mass. The static limit of the Einstein equation in the weak field is given by

$$M = \frac{4\pi}{3}(\rho + 3p)r^3 \tag{4.2}$$

ρ is the density of the fluid and p its pressure. Combining equations (4.1) and (4.2) yields

$$\ddot{r} = \frac{4\pi G r}{3}(\rho + 3p) \tag{4.3}$$

By conservation of energy, equation (4.3) gives

$$\frac{d\rho}{(\rho+p)} = -\frac{3dr}{r} \tag{4.4}$$

It is obvious to write $r = Dy$, D denotes a constant length while y depends on the scale factor which is taken to be 1 at the present time. For matter with no pressure $(p = 0)$ $p_r = \frac{1}{3}\rho_r$ for radiation and the vacuum energy density is given by $\rho_v = -p_v = -3\lambda H_0^2/8\pi G$ upon performing integration on (4.3) the evolution of the scale factor is given by a dimensionless Friedmann equation as

$$H^2 = \left(\frac{\dot{y}}{y}\right)^2 = \Omega_0 y^{-3} + \Omega_r y^{-4} - y^{-2}(\Omega_0 + \Omega_r + \lambda - 1) + \lambda \tag{4.5}$$

Here, λ is the cosmological constant which is a dimensionless quantity. The density parameter for non-relativistic matter with present density ρ_0 is

$$\Omega_0 = \frac{8\pi G\rho_0}{3H_0^2} \tag{4.6}$$

For CMB radiation, with a temperature of blackbody radiation T_0, the density parameter is

$$\Omega_r = \frac{8\pi G a T_0^4}{3H_0^2 c^2} \tag{4.7}$$

In equation (4.5), the integration constant is $(\Omega_0 + \Omega_r + \lambda - 1)$ which is evaluated with the curvature of space-time and, h denotes the Hubble parameter.

According to the theory of Modified Newton Dynamics, for accelerations below the threshold $\boldsymbol{a}_0$, the actual gravitational acceleration g relates with Newton's force of gravity g_n, by

$$g\mu(\sigma) = g_n \tag{4.8}$$

Where $\sigma = \frac{g}{a_o}$

$$F = ma\mu(a/a_0) \tag{4.9}$$

$\mu(y)$ is a function that is not specified and approaches unity whenever such that $\mu(y) \to 1 \quad y \gg 1$ also, $\mu(y) = y$ whenever $y \ll 1$. The force of gravity tends to Newtonian force as acceleration increases and low acceleration limits are $= \sqrt{g_n a_0}$. We assumed that change in $\mu(y)$ about the two limits that happen at $y = 1$,[27].

The MOND in low-acceleration is characterized by

$$\ddot{r} = -\left[\frac{4\pi G a_0}{3}(\rho + 3p)r\right]^{1/2} \tag{4.10}$$

Considering the equation of state only for pressureless and non-relativistic matter only, the conservation equation says

$$\rho = \frac{\rho_0}{(r/r_0)^3} \tag{4.11}$$

The region of the sphere has a comoving radius denoted by r_0 and the acceleration equation (4.9) is given by

$$\ddot{r} = -\left[\frac{\Omega_0}{2} H_0^2 r_0^3 a_0\right]^{1/2} r^{-1} \tag{4.12}$$

Taking the integral of this equation gives the Friedmann equation in the form

$$\dot{r}^2 = u_i^2 - [2\Omega_0 H_0^2 r_0^3 a_0]^{1/2} ln(r/r_i) \quad (4.13)$$

with r_i taken as the initial radius of the sphere which expands with velocity u_i. Equation (4.12) reveals recollapsing region characterized by end of the expansion that the expansion at a maximum radius r_m given by

$$r/r_i = e^{q^2} \quad (4.14)$$

Where

$$q^2 = u_i^2 (2\Omega_0 H_0^2 r_0^3 a_0)^{-1/2} \quad (4.15)$$

[28] derived this expression.

4.2 Physical Foundation of the MOND

MOND is a modified Newton's Dynamics which is not linked to a unified theoretical framework which is at present, it's the only shortcoming. Physical problems like non-conserving linear momentum are usually the case when we try to apply Milgrom's inertial idea to the N-body system [28]. Based on this ground, Bekenstein and Milgrom in 1984 proposed Lagrangian-based model as a modified theory of the Newtonian's gravitational theory [29]. If $\emptyset$ denotes a scalar potential, the field action is defined

$$S_f = -\int \left[\rho\emptyset + (8\pi G)^{-1} {a_0}^2 F\left(\frac{\nabla\emptyset^2}{{a_0}^2}\right)\right] d^3r \quad (4.16)$$

Assuming the concept of stationary action, the field equation takes the form

$$\nabla.\left[\mu\left(\frac{|\nabla\emptyset|}{a_0}\right)\nabla\emptyset\right] = 4\pi G\rho \quad (4.17)$$

$\mu(x) = dF/dx^2$ is a function that behaves asymptotically for MOND description. Due to symmetry, this theory does not violate the law of conservation of angular momentum and energy. The motion stars and other compound objects do not depend on their internal acceleration in an external field [29]. Furthermore, in systems, its internal dynamics are independent of its external acceleration.

In general, most physical equations (for example, the Maxwellian equations) becomes invariantly transform as

$$\nabla.\left\{[(\nabla\emptyset)^2]^{D/2-1}\nabla\emptyset\right\} = \alpha_D G\rho \tag{4.18}$$

Equation (4.18) becomes Poisson's equation if $D = 2$, and becomes an equation of the MOND when $D = 3$. That is to say, in three-dimensional space, the Bekenstein-Milgrom field equation is invariant in the MOND limit. Although this modified theory is not covariant it can be founded on a theoretical basis. Also, some MOND phenomena such as the external field can be considered by this theory. This non-linear equation is appearing to have no physical solution; however, due to symmetry, it can have a solution to a simple algorithm.

The numerical method developed by Brada in 1997 was applied by Brada and Milgrom to solve an essential problem of stability of disk galaxies [30]. This method is now used in calculating different effects associated with the external field. Examples of such effect warp in the galactic plane that influences satellites. Brada and Milgrom have also considered how the acceleration field of a big galaxy affects a dwarf satellite. Due to the expansion caused by an external field as it approaches a parent galaxy, tides force the satellite to become vulnerable. As indicated by [31], this hypothesis drives itself to a covariant generalization as a non-linear scalar-tensor hypothesis of gravity.

4.3 Modification of Newtonian Inertia - MOND

In another approach [32], considered MOND analogous to modifying the particle's inertia At levels that are not relativistic, usual particle action ($\int v^2/2\, dt$) is replaced by the somewhat complicated object $A_m S[r(t), a_0]$. A_m is associated with the mass of the particle while S is characterized by a_0 is a function of a particle trajectory

$r(t)$. According to Milgrom, such action should have an accurate limiting property (Newtonian as $a_0 \to 0$ while the Modified Newtonian Dynamics has $a_0 \to \infty$). That is, it is invariant under Galilean transformation.

Cosmology does not have a direct impact of particle motion however, cosmic phenomena such as cosmological constant may impact both cosmology and dynamics of particles; for instance, the interaction of accelerating particles in a vacuum. Since the concept of the cosmological constant is a characteristic of a vacuum, a non-trivial effect of particle acceleration can be of the form, $t_0 \approx c\sqrt{\Lambda}$.

The Unruh radiation principle gives us a glimpse of how it happens. For a uniform accelerating observer in a Minkowski space, the vacuum field as a thermal bath with temperature T is described by

$$kT = \frac{\hbar}{2\pi c} a \tag{4.19}$$

The acceleration a is the gravitational acceleration at the event horizon and analogous to Hawking radiation. For an accelerating observer via de Sitter space, the modified thermal bath is now seen as [33]

$$kT_\Lambda = \frac{\hbar}{2\pi c}\sqrt{a^2 + \frac{c^2\Lambda}{3}} \tag{4.20}$$

Because Unruh's radiation is too tiny, it may not have a direct impact on the field providing the inertia. Hence, the useful quantity for identifying inertia is $\Delta T = T_\Lambda - T$, and we can now write

$$\frac{2\pi c}{\hbar} k\Delta T = a\mu(a/a_0) \tag{4.21}$$

$$\mu(y) = [1 + (2y)^{-2}]^{1/2} - (2y)^{-1} \tag{4.22}$$

where $\quad \mathrm{a}_0 = 2\mathrm{c}\left(\frac{\Lambda}{3}\right)^{1/2}$

This is only a suggestive line of argument and may not be considered of MOND with modified inertia. Once more, this is not a hypothesis of MOND an alteration of inertia, however just an intriguing line of contention. The best theory of inertia would be the one derived from interaction with the vacuum field like in induced gravity in which spacetime curvature modified the vacuum field by providing the needed action the metric field [34].

Chapter 5

NEWTONIAN COSMOLOGY WITH QUANTUM BOUNCE

It was a remarkable achievement when McCrea and Milne applied Newtonian mechanics to obtain the Friedmann equations in Cosmology [13]; something that was only feasible via general relativity. This evolution gave birth to attempts already in the literature to modify the Friedmann Equations. Such modifications could include Modified Friedmann Equations with Quantum Corrections to the Newtonian potential [35-44].

5.1 Friedmann equation in the Newtonian Dynamics

Different approaches have been taken to derive the cosmological Friedmann equation from Newtonian dynamics using different concepts; in the end, they all converge at the same Friedmann equation [37]. The simplest step is to start an original point of view that describes the expansion of the universe. For instance, consider an object such as a galaxy of mass m, revolving around the earth of mass M, the total energy is

$$E = \frac{1}{2} m v^2 - \frac{mMG}{D} \tag{5.1}$$

The real equations of distance and velocity are $D = Rx \quad and \quad v = \dot{R}x$. R(t) is a scale factor and the distance (x) between the center point and a galaxy is an equal one, then the total energy reads

$$E = \frac{1}{2} m \dot{R}^2 - \frac{mMG}{R} \tag{5.2}$$

$$\frac{2E}{m} = \dot{R}^2 - \frac{2MG}{R} \tag{5.3}$$

Since E and m are constant then $= \frac{2E}{m}$. Writing the mass inside the sphere of radius R, as $M = \frac{4\pi}{3} R^3 \rho$, where ρ is the density of the universe the first Friedmann equation is obtained as

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \qquad (5.4)$$

We assumed that the universe has volume (V) and its expansion is triggered by the work done by pressure given by pdV. From the adiabatic $dQ = 0$ the first law of thermodynamic reads

$$dQ = dE + PdV = 0 \qquad (5.5)$$

Volume and energy for the spherical universe is $= \frac{4\pi}{3}(R)^3$, $E = V\rho$. Taking the derivatives of these equations and put into an equation (5.5), the fluid equation becomes

$$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + P) \qquad (5.6)$$

By taking a derivative of the first Friedman equation and plug the fluid equation in it, we get the Second Friedmann equation

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P) \qquad (5.7)$$

5.2 Friedmann equation with quantum bounce correction

Taking gravity as a powerful hypothesis and performing a one-loop graviton calculation, ħ correction to the Newtonian potential has been obtained by a few authors and already formed parts of the literature [38-44]. This potential is of the form

$$\emptyset(r) = -\frac{GM_1M_2}{r} + \frac{G^2M_1M_2\hbar\gamma_q}{r^3c^3} \qquad (5.8)$$

In other cases, the consequence of the quantum correction to the Newtonian potential is given in an alternate structure by

$$\emptyset(r) == -\frac{GM_1M_2}{r}\left[1+\lambda\frac{G(M_1+M_2)}{rc^2}-\tilde{\gamma}\frac{G\hbar}{r^2c^3}+\cdots\right] \qquad (5.9)$$

Where λ and $\tilde{\gamma}$ are constants whose values are taken at the author's satisfaction. For uncertainty of this potential due to the coordinates are not evident some articles have discussed this issue [40-41]. They explained that a redefinition $r \to r' = r(1 + aGM/r)$ may change the parameter λ without affecting the observable. Then we can use equation (5.8) for correction potential. Observations discovered different values for γ_q with different signs such as $\left(\gamma_q = \frac{107}{30\pi}\right)$ this value found by [42], but in 2015 found it with a different sign as $\left(\gamma_q = \frac{-41}{10}\right)$ by [43].

