Important Relations of Classical Orthogonal Polynomials

Ertan Akacan

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Approval of the Institute of Graduate Studies and Research

Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Sonuç Zorlu Oğurlu
Supervisor

Examining Committee

1. Prof. Dr. Sonuç Zorlu Oğurlu
2. Asst. Prof. Dr. Pembe Sabancıgil Özder
3. Asst. Prof. Dr. Ceren Ustaoğlu

ABSTRACT

In this thesis, the theory of classical orthogonal polynomials which are Hermite, Laguerre and Jacobi polynomials will be studied. To begin with, we will supply an outline regarding the special functions. Followed by examples of properties for orthogonal polynomials in Chapter 2. In the third chapter, we begin classical orthogonal polynomials. To start with, we collate the orthogonal relation, Rodrigues formulas followed by the norm of the classical orthogonal polynomials. In the same chapter, the division of the collected examples of classical orthogonal polynomials into three chapters and assign them the weight function, intermission of the orthogonality, followed by differential equations, hypergeometric representation. To finalise we explain limit relations between polynomials.

Keywords: Classical orthogonal polynomials, hypergeometric functions, second order differential equations, Rodrigues formula.

ÖZ

Bu tezde Hermite, Laguerre ve Jacobi olan klasik ortogonal polinomlar açıklanmıştır. Öncelikli olarak özel fonksiyonlar hakkında bilgi verilmiştir. İlerleyen bölümlerinde ise ortogonal polinomların özelikleri anlatılmıştır. Daha sonraki bölümde de klasik ortogonal polinomlar tanımlanarak ortogonallik ilişkisi anlatılmıştır. Rodrigues formülü ile klasik ortogonal polinomlar için norm hesabı yapılmıştır. Daha sonra ise klasik ortogonal polinom örneklerinin üç bölüme ayrıldığını görürüz. Bunların her biri için ayrı ayrı ağırlık fonksiyonları, ortogonallik aralığı, ikinci dereceden diferensiyel denklemi ve hipergeometrik gösterimi verilerek anlatılmıştır. Tezin son bölümünde de polinomlar arasındaki limit ilişkileri açıklanmıştır.

Anahtar Kelimeler: Klasik ortogonal polinomlar, hipergeometrik fonksiyon, ikinci dereceden diferansiyel denklem, Rodrigues formülü.

DEDICATION

To My Family

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Chapter 1

INTRODUCTION

1.1 Preliminaries

Definition 1.1 (Inner Product Space) Assume Y is a vector space then scalar valued function $<,>: Y * Y \to L$ with $L = R\ or\ C$ is said to be inner product space if it fulfills the following axioms which is denoted by $(Y, <>)$;

1) $\forall\, u, v, w \in y \quad < u + v, w > = < u, w > + < v, w >$,

2) $\forall\, u, v \in Y\ \ and\ l \in L\ \ < Lu, v > = l < u, v >$,

3) $\forall\, u, v \in Y\ \ < u, v > = \overline{< v, u >}$,

4) $\forall u \in Y\ \ < u, u > \geq 0 \quad or \quad < u, u > = 0 \iff u = 0$.

Example 1.2 Let $C[m, n]$ be an inner product space where being the space of all real-valued continuous functions on a closed interval $[m, n]$. The inner product is defined as follows;

$< h, w > = \int_m^n h(x).w(x)dx$ where $h, w \in C[m, n]$.

Theorem 1.3 Let u and v be any elements of Y and let $(Y, <>)$ be inner product space. Then u and v are orthogonal to each other if and only if $< u, v > = 0$.

Example 1.4 Assume we have two functions $g(x)$ and $w(x)$ and they are defined on closed interval $[m, n]$. As stated in theorem 1.3, $g(x)$ and $w(x)$ are orthogonal on a closed interval $[m, n]$ provided that their inner product is zero

$$\int_a^b g(x).w(x)dx = 0.$$

Definition 1.5 (Hypergeometric Equations) The 2nd order differential equation $D(x)k''(x) + E(x)k'(x) + \gamma k(x) = 0$, is said to be hypergeometric equation where degree of $D(x)$ is at most 2, degree of $E(x)$ is at most 1 and γ is a constant.

Theorem 1.6 Hypergeometric equations provides all the derivatives of the solution of hypergeometric equations.

Definition 1.7 (Gamma Function) Let x be a Gamma function which is defined as follows

$$\Gamma(x) = \int_0^\infty e^{-k} k^{x-1} dk \ , \qquad \forall x \in \mathbb{R} - \{\dots, -2, -1, 0\}. \qquad (1.1)$$

Gamma functions have some properties as follows:

1. $\Gamma(x + 1) = x\Gamma(x)$,

2. $\Gamma(x + 1) = x!$,

3. $\Gamma(x + m) = (x)_m \Gamma(x)$,

4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Definition 1.8 (Beta Function) The Beta function is defined as follows;

$$\boldsymbol{B}(w,y) = \int_0^1 k^{w-1}(1-k)^{y-1}dk \qquad where\ Re(w), Re(y) > 0\ . \tag{1.2}$$

Gamma function can be written in terms of the beta function as follows;

$$\boldsymbol{B}(w,y) = \frac{\Gamma(w)\Gamma(y)}{\Gamma(w+y)}. \tag{1.3}$$

Definition 1.9 (Pochhammer Symbol) The Pochhammer symbol is denoted by $(x)_n$, where $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}\ or\ \mathbb{C}$

$$(x)_n = x(x+1)(x+2)\ldots(x+n-1)\ . \tag{1.4}$$

Pochhammer Symbols have some properties as follows:

1. $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$,

2. $(-x)_n = (-1)^n(x-n+1)_n$,

3. $(x)_{2n} = 2^{2n}\left(\frac{x}{2}\right)_n\left(\frac{x+1}{2}\right)_n$, (1.5)

4. $(x)_{n+p} = (x)_n(x+p)_n$,

5. $(x)_n = \frac{(-1)^n(-x)!}{(-x-n)!}$,

6. $(x)_{-n} = \frac{(-1)^n}{(1-x)_n}$,

7. $(p-l)! = \frac{(-1)^l p!}{(-p)_l}$, (1.6)

8. $\binom{\boldsymbol{p}}{\boldsymbol{l}} = \frac{\boldsymbol{p}!}{\boldsymbol{l}!(\boldsymbol{p}-\boldsymbol{l})!} = \frac{(-\mathbf{1})^{\boldsymbol{n}}(-\boldsymbol{p})_{\boldsymbol{l}}}{\boldsymbol{l}!}$.

Definition 1.10 (Hypergeometric Functions) Let ${}_qF_r$ be generalized hypergeometric series, which is defined as follows;

$${}_qF_r\left(\alpha_1, \alpha_2 \ldots \alpha_q; \beta_1, \beta_2 \ldots \beta_r; y\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_q)_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_r)_k} \frac{y^k}{k!}\ . \tag{1.7}$$

Properties of Hypergeometric Functions:

1. $_2F_1(\alpha_1, \alpha_2; \beta_1; 1) = \frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)}$,

2. $_2F_1(-r, \alpha; \beta; 1) = \frac{(\beta-\alpha)_r}{(\beta)_r}$,

3. $_1F_0(\alpha; -; y) = (1-y)^{-\alpha}$.

