On Impulsive Sequential Fractional Differential Equations (ISFDE's)

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ABSTRACT

In this thesis, we study the existence and uniqueness of a nonlinear impulsive sequential fractional differential equations of order $\alpha \in (1,2]$ involving Liouville-Caputo fractional derivative supplemented with the separate boundary value conditions. The subject of boundary value problem and fractional differential equations are very important in many fields of science and engineering. In fact, both sequential fractional differential equations and Impulsive fractional differential equations are studied individually from various perspectives. However, this topic combining both of them to produce a wider case namely, impulsive sequential fractional differential equations. By doing so, a new existence and uniqueness results of solutions are provided for the problems.

Keywords: Nonlinear impulsive sequential fractional differential equations, Caputo fractional derivative, Banach fixed point theorem.

ÖZ

Bu tezde, ayrı sınır-değer koşullarıyla desleklenmiş, Liouville-Caputo kesirli türevi içeren, $\alpha \in (1,2]$ mertebeli doğrusal olmayan Impılsif sıralı kesirli diferansiyel denklemlerin varlık ve tekliği çalışılacaktır. Sınır-değer problemleri ve kesirli diferansiyel denklemler, temel bilimler ve mühendisliğin birçok alanında çok büyük önem taşımaktadır. Hem sıralı kesirli diferansiyel denklemler hem de impulsif kesirli diferansiyel denklemler ayrı ayrı farklı perspektiflerden çalışılmıştır. Ancak, bu iki tip diferansiyel denklemin birleştirilmesiyle elde edilen ve daha geniş bir sınıf oluşturan impulsif sıralı kesirli diferansiyel denklemler sadece bu tezde çalışılmıştır. Bu çalışmada, bu özel tipteki diferansiyel denklemlerin çözümü için yeni varlık ve teklik sonuçları elde edilmiştir.

Anahtar Kelimeler: doğrusal olmayan impulsif sıralı kesirli diferansiyel denklemler, Caputo kesirli türevi, Banach Sabit nokta teoremi.

DEDICATION

I am dedicating this thesis to my family

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LIST OF ABBREVIATIONS

BVP	Boundary Value Problem
BC’s	Boundary Conditions
GF	Gamma Function
CD	Caputo derivative
IC’s	Impulsive Conditions
FDE’s	Fractional Differential Equations
FC	Fractional Calculus
L-C	Liouville-Caputo
FP	Fixed Point
FPT	Fixed Point Theorem
R&L- HL’S	Right and Left–Hand Limit’s
R-L	Riemann-Liouville
LHS	Left Hand Side
RHS	Right Hand Side

Chapter 1

INTRODUCTION

The 1695 curiosity of L'Hopital in response to a letter by Leibniz's regarding the meaning of derivatives with integer order being generalizing to derivatives with non-integers has been an old but developed topic FC as a sect of calculus is a process of investigation as well as application of integrals and derivatives into a relative order [1], [2].

Recent work on BVP with FDE's have successfully been adopted and implemented in various scientific and engineering fields. The results and impact of BVP and FDE's have led to continuous interest and further research works on the field. BVP and FDE'S including the R–L fractional derivative or the Caputo fractional derivative have been gaining much importance to see [3]- [28].

On the other hand, the subjects of impulsive-FDR's and sequential -FDE's have been recently addressed by many researchers and it is paid more and more attention [29]- [40]. However, this thesis ties both of them to produce a wider case namely, impulsive sequential fractional differential equations.

A couple of FPT, which are Leray-Schauder's and Altman's were used, in 2011 [29], to find the existence of the solutions of the problem which is given as follows:

$$
\begin{gathered}
{}^{c}D^{q}u(t) = f\left(t, u(t)\right), \mathrm{t} \in [0,1], 1<\mathrm{q} \leq 2, \\
au(0) - bu'(0) = x_0 \quad cu(1) + du'(1) = x_1, \\
\Delta u(t_k) = u(t_k^+) - u(t_k^-) = Q_k\left(u(t_k)\right), \\
\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k\left(u(t_k)\right),\ k = 1, 2, \ldots, p,
\end{gathered}
$$

with ${}^{c}D^{q}$ Caputo derivative of order $q \in (1,2]$.

In [30], Liu and Haibo studied the following problem of nonlocal BVP with IFDE's:

$$
\begin{gathered}
{}^{c}D^{q}u(t) = f\left(t, u(t), u'(t)\right), t \in [0,1], 1 < q \leq 2, \\
\alpha u(0) + \beta u'(0) = g_1(u), \quad \alpha u(1) + \beta u'(1) = g_2(u), \\
\Delta u(t_k) = I_k\left(u(t_k)\right), \Delta u'(t_k) = J_k\left(u(t_k)\right), t_k \in (0,1)\ k = 1, 2, \ldots, p,
\end{gathered}
$$

where $J = [0,1]$, $F : J \times R \times R \to R$ is a continuous function.

In [31], Jianxin and Haibo studied the problem of nonlinear IFDE's with BC's that consider as follows:

$$
\begin{gathered}
{}^{c}D^{q}u(t) = f\left(t, u(t), u(\omega(t))\right), t \in J', \\
u(t) = 0, t \in [-r, 0] \\
\Delta u|_{t=t_k} = I_k\left(u(t_k)\right), \quad \Delta u|_{t=t_k} = I^{*}{}_{k}\left(u(t_k)\right), \ k = 1, 2, \ldots, m,
\end{gathered}
$$

subject to the nonlinear BC's as follows:

$$
g_0\left(u(t), u(T)\right) = 0, g_0\left(u'(t), u'(T)\right) = 0
$$

with ${}^{c}D^{q}$ Caputo derivative of order q.

In [32], Xiaoping, Fulai & Xuezhu studied the following anti-periodic BVP of IFDE's:

$$
\begin{gathered}
{}^{c}D_0^{q}u(t) = f\left(t, u(t)\right), t \in J' := J \setminus \{t_1, t_2, \ldots, t_m\}, J = [0,1], \\
\alpha u(0) + bu(1) = 0, \quad \alpha u'(0) + bu'(1) = \sigma_2, \\
\Delta u(t_k) = I_k, \quad \Delta u'(t_k) = J_k, \ k = 1, 2, \ldots, m,
\end{gathered}
$$

where ${}^{c}D_t^{q}$ the Caputo fractional derivative of order $q \in (1,2]$.

In [33], Peiluan & Youlin studied the following impulsive-FDE with nonlocal BVP's:

$$
\begin{gathered}
{}^{c}D_t^{q}u(t) = f\left(t, u(t)\right), t \in J = [0,1] \setminus \{t_1, t_2, \ldots, t_p\}, \\
au(0) + bu(1) = g_1(u), \quad au'(1) + bu'(1) = g_2(u), \\
\Delta u(t_k) = I_k u\left(t_k^{-}\right), \quad \Delta u'(t_k) = J_k u\left(t_k^{-}\right), \ k = 1, 2, \ldots, p,
\end{gathered}
$$

where ${}^{c}D_t^{q}$ the Caputo fractional derivative of order $q \in (1,2]$.

In 2016 [34], Shuai & Shuqin, discussed a IFDE's with BVP. They transferred the BVP into the equivalent integral equation. Banach- FPT and Schauder- FPT. They are used to acquire the existence of the solutions of the following problem:

$$
\begin{gathered}
{}^{c}D^{\alpha}x(t) = f\left(t, x(t)\right), \alpha \in (1,2], t \in J = [0,1], t = t_k, \\
x(0) = h(x), x(1) = g(x) \\
\Delta x|_{t=t_k} = I_k\left(x(t_k)\right),\ \Delta x|_{t=t_k} = I_k^*\left(x(t_k)\right),\ k = 1,2,...,m; t_k = (0,1) \\
h(x) = \min{}_j \frac{\left|x(\zeta_j)\right|}{\kappa + \left|x(\zeta_j)\right|}, g(x) = \max{}_j \frac{\left|x(\xi_j)\right|}{\lambda + \left|x(\xi_j)\right|},
\end{gathered}
$$

where $\zeta_j < 1,\ 0 < \xi_j,\ \zeta_j \neq t_i,\ j = 1,2,...,n,$ and κ, λ are given positive constants.

In [35], Nazim and Unul provided existence of solutions for the following IFDE's of order q with mixed BVP:

$$
\begin{gathered}
{}^{c}D_0^{q}x(t) = f\left(t, x(t)\right), t \in [0,1], 1 < q \leq 2, \\
x(0) + \mu_1 x'(1) = \sigma_1,\ x(0) + \mu_2 x'(1) = \sigma_2, \\
\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x(t_k)\right), \\
\Delta x'(t_k) = x'\left(t_k^+\right) - x'\left(t_k^-\right) = J_k\left(x(t_k)\right),\ k = 1,2,...,q,
\end{gathered}
$$

with ${}^{c}D^{q}$ Caputo derivative of order $q \in (1,2],\ f \in (J \times R, R),\ \varphi_k, I_k \in R \to R$ is a continuous function, , $0 = t_0 < t_1 < ... < t_k < ... < t_p < t_{p+1} = 1.$

In 2013 [36], Bashir & Juan FPT used to establish the existence results for a sequential integro-differential equation of fractional order with some BC's, which is given as:

$$
\begin{gathered}
\left({}^{c}D^{\alpha} + \lambda {}^{c}D^{\alpha-1}\right)u(t) = pf\left(t, u(t)\right) + qI^{\alpha}g\left(t, u(t)\right), t \in [0,1], \\
u(t) = 0, u(1) = 0,
\end{gathered}
$$

where ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α.

In [37], Bashir & Sotiris obtained a new existence result by using standard FPT:

$$
\begin{gathered}
{}^{c}D^{q}\left(D + \lambda\right)x(t) \in f_1\left(t, x(t),\right), \mathrm{q} \in (2,3], 0 < t < 1, n < \xi < n-1, \\
x(0) = 0, x'(1) = 0, \\
x'(0) = 0, ..., x^{n-1}(0) = 0, x(1) = \alpha x(\sigma),
\end{gathered}
$$

where ${}^{c}D^{q}$ is the Caputo fractional derivatives, $f:[0,1]\times R\to\mathcal{F}(R)$ is a multivalued map, $\mathcal{F}(R)$ is the family of all subsets of R.

In [38], the standard FPT has been used to obtain some existence results of the solutions for following problem:

$$\left({}^{c}D^{\alpha}+k\,{}^{c}D^{\alpha-1}\right)x(t)=f\left(t,x(t)\right),t\in[0,1],2<\alpha\leq 3$$
$$x(t)=0,x'(0)=0,x(\zeta)=a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}x(s)ds,\beta>0,$$

where ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α. $0<\eta<\zeta<1$.

In [39], B, Ahmad analyzed the existence and uniqueness results of three-BVP and sequential fractional integral-differential, given as the following:

$$\left({}^{c}D^{q}+\lambda\,{}^{c}D^{q-1}\right)x(t)=f\left(t,{}^{c}D^{\beta}x(t),I^{\gamma}x(t)\right),\mathrm{q}\in(2,3],t=[0,1],k>0,\gamma<1,$$
$$x(0)=0,x'(1)=0,$$
$$\sum_{i=1}^{m}a_ix(\zeta_i)=\lambda\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}x(s)ds,\delta\geq 1,0<\eta<\zeta<..<\zeta<1,$$

Here $D(.)$ is the Caputo derivatives of fractional order$(.)$, $f:[0,1]\times R^{3}\to R$ is a given continuous function.

In [40], Alsaed, et al used Banach-FPT to develop the existence theory for the following problem:

$$\left({}^{c}D^{q}+k\,{}^{c}D^{q-1}\right)u(t)=f\left(t,u(t)\right),1<q\leq 2,t\in[0,T],$$
$$\alpha_1u(0)+\sum_{i=1}^{m}a_iu(\eta_i)+\gamma_1u(T)=\beta_1,$$
$$\alpha_2u'(0)+\sum_{i=1}^{m}b_iu'(\eta_i)+\gamma_2u'(T)=\beta_2,$$
$$\alpha_3u''(0)+\sum_{i=1}^{m}c_iu''(\eta_i)+\gamma_3u''(T)=\beta_3,$$

with ${}^{c}D$ Caputo fractional derivative.

Moreover, in chapter 2 we will consider some concept of FC and FDE’s and in chapter 3 we will consider the problems involving an Impulsive (SFDE’s) with different BCs. The existence and uniqueness results of the solution of an Impulsive- SFDEs with BC’s are obtained by using some theorems such as FPT, contraction mapping and Krasnoselskii’s-FPT as we will see that in chapter 4. Next, we demonstrate the result of the existence and uniqueness by introducing some examples.

Chapter 2

GENERAL CONCEPTS OF FC AND FDE

This chapter consists of fundamental concepts, definitions and some theories of FC and FDE’s that were employed to provide a solution to the upcoming problem in the next chapters.

2.1 Preliminaries of FC

In this section, we will review the definitions and properties of FC [1], [2].

2.1.1 GF

The GF is the one of basic mathematical function and it is used in many fields in applied sciences.

Theorem 2.1 1 If $\operatorname{Re}(z) > 0,$ then

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{2.1}$$

Definition 2.2 Let $z > 0$ $, n \in N,$ define:

$$\Gamma(z) = Lim_{n\to\infty} \frac{(n-1)!n^z}{z(z+1)(z+2)...(z+n-1)}$$

$$= Lim_{n\to\infty} \frac{(n-1)!n^z}{(z)_n}. \tag{2.2}$$

Definition 2.3 The Euler Maschoroni Constant (EMC)

The constant γ defined by

$$\gamma = \lim\left(\sum_{r=1}^{p} \frac{1}{r} \log p\right), \tag{2.3}$$

which is equal to 0.5772.

Properties 2.4 Some properties of the GF .

Let $z \neq 0, n \in N$, we have

$(1)\Gamma(n+1) = n!,$

$(2)\ \Gamma(z) = \Gamma(z+1)\frac{1}{z}$, for negative value of z ,

$(3)\ (z)_n = \frac{\Gamma(z+n)}{\Gamma(z)},$

$(4)\ \Gamma(z)\Gamma(z-1) = \frac{\pi}{\sin(z\pi)},$

$(5)\ \Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \pi\sec(z\pi),$

$(6)\ \Gamma(z) = z^{-1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1},$

$(9)\ \frac{1}{\Gamma(z)} = z\lim_{q\to\infty}\left(n^{-z}\prod_{k=1}^{z}\left(1+\frac{z}{k}\right)\right).$

By using the above equations, we have

$(1)\ \Gamma\left(\frac{1}{2}\right) = \pi,$

$(2)\ \Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi,$

$(3)\ \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$

2.1.2 R-L Fractional Integration

Fractional integration can be defined in [29]:

Definition 2.5

We define the RHS of R-L Fractional integrals $I_{a^+}^{\alpha}$ of order $\alpha > 0$ of a function $f:[a,+\infty) \to R$ by

$$I_{a^+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t f(r)(t-r)^{\alpha-1}\,dr,\ a>0, t\in(a,b], \tag{2.4}$$

provided that the R-HS is point-wise defined on $[a,\infty)$, where $\Gamma(.)$ denotes GF.

2.1.3 R-L Fractional Derivative

Let us the fractional derivative as follows:

Definition 2.6

The RHS of R–L Fractional derivative of order $\alpha > 0,\ n-1 < \alpha < n,\ n \in N,$ is defined as

$$D_{a^+}^{\alpha} f(t) := \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t f(r)(t-r)^{n-\alpha-1} dr,\ t > 0, \tag{2.5}$$

where the function f has absolutely continuous derivative up to the order $n-1$.

Definition 2.7

The CD of order α for a function $f:[0,\infty) \to R$ can be written as

$$^{c}D^{\alpha} f(t) = D_0^{\alpha}\left(f(t) - \sum_{x=0}^{m-1} t^x \frac{f^x(0)}{k} \right),\ t > 0, m-1 < \alpha < m.\ t > 0. \tag{2.6}$$

Remark If $f \in C^m[0,\infty),$ then

$$^{c}D^{\alpha} f(t) := \int_0^t \frac{f^m(r)}{\Gamma(m-\alpha)}(t-r)^{-\alpha+m-1} dr$$

$$= I^{\alpha+m} f^n(t), m-1 < \alpha < m, t > 0.$$

Lemma 2.8

For $\alpha > 0$, the general solution of the FDE's $^{c}D^{\alpha} v(t) = 0$ is given by

$$x(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1},$$

where $a_j \in R,\ j = 0,1,2,\ldots,n-1, n = -[-\alpha].$

In view of Lemma 2.8, it follows that

$$I_0^{\alpha}\left({}^{C}D_0^{\alpha} v\right)(t) = x(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1},$$

where $a_j \in R,\ j = 0,1,2,\ldots,n-1, n = [-\alpha].$

Definition 2.9

The sequential derivative for a sufficiently smooth function due to Miller-Ross is defined

as

$$D^{\delta} f(t) = D^{\delta_1} D^{\delta_2} ... D^{\delta_k} f(t),\ t > 0, \tag{2.7}$$

where $\delta = (\delta_1, ..., \delta_k)$, is a multi-index.

In general, the operator D^{δ} at (2.7) can either be R-L or Caputo or any other kind of integro-differential operator. For instance,

$$^{c}D^{\alpha} f(t) = D^{-(n-\alpha)} \left(\frac{d}{dt} \right) f(t), n-1 < \alpha < n,$$

where $D^{-(n-\alpha)}$ is a fractional integral operator of order $n-\alpha$. Here we emphasize that

$$D^{-b} f(t) = I^{b} f(t), b = n - \alpha.$$

2.2 Reviews of the FDE's

In this section we will consider the basic definitions and theorems for the FDE's necessary for use in the subsequence chapters.

2.2.1 Contraction Mapping

Definition 2.10

Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction mapping, or contraction, if there exist a constant c, with $0 \le c < 1$, such that

$$d\left(\left(T(x), T(y)\right)\right) \le cd(x, y); x, y \in X.$$

Theorem 2.11

If $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ is a contraction mapping on a complete metric space $(\mathrm{X,d})$, then there is exactly one solution $x \in X$.

2.2.2 FPT

Definition 2.12 Given a set E and a function $f : \mathrm{E} \to \mathrm{E}$, $y* \in E$ is a FP of f if and only if $f(y^*) = \mathrm{y}^*$.

Theorem 2.13 If $E = [a, b] \subseteq R$ and $f : \mathrm{E} \to \mathrm{E}$ is continuous then f has a FP.

2.2.3 Banach- FPT

Let (E,d) be a non-empty complete metric space with a contraction mapping $f:$ E$\rightarrow$ E . Then f admits a UFP y^* in E (i.e.) $f(y^*)$ = y*. Furthermore, x^* can be found as follows: start with an arbitrary element y_0 in X and define a sequence $\{y_m\}$ by $y_m = f(y_{m-1})$, then $y_m \rightarrow$ y*.

2.2.4 Krasnoselskii's-FPT

Let E be a closed convex and a nonempty subset of a Banach space X . Let T_1, T_2 be the operators such that:

1- $T_1x + T_2y \in E$ whenever $x, y \in E$;

2- T_1 is compact and continuous;

3- T_2 is a contraction mapping.

Then there exists $z \in E$ such that $z = T_1$x$+T_2$y .

2.2.5 Schauder's-FPT

If B is a non-empty, convex and compact subset Banach space X and $T: B \rightarrow B$ a continuous function, then T has a FP in B.

2.2.6 Leary Shauder's-FPT

If B is a non-empty, convex, bounded and closed subset of Banach space X and $T: B \rightarrow B$ compact and continuous map, then T has a FP in B.

Lemma 2.14

The set $F \subset PC([0,1], R^n)$ is relatively compact if and only if

$1-F$ is bounded , that is $\| x \| \leq C$ for each $x \in F$ and some $C > 0$;

$2-F$ is quasi-equicontinuous in $[0,T]$. That is to say that for any $\varepsilon > 0$ there exist $\gamma > 0$ such that if $x \in F; k \in N; s_1, s_2 \in (t_{k-1}, t_k]$, and $|s_1 - s_2| < \gamma, |x(s_1) - x(s_2)| < \varepsilon$.

Chapter 3

SOLUTIONS OF NONLINEAR IMPULSIVE- SFDEs

Now, this chapter examine the solution of the nonlinear impulsive- SFDE's with diverse BC's. Using the BC's on nonlinear impulsive- SFDE's we reached the solution in section 3.1.1. Furthermore, we arrived at the solutions of section 3.1.2 using the BC's (3.4) on nonlinear impulsive- SFDE's (3.1) and section 3.1.3 using the BC's (3.5) on nonlinear impulsive- SFDE's (3.2) .