Having introduced the quantum correction, we follow the same procedure afore to obtain the modified equations. The total energy gets another contribution because of the quantum rectification in the Newtonian potential [44]

$$E = \frac{1}{2}m\dot{R}^2 - \frac{mMG}{R} + \frac{G^2Mm\hbar\gamma_q}{R^3c^3} \qquad (5.10)$$

Taking $l_p = \sqrt{\frac{G\hbar}{c^3}}$ (the plank length) and ρ, equation (5.10) transformed into

$$\frac{2E}{mR^2} = \frac{\dot{R}^2}{R^2} - \frac{8\pi G\rho}{3} + \frac{8\pi G\rho l_p^2\gamma_q}{3R^2} \qquad (5.11)$$

Introducing the Hubble parameter and constant curvature (k) the first Friedmann equation with a $\hbar$-correction can be given as

$$H^2 = \frac{8\pi G\rho}{3} - \frac{8\pi G\rho l_p^2\gamma_q}{3R^2} - \frac{k}{R^2} \qquad (5.12)$$

Again, take the derivative of the above equation, and plug it with the fluid equation, the second corrected Friedmann equation reads [44]

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P) + 4\pi G l_P^2\gamma_q\frac{(\rho+P)}{R^2} \qquad (5.13)$$

The equations (5.12) and (5.13) are the quantum corrected Friedmann equations derived within the framework of Newtonian mechanics. The cosmological constant (Λ) model and flat universe $k = 0$ is used to better understanding of similarities and differences between the two standards Friedmann equation.

$$H^2 = \frac{8\pi G\rho}{3} - \frac{8\pi G\rho l_p^2 \gamma_q}{3R^2} + \frac{\Lambda}{3} \qquad (5.14)$$

We note that it is adequate to put $\hbar \to 0$ to recover the Friedman equations from general relativity. Making use of the standard definition $\rho_{cr.(t)} = \frac{3H^2}{8\pi G}$ and $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ the First Friedman equation with the cosmological constant and the $\hbar$ corrections is simply

$$\rho_{cr.} = \rho\left(1 - \frac{l_p^2 \gamma_q}{a^2}\right) + \rho_\Lambda \qquad (5.15)$$

Dividing the energy density by the critical density $(\rho_{cr.})$, we introduce the density parameter $\Omega_i = \frac{\rho_i}{\rho_{cr.}}$ the above equation becomes

$$1 = \Omega_m\left(1 - \frac{l_p^2 \gamma_q}{R^2}\right) + \Omega_\Lambda \qquad (5.16)$$

Also, we defined another form of $\rho_{cr.}$ by dividing $\left(1 - \frac{l_p^2 \gamma_q}{R^2}\right)$, where $\tilde{\rho}_{cr.} = \frac{\rho_{cr.}}{\left(1 - \frac{l_p^2 \gamma_q}{R^2}\right)}$ and $\widetilde{\Omega}_m = \frac{\rho}{\rho_{cr.}}$ then the first Friedmann equation will simply be

$$1 = \widetilde{\Omega}_m + \Omega_\Lambda \qquad (5.17)$$

The equation of state of radiation is

$$\frac{1}{a^4} = \frac{\rho}{\rho_0} \qquad (5.18)$$

With $a = R/R_0$ and $\beta' = l_p^2 \gamma_q / R_0^2$, it is simple to get the first Friedmann equation

$$H^2 = \frac{8\pi G\rho}{3}\left(1 - \beta' \frac{R_0^2}{R^2}\right) = \frac{8\pi G\rho}{3}\left(1 - \beta' \frac{1}{a^2}\right) = \frac{8\pi G\rho}{3}\left(1 - \beta' \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}}\right) \qquad (5.19)$$

On account of positive β' it allows to introduce a critical energy

$$\tilde{\rho}_{cr.} = \frac{\rho_0}{(\beta')^2} = \frac{\rho_0 R_0^4}{l_p^4 \gamma_q} = constant \tag{5.20}$$

For if $\rho = \tilde{\rho}_{cr.}$ then$H = 0$. Expressing in terms of loop quantum gravity we rewrite

$$H^2 = \frac{8\pi G\rho}{3}\left(1 - \frac{\rho}{\rho_{cr}}\right) \tag{5.21}$$

Going ahead, we discuss some standard solutions to the Friedmann equations via the tool of the classical-quantum universe without taking into account the effect of quantum corrections. Considering the case where $\gamma_q = 0$, Λ=0 and $k = 0$, i.e the universe filled with radiation. We can show that the solution to the first Friedmann equation (5.12) for $a = \frac{R}{R_0}$ is $\frac{(a^2-1)}{2} = \pm\tau \equiv \sqrt{\frac{8\pi G\rho_0}{3}}(t - t_0)$ with the two branches corresponding to

$$\left.\begin{array}{ll} a_+ = \sqrt{2\tau + 1}, & \tau > -1/2 \\ a_- = \sqrt{1 - 2\tau}\,, & 1/2 > \tau \end{array}\right| \tag{5.22}$$

In the matter case the first Friedmann equation (5.12) for $a = \frac{R}{R_0}$ is $\frac{(a^3-1)}{3} = \pm\tau \equiv \sqrt{\frac{8\pi G\rho_0}{3}}(t - t_0)$ where $R_0^{3/2} = 1$ the two branches read as

$$\left.\begin{array}{l} a_+ = (3\tau + 1)^{1/3} \\ a_- = (1 - 3\tau)^{1/3} \end{array}\right| \tag{5.23}$$

Considering the case where$\gamma_q = 0$, Λ=0 and$k = 1$, is a universe dominated with radiation. Since H_0^2 can be zero which corresponds to a local minimum, the solution of the first friedmann equation (5.12) given by

$$\left.\begin{array}{l} a_-(\tau) = \sqrt{-\frac{3\tau^2}{8\pi G\rho_0 R_0^2} - \frac{2\tau}{R_0^2}\sqrt{R_0^4 - \frac{3R_0^2}{8\pi G\rho_0}} + 1} \\ a_+(\tau) = \sqrt{-\frac{3\tau^2}{8\pi G\rho_0 R_0^2} + \frac{2\tau}{R_0^2}\sqrt{R_0^4 - \frac{3R_0^2}{8\pi G\rho_0}} + 1} \end{array}\right| \tag{5.24}$$

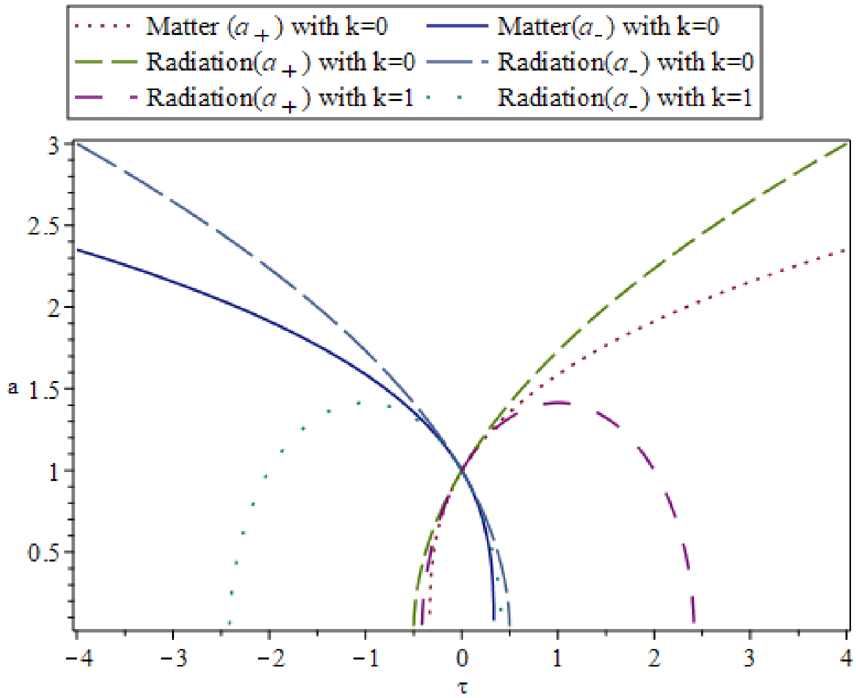

Figure 5.1: cosmological solution for the scale factor from the standard Friedmann equations.

The measure of the present distance R_0, is restricted within the limit $R_0 \geq \left(\frac{3}{8\pi G\rho_0}\right)^{1/2}$ and the real distance R is $R \leq R_{max}$. Where $R_{max} = \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} {R_0}^2$, causing the Hubble Parameter to vanish. We obtain plots from equations (5.22), (5.23) and (5.24) as shown in figure 5.1.According to the graph showed that we have positive and negative proper time $(\pm\tau)$, it denotes the cosmic time which before and after the Big Bang. It is observed that the universe is expanding in both the cases of radiation and matter with curvature k = 0. When the curvature increases to k = 1, the universe collapses for radiation dominated case.

5.3 Newtonian Quantum Universe

To discover the impact of the new term proportional $(\hbar)$ in the Friedmann, equation let uses the first an equation of state from the energy conservation equation [44].

$$P = \omega\rho \tag{5.25}$$

ω is the number denoted by $\omega = (\gamma - 1)$ and γ is different from γ_q , the standard solution for energy density (ρ) in terms of R is given

$$\rho(R) = \frac{\rho_0}{a^{3\gamma}} \tag{5.26}$$

When $= R/R_0$, and $R = R_0$ the energy density $\rho(R_0) = \rho_0$. Putting this into the First Friedmann equation with $k = 0$, reads

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho(R) - \frac{8\pi G l_p^2 \gamma_q}{3R^2}\rho(R) \tag{5.27}$$

If $\varphi = l_p^2 \gamma_q$ and multiply through with R^2, we get

$$\dot{R}^2 = \frac{8\pi G \rho_0 R_0^{3\gamma}}{3R^{3\gamma}}[R^2 - \beta] \tag{5.28}$$

By taking integral form this reads

$$t - t_0 = \pm \frac{R_0^{-3\gamma/2}}{\sqrt{\frac{8\pi G\rho_0}{3}}} \int_{R_0}^{R} \frac{\bar{R}^{3\gamma/2}}{\sqrt{\bar{R}^2 - \beta}} d\bar{R} \tag{5.29}$$

The solution of this integration strongly depends on the sign of the equation of state (γ) and β which is the same sign of γ_q. To discuss this solution needs some different cases separately.