Definition 1.11 (Differential Equations of Hypergeometric Functions) The hypergeometric function, which is defined as

$$_2F_1(\alpha_1, \alpha_2; \beta; y) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k}{(\beta)_k} \frac{y^k}{k!}$$

has the differential equation which is given below ;

$$y(1-y)r'' + [\beta - (\alpha_1 + \alpha_2 + 1)y]r' - \alpha_1\alpha_2 r = 0. \tag{1.8}$$

Definition 1.12 (Linear Functional) A linear functional is a linear operator that consists of a vector space and L ($\mathbb{R}$ or $\mathbb{C}$) which is stated as follows;

$g: Y \to L$ ($\mathbb{R}$ or $\mathbb{C}$)

Chapter 2

ORTHOGONAL POLYNOMIALS

Definition 2.1 (Orthogonality on Intervals) Assume (a,b) is a finite or infinite open interval in the real line $\mathbb{R}$. $\{p_m(x)\}$, m=0,1,2,…, is called an orthogonal set of polynomials on (a,b) with respect to the weight function $\omega(x)$ (≥ 0) if

$$\int_a^b p_m(x) p_s\, \omega(x)\, dx = 0\,, \qquad m \neq s, \tag{2.1}$$

with $\omega(x)$ continuous or piecewise continuous or integrable, satisfying

$0 < \int_a^b x^{2m}\, \omega(x)\, dx < \infty$ for all m.

Throughout the thesis, we will assume that for every orthogonal polynomials $p_m(x)$ the variable x is restricted with the closure of the interval (a,b) unless otherwise specified.

Definition 2.2 (Orthogonality on Finite Point Sets) Assume X is a finite set of distinct points, or a countable infinite set of distinct points on the real line $\mathbb{R}$, with ω_x, $x \epsilon X$, a set of non-negative constants. In that case, a system of polynomials $\{p_m(x)\}$, m=0,1,2,…, on X with respect to the weight function ω_x is said to be orthogonal if

$$\sum_{x \in X} p_m(x) p_s(x) \omega_x = 0\,, \qquad m \neq s \tag{2.2}$$

When X is infinite provided that ,

$$\sum_{x \in X} x^{2m} \omega_x < \infty\,, \qquad m, s = 0,1, \ldots, N;\ m \neq s, \tag{2.3}$$

and

$$\sum_{x \in X} p_m(x) p_s(x) \omega_x = 0 ; \qquad m , s = 0,1, \ldots , N; \; m \neq s, \tag{2.4}$$

when X is a finite set of N+1 distinct points, whereas in the second case the system $\{p_m(x)\}$ is finite : $m = 0,1, \ldots, N$.

Definition 2.3 (X-Difference Operators) Provided that the orthogonality discrete set X is $\{0,1,2, \ldots N\}$ or $\{0,1,2, \ldots\}$, in that instance the role of the differentiation operator $\frac{d}{dx}$ in the case of classical orthogonal polynomials is placed by Δ_x, the forward difference operator , or by ∇_x, the backward difference operator.

If the orthogonality interval is $(-\infty, \infty)$ or $(0, \infty)$, then the role of $\frac{d}{dx}$ can be placed by δx, the central difference operator in the imaginary direction .

Definition 2.4 (Normalization) The orthogonality relations from (2.1) to (2.4) determine the polynomials $p_n(x)$ uniquely up to constant factors, which can be fixed by suitable normalization.

To clarify

$$h_n = \int_a^b (p_n(x))^2 \, \omega(x) \, dx \;\; or \;\; \sum_{x \epsilon X} (p_n(x))^2 \omega_x , \tag{2.5}$$

$$\overline{h_n} = \int_a^b x (p_n(x))^2 \omega(x) \, dx \;\; or \;\; \sum_{x \epsilon X} x (p_n(x))^2 \omega_x , \tag{2.6}$$

and

$$p_n(x) = k_n x^n + \overline{k_n} \, x^{n-1} + \overline{\overline{k_n}} x^{n-2} + \cdots, \tag{2.7}$$

In that case there are two special types of normalization:

1. Orthonormal Orthogonal Polynomials with $h_n = 1$, $k_n > 0$;
2. Monic Orthogonal Polynomials with $k_n = 1$

Definition 2.5 (Normalization Functions) γ_m is called a normalization function defined as follows;

$$\gamma_m = \int_a^b \left(p_m(x)\right)^2 \omega(x)dx,$$

for continuous orthogonal polynomials with $m = 1,2,...$

Also,

$$\gamma_m = \sum_{x \in X} \left(p_m(x)\right)^2 \omega(x),$$

for discrete orthogonal polynomials $m = 1,2,... , M$ where $M \rightarrow \infty$.

Definition 2.6 (Recurrence Relations) The polynomials $\{p_0(x), p_1(x), ...\}$ is an orthogonal set which has a three-term recurrence relation which is given below,

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \qquad n = 1,2,...$$

Obviously the coefficients appearing in the relation depend on *n*.

1) First Form

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \tag{2.8}$$

A_n, B_n $(n \geq 0)$, and C_n $(n \geq 1)$ appearing in (2.8) are real constants, and $A_{n-1}A_nC_n > 0$ for $n \geq 1$. We then have,

$$A_n = \frac{k_{n+1}}{k_n},$$

$$B_n = \left(\frac{\overline{k_{n+1}}}{k_{n+1}} - \frac{\overline{k_n}}{k_n}\right) A_n = -\frac{\overline{h_n}}{h_n} A_n\,, \tag{2.9}$$

$$C_n = \frac{A_n \overline{\overline{k_n}} + B_n \overline{k_n} - \overline{\overline{k_{n+1}}}}{k_{n-1}} = \frac{A_n}{A_{n-1}} \frac{h_n}{h_{n-1}}$$

2) Second Form

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \qquad n \geq 0 \tag{2.10}$$

Here again the coefficients a_n , b_n $(n \geq 0)$, and c_n $(n \geq 1)$ are real constants, and $a_{n-1}c_n \geq 0$ when $n \geq 1$. Thus,

$$a_n = \frac{k_n}{k_{n+1}},$$

$$b_n = \frac{\bar{k}_n}{k_n} - \frac{\bar{k}_{n+1}}{k_{n+1}} = \frac{\overline{h_n}}{h_n}, \tag{2.11}$$

$$c_n = \frac{\bar{\bar{k}}_n - a_n \bar{\bar{k}}_{n+1} - b_n \bar{k}_n}{k_{n-1}} = a_{n-1}\frac{h_n}{h_{n-1}}$$

- If the orthogonal polynomials are orthonormal, then $c_n = a_{n-1}$ $(n \geq 1)$.
- If the orthogonal polynomials are monic, then $a_n = 1$ $(n \geq 0)$.

On the contrary, if a system of polynomials $\{p_n(x)\}$ satisfies (2.10) with $a_{n-1}c_n > 0$ $(n \geq 1)$, then $\{p_n(x)\}$ is orthogonal with respect to some positive measure on $\mathbb{R}$ (see Favard's theorem). It is not necessarily for this measure to be of the form $\omega(x)\,dx$ and it need not necessarily be unique.

Proof: Any polynomial $T_n(x)$ of degree, can be expressed in terms of $p_0, p_1, \dots, p_n$ and with coefficients β_{in} such that

$$T_n(x) = \sum_{i=0}^{n} \beta_{in} p_i(x)$$

where

$$\beta_{in} = \frac{1}{\sigma_n(x)} \int_a^b T_n(x) p_i(x) w(x)\, d_x \tag{2.12}$$

$$xp_{n+1}(x) = A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x)$$

Then we can write

$$xp_n(x) = \sum_{i=0}^{n+1} \beta_{in} p_i(x),$$

$$xp_n(x) = \beta_{0n} p_0(x) + \beta_{1n} p_1(x) + \cdots + \beta_{n+1,n} p_{n+1}(x) .$$

The set of polynomials $\{p_0(x), p_1(x), \dots .\}$ is orthogonal and every $p_n(x)$ is orthogonal to any polynomial with degree < n.