3.1 Methods of Solving ISFDE's with Separated BC's.

We will consider different BC's and L-C type nonlinear impulsive- SFDE's as follows:

$$\left({}^{c}D^{\alpha}+\lambda\,{}^{c}D^{\alpha-1}\right)v(t)=f\left(t,v(t)\right),\ 1<\alpha\le 2,\ 0<t<1, \tag{3.1}$$

$$\left({}^{c}D^{\beta_m}_{t_m^+}+\lambda\,{}^{c}D^{\beta_m-1}_{t_m^+}\right)v(t)=f\left(t,v(t)\right),\ \ 0<t<1,\ 1<\beta_m\le 2. \tag{3.2}$$

The first BC's can expressed as

$$z_1v(0)+w_1v(0)=y_1,\ \ z_2v(1)+w_2v(1)=y_2. \tag{3.3}$$

The second BC's can expressed as:

$$cv(0)+{}^{c}D^{\alpha-1}v(0)=x_1,\quad dv(1)+{}^{c}D^{\alpha-1}v(1)=x_2, \tag{3.4}$$

and the third BC's can expressed as:

$$v(0)=\sum_{m=0}^{p}\lambda_m I^{\alpha_m}_{t_m^+}v(\eta),\ \ v'(0)=0, \tag{3.5}$$

Supplemented with IC's

$$\Delta v(t_m)=v\left(t_m^+\right)-v\left(t_+^-\right)=Q_m\left(v(t_m)\right),\ \ \Delta v'(t_m)=v'\left(t_m^+\right)-v'\left(t_m^-\right)=Q_m^*\left(v(t_m)\right),$$

$m=1,2,...,p$ where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are the CD of order $\alpha(1,2]$ and $\beta\in(1,2]$; and

$f\in[0,1]\times R\times R,\ \alpha,\beta,z_1,z_2,w_1,w_2,x_1,x_2,y_1,y_2\in R,\ \lambda\in R^+,\lambda_m\in R,Q,Q^*\in C(R,R)$

$t_0=0,t_{p+1}=1,\ J=[0,1]\ ,J_0=[0,t_1],\ J_1=(t_1,t_2],...,J_p=[t_p,1],\ J'=J\setminus\{t_1,...,t_p\},$

$0=t_0<t_1<...<t_p<t_{p+1}=1.\ \Delta v(t_m)=v\left(t_m^+\right)-v\left(t_m^-\right),\ \Delta v'(t_m)=v'\left(t_m^+\right)-v'\left(t_m^-\right).$

$v\left(t_m^+\right)$ and $v\left(t_m^-\right)$ represent the R and L –HL'S of the function $v(t)$ at $t=t_m$.

The Banach space can be written as:

$PC(J)=\{v:J\rightarrow R\,|\,v\in C(J'),$ and $v(t_m^+),v(t_m^-)$ exist, and $v(t_m^+)=v(t_m^-)$ $,1\le m\le p\}$

and the norm can be written as

$$\|v\|_{PC}=\sup_{t\in J}|v(t)|.$$

3.1.1 Results of Nonlinear Impulsive- SFDE's

We use the Lemma 2.8 associated with the nonlinear different type of equations (3.1)-(3.3), (3.1)-(3.4), (3.2)-(3.5).

Lemma 3.1 For $\alpha\in(1,2]$ and the continuous function $f:J\rightarrow R,$ the solution of the following problem:

$$\left({}^cD^{\alpha}+\lambda\,{}^cD^{\alpha-1}\right)v(t)=f(t,v(t)),1<\alpha\le 2,0<t<1,$$

$$z_1v(0)+w_1v(0)=y_1,z_2v'(1)+w_2v'(1)=y_2,$$

$$\Delta v(t_m)=v(t_m^+)-v(t_m^-)=Q_m(v(t)),\Delta v'(t_m)=v'(t_m^+)-v'(t_m^-)=Q^*{}_m(v(t))$$

can be formulated

$$\begin{aligned}v(t)=&\int_0^t e^{-m(t-r)}I^{\alpha-1}w(r)\,dr+h_1(t)\int_0^1 e^{-m(1-r)}I^{\alpha-1}w(r)\,dr+h_2(t)I^{\alpha-1}w(1)\\&+h_3(t)\sum_{n=1}^{p}Q_n(v(t_n))+h_4(t)\sum_{n=1}^{p}Q_n^*(v(t_n))+\sum_{n=1}^{p}N_{2,n}Q_n(v(t_n))\\&+\sum_{n=m+1}^{p}N_{2,n}Q_n^*(v(t_n))-\sum_{n=m+1}^{p}Q_n^*(v(t_n))+N_3(t),\\&t\in[t_m,t_{m+1}),m=0,1,...,p,\end{aligned}$$

where

$$\Delta=(z_1-\lambda w_1)z_2-e^{-m}(z_2-\lambda w_2)z_1\ne 0,\ h_1(t)=\frac{\left(z_1e^{-mt}-z_1+\lambda w_1\right)(w_2-\lambda z_2)}{\Delta},$$

$$h_2(t)=\frac{\left(z_1e^{-mt}-z_1+\lambda w_1\right)w_2}{\Delta},\ \ h_3(t)=\frac{z_1z_2e^{-mt}}{\Delta}-\frac{e^{-m}z_1(z_2-\lambda w_1)w_2}{\Delta},$$

$$h_4(t)=\frac{z_1z_2e^{-mt}}{\Delta m}-\frac{e^{-m}z_1(z_2-\lambda w_1)w_2}{\Delta m},$$

$$N_{1,n}(t) = -\frac{e^{\lambda t_n} e^{-\lambda t} z_2 (z_1 - \lambda w_1)}{\Delta\lambda} + \frac{e^{\lambda t_n} e^{-\lambda}(z_1 - \lambda w_1)(z_2 - \lambda w_1)}{\Delta\lambda},\ N_{2,n}(t) = \left(\frac{e^{\lambda t_n} e^{-\lambda t}}{\lambda} - \frac{1}{\lambda}\right),$$

$$N_3(t) = \left(\frac{e^{-\lambda t} z_1}{\Delta} - \frac{(z_2 e^{-\lambda} - \lambda w_1 e^{-\lambda})}{\Delta}\right) y_1 - \left(\frac{e^{-\lambda t} z_1}{\Delta} - \frac{(z_1 - \lambda w_1)}{\Delta}\right) y_2.$$

Proof: The equation (3.1) has a solution v on $t \in J$,

$$({}^c D^{\alpha} + \lambda\, {}^c D^{\alpha-1}) v(t) = w(t).$$

Applying the operator $\mathrm{I}^{\alpha-1}$ on both sides of the above equation, we get

$$\mathrm{I}^{\alpha-1}({}^c D^{\alpha} + \lambda\, {}^c D^{\alpha-1}) v(t) = \mathrm{I}^{\alpha-1} w(t),$$

$$(D + \lambda) v(t) = c_0 + \mathrm{I}^{\alpha-1} w(t),$$

which can be expressed as

$$D\left(e^{\lambda t} v(t)\right) = (c_0 + \mathrm{I}^{\alpha-1} w(t)) e^{\lambda t}.$$

Integrating both sides from 0 to t, we have

$$\int_0^t D e^{\lambda r} v(r) dr = \int_0^t c_0 e^{\lambda r} dr + \int_0^t e^{\lambda r} \mathrm{I}^{\alpha-1} w(r) dr,$$

$$e^{\lambda t} v(t) = v(0) + \frac{c_0}{\lambda}\left(e^{\lambda t} - 1\right) + \int_0^t e^{\lambda r} \mathrm{I}^{\alpha-1} w(r) dr,$$

$$v(t) = \frac{1}{e^{\lambda t}}\left(v(0) + \frac{c_0}{\lambda}\left(e^{\lambda t} - 1\right) + \int_0^t e^{\lambda r} \mathrm{I}^{\alpha-1} w(r) dr \right),$$

$$v(t) = \frac{v(0)}{e^{\lambda t}} + \frac{c_0}{\lambda}[1 - e^{\lambda t}] + \int_0^t e^{-\lambda t} e^{\lambda r} \mathrm{I}^{\alpha-1} w(r) dr,$$

$$v(t) = e^{-\lambda t}\left(v(0) - \frac{c_0}{\lambda}\right) + \frac{c_0}{\lambda} + \int_0^t e^{-\lambda t} e^{\lambda r} \mathrm{I}^{\alpha-1} w(r) dr,$$

$$v(t) = e^{-\lambda t}\left(v(0) - \frac{c_0}{\lambda}\right) + \frac{c_0}{\lambda} + \int_0^t e^{-\lambda(t-r)} \mathrm{I}^{\alpha-1} w(r) dr,$$

where

$$A_m = \left(v(0) - \frac{c_0}{\lambda}\right), \quad B_m = \frac{c_0}{\lambda}.$$

The general solution v of (3.1) on each interval $(t_m, t_{m+1}]$, $m = 0,1,..., p$, can be written as

$$v(t) = e^{-\lambda t} A_m + B_m + \int_0^t e^{-\lambda(t-r)} \mathrm{I}^{\alpha-1} w(r) dr \text{ , for } t \in J. \tag{3.6}$$

Next, solving the obtained linear equation (3.6) on J_0, we get

$$v(t) = e^{-\lambda t} A_0 + B_0 + \int_0^t e^{-\lambda(t-r)} \mathrm{I}^{\alpha-1} w(r) dr, t \in J_0. \tag{3.7}$$

where A_0 and B_0 are arbitrary constants. Taking the derivative to (3.7), we get

$$v'(t) = -\lambda e^{-\lambda t} A_0 - \lambda \int_0^t e^{-\lambda(t-r)} \mathrm{I}^{\alpha-1} w(r) dr + \mathrm{I}^{\alpha-1} w(t),$$

(3.8)

Now, applying the BCs $z_1 v(0) + w_1 v(0) = y_1$, we have

$$z_1 v(0) = z_1 (A_0 + B_0),$$

$$w_1 v'(0) = -\lambda w_1 A_0,$$

then

$$\left(z_1 - \lambda w_1\right) A_0 + z_1 B_0 = y_1. \tag{3.9}$$

In general, for $t \in \left(t_m, t_{m+1}\right]$, we find that

$$v(t) = e^{-\lambda t} A_m + B_m + \int_0^t e^{-\lambda(t-r)} \mathrm{I}^{\alpha-1} w(r) dr,$$

$$v'(t) = -\lambda e^{-\lambda t} A_m - \lambda \int_0^t e^{-\lambda(t-r)} \mathrm{I}^{\alpha-1} w(r) ds + \mathrm{I}^{\alpha-1} w(t),$$

Now, applying the BCs $z_2 v'(1) + z_2 v'(1) = y_2$, at $t_{m+1} = 1$, we have

$$v(1) = e^{-\lambda} A_p + B_p + \int_0^1 e^{-\lambda(1-r)} \mathrm{I}^{\alpha-1} w(r) dr,$$

$$v(1) = -\lambda e^{-\lambda} A_p - \lambda \int_0^1 e^{-\lambda(1-r)} \mathrm{I}^{\alpha-1} w(r) dr + \mathrm{I}^{\alpha-1} w(1),$$

$$\begin{gathered}\left(z_2 e^{-\lambda} - \lambda w_2 e^{-\lambda}\right) A_p + z_2 B_p = y_2 - \left(z_2 - \lambda w_2\right) \int_0^1 e^{-\lambda(1-r)} \mathrm{I}^{\alpha-1} w(r) dr \\ -w_2 \mathrm{I}^{\alpha-1} z(1).\end{gathered} \tag{3.10}$$

Next, we have to find the current IC's as follows:

From $\Delta v'\left(t_m\right) = v'\left(t_m^+\right) - v'\left(t_m^+\right) = Q_m^*\left(v\left(t_m\right)\right)$, we have

$$Q_m^*\left(v\left(t_m\right)\right) = -\lambda e^{-\lambda t_m} A_m + \lambda e^{-\lambda t_{m-1}} A_{m-1},$$

$$A_m - A_{m-1} = -\frac{1}{\lambda} e^{\lambda t_m} Q_m^*\left(v\left(t_m\right)\right), m = 1, 2, \ldots, p. \tag{3.11}$$

Similarly, from $\Delta v\left(t_m\right) = v\left(t_m^+\right) - v\left(t_m^-\right) = Q_m\left(v\left(t_m\right)\right)$, we get

$$Q_m\left(v\left(t_m\right)\right) = e^{-\lambda t_m} A_m - e^{-\lambda t_{m-1}} A_{m-1} + B_m - B_{m-1},$$

$$B_m - B_{m-1} = Q_m\left(v\left(t_m\right)\right) + \frac{1}{\lambda} Q_m^*\left(v\left(t_m\right)\right), m = 1, 2, \ldots, p. \tag{3.12}$$

Next, it follows from (3.11) and (3.12)

$$A_p - A_m = -\frac{1}{\lambda} \sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right), m = 1, 2, \ldots, p-1. \tag{3.13}$$

$$B_p - B_m = \sum_{n=m+1}^{p} Q_n\left(v\left(t_n\right)\right) + \frac{1}{\lambda} \sum_{n=m+1}^{p} Q_n^*\left(v\left(t_n\right)\right), m = 1, 2, \ldots, p-1. \tag{3.14}$$

It follows that from $m = 0$, from $\left(z_1 - \lambda w_1\right) A_0 + z_1 B_0 = y_1$, that

$$\left(z_1 - \lambda w_1\right) A_p + \left(z_1 - \lambda w_1\right) A_0 = -\frac{1}{\lambda}\left(z_1 - \lambda w_1\right) \sum_{n=1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right),$$

$$z_1 B_p - z_1 B_0 = z_1 \sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right) + \frac{z_1}{\lambda} \sum_{n=1}^{p} Q_m^*\left(v\left(t_m\right)\right),$$

then

$$\begin{aligned}\left(z_1 - \lambda w_1\right) A_p + z_1 B_p = y_1 &- \frac{1}{\lambda}\left(z_1 - \lambda w_1\right) \sum_{n=1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right) \\ &+ z_1 \sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right) + \frac{z_1}{\lambda} \sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right).\end{aligned}$$

Solving the last equation together (3.10), for A_p and B_p, we get

$$A_p = \frac{z_1\left(z_2-\lambda w_2\right)}{\Delta}\int_0^1 e^{\lambda(1-r)} I^{\alpha-1} w(r)\,dr + \frac{z_1 w_2}{\Delta} I^{\alpha-1} w(1)$$
$$+\frac{z_1 z_2}{\Delta}\sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right) + \frac{z_1 z_2}{\Delta\lambda}\sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right)$$
$$-\frac{z_2\left(z_1-\lambda z_1\right)}{\Delta\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right) + \frac{z_2}{\Delta} y_1 - \frac{z_1}{\Delta} y_2,$$

and

$$B_p = -\frac{\left(z_1-\lambda w_1\right)\left(z_2-\lambda w_2\right)}{\Delta}\int_0^1 e^{\lambda(1-r)} I^{\alpha-1} w(r)\,dr$$
$$-\frac{\left(z_1-\lambda w_1\right) w_2}{\Delta} I^{\alpha-1} w(1) - \frac{e^{-\lambda} z_1\left(z_2-\lambda w_2\right)}{\Delta}\sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right)$$
$$-\frac{e^{-m} z_1\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right)$$
$$+\frac{\left(z_1-\lambda w_1\right) e^{-m}\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right)$$
$$-\frac{e^{-\lambda}\left(z_2-\lambda w_2\right)}{\Delta} y_1 + \frac{\left(z_1-\lambda w_1\right)}{\Delta} y_2,$$

where $\Delta = \left(z_1-\lambda w_1\right) z_2 - e^{-\lambda}\left(z_2-\lambda w_2\right) z_1 \neq 0.$ Now, from the equation (3.13) and (3.14) it follows that

$$A_m = A_p + \frac{1}{\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right), m = 1, 2, ..., p-1.$$

$$B_m = B_p - \sum_{n=m+1}^{p} Q_n\left(v\left(t_n\right)\right) - \frac{1}{\lambda}\sum_{n=m+1}^{p} Q_n^*\left(v\left(t_n\right)\right), m = 1, 2, ..., p-1.$$

So

$$A_m = -\frac{\left(z_1-\lambda w_1\right)\left(z_2-\lambda w_2\right)}{\Delta}\int_0^1 e^{\lambda(1-r)} I^{\alpha-1} w(r)\,dr$$
$$-\frac{\left(z_1-\lambda w_1\right) w_2}{\Delta} I^{\alpha-1} w(1)$$
$$-\frac{e^{-\lambda} z_1\left(z_2-\lambda w_2\right)}{\Delta}\sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right)$$
$$-\frac{e^{-\lambda} z_1\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right) \tag{3.15}$$
$$+\frac{\left(z_1-\lambda w_1\right) e^{-m}\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right)$$
$$+\frac{1}{\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right) - \frac{\left(z_2 e^{-\lambda} - e^{-\lambda}\lambda w_2\right)}{\Delta} y_1 + \frac{\left(z_1-\lambda w_1\right)}{\Delta} y_2,$$

and

$$
\begin{aligned}
B_m = &-\frac{(z_1-\lambda w_1)(z_2-\lambda w_2)}{\Delta}\int_0^1 e^{\lambda(1-r)} I^{\alpha-1} w(r)\,dr \\
&-\frac{(z_1-\lambda w_1)\beta_2}{\Delta} I^{\alpha-1} w(1) - \frac{z_1\left(z_2 e^{-\lambda} - e^{-\lambda}\lambda w_2\right)}{\Delta}\sum_{n=1}^{p} Q_n\left(v(t_n)\right) \\
&-\frac{e^{-\lambda} z_1 (z_2-\lambda w_2)}{\Delta\lambda}\sum_{n=1}^{p} Q_n^*\left(v(t_n)\right) \\
&+\frac{(z_1-\lambda w_1)e^{-m}(z_2-\lambda w_2)}{\Delta\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v(t_n)\right) \\
&-\sum_{n=m+1}^{p} Q_n\left(v(t_n)\right) - \frac{1}{\lambda}\sum_{n=m+1}^{p} Q_n^*\left(v(t_n)\right) \\
&-\frac{e^{-m}(z_2-\lambda w_2)}{\Delta} y_1 + \frac{(z_1-\lambda w_1)}{\Delta} y_2.
\end{aligned}
\tag{3.16}
$$

Multiplying the equation (4.15) by $e^{-\lambda t}$, we get

$$
\begin{aligned}
e^{-\lambda t} A_m = &-\frac{e^{-\lambda t}(z_1-\lambda w_1)(z_2-\lambda w_2)}{\Delta}\int_0^1 e^{\lambda(1-r)} I^{\alpha-1} w(r)\,dr \\
&-\frac{e^{-\lambda t}(z_1-\lambda w_1) w_2}{\Delta} I^{\alpha-1} w(1) \\
&-\frac{e^{-\lambda t} e^{-\lambda} z_1 (z_2-\lambda w_2)}{\Delta}\sum_{n=1}^{p} Q_n\left(v(t_n)\right) \\
&-\frac{e^{-\lambda t} e^{-\lambda} z_1 (z_2-\lambda w_2)}{\Delta\lambda}\sum_{n=1}^{p} Q_n^*\left(v(t_n)\right) \\
&+\frac{e^{-mt}(z_1-\lambda w_1)e^{-m}(z_2-\lambda w_2)}{\Delta m}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v(t_n)\right) \\
&+\frac{e^{-\lambda t}}{\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v(t_n)\right) - \frac{e^{-\lambda t} e^{-\lambda}(z_2-\lambda w_2)}{\Delta} y_1 + \frac{e^{-\lambda t}(z_1-\lambda w_1)}{\Delta} y_2.
\end{aligned}
$$

Combining the last two equations, we get

$$
\begin{aligned}
e^{-\lambda t}A_m+B_m=&-\frac{e^{-\lambda t}\left(z_1-\lambda w_1\right)\left(z_2-\lambda w_2\right)}{\Delta}\int_0^1 e^{\lambda(1-r)}I^{\alpha-1}w(r)\,dr\\
&-\frac{e^{-\lambda t}\left(z_1-\lambda w_1\right)w_2}{\Delta}I^{\alpha-1}w(1)\\
&-\frac{e^{-\lambda t}e^{-\lambda}z_1\left(z_2-\lambda w_2\right)}{\Delta}\sum_{n=1}^{p}Q_n\left(v\left(t_n\right)\right)\\
&-\frac{e^{-\lambda t}e^{-\lambda}z_1\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=1}^{p}Q_n^*\left(v\left(t_n\right)\right)\\
&+\frac{e^{-\lambda t}\left(z_1-\lambda w_1\right)e^{-\lambda}\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=m+1}^{p}e^{\lambda t_n}Q_n^*\left(v\left(t_n\right)\right)\\
&+\frac{e^{-\lambda t}}{\lambda}\sum_{n=m+1}^{p}e^{\lambda t_n}Q_n^*\left(v\left(t_n\right)\right)-\frac{e^{-\lambda t}e^{-\lambda}\left(z_2-\lambda w_2\right)}{\Delta}y_1\\
&+\frac{e^{-\lambda t}\left(z_1-\lambda w_1\right)}{\Delta}y_2-\frac{\left(z_1-\lambda w_1\right)\left(z_2-\lambda w_2\right)}{\Delta}\int_0^1 e^{\lambda(1-r)}I^{\alpha-1}w(r)\,dr\\
&-\frac{\left(z_1-\lambda w_1\right)w_2}{\Delta}I^{\alpha-1}w(1)-\frac{e^{-\lambda}z_1\left(z_2-\lambda w_2\right)}{\Delta}\sum_{n=1}^{p}Q_n\left(v\left(t_n\right)\right)\\
&-\frac{e^{-\lambda}z_1\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=1}^{p}Q_n^*\left(v\left(t_n\right)\right)\\
&+\frac{\left(z_1-\lambda w_1\right)e^{-\lambda}\left(z_2-\lambda w_2\right)}{\Delta\lambda}\sum_{n=m+1}^{p}e^{\lambda t_n}Q_n^*\left(v\left(t_n\right)\right)-\sum_{n=m+1}^{p}Q_n\left(v\left(t_n\right)\right)\\
&-\frac{1}{\lambda}\sum_{n=m+1}^{p}Q_n^*\left(v\left(t_n\right)\right)-\frac{e^{-\lambda}\left(z_2-\lambda w_2\right)}{\Delta}y_1+\frac{\left(z_1-\lambda w_1\right)}{\Delta}y_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
e^{-\lambda t}A_m+B_m=&\,h_1(t)\int_0^1 e^{-\lambda(1-r)}I^{\alpha-1}w(r)\,dr+h_2(t)I^{\alpha-1}w(1)\\
&+h_3(t)\sum_{n=1}^{p}Q_n\left(v\left(t_n\right)\right)+h_4(t)\sum_{n=1}^{p}Q_n^*\left(v\left(t_n\right)\right)\\
&+\sum_{n=1}^{p}N_{1,n}Q_n\left(v\left(t_n\right)\right)+\sum_{n=m+1}^{p}N_{2,n}Q_n^*\left(v\left(t_n\right)\right)\\
&-\sum_{n=m+1}^{p}Q_n^*\left(v\left(t_n\right)\right)+N_3(t).
\end{aligned}
\tag{3.17}
$$

Inserting (3.17) into (3.6), we get

$$
\begin{aligned}
v(t) &= \int_0^t e^{-\lambda(t-r)} I^{\alpha-1} w(r)\,dr + h_1(t)\int_0^1 e^{-\lambda(1-r)} I^{\alpha-1} w(r)\,dr \\
&+ h_2(t) I^{\alpha-1} w(1) + h_3(t)\sum_{n=1}^{p} Q_n\left(v(t_n)\right) \\
&+ h_4(t)\sum_{n=1}^{p} Q_n^*\left(v(t_n)\right) + \sum_{n=1}^{p} N_{1,n} Q_n\left(v(t_n)\right) \\
&+ \sum_{n=m+1}^{p} N_{2,n} Q_n^*\left(v(t_n)\right) - \sum_{n=m+1}^{p} Q_n^*\left(v(t_n)\right) + N_3(t), \\
& t\in\left[t_m, t_{m+1}\right), m = 0,1,\dots, p,
\end{aligned}
\tag{3.18}
$$

The reverse (other side) of the Lemma follows by explicit computation. Prove of the Lemma is completed.