Case $\boldsymbol{\beta} < 0$:

Radiation dominated $(\gamma = 4/3)$.In this situation, the solution can be given by a function namely

$$t - t_0 = \pm \frac{R_0^{-2}}{\sqrt{\frac{8\pi G\rho_0}{3}}} \int_{R_0}^{R} \frac{R^2}{\sqrt{R^2 + \beta}} dR \tag{5.30}$$

By taking integrate it becomes

$$\tau = \pm \frac{1}{2R_0^2}\left[\begin{matrix} -R_0\sqrt{R_0^2 + |\beta|} + \\ \beta \ln\left(R_0 + \sqrt{R_0^2 + |\beta|}\right) + R\sqrt{R^2 + |\beta|} - \beta \ln\left(R + \sqrt{R^2 + |\beta|}\right) \end{matrix}\right] + D \tag{5.31}$$

Where $\sqrt{\frac{8\pi G\rho_0}{3}}(t-t_0)=\tau$, and D takes care of the initial value $R(t_0)=R_0$. If $\beta' = \beta/R_0^2$ and put it into the above equation with the initial value, it reads

$$\tau = \pm\frac{1}{2}\left[\sqrt{a^2+|\beta'|}-\sqrt{1+|\beta'|}\right]\mp\frac{1}{2}|\beta'|\ln\left[\frac{a+\sqrt{a^2+|\beta'|}}{1+\sqrt{1+|\beta'|}}\right] \tag{5.32}$$

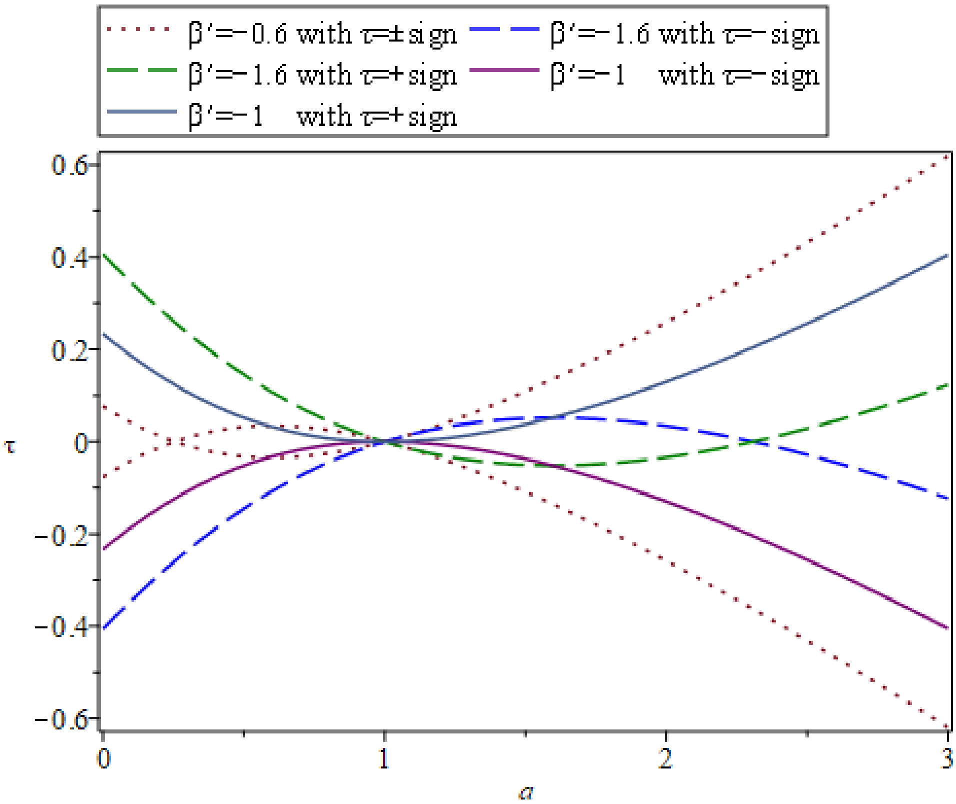

Figure 5.2: cosmological solution for the scale factor from the modified Friedmann equation, for the radiation case whenβ′<0.

Using equations (B.1, B.2, and B.3) as shown in the appendix, we obtain a solution as shown in figure (5.2) Due to the $\pm$ signs in the equation, we obtain two different branches. As shown in the figure, they both converge at $a = 1$ and $\tau = 0$.from present time $a_0 = 1$, the universe divided for two parts one being a mirror image of the other. For values of the $|\beta'| > 1$, we obtain a singular universe starting at $a = 0$ and ending at some $a_{max} > a_0$. For $|\beta'| < 1$,we get always a non-singular universe , the expanding universe starts at a non-zero value $a_{min} < a_0$ and expanding universe. The universe has a Critical value at $\beta' = -1$ after which it will either expand or collapse.

Dust filled Universe ($\gamma = 1$):

The integral to solve reads

$$t - t_0 = \pm \frac{R_0^{-3}}{\sqrt{\frac{8\pi G\rho_0}{3}}} \int_{R_0}^{R} \frac{\bar{R}^3}{\sqrt{\bar{R}^2 - \beta}} d\bar{R} \tag{5.33}$$

The above equation can be rewritten in terms of ($a = R/R_0$) as

$$\left.\begin{aligned} \sqrt{\frac{8\pi G\rho_0}{3}}(t - t_0) = \pm f(a) \\ f(a) = \int_1^a \sqrt{\frac{\tau^3}{\tau^2 + |\beta'|}} d\tau \quad , \quad \beta' = \beta/R_0^2 \end{aligned}\right| \tag{5.34}$$

To obtain a solution of the integral in the above equation, we change it to

$$f(a) = \int_1^a \frac{\tau^2}{\sqrt{\tau(\tau^2 + |\beta'|)}} d\tau \tag{5.35}$$

To integrate this equation look the equation (B.4) is calculated by some steps as shown in the appendix, then the above integration becomes as

$$f = \frac{2\sqrt{a^3 + a|\beta'|}}{3} - \frac{I|\beta|^{3/2}\sqrt{\frac{-I(a+I\sqrt{|\beta'|})}{\sqrt{|\beta|}}}(\sqrt{2})\sqrt{\frac{I(a-I\sqrt{|\beta'|})}{\sqrt{|\beta'|}}\left(\sqrt{\frac{Ia}{\sqrt{|\beta'|}}}\right)} EllipticF\left(\frac{-I(a+I\sqrt{|\beta'|})}{\sqrt{|\beta'|}}, \frac{\sqrt{2}}{2}\right)}{3\sqrt{a^3 + a|\beta'|}} -$$

$$\frac{2\sqrt{1+|\beta'|}}{3} + \frac{I|\beta|^{3/2}\sqrt{\frac{-I(1+I\sqrt{|\beta'|})}{\sqrt{|\beta'|}}}(\sqrt{2})\sqrt{\frac{I(1-I\sqrt{|\beta'|})}{\sqrt{|\beta'|}}\left(\sqrt{\frac{I}{\sqrt{|\beta'|}}}\right)} EllipticF\left(\frac{-I(1+I\sqrt{|\beta|})}{\sqrt{|\beta'|}}, \frac{\sqrt{2}}{2}\right)}{3\sqrt{1+|\beta'|}} \tag{5.36}$$

After substitute the value of $(\beta' = -0.3)$ into the above equation the result given by f_1 as determined in the appendix by equation B.5.

$$f_1$$
$$= \frac{2\sqrt{a^3 + 0.3a}}{3}$$
$$- \frac{1}{\sqrt{a^3 + 0.3a}} \begin{pmatrix} 0.1351200155I\sqrt{-1(a + 0.54772255I)}\sqrt{2}\sqrt{I(a - 0.5477225575I)}\sqrt{Ia} \\ EllipticF\left(1.351200155\sqrt{-1(a + 0.54772255751)}, \frac{\sqrt{2}}{2}\right) \end{pmatrix}$$
$$- 0.76011699500 + (0.039513822205 + 0.17714622861I)\sqrt{2}, - \frac{2\sqrt{a^3 + 0.3a}}{3}$$
$$+ \frac{1}{\sqrt{a^3 + 0.3a}} \begin{pmatrix} 0.1351200155I\sqrt{-1(a + 0.54772255I)}\sqrt{2}\sqrt{I(a - 0.5477225575I)}\sqrt{Ia} \\ EllipticF\left(1.351200155\sqrt{-1(a + 0.54772255751)}, \frac{\sqrt{2}}{2}\right) \end{pmatrix}$$
$$+ 0.76011699500 - (0.039513822205 + 0.17714622861I)\sqrt{2}$$

For the value of $(\beta' = -0.6 \text{ and} - 1.1)$ in to the equation 5.36 , the result can be obtained as showed by the equations (B.6, and B.7) , then these two equations we explained details in the appendix, to get the solution for these elliptic integral, the MAPLE codes used for the plot these elliptic integral equations as experienced in the below figure (5.3).

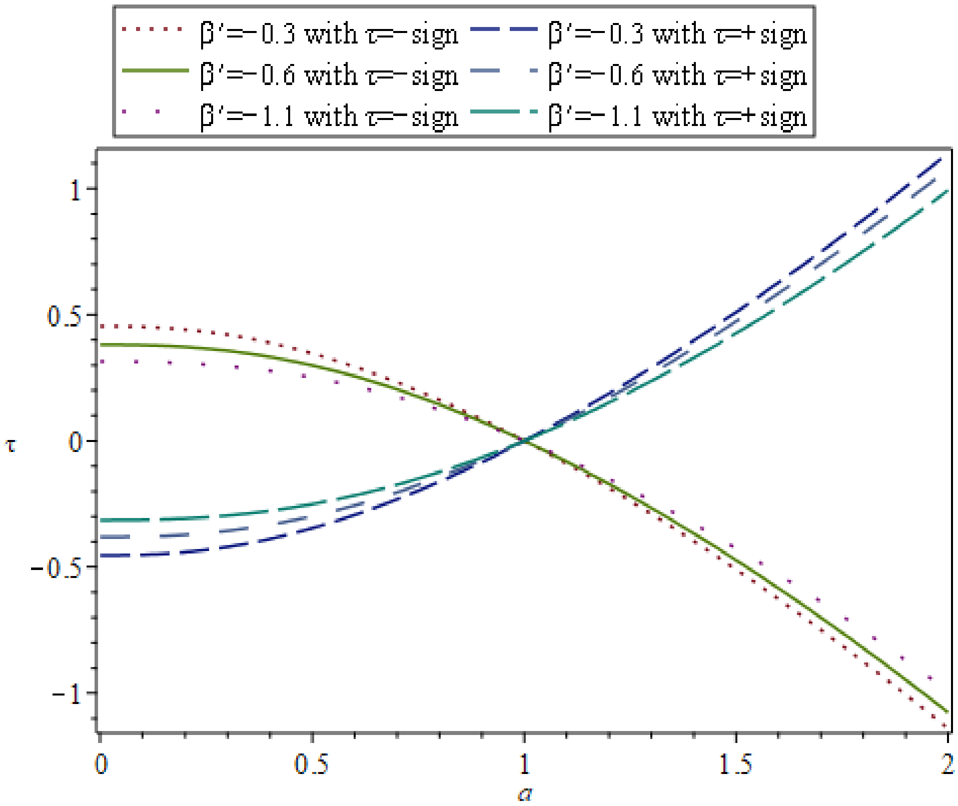

Figure 5.3: cosmological solution for the scale factor from the modified Friedmann equation, for the Dust case whenβ′<0.

From figure (5.3), it is seen that for both ±signs, all the branches start from zero. We, conclude that the all universe is singular as they start and end at zero. Then, the universe for this case of Dust dominated universe is singularity while all of them started from zero. To avoid singularity in the case of -β', the equation of state is a need.

Case $\beta > 0$

Radiation ($\gamma = 4/3$).The integral form becomes

$$t - t_0 = \pm \frac{R_0^{-2}}{\sqrt{\frac{8\pi G\rho_0}{3}}} \int_{R_0}^{R} \frac{R^2}{\sqrt{R^2-\beta}} dR \qquad (5.37)$$

By taking integral, it becomes

$$\tau = \pm \frac{1}{2R_0^2}\left[-R_0\sqrt{R_0^2 - \beta} - \beta \ln\left(R_0 + \sqrt{R_0^2 - \beta}\right) + R\sqrt{R^2 - \beta} + \beta \ln\left(R + \sqrt{R^2 - \beta}\right)\right] + C \qquad (5.38)$$

C takes care of the initial value$R(t_0) = R_0$. If $\beta' = \beta/R_0^2$ and put it into the above equation with the initial value, it reads

$$\tau = \pm\frac{1}{2}\left[\sqrt{a^2 - \beta'} - \sqrt{1 - \beta'}\right] \pm \frac{1}{2}\beta' \ln\left[\frac{a + \sqrt{a^2 - \beta'}}{1 + \sqrt{1 - \beta'}}\right] \qquad (5.39)$$

By putting the β' value into the above equation the equations (B.8, B.9, and B.10) can be obtained and solve by the below figure.