From (2.12)

$$\beta_{in} = \frac{1}{\sigma_n(x)} \int_a^b p_i(x) x p_n(x) w(x) d_x,$$

$$xp_n(x) = \beta_{on} p_0(x) + \beta_{1n} p_1(x) + \cdots + \beta_{n-1,n} p_{n-1}(x) + \beta_{nn} p_n(x)$$
$$+ \beta_{n+1,n} p_{n+1}(x).$$

Since the degree of $xp_i(x)$ is $i+1$, the orthogonality property of $p_n(x)$ yields the coefficients β_{in} to be all zero when $i+1$ is less than *n* which has the form below;

$xp_n(x)$=$\beta_{n-1,n} p_{n-1}(x) + \beta_{nn} p_n(x) + \beta_{n+1,n} p_{n+1}(x)$.

Let us compare the coefficients of $p_{n-1}(x), p_n(x), p_{n+1}(x)$ of the equations;

$A_n = \beta_{n+1,n}$,

$B_n = \beta_{nn}$,

$C_n = \beta_{n-1,n}$.

Change the index and write β_{in} one more time

$$\beta_{in} = \frac{1}{\sigma_n(x)} \int_a^b p_i(x) x p_n(x) w(x) d_x$$

$$\beta_{ni} = \frac{1}{\sigma_i(x)} \int_a^b p_i(x) x p_n(x) w(x) d_x$$

$$\beta_{in} \sigma_n(x) = \beta_{ni} \sigma_i(x)$$

$$\beta_{ni} = \frac{\beta_{in} \sigma_n(x)}{\sigma_i(x)}$$

Turn back to;

$A_n = \beta_{n+1,n}$ and take $n \to n-1$

$A_{n-1} = \beta_{n,n-1}$ define $n-1 = i$

$A_i = \beta_{ni} = \frac{\beta_{in}\sigma_n(x)}{\sigma_i(x)}$

$A_i = \frac{\sigma_n(x)}{\sigma_i(x)}\beta_{in} \quad \to \quad A_{n-1} = \frac{\sigma_n(x)}{\sigma_{n-1}(x)}\beta_{n-1n}$ where $C_n = \beta_{n-1,n}$

$$A_{n-1} = \frac{\sigma_{n-1}(x)}{\sigma_n(x)} C_n$$

$$C_n = A_{n-1}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}$$

Now let us check the representation of $p_n(x)$ with respect to the three terms recurrence relation;

$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x)$

$$\begin{aligned} a_n x^{n+1} + a_{n-1}x^n + \cdots + a_0 \\ &= A_n[a_{n+1}x^{n+1} + a_n x^n + \cdots + a_0] \\ &+ B_n[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0] \\ &+ C_n[a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0]. \end{aligned}$$

Compare the coefficients of the terms x^{n+1} and x^n;

$a_n = A_n a_{n+1}$

$$A_n = \frac{a_n}{a_{n+1}} \qquad (2.13)$$

$a_{n-1} = A_n a_n + B_n a_n \qquad a_{n-1} = a_n[\frac{a_n}{a_{n+1}} + B_n]$

$$B_n = \frac{a_{n-1}}{a_n} - \frac{a_n}{a_{n+1}}$$

Let $a_{n-1} = C_n$, $a_n = C_{n+1}$.

$$B_n = \frac{C_n}{a_n} - \frac{C_{n+1}}{a_{n+1}} \tag{2.14}$$

Since we have; $C_n = \quad A_{n-1} \frac{\sigma_n(x)}{\sigma_{n-1}(x)}$ where $A_{n-1} = \frac{a_{n-1}}{a_n}$

$$C_n = \frac{a_{n-1}}{a_n} \frac{\sigma_n(x)}{\sigma_{n-1}(x)} \tag{2.15}$$

Theorem 2.7 (Christoffel-Darboux Formula) In mathematics, the Christoffel–Darboux theorem is an identity for a sequence of orthogonal polynomials, introduced by Elwin Bruno Christoffel (1858) and Jean Gaston Darboux (1878).

The theorem states that

$$\sum_{l=0}^{n} \frac{p_l(x)p_l(y)}{h_l} = \frac{k_n}{h_n k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \qquad x \neq y$$

where $p_l(x)$ is the l^{th} term of a set of orthogonal polynomials of squared norm h_l and leading coefficient k_l.

There is also a "confluent form" of this identity:

$$\sum_{l=0}^{n} \frac{(p_l(x))^2}{h_l} = \frac{k_n}{h_n k_{n+1}} (p'_{n+1}(x)p_n(x) - p_n{}'(x)p_{n+1}(x))$$

Proof: Write the 3 three-term recurrence relation with the terms x and ;

$$xp_n(x) = \frac{a_n}{a_{n+1}} p_{n+1}(x) + \beta_n p_n(x) + \frac{a_{n-1}}{a_n} \frac{\sigma_n(x)}{\sigma_{n-1}(x)} p_{n-1}(x)$$

$$yp_n(y) = \frac{a_n}{a_{n+1}} p_{n+1}(y) + \beta_n p_n(y) + \frac{a_{n-1}}{a_n} \frac{\sigma_n(x)}{\sigma_{n-1}(x)} p_{n-1}(y)$$

Multiply the 1st equation with $p_n(y)$ and then the 2nd equation with $p_n(x)$ to get

$$xp_n(x)p_n(y) = \frac{a_n}{a_{n+1}}p_{n+1}(x)p_n(y) + \beta_n p_n(x)p_n(y)$$

$$+\frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)p_n(y)$$

$$yp_n(y)p_n(x) = \frac{a_n}{a_{n+1}}p_{n+1}(y)p_n(x) + \beta_n p_n(y)p_n(x)$$

$$+\frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(y)p_n(x).$$

Subtract the equations

$$(x-y)p_n(x)p_n(y) =$$

$$\frac{a_n}{a_{n+1}}[p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)] + \frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}[p_{n-1}(x)p_n(y) -$$

$$p_{n-1}(y)p_n(x)]$$

$$p_n(x)p_n(y) = \frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)} +$$

$$\frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}\frac{p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)}{(x-y)}$$

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)} =$$

$$p_n(x)p_n(y) - \frac{\sigma_n(x)}{\sigma_{n-1}(x)}\frac{a_{n-1}}{a_n}\frac{p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)}{(x-y)} \tag{2.16}$$

Replace n by $n-1$ in (2.16), then right hand side can be easily seen as the following;

$$\frac{a_{n-1}}{a_n}\frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{(x-y)} = p_{n-1}(x)p_{n-1}(y) -$$

$$\frac{\sigma_{n-1}(x)}{\sigma_{n-2}(x)}\frac{a_{n-2}}{a_{n-1}}\frac{p_{n-2}(x)p_{n-1}(y) - p_{n-2}(y)p_{n-1}(x)}{(x-y)}$$

Multiplying the equation by -1 and substituting it into the equation we have,

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y)-p_{n+1}(y)p_n(x)}{(x-y)}$$

$$=p_n(x)p_n(y)-\frac{\sigma_n(x)}{\sigma_{n-1}(x)}[-p_{n-1}(x)p_{n-1}(y)+$$

$$\frac{\sigma_{n-1}(x)}{\sigma_{n-2}(x)}\frac{a_{n-2}}{a_{n-1}}\frac{p_{n-2}(x)p_{n-1}(y)-p_{n-2}(y)p_{n-1}(x)}{(x-y)}$$