Lemma 3.2 for $\alpha\in(1,2]$ and the continuous function $f : J \to R,$ the solution of the following problem:

$$
\begin{gathered}
\left({}^cD^{\alpha} + \lambda\, {}^cD^{\alpha-1}\right) v(t) = w(t),\ 1<\alpha\le 2,\ 0<t<1, \\
cv(0) + {}^cD^{\alpha-1} v(0) = x_1,\ \ dv(1) + {}^cD^{\alpha-1} v(1) = x_2, \\
\Delta v(t_m) = v\left(t_m^+\right) - v\left(t_m^-\right) = Q_m\left(v(t)\right), \Delta v'(t_m) = v'\left(t_m^+\right) - v'\left(t_m^-\right) = Q^*{}_m\left(v(t)\right),
\end{gathered}
$$

is given by

$$
\begin{aligned}
v(t) &= \int_0^t M_1(t,\tau) w(\tau)\,d\tau + d_1(t)\int_0^1 M_1(t,\tau) w(\tau)\,d\tau \\
&+ d_2(t)\int_0^1 M_2(t,\tau) w(\tau)\,d\tau + d_3(t)\int_0^1 w(\tau)\,d\tau \\
&+ d_4(t)\sum_{n=1}^{p} Q_n\left(v(t_n)\right) + d_5(t)\sum_{n=1}^{p} Q_n^*\left(v(t_n)\right) \\
&+ \sum_{n=1}^{p} p_{1,n} Q_n\left(v(t_n)\right) + \sum_{n=m+1}^{p} p_{2,n} Q_n^*\left(v(t_n)\right) \\
&- \sum_{n=m+1}^{p} Q_n^*\left(v(t_n)\right) + d_6(t), \\
& t\in\left[t_m, t_{m+1}\right), m = 0,1,\dots, p,
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{gathered}
d_1(t) = \frac{e^{-\lambda t} d - d}{d-\eta},\ d_2(t) = \frac{\lambda - \lambda e^{-\lambda t}}{d-\eta}, \\
d_3(t) = \frac{e^{-\lambda t} - 1}{d-\eta},\ d_4(t) = \frac{e^{-\lambda t} d - \eta}{d-\eta}, \\
d_5(t) = \frac{e^{-mt} d - \eta}{(d-\eta)\lambda},\ p_{1,n} = \frac{\eta e^{\lambda t_n} - e^{\lambda t_n} e^{-\lambda t} d}{(d-\eta)\lambda},
\end{gathered}
$$

$$p_{2,n}=\left(\frac{e^{-\lambda t}e^{\lambda t_n}}{\lambda}-\frac{1}{\lambda}\right),$$

$$d_6(t)=\left(\frac{e^{-\lambda t}d-\eta}{cd-c\eta}\right)x_1+\left(\frac{1-e^{-\lambda t}}{d-\eta}\right)x_2,$$

and

$$M_1(t,\tau)=\frac{1}{\Gamma(\alpha-1)}\int_s^t e^{-\lambda(t-s)}(s-\tau)^{\alpha-2}\,ds,$$

$$\int_0^t M_1(t,\tau)w(\tau)\,d\tau=\int_0^t e^{-\lambda(t-s)}I^{\alpha-1}w(s)\,ds,$$

$$\begin{aligned}M_2(t,s)&=\frac{1}{\Gamma(2-\alpha)}\int_s^t (t-\tau)^{1-\alpha}M_1(\tau,s)\,d\tau\\&=\frac{1}{\Gamma(2-\alpha)}\int_0^1(1-s)^{1-\alpha}\left(\int_0^s e^{-\lambda(s-\tau)}I^{1-\alpha}w(\tau)\,d\tau\right)ds\\&=\frac{1}{\Gamma(2-\alpha)}\int_0^1(1-s)^{1-\alpha}\left(\int_0^s M_1(s,\tau)w(\tau)\,d\tau\right)ds\\&=\int_0^1 M_2(1,\tau)w(\tau)\,d\tau,\end{aligned}$$

$$\Upsilon=\left(de^{-\lambda}-\lambda\int_0^1\frac{(1-s)}{\Gamma(2-\alpha)}e^{-\lambda s}ds\right).$$

Proof: The general solution of (3.1) and (3.4) on each interval $(t_m,t_{m+1}]$, $(m=0,1,2,...,p)$, can be written as

$$v(t)=e^{-\lambda t}a_m+b_m+\int_0^t e^{-\lambda(t-s)}\mathrm{I}^{\alpha-1}w(s)ds\text{ , for } t\in(t_m,t_{m+1}], \tag{3.20}$$

where a_m and b_m are arbitrary constants. Now, consider (3.20) on $t\in[t_0,t_1]$ and take the Caputo derivative by using (3.20) on $t\in[t_0,t_1]$, we get that

$$v(t)=e^{-\lambda t}a_0+b_0+\int_0^t e^{-\lambda(t-s)}\mathrm{I}^{\alpha-1}w(s)ds, t\in J_0, \tag{3.21}$$

$$\begin{aligned}{}^cD^{\alpha-1}v(t)=&-\frac{\lambda a_0}{\Gamma(2-\alpha)}\int_0^t e^{-\lambda s}(t-s)^{1-\alpha}ds\\&-\frac{m}{\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}\left(\int_0^s e^{-\lambda(s-\tau)}\mathrm{I}^{\alpha-1}w(\tau)d\tau\right)ds\\&+\int_0^t\frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}\mathrm{I}^{\alpha-1}w(s)\,ds, t\in J_0.\end{aligned} \tag{3.22}$$

Next, for $t_{m+1}=1$ we can write equation (3.20) as following

$$v(t)=e^{-\lambda}a_p+b_p+\int_0^1 e^{-\lambda(1-s)}\mathrm{I}^{\alpha-1}w(s)ds. \tag{3.23}$$

To find the Caputo derivative by using equation (3.23), we can get

$$
\begin{aligned}
{}^{c}D^{\alpha-1}v(1) = &-\frac{\lambda a_p}{\Gamma(2-\alpha)}\int_0^1 e^{-\lambda s}(1-s)^{1-\alpha}\,ds \\
&-\lambda\int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\left(\int_0^s e^{-\lambda(s-\tau)}\mathrm{I}^{\alpha-1}w(\tau)d\tau\right)ds \\
&+\int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\mathrm{I}^{\alpha-1}w(s)\,ds, t\in J_0.
\end{aligned} \tag{3.24}
$$

Now, solving the first BC's, $cv(0)+{}^{c}D^{\alpha-1}v(0)=x_1$, $dv(1)+{}^{c}D^{\alpha-1}v(1)=x_2$, by using the following equations, we have

$$cv(0)=c(a_0+b_0),$$

$${}^{C}D^{\alpha-1}v(0)=0,$$

$$dv(1)=de^{-\lambda}a_p+db_p+d\int_0^1 e^{-\lambda(1-s)}\mathrm{I}^{\alpha-1}w(s)ds,$$

$$
\begin{aligned}
{}^{c}D^{\alpha-1}v(1) = &-\frac{\lambda a_p}{\Gamma(2-\alpha)}\int_0^1 e^{-\lambda s}(1-s)^{1-\alpha}ds-\lambda\int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\left(\int_0^s e^{-\lambda(s-\tau)}\mathrm{I}^{\alpha-1}w(\tau)d\tau\right)ds \\
&+\int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)}\mathrm{I}^{\alpha-1}w(s)\,ds,
\end{aligned}
$$

then the BC's of (3.4) can be formulated as follows

$$
\begin{aligned}
&c(a_0+b_0)=x_1 \\
&\Upsilon a_p+db_p=x_2-d\int_0^1 M_1(1,\tau)w(\tau)d\tau \\
&\qquad+\lambda\int_0^1 M_2(1,\tau)w(\tau)d\tau-\int_0^1 w(\tau)d\tau.
\end{aligned} \tag{3.25}
$$

Furthermore, at this point we find the IC's by using the obtained linear equation (3.20).

For first IC's $\Delta v'(t_m)=v'(t_m^+)-v'(t_m^-)=Q_m^*(v(t))$, we have that

$$v'(t_m^+)=-\lambda e^{-\lambda t_m}a_m-\lambda\int_0^t e^{-\lambda(t_m-s)}\mathrm{I}^{\alpha-1}w(s)ds+w(t_m)\mathrm{I}^{\alpha-1},$$

$$v'(t_{m-1}^-)=-\lambda e^{-\lambda t_{m-1}}a_{m-1}-\lambda\int_0^t e^{-\lambda(t_{m-1}-s)}\mathrm{I}^{\alpha-1}w(s)ds+\mathrm{I}^{\alpha-1}w(t_{m-1}).$$

Next, we have

$$\Delta v'(t_m)=-\lambda e^{-\lambda t_m}a_m+\lambda e^{-\lambda t_{m-1}}a_{m-1}=Q_m^*(v(t_m)),$$

$$a_m-a_{m-1}=-\frac{e^{\lambda t_m}}{\lambda}Q_m^*(v(t_m)),$$

$$a_m=a_{m-1}-\frac{e^{\lambda t_m}}{\lambda}Q_m^*(v(t_m)),$$

$$a_{m+1}=a_m-\frac{e^{\lambda t_{m+1}}}{\lambda}Q_{m+1}^*(v(t_{m+1})),$$

$$a_P=a_{m-1}-\frac{1}{\lambda}\sum_{n=m+1}^{p}e^{\lambda t_n}Q_n^*(v(t_n)),$$

$$a_m = a_p + \frac{1}{\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right), m = 1,\dots, p.$$

In the same process we find the second IC's $\Delta v\left(t_m\right) = v\left(t_m^+\right) - v\left(t_m^-\right) = Q_m\left(v\left(t\right)\right)$, we have

$$b_m = b_{m-1} + Q_m\left(v\left(t_m\right)\right) + \frac{1}{\lambda} Q_m^*\left(v\left(t_m\right)\right),$$

$$b_m = b_p - \sum_{n=m+1}^{p} Q_n\left(v\left(t_n\right)\right) - \frac{1}{\lambda}\sum_{n=m+1}^{p} Q_n^*\left(v\left(t_n\right)\right), m = 1,\dots, p.$$

By the above equations the IC's can be written as follows

$$a_m = a_p + \frac{1}{\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right), m = 1,\dots, p, \tag{3.26}$$

$$b_m = b_p - \sum_{n=m+1}^{p} Q_n\left(v\left(t_n\right)\right) - \frac{1}{\lambda}\sum_{n=m+1}^{p} Q_n^*\left(v\left(t_n\right)\right), m = 1,\dots, p. \tag{3.27}$$

If $m = 0$ then from $c\left(a_0 + b_0\right) = x_1$, we get that

$$c\left(a_p + b_p\right) = x_1 + c\sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right) + \frac{c}{\lambda}\sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right) - \frac{c}{\lambda}\sum_{n=1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right).$$

Now, from last equation and (3.25), we have

$$b_p = -a_p + \frac{1}{c}x_1 + \sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right) + \frac{1}{\lambda}\sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right) - \frac{1}{\lambda}\sum_{n=1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right),$$

$$b_p = -\frac{\Upsilon}{d}a_p + \frac{1}{d}x_2 - \int_0^1 M_1\left(1,\tau\right)w\left(\tau\right)d\tau + \frac{\lambda}{d}\int_0^1 w\left(\tau\right)M_2\left(1,\tau\right)d\tau - \frac{1}{d}\int_0^1 w\left(\tau\right)d\tau.$$

$$-a_p + \frac{1}{c}x_1 + \sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right) + \frac{1}{\lambda}\sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right) - \frac{1}{\lambda}\sum_{n=1}^{p} e^{v t_n} Q_n^*\left(v\left(t_n\right)\right)$$

$$= -\frac{\Upsilon}{d}a_p + \frac{1}{d}x_2 - \int_0^1 M_1\left(1,\tau\right)w\left(\tau\right)d\tau + \frac{\lambda}{d}\int_0^1 w\left(\tau\right)M_2\left(1,\tau\right)d\tau - \frac{1}{d}\int_0^1 w\left(\tau\right)d\tau,$$

then for a_p and b_p, we can get

$$\begin{aligned} a_p = {} & \frac{d}{\left(d-\Upsilon\right)c}x_1 - \frac{1}{\left(d-\Upsilon\right)}x_2 + \frac{d}{\left(d-\Upsilon\right)}\int_0^1 M_1\left(1,\tau\right)w\left(\tau\right)d\tau \\ & - \frac{\lambda}{\left(d-\Upsilon\right)}\int_0^1 w\left(\tau\right)M_2\left(1,\tau\right)d\tau + \frac{1}{\left(d-\Upsilon\right)}\int_0^1 w\left(\tau\right)d\tau \\ & - \frac{d}{\left(d-\Upsilon\right)\lambda}\sum_{n=1}^{p} e^{\lambda t_n} Q_n^*\left(v\left(t_n\right)\right) + \frac{d}{\left(d-\Upsilon\right)}\sum_{n=1}^{p} Q_n\left(v\left(t_n\right)\right) \\ & + \frac{d}{\left(d-\Upsilon\right)\lambda}\sum_{n=1}^{p} Q_n^*\left(v\left(t_n\right)\right), \end{aligned} \tag{3.28}$$

$$
\begin{aligned}
b_p = & -\frac{\Upsilon}{(d-\Upsilon)c}x_1 + \frac{1}{(d-\Upsilon)}x_2 - \frac{d}{(d-\Upsilon)}\int_0^1 M_1(1,\tau)w(\tau)d\tau \\
& + \frac{m}{(d-\Upsilon)}\int_0^1 M_2(1,\tau)w(\tau)d\tau - \frac{1}{(d-\Upsilon)}\int_0^1 w(\tau)d\tau \\
& + \frac{\Upsilon}{(d-\Upsilon)\lambda}\sum_{n=1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)) - \frac{\Upsilon}{(d-\Upsilon)}\sum_{n=1}^{p} Q_n(v(t_n)) \\
& - \frac{\Upsilon}{(d-\Upsilon)\lambda}\sum_{n=1}^{p} Q_n^*(v(t_n)).
\end{aligned} \tag{3.29}
$$

Now, by (3.26) and (3.28), we get

$$
\begin{aligned}
a_m = & \frac{d}{(d-\Upsilon)c}x_1 - \frac{1}{(d-\Upsilon)}x_2 \\
& + \frac{d}{(d-\Upsilon)}\int_0^1 M_1(1,\tau)w(\tau)d\tau \\
& - \frac{\lambda}{(d-\Upsilon)}\int_0^1 M_2(1,\tau)w(\tau)d\tau + \frac{1}{(d-\Upsilon)}\int_0^1 w(\tau)d\tau \\
& - \frac{d}{(d-\Upsilon)\lambda}\sum_{n=1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)) + \frac{d}{(d-\Upsilon)}\sum_{n=1}^{p} Q_n(v(t_n)) \\
& + \frac{d}{(d-\Upsilon)\lambda}\sum_{n=1}^{p} Q_n^*(v(t_n)) + \frac{1}{\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)),
\end{aligned} \tag{3.30}
$$

also from (3.27) and (3.29), we get

$$
\begin{aligned}
b_m = & -\frac{\Upsilon}{(d-\Upsilon)c}x_1 + \frac{1}{(d-\Upsilon)}x_2 \\
& - \frac{d}{(d-\Upsilon)}\int_0^1 M_1(1,\tau)w(\tau)d\tau \\
& + \frac{\lambda}{(d-\Upsilon)}\int_0^1 M_2(1,\tau)w(\tau)d\tau - \frac{1}{(d-\Upsilon)}\int_0^1 w(\tau)d\tau \\
& + \frac{\Upsilon}{\lambda(d-\Upsilon)}\sum_{n=1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)) - \frac{\Upsilon}{(d-\Upsilon)}\sum_{n=1}^{p} Q_n(v(t_n)) \\
& - \frac{\Upsilon}{\lambda(d-\Upsilon)}\sum_{n=1}^{p} Q_n^*(v(t_n)). - \sum_{n=m+1}^{p} Q_n(v(t_n)) \\
& - \frac{1}{\lambda}\sum_{n=m+1}^{p} Q_n^*(v(t_n)).
\end{aligned} \tag{3.31}
$$

Multiplying equation (3.30) by ($e^{-\lambda t}$), we get

$$
\begin{aligned}
e^{-\lambda t}a_m &= \frac{de^{-\lambda t}}{(d-\Upsilon)c}x_1 - \frac{e^{-\lambda t}}{(d-\Upsilon)}x_2 \\
&+\frac{de^{-\lambda t}}{(d-\Upsilon)}\int_0^1 M_1(1,\tau)w(\tau)d\tau \\
&-\frac{me^{-\lambda t}}{(d-\Upsilon)}\int_0^1 M_2(1,\tau)w(\tau)d\tau + \frac{e^{-\lambda t}}{(d-\Upsilon)}\int_0^1 w(\tau)d\tau \\
&-\frac{e^{-\lambda t}}{(d-\Upsilon)\lambda}\sum_{n=1}^{p} e^{\lambda t_n}Q_n^*(x(t_n)) + \frac{e^{-\lambda t}}{(d-\Upsilon)}\sum_{n=1}^{p} Q_n(v(t_n)) \\
&+\frac{e^{-\lambda t}}{(d-\Upsilon)\lambda}\sum_{n=1}^{p} Q_n^*(v(t_n)) + \frac{e^{-\lambda t}}{\lambda}\sum_{n=m+1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)).
\end{aligned} \tag{3.32}
$$

We can find from (3.31) and (3.32) that

$$
\begin{aligned}
e^{-\lambda t}a_m + b_m &= \frac{de^{-\lambda t}}{(d-\Upsilon)c}x_1 - \frac{e^{-\lambda t}}{(d-\Upsilon)}x_2 + \frac{de^{-\lambda t}}{(d-\Upsilon)}\int_0^1 M_1(1,\tau)w(\tau)d\tau \\
&-\frac{\lambda e^{-\lambda t}}{(d-\Upsilon)}\int_0^1 w(\tau)M_2(1,\tau)d\tau + \frac{e^{-\lambda t}}{(d-\Upsilon)}\int_0^1 w(\tau)d\tau \\
&-\frac{e^{-vt}}{\lambda(d-\Upsilon)}\sum_{n=1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)) + \frac{e^{-\lambda t}}{(d-\Upsilon)}\sum_{n=1}^{p} Q_n(v(t_n)) \\
&+\frac{e^{-mt}}{\lambda(d-\Upsilon)}\sum_{n=1}^{p} Q_n^*(v(t_n)) + \frac{e^{-\lambda t}}{v}\sum_{n=m+1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)) \\
&-\frac{\Upsilon}{(d-\Upsilon)c}x_1 + \frac{1}{(d-\Upsilon)}x_2 - \frac{d}{(d-\Upsilon)}\int_0^1 w(\tau)M_1(1,\tau)d\tau \\
&+\frac{\lambda}{(d-\Upsilon)}\int_0^1 M_2(1,\tau)w(\tau)d\tau - \frac{1}{(d-\Upsilon)}\int_0^1 w(\tau)d\tau \\
&+\frac{\Upsilon}{\lambda(d-\Upsilon)}\sum_{n=1}^{p} e^{\lambda t_n}Q_n^*(v(t_n)) - \frac{\Upsilon}{(d-\Upsilon)}\sum_{n=1}^{p} Q_n(v(t_n)) \\
&-\frac{\Upsilon}{\lambda(d-\Upsilon)}\sum_{n=1}^{p} Q_n^*(v(t_n)) - \sum_{n=m+1}^{p} Q_n(v(t_n)) - \frac{1}{\lambda}\sum_{n=m+1}^{p} Q_n^*(v(t_n)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
e^{-\lambda t}a_m + b_m &= \int_0^1 M_1(t,\tau)w(\tau)d\tau + d_2(t)\int_0^1 M_2(t,\tau)w(\tau)d\tau \\
&+d_3(t)\int_0^1 w(\tau)d\tau + d_4(t)\sum_{n=1}^{p} Q_n(v(t_n)) \\
&+d_5(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) + \sum_{n=1}^{p} p_{1,n}Q_n(v(t_n)) \\
&+\sum_{n=m+1}^{p} p_{2,n}Q_n^*(v(t_n)) - \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + d_6(t).
\end{aligned} \tag{3.33}
$$

Now, taking (3.33) into (3.20), we can get

$$
\begin{aligned}
x(t) &= \int_0^t M_1(t,\tau)w(\tau)d\tau + d_1(t)\int_0^1 M_1(t,\tau)w(\tau)d\tau \\
&+ d_2(t)\int_0^1 M_2(t,\tau)w(\tau)d\tau + d_3(t)\int_0^1 w(\tau)d\tau \\
&+ d_4(t)\sum_{n=1}^{p} Q_n(v(t_n)) + d_5(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) \qquad (3.34)\\
&+ \sum_{n=1}^{p} p_{1,n}Q_n(v(t_n)) + \sum_{n=m+1}^{p} p_{2,n}Q_n^*(v(t_n)) \\
&- \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + d_6(t),
\end{aligned}
$$

Where $t \in [t_m, t_{m+1}), m = 0,1,..., p.$

Proofe of the Lemma is completed.