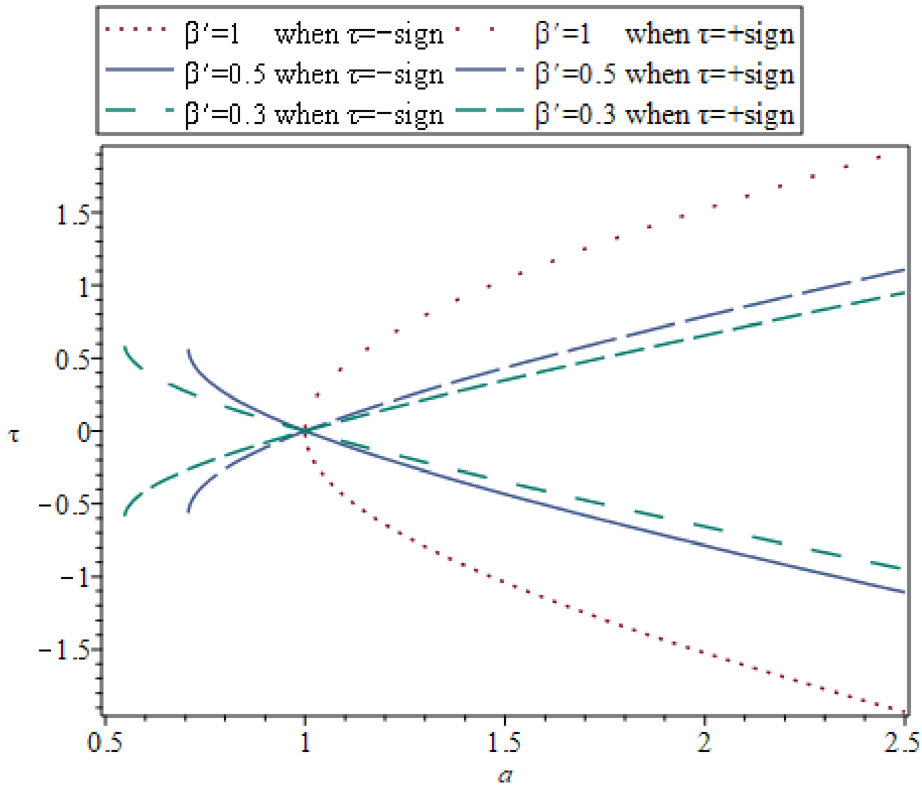

Figure 5.4: cosmological solution for the scale factor from the modified Friedmann equation, for the radiation case whenβ′>0.

The case of generality is one in which we impose $\gamma_q > 0$ such that $R^2 \geq {R^2}_{min} \equiv \beta'$ or $a^2 \geq \beta'$. This is explicitly shown in figure 5.4. From equation (5.39), we obtain only if $\beta' \leq 1$, we infer that all universes either originate or terminate at R_{min} as long as β' is less than 1. For $\beta' = 1$, R_{min} corresponds to a point of the local minimum that joins two branches with different signs at $a = 1$ to give a unique solution. As can be seen from the figure, the universe comes from infinity to a minimum point and expands to infinity.

Dust ($\gamma = 1$):

For this case, the integral to solve reads

$$t - t_0 = \pm \frac{R_0^{-3}}{\sqrt{\frac{8\pi G \rho_0}{3}}} \int_{R_0}^{R} \frac{\bar{R}^3}{\sqrt{\bar{R}^2 - \beta}} d\bar{R} \tag{5.40}$$

The above equation can be written in terms of ($a = R/R_0$) as

$$\left.\begin{aligned} &\sqrt{\frac{8\pi G\rho_0}{3}}(t - t_0) = \pm Q(a) \\ &Q(a) = \int_1^a \sqrt{\frac{\tau^3}{\tau^2 - \beta'}} d\tau \quad , \quad \beta' = \beta / R_0^2 \end{aligned}\right| \tag{5.41}$$

So as to explain the necessary showing up in the above expression let change it as follow

$$Q(a) = \int_1^a \frac{\tau^2}{\sqrt{\tau(\tau^2 - \beta')}} d\tau \tag{5.42}$$

To integrate this equation look the equation (B.11) is calculated by some steps as shown in the appendix, then the above integration becomes as

$$Q \ = \frac{2\sqrt{a^3 - a\beta}}{3} + \frac{\beta^{3/2}\sqrt{\frac{(a+\sqrt{\beta})}{\sqrt{\beta}}}\sqrt{-\frac{(a-\sqrt{\beta})}{\sqrt{\beta}}}\left(\sqrt{\frac{-a}{\sqrt{\beta}}}\right) EllipticF\left(\sqrt{\frac{(a+\sqrt{\beta})}{\sqrt{\beta}}}, \frac{\sqrt{2}}{2}\right)}{3\sqrt{a^3 + a|\beta'|}} - \frac{2\sqrt{1-\beta}}{3} -$$

$$\frac{\beta^{3/2}\sqrt{\frac{(1+\sqrt{\beta})}{\sqrt{\beta}}}\sqrt{\frac{-2(1-\sqrt{\beta})}{\sqrt{\beta}}}\left(\sqrt{\frac{-1}{\sqrt{\beta}}}\right) EllipticF\left(\sqrt{\frac{(1+\sqrt{\beta})}{\sqrt{\beta}}}, \frac{\sqrt{2}}{2}\right)}{3\sqrt{1-\beta}} \tag{5.43}$$

After substitute, the β' value into the above equation as showed in the equations (B.12, B.13, and B.14), then the solution of these elliptic integral equations showed in the below figure (5.5).

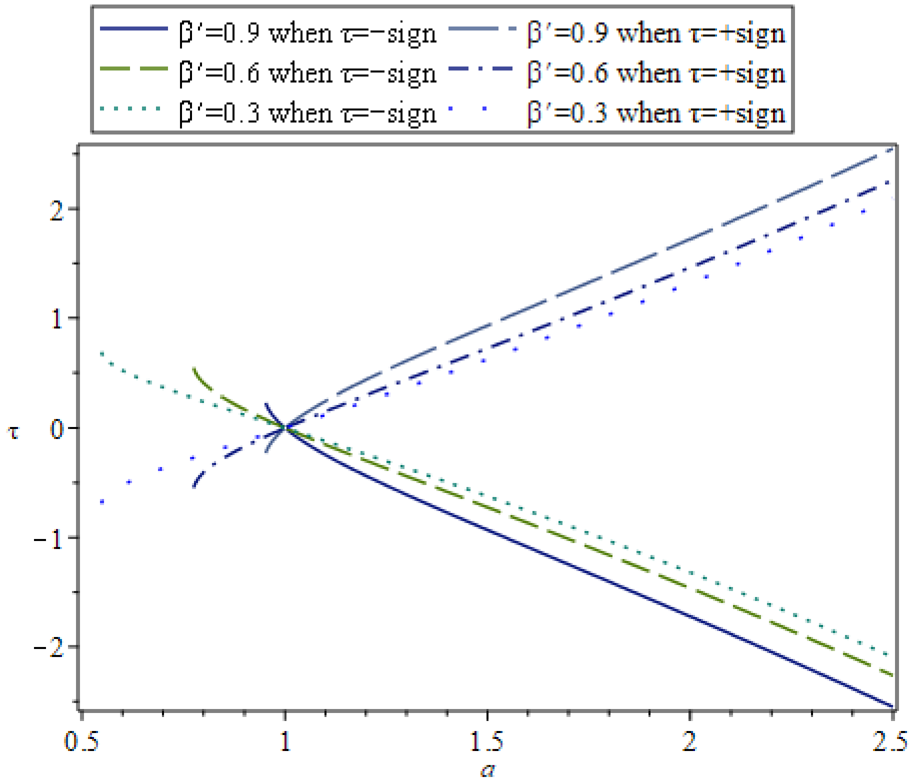

Figure 5.5: cosmological solution for the scale factor from the modified Friedmann equation, for the Dust case when β′>0.

Figure 5.5 is the case where every value of β' is positive. As shown, the interpretation is similar to the radiation case in figure 5.4 when $\beta' = 1$ was considered. Then, changing the equation of state doesn't have effect on this case.

5.4 The case with a cosmological constant

The quantum correction of Friedmann equations has been calculated in the flat curvature [44]. Here, the first Friedmann equations with the cosmological constant read

$$H^2 = \frac{8\pi G\rho}{3} - \frac{8\pi G\rho l_p^2 \gamma_q}{3R^2} + \frac{\Lambda}{3} \tag{5.44}$$

Put the energy density $\rho = \rho_0 a^{-3\gamma}$ when $a = R/R_0$ into the above Friedmann equations, the integral form reads

$$t - t_0 = \pm \int_{R_0}^{R} \frac{1}{\sqrt{\frac{8\pi G\rho_0 R_0^{3\gamma}}{3R^{3\gamma}}(R^2-\beta)+\frac{\Lambda R^2}{3}}} dR \tag{5.45}$$

This integral form can be calculated in some cases:

In the radiation case where ($\gamma = 4/3$):

This is the case with non-zero cosmological constant and the integral becomes

$$t - t_0 = \pm \frac{1}{R_0^2} \int_{R_0}^{R} \frac{R^2}{\sqrt{\frac{8\pi G\rho_0}{3}(R^2 - \beta) + \frac{\Lambda R^6}{3R_0^4}}} dR \tag{5.46}$$

If $\varphi' = \varphi/R_0^2$ and $\rho_{vac} \equiv \Lambda/8\pi G$, the integral can be rewritten in terms of $a(t)$

$$\left. \begin{aligned} I(a) &= \int_1^a g(\tau) d\tau \\ where \quad g(\tau) &= \frac{\tau^2}{\sqrt{\tau^2 - \beta' + \epsilon\tau^6}} \end{aligned} \right] \tag{5.47}$$

Where $\sqrt{\frac{8\pi G\rho_0}{3}}(t - t_0) = \pm I(a)$ and$\epsilon = \rho_{vac}/\rho_0$. The function of $g(\tau)$ can be expanded in the very small parameter ϵ. This series expansion calculated details in the appendix from equation (B.15). Thus, it becomes

$$g(\tau) = \frac{\tau^2}{\sqrt{\tau^2 - \beta'}} - \frac{\tau^8}{\sqrt{(\tau^2 - \beta')^3}} \frac{\epsilon}{2} + O(\epsilon^2) \tag{5.48}$$

Taking the integrals for the first terms from the above equation, it reads

$$Q_1(a) = \int_1^a \frac{\tau^2}{\sqrt{\tau^2 - \beta'}} d\tau = +\frac{1}{2}\left[a\sqrt{a^2 - \beta'} - \sqrt{1 - \beta'}\right] + \frac{1}{2}\beta' \ln\left[\frac{a + \sqrt{a^2 - \beta'}}{1 + \sqrt{1 - \beta'}}\right] \tag{5.49}$$

And the integral for the second terms of the equation (5.48), it shows

$$W_1(a) = -\frac{1}{2}\int_1^a \frac{\tau^8}{\sqrt{(\tau^2 - \varphi')^{3/2}}} d\tau = \frac{35}{32}(\beta')^3 \times \ln\frac{1 + \sqrt{1 - \beta'}}{a + \sqrt{a^2 - \beta'}} - \left[\frac{a^6}{3} + \frac{7\beta' a^4}{12} + \right.$$

$$\left. \frac{35(\beta')^2 a^2}{24} - \frac{35(\beta')^3}{8}\right]\left(\frac{a}{4\sqrt{a^2 - \beta'}}\right) - \frac{1}{4\sqrt{1 - \beta'}}\left[\frac{35(\beta')^3}{8} - \frac{35(\beta')^2}{24} - \frac{7\beta'}{12} - \frac{1}{3}\right] \tag{5.50}$$

These two integrations calculated details in the appendix (B.16 and B.17).The integration function of $I(a)$ is given at the first order in ϵ by the following expression

$$I(a) = \pm[Q_1(a) + W_1(a)\epsilon] + O(\epsilon^2) \tag{5.51}$$

The case of dust ($\gamma = 1$).In this case:

the integration of (5.45), it becomes

$$\left.\begin{array}{r} S(a) = \int_1^a f(\tau)d\tau \\ where \quad f(\tau) = \sqrt{\frac{\tau^3}{\tau^2-\beta'+\epsilon\tau^5}} \end{array}\right] \tag{5.52}$$

Where $\sqrt{\frac{8\pi G\rho_0}{3}}(t-t_0) = \pm S(a)$ and$\epsilon = \rho_{vac}/\rho_0$. Expanding the function of $f(\tau)$ in the very small parameter ϵ, reads

$$f(\tau) = \sqrt{\frac{\tau^3}{\tau^2-\beta'}} - \frac{\tau^5}{2(\tau^2-\beta')}\sqrt{\frac{\tau^3}{\tau^2-\beta'+}}\frac{\epsilon}{2} + O(\epsilon^2) \tag{5.53}$$