Taking $n\to n-2$ in equation (2.16) again we can find the last term of this equation by

$$\frac{a_{n-2}}{a_{n-1}}\frac{p_{n-1}(x)p_{n-2}(y)-p_{n-1}(y)p_{n-2}(x)}{(x-y)}$$

$$=p_{n-2}(x)p_{n-2}(y)$$

$$-\frac{\sigma_{n-2}(x)}{\sigma_{n-3}(x)}\frac{a_{n-3}}{a_{n-2}}\frac{p_{n-3}(x)p_{n-2}(y)-p_{n-3}(y)p_{n-2}(x)}{(x-y)}$$

Putting this equation to (2.15)

$$\frac{a_{n-1}}{a_n}\frac{p_n(x)p_{n-1}(y)-p_n(y)p_{n-1}(x)}{(x-y)}$$

$$=p_{n-1}(x)p_{n-1}(y)+\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)p_{n-1}(y)-$$

$$\frac{\sigma_n(x)}{\sigma_{n-1}(x)}\frac{\sigma_{n-1}(x)}{\sigma_{n-2}(x)}\Bigg[-p_{n-2}(x)p_{n-2}(y)$$

$$+\frac{\sigma_{n-2}(x)}{\sigma_{n-3}(x)}\frac{a_{n-3}}{a_{n-2}}\frac{p_{n-3}(x)p_{n-2}(y)-p_{n-3}(y)p_{n-2}(x)}{(x-y)}\Bigg]$$

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y)-p_{n+1}(y)p_n(x)}{(x-y)}=$$

$$p_n(x)p_n(y)+\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)p_{n-1}(y)+\frac{\sigma_n(x)}{\sigma_{n-2}(x)}p_{n-2}(x)p_{n-2}(y)-$$

$$\frac{\sigma_n(x)}{\sigma_{n-3}(x)}\frac{a_{n-3}}{a_{n-2}}\frac{p_{n-3}(x)p_{n-2}(y)-p_{n-3}(y)p_{n-2}(x)}{(x-y)}$$

If we continue to replicate the equation we get;

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y)-p_{n+1}(y)p_n(x)}{(x-y)}$$

$$=p_n(x)p_n(y)+\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)p_{n-1}(y)+\frac{\sigma_n(x)}{\sigma_{n-2}(x)}p_{n-2}(x)p_{n-2}(y)+\cdots$$

$$=\frac{\sigma_n(x)}{\sigma_0(x)}p_0(x)p_0(y).$$

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y)-p_{n+1}(y)p_n(x)}{(x-y)}=\sum_{k=0}^{n}\frac{\sigma_n(x)}{\sigma_k(x)}p_k(x)p_k(y)$$

$$\sum_{k=0}^{n}\frac{1}{\sigma_k(x)}p_k(x)p_k(y)=\frac{1}{\sigma_n(x)}\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y)-p_{n+1}(y)p_n(x)}{(x-y)}.$$

Definition 2.8 (Zeros) Every n zeros of an orthogonal polynomials $\mathrm{p_n(x)}$ are simple, and located on (a,b), the orthogonality interval. The zeros of $\mathrm{p_n(x)}$ and $\mathrm{p_{n+1}(x)}$ distinquish each other, and if $m<n$, there is at least one zero of $\mathrm{p_n(x)}$ between any two zeros of $\mathrm{p_m(x)}$.

Definition 2.9 (Explicit Representation)

- Trigonometric Functions
- Rodrigues Formulas
- Finite Power Series, the Hypergeometric Functions and Generalized Hypergeometric Functions
- Numerical Coefficient

2.1 Types of Orthogonal Polynomials

2.1.1 Discrete Orthogonal Polynomials

Definition 2.10 (Meixner polynomials) The Meixner polynomials $M_N^{\gamma,\mu}(x)$ are orthogonal with respect to the weight function $\omega(x)$ on $[0,\infty]$with

$\sigma(x) = x, \quad \tau(x) = \gamma\mu - x(1-\mu), \quad 0 < \mu < 1\,, \ \gamma > 0, \ \ \lambda_n = n(1-\mu),$

and

$$R_n = \frac{1}{(\mu-1)^n} \quad , \quad \rho(x) = \frac{\mu^x (1-\mu)^\gamma \Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(1+x)}, \quad d_n^2 = \frac{n!\,(\gamma)_n \mu^n}{(1-\mu)^{2n}}$$

Definition 2.11 (Kravchuk polynomials) The Kravchuk polynomials $K_n^p(x)$ are orthogonal with respect to the weight function $\omega(x)$ on $[0,N]$with $n \le N$

$$\sigma(x) = x, \quad \tau(x) = \frac{N_p - x}{1-p}, \quad 0 < p < 1\,, \quad \lambda_n = \frac{n}{1-p},$$

and

$$R_n = (p-1)^n \quad , \quad \rho(x) = \frac{p^x N!\,(1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(1+x)}, \quad d_n^2 = \frac{n!\,N!\,p^n(1-p)^n}{(N-n)!}$$

Definition 2.12 (Charlier polynomials) The Charlier polynomials $C_n^\mu(x)$ are orthogonal with respect to the weight function $\omega(x)$ on $[0,\infty]$with

$$\sigma(x) = x, \quad \tau(x) = \mu - x, \quad \mu > 0\,, \quad \lambda_n = n,$$

and

$$R_n = (-1)^n \quad , \quad \rho(x) = \frac{\mu^x (e)^{-\mu}}{\Gamma(1+x)}, \quad d_n^2 = n!\,\mu^n$$

Definition 2.13 (Hahn polynomials) The Hahn polynomials $h_n^{\alpha,\beta}(x,N)$ are orthogonal with respect to the weight function $\omega(x)$ on $[0,N)$ with

$(\alpha > -1, \beta > -1)$

$$\sigma(x) = x(x+\alpha-N), \qquad \tau(x) = (\beta+1)(N-1) - x(\alpha+\beta+2),$$

$$\lambda_n = n(\alpha+\beta+N+1),$$

and

$$R_n = \frac{(-1)^n}{(\alpha+\beta+n+1)_n} \ ,$$

$$\rho(x) = \frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-x)\Gamma(\beta+1+x)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-x)\Gamma(1+x)}$$

$$d_n^2 = \frac{\Gamma(N)\Gamma(\alpha+\beta+2)n!\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+N+n+1)(\alpha+\beta+n+1)_n^{-2}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)(\alpha+\beta+2n+1)(N-n-1)!\Gamma(\alpha+\beta+n+1)}$$

with the symmetry property

$$h_n^{\beta,\alpha}(N-1-x,N) = (-1)^n h_n^{\alpha,\beta}(x,N).$$

2.1.2 Continuous Orthogonal Polynomials

Definition 2.14 (Jacobi polynomials) The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ are orthogonal with respect to the weight function $\omega(x)$ on $[-1,1]$ with

$\sigma(x) = 1 - x^2$, $\tau(x) = -(\alpha+\beta+2)x + \beta - \alpha$, $\lambda_n = (n+\alpha+\beta+1)$, and

$$R_n = \frac{(-1)^n}{(n+\alpha+\beta+1)_n} \ ,$$

$$\rho(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha 1 + (1+x)^\beta \ , \alpha > -1 \ , \beta > -1$$

$$d_n^2 = \frac{2^{2n} n!\,\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)(n+\alpha+\beta+1)_n^2}$$

with the symmetry property

$$P_n^{\beta,\alpha}(-x) = (-1)^n P_N^{\alpha,\beta}(x).$$

Definition 2.15 (Laguerre polynomials) The Laguerre polynomials $L_n^{\alpha}(x)$ are orthogonal with respect to the weight function $\omega(x)$ on [0,∞) with

$\sigma(x) = x, \quad \tau(x) = -x + \alpha + 1, \quad \lambda_n = n,$

and

$$R_n = (-1)^n \ , \qquad \rho(x) = \frac{x^0 e^{-x}}{\Gamma(\alpha+1)}, \qquad \alpha > -1 \ , \qquad d_n^2 = \frac{\Gamma(n+\alpha+1)n!}{\Gamma(\alpha+1)}$$

It was mentioned that the weight functions $\omega(x)$ in the above formulas correspond to probability measures, for example total weight equal to 1. This will be useful in obtaining the correct limit relations between the corresponding generalized polynomials.