Lemma 3.3 For a given $\alpha \in (1,2]$ and a continuous function $f : J \to R,$ the solution of the following problem:

$$
\left({}^cD_{t_m^+}^{\beta_m} + \lambda\, {}^cD_{t_m^+}^{\beta_m - 1}\right)v(t) = w(t),\ \ 0 < t < 1,\ 1 < \beta_m \le 2,
$$

$$
v(0) = \sum_{m=0}^{p} \lambda_m I_{t_m^+}^{\alpha_m} v(\eta), v'(0) = 0,\ v'(0) = 0,
$$

$$
\Delta v(t_m) = Q_m(v(t)),\ \ \Delta Q'(t_m) = Q_m^*(v(t)),\ m = 1,2,..., p
$$

is given by

$$
v(t) = \begin{cases}
\int_0^t e^{-\lambda(t-s)} I_{0^+}^{\beta_0 - 1} w(s)ds + \wp, \\
\int_{t_m}^t e^{-\lambda(t-s)} I_{t_m}^{\beta_{t_m} - 1} w(r)dr + \sum_{n=1}^{m} e^{-m(t-t_n)} \\
\times\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n - s)} I_{t_{n-1}}^{\beta_{n-1} - 1} w(s)dr - \frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1} - 1} w(t) - \frac{1}{\lambda} Q^*(v(t_n))\right] \\
+\sum_{n=1}^{m} e^{-\lambda(t-t_n)}\left[\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1} - 1} w(t) + Q(v(t_n)) + \frac{1}{\lambda} Q^*(v(t_n))\right] + \wp, \\
t \in J, m = 1,2,..., p
\end{cases} \qquad (3.35
$$

where

$$
\begin{aligned}
\wp &= \left(1 - \sum_{m=0}^{p} \frac{\lambda_m(\eta_m - t_m)^{\alpha_m}}{\Gamma(\alpha_m + 1)}\right)^{-1} \left\{\sum_{m=0}^{p} \lambda_m I_{t_m}^{\alpha_m}\left(\int_{t_m}^{\eta_m} e^{-\lambda(r-s)} I_{t_{n-1}}^{\beta_{n-1} - 1} w(s)ds\right)(\eta_m)\right. \\
&+ \sum_{m=0}^{p}\sum_{n=1}^{m} \lambda_m I_{t_m}^{\alpha_m} e^{-\lambda(\eta_m - t_n)}\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n - s)} I_{t_{n-1}}^{\beta_{n-1} - 1} w(s)ds + I_{t_{n-1}}^{\beta_{n-1} - 1} w(t_n) - \frac{1}{\lambda} Q^*(v(t_n))\right] \\
&\left.+ \sum_{m=0}^{p}\sum_{n=1}^{m} \frac{\lambda_m(\eta_m - t_m)^{\alpha_m}}{\Gamma(\alpha_m + 1)}\left[\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1} - 1} w(t_n) + Q(v(t_n)) + \frac{1}{\lambda} Q^*(v(t_n))\right]\right\}.
\end{aligned}
$$

Proof: Assume that v is a solution of (3.2)-(3.5). For any $t \in J_0$, we have

$$v(t)=\int_0^t e^{-\lambda(t-s)} I_{t_0}^{\beta_0-1} w(s)\,ds+e^{-\lambda t}a_1+a_2, t\in J_0, \tag{3.36}$$

where a_1 and $a_2 \in R$. Differentiating the obtained linear equation (3.36) on J_0, leads to

$$v'(t)=-\lambda\int_0^t e^{-\lambda(t-s)} I_{t_0}^{\beta_0-1} w(s)\,ds+I_{t_0}^{\beta_0-1}w(t)-\lambda e^{-\lambda t}a_1, t\in J_0. \tag{3.37}$$

If $t \in J_1$, then

$$v(t)=\int_{t_1}^t e^{-\lambda(t-s)} I_{t_1}^{\beta_1-1} w(s)\,ds+e^{-\lambda(t-t_1)}b_1+b_2,$$

$$v(t)=-\lambda\int_{t_1}^t e^{-\lambda(t-s)} I_{t_1}^{\beta_1-1} w(s)\,ds+I_{t_1}^{\beta_1-1}w(t)-\lambda e^{-\lambda(t-t_1)}b_1,$$

for some b_1 and $b_2 \in R$. Thus,

$$v(t_1^-)=\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds+e^{-\lambda t_1}a_1+a_2,$$

and

$$v'(t_1^-)=-\lambda\int_0^{t_1} e^{-(t\lambda_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds+I_{t_0}^{\beta_0-1}w(t_1)-\lambda e^{-\lambda t_1}a_1,$$

$$v(t_1^+)=b_1+b_2,$$

$$v'(t_1^+)=-\lambda b_1.$$

Now, by using the IC's $\Delta v(t_1^+)=v(t_1^+)-(t_1^-)=Q(v(t_1))$ we have

$$\Delta v(t_1)=b_1+b_2-\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds+e^{-\lambda t_1}a_1+a_2=Q_1(v(t_1))$$

$$b_2=-b_1+\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds-e^{-\lambda t_1}a_1-a_2+Q_1(v(t_1)) \tag{3.38}$$

also by using the IC's $\Delta v'(t_1^+)=v'(t_1^+)-v'(t_1^-)=Q_1^*(v(t_1))$, we have

$$\Delta v'(t_1^+)=-\lambda b_1+\lambda\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds+I_{t_0}^{\beta_0-1}w(t_1)-\lambda e^{-\lambda t_1}a_1=Q_1^*(v(t_1))$$

$$b_1=\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds-\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_1)+e^{\lambda t_1}a_1-\frac{1}{\lambda}Q_1^*(v(t_1)) \tag{3.39}$$

Now, taking (3.39) into (3.38), we can get

$$b_2=-\left(\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds-\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_1)+e^{-\lambda t_1}a_1-\frac{1}{\lambda}Q_1^*(v(t_1))\right)$$
$$+\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds-e^{-\lambda t_1}a_1-a_2+Q_1(v(t_1))$$

$$b_2=\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_1)+Q_1(v(t_1))+\frac{1}{\lambda}Q_1^*(v(t_1))+a_2,$$

Consequently,

$$\begin{aligned} v(t)=&\int_{t_1}^t e^{-\lambda(t-s)} I_{t_1}^{\beta_1-1} w(s)\,ds+e^{-\lambda(t-t_1)}\\ &\times\left[\int_0^{t_1} e^{-\lambda(t_1-s)} I_{t_0}^{\beta_0-1} w(s)\,ds-\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_1)-\frac{1}{\lambda}Q_1^*(v(t_1))\right]\\ &+\left[\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_1)+Q_1(v(t_1))+\frac{1}{\lambda}Q_1^*(v(t_1))\right]+e^{-\lambda t_1}a_1+a_2,\\ &t\in J_1. \end{aligned}$$

If $t\in J_2$, then

$$v(t)=\int_{t_2}^{t}e^{-\lambda(t-s)}I_{t_2}^{\beta_2-1}w(s)\,ds+e^{-\lambda(t-t_2)}c_1+c_2,$$

$$v(t)=-\lambda\int_{t_2}^{t}e^{-\lambda(t-s)}I_{t_2}^{\beta_2-1}w(s)\,ds+I_{t_2}^{\beta_2-1}w(t)-\lambda e^{-\lambda(t-t_2)}c_1,$$

For some $c_1,c_2\in R$, thus

$$v(t_2^-)=\int_{t_1}^{t_2}e^{-\lambda(t_2-s)}I_{t_1}^{\beta_1-1}w(s)\,ds+e^{-\lambda(t_2-t_1)}b_1+b_2,$$

$$v'(t_2^-)=-\lambda\int_{t_1}^{t_2}e^{-\lambda(t_2-s)}I_{t_1}^{\beta_1-1}w(s)\,ds+I_{t_1}^{\beta_1-1}w(t_2)-\lambda e^{-\lambda(t_2-t_1)}b_1,$$

$$v(t_2^+)=c_1+c_2,$$

$$v'(t_2^+)=-\lambda c_1.$$

In the same way we have to find the following IC's

$$\Delta v(t_2^+)=\mathrm{v}(t_2^+)-v(t_2^-)=Q_2(v(t_2)),$$

$$\Delta v'(t_2^+)=\mathrm{v}'(t_2^+)\text{-}\mathrm{v}'(t_2^-)=Q_1^*(v(t_2)).$$

We can obtain

$$c_1=\int_{t_1}^{t_2}e^{-\lambda(t_2-s)}I_{t_1}^{\beta_1-1}w(s)\,ds-\frac{1}{\lambda}I_{t_1}^{\beta_1-1}w(t_2)+e^{-\lambda(t_2-t_1)}a_1-\frac{1}{\lambda}Q_2^*(v(t_2))$$

$$c_2=\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_2)+Q_2(v(t_2))+\frac{1}{\lambda}Q_2^*(v(t_2))+b_2.$$

Consequently,

$$\begin{aligned}v(t)=&\int_{t_2}^{t}e^{-\lambda(t-s)}I_{t_2}^{\beta_1-1}w(s)\,dr+e^{-\lambda(t-t_2)}\\&\times\left[\int_{t_1}^{t_2}e^{-\lambda(t_2-s)}I_{t_1}^{\beta_1-1}w(s)\,ds-\frac{1}{\lambda}I_{t_1}^{\beta_1-1}w(t_2)-\frac{1}{\lambda}Q_2^*(v(t_2))\right]\\&+\frac{1}{\lambda}I_{t_1}^{\beta_1-1}w(t_2)+Q_2(v(t_2))+\frac{1}{\lambda}Q_2^*(v(t_2))+e^{-\lambda(t-t_1)}b_1+b_2,\end{aligned}$$

$$\begin{aligned}v(t)=&\int_{t_2}^{t}e^{-\lambda(t-s)}I_{t_2}^{\beta_1-1}w(s)\,ds+e^{-\lambda(t-t_2)}\\&\times\left[\int_{t_1}^{t_2}e^{-\lambda(t_2-s)}I_{t_1}^{\beta_1-1}w(s)\,ds-\frac{1}{\lambda}I_{t_1}^{\beta_1-1}w(t_2)-\frac{1}{\lambda}Q_2^*(v(t_2))\right]\\&+\frac{1}{\lambda}I_{t_1}^{\beta_1-1}w(t_2)+Q_2(v(t_2))+\frac{1}{\lambda}Q_2^*(v(t_2))+e^{-\lambda(t-t_1)}\\&\times\left[\int_{0}^{t_1}e^{-\lambda(t_1-s)}I_{t_0}^{\beta_0-1}w(s)\,ds-\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_1)-\frac{1}{\lambda}Q_1^*(v(t_1))\right]\\&+\frac{1}{\lambda}I_{t_0}^{\beta_0-1}w(t_1)+Q_1(v(t_1))+\frac{1}{\lambda}Q_1^*(v(t_1))+e^{-\lambda t}a_1+a_2,\end{aligned}$$

Where $t\in J_2$. Repeating the process in this way, we get

$$
\begin{aligned}
v(t) &= \int_{t_m}^{t} e^{-\lambda(t-s)} I_{t_m}^{\beta_{t_m}-1} w(s)\,ds + \sum_{n=1}^{m} e^{-\lambda(t-t_n)} \\
&\times \left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} w(s)\,ds - \frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} w(t) - \frac{1}{\lambda} Q^*\left(v(t_n)\right) \right] \quad (3.40) \\
&+ \sum_{n=1}^{m} e^{-\lambda(t-t_n)} \left[\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} w(t) + Q\left(v(t_n)\right) + \frac{1}{\lambda} Q^*\left(v(t_n)\right) \right].
\end{aligned}
$$

Applyinh the BC's, $v(0)=0$, implies $a_1 = 0$. For $t \in J_m$, we have

$$
\begin{aligned}
I_{t_m^+}^{\alpha_m} v(t) &= I_{t_m^+}^{\alpha_m} \left(\int_{t_m}^{r} e^{-m(r-s)} I_{t_k}^{\beta_{t_k}-1} w(s)\,ds \right)(t) + \sum_{n=1}^{m} e^{-\lambda(t-t_n)} I_{t_m^+}^{\alpha_m} \\
&\times \left[\int_{t_{n-1}}^{t_n} e^{-m(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} w(s)\,ds - \frac{1}{m} I_{t_{n-1}}^{\beta_{n-1}-1} w(t) - \frac{1}{m} Q^*\left(v(t_n)\right) \right] \\
&+ \sum_{n=1}^{m} \frac{(t-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)} e^{-m(t-t_n)} \left[\frac{1}{m} I_{t_{n-1}}^{\beta_{n-1}-1} w(t) + Q\left(v(t_n)\right) + \frac{1}{m} Q^*\left(v(t_n)\right) \right] \\
&+ \frac{(t-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)} a_2,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{m=0}^{p} \lambda_m I_{t_m^+}^{\alpha_m} x(t) &= \sum_{m=0}^{p} \lambda_m I_{t_m^+}^{\alpha_m} \left(\int_{t_m}^{r} e^{-\lambda(r-s)} I_{t_n}^{\beta_{t_n}-1} w(s)\,ds \right)(t) + \sum_{m=0}^{p} \sum_{n=1}^{m} \lambda_m e^{-\lambda(t-t_n)} I_{t_m^+}^{\alpha_m} \\
&\times \left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} w(s)\,ds - \frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} w(t) - \frac{1}{\lambda} Q^*\left(v(t_n)\right) \right] \\
&+ \sum_{m=0}^{p} \sum_{n=1}^{m} \frac{\lambda_m (t-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)} e^{-\lambda(t-t_n)} \left[\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} w(t) + Q\left(v(t_n)\right) + \frac{1}{\lambda} Q^*\left(v(t_n)\right) \right] \\
&+ \sum_{m=0}^{p} \frac{\lambda_m (t-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)} a_2,
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= \left(1 - \sum_{m=0}^{p} \frac{\lambda_m (\eta_m - t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)} \right)^{-1} \left\{ \sum_{m=0}^{p} \lambda_m I_{t_m}^{\alpha_m} \left(\int_{t_m}^{\eta_m} e^{-\lambda(r-s)} I_{t_{n-1}}^{\alpha_{n-1}-1} w(s)\,ds \right)(\eta_m) \right. \\
&+ \sum_{m=0}^{p} \sum_{n=1}^{m} \lambda_m I_{t_m}^{\alpha_m} e^{-\lambda(\eta_m-t_n)} \left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} w(s)\,ds + I_{t_{n-1}}^{\beta_{n-1}-1} w(t_n) - \frac{1}{\lambda} Q^*\left(v(t_n)\right) \right] \\
&\left. + \sum_{m=0}^{p} \sum_{n=1}^{m} \frac{\lambda_m (\eta_m - t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)} \left[\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} w(t_n) + Q\left(v(t_n)\right) + \frac{1}{\lambda} Q^*\left(v(t_n)\right) \right] \right\}.
\end{aligned}
$$

Substituting the value of $a_j, (j = 1, 2)$ in (3.36) and (3.40), we obtain (3.35).

Conversely, assume that v is a solution of the impulsive sequential fractional integral equation (3.35), then by a direct computation, it follows that the solution given by (3.35) satisfies (3.5). The proof is completed.

Chapter 4

EXISTENCE AND UNIQUENESS

This chapter answers the existence & uniqueness of the equations (3.1)-(3.3), (3.1)-(3.4) and (3.2)-(3.5) by using some theories such as Banach-FPT and Krasnoselskii's- FPT.

4.1 Existence and Uniqueness Results

We are going to show the solution of problems (3.1)-(3.3), (3.1)-(3.4) and (3.2)-(3.5) by using the existence and uniqueness theorem. To start, we will state and prove the main results using the following hypotheses.

(H_1) $f: J \times R \to R$ is a jointly continuous function.

(H_2) $\exists$ a constant $L_f > 0$ such that

$$| f(t,v) - f(t,u) | \leq | v-u | L_f, \ \mathrm{v}, u \in R, \ t \in J.$$

(H_3) $\exists$ a positive constants $K_Q, K_{Q^*}, L_Q, L_{Q^*}$, such that

$$| Q_m^*(v) - Q_m^*(u) | \leq L_{Q^*} | v-u |, \ | Q_m(v) - Q_m(u) | \leq L_Q | v-u |,$$

$$\left| Q_m(v) \right| \leq K_Q, \ \left| Q_m^*(v) \right| \leq K_{Q^*}.$$

From (H_1)-(H_3) we have that

$$| f(t,v) | \leq K_f + L_f | v |, \ \ \mathrm{v} \in R, \ \ t \in J, \ \ \mathrm{K}_f := \sup\left\{ \left| f(t,0) : 0 < t \leq 1 \right| \right\},$$

$$\left| Q_m^*(v) \right| \leq L_Q . |v| + K_{Q^*}, \ \left| Q_m(v) \right| \leq L_Q |v| + K_Q.$$

(H_4) $| f(t,v) | \leq \Phi(t)$, for $(t,v) \in J \times R$ where $\Phi \in L^{\frac{1}{\rho}}(J), \rho(0, \alpha - 1)$.

(H_5) $\exists$ $\vartheta_f \in PC(J,R)$ and $\Psi : R^* \to R^*$ continuous and nondecreasing such that

$$| f(t,v) | \leq \vartheta(t) \mu(\|v\|), \ \ for\, all \ \ (t,v) \in J \times R,$$

(H_6) $\exists$ an a number $N > 0$ such that

$$\frac{N}{L_{\mathrm{T}} \|\vartheta\| \mu(N)} > 1 \ .$$

(H_7) $\exists$ a nonnegative function $a(t) \in C(0,1)$ such that

$$| f(t,v) | \leq a(t) + \xi |v|^{\sigma}, \ \sigma > 0.$$

4.1.1 Existence results of the problem (3.1)-(3.2)

In view of the lemma 3.1, we can reconstruct the problem (3.1)-(3.3) as a FP problem.

Consider the operator $T: PC(J,R) \to PC(J,R)$ defined by

$$
\begin{aligned}
v(t) &= \int_0^t e^{-\lambda(t-r)} I^{\alpha-1} f(r,v(r))\,dr + h_1(t)\int_0^1 e^{-\lambda(1-r)} I^{\alpha-1} f(r,v(r))\,dr \\
&\quad + h_2(t) I^{\alpha-1} f(1,v(1)) + h_3(t)\sum_{n=1}^{p} Q_n(v(t_n)) + h_4(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) \\
&\quad + \sum_{n=1}^{p} N_{1,n} Q_n(v(t_n)) + \sum_{n=m+1}^{p} N_{2,n} Q_n^*(v(t_n)) - \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + N_3(t), \\
&\quad t \in [t_m, t_{m+1}),\ m = 0,1,\ldots,p,
\end{aligned}
\tag{4.1}
$$

It is clear that T is well defined due to (H_1) and $PC(J,R)$ into itself.