Integral in the first term of the $f(\tau)$ function reads

$$\int_1^a \sqrt{\frac{\tau^3}{\tau^2-\beta'}}\, d\tau = \int_1^a \frac{\tau^2}{\sqrt{\tau(\tau^2-\beta')}} d\tau = N_1(a) \tag{5.54}$$

And integral for the second terms is

$$-\frac{1}{2}\int_1^a \frac{\tau^7}{(\tau^2-\beta')\sqrt{\tau(\tau^2-\beta')}} d\tau = M_1(a) \tag{5.55}$$

The way to find $N_1(a)$ and $M_1(a)$ is long, then they integrated details in the appendix where they showed in the equations (B.18 and B.19). Then the solution becomes

$$S(a) = \pm[N_1(a) + M_1(a)\epsilon] + O(\epsilon^2) \tag{5.56}$$

5.5 Comparison of loop quantum gravity with Friedmann equation

Consider the mini-superspace method to deal with classical general relativity for the flat case. The Ashtekar variables, $c\ and\ p$, are given by the poison bracket $\{c, p\} = \frac{8}{3}\pi G\beta_{BI}$, when G is Newton's constant and β_{BI} is the barber-Immirzi parameter, the gravitational Hamiltonian is

$$\mathcal{H}_G = \frac{-6c^2}{\beta_{BI}^2}\sqrt{|P|} \tag{5.57}$$

Where $a^2 = P\ and\ c = \dot{a}$. The Hamiltonian constraint for a massless and free scalar field showed by

$$\mathcal{H}_\emptyset = \frac{8\pi G P_\emptyset^2}{\sqrt{|P|^3}} \tag{5.58}$$

The Hubble parameter in this comparison is equal to $H = \dot{P}/2P$ and the matter density for the scale field as $= P_\emptyset^2/2|P|^3$. From the usual Friedmann equation get the total Hamiltonian constraint as $16\pi G(\mathcal{H}_G + \mathcal{H}_\emptyset)$,

$$H^2 = \frac{8\pi G\rho}{3} \tag{5.59}$$

According to this equation the volume of the universe $V = 0$ at $t = 0$, it suggests the usual big-bang singularity. The significant point is that the equation of motion determined from the Hamiltonian effective showed by $\dot{P} = \{P, \mathcal{H}_{eff}\}$, can be expressed as a modified Friedmann equation, in the form

$$H^2 = \frac{8\pi G\rho}{3}\left(1 - \frac{\rho}{\rho_{cr}^{LQG}}\right) \tag{5.60}$$

The critical density is equal as

$$\left.\begin{aligned} \rho_{cr}^{LQG} &= \frac{3}{8\pi G\beta_{BI}^2\mu_0^2} \approx 0.41\ \rho_P \\ and \quad \rho_{cr}^{LQG} &= \frac{\sqrt{3}}{32\pi^2 G\beta_{BI}^3 l_p^2} \approx 0.41\ \rho_P \end{aligned}\right] \tag{5.61}$$

Where $\beta_{BI} \approx 0.2375$, as expected by black hole physics. ρ_P is the Plank density. In the restriction $\mu_0 \to 0$, which identity to $G\hbar \to 0$, the critical energy density reads singularity and the classical singularity appears.

The interesting point is that there is a non-singular universe in the modified Friedmann equation. Therefore, the critical energy density does not denote $\dot{a}$, it means that the universe bounces.

$$H^2 = \frac{8\pi G\rho}{3}\left(1 - \frac{\beta'}{a^2}\right) \tag{5.62}$$

Equation (5.60) calls it the LQG-corrected Friedmann equation with some similarities and differences are present. For positive γ_q, as explained in the equation (5.19), with the radiation case the critical density can be gotten, but the critical density is different as obtained from the Friedmann equation. For negative γ_q, the equation (5.62) it has given different sign which appears in it, let's compare it with equation (5.60), directly there is not obvious between this comparison for the critical density [45-46].

Chapter 6

CONCLUSION

The rate at which our universe is expanding is studied with the Friedmann equation by determining the scale factor for a particular universe under consideration. Many authors in the works of literature have attempted to modify the Friedmann Equations, depending on their interest. Here, we applied the quantum correction to the Newtonian potential and used it to derived the Friedmann equations within the Newtonian formalism to obtain the corrected Friedmann equations [47]. As it is with other models of the quantum universe, the choice of the sign also plays an important role in this corrected model. As a matter of fact, a collapsing universe always bounces off a minimum length that varies directly with the Planck length then begins to expand again [44].

To understand and describe the behavior of the universe under this modification, we plotted graphs of the invariant τ with the scale factor of the universe for various values of β'. We observed that the quantum correction effect depends on the sign in the correction. The MAPLE codes used for the plot are shown in the appendix.

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APPENDIX

Maple Codes

in the Radiation case $(\beta < 0)$, After put the values of $(\beta' = -0.6, -1\ and\ -1.6)$ into equation (5.32), we get tau equations with positive and negative signs.

$subs(\mathrm{beta} = 0.6, \mathrm{tau})$

$$\frac{\sqrt{a^2+0.6}}{2} - 0.6324555320 - \frac{0.6 \cdot \ln\left(0.4415184401\,a + 0.4415184401\sqrt{a^2+0.6}\right)}{2}$$

$$\tau 1 := \frac{\sqrt{a^2+0.6}}{2} - 0.6324555320 - 0.3000000000\ln\left(0.4415184401\,a + 0.4415184401\sqrt{a^2+0.6}\right), -\frac{\sqrt{a^2+0.6}}{2} + 0.6324555320 + 0.3000000000\ln\left(0.4415184401\,a + 0.4415184401\sqrt{a^2+0.6}\right) \tag{B.1}$$

$subs(\mathrm{beta} = 1, \mathrm{tau})$

$$\frac{\sqrt{a^2+1}}{2} - \frac{\sqrt{2}}{2} - \frac{1 \cdot \ln\left(\frac{a+\sqrt{a^2+1}}{1+\sqrt{2}}\right)}{2}$$

$$\tau 2 := \frac{\sqrt{a^2+1}}{2} - \frac{\sqrt{2}}{2} - \frac{\ln\left(\frac{a+\sqrt{a^2+1}}{1+\sqrt{2}}\right)}{2}, -\frac{\sqrt{a^2+1}}{2} + \frac{\sqrt{2}}{2} + \frac{\ln\left(\frac{a+\sqrt{a^2+1}}{1+\sqrt{2}}\right)}{2} \tag{B.2}$$

$subs(\mathrm{beta} = 1.6, \mathrm{tau})$

$$\frac{\sqrt{a^2+1.6}}{2} - 0.8062257750 - \frac{1.6 \cdot \ln\left(0.3827822185\,a + 0.3827822185\sqrt{a^2+1.6}\right)}{2}$$

$$\tau 3 := \frac{\sqrt{a^2+1.6}}{2} - 0.8062257750 - 0.8000000000\ln\left(0.3827822185\,a + 0.3827822185\sqrt{a^2+1.6}\right), -\frac{\sqrt{a^2+1.6}}{2} + 0.8062257750 + 0.8000000000\ln\left(0.3827822185\,a + 0.3827822185\sqrt{a^2+1.6}\right) \tag{B.3}$$

in the Dust case $(\beta < 0)$, by taking integrate for the equation (5.35), and substitute the integral amplitude value of $(\tau = 1\ and\ a)$ into the integration equation, it reads as

$$\frac{\mathrm{tau}^2}{\sqrt{\mathrm{tau}\cdot\left(\mathrm{tau}^2 + |\mathrm{beta}|\right)}}$$

$$\frac{\tau^2}{\sqrt{\tau\left(\tau^2+|\beta|\right)}}$$

$$f := int\left(\frac{\tau^2}{\sqrt{\tau\left(\tau^2+|\beta|\right)}}, \text{tau}\right)$$

$$f := \frac{2\sqrt{\tau^3+|\beta|\,\tau}}{3} - \frac{1}{3\sqrt{\tau^3+|\beta|\,\tau}}\left(\mathrm{I}\,|\beta|^{3/2}\sqrt{\frac{-\mathrm{I}\left(\tau+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{2}\sqrt{\frac{\mathrm{I}\left(\tau-\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{\frac{\mathrm{I}\,\tau}{\sqrt{|\beta|}}}\right.$$
$$\left.\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(\tau+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}, \frac{\sqrt{2}}{2}\right)\right)$$

$$L2 := subs(\text{tau}=a, f)$$

$$L2 := \frac{2\sqrt{a^3+|\beta|\,a}}{3} - \frac{1}{3\sqrt{a^3+|\beta|\,a}}\left(\mathrm{I}\,|\beta|^{3/2}\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{2}\sqrt{\frac{\mathrm{I}\left(a-\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{\frac{\mathrm{I}\,a}{\sqrt{|\beta|}}}\right.$$
$$\left.\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}, \frac{\sqrt{2}}{2}\right)\right)$$

$$L1 := subs(\text{tau}=1, f)$$

$$L1 := \frac{2\sqrt{1+|\beta|}}{3} - \frac{1}{3\sqrt{1+|\beta|}}\left(\mathrm{I}\,|\beta|^{3/2}\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{2}\sqrt{\frac{\mathrm{I}\left(1-\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{\frac{\mathrm{I}}{\sqrt{|\beta|}}}\right.$$
$$\left.\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}, \frac{\sqrt{2}}{2}\right)\right)$$

$$f := L2 - L1$$

$$f := \frac{2\sqrt{a^3+|\beta|\,a}}{3} - \frac{1}{3\sqrt{a^3+|\beta|\,a}}\left(\mathrm{I}\,|\beta|^{3/2}\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{2}\sqrt{\frac{\mathrm{I}\left(a-\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{\frac{\mathrm{I}\,a}{\sqrt{|\beta|}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}},\frac{\sqrt{2}}{2}\right)\right) - \frac{2\sqrt{1+|\beta|}}{3} + \frac{1}{3\sqrt{1+|\beta|}}\left(\mathrm{I}\,|\beta|^{3/2}\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{2}\sqrt{\frac{\mathrm{I}\left(1-\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}}\sqrt{\frac{\mathrm{I}}{\sqrt{|\beta|}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|\beta|}\right)}{\sqrt{|\beta|}}},\frac{\sqrt{2}}{2}\right)\right) \tag{B.4}$$

then put the value ($\beta' = -0.3, -0.61\ and -1.1$) into (B.4), three equations with different signs can be gotten as

$subs(\mathrm{beta}=-0.3, f)$

$$f1 := \frac{2\sqrt{a^3+0.3\,a}}{3} - \frac{1}{\sqrt{a^3+0.3\,a}}\left(0.1351200155\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+0.5477225575\,\mathrm{I})}\sqrt{2}\sqrt{\mathrm{I}\,(a-0.5477225575\,\mathrm{I})}\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\left(1.351200155\sqrt{-\mathrm{I}\,(a+0.5477225575\,\mathrm{I})},\frac{\sqrt{2}}{2}\right)\right) - 0.7601169500 + (0.03951382205 + 0.1771462286\,\mathrm{I})\sqrt{2}, -\frac{2\sqrt{a^3+0.3\,a}}{3} + \frac{1}{\sqrt{a^3+0.3\,a}}\left(0.1351200155\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+0.5477225575\,\mathrm{I})}\sqrt{2}\sqrt{\mathrm{I}\,(a-0.5477225575\,\mathrm{I})}\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\left(1.351200155\sqrt{-\mathrm{I}\,(a+0.5477225575\,\mathrm{I})},\frac{\sqrt{2}}{2}\right)\right) + 0.7601169500 - (0.03951382205 + 0.1771462286\,\mathrm{I})\sqrt{2} \tag{B.5}$$