For all monic polynomials we also have,

$$M_m^{\gamma,\mu}(0) = \frac{\mu^n}{(\mu-1)^n}\frac{\Gamma(n+\gamma)}{\Gamma(\gamma)}, \qquad K_n^p(0) = \frac{(-p)^n N!}{(N-n)!}, \qquad C_n^{\mu}(0) = (-\mu)^n,$$

$$h_n^{\alpha,\beta}(0,N) = \frac{(-1)^n \Gamma(\beta+n+1)(N-1)!}{\Gamma(\beta+1)(N-n-1)!\,(n+\alpha+\beta+1)_n},$$

$$h_n^{\alpha,\beta}(N-1,N) = \frac{\Gamma(\alpha+n+1)(N-1)!}{\Gamma(\alpha+1)(N-n-1)!\,(n+\alpha+\beta+1)_n},$$

$$P_n^{\alpha,\beta}(1) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n} \ , \qquad P_n^{\alpha,\beta}(-1) = \frac{2^n(-1)^n(\beta+1)_n}{(n+\alpha+\beta+1)_n},$$

$$L_n^{\alpha}(0) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}$$

Chapter 3

CLASSICAL ORTHOGONAL POLYNOMIALS

Definition 3.1 A polynomial $p_m(x)$ form of classical orthogonal set, if it satisfies the hypergeometric type differential equations

$U(x)p''_m(x) + V(x)p'_m(x) + \gamma_m p_m(x) = 0$ with respect to the Pearson equation $\frac{d}{dx}[U(x).w(x)] = V(x).w(x)$.

Classical orthogonal polynomials have properties as follows:

- Orthogonality relation is satisfied.
- Rodrigues formula is satisfied as follows:

  $p_s(x) = \frac{K_s}{w(x)} \frac{d^s}{d_x^s}[w(x).U^s(x)].$
- They can be written in hypergeometric form.
- Their derivatives ($\{p'_m(x)\}$) also form an orthogonal system.
- Hypergeometric type differential equation is satisfied as follows:

  $U(x)p''_m(x) + V(x)p'_m(x) + \gamma_m p_m(x) = 0.$

3.1 Examples for the Classical Orthogonal Polynomials

- Hermite Polynomials : $H_m(x)$.
- Laguerre Polynomials: $L_m^{\gamma}(x)$ where $\gamma > -1$.
- Jacobi Polynomials: $P_m^{(\gamma,\delta)}(x)$ where $\gamma > -1$, $\delta > -1$.

3.2 Hermite Polynomials

3.2.1 Generating Function for Hermite Polynomials

We can define the Hermite Polynomials as ,

$$H_m(x) = \sum_{l=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^l (2x)^{m-2l} m!}{(m-2l)!\, l!}.$$

Thus,

$H_m(x) = (2x)^m - \frac{m!(2x)^{m-2}}{(m-2)!2!} + \frac{m!(2x)^{m-4}}{(m-4)!3!} - \cdots$,

where m is the highest degree of $H_m(x)$.

The polynomial is defined by

$H_m(x) = 2^m x^m + \tau_{m-2}(x)$, where the degree of polynomial $\tau_{m-2}(x)$ is $(m-2)$ in x.

- Assume that m is even, then the polynomial of $H_m(x)$ is also even polynomial.
- Assume that m is odd, then the polynomial of $H_m(x)$ is also odd polynomial.

3.2.2 Hypergeometric Representation of Hermite Polynomials

Let

$$H_m(x) = \sum_{l=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^l (2x)^{m-2l} m!}{(m-2l)!\, l!}.$$

In the equation above $(2x)^m$ does not depend on l so we can take $(2x)^m$ to the outside of the summation. Then we have

$H_m(x) = (2x)^m \sum_{l=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^l m!}{(m-2l)!l!} \left(\frac{1}{2x}\right)^{2l} = (2x)^m \sum_{l=0}^{\left[\frac{m}{2}\right]} \frac{m!}{2^{2l}(m-2l)!l!} \left(\frac{-1}{x^2}\right)^l.$

Now we can use pochhammer symbol property (1.5) and (1.6). From (1.5) and (1.6), we get the following;

$$(m-2l)! = \frac{(-1)^{2l} m!}{(-m)_{2l}}$$

$$\frac{m!}{(m-2l)!} = (-m)_{2l} = 2^{2l}\left(\frac{-m}{2}\right)_l \left(\frac{-m+1}{2}\right)_l$$

$$H_m(x) = (2x)^m \sum_{l=0}^{\left[\frac{m}{2}\right]} \frac{2^{2l}\left(\frac{-m}{2}\right)_l \left(\frac{-m+1}{2}\right)_l}{2^{2l} l!} \left(\frac{-1}{x^2}\right)^l$$

$$H_m(x) = (2x)^m \quad {}_2F_0\left(\frac{-m}{2}, \frac{-m+1}{2}; -; \frac{-1}{x^2}\right).$$

3.2.3 Reccurrence relations for Hermite Polynomials

We can define recurrence relation for Hermite polynomials as

$$2(m+1)H_m(x) = 2H_m(x) + 2xH_m'(x) - H_m''(x).$$

Now send all terms into the one side;

$$H_m''(x) - 2xH_m'(x) + 2mH_m(x) = 0\ .$$

We get the hypergeometric equation with;

$U(x) = 1,$

$V(x) = -2x,$

$\varphi_n = 2m.$

3.2.4 Orthogonal Relations for Hermite Polynomials

Ortogonality of Hermite polynomials is defined as follows ;

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_s(x) = 0,$$

where the left hand side will be zero because of the orthogonality conditions.

The orthogonality interval is $(-\infty, \infty)$ with weight functions : $w(x) = e^{-x^2}$.