Theorem 4.1 Suppose that (H_1), (H_2) and (H_3) are holds. If

$$
\begin{aligned}
L_T &= \left(\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}(1+\|h_1\|) + \frac{1}{\lambda\Gamma(\alpha)}\|h_2\| \right) L_f + (1+\|h_3\|)\, p(L_Q) \\
&\quad + (\|h_4\| + \|N_{1,n}\| + \|N_{2,n}\|)\, p(L_{Q^*}) + \|N_3\|.
\end{aligned}
\tag{4.2}
$$

Then the equation (3.1)-(3.3) has a unique solution on J.

Proof: Step1: T maps $B_r = \{v \in PC([0,1],R), \|v\| \le r\}$ into itself for some $r > 0$.

$$
\begin{aligned}
r &> (1-L_T)^{-1} \left(\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}(1+\|h_1\|) + \frac{1}{\lambda\Gamma(\alpha)}\|h_2\| \right) L_f \\
&\quad + (1+\|h_3\|)\, p(L_Q r + K_Q) + (\|h_4\| + \|N_{1,n}\| + \|N_{2,n}\|) \\
&\quad p(L_{Q^*} r + K_{Q^*}) + \|N_3\|.
\end{aligned}
$$

For $t \in J_m, m = 0,1,\ldots,p,$ we have

$$
\begin{aligned}
|Tv(t)| &= \Bigg| \int_0^t e^{-\lambda(t-r)} I^{\alpha-1} f(r,v(r))\,dr + h_1(t)\int_0^1 e^{-\lambda(1-r)} I^{\alpha-1} f(r,v(r))\,dr \\
&\quad + h_2(t) I^{\alpha-1} f(1,v(1)) + h_3(t)\sum_{n=1}^{p} Q_n(v(t_n)) + h_4(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) \\
&\quad + \sum_{n=1}^{p} N_{1,n} Q_n(v(t_n)) + \sum_{n=m+1}^{p} N_{2,n} Q_n^*(v(t_n)) + \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + N_3(t) \Bigg|,
\end{aligned}
$$

$$\left|Tv(t)\right| \le \int_0^t e^{-\lambda(t-r)} I^{\alpha-1} \left| f\left(r, v(r)\right)\right| dr + \left|h_1(t)\right| \int_0^1 e^{-\lambda(1-r)} I^{\alpha-1} \left| f\left(r, v(r)\right)\right| dr$$

$$+ \left|h_2(t)\right| I^{\alpha-1} \left| f\left(1, x(1)\right)\right| + \left|h_3(t)\right| \sum_{n=1}^{p} \left|Q_n\left(v(t_n)\right)\right| + \left|v_4(t)\right| \sum_{n=1}^{p} \left|Q_n^*\left(v(t_n)\right)\right|$$

$$+ \sum_{n=1}^{p} \left|N_{1,n}\right| \left|Q_n\left(v(t_n)\right)\right| + \sum_{n=m+1}^{p} \left|N_{2,n}\right| \left|Q_n^*\left(v(t_n)\right)\right| + \sum_{n=m+1}^{p} \left|Q_n^*\left(v(t_n)\right)\right| + \left|N_3(t)\right|,$$

and then

$$\left|Tv(t)\right| \le \int_0^t e^{-\lambda(t-r)} I^{\alpha-1} \left| f\left(r, v(r)\right) - f(r,0)\right| + f(r,0)\, dr$$

$$+ \left|h_1(t)\right| \int_0^1 e^{-\lambda(1-r)} I^{\alpha-1} \left| f\left(r, v(r)\right) - f(r,0)\right| + f(r,0)\, dr$$

$$+ \left|h_2(t)\right| I^{\alpha-1} \left| f\left(1, v(1)\right) - f(r,0)\right| + f(r,0) + \left|h_3(t)\right| \sum_{n=1}^{p} \left|Q_n\left(v(t_n)\right)\right|$$

$$+ \left|h_4(t)\right| \sum_{n=1}^{p} \left|Q_n^*\left(v(t_n)\right)\right| + \sum_{n=1}^{p} \left|N_{1,n}\right| \left|Q_n\left(v(t_n)\right)\right| + \sum_{n=m+1}^{p} \left|N_{2,n}\right| \left|Q_n^*\left(v(t_n)\right)\right|$$

$$+ \sum_{n=m+1}^{p} \left|Q_n^*\left(v(t_n)\right)\right| + \left|N_3(t)\right|,$$

thus

$$\left|(Tv)(t)\right| \le \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda t}\right)\left(L_f r + K_f\right) + \left|h_1(t)\right| \frac{1^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda}\right)\left(L_f r + K_f\right)$$

$$+ \left|h_2(t)\right| \frac{1^{\alpha-1}}{\Gamma(\alpha)} \left(L_f r + K_f\right) + \left|h_3(t)\right| p\left(L_Q r + K_Q\right) + \left|h_4(t)\right| p\left(L_{Q^*} r + K_{Q^*}\right)$$

$$+ \left|N_{1,n}(t)\right| p\left(L_{Q^*} r + N_{Q^*}\right) + \left|N_{2,n}(t)\right| p\left(L_{Q^*} r + N_{Q^*}\right) + \left|N_3(t)\right|.$$

We use the following estimation in what follows

$$\left|\frac{1}{\Gamma(\alpha-1)} \int_0^t e^{-\lambda(t-r)} \left(\int_0^r (r-\tau)^{(\alpha-1)} v(\tau)\, d\tau\right) dr\right| \le \frac{\left(1 - e^{-\lambda t}\right) t^{(\alpha-1)}}{\lambda\Gamma(\alpha)} \|v\|_{PC}$$

$$= \frac{\left(1 - e^{-\lambda t}\right)}{\lambda\Gamma(\alpha)} \|v\|_{PC}, v \in PC(J, R).$$

We obtain that

$$\left|(Tv)(t)\right| \le \left(\frac{\left(1 - e^{-\lambda t}\right)}{\lambda\Gamma(\alpha)} \left(\left|h_1(t)\right| + 1\right) + \frac{\left|h_2(t)\right|}{\Gamma(\alpha)}\right)\left(L_f r + K_f\right)$$

$$+ \left(1 + \left|h_3(t)\right|\right) p\left(L_Q r + K_Q\right) + \left(\left|h_4(t)\right| + \left|N_{1,n}(t)\right| + \left|N_{2,n}(t)\right|\right)$$

$$p\left(L_{Q^*} r + K_{Q^*}\right) + \left|N_3(t)\right|,$$

which implies that $Tv \in B_r$. Thus $TB_r \in B_r$.

Step2. T is a contraction operator on $PC(J,R)$. Let $v,u\in B_r$. Then $\forall\ t\in J$, we have

$$\begin{aligned}|Tv(t)-Tu(t)| = &\Bigg|\int_0^t e^{-\lambda(t-r)} I^{\alpha-1} f(r,v(r))dr + h_1(t)\int_0^1 e^{-\lambda(1-r)} I^{\alpha-1} f(r,v(r))dr \\ &+h_2(t) I^{\alpha-1} f(1,v(1)) + h_3(t)\sum_{n=1}^{p} Q_n(v(t_n)) + h_4(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) \\ &+\sum_{n=1}^{p} N_{1,n} Q_n(v(t_n)) + \sum_{n=m+1}^{p} N_{2,n} Q_n^*(v(t_n)) - \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + N_3(t)\Bigg| \\ &-\Bigg|\int_0^t e^{-\lambda(t-s)} I^{\alpha-1} f(r,u(r))dr + h_1(t)\int_0^1 e^{-\lambda(1-r)} I^{\alpha-1} f(r,u(r))dr \\ &+h_2(t) I^{\alpha-1} f(1,u(1)) + h_3(t)\sum_{n=1}^{p} Q_n(u(t_n)) + h_4(t)\sum_{n=1}^{p} Q_n^*(u(t_n)) \\ &+\sum_{n=1}^{p} N_{1,n} Q_n(u(t_n)) + \sum_{n=m+1}^{p} N_{2,n} Q_n^*(u(t_n)) - \sum_{n=m+1}^{p} Q_n^*(u(t_n)) + N_3(t)\Bigg|,\end{aligned}$$

$$\begin{aligned}|Tv(t)-Tu(t)| \le &\int_0^t e^{-\lambda(t-r)} I^{\alpha-1} \left| f(r,v(r)) - f(r,u(r))\right| dr \\ &+|h_1(t)|\int_0^1 e^{-\lambda(1-s)} I^{\alpha-1} \left| f(r,v(r)) - f(r,u(r))\right| dr \\ &+|h_2(t)| I^{\alpha-1} \left| f(1,v(1)) - f(1,u(1))\right| \\ &+|h_3(t)|\sum_{n=1}^{p}\left|Q_n(v(t_n)) - Q_n(u(t_n))\right| \\ &+|h_4(t)|\sum_{n=1}^{p}\left|Q_n^*(v(t_n)) - Q_n^*(u(t_n))\right| \\ &+\sum_{n=1}^{p}|N_{1,n}|\left|Q_n(v(t_n)) - Q_n(u(t_n))\right| \\ &+\sum_{n=m+1}^{p}|N_{2,n}|\left|Q_n^*(v(t_n)) - Q_n^*(u(t_n))\right| \\ &+\sum_{n=m+1}^{p}\left|Q_n^*(v(t_n)) - Q_n^*(u(t_n))\right|.\end{aligned}$$

Thus

$$\begin{aligned}|Tv(t)-Tu(t)| \le &\Bigg(\left(\frac{\left(1-e^{-\lambda t}\right)}{\lambda\Gamma(\alpha)}\left(1+|h_1(t)|\right) + \frac{|h_2(t)|}{\Gamma(\alpha)}\right)(L_f) \\ &+\left(1+|h_3(t)|\right) p(L_Q) + \left(|h_4(t)| + |N_{1,n}(t)| + |N_{2,n}(t)|\right) \\ &\times p\left(L_{Q^*}\right) + |N_3(t)|\Bigg)\|v-u\|_{PC} \\ &= L_T\|v-u\|_{PC}.\end{aligned}$$

Thus, T is a contraction mapping on $PC(J,R)$ due to condition (4.2). By applying the well-known Banach-contraction mapping principle we see that the operator T has a unique- FP on. Therefore, the problem (3.1)-(3.3) has a unique solution.

Theorem 4.2 Suppose that (H_1),(H_3) and (H_4) holds. If

$$\left(1+|h_3(t)|\right)p\left(L_Q\right)+\left(|h_4(t)|+|N_{1,n}(t)|+|N_{2,n}(t)|\right)p\left(L_{Q^*}\right)+|N_3(t)|\Big)<1.$$

Then the equation (3.1)-(3.3) has a unique solution on J.

Proof: Let $B_r=\{v\in PC(J,R),\|v\|_{PC}\le r\}$. We can choose

$$r\ge\|\Phi\|_{L^{\frac{1}{\sigma}}}\left(\frac{\left(1-e^{-\lambda}\right)1^{\alpha\text{-}\rho\text{-}1}}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{\rho-1}\right)}\left(1+\|h_1\|\right)+\frac{1^{\alpha\text{-}\rho\text{-}1}}{\lambda\left(\frac{\alpha-\rho-1}{\rho-1}\right)}\|h_2\|\right)$$
$$+\left(1+\|h_3\|\right)pL_Q+\left(\|h_4\|+\|N_{1,n}\|+\|N_{2,n}\|\right)pL_{Q^*}+\|N_3\|.$$

The operators T_1 and T_2 on B_r are defined as:

$$(T_1v)(t)=\int_0^t e^{-\lambda(t-r)}I^{\alpha-1}f\left(r,v(r)\right)dr+h_1(t)\int_0^1 e^{-\lambda(1-r)}I^{\alpha-1}f\left(r,v(r)\right)dr$$
$$+h_2(t)I^{\alpha-1}f\left(1,v(1)\right),$$

$$(T_2x)(t)=h_3(t)\sum_{n=1}^{p}Q_n\left(v(t_n)\right)+h_4(t)\sum_{n=1}^{p}Q_n^*\left(v(t_n)\right)+\sum_{n=1}^{p}N_{1,n}Q_n\left(v(t_n)\right)$$
$$+\sum_{n=m+1}^{p}N_{2,n}Q_n^*\left(v(t_n)\right)-\sum_{n=m+1}^{p}Q_n^*\left(v(t_n)\right)+N_3(t).$$

Step1 $T_1v+T_2u\in B_r$. For $v,u\in B_r$.

For any $v,u\in B_r$ and $t\in J_m$, using Holder is inequality with the assumption (H_1) we get

$$\int_0^t\left|e^{-\lambda(t-r)}I^{\alpha-1}f(r,v(r))\right|dr\le\int_0^t\left|e^{-\lambda(t-r)}\left(\frac{1}{\Gamma(\alpha-1)}\int_0^r(r-\tau)^{\alpha-2}f(\tau,v(\tau))d\tau\right|dr$$
$$\le\left(\frac{(1-e^{-\lambda t})}{\lambda\Gamma(\alpha-1)}\int_0^r(r-\tau)^{\frac{\alpha-2}{1-\rho}}\right)^{1-\rho}\left(\int_0^r(\Phi(r))^{\frac{1}{\rho}}\right)\le\frac{t^{\alpha-\rho-1}(1-e^{-\lambda t})\|\Phi\|_{L^{\frac{1}{\rho}}}(J)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}.$$

$$I^{\alpha-1}v(1)=\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-r)^{\alpha-2}\left|f(r,v(r))\right|dr$$
$$\le\frac{1}{\Gamma(\alpha-1)}\left(\int_0^1(1-r)^{\frac{\alpha-2}{1-\rho}}\right)^{1-\rho}\left(\int_0^1(\Phi(r))^{\frac{1}{\rho}}\right)$$
$$\le\frac{1^{\alpha-\rho-1}\|\Phi\|_{L^{\frac{1}{\rho}}}(J)}{\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}.$$

$$\int_0^1 \left| e^{-\pi(1-r)} (\int_0^r \frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f(\tau, v(\tau)) d\tau \right| dr \le \frac{1^{\alpha-\rho-1}(1-e^{-\lambda}) \|\Phi\|_{L^{\frac{1}{\rho}}}}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}.$$

Therefore,

$$\|T_1 v + T_2 u\|_{PC} \le \frac{\|\Phi\|_{L^{\frac{1}{\rho}}}(J)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} + \frac{1^{\alpha-\rho-1}(1-e^{-\lambda})\|\Phi\|_{L^{\frac{1}{\rho}}}}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} h_1(t) + \frac{1^{\alpha-\rho-1}\|\Phi\|_{L^{\frac{1}{\rho}}}}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} h_2(t)$$

$$+\left(1+\|h_3\|\right) pL_Q + \left(\|h_4\| + \|N_{1,n}\| + \|N_{2,n}\|\right) pL_{Q^*} + \|N_3\|.$$

$$\|T_1 x + T_2 y\|_{PC} \le \|\Phi\|_{L^{\frac{1}{\rho}}} \left(\frac{1}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} + \left(1+h_1(t)\right) + \frac{1^{\alpha-\sigma-1}\|\Phi\|_{L^{\frac{1}{\sigma}}}}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} h_2(t) \right)$$

$$+\left(1+\|h_3\|\right) pL_Q + \left(\|h_4\| + \|N_{1,n}\| + \|N_{2,n}\|\right) pL_{Q^*} + \|N_3\|.$$

Thus,

$\|T_1 v + T_2 u\|_{PC} \le a$, $T_1 v + T_2 u \in B_r$.

Step 2. T_1 is compact and continuous. The continuity of f implies that T_1 is continuous, also T_1 is uniformly bounded on B_a as

$$\|T_1 v\|_{PC} \le \|\Phi\|_{L^{\frac{1}{\sigma}}} \left(\frac{1}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} + \left(1+h_1(t)\right) + \frac{1^{\alpha-\sigma-1}\|\Phi\|_{L^{\frac{1}{\rho}}}}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\sigma}} h_2(t) \right) \le r.$$

For equicontinuity on $[0,t_1]$, let $v \in B_r$ and for any $s_1, s_2 \in [0,t_1], s_1 < s_2$, we have

$$\begin{aligned}
\left|(T_1 v)(s_2) - (T_1 v)(s_1)\right| = & \int_0^{s_2} e^{-\lambda(s_2-r)} \left(\int_0^r \frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(\tau, v(\tau)) d\tau \right| \right) dr \\
& + v_1 \int_0^1 e^{-\lambda(1-r)} \left(\int_0^r \frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(\tau, v(\tau)) d\tau \right| \right) dr \\
& + \frac{h_2}{\Gamma(\alpha-1)} \int_0^1 (1-r)^{\alpha-2} \left| f(r, v(r)) \right| dr \\
& - \int_0^{s_1} e^{-\lambda(s_1-r)} \left(\int_0^r \frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(\tau, v(\tau)) d\tau \right| \right) dr \\
& + v_1 \int_0^1 e^{-\lambda(1-r)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(\tau, v(\tau)) d\tau \right| \right) dr \\
& + \frac{h_2}{\Gamma(\alpha-1)} \int_0^1 (1-r)^{\alpha-2} \left| f(r, v(r)) \right| dr,
\end{aligned}$$

$$\left|(T_1v)(s_2)-(T_1v)(s_1)\right|\le\left(e^{-m(s_2)}-e^{-m(s_1)}\right)\int_0^{s_2}e^{mr}\left(\int_0^r\frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau\right)dr$$

$$\int_{s_1}^{s_2}e^{-m(s_2-r)}\left(\int_0^r\frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau\right)dr$$

$$+\left|h_1(t)-h_1(t)\right|\int_0^1e^{-k(1-r)}\left(\int_0^r\frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau\right)dr$$

$$+\left|h_2(t)-h_2(t)\right|\int_0^1\frac{(1-r)^{\alpha-2}}{\Gamma(\alpha-1)}dr.$$

It tends to zero as $s_1\to s_2$. This implies that T_1 is equicontinuous on the interval $[0,t_1]$. In general, for the time $\left(t_m,t_{m+1}\right]$, similarly one can obtain the same inequality, which yields that T_1 is equicontinuous on $\left(t_m,t_{m+1}\right]$. Together with the PC-type Arzela-Ascoli (Lemma 3.14) theorem, we can conclude that $T_1:B_r\to B_r$, T_1 is continuous and compact.

Step 3. It is clear that T_2 is contraction mapping. Thus, all the assumptions of the Krasnoselskii's theorem are satisfied. In consequence, the Krasnoselskii's theorem is applied and hence the problem (3.1)-(3.3) has at least one solution on J.

Theorem 4.3 Suppose that (H_5) and (H_6) holds. Then our BVP (3.1)-(3.3) has at least one solution on J.

Proof: Consider the operator $T\;:\;PC\left(J,R\right)\to\;PC(J,R)$ defined by (4.1). Clearly, it is obvious that T is continuous and compact.