$subs(\mathrm{beta}=-0.6, f)$

$$\frac{2\sqrt{a^3+|-0.6|\,a}}{3} - \frac{1}{3\sqrt{a^3+|-0.6|\,a}}\left(\mathrm{I}\,|-0.6|^{3/2}\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|-0.6|}\right)}{\sqrt{|-0.6|}}}\,\sqrt{2}\,\sqrt{\frac{\mathrm{I}\left(a-\mathrm{I}\sqrt{|-0.6|}\right)}{\sqrt{|-0.6|}}}\,\sqrt{\frac{\mathrm{I}\,a}{\sqrt{|-0.6|}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|-0.6|}\right)}{\sqrt{|-0.6|}}}, \frac{\sqrt{2}}{2}\right)\right) - \frac{2\sqrt{1+|-0.6|}}{3} + \frac{1}{3\sqrt{1+|-0.6|}}\left(\mathrm{I}\,|-0.6|^{3/2}\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|-0.6|}\right)}{\sqrt{|-0.6|}}}\,\sqrt{2}\,\sqrt{\frac{\mathrm{I}\left(1-\mathrm{I}\sqrt{|-0.6|}\right)}{\sqrt{|-0.6|}}}\,\sqrt{\frac{\mathrm{I}}{\sqrt{|-0.6|}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|-0.6|}\right)}{\sqrt{|-0.6|}}}, \frac{\sqrt{2}}{2}\right)\right)$$

$$f2 := \frac{2\sqrt{a^3+0.6\,a}}{3} - \frac{1}{\sqrt{a^3+0.6\,a}}\left(0.2272438736\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-0.7745966692\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\left(1.136219367\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})}, \frac{\sqrt{2}}{2}\right)\right) - 0.8432740427 + (0.02886450778 + 0.2979232575\,\mathrm{I})\sqrt{2}, -\frac{2\sqrt{a^3+0.6\,a}}{3} + \frac{1}{\sqrt{a^3+0.6\,a}}\left(0.2272438736\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-0.7745966692\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\left(1.136219367\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})}, \frac{\sqrt{2}}{2}\right)\right) + 0.8432740427 - (0.02886450778 + 0.2979232575\,\mathrm{I})\sqrt{2}$$

$$f2 := \frac{2\sqrt{a^3+0.6\,a}}{3} - \frac{1}{\sqrt{a^3+0.6\,a}}\Big(0.2272438736\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-0.7745966692\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\;\mathrm{EllipticF}\Big(1.136219367\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})},\frac{\sqrt{2}}{2}\Big)\Big) - 0.8432740427 + (0.02886450778 + 0.2979232575\,\mathrm{I})\sqrt{2}, -\frac{2\sqrt{a^3+0.6\,a}}{3} + \frac{1}{\sqrt{a^3+0.6\,a}}\Big(0.2272438736\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-0.7745966692\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\;\mathrm{EllipticF}\Big(1.136219367\sqrt{-\mathrm{I}\,(a+0.7745966692\,\mathrm{I})},\frac{\sqrt{2}}{2}\Big)\Big) + 0.8432740427 - (0.02886450778 + 0.2979232575\,\mathrm{I})\sqrt{2} \tag{B.6}$$

$subs(\mathrm{beta}=-1.1, f)$

$$\frac{2\sqrt{a^3+|-1.1|\,a}}{3} - \frac{1}{3\sqrt{a^3+|-1.1|\,a}}\left(\mathrm{I}\,|-1.1|^{3/2}\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|-1.1|}\right)}{\sqrt{|-1.1|}}}\,\sqrt{2}\,\sqrt{\frac{\mathrm{I}\left(a-\mathrm{I}\sqrt{|-1.1|}\right)}{\sqrt{|-1.1|}}}\,\sqrt{\frac{\mathrm{I}\,a}{\sqrt{|-1.1|}}}\;\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(a+\mathrm{I}\sqrt{|-1.1|}\right)}{\sqrt{|-1.1|}}},\frac{\sqrt{2}}{2}\right)\right) - \frac{2\sqrt{1+|-1.1|}}{3} + \frac{1}{3\sqrt{1+|-1.1|}}\left(\mathrm{I}\,|-1.1|^{3/2}\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|-1.1|}\right)}{\sqrt{|-1.1|}}}\,\sqrt{2}\,\sqrt{\frac{\mathrm{I}\left(1-\mathrm{I}\sqrt{|-1.1|}\right)}{\sqrt{|-1.1|}}}\,\sqrt{\frac{\mathrm{I}}{\sqrt{|-1.1|}}}\;\mathrm{EllipticF}\left(\sqrt{\frac{-\mathrm{I}\left(1+\mathrm{I}\sqrt{|-1.1|}\right)}{\sqrt{|-1.1|}}},\frac{\sqrt{2}}{2}\right)\right)$$

$$f3 := \frac{2\sqrt{a^3+1.1\,a}}{3} - \frac{1}{\sqrt{a^3+1.1\,a}}\Bigg(0.3580331660\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-1.048808848\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\Big(0.9764540898\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}, \frac{\sqrt{2}}{2}\Big)\Bigg) - 0.9660917833 + (-0.008529437557 + 0.4693917839\,\mathrm{I})\sqrt{2}, -\frac{2\sqrt{a^3+1.1\,a}}{3} + \frac{1}{\sqrt{a^3+1.1\,a}}\Bigg(0.3580331660\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-1.048808848\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\Big(0.9764540898\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}, \frac{\sqrt{2}}{2}\Big)\Bigg) + 0.9660917833 - (-0.008529437557 + 0.4693917839\,\mathrm{I})\sqrt{2}$$

$$f3 := \frac{2\sqrt{a^3+1.1\,a}}{3} - \frac{1}{\sqrt{a^3+1.1\,a}}\Bigg(0.3580331660\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-1.048808848\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\Big(0.9764540898\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}, \frac{\sqrt{2}}{2}\Big)\Bigg) - 0.9660917833 + (-0.008529437557 + 0.4693917839\,\mathrm{I})\sqrt{2}, -\frac{2\sqrt{a^3+1.1\,a}}{3} + \frac{1}{\sqrt{a^3+1.1\,a}}\Bigg(0.3580331660\,\mathrm{I}\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}\,\sqrt{2}\,\sqrt{\mathrm{I}\,(a-1.048808848\,\mathrm{I})}\,\sqrt{\mathrm{I}\,a}\,\mathrm{EllipticF}\Big(0.9764540898\sqrt{-\mathrm{I}\,(a+1.048808848\,\mathrm{I})}, \frac{\sqrt{2}}{2}\Big)\Bigg) + 0.9660917833 + (0.008529437557 - 0.4693917839\,\mathrm{I})\sqrt{2} \tag{B.7}$$

in the Radiation case $(\beta > 0)$, After put the values of $(\beta' = 0.3, 0.5\ and\ 1)$ into equation (5.38), we get tau equations with positive and negative signs.

$subs(\mathrm{beta} = 0.3, \mathrm{tau})$

$$\frac{\sqrt{a^2-0.3}}{2} - 0.4183300132 + \frac{0.3 \cdot \ln\left(0.5444665784\,a + 0.5444665784\sqrt{a^2-0.3}\right)}{2}$$

$$tau1 := \frac{\sqrt{a^2-0.3}}{2} - 0.4183300132 + \frac{0.3 \cdot \ln\left(0.5444665784\,a + 0.5444665784\sqrt{a^2-0.3}\right)}{2}, -\frac{\sqrt{a^2-0.3}}{2} + 0.4183300132 - \frac{0.3 \cdot \ln\left(0.5444665784\,a + 0.5444665784\sqrt{a^2-0.3}\right)}{2}$$

$$\tau 1 := \frac{\sqrt{a^2-0.3}}{2} - 0.4183300132 + 0.1500000000\ln\left(0.5444665784\,a + 0.5444665784\sqrt{a^2-0.3}\right), -\frac{\sqrt{a^2-0.3}}{2} + 0.4183300132 - 0.1500000000\ln\left(0.5444665784\,a + 0.5444665784\sqrt{a^2-0.3}\right) \tag{B.8}$$

$subs(\mathrm{beta} = 0.5, \mathrm{tau})$

$$\frac{\sqrt{a^2-0.5}}{2} - 0.3535533906 + \frac{0.5 \cdot \ln\left(0.5857864377\,a + 0.5857864377\sqrt{a^2-0.5}\right)}{2}$$

$$tau2 := \frac{\sqrt{a^2-0.5}}{2} - 0.3535533906 + \frac{0.5 \cdot \ln\left(0.5857864377\,a + 0.5857864377\sqrt{a^2-0.5}\right)}{2}, -\frac{\sqrt{a^2-0.5}}{2} + 0.3535533906 - \frac{0.5 \cdot \ln\left(0.5857864377\,a + 0.5857864377\sqrt{a^2-0.5}\right)}{2}$$

$$\tau 2 := \frac{\sqrt{a^2-0.5}}{2} - 0.3535533906 + 0.2500000000\ln\left(0.5857864377\,a + 0.5857864377\sqrt{a^2-0.5}\right), -\frac{\sqrt{a^2-0.5}}{2} + 0.3535533906 - 0.2500000000\ln\left(0.5857864377\,a + 0.5857864377\sqrt{a^2-0.5}\right) \tag{B.9}$$

$subs(\mathrm{beta} = 1, \mathrm{tau})$

$$\frac{\sqrt{a^2-1}}{2} + \frac{1 \cdot \ln\left(a + \sqrt{a^2-1}\right)}{2}$$

$$tau3 := \frac{\sqrt{a^2-1}}{2} + \frac{\ln\left(a + \sqrt{a^2-1}\right)}{2}, -\frac{\sqrt{a^2-1}}{2} - \frac{\ln\left(a + \sqrt{a^2-1}\right)}{2}$$

$$\tau 3 := \frac{\sqrt{a^2-1}}{2} + \frac{\ln\left(a + \sqrt{a^2-1}\right)}{2}, -\frac{\sqrt{a^2-1}}{2} - \frac{\ln\left(a + \sqrt{a^2-1}\right)}{2} \tag{B.10}$$

in the Dust case $(\beta > 0)$, by taking integrate for the equation (5.35), and substitute the integral amplitude value of $(\tau = 1 \; and \; a)$ into the integration equation, it reads as

$$\frac{\mathrm{tau}^2}{\sqrt{\mathrm{tau}\cdot(\mathrm{tau}^2 - \mathrm{beta})}}$$

$$\frac{\tau^2}{\sqrt{\tau\left(\tau^2 - \beta\right)}}$$

$$Q := int\left(\frac{\tau^2}{\sqrt{\tau\left(\tau^2 - \mathrm{beta}\right)}}, \mathrm{tau}\right)$$

$$Q := \frac{2\sqrt{\tau^3 - \beta\tau}}{3} + \frac{1}{3\sqrt{\tau^3 - \beta\tau}}\left(\beta^{3/2}\sqrt{\frac{\tau + \sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{2\left(\tau - \sqrt{\beta}\right)}{\sqrt{\beta}}}\sqrt{-\frac{\tau}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{\tau + \sqrt{\beta}}{\sqrt{\beta}}}, \frac{\sqrt{2}}{2}\right)\right)$$

$$L2 := subs(\mathrm{tau} = a, Q)$$

$$L2 := \frac{2\sqrt{a^3 - a\beta}}{3} + \frac{1}{3\sqrt{a^3 - a\beta}}\left(\beta^{3/2}\sqrt{\frac{a + \sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{2\left(a - \sqrt{\beta}\right)}{\sqrt{\beta}}}\sqrt{-\frac{a}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{a + \sqrt{\beta}}{\sqrt{\beta}}}, \frac{\sqrt{2}}{2}\right)\right)$$

$$L1 := subs(\mathrm{tau} = 1, Q)$$

$$L1 := \frac{2\sqrt{-\beta+1}}{3} + \frac{1}{3\sqrt{-\beta+1}}\left(\beta^{3/2}\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{2\left(1-\sqrt{\beta}\right)}{\sqrt{\beta}}}\sqrt{-\frac{1}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}, \frac{\sqrt{2}}{2}\right)\right)$$

$$Q := L2 - L1$$

$$Q := \frac{2\sqrt{a^3-a\beta}}{3} + \frac{1}{3\sqrt{a^3-a\beta}}\left(\beta^{3/2}\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{2\left(a-\sqrt{\beta}\right)}{\sqrt{\beta}}}\sqrt{-\frac{a}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}}, \frac{\sqrt{2}}{2}\right)\right) - \frac{2\sqrt{-\beta+1}}{3} - \frac{1}{3\sqrt{-\beta+1}}\left(\beta^{3/2}\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{2\left(1-\sqrt{\beta}\right)}{\sqrt{\beta}}}\sqrt{-\frac{1}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}, \frac{\sqrt{2}}{2}\right)\right) \tag{B.11}$$

then put the value ($\beta' = 0.3, 0.6\ and\ 1$) into (B.11), three equations with different signs can be gotten as