3.2.5 Rodrigues Formula for Hermite Polynomials

For Hermite polynomials , we can give the Rodrigues formula;

Let $L_m = (-1)^m$ then ,

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}(e^{-x^2}) .$$

3.2.6 Derivative of Hermite Polynomials

We define the derivative of Hermite polynomials as follows;

$$\frac{d}{dx} H_m(x) = 2mH_{m-1}(x).$$

3.2.7 Finding the Coefficients a_m and c_m for Hermite Polynomials

- ✓ For the coefficient $\boldsymbol{a_m}$, we can define the formula as follows;

$$a_m = L_m \prod_{k=0}^{m-1} [V'(x) + \frac{m+k-1}{2} U''(x)],$$

$$a_m = (-1)^m \prod_{k=0}^{m-1} [-2] = 2^m$$

$$a_m = 2^m.$$

- ✓ For the coefficient $\boldsymbol{c_m}$, we can define the formula as follows;

$$c_m = ma_n \frac{V_{m-1}(0)}{V'_{m-1}(x)},$$

where

$$V_m(x) = V(x) + mU'(x)$$

$$V_m(x) = -2x \quad V_{m-1}(0) = 0$$

$$V'_m(x) = -2 \quad V'_{m-1}(x) = -2$$

$$c_m = m2^m 0 \;\; \rightarrow \;\; c_m = 0$$

3.2.8 Normalization Function for Hermite Polynomials

$$\int_a^b p_m(x)p_s(x)w(x)dx = \sigma_m \delta_{sm}$$

is generalized form for the orthogonality equations. We can get these equations by using the norm of Hermite Polynomials as follows ;

$$\int_{-\infty}^{\infty} H_s(x)H_m(x)e^{-x^2}dx = 2^m m!\sqrt{\pi}\delta_{sm}.$$

3.3 Laguerre Polynomials

3.3.1 Rodrigues Formula and Hypergeometric Representation of Laguerre Polynomials

The Rodrigues formula for Laguerre polynomials are defined as follows ;

$$L_m^{\gamma}(x) = \frac{x^{-\gamma}e^x}{m!}\frac{d^m}{dx^m}(e^{-x}x^{m+\gamma})$$

and also one can define the hypergeometric representation of Laguerre polynomials as

$$L_m^{\gamma}(x) = \frac{(\gamma+1)_m}{m!} \quad {}_1F_1(-m;\gamma+1;x)$$

where

$$\sum_{l=0}^{m}\frac{(-m)_l}{k!\,(\gamma+1)_l}x^l = \quad {}_1F_1(-m;\gamma+1;x)$$

3.3.2 Gamma Functions Representation of Laguerre Polynomials

Representation of Laguerre polynomials defined as follows ;

$$L_m^{\gamma}(x) = \sum_{l=0}^{m}\frac{\Gamma(\gamma+1+m)}{\Gamma(\gamma+1+l)}\frac{(-x)^l}{(m-l)!\,l!} .$$

3.3.3 Generating Function for Laguerre Polynomials

The generating function of Laguerre polynomials has the form ;

$$\sum_{m=0}^{\infty} L_m^{\gamma}(x) y^m = \frac{1}{(1-y)^{\gamma+1}} e^{\frac{-xy}{1-y}}.$$

3.3.4 Recurrence Relation for Laguerre Polynomials

3 term recurrence relation has the form ;

$$(m+1)L_{m+1}^{\gamma}(x) + (x - 2m - \gamma - 1)L_m^{\gamma}(x) + (m+\gamma)L_{m-1}^{\gamma}(x) = 0. \quad m = 1,2,...$$

3.3.5 Orthogonality Relation for Laguerre Polynomials

Orthogonality of Laguerre polynomials is defined as follows ;

$$\int_0^{\infty} e^{-x} x^{\gamma} L_m^{\gamma}(x)\, L_s^{\gamma}(x) d_x = 0\,.$$

where the left hand side will be zero because of the orthogonality conditions.

The orthogonality interval is $[0, \infty)$ with weight function: $w(x) = e^{-x}x^{\gamma}$.

For the Laguerre polynomials, we can give the rodrigues formula

where $L_m = \frac{1}{m!}$

$$L_m^{\gamma}(x) = \frac{x^{-\gamma} e^{x}}{m!} \frac{d^m}{dx^m} (e^{-x} x^{m+\gamma})\,.$$

3.3.6 Derivative of Laguerre Polynomials

Derivative of Laguerre Polynomials has the form;

$$\frac{d}{dx} L_m^{\gamma}(x) = -L_{m-1}^{\gamma+1}(x).$$

3.3.7 Finding the Coefficients a_m and c_m for Laguerre Polynomials

✓ For the coefficient $\boldsymbol{a_m}$, we can define the formula as follows;

$$a_m = L_m \prod_{k=0}^{m-1} [V'(x) + \frac{m+k-1}{2} U''(x)].$$

$$a_m = \frac{1}{m!} \prod_{k=0}^{m-1} (-1)$$

$$a_m = \frac{(-1)^m}{m!}.$$

✓ For the coefficient $\boldsymbol{c_m}$, we can define the formula as follows;

$$c_m = m a_m \frac{V_{m-1}(0)}{V'_{m-1}(x)},$$

where

$V_m(x) = V(x) + mU'(x)$. $V_m(x) = 1 + \gamma - x + m$

$V_{m-1}(x) = 1 + \gamma - x + m - 1 = \gamma - x + m$

$V_{m-1}(0) = \gamma + m$ $V'_{m-1}(x) = -1$

$$c_m = m \frac{(-1)^m}{m!} \frac{\gamma + m}{-1}$$

$$c_m = \frac{(-1)^{m-1}(\gamma + m)}{(m-1)!} \ .$$

3.3.8 Normalization Function for Laguerre Polynomials

Normalization function for Laguerrre Polynomials are defined as follows;

$$\beta_m = \frac{\Gamma(m + \gamma + 1)}{m!} \ .$$

3.4 Jacobi Polynomials

3.4.1 Rodrigues Formula and Hypergeometric Representation of Jacobi Polynomials

Definition3.4.1: For Jacobi Polynomials, the rodrigues formula is defined as follows;

$$P_m^{(\gamma,\delta)}(x) = \frac{(-1)^m}{2^m m!}(1-x)^{-\gamma}(1+x)^{-\delta}\frac{d^m}{d_x^m}\left[(1-x)^{\gamma+m}(1+x)^{\delta+m}\right].$$

For Jacobi Polynomials, there are 4 different hypergeometric representations;

1. The 1st representation of Jacobi Polynomials is ;

$$P_m^{(\gamma,\delta)}(x) = \left(\frac{x-1}{2}\right)^m \frac{(1+\delta)_m}{m!} \quad {}_2F_1\left(-m, -m-\gamma; \delta+1; \frac{1+x}{x-1}\right).$$

2. The 2nd representation of Jacobi Polynomials is ;

$$P_m^{(\gamma,\delta)}(x) = \left(\frac{1+x}{2}\right)^m \frac{(1+\gamma)_n}{m!} \quad {}_2F_1\left(-m, -m-\delta; \gamma+1; \frac{1+x}{x-1}\right).$$

3. The 3rd representation of Jacobi Polynomials is ;

$$P_m^{(\gamma,\delta)}(x) = \frac{(\gamma+1)_m}{m!} \quad {}_2F_1\left(-m, \delta+m+\gamma+1; \gamma+1; \frac{1-x}{2}\right).$$

4. The 4th representation of Jacobi Polynomials is ;

$$P_m^{(\gamma,\delta)}(x) = \frac{(-1)^m(\delta+1)_m}{m!} \quad {}_2F_1\left(-m, \delta+m+\gamma+1; \delta+1; \frac{1+x}{2}\right).$$

3.4.2 Symmetry Property of Jacobi Polynomials

Symmetry relation for Jacobi polynomials is defined as

$$P_m^{(\gamma,\delta)}(-x) = (-1)^m P_m^{(\delta,\gamma)}(x).$$

3.4.3 Generating Function of Jacobi Polynomials

Generating function of Jacobi polynomials has the form

$$\sum_{m=0}^{\infty} \frac{(\gamma+\delta+1)_m P_m^{(\gamma,\delta)}(x) t^m}{(\gamma+1)_m}$$

$$= \frac{1}{(1-t)^{\gamma+\delta+1}} *_2 F_1 \left(\frac{1}{2}(\delta+\gamma+1), \frac{1}{2}(\delta+\gamma+2); \gamma + 1; \frac{2t(x-1)}{(1-t)^2}\right)$$

3.4.4 Orthogonal Relation for Jacobi Polynomials

Orthogonality of Jacobi polynomials is defined as follows ;

$$\int_{-1}^{1} (1-x)^{\gamma}(1+x)^{\delta} P_m^{(\gamma,\delta)}(x)\, P_s^{(\gamma,\delta)}(x) = 0.$$

where the left hand side will be zero because of the orthogonality conditions.