T maps bounded sets into bounded sets in $PC(J,R)$. Repeating the same process in Step2 Theorem 4.2 , we get

$$\left|Tv(t)\right|=\left|\int_0^t e^{-\lambda(t-r)}I^{\alpha-1}f\left(r,v(r)\right)ds+h_1(t)\int_0^1 e^{-\lambda(1-r)}I^{\alpha-1}f\left(r,v(r)\right)dr\right.$$

$$+h_2(t)I^{\alpha-1}f\left(1,v(1)\right)+h_3(t)\sum_{n=1}^{p}Q_n\left(v(t_n)\right)+h_4(t)\sum_{n=1}^{p}Q_n^*\left(v(t_n)\right)$$

$$\left.+\sum_{n=1}^{p}N_{1,n}Q_n\left(v(t_n)\right)+\sum_{n=m+1}^{p}N_{2,n}Q_n^*\left(v(t_n)\right)+\sum_{n=m+1}^{p}Q_n^*\left(v(t_n)\right)+N_3(t)\right|,$$

$$\left|Tv(t)\right| \le \int_0^t e^{-\lambda(t-r)} I^{\alpha-1}\left|f\left(r,v(r)\right)\right|dr + \left|h_1(t)\right|\int_0^1 e^{-\lambda(1-r)} I^{\alpha-1}\left|f\left(r,v(r)\right)\right|dr$$
$$+\left|h_2(t)\right| I^{\alpha-1}\left|f\left(1,v(1)\right)\right| + \left|h_3(t)\right|\sum_{n=1}^{p}\left|Q_n\left(v(t_n)\right)\right| + \left|h_4(t)\right|\sum_{n=1}^{p}\left|Q_n^*\left(v(t_n)\right)\right|$$
$$+\sum_{n=1}^{p}\left|Q_n\left(v(t_n)\right)\right|\left|N_{1,n}\right| + \sum_{n=m+1}^{p}\left|Q_n^*\left(v(t_n)\right)\right|\left|N_{2,n}\right| + \sum_{n=m+1}^{p}\left|Q_n^*\left(v(t_n)\right)\right| + \left|N_3(t)\right|,$$

$$\left|Tv(t)\right| \le \int_0^t e^{-\lambda(t-r)} I^{\alpha-1}\vartheta\mu\left(|v|\right)dr + \left|h_1(t)\right|\int_0^1 e^{-\lambda(1-r)} I^{\alpha-1}\vartheta\mu\left(|v|\right)dr$$
$$+\left|h_2(t)\right| I^{\alpha-1}\vartheta\mu\left(|v|\right) + \left|h_3(t)\right|\sum_{n=1}^{p}\left|Q_n\left(v(t_n)\right)\right| + \left|h_4(t)\right|\sum_{n=1}^{p}\left|Q_n^*\left(v(t_n)\right)\right|$$
$$+\sum_{n=1}^{p}\left|N_{1,n}\right|\left|Q_n\left(v(t_n)\right)\right| + \sum_{n=m+1}^{p}\left|N_{2,n}\right|\left|Q_n^*\left(v(t_n)\right)\right| + \sum_{n=m+1}^{p}\left|Q_n^*\left(v(t_n)\right)\right| + \left|N_3(t)\right|,$$
$$\left\|T_1v + T_2u\right\|_{PC} \le \frac{1^{\alpha-\rho-1}(1-e^{-\lambda})}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} + \frac{1^{\alpha-\rho-1}(1-e^{-\lambda})\vartheta\mu\left(|v|\right)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}h_1(t) + \frac{1^{\alpha-\rho-1}\vartheta\mu\left(|v|\right)}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}h_2(t)$$
$$+\left(1+\|h_3\|\right)pL_Q + \left(\|h_4\| + \|N_{1,n}\| + \|N_{2,n}\|\right)pL_{Q^*} + \|N_3\|.$$
$$\left\|T_1v + T_2u\right\|_{PC} \le \vartheta\mu\left(|v|\right)\left(\frac{1}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}} + \left(1+h_1(t)\right) + \frac{1}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}h_2(t)\right)$$ Now,
$$+\left(1+\|h_3\|\right)pL_Q + \left(\|h_4\| + \|N_{1,n}\| + \|N_{2,n}\|\right)pL_{Q^*} + \|N_3\|.$$

construct the set $\Lambda = \{v \in PC(J,R) : \|v\| < N\}$. The operator $T : \Lambda \to PC(J,R)$ is continuous and completely continuous. From the choice of Λ, there is no $v \in \partial\Lambda$ such that $v = \lambda Tv$, $0 < \lambda < 1$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that T has a FP $v \in \partial\Lambda$, which concludes that the problem (3.1)-(3.3) has at least one solution.

4.1.2 Existence results of the problem (3.1)-(3.3)

In view of the Lemma 3.2, we can transform the problem (3.1)-(3.4) into a FP problem.

Consider the operator $T : PC(J,R) \to PC(J,R)$ defined by

$$
\begin{aligned}
(Tv)(t) &= \int_0^t M_1(t,\tau)\, f(\tau, v(\tau))\, d\tau + d_1(t)\int_0^1 M_1(t,\tau)\, f(\tau, v(\tau))\, d\tau \\
&\quad + d_2(t)\int_0^1 M_2(t,\tau)\, f(\tau, v(\tau))\, d\tau + d_3(t)\int_0^1 f(\tau, v(\tau))\, d\tau \\
&\quad + d_4(t)\sum_{n=1}^{p} Q_n(v(t_n)) + d_5(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) + \sum_{n=1}^{p} p_{1,n} Q_n(v(t_n)) \\
&\quad + \sum_{n=m+1}^{p} p_{2,n} Q_n^*(v(t_n)) - \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + d_6(t),
\end{aligned} \tag{4.3}
$$

where $t \in [t_m, t_{m+1}), m = 0,1,...,n$. It is obvious that T is well defined due to (H_1) and sends $PC(J,R)$ into itself.

Theorem 4.4 Suppose that (H_1),(H_2) and (H_3)hold. If

$$
\begin{aligned}
L_T &= \left(\frac{(1-e^{-\lambda})}{\lambda\Gamma(\alpha)}(1+\|d_1\|) + \frac{1}{\lambda}(\lambda + e^{-\lambda} - 1)\|d_2\| + \|d_3\| \right) L_f \\
&\quad + (1+\|d_4\|)\, pL_Q + (\|d_5\| + \|p_{1,n}\| + \|p_{2,n}\|)\, pL_{Q^*} + \|d_6\| \\
&< 1,
\end{aligned} \tag{4.4}
$$

then the problem (3.1)-(3.4) has a unique solution on J.

Proof: Step1: T maps $B_r = \{v \in PC([0,1], R), \|v\| \le r\}$ into itself for some $r > 0$.

$$
\begin{aligned}
r > (1-L_T)^{-1} &\left(\frac{1}{\lambda\Gamma(\alpha)}(1-e^{-\lambda})(1+\|d_1\|) + \frac{1}{\lambda}(\lambda + e^{-\lambda} - 1)\|d_2\| + \|d_3\| \right) L_f \\
&+ (1+\|d_4\|)\, p(L_Q r + K_Q) + (\|d_5\| + \|p_{1,n}\| + \|p_{2,n}\|)\, p(L_{Q^*} r + K_{Q^*}) + \|d_6\|.
\end{aligned}
$$

For $t \in J_m, m = 0,1,...p,$ we have

$$
\begin{aligned}
|(Tv)(t)| = \Bigg| &\int_0^t M_1(t,\tau)\, f(\tau, v(\tau))\, d\tau + d_1(t)\int_0^1 M_1(t,\tau)\, f(\tau, v(\tau))\, d\tau \\
&+ d_2(t)\int_0^1 M_2(t,\tau)\, f(\tau, v(\tau))\, d\tau + d_3(t)\int_0^1 f(\tau, v(\tau))\, d\tau \\
&+ d_4(t)\sum_{n=1}^{p} Q_n(v(t_n)) + d_5(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) + \sum_{n=1}^{p} p_{1,n} Q_n(v(t_n)) \\
&+ \sum_{n=m+1}^{p} p_{2,n} Q_n^*(v(t_n)) + \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + d_6(t) \Bigg|,
\end{aligned}
$$

$$
\begin{aligned}
\left|(Tv)(t)\right| \le & \int_0^1 M_1(t,\tau)\left|f(\tau,v(\tau))\right|d\tau + \left|d_1(t)\right|\int_0^1 M_1(t,\tau)\left|f(\tau,v(\tau))\right|d\tau \\
& + \left|d_2(t)\right|\int_0^1 M_2(t,\tau)\left|f(\tau,v(\tau))\right|d\tau + \left|d_3(t)\right|\int_0^1 \left|f(\tau,v(\tau))\right|d\tau \\
& + \left|d_4(t)\right|\sum_{n=1}^{p}\left|Q_n(v(t_n))\right| + \left|d_5(t)\right|\sum_{n=1}^{p}\left|Q_n^*(v(t_n))\right| + \sum_{n=1}^{p}\left|p_{1,n}\right|\left|Q_n(v(t_n))\right| \\
& + \sum_{n=m+1}^{p}\left|p_{2,n}\right|\left|Q_n^*(v(t_n))\right| + \sum_{n=m+1}^{p}\left|Q_n^*(v(t_n))\right| + \left|d_6(t)\right|,
\end{aligned}
$$

and then

$$
\begin{aligned}
\left|(Tv)(t)\right| \le & \int_0^t M_1(t,\tau)\left|-f(\tau,0)+f(\tau,v(\tau))\right| + \left|f(\tau,0)\right|d\tau \\
& + \left|d_1(t)\right|\int_0^1 M_1(t,\tau)\left|-f(\tau,0)+f(\tau,v(\tau))\right| + \left|f(\tau,0)\right|d\tau \\
& + \left|d_2(t)\right|\int_0^1 M_2(t,\tau)\left|f(\tau,x(\tau))-f(\tau,0)\right| + \left|f(\tau,0)\right|d\tau \\
& + \left|d_3(t)\right|\int_0^1 \left|f(\tau,v(\tau))-f(\tau,0)\right| + \left|f(\tau,0)\right|d\tau \\
& + \left|d_4(t)\right|\sum_{n=1}^{p}\left|Q_n(v(t_n))\right| + \left|d_5(t)\right|\sum_{n=1}^{p}\left|Q_n^*(v(t_n))\right| \\
& + \sum_{n=1}^{p}\left|p_{1,n}\right|\left|Q_n(v(t_n))\right| + \sum_{n=m+1}^{p}\left|p_{2,n}\right|\left|Q_n^*(v(t_n))\right| \\
& + \sum_{n=m+1}^{p}\left|Q_n^*(v(t_n))\right| + \left|d_6(t)\right|,
\end{aligned}
$$

$$
\begin{aligned}
\left|(Tv)(t)\right| \le & \frac{1^{\alpha-1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)}\left(L_f r + K_f\right) + \left|d_1(t)\right|\frac{1^{\alpha-1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)}\left(L_f r + K_f\right) \\
& + \left|d_2(t)\right|\frac{1^{\alpha-1}\left(\lambda+e^{-\lambda}-1\right)}{\Gamma(\alpha)}\left(L_f r + K_f\right) + \left|d_3(t)\right| p\left(L_Q r + K_Q\right) \\
& + \left|d_4(t)\right| p\left(L_{Q^*} r + K_{Q^*}\right) + \left|p_{1,n}(t)\right| p\left(L_{Q^*} r + K_{Q^*}\right) + \left|p_{2,n}\right| p\left(L_{Q^*} r + K_{Q^*}\right) \\
& + p\left(L_Q r + K_Q\right) + \left|d_6(t)\right| \\
& < r, \\
& \le \left(\frac{1^{\alpha-1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)}\left(1+\left\|d_1\right\|\right) + \frac{1}{\lambda}\left(\lambda+e^{-\lambda}-1\right)\left\|d_2\right\| + \left\|d_3\right\|\right)\left(L_f r + K_f\right) \\
& + \left(1+\left\|d_4\right\|\right) p\left(L_Q r + K_Q\right) + \left(\left\|d_5\right\| + \left\|p_{1,n}\right\| + \left\|p_{2,n}\right\|\right) p\left(L_{Q^*} r + K_{Q^*}\right) + \left\|d_6\right\|
\end{aligned}
$$

Then

$$\left|(Tv)(t)\right| \le \left(\frac{1^{\alpha-1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)}\left(1+\|d_1\|\right) + \frac{\left(\lambda+e^{-\lambda}-1\right)}{\lambda}\|d_2\| + \|d_3\| \right)\left(L_f r + K_f\right)$$

$$+\left(1+\|d_4\|\right) p\left(L_Q r + K_Q\right) + \left(\|d_5\| + \|p_{1,n}\| + \|p_{2,n}\|\right) p\left(L_{Q^*} r + K_{Q^*}\right) + \|d_6\|$$

$$< r.$$

This implies that $Tv \in B_r$. Thus $TB_r \in B_r$.

Step2: T is a contraction operator on Let $v,u \in B_r$. Then for each $t \in J$, we have

$$\left|(Tv)(t)-(Tu)(t)\right| = \left| \int_0^t M_1(t,\tau)\, f\left(\tau, v(\tau)\right) d\tau + d_1(t)\int_0^1 M_1(t,\tau)\, f\left(\tau, v(\tau)\right) d\tau \right.$$

$$+d_2(t)\int_0^1 M_2(t,\tau)\, f\left(\tau, v(\tau)\right) d\tau + d_3(t)\int_0^1 f\left(\tau, v(\tau)\right) d\tau$$

$$+d_4(t)\sum_{n=1}^{p} Q_n\left(v(t_n)\right) + d_5(t)\sum_{n=1}^{p} Q_n^*\left(v(t_n)\right) + \sum_{n=1}^{p} p_{1,n} Q_n\left(v(t_n)\right)$$

$$\left. + \sum_{n=k+1}^{p} p_{2,n} Q_n^*\left(v(t_n)\right) + \sum_{n=k+1}^{p} Q_n^*\left(v(t_n)\right) + d_6(t) \right|$$

$$-\left| \int_0^t M_1(t,\tau)\, f\left(\tau, u(\tau)\right) d\tau + d_1(t)\int_0^1 M_1(t,\tau)\, f\left(\tau, u(\tau)\right) d\tau \right.$$

$$+d_2(t)\int_0^1 M_2(t,\tau)\, f\left(\tau, u(\tau)\right) d\tau + d_3(t)\int_0^1 f\left(\tau, u(\tau)\right) d\tau$$

$$+d_4(t)\sum_{n=1}^{p} Q_n\left(u(t_n)\right) + d_5(t)\sum_{n=1}^{p} Q_n^*\left(u(t_n)\right)$$

$$\left. +\sum_{n=1}^{p} p_{1,n} Q_n\left(u(t_n)\right) + \sum_{n=m+1}^{p} p_{2,n} Q_n^*\left(u(t_n)\right) + \sum_{n=m+1}^{p} Q_n^*\left(u(t_n)\right) + d_6(t) \right|$$

$$
\begin{aligned}
\left|(Tx)(t)-(Ty)(t)\right| &= \int_0^t M_1(t,\tau)\left|f\left(\tau,v(\tau)\right)-f\left(\tau,u(\tau)\right)\right|d\tau \\
&+\left|d_1(t)\right|\int_0^1 M_1(t,\tau)\left|f\left(\tau,v(\tau)\right)-f\left(\tau,u(\tau)\right)\right|d\tau \\
&+\left|d_2(t)\right|\int_0^1 M_2(t,\tau)\left|f\left(\tau,v(\tau)\right)-f\left(\tau,u(\tau)\right)\right|d\tau \\
&+\left|d_3(t)\right|\int_0^1 \left|f\left(\tau,v(\tau)\right)-f\left(\tau,u(\tau)\right)\right|d\tau \\
&+\left|d_4(t)\right|\sum_{n=1}^{p}\left|Q_n\left(v(t_n)\right)-Q_n\left(u(t_n)\right)\right| \\
&+\left|d_5(t)\right|\sum_{n=1}^{p}\left|Q_n^*\left(v(t_n)\right)-Q_n^*\left(u(t_n)\right)\right| \\
&+\sum_{n=1}^{p}\left|p_{1,n}\right|\left|Q_n\left(v(t_n)\right)-Q_n\left(u(t_n)\right)\right| \\
&+\sum_{n=m+1}^{p} p_{2,n}\left|Q_n^*\left(v(t_n)\right)-Q_n^*\left(u(t_n)\right)\right| \\
&+\sum_{n=m+1}^{p}\left|Q_n^*\left(v(t_n)\right)-Q_n^*\left(u(t_n)\right)\right|+\left|d_6(t)\right|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left|(Tv)(t)-(Tu)(t)\right| \le & \Bigg(\left(\frac{\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)}\left(1+\|d_1\|\right)+\frac{\left(\lambda+e^{-\lambda}-1\right)}{\lambda}\|d_2\|+\|d_3\|\right)\left(L_f r+K_f\right) \\
&+\left(1+\|d_4\|\right)p\left(L_Q r+K_Q\right)+\left(\|d_5\|+\|p_{1,n}\|+\|p_{2,n}\|\right)p\left(L_{Q^*}r+K_{Q^*}\right)\Bigg)\|v-u\|_{PC} \\
&= L_T\,\|v-u\|_{PC}\,.
\end{aligned}
$$

Thus, T is a contraction mapping on $PC\left(J,R\right)$ due to condition (4.4). By applying the well-known Banach-contraction mapping we see that the operator T has a unique -FP on $PC\left(J,R\right)$. Therefore, the problem (3.1)-(3.4) has a unique solution.

The second result is based on Krasnoselskii's- FPT. We state a known result due to Krasnoselskii's which is needed to prove the existence of at least one solution of (3.1)-(3.4).

Theorem 4.5 Assume that (H_1)**,** (H_3) and (H_4)hold. If

$$
\left(1+\|d_4\|\right)pL_Q+\left(\|d_5\|+\|p_{1,n}\|+\|p_{2,n}\|\right)pL_{Q^*}<1,
$$

then the BVP (3.1)-(3.3) has at least one solution on J.

Proof: Let $B_r = \{v \in PC(J,R), \|v\|_{PC} \le r\}$. We choose

$$r \ge \|\Phi\|_{L^{\frac{1}{\rho}}} \left(\frac{1^{\alpha\text{-}\rho\text{-}1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{\rho-1}\right)^{1-\rho}}\left(1+\|d_1\|\right) + \frac{1^{\alpha\text{-}\rho\text{-}1}\left(\lambda+e^{-\lambda}-1\right)}{\lambda\left(\frac{\alpha-\rho-1}{\rho-1}\right)^{1-\rho}}\|d_2\| \right.$$

$$\left. + \frac{1^{\alpha\text{-}\rho\text{-}1}}{\left(\frac{\alpha-\rho-1}{\rho-1}\right)}\|d_3\| \right) + \left(1+\|d_4\|\right) pL_Q + \left(\|d_5\| + \|p_{1,n}\| + \|p_{2,n}\|\right) pL_{Q^*} + \|d_6\|.$$

The operators T_1 and T_2 on B_r are defined as:

$$(T_1 v)(t) = \int_0^t M_1(t,\tau) f(\tau, v(\tau)) d\tau + d_1(t) \int_0^1 M_1(t,\tau) f(\tau, v(\tau)) d\tau$$

$$+ d_2(t) \int_0^1 M_2(t,\tau) f(\tau, v(\tau)) d\tau + d_3(t) \int_0^1 f(\tau, v(\tau)) d\tau.$$

$$(T_2 v)(t) = d_4(t) \sum_{n=1}^{p} Q_n(v(t_n)) + d_5(t) \sum_{n=1}^{p} Q_n^*(v(t_n)) + \sum_{n=1}^{p} p_{1,n} Q_n(v(t_n))$$

$$+ \sum_{n=m+1}^{p} p_{2,n} Q_n^*(v(t_n)) \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + d_6(t).$$

Step 1. For any $v,u \in B_r$ and $t \in J_m$, using the assumption (H_4) with the Holder inequality we get

$$\left| \int_0^1 M_2(1,\tau) w(\tau) d\tau \right| \le \frac{1^{\alpha\text{-}\rho\text{-}1}\left(1-e^{-\lambda}\right)}{\lambda\left(\frac{\alpha-\rho-1}{\rho-1}\right)^{\rho\text{-}1}} \left(\lambda + e^{-\lambda} - 1\right),$$

$$\left| \int_0^t M_1(t,\tau) w(\tau) d\tau \right| \le \int_0^t \left| e^{-\lambda(t-r)} I^{\alpha-1} f(r, v(r)) \right| dr$$

$$\le \int_0^t \left| e^{-\lambda(t-r)} \left(\int_0^r \frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f(r, v(\tau)) d\tau\right) \right| dr$$

$$\le \left(\frac{(1-e^{-\lambda t})}{\lambda\Gamma(\alpha-1)} \int_0^r (r-\tau)^{\frac{\alpha-2}{1-\rho}} \right)^{1-\rho} \left(\int_0^s (\Phi(s))^{\frac{1}{\rho}} \right)$$

$$\le \frac{(1-e^{-\lambda t}) \|\Phi\|_{L^{\frac{1}{\rho}}}(J)}{\lambda\Gamma(\alpha-1)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}.$$

Therefore,

$$\left\|T_1v+T_2u\right\|_{PC}\le \frac{1^{\alpha\text{-}\rho\text{-}1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}+\left\|d_1\right\|\frac{1^{\alpha\text{-}\rho\text{-}1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}+\left\|d_2\right\|\frac{1^{\alpha\text{-}\rho\text{-}1}\left(\lambda+e^{-\lambda}-1\right)}{\lambda\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}$$

$$+\frac{1^{\alpha\text{-}\rho\text{-}1}}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}\left\|d_3\right\|+\left(1+\left\|d_4\right\|\right)pL_Q+\left(\left\|d_5\right\|+\left\|p_{1,n}\right\|+\left\|p_{2,n}\right\|\right)pL_{Q^*}$$

$$<1,$$

$$\left\|T_1v+T_2u\right\|_{PC}\le \left\|\Phi\right\|_{L^{\frac{1}{\rho}}}\left(\frac{1^{\alpha\text{-}\rho\text{-}1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}\left(1+\left\|d_1\right\|\right)+\left\|d_2\right\|\frac{1^{\alpha\text{-}\rho\text{-}1}\left(\lambda+e^{-\lambda}-1\right)}{\lambda\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}\right.$$

$$\left.+\frac{\left\|d_3\right\|1^{\alpha\text{-}\rho\text{-}1}}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}\right)+\left(1+\left\|d_4\right\|\right)pL_Q+\left(\left\|d_5\right\|+\left\|p_{1,n}\right\|+\left\|p_{2,n}\right\|\right)pL_{Q^*}+\left\|d_6\right\|.$$

Thus,

$$\left\|T_1v+T_2u\right\|_{PC}\le r,\ \text{so}\ T_1v+T_2u\in B_r.$$

Step 2. T_1 is compact and continuous. The continuity of f implies T_1 is continuous, also T_1 is uniformly bounded on B_r as

$$\left\|T_1v\right\|_{PC}\le \left\|\Phi\right\|_{L^{\frac{1}{\rho}}}\left(\frac{1^{\alpha-\rho-1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}\left(1+\left\|d_1\right\|\right)+\left\|d_2\right\|\frac{1^{\alpha-\rho-1}\left(\lambda+e^{-\lambda}-1\right)}{\lambda\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}+\left\|d_3\right\|\frac{1^{\alpha-\rho-1}}{\left(\frac{\alpha-\rho-1}{1-\rho}\right)^{1-\rho}}\right).$$

For equicontinuity on $[0,t_1]$, let $v\in B_r$ and for any $s_1,s_2\in[0,t_1], s_1<s_2$, we have

$$\left|\left(T_1v\right)\left(s_2\right)-\left(T_1v\right)\left(s_1\right)\right|=\left|\int_0^{s_2}M_1\left(s_2,\tau\right)f\left(\tau,v(\tau)\right)d\tau+\int_{s_1}^{s_2}M_1\left(s_2,\tau\right)f\left(\tau,v(\tau)\right)d\tau\right.$$

$$+d_1(t)\int_0^1 M_1(t,\tau)f\left(\tau,v(\tau)\right)d\tau$$

$$\left.+d_2(t)\int_0^1 M_2(t,\tau)f\left(\tau,v(\tau)\right)d\tau+d_3(t)\int_0^1 f\left(\tau,v(\tau)\right)d\tau\right|$$

$$-\left|\int_0^{s_1}M_1\left(s_1,\tau\right)f\left(\tau,v(\tau)\right)d\tau+d_1(t)\int_0^1 M_1(t,\tau)f\left(\tau,v(\tau)\right)d\tau\right.$$

$$\left.+d_2(t)\int_0^1 M_2(t,\tau)f\left(\tau,v(\tau)\right)d\tau+d_3(t)\int_0^1 f\left(\tau,v(\tau)\right)d\tau\right|,$$

$$
\begin{aligned}
\left|(T_1v)(s_2)-(T_1v)(s_1)\right| \le & \int_0^{s_1}\left|M_1(s_2,\tau)-M_1(s_1,\tau)\right| f\left(\tau,v(\tau)\right)d\tau \\
& +\int_{s_1}^{s_2} M_1(s_2,\tau)\, f\left(\tau,v(\tau)\right)d\tau \\
& +\left|d_1(t)-d_1(t)\right|\int_0^1 M_1(t,\tau)\, f\left(\tau,v(\tau)\right)d\tau \\
& +\left|d_2(t)-d_2(t)\right|\int_0^1 M_2(t,\tau)\, f\left(\tau,v(\tau)\right)d\tau \\
& +\left|d_3(t)-d_3(t)\right|\int_0^1 f\left(\tau,v(\tau)\right)d\tau.
\end{aligned}
$$

It tends to zero as $s_1 \to s_2$.