$subs(\mathrm{beta} = 0.3, Q)$

$$\frac{2\sqrt{a^3-0.3\,a}}{3} + \frac{1}{\sqrt{a^3-0.3\,a}}\left(0.05477225573\sqrt{1.825741858\,a+0.9999999998}\sqrt{-3.651483716\,a+2.000000000}\sqrt{-1.825741858\,a}\,\mathrm{EllipticF}\left(\sqrt{1.825741858\,a+0.9999999998}, \frac{\sqrt{2}}{2}\right)\right) - 0.5577733510 + 0.1910885584\,\mathrm{EllipticF}\left(1.680994307, \frac{\sqrt{2}}{2}\right)$$

$$
\begin{aligned}
Q1 := {} & \frac{2\sqrt{a^3 - 0.3\,a}}{3} \\
& + \frac{1}{\sqrt{a^3 - 0.3\,a}} \Big(0.05477225573 \sqrt{1.825741858\,a + 0.9999999998} \\
& \sqrt{-3.651483716\,a + 2.000000000}\ \sqrt{-1.825741858\,a}\ \mathrm{EllipticF}\Big(\\
& \sqrt{1.825741858\,a + 0.9999999998}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.5577733510 \\
& + 0.1910885584\,\mathrm{EllipticF}\left(1.680994307, \frac{\sqrt{2}}{2}\right), - \frac{2\sqrt{a^3 - 0.3\,a}}{3} \\
& - \frac{1}{\sqrt{a^3 - 0.3\,a}} \Big(0.05477225573 \sqrt{1.825741858\,a + 0.9999999998} \\
& \sqrt{-3.651483716\,a + 2.000000000}\ \sqrt{-1.825741858\,a}\ \mathrm{EllipticF}\Big(\\
& \sqrt{1.825741858\,a + 0.9999999998}, \frac{\sqrt{2}}{2}\Big)\Big) + 0.5577733510 \\
& - 0.1910885584\,\mathrm{EllipticF}\left(1.680994307, \frac{\sqrt{2}}{2}\right)
\end{aligned}
$$

(B.12)

$$
\begin{aligned}
Q1 := {} & \frac{2\sqrt{a^3 - 0.3\,a}}{3} \\
& + \frac{1}{\sqrt{a^3 - 0.3\,a}} \Big(0.05477225573 \sqrt{1.825741858\,a + 0.9999999998} \\
& \sqrt{-3.651483716\,a + 2.000000000}\ \sqrt{-1.825741858\,a}\ \mathrm{EllipticF}\Big(\\
& \sqrt{1.825741858\,a + 0.9999999998}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.3508600549 - 0.3542924574\,\mathrm{I}, \\
& - \frac{2\sqrt{a^3 - 0.3\,a}}{3} \\
& - \frac{1}{\sqrt{a^3 - 0.3\,a}} \Big(0.05477225573 \sqrt{1.825741858\,a + 0.9999999998} \\
& \sqrt{-3.651483716\,a + 2.000000000}\ \sqrt{-1.825741858\,a}\ \mathrm{EllipticF}\Big(\\
& \sqrt{1.825741858\,a + 0.9999999998}, \frac{\sqrt{2}}{2}\Big)\Big) + 0.3508600549 + 0.3542924574\,\mathrm{I}
\end{aligned}
$$

$subs(\mathrm{beta} = 0.6, Q)$

$$\frac{2\sqrt{a^3 - 0.6\,a}}{3} + \frac{1}{\sqrt{a^3 - 0.6\,a}}\Big(0.1549193338\sqrt{1.290994449\,a + 1.000000000}\,\sqrt{-2.581988898\,a + 2.000000000}\,\sqrt{-1.290994449\,a}\,\mathrm{EllipticF}\Big(\sqrt{1.290994449\,a + 1.000000000}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.4216370213 + 0.3213713678\,\mathrm{EllipticF}\Big(1.513603135, \frac{\sqrt{2}}{2}\Big)$$

$$Q2 := \frac{2\sqrt{a^3 - 0.6\,a}}{3} + \frac{1}{\sqrt{a^3 - 0.6\,a}}\Big(0.1549193338\sqrt{1.290994449\,a + 1.000000000}\,\sqrt{-2.581988898\,a + 2.000000000}\,\sqrt{-1.290994449\,a}\,\mathrm{EllipticF}\Big(\sqrt{1.290994449\,a + 1.000000000}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.4216370213 + 0.3213713678\,\mathrm{EllipticF}\Big(1.513603135, \frac{\sqrt{2}}{2}\Big), -\frac{2\sqrt{a^3 - 0.6\,a}}{3} - \frac{1}{\sqrt{a^3 - 0.6\,a}}\Big(0.1549193338\sqrt{1.290994449\,a + 1.000000000}\,\sqrt{-2.581988898\,a + 2.000000000}\,\sqrt{-1.290994449\,a}\,\mathrm{EllipticF}\Big(\sqrt{1.290994449\,a + 1.000000000}, \frac{\sqrt{2}}{2}\Big)\Big) + 0.4216370213 - 0.3213713678\,\mathrm{EllipticF}\Big(1.513603135, \frac{\sqrt{2}}{2}\Big)$$

(B.13)

$$Q2 := \frac{2\sqrt{a^3 - 0.6\,a}}{3} + \frac{1}{\sqrt{a^3 - 0.6\,a}}\Big(0.1549193338\sqrt{1.290994449\,a + 1.000000000}\,\sqrt{-2.581988898\,a + 2.000000000}\,\sqrt{-1.290994449\,a}\,\mathrm{EllipticF}\Big(\sqrt{1.290994449\,a + 1.000000000}, \frac{\sqrt{2}}{2}\Big)\Big) + 0.0119697124 - 0.5958465153\,\mathrm{I},$$

$$-\frac{2\sqrt{a^3 - 0.6\,a}}{3} - \frac{1}{\sqrt{a^3 - 0.6\,a}}\Big(0.1549193338\sqrt{1.290994449\,a + 1.000000000}\,\sqrt{-2.581988898\,a + 2.000000000}\,\sqrt{-1.290994449\,a}\,\mathrm{EllipticF}\Big(\sqrt{1.290994449\,a + 1.000000000}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.0119697124 + 0.5958465153\,\mathrm{I}$$

$subs(\mathrm{beta} = 0.9, Q)$

$$\frac{2\sqrt{a^3 - 0.9\,a}}{3} + \frac{1}{\sqrt{a^3 - 0.9\,a}}\Big(0.2846049895\sqrt{1.054092553\,a + 0.9999999997}\,\sqrt{-2.108185106\,a + 1.999999999}\,\sqrt{-1.054092553\,a}\,\mathrm{EllipticF}\Big(\sqrt{1.054092553\,a + 0.9999999997}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.2108185107 + 0.4355877170\,\mathrm{EllipticF}\Big(1.433210575, \frac{\sqrt{2}}{2}\Big)$$

$$Q3 := \frac{2\sqrt{a^3 - 0.9\,a}}{3} + \frac{1}{\sqrt{a^3 - 0.9\,a}}\Big(0.2846049895\sqrt{1.054092553\,a + 0.9999999997}\,\sqrt{-2.108185106\,a + 1.999999999}\,\sqrt{-1.054092553\,a}\ \mathrm{EllipticF}\Big(\sqrt{1.054092553\,a + 0.9999999997}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.2108185107 + 0.4355877170\,\mathrm{EllipticF}\Big(1.433210575, \frac{\sqrt{2}}{2}\Big), -\frac{2\sqrt{a^3 - 0.9\,a}}{3} - \frac{1}{\sqrt{a^3 - 0.9\,a}}\Big(0.2846049895\sqrt{1.054092553\,a + 0.9999999997}\,\sqrt{-2.108185106\,a + 1.999999999}\,\sqrt{-1.054092553\,a}\ \mathrm{EllipticF}\Big(\sqrt{1.054092553\,a + 0.9999999997}, \frac{\sqrt{2}}{2}\Big)\Big) + 0.2108185107 - 0.4355877170\,\mathrm{EllipticF}\Big(1.433210575, \frac{\sqrt{2}}{2}\Big) \tag{B.14}$$

$$Q3 := \frac{2\sqrt{a^3 - 0.9\,a}}{3} + \frac{1}{\sqrt{a^3 - 0.9\,a}}\Big(0.2846049895\sqrt{1.054092553\,a + 0.9999999997}\,\sqrt{-2.108185106\,a + 1.999999999}\,\sqrt{-1.054092553\,a}\ \mathrm{EllipticF}\Big(\sqrt{1.054092553\,a + 0.9999999997}, \frac{\sqrt{2}}{2}\Big)\Big) + 0.4968213928 - 0.8076121561\,I, -\frac{2\sqrt{a^3 - 0.9\,a}}{3} - \frac{1}{\sqrt{a^3 - 0.9\,a}}\Big(0.2846049895\sqrt{1.054092553\,a + 0.9999999997}\,\sqrt{-2.108185106\,a + 1.999999999}\,\sqrt{-1.054092553\,a}\ \mathrm{EllipticF}\Big(\sqrt{1.054092553\,a + 0.9999999997}, \frac{\sqrt{2}}{2}\Big)\Big) - 0.4968213928 + 0.8076121561\,I$$

Series expansion for equation (5.46) reads as

$$g := \frac{\mathrm{tau}^2}{\sqrt{\mathrm{tau}^2 - \beta + \epsilon \cdot \mathrm{tau}^6}}$$

$$g := \frac{\tau^2}{\sqrt{\epsilon\,\tau^6 + \tau^2 - \beta}}$$

$$series\left(\frac{\tau^2}{\sqrt{\epsilon\,\tau^6 + \tau^2 - \beta}}, \epsilon, 2\right)$$

$$\frac{\tau^2}{\sqrt{\tau^2 - \beta}} - \frac{1}{2}\,\frac{\tau^8}{\left(\tau^2 - \beta\right)^{3/2}}\,\epsilon + \mathrm{O}(\epsilon^2) \tag{B.15}$$

By taking integration for each terms in the above equation , they read as

$$\frac{\tau^2}{\sqrt{\tau^2 - \beta}}$$

$$\frac{\tau^2}{\sqrt{\tau^2 - \beta}}$$

$$Q := int\left(\frac{\tau^2}{\sqrt{\tau^2 - \beta}}, \mathrm{tau}\right)$$

$$Q := \frac{\tau\sqrt{\tau^2 - \beta}}{2} + \frac{\beta \ln\left(\tau + \sqrt{\tau^2 - \beta}\right)}{2}$$

$$L2 := subs(\mathrm{tau} = a, Q)$$

$$L2 := \frac{a\sqrt{a^2 - \beta}}{2} + \frac{\beta \ln\left(a + \sqrt{a^2 - \beta}\right)}{2}$$

$$L1 := subs(\mathrm{tau} = 1, Q)$$

$$L1 := \frac{\sqrt{-\beta + 1}}{2} + \frac{\beta \ln\left(1 + \sqrt{-\beta + 1}\right)}{2}$$

$$Q1 := L2 - L1$$

$$Q1 := \frac{a\sqrt{a^2 - \beta}}{2} + \frac{\beta \ln\left(a + \sqrt{a^2 - \beta}\right)}{2} - \frac{\sqrt{-\beta + 1}}{2} - \frac{\beta \ln\left(1 + \sqrt{-\beta + 1}\right)}{2} \tag{B.16}$$

$$\frac{\tau^8}{\left(\tau^2 - \beta\right)^{3/2}}$$

$$\frac{\tau^8}{\left(\tau^2 - \beta\right)^{3/2}}$$

$$W := int\left(\frac{\tau^8}{\left(\tau^2 - \beta\right)^{3/2}}, \mathrm{tau}\right)$$

$$W := \frac{\tau^7}{6\sqrt{\tau^2-\beta}} + \frac{7\,\beta\,\tau^5}{24\sqrt{\tau^2-\beta}} + \frac{35\,\beta^2\,\tau^3}{48\sqrt{\tau^2-\beta}} - \frac{35\,\beta^3\,\tau}{16\sqrt{\tau^2-\beta}} + \frac{35\,\beta^3\ln\left(\tau+\sqrt{\tau^2-\beta}\right)}{16}$$