The orthogonality interval is $[-1,1]$ with weight function:

$w(x) = (1-x)^{\gamma}(1+x)^{\delta}$.

3.4.5 Three Term Recurrence Relation for Jacobi Polynomials

3 term recurrence relation for Jacobi polynomials has the form ;

$$xP_m^{(\gamma,\delta)}(x) = \frac{2(m+1)(m+\gamma+\delta+1)}{(2m+2+\gamma+\delta)(2m+1+\gamma+\delta)} P_{m+1}^{(\gamma,\delta)}(x) + \frac{\delta^2-\gamma^2}{(\gamma+\delta+2m)(\gamma+\delta+2m+2)} P_m^{(\gamma,\delta)}(x) + \frac{2(m+\delta)(m+\gamma)}{(2m+\gamma+\delta+2)(2m+\gamma+\delta+1)} P_{m-1}^{(\gamma,\delta)}(x).$$

3.4.6 Derivative of Jacobi Polynomials

The derivative of Jacobi polynomials is defined as follows;

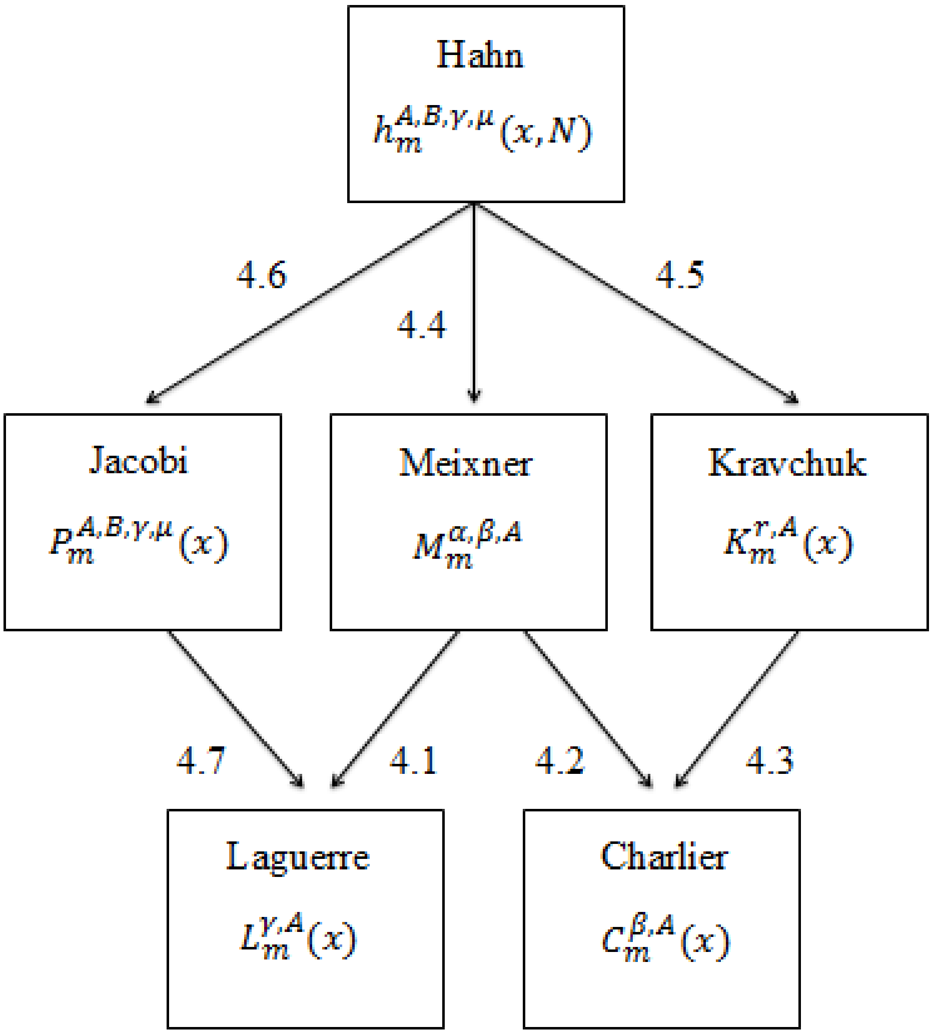
$$\frac{d}{d_x} P_m^{(\gamma,\delta)}(x) = \frac{(\gamma+\delta+m+1)}{2} P_{m-1}^{(\gamma+1,\delta+1)}(x).$$

Chapter 4

LIMIT RELATIONS

In this chapter we contemplate the dissimilar limit transition for moderation of the classical polynomials acquired by the addition of one or two point masses at the end of the interval of orthogonality. The relationship between Jacobi, Laguerre, Charlier, Meixner, Kravchuk and Hahn generalized polynomials are proved.

Limit Relations involving the generalized polynomials



4.1 Limit relation between Meixner and Laguerre Polynomials

Limit relation between Meixner and Laguerre polynomials is defined as follows;

$$\lim_{h\to 0} h^m M_m^{\gamma+1,1-h}\left(\frac{x}{h}\right) = L_m^{\gamma}(x)$$

Proof: We need to prove that

$$\lim_{h\to 0} h^m M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) = L_m^{\gamma,A}(x)$$

$$M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) = \lim_{h\to 0} h^m [(M_m^{\gamma+1,1-h}\left(\frac{x}{h}\right)$$

$$+ A\frac{(\gamma+1)_m(1-h)^m}{n!\,(1+AKer_{m-1}^{L}(0,0)}\frac{M_m^{\gamma+1,1-h,A(\frac{x}{h})} - M_m^{\gamma+1,1-h,A}(\frac{x-h}{h})}{h}$$

We know that $\lim_{h\to 0} h^m M_m^{\gamma+1,1-h}\left(\frac{x}{h}\right) = L_m^{\gamma}(x)$, if we substitute in the above formula, we get

$$M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right)$$

$$= L_m^{\gamma}(x) + \lim_{h\to 0} h^m A\frac{(\gamma+1)_m(1-h)^m}{n!\,(1+AKer_{m-1}^{L}(0,0)}\frac{M_m^{\gamma+1,1-h,A(\frac{x}{h})} - M_m^{\gamma+1,1-h,A}(\frac{x-h}{h})}{h}$$

From generalized Laguerre case we know that $Ker_{m-1}^{L}(0,0) = \sum_{S=0}^{m-1}\frac{(\gamma+1)_S}{s!}$

Then we have

$$M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) = L_m^{\gamma}(x)$$

$$+ \lim_{h\to 0} h^m A\frac{(\gamma+1)_m(1-h)^m}{n!\,(1+A\sum_{S=0}^{m-1}\frac{(\gamma+1)_s(1-h)^s}{s!}}\frac{M_m^{\gamma+1,1-h,A}(\frac{x}{h}) - M_m^{\gamma+1,1-h,A}(\frac{x-h}{h})}{h}$$