This implies that T_1 is equicontinuous on the interval $[0,t_1]$. In general, for the time $\left(t_m,t_{m+1}\right]$, we similarly obtain the same inequality, which yields that T_1 is equicontinuous on the interval $\left(t_m,t_{m+1}\right]$. Together with the PC-type Arzela-Ascoli (Lemma 3.14) theorem, we can conclude that $T_1 : B_r \to B_r$, T_1 is continuous and compact.

Step 3. It is clear that T_2 is a contraction mapping. Thus, all the assumptions of the Krasnoselskii's theorem are satisfied. In consequence, the Krasnoselskii's theorem is applied and hence the problem (3.1)-(3.4) has at least one solution on J.

Theorem 4.6 Suppose that (H_5) and (H_6) hold. Then the BVP (3.1)-(3.4) has at least one solution on J.

Proof: Consider the operator $T : PC\left(J,R\right) \to PC(J,R)$ defined by (4.3). Clearly, it is obvious that T is continuous and compact. T Maps bounded sets into bounded sets in $PC(J,R)$. Repeating the same process in Step2 theorem 4.6**.**

For a positive number r, let $B_r = \{v \in PC\left(J,R\right), \|v\|_{pc} \le r\}$ be bounded sets in $PC(J,R)$. Then

$$\left|(Tv)(t)\right| = \left|\int_0^t M_1(t,\tau) f(\tau, v(\tau)) d\tau + d_1(t)\int_0^1 M_1(t,\tau) f(\tau, v(\tau)) d\tau\right.$$

$$+ d_2(t)\int_0^1 M_2(t,\tau) f(\tau, v(\tau)) d\tau + d_3(t)\int_0^1 f(\tau, v(\tau)) d\tau$$

$$+ d_4(t)\sum_{n=1}^{p} Q_n(v(t_n)) + d_5(t)\sum_{n=1}^{p} Q_n^*(v(t_n)) + \sum_{n=1}^{p} p_{1,n} Q_n(v(t_n))$$

$$\left. + \sum_{n=m+1}^{p} p_{2,n} Q_n^*(v(t_n)) + \sum_{n=m+1}^{p} Q_n^*(v(t_n)) + d_6(t)\right|,$$

$$\left|(Tv)(t)\right| \le \int_0^t M_1(t,\tau)\vartheta\mu(\|v\|) d\tau + |d_1(t)|\int_0^1 M_1(t,\tau)\vartheta\mu(\|v\|) d\tau$$

$$+ |d_2(t)|\int_0^1 M_2(t,\tau)\vartheta\mu(\|v\|) d\tau + |d_3(t)|\int_0^1 \vartheta_f \mu(\|v\|) d\tau$$

$$+ |d_4(t)|\sum_{n=1}^{p}|Q_n(v(t_n))| + |d_5(t)|\sum_{n=1}^{p}|Q_n^*(v(t_n))| + \sum_{n=1}^{p}|p_{1,n}||Q_n(v(t_n))|$$

$$+ \sum_{n=m+1}^{p}|p_{2,n}||Q_n^*(v(t_n))| + \sum_{n=m+1}^{p}|Q_n^*(v(t_n))| + |d_6(t)|,$$

$$\left|(Tx)(t)\right| \le \left(\frac{1^{\alpha\text{-}\sigma\text{-}1}\left(1-e^{-\lambda}\right)\|\vartheta\|\mu(\|v\|)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{\rho-1}\right)} + \frac{1^{\alpha\text{-}\sigma\text{-}1}\left(1-e^{-\lambda}\right)\|\vartheta\|\mu(\|v\|)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{\rho-1}\right)}\|d_1\|\right.$$

$$\left. + \frac{1^{\alpha\text{-}\sigma\text{-}1}\|\vartheta\|\mu(\|v\|)}{\lambda\left(\frac{\alpha-\rho-1}{\rho-1}\right)}\left(\lambda + e^{-\lambda} - 1\right)\|d_2\| + \frac{1^{\alpha\text{-}\sigma\text{-}1}\|\vartheta\|(\|v\|)}{\left(\frac{\alpha-\rho-1}{\rho-1}\right)}\|d_3\|\right)$$

$$+ \left(1+\|d_4\|\right) pK_Q + \left(\|d_5\| + \|p_{1,n}\| + \|p_{2,n}\|\right) pK_{Q^*} + \|d_6\|.$$

$$\left|(Tv)(t)\right| \le \left(\frac{1^{\alpha\text{-}\rho\text{-}1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)\left(\frac{\alpha-\rho-1}{\rho-1}\right)^{\rho-1}}\left(1+\|d_1\|\right) + \|d_2\|\frac{1^{\alpha\text{-}\rho\text{-}1}\left(\lambda + e^{-\lambda} - 1\right)}{\lambda\left(\frac{\alpha-\rho-1}{\rho-1}\right)^{\rho-1}}\right.$$

$$\left. + \|d_3\|\frac{1^{\alpha\text{-}\rho\text{-}1}}{\left(\frac{\alpha-\rho-1}{\rho-1}\right)^{\rho-1}}\right)\|\vartheta\|\mu(\|v\|) + \left(1+\|d_4\|\right) pK_Q$$

$$+ \left(\|d_5\| + \|p_{1,n}\| + \|p_{2,n}\|\right) pK_{Q^*} + \|d_6\|.$$

Now, construct the set $\Lambda = \{v \in PC(J,R) : \|v\| < N\}$.The operator $T : \Lambda \to PC(J,R)$ is continuous and completely continuous. From the choice of Λ, there is no $v \in \partial\Lambda$ such that $v=\lambda Tv$, $0<\lambda<1$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that T has a FP $v\in\partial\Lambda$, which concludes that the problem (3.1)-(3.4) has at least one solution.

4.1.3 Existence results of the problem (3.2)-(3.5)

In view of the Lemma 3.3, we can transform the problem (3.2)-(3.5) into a FP problem. Define an operator $T:PC(J)\to PC(J)$ by

$$\begin{aligned}
Tv(t)=&\int_{t_m}^{t} e^{-\lambda(t-s)} I_{t_m}^{\beta_{t_m}-1} f\left(s,v(s)\right)ds+\sum_{n=1}^{m} e^{-\lambda(t-t_n)}\\
&\times\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} f\left(s,v(s)\right)ds-\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} f\left(s,v(s)\right)-\frac{1}{\lambda}Q^{*}\left(v(t_n)\right)\right]\\
&+\sum_{n=1}^{m} e^{-\lambda(t-t_n)}\left[\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} f\left(t_n,v(t_n)\right)+Q\left(v(t_n)\right)+\frac{1}{\lambda}Q^{*}\left(v(t_n)\right)\right]\\
&+\left(1-\sum_{m=0}^{p}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\right)^{-1}\left\{\sum_{m=0}^{p}\lambda_m I_{t_m}^{\alpha_m}\left(\int_{t_m}^{\eta_m} e^{-\lambda(r-s)} I_{t_{n-1}}^{\alpha_{n-1}-1} f\left(s,v(s)\right)ds\right)(\eta_m)\right.\\
&+\sum_{m=0}^{p}\sum_{n=1}^{m}\lambda_m I_{t_m}^{\alpha_m} e^{-\lambda(\eta_m-t_n)}\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} f\left(t_n,v(t_n)\right)ds+I_{t_{n-1}}^{\beta_{n-1}-1} f\left(t_n,v(t_n)\right)\right)\\
&\left(\frac{1}{\lambda}Q^{*}\left(v(t_n)\right)\right]+\sum_{m=0}^{p}\sum_{n=1}^{m}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\left[\frac{1}{\lambda} I_{t_{n-1}}^{\beta_{n-1}-1} f\left(t_n,v(t_n)\right)+Q\left(v(t_n)\right)\right.\\
&\left.\left.+\frac{1}{\lambda}Q^{*}\left(v(t_n)\right)\right]\right\}.
\end{aligned}\tag{4.6}$$

Theorem 4.7 Suppose that (H_1), (H_3) and (H_7) hold. If

$$L_T<L_f\,\frac{\max\limits_{1\le m\le p}(t-t_m)}{m\min\limits_{1\le m\le p}\Gamma(\beta_m)}+\Lambda_0+\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4<1 \tag{4.7}$$

then the problem (3.2)-(3.5) has a unique solution on J.

Proof: Show that $T:PC(J)\to PC(J)$ is a completely continuous operator

$$\left|Tv(t)\right| = \int_{t_m}^{t} e^{-\lambda(t-s)} I_{t_m}^{\beta_{t_m}-1} \left| f\left(s,v(s)\right)\right| ds + \sum_{n=1}^{m} e^{-\lambda(t-t_n)}$$

$$\times\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} \left| f\left(s,v(s)\right)\right| ds + \left|\frac{1}{\lambda}\right| I_{t_{n-1}}^{\beta_{n-1}-1} \left| f\left(s,v(s)\right)\right| + \left|\frac{1}{\lambda}\right| \left|Q^*\left(v(t_n)\right)\right|\right]$$

$$+\sum_{n=1}^{m} e^{-\lambda(t-t_n)} \left[\left|\frac{1}{\lambda}\right| I_{t_{n-1}}^{\beta_{n-1}-1} \left| f\left(t_n,v(t_n)\right)\right| + \left|Q\left(v(t_n)\right)\right| + \left|\frac{1}{\lambda}\right| \left|Q^*\left(v(t_n)\right)\right|\right]$$

$$+\left(1-\sum_{m=0}^{p}\frac{\lambda_m(\eta_k-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\right)^{-1}\left\{\sum_{m=0}^{p}\lambda_m I_{t_m}^{\alpha_m}\left(\int_{t_m}^{\eta_m} e^{-\lambda(r-s)} I_{t_{n-1}}^{\alpha_{n-1}-1}\left| f\left(s,v(s)\right)\right| ds\right)(\eta_m)\right.$$

$$+\sum_{m=0}^{p}\sum_{n=1}^{m}\lambda_m I_{t_m}^{\alpha_m} e^{-\lambda(\eta_m-t_n)}\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1}\left| f\left(s,v(s)\right)\right| ds + I_{t_{n-1}}^{\beta_{n-1}-1}\left| f\left(t_n,v(t_n)\right)\right|\right.$$

$$+\left|\frac{1}{\lambda}\right|\left|Q^*\left(v(t_n)\right)\right| + \sum_{m=0}^{p}\sum_{n=1}^{m}\frac{\lambda_m(\eta_m-t_m)^{\alpha_k}}{\Gamma(\alpha_m+1)}$$

$$\left.\left.\times\left[\left|\frac{1}{\lambda}\right| I_{t_{n-1}}^{\beta_{n-1}-1}\left| f\left(t_n,v(t_n)\right)\right| + \left|Q\left(v(t_n)\right)\right| + \left|\frac{1}{\lambda}\right|\left|Q^*\left(x(t_n)\right)\right|\right]\right]\right\}.$$

$$\left|Tv(t)\right| \le L_f\,\frac{(t-t_m)}{\lambda\Gamma(\beta_m)} + \sum_{n=1}^{m} e^{-\lambda(t-t_n)}\left[L_f\,\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_f\,\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_{Q^*}\,\frac{1}{\lambda}\right]$$

$$+\sum_{n=1}^{m}\left[L_f\,\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{m\Gamma(\beta_{n-1})} + L_Q + L_{Q^*}\,\frac{1}{\lambda}\right] + \Delta\left\{\left(L_f\sum_{m=0}^{p}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m+\beta_m+1}}{\Gamma(\alpha_m+1)}\right)^{-1}\right.$$

$$+\sum_{m=1}^{p}\sum_{n=0}^{m}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\left[L_f\,\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_f\,\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_{Q^*}\,\frac{1}{\lambda}\right]$$

$$\left.+\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\left[L_f\,\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_Q + L_{Q^*}\,\frac{1}{\lambda}\right]\right\}.$$

$$\left|Tv(t)\right| \leq L_f \frac{\max\limits_{1\leq m\leq p}\left(t-t_m\right)}{m\min\limits_{1\leq m\leq p}\Gamma\left(\beta_m\right)}$$

$$+\left[L_f\sum_{n=0}^{m}e^{-\lambda(t-t_n)}\frac{\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\beta_{n-1}\right)}+L_f\sum_{n=0}^{m}e^{-\lambda(t-t_n)}\frac{\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\beta_{n-1}\right)}+\frac{1}{\lambda}\sum_{n=0}^{m}e^{-\lambda(t-t_n)}L_{Q^*}\right]$$

$$+\left[L_f\sum_{n=0}^{m}\frac{\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\beta_{n-1}\right)}+\sum_{n=0}^{m}L_Q+\frac{1}{\lambda}\sum_{n=0}^{m}L_{Q^*}\right]+\Delta\left(L_f\sum_{m=0}^{p}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m+\beta_m+1}}{\Gamma\left(\alpha_m+1\right)}\right)^{-1}$$

$$+\left[L_f\Delta\sum_{m=1}^{p}\sum_{n=0}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\alpha_m+1\right)\Gamma\left(\beta_{n-1}\right)}+L_f\Delta\sum_{m=1}^{p}\sum_{n=0}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\alpha_m+1\right)\Gamma\left(\beta_{n-1}\right)}\right.$$

$$\left.+\frac{\Delta}{\lambda}\sum_{m=0}^{p}\sum_{n=0}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}}{\Gamma\left(\alpha_m+1\right)}L_{Q^*}\right]+\left[L_f\Delta\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\alpha_m+1\right)\Gamma\left(\beta_{n-1}\right)}\right.$$

$$\left.+\Delta\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}}{\Gamma\left(\alpha_m+1\right)}L_Q+\frac{\Delta}{\lambda}\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}}{\Gamma\left(\alpha_m+1\right)}L_{Q^*}\right],$$

thus

$$\left|Tv(t)\right| \leq L_f \frac{\max\limits_{1\leq m\leq p}\left(t-t_m\right)}{m\min\limits_{1\leq m\leq p}\Gamma\left(\beta_m\right)}+\Lambda_0+\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4=\Omega,$$

where

$$\Lambda_0=\left[L_f\sum_{n=0}^{m}e^{-\lambda(t-t_n)}\frac{\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\beta_{n-1}\right)}+L_f\sum_{n=0}^{m}e^{-\lambda(t-t_n)}\frac{\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\beta_{n-1}\right)}+\frac{1}{\lambda}\sum_{n=0}^{m}e^{-\lambda(t-t_n)}L_{Q^*}\right]$$

$$\Lambda_1=\left[L_f\sum_{n=0}^{m}\frac{\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\beta_{n-1}\right)}+\sum_{n=0}^{m}L_Q+\frac{1}{\lambda}\sum_{n=0}^{m}L_{Q^*}\right],\Lambda_2=\Delta\left(L_f\sum_{m=0}^{p}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m+\beta_m+1}}{\Gamma\left(\alpha_m+1\right)}\right)^{-1},$$

$$\Lambda_3=\left[L_f\Delta\sum_{m=1}^{p}\sum_{n=0}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\alpha_m+1\right)\Gamma\left(\beta_{n-1}\right)}+L_f\Delta\sum_{m=1}^{p}\sum_{n=0}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\alpha_m+1\right)\Gamma\left(\beta_{n-1}\right)}\right.$$

$$\left.+\frac{\Delta}{\lambda}\sum_{m=1}^{p}\sum_{n=0}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}}{\Gamma\left(\alpha_m+1\right)}L_{Q^*}\right],$$

$$\Lambda_4=\left[L_f\Delta\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}\left(t_n-t_{n-1}\right)^{\beta_{n-1}-1}}{\lambda\Gamma\left(\alpha_m+1\right)\Gamma\left(\beta_{n-1}\right)}+\Delta\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\pi_m\left(\eta_m-t_m\right)^{\alpha_m}}{\Gamma\left(\alpha_m+1\right)}L_Q+\right.$$

$$\left.\frac{\Delta}{\lambda}\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\lambda_m\left(\eta_m-t_m\right)^{\alpha_m}}{\Gamma\left(\alpha_m+1\right)}L_{Q^*}\right],$$

which implies that $Tv \in B$. Thus $TB \in B$. On the other hand, for any $t \in J_m,\ 0 < m < p,$ we have

$$\left|(Tv)^{'}(t)\right| \le \lambda \int_{t_m}^{t} e^{-\lambda(t-s)} I_{t_m}^{\beta_{t_m}-1} \left| f\left(s, v(s)\right)\right| ds + \int_{t_m}^{t} \frac{(t-s)^{\beta_m -1}}{\Gamma(\beta_m - 1)} \left| f\left(s, v(s)\right)\right| ds$$
$$+\sum_{n=1}^{p}\left[\int_{t_{n-1}}^{t} I_{t_{n-1}}^{\beta_{n-1}-1} \left| f\left(s, v(s)\right)\right| ds + \left|\frac{1}{\lambda}\right| \int_{t_{n-1}}^{t_n} \frac{(t_n - s)^{\beta_m -1}}{\Gamma(\beta_{n-1} - 1)} \left| f\left(s, v(s)\right)\right| ds + \left|\frac{1}{\lambda}\right| \left|Q^{*}\left(v(t_n)\right)\right|\right],$$

$$\left|(Tv)^{'}(t)\right| \le L_f \left|\lambda\right| \int_{t_m}^{t} e^{-\lambda(t-s)} I_{t_m}^{\beta_{t_m}-1} ds + L_f \int_{t_k}^{t} \frac{(t-s)^{\beta_m -1}}{\Gamma(\beta_m - 1)} ds$$
$$+\sum_{n=1}^{p}\left[L_f \int_{t_{n-1}}^{t} I_{t_{n-1}}^{\beta_{n-1}-1} ds + \left|\frac{1}{\lambda}\right| L_f \int_{t_{n-1}}^{t_n} \frac{(t_n - s)^{\beta_n -1}}{\Gamma(\beta_{n-1} - 1)} ds + \left|\frac{1}{\lambda}\right| L_{Q^*} \right],$$

Hence, for $s_1, s_2 \in J_m$ with $s_1 < s_2$ and $0 < m < p$, we have

$$\left|(Tv)(s_1) - (Tv)(s_2)\right| \le \int_{s_1}^{s_2} (Tv)^{'}(s) ds \le \ell (s_2 - s_1).$$