$P2 := subs(\mathrm{tau} = a, W)$

$$P2 := \frac{a^7}{6\sqrt{a^2-\beta}} + \frac{7\,\beta\,a^5}{24\sqrt{a^2-\beta}} + \frac{35\,\beta^2\,a^3}{48\sqrt{a^2-\beta}} - \frac{35\,\beta^3\,a}{16\sqrt{a^2-\beta}} + \frac{35\,\beta^3\ln\left(a+\sqrt{a^2-\beta}\right)}{16}$$

$P1 := subs(\mathrm{tau} = 1, W)$

$$P1 := \frac{1}{6\sqrt{-\beta+1}} + \frac{7\,\beta}{24\sqrt{-\beta+1}} + \frac{35\,\beta^2}{48\sqrt{-\beta+1}} - \frac{35\,\beta^3}{16\sqrt{-\beta+1}} + \frac{35\,\beta^3\ln\left(1+\sqrt{-\beta+1}\right)}{16}$$

$W1 := P2 - P1$

$$W1 := \frac{a^7}{6\sqrt{a^2-\beta}} + \frac{7\,\beta\,a^5}{24\sqrt{a^2-\beta}} + \frac{35\,\beta^2\,a^3}{48\sqrt{a^2-\beta}} - \frac{35\,\beta^3\,a}{16\sqrt{a^2-\beta}} + \frac{35\,\beta^3\ln\left(a+\sqrt{a^2-\beta}\right)}{16} - \frac{1}{6\sqrt{-\beta+1}} - \frac{7\,\beta}{24\sqrt{-\beta+1}} - \frac{35\,\beta^2}{48\sqrt{-\beta+1}} + \frac{35\,\beta^3}{16\sqrt{-\beta+1}} - \frac{35\,\beta^3\ln\left(1+\sqrt{-\beta+1}\right)}{16} \tag{B.17}$$

$N_1(a)$ and $G_1(a)$can be integrated such as

$$\sqrt{\frac{\tau^3}{\tau^2-\beta}}$$

$$\sqrt{\frac{\tau^3}{\tau^2-\beta}}$$

$$N := int\left(\sqrt{\frac{\tau^3}{\tau^2-\beta}}, \mathrm{tau}\right)$$

$$N := \frac{1}{3\,\tau\sqrt{\tau\left(\tau^2-\beta\right)}\,\sqrt{\tau^3-\beta\,\tau}}\left(\sqrt{\frac{\tau^3}{\tau^2-\beta}}\,\left(\tau^2-\beta\right)\left(\beta^{3/2}\sqrt{\frac{\tau+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{2}\,\sqrt{\frac{-\tau+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{-\frac{\tau}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{\tau+\sqrt{\beta}}{\sqrt{\beta}}},\frac{\sqrt{2}}{2}\right)+2\,\tau^3-2\,\beta\,\tau\right)\right)$$

$P2 := subs(\mathrm{tau} = a, N)$

$$P2 := \frac{1}{3\,a\sqrt{a\left(a^2-\beta\right)}\,\sqrt{a^3-\beta\,a}}\left(\sqrt{\frac{a^3}{a^2-\beta}}\,\left(a^2-\beta\right)\left(\beta^{3/2}\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{2}\,\sqrt{\frac{-a+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{-\frac{a}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}},\frac{\sqrt{2}}{2}\right)+2\,a^3-2\,\beta\,a\right)\right)$$

$P1 := subs(\mathrm{tau} = 1, N)$

$$P1 := \frac{1}{3}\left(\sqrt{\frac{1}{-\beta+1}}\left(\beta^{3/2}\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{2}\,\sqrt{\frac{-1+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{-\frac{1}{\sqrt{\beta}}}\,\mathrm{EllipticF}\left(\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}},\frac{\sqrt{2}}{2}\right)+2-2\,\beta\right)\right)$$

$N1 := P2 - P1$

$$N1 := \frac{1}{3\,a\,\sqrt{a\left(a^2-\beta\right)}\,\sqrt{a^3-\beta\,a}}\left(\sqrt{\frac{a^3}{a^2-\beta}}\,\left(a^2-\beta\right)\left(\beta^{3/2}\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{2}\,\sqrt{\frac{-a+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{-\frac{a}{\sqrt{\beta}}}\;\mathrm{EllipticF}\left(\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}},\frac{\sqrt{2}}{2}\right)+2\,a^3-2\,\beta\,a\right)\right) - \frac{1}{3}\left(\sqrt{\frac{1}{-\beta+1}}\left(\beta^{3/2}\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{2}\,\sqrt{\frac{-1+\sqrt{\beta}}{\sqrt{\beta}}}\,\sqrt{-\frac{1}{\sqrt{\beta}}}\;\mathrm{EllipticF}\left(\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}},\frac{\sqrt{2}}{2}\right)+2-2\,\beta\right)\right) \tag{B.18}$$

for $G_1(a)$ is

$$\frac{-\tau^5}{2.\left(\tau^2-\mathrm{beta}\right)}\sqrt{\frac{\tau^3}{\tau^2-\mathrm{beta}}}$$

$$-\frac{\tau^5\sqrt{\frac{\tau^3}{\tau^2-\beta}}}{2.\,\tau^2-2.\,\beta}$$

$$M := int\left(-\frac{\tau^5\sqrt{\frac{\tau^3}{\tau^2-\beta}}}{2.\,\tau^2-2.\,\beta},\ \mathrm{tau}\right)$$

$M :=$

$$\frac{1}{\tau^2 \sqrt{\tau^3 - \beta\tau}} \left(0.005555555556 \sqrt{\frac{\tau^3}{\tau^2 - 1.\beta}} \left(653.3666658\,\beta^3 \sqrt{\frac{\tau + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{\frac{-1.\tau + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{-\frac{1.\tau}{\sqrt{\beta}}} \sqrt{\tau(\tau^2 - 1.\beta)}\, \mathrm{EllipticE}\left(\sqrt{\frac{\tau + \sqrt{\beta}}{\sqrt{\beta}}}, 0.7071067812\right) - 326.6833329\,\beta^3 \sqrt{\frac{\tau + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{\frac{-1.\tau + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{-\frac{1.\tau}{\sqrt{\beta}}} \sqrt{\tau(\tau^2 - 1.\beta)}\, \mathrm{EllipticF}\left(\sqrt{\frac{\tau + \sqrt{\beta}}{\sqrt{\beta}}}, 0.7071067812\right) - 20.\tau^6 \sqrt{\tau(\tau^2 - 1.\beta)} - 44.\sqrt{\tau(\tau^2 - 1.\beta)}\,\tau^4\beta + 90.\sqrt{\tau^3 - \beta\tau}\,\tau^2\beta^2 + 64.\sqrt{\tau(\tau^2 - 1.\beta)}\,\tau^2\beta^2 \right)\right)$$

$L4 := subs(\mathrm{tau} = a, M)$

$L4 :=$

$$\frac{1}{a^2 \sqrt{a^3 - \beta a}} \left(0.005555555556 \sqrt{\frac{a^3}{a^2 - 1.\beta}} \left(653.3666658\,\beta^3 \sqrt{\frac{a + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{\frac{-1.a + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{-\frac{1.a}{\sqrt{\beta}}} \sqrt{a(a^2 - 1.\beta)}\, \mathrm{EllipticE}\left(\sqrt{\frac{a + \sqrt{\beta}}{\sqrt{\beta}}}, 0.7071067812\right) - 326.6833329\,\beta^3 \sqrt{\frac{a + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{\frac{-1.a + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{-\frac{1.a}{\sqrt{\beta}}} \sqrt{a(a^2 - 1.\beta)}\, \mathrm{EllipticF}\left(\sqrt{\frac{a + \sqrt{\beta}}{\sqrt{\beta}}}, 0.7071067812\right) - 20.a^6 \sqrt{a(a^2 - 1.\beta)} - 44.\sqrt{a(a^2 - 1.\beta)}\,a^4\beta + 90.\sqrt{a^3 - \beta a}\,a^2\beta^2 + 64.\sqrt{a(a^2 - 1.\beta)}\,a^2\beta^2 \right)\right)$$

$L3 := subs(\mathrm{tau} = 1, M)$

$L3 :=$

$$\frac{1}{\sqrt{1 - 1.\beta}} \left(0.005555555556 \sqrt{\frac{1}{1 - 1.\beta}} \left(653.3666658\,\beta^3 \sqrt{\frac{1 + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{\frac{-1. + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{-\frac{1.}{\sqrt{\beta}}} \sqrt{1 - 1.\beta}\, \mathrm{EllipticE}\left(\sqrt{\frac{1 + \sqrt{\beta}}{\sqrt{\beta}}}, 0.7071067812\right) - 326.6833329\,\beta^3 \sqrt{\frac{1 + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{\frac{-1. + \sqrt{\beta}}{\sqrt{\beta}}} \sqrt{-\frac{1.}{\sqrt{\beta}}} \sqrt{1 - 1.\beta}\, \mathrm{EllipticF}\left(\sqrt{\frac{1 + \sqrt{\beta}}{\sqrt{\beta}}}, 0.7071067812\right) - 20.\sqrt{1 - 1.\beta} - 44.\sqrt{1 - 1.\beta}\,\beta + 154.\sqrt{1 - 1.\beta}\,\beta^2 \right)\right)$$

$M1 := L4 - L3$

(B.19)

$$
\begin{aligned}
M1 :=&\\
&\frac{1}{a^2\sqrt{a^3-\beta a}}\left(0.005555555556\sqrt{\frac{a^3}{a^2-1.\,\beta}}\left(653.3666658\,\beta^3\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}}\right.\right.\\
&\sqrt{\frac{-1.\,a+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{1.\,a}{\sqrt{\beta}}}\sqrt{a\left(a^2-1.\,\beta\right)}\,\mathrm{EllipticE}\left(\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}},0.7071067812\right)\\
&-326.6833329\,\beta^3\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{\frac{-1.\,a+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{1.\,a}{\sqrt{\beta}}}\sqrt{a\left(a^2-1.\,\beta\right)}\\
&\mathrm{EllipticF}\left(\sqrt{\frac{a+\sqrt{\beta}}{\sqrt{\beta}}},0.7071067812\right)-20.\,a^6\sqrt{a\left(a^2-1.\,\beta\right)}\\
&\left.\left.-44.\sqrt{a\left(a^2-1.\,\beta\right)}\,a^4\beta+90.\sqrt{a^3-\beta a}\,a^2\beta^2+64.\sqrt{a\left(a^2-1.\,\beta\right)}\,a^2\beta^2\right)\right)\\
&-\frac{1}{\sqrt{1-1.\,\beta}}\left(0.005555555556\sqrt{\frac{1}{1-1.\,\beta}}\left(653.3666658\,\beta^3\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}\right.\right.\\
&\sqrt{\frac{-1.+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{1.}{\sqrt{\beta}}}\sqrt{1-1.\,\beta}\,\mathrm{EllipticE}\left(\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}},0.7071067812\right)\\
&-326.6833329\,\beta^3\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{\frac{-1.+\sqrt{\beta}}{\sqrt{\beta}}}\sqrt{-\frac{1.}{\sqrt{\beta}}}\sqrt{1-1.\,\beta}\,\mathrm{EllipticF}\left(\right.\\
&\left.\sqrt{\frac{1+\sqrt{\beta}}{\sqrt{\beta}}},0.7071067812\right)-20.\sqrt{1-1.\,\beta}-44.\sqrt{1-1.\,\beta}\,\beta\\
&\left.\left.+154.\sqrt{1-1.\,\beta}\,\beta^2\right)\right)
\end{aligned}
$$