Assume that $\Gamma_m = \frac{A(\gamma+1)_m}{n!\left(1+AKer_{m-1}^{L}(0,0)\right)}$

$$\Rightarrow \Gamma_m(h^m(1-h)^m(\frac{M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right)-M_m^{\gamma+1,1-h,A}\left(\frac{x-h}{h}\right)}{h})$$

$$\lim_{h\to 0} M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) = L_m^{\gamma}(x)$$

$$+ \lim_{h\to 0}\Gamma_m(h^m(1-h)^m(\frac{M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) - M_m^{\gamma+1,1-h,A}\left(\frac{x-h}{h}\right)}{h})$$

It is clear that $\left(\frac{M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) - M_m^{\gamma+1,1-h,A}\left(\frac{x-h}{h}\right)}{h}\right) = (\frac{d}{dx} M^{\gamma+1,1-h,A}\left(\frac{x}{h}\right))$

$$\lim_{h\to 0} M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) = L_m^{\gamma}(x) + \lim_{h\to 0}\Gamma_m(h^m(1-h)^m\left(\frac{d}{dx} M^{\gamma+1,1-h,A}\left(\frac{x}{h}\right)\right))$$

Now , according to the Meixner representation formulas

$$M_m^{\alpha,\beta,A}(x) = M_m^{\alpha,\beta}(x) + B_m \nabla M_m^{\alpha,\beta}(x) = (I + B_m \nabla) M_m^{\alpha,\beta}(x), \tag{4.1}$$

$B_m = A \frac{\beta^m (1-\beta)^{-1}(\alpha)_m}{m!(1+AKer_{m-1}^M(0,0))}$,

we find $\alpha = \gamma + 1 \quad \beta = 1 - h.$

Substitute α and β in Meixner representation formulas (4.1) and multiply both sides with h^m.

We find

$$\lim_{h\to 0} h^m M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) = h^m M_m^{\gamma+1,1-h}\left(\frac{x}{h}\right) + h^m B_m \nabla M_m^{\alpha,\beta}(x)$$

$$\lim_{h\to 0} h^m M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right)$$

$$= h^m M_m^{\gamma+1,1-h}\left(\frac{x}{h}\right)$$

$$+ h^m A \frac{(1-h)^m (1-(1-h))^{-1}(\gamma+1)_m}{m!\,(1 + AKer_{m-1}^M(0,0))} \nabla M_m^{\alpha,\beta}(x)$$

Since $\Gamma_m = \frac{A(\gamma+1)_m}{n!\left(1+AKer_{m-1}^L(0,0)\right)}$,

$$\lim_{h\to 0} M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right) = L_m^{\gamma}(x) + \lim_{h\to 0} h^m \Gamma_m (1-h)^m \frac{\nabla M_m^{\gamma+1,1-h}\left(\frac{x}{h}\right)}{h}$$

$$\Longrightarrow \lim_{h\to 0}\Gamma_m(1-h)^m \lim_{h\to 0} \frac{h^m \nabla M_m^{\gamma+1,1-h}\left(\frac{x}{h}\right)}{h}$$

$$\Longrightarrow \lim_{h\to 0}\Gamma_m(1-h)^m \lim_{h\to 0}\nabla L_m^{\gamma}(x)$$

$$\Longrightarrow \lim_{h\to 0}\Gamma_m(1-h)^m \frac{dL_m^{\gamma}(x)}{dx}$$

$$=L_m^{\gamma}(x)+(\lim_{h\to 0}\Gamma_m(1-h)^m)\frac{dL_m^{\gamma}(x)}{dx}$$

Now, according to the Laguerre representation formulas

$$L_m^{\gamma,A}(x)=L_m^{\gamma}(x)+\Gamma_m\frac{d}{dx}L_m^{\gamma}(x)=\left(I+\Gamma\frac{d}{dx}\right)L_m^{\gamma}(x),$$

$$\Gamma_m=\frac{A(\gamma+1)_m}{m!\,(1+AKer_{m-1}^{L}(0,0))}=\frac{A(\gamma+1)_m}{m!\,(1+A(\frac{(\gamma+2)_{m-1}}{(m-1)!}))}$$

We find

$$\lim_{h\to 0}M_m^{\gamma+1,1-h,A}\left(\frac{x}{h}\right)=L_m^{\gamma}(x)+\Gamma_m\frac{d}{dx}L_m^{\gamma}(x)=L_m^{\gamma,A}(x)$$

4.2 Limit relation between Meixner and Charlier Polynomials

Limit relation between Meixner and Charlier polynomials is defined as follows;

$$\lim_{\alpha\to\infty}M_m^{\alpha,\left(\frac{\beta}{\beta+\alpha}\right)}(x)=C_m^{\beta}(x)$$

4.3 Limit relation between Kravchuk and Charlier Polynomials

Limit relation between Kravchuk and Charlier polynomials is defined as follows;

$$\lim_{N\to\infty}K_m^{\beta/N}(x)=C_m^{\beta}(x)$$

4.4 Limit relation between Hahn and Meixner Polynomials

Limit relation between Hahn and Meixner polynomials is defined as follows;

$$\lim_{N\to\infty} h_m^{((1-\beta)/\beta)N,\alpha-1}(x,N) = M_m^{\alpha,\beta}(x)$$

where

$$h_m^{\gamma,\mu}(x,N) = \frac{(-1)^m (N-1)!\Gamma(\mu+m+1)}{m!(N-m-1)!\Gamma(\mu-1)} \quad {}_3F_2\left({-x,\gamma+\mu+m+1,-m \atop 1-N,\mu+1}; 1\right)$$

and

$$M_m^{\alpha,\beta}(x) = (\alpha)_m \frac{\beta^m}{(\beta-1)^m} \quad {}_2F_1\left({-m,-x \atop \alpha}; 1-\frac{1}{\beta}\right)$$

4.5 Limit relation between Hahn and Kravchuk Polynomials

Limit relation between Hahn and Kravchuk polynomials is defined as follows;

$$\lim_{t\to\infty} h_m^{(1-r)t,rt}(x,N) = K_m^r(x,N-1)$$

4.6 Limit relation between Hahn and Jacobi Polynomials

Limit relation between Hahn and Jacobi polynomials is defined as follows;

$$\lim_{N\to\infty} \frac{2^m}{N^m} h_m^{\gamma,\mu}((N-1)x,N) = P_m^{\gamma,\mu}(2x-1)$$

4.7 Limit relation between Jacobi and Laguerre Polynomials

Limit relation between Jacobi and Laguerre polynomials is defined as follows;

$$\lim_{\mu\to\infty} \frac{(-1)^m \mu^m}{2^m} P_m^{\gamma,\mu}\left(1-\frac{2x}{\mu}\right) = L_m^{\gamma}(x)$$

Chapter 5

CONCLUSION

The first two chapters in this thesis, attempt to introduce the orthogonal polynomials. They were written in order to supply conclusive information regarding any set of orthogonal polynomials. A frequentative progression to create a collection of polynomials which are orthogonal in relation to each other, are then supplied with several attributes agreeable by any assemblage of orthogonal polynomials as detailed. The classical orthogonal polynomials become apparent when the weight functions connected to the orthogonality status becomes a certain form.

These polynomials acquire an additional accumulation of contributing factors and directly fulfill a secondary differential equation, which are detailed in chapter three. The chapters that follow delve into the breakdown of specific polynomial collections beginning at the differential equation. Known as classical orthogonal polynomials referred to as Hermite polynomials, Laguerre polynomials and Jacobi polynomials.

In the following sections significant characteristics of classical orthogonal polynomials relating to the weight function, time lapse of the orthogonality and the secondary direction of differential equation, Rodrigues formula, hypergeometric representations are given.

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