This implies that Tv is equicontinuous on all J_m, $m = 0,1,..,p$. Consequently, Arzela-Ascoli theorem ensures that the operator $T : PC(J) \to PC(J)$ is a completely continuous operator. Next show that the operator maps B into B. For that, let us choos $R \ge \max\left\{2\mu, (2L_\sigma)^{\frac{1}{1-\sigma}}\right\}$ and define a ball $B = \left\{v \in PC(J, R) : \|v\| \le R\right\}$. For any $v \in B$, by the conditions (H_2) and (H_6), we have

$$
\begin{aligned}
&|Tv(t)| \le \\
&\quad \int_{t_m}^{t} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1}\left[a(s)+\xi|v(s)|\right]ds + \sum_{n=1}^{m} e^{-\lambda(t-t_n)} \\
&\quad \times\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1}\left[a(s)+\xi|v(s)|\right]ds + \left|\frac{1}{\lambda}\right| I_{t_{n-1}}^{\beta_{n-1}-1}\left[a(s)+\xi|v(s)|\right] + \left|\frac{1}{\lambda}\right|\left|Q^*(v(t_n))\right|\right] \\
&\quad +\sum_{n=1}^{m} e^{-\lambda(t-t_n)}\left[\left|\frac{1}{\lambda}\right| I_{t_{n-1}}^{\beta_{n-1}-1}\left[a(t_n)+\xi|v(t_n)|\right] + \left|Q(v(t_n))\right| + \left|\frac{1}{\lambda}\right|\left|Q^*(v(t_n))\right|\right] \\
&\quad +\left(1-\sum_{m=0}^{p}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\right)^{-1}\left\{\sum_{m=0}^{p}\lambda_m I_{t_m}^{\alpha_m}\left(\int_{t_m}^{\eta_m} e^{-m(r-s)} I_{t_{n-1}}^{\alpha_{n-1}-1}\left[a(s)+\xi|v(s)|\right]ds\right)(\eta_m)\right. \\
&\quad +\sum_{m=0}^{p}\sum_{n=1}^{m}\lambda_m I_{t_m}^{\alpha_m} e^{-m(\eta_m-t_n)} \\
&\quad \times\left[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1}\left[a(s)+\xi|v(s)|\right]ds + I_{t_{n-1}}^{\beta_{n-1}-1}\left[a(t_n)+\xi|v(t_n)|\right] + \left|\frac{1}{\lambda}\right|\left|Q^*(v(t_n))\right|\right] \\
&\quad \left.+\sum_{m=0}^{p}\sum_{n=1}^{m}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\left[\left|\frac{1}{\lambda}\right| I_{t_{n-1}}^{\beta_{n-1}-1}\left[a(t_n)+\xi|v(t_n)|\right] + \left|Q(v(t_n))\right| + \left|\frac{1}{\lambda}\right|\left|Q^*(v(t_n))\right|\right]\right\}.
\end{aligned}
$$

$$
\begin{aligned}
|Tv(t)| &\le \left[a(s)+\xi|v(s)|\right]\frac{(t-t_m)}{\lambda\Gamma(\beta_m)} + \sum_{n=0}^{m} e^{-\lambda(t-t_n)} \\
&\quad \times\left[\left[a(s)+\xi|v(s)|\right]\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + \left[a(s)+\xi|v(s)|\right]\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_{Q^*}\frac{1}{\lambda}\right] \\
&\quad +\sum_{n=0}^{m}\left[\left[a(s)+\xi|v(s)|\right]\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_Q + L_{Q^*}\frac{1}{\lambda}\right] \\
&\quad +\Delta\left(\left[a(s)+\xi|v(s)|\right]\sum_{m=0}^{p}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m+\beta_m+1}}{\Gamma(\alpha_m+1)}\right)^{-1} + \sum_{m=0}^{p}\sum_{n=0}^{m}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
&\quad \times\left[\left[a(s)+\xi|v(s)|\right]\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + \left[a(s)+\xi|v(s)|\right]\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_{Q^*}\frac{1}{\lambda}\right] \\
&\quad \left.+\sum_{m=0}^{p}\sum_{n=1}^{m}\frac{\lambda_m(\eta_m-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\left[\left[a(s)+\xi|v(s)|\right]\frac{(t_n-t_{n-1})^{\beta_{n-1}-1}}{\lambda\Gamma(\beta_{n-1})} + L_Q + L_{Q^*}\frac{1}{\lambda}\right]\right\}.
\end{aligned}
$$

$$
|Tv(t)| \le \Omega\left(\|a\|_C + \xi R\right).
$$

This implies $T: B \to B$. Hence, we conclude that $T: B \to B$ is completely continuous. It is follows from the Schauder-FPT that the operator T has at least one fixed point. That is problem (3.2)-(3.5) has at least one solution in B.

Theorem 4.8 Assume that there exist a nonnegative function $W \in C(J, R^+)$ and nonnegative constants M, Z such that

$$\left| f(t,v) - f(t,u) \right| \le W(t) \left| v-u \right|, \ t \in J, v,u \in R,$$
$$\left| Q_m(v) - Q_m(u) \right| \le M \left| v-u \right|, \ \left| Q_m^*(v) - Q_m^*(u) \right| \le Z \left| v-u \right|,$$

for $t \in J, v,u \in R$ and $m = 1, 2, ..., p$. Furthermore, the assumption μ(W)<1 holds. Then the problem (3.2)-(3.5) has a unique solution on J.

Proof: For $v,u \in B$ and for each $t \in J$ we have

$$\begin{aligned}
\left| Tv(t) - Tu(t) \right| \le & \int_{t_m}^{t} e^{-\lambda(t-s)} I_{t_m}^{\beta_{t_m}-1} \left| f(s,v(s)) - f(s,u(s)) \right| ds + \sum_{n=1}^{m} e^{-\lambda(t-t_n)} \\
& \times \Bigg[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} \left| f(s,v(s)) - f(s,u(s)) \right| ds \\
& + \left| \frac{1}{\lambda} \right| I_{t_{n-1}}^{\beta_{n-1}-1} \left| f(s,v(s)) - f(s,u(s)) \right| + \left| \frac{1}{\lambda} \right| \left| Q^*(v(t_n)) - Q^*(u(t_n)) \right| \Bigg] \\
& + \sum_{n=1}^{m} e^{-\lambda(t-t_n)} \Bigg[\left| \frac{1}{\lambda} \right| I_{t_{n-1}}^{\beta_{n-1}-1} \left| f(t_n,v(t_n)) - f(t_n,u(t_n)) \right| \\
& + \left| Q(v(t_n)) - Q(u(t_n)) \right| + \left| \frac{1}{\lambda} \right| \left| Q^*(v(t_n)) - Q^*(u(t_n)) \right| \Bigg] \\
& + \left(1 - \sum_{m=0}^{p} \frac{\lambda_m (\eta_m - t_m)^{\alpha_m}}{\Gamma(\alpha_m + 1)} \right)^{-1} \\
& \times \Bigg\{ \sum_{m=0}^{p} \lambda_m I_{t_m}^{\alpha_m} \left(\int_{t_m}^{\eta_m} e^{-k(r-s)} I_{t_{n-1}}^{\alpha_{n-1}-1} \left| f(s,v(s)) - f(s,u(s)) \right| ds \right)(\eta_m) \\
& + \sum_{m=0}^{p} \sum_{n=1}^{m} \lambda_m I_{t_m}^{\alpha_m} e^{-\lambda(\eta_m - t_n)} \Bigg[\int_{t_{n-1}}^{t_n} e^{-\lambda(t_n-s)} I_{t_{n-1}}^{\beta_{n-1}-1} \left| f(s,v(s)) - f(s,u(s)) \right| ds \\
& + I_{t_{n-1}}^{\beta_{n-1}-1} \left| f(t_n,v(t_n)) - f(t_n,u(t_n)) \right| + \left| \frac{1}{\lambda} \right| \left| Q^*(v(t_n)) - Q^*(u(t_n)) \right| \Bigg] \\
& + \sum_{m=0}^{p} \sum_{n=1}^{m} \frac{\lambda_m (\eta_m - t_m)^{\alpha_m}}{\Gamma(\alpha_m + 1)} \Bigg[\left| \frac{1}{\lambda} \right| I_{t_{n-1}}^{\beta_{n-1}-1} \left| f(t_n,v(t_n)) - f(t_n,u(t_n)) \right| \\
& + \left| Q(v(t_n)) - Q(u(t_n)) \right| + \left| \frac{1}{\lambda} \right| \left| Q^*(v(t_n)) - Q^*(u(t_n)) \right| \Bigg] \Bigg\}.
\end{aligned}$$

As $\mu(W) \le 1$, we have $\left| Tv(t) - Tu(t) \right| \le \Omega \left\| v-u \right\|$. Therefore, T is a contraction mapping on $PC(J, R)$ due to condition (4.5). By applying the well-known Banach's contraction mapping we see that the operator T has a unique FP on $PC(J, R)$. Therefore, the problem (3.2)-(3.5) has a unique solution.

4.2 Examples

4.2.1 Example of the problem (3.1)-(3.3)

Consider the problem (3.1)-(3.2):

$$\left({}^{c}D^{\frac{3}{2}}+2\,{}^{c}D^{\frac{1}{2}}\right)v(t)=L\left(t^{2}+\sin t+1+\tan^{-1}v\right),\ \ 0<t<1,\ \ 1<\frac{3}{2}\le 2,$$

$$v(0)+v(0)=0,\ \ v'(1)+v'(1)=0,$$

$$\Delta v\left(\frac{1}{4}\right)=\frac{\left\|v\left(\frac{1}{4}\right)\right\|^{2}}{1+\left\|v\left(\frac{1}{4}\right)\right\|^{2}},\ \ \Delta v'(t_{m})=\frac{\left\|v\left(\frac{1}{4}\right)\right\|^{2}}{1+\left\|v\left(\frac{1}{4}\right)\right\|^{2}},$$

Here $t\in[0,1]$, let $z_1=1, z_2=1, w_1=1, w_2=1,\ \alpha=(3/2),\ \lambda=2, y_1, y_2=0$,

$L_Q, L_{Q^*}, L_f=0.01$, $f\left(t,v(t)\right)=L\left(t^{2}+\sin t+1+\tan^{-1}v\right)$ and since $0.88<\Gamma\left(\frac{3}{2}\right)<0.89$.

Solution:

$$\Delta=(1-2)-\left(e^{-2}-2e^{-2}\right)=-0.865,\ \ h_1(0)=\frac{(1-1+2)(1-2)}{-0.865}=2.312,$$

$$h_2(0)=\frac{(1-1+2)}{-0.865}=-2.312,\ \ h_3(0)=\frac{1}{-0.865}-\frac{\left(e^{-2}-2e^{-2}\right)}{-0.865}=1.312,$$

$$h_4(0)=\frac{1}{2\Delta}-\frac{\left(e^{-2}-2e^{-2}\right)}{2\Delta}=0.656,\ \ N_{1,n}(0)=-\frac{(1-2)}{2\Delta}+\frac{(1-2)\left(e^{-2}-2e^{-2}\right)}{2\Delta}=0.656,$$

$$N_{2,n}(t)=\left(\frac{1}{2}-\frac{1}{2}\right)=0,\ \ N_3(0)=\left(\frac{1}{\Delta}-\frac{\left(e^{-2}-2e^{-2}\right)}{\Delta}\right)0-\left(\frac{1}{\Delta}-\frac{(1-2)}{\Delta}\right)0=0,$$

and

$$h_1(1)=\frac{\left(e^{-2}-1+2\right)(1-2)}{-0,865}=1.135,\ \ h_2(1)=\frac{\left(e^{-2}-1+2\right)}{\Delta}=1.312,$$

$$h_3(1)=\frac{e^{-2}}{\Delta}-\frac{\left(e^{-2}-2e^{-2}\right)}{\Delta}=-0.33,\ \ h_4(t)=\frac{e^{-2}}{2\Delta}-\frac{\left(e^{-2}-2e^{-2}\right)}{2\Delta}=0.156,$$

$$N_{1,n}(t)=-\frac{e^{2}e^{-2}(1-2)}{2\Delta}+\frac{e^{2}(1-2)\left(e^{-2}-2e^{-2}\right)}{2\Delta}=1.152,$$

$$N_{2,n}(t)=\left(\frac{e^{2}e^{-2}}{2}-\frac{1}{2}\right)=0.002,$$

$$\left|f(t,v)-f(t,)\right|\leq L\left|v-u+tan^{-1}v-tan^{-1}u\right|\leq L\left|v-u\right|.$$

$$L_T=\left(\frac{1^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda}\right)\left(1+\|h_1\|\right)+\frac{1^{\alpha-1}}{\lambda\Gamma(\alpha)}\|h_2\|\right)L_f$$
$$+\left(1+\|h_3\|\right)p\left(L_Q\right)+\left(\|h_4\|+\|N_{1,n}\|+\|N_{2,n}\|\right)p\left(L_Q\right)L_{Q^*}+\|N_3\|.$$

$$L_T=\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(1-e^{-2}\right)\left(1+2.312\right)+\frac{1}{\Gamma\left(\frac{3}{2}\right)}2.312\right)0.01$$
$$+(1+1.312)0.01+(0.656+1.152+0.002)0.01,$$

$$L_T=0.042+0.248<1.$$

Therefore, by (4.2), Impulsive (SFDE's) with (BVP'S)has a unique solution on [0,1].

4.2.2 Example of the problem (3.1)-(3.3)

Consider the problem (3.1)-(3.3):

$$\left({}^cD^{\frac{5}{3}}+\frac{2}{3}\,{}^cD^{\frac{2}{3}}\right)v(t)=\frac{1}{(t+121)^{\frac{1}{2}}}\frac{|v(t)|}{1+|v(t)|}+\tan^{-1}v(t),\ \ 0<t<1,\ 1<\frac{5}{3}\leq 2,$$
$$v(0)+v(0)=0,\ \ v(1)+v(1)=0,$$
$$\Delta v\left(\frac{1}{2}\right)=\frac{\left\|v\left(\frac{1}{2}\right)\right\|^2}{1+\left\|v\left(\frac{1}{2}\right)\right\|^2},\quad \Delta v'\left(\frac{1}{2}\right)=\frac{\left\|v\left(\frac{1}{2}\right)\right\|^2}{1+\left\|v\left(\frac{1}{2}\right)\right\|^2},$$

Here $t\in[0,1]$, let $z_1=1, z_2=1, w_1=1, w_2=1,\ \alpha=(8/3),\ \lambda=\frac{2}{3}, y_1, y_2=0$,

$L_Q, L_{Q^*}, L_f=0.02$, $f(t,v(t))=\frac{1}{(t+121)^{\frac{1}{2}}}\frac{|v(t)|}{1+|v(t)|}+\tan^{-1}v(t).$

Solution:

$$\left|f(t,x)-f(t,y)\right|\leq\left|tan^{-1}x-tan^{-1}y\right|\leq|x-y|.$$

$$L_T=\left(\frac{1^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-m}\right)\left(1+\|h_1\|\right)+\frac{1^{\alpha-1}}{\lambda\Gamma(\alpha)}\|h_2\|\right)L_f$$
$$+\left(1+\|h_3\|\right)p\left(L_Q\right)+\left(\|v_4\|+\|N_{1,n}\|+\|N_{2,n}\|\right)p\left(L_Q\right)L_{Q^*}+\|N_3\|.$$

$$L_T=\left(\frac{1}{\frac{3}{2}(1.5)}\left(1-e^{-\frac{2}{3}}\right)(1+0.190)+\frac{0.570}{(1.5)}2.312\right)0.02$$

$$+(1+0.709)0.02+(1.155+0.358+0.002)0.02,$$

$$L_T=0.019+0.034+0.038=0.397<1.$$

Therefore, by (4.2), Impulsive -SFDE's with BVP has a unique solution on [0,1].

4.2.3 Example of the problem (3.1)-(3.4)

Consider the problem (3.1)-(3.4):

$$\left({}^cD^{\frac{3}{2}}+2\,{}^cD^{\frac{1}{2}}\right)v(t)=L\left(t^2+\sin t+1+\tan^{-1}v\right),\ 0<t<1,\ 1<\frac{3}{2}\le 2,$$

$$v(0)+{}^cD^{\frac{3}{2}}v(0)=0,\ v(1)+{}^cD^{\frac{3}{2}}v(1)=0,$$

$$\Delta v\left(\frac{1}{2}\right)=\frac{\left\|v\left(\frac{1}{2}\right)\right\|^2}{1+\left\|v\left(\frac{1}{2}\right)\right\|^2},\quad \Delta v'\left(\frac{1}{2}\right)=\frac{\left\|v\left(\frac{1}{2}\right)\right\|^2}{1+\left\|v\left(\frac{1}{2}\right)\right\|^2},$$

Here $t\in[0,1]$, let $z_1=1, z_2=1, w_1=1, w_2=1,\ \alpha=(3/2),\ \lambda=2, y_1, y_2=0$,

$L_Q, L_{Q^*}, L_f=0.01$, $f(t,v(t))=L\left(t^2+\sin t+1+\tan^{-1}v\right)$ and since $0.88<\Gamma\left(\frac{3}{2}\right)<0.89$.

Solution:

$$|f(t,v)-f(t,u)|\le\left|v-u+tan^{-1}v-tan^{-1}u\right|\le L_T|v-u|.$$

$$L_T=\left(\frac{1^{\alpha-1}\left(1-e^{-\lambda}\right)}{\lambda\Gamma(\alpha)}\left(1+\|d_1\|\right)+\frac{\left(\lambda+e^{-\lambda}-1\right)}{\lambda}\|d_2\|+\|d_3\|\right)L_f$$

$$+\left(1+\|d_4\|\right)pL_Q+\left(\|d_5\|+\|p_{1,n}\|+\|p_{2,n}\|\right)pL_{Q^*}+\|d_6\|$$

$$L_T=\left(\frac{1}{2(0.902)}(1-0.135)\left(1+\|0.471\|\right)+\frac{1}{2}(1\ -\ 0.135)\|1.277\|+\|0.471\|\right)L_f$$

$$+\ \left(1\ +\ \|1.652\|\right)0.02+\left(\|0.826\|+\|6.054\|+\|0.002\|\right)\ 0.02+\|0\|$$

$$L_T=0.224<1.$$

Therefore, by (4.4), Impulsive- SFDE's with BVP has a unique solution on [0.1].

4.2.4 Example of the problem (3.2)- (3.5)

Consider the problem (3.2)-(3.5):

$$\left({}^{c}D_{t_m^+}^{\alpha_m} + m\,{}^{c}D_{t_m^+}^{\alpha_m - 1}\right)v(t) = \frac{\left[3v(t) + e^{\left(\frac{1}{2}\right)v(t)}\right]e^{t}}{2 + v^{4}(t)} + \frac{\cos(2t+5)}{\sqrt{3+v(t)}}\left|v(t)^{\sigma}\right| \qquad (4.8)$$

$$,0<t \le 1, t \ne \frac{3}{4}, m = 1, 2, \dots, p$$

$$\Delta v\left(\frac{3}{4}\right) = 11\sin^{2} v\left(\frac{3}{4}\right), \Delta v'\left(\frac{3}{4}\right) = \frac{v\left(\frac{3}{4}\right)}{2\left(1 + \left|v\left(\frac{3}{4}\right)\right|\right)},$$

$$\Delta v(0) = \sum_{m=0}^{1} \lambda_m I_{t_m}^{\alpha_m} v(\eta_m) + \frac{1}{2}, \Delta v'(0) = 0,$$

where $t \in [0,1]$, $\beta_0 = (5/4), \beta_1 = 1, \beta = (8/5), \alpha_0 = (1/2), \alpha_1 = (5/3), m_1 = (2/5)$, $m_1 = (3/7), \eta_1 = (1/2), \eta_2 = (4/5)$.

$$|f(t,v,u)| = \left| \frac{\left[3v(t) + e^{\left(\frac{1}{2}\right)v(t)}\right]e^{t}}{2 + v^{4}(t)} + \frac{\cos(2t+5)}{\sqrt{3+v(t)}}\left|v(t)^{\sigma}\right| \right|$$

$$\le \frac{e^{2}}{2} + \frac{1}{\sqrt{3}}|v|^{\sigma}.$$

Clearly, $a(t) = \frac{e^{2}}{2}$, $\xi = \frac{1}{\sqrt{3}}$, $L_Q = 11, L_{Q^*} = \frac{1}{2}$, and the conditions of Theorem 4.8 hold.

Thus, by Theorem 4.7 impulsive- SFDE's with BVP (4.7) has at least one solution [0,1].

Chapter 5

CONCLUSION

In this thesis, the existence (and uniqueness) results of a nonlinear impulsive sequential fractional differential equations of order $\alpha \in (1, 2]$ involving Liouville-Caputo fractional derivative supplemented with the separate boundary value conditions are studied. Both sequential fractional differential equations and impulsive fractional differential equations are studied individually from various perspectives. A new result on the existence of a solution is established by using different fixed point theorems. An example is presented to illustrate the result. Using the technique based on the concept of measure of noncompactness and the fixed point theory a new existence results will be established in future works.

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