Numerical Solutions of Fractional Differential Equations

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ABSTRACT

In this thesis, we focus on numerical solutions of general linear multi-term fractional differential equations (FDEs) with fractional derivatives defined in the Caputo sense. Multi-term fractional order differential equations are involving both ordinary and fractional derivative operators. Numerical methods plays very crucial role for solving fractional differential equations, since analytical solutions are not always possible for solving them. Memory trait of fractional calculus is one of the main reason for difficulty of developing analytical techniques for such a equations. Therefore, there has been considerable interest in solving FDEs numerically in recent years and many powerful schemes have been developed. Essentially, most of the developed methods are modified from original versions for classical differential equations and applied to FDEs.

In this study, we introduce a numerical technique based on the fractional Taylor vector and we construct fractional Taylor operational matrix of fractional integration to solve multi-term FDEs. The main characteristic of this technique is to reduce the given IVP of fractional order to a system of algebraic equations by employing the fractional Taylor operational matrix of fractional integration. Finally, this set of algebraic equations can be solved easily and efficiently for unknown coefficients by using computer programming. Consequently, by using these coefficients, the approximate solution of the given problem can be obtained. Some numerical examples are presented to demonstrate the accuracy and applicability of given method. The approximate solutions obtained by use of given technique are compared with numerical results of some other methods in literature and exact solutions of given problems. From these results, we can conclude that the presented technique is efficient and applicable for solving high order multi-term fractional order differential equations numerically.

Keywords: numerical solutions, fractional Taylor vector,fractional differential equations, spectral method, Caputo fractional derivative, Riemann-Liouville fractional integral, operational matrices.

ÖZ

Bu tez çalışmasında, Caputo kesirli türevlerine sahip, genel lineer çok terimli kesirli diferansiyel denklemlerin sayısal yöntem ile çözümlerine odaklanılmıştır. Çok terimli kesirli türevlere sahip diferansiyel denklemler, hem klasik hem kesirli türev operatörleri içeren denklemlerdir. Analitik metodlar ile kesirli türevlere sahip diferansiyel denklemlerin çözümlerine ulaşmak her zaman mümkün olmadığından, sayısal metodlar bu tür denklemlerin çözümlerinde çok önemli bir rol oynamaktadır. Kesirli analizin uzun hafıza özelliği, bu tür diferansiyel denklemlerin çözümü için analitik yöntemler geliştirmeyi zorlaştıran en önemli sebeplerden biridir. Bu nedenle, kesirli türevli diferansiyel denklemlerin sayısal yöntemler kullanılarak çözümü son yıllarda büyük ilgi görmektedir ve bunun sonucu olarak birçok güçlü teknik geliştirilmiştir. Aslında, geliştirilen yöntemlerin çoğu, klasik diferansiyel denklemlerin çözümü için kullanılan orijinal versiyonlardan değiştirilip güncellenerek kesirli diferansiyel denklemlere uygulanan yöntemlerdir.

Bu çalışmada, çok terimli kersirli diferansiyel denklemlerin sayısal çözümleri için, kesirli Taylor vektörüne dayanan bir yöntem sunulmaktadır. Sunulan yöntemin ana amacı, kesirli Taylor vektöründen yararlanarak kesirli integrasyonun operasyonel matrisini oluşturmak ve bu matrisi kullanarak, verilen çok terimli kesirli diferansiyel denklemin bir cebirsel denklem sistemine indirgenmesini sağlamaktır. Son olarak, elde edilen bu cebirsel denklem sistemi, bilgisayar programlaması kullanılarak, bilinmeyen katsayı için verimli bir biçimde çözülebilmektedir. Sonuç olarak, elde edilecek katsayılar kullanılarak, verilen problemin yaklaşık çözümü elde edilmektedir. Sunulan yöntemin verimliliğini ve uygulanabilirliğini test edebilmek için bazı örnekler verilmiştir. Sunulan yöntem kullanılarak elde edilen yaklaşık çözümler, verilen problemlerin kesin çözümleri ve literatürde bulunan bazı diğer sayısal yöntemler ile karşılaştırılmıştır. Elde edilen sonuçlar ve karşılaştırmalar, sunulan yöntemin, çok terimli kesirli diferansiyel denklemlerin yaklaşık çözümlerine ulaşmakta çok başarılı ve verimli olduğunu kanıtlamaktadır.

Anahtar Kelimeler: sayısal çözümler, kesirli diferansiyel denklem, spektral metod, Caputo kesirli türevi, Riemann-Liouville kesirli integrali, kesirli Taylor vektörü, operasyonel matrix.

DEDICATION

To My Family

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LIST OF SYMBOLS AND ABBREVIATIONS

BVP	Boundary Value Problem
FDE	Fractional Differential Equation
IVP	Initial Value Problem
ML	Mittag–Leffler Function
RL	Riemann–Liouville
$\beta(n,m)$	Beta Function
$D^{\alpha}y$	Caputo Fractional Derivative
$\Gamma(k)$	Gamma Function
$E_n(k)$	Mittag–Leffler Function of one parameter
$E_{n,m}(k)$	Mittag–Leffler Function of two parameter
$I^{\alpha}y$	Riemann–Liouville Fractional Integral
$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of natural numbers
$\|\cdot\|$	Norm of $\cdot$
$\lfloor\cdot\rfloor$	The floor function, i.e. greatest integer less than or equal to the $\cdot$

Chapter 1

INTRODUCTION

1.1 Fractional calculus: A brief history and some applications

An emerging field of mathematical analysis; fractional calculus, which can be described as generalisation of ordinary differentiation and integration to arbitrary non-integer orders. Although the title ”integration and differentiation of arbitrary order” being more proper for this topic, a misnomer designation; ”fractional calculus” is in use from the days of L'Hospital. The history of fractional calculus is almost as long as the history of traditional calculus, beginning with some speculations of G.W. Leibniz (1695) and L. Euler (1730). However, fractional calculus and fractional differential equations (FDEs) are rapidly developing and increasingly becoming popular in recent years. Some of famous mathematicians, who have provided crucial contributions for fractional calculus, contains P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832-1873), B. Riemann (1847), A.K. Grünwald (1867-1872), A.V. Letnikov (1868-1872), H. Laurent (1884), J. Hadamard (1892), S. Pincherle (1902), H. Weyl (1917), H.T. Davis (1924-1936), A. Zygmund (1935-1945), E.R. Love (1938-1996), D.V. Widder (1941), M. Riesz (1949) and so on [13]. Since fractional derivative is not necessarily unique, there are some different well-known definitions in the literature, i.e. Grünwald–Letnikov derivative, RL fractional derivative and Caputo derivative are some important ones. The progressively developing history of this old and yet novel topic can be found in [1]- [5]. In fact, fractional calculus provides the mathematical modeling of some crucial phenomena like social and natural in a more powerful way than the ordinary one. Because, if we compare with the classical calculus, the fractional calculus has the long interaction features, namely memory effects. Therefore, this memory treat of fractional calculus can better illustrate different kinds of nonlinear dynamics in both theories and mathematical modeling of engineering problems. Over the last few decades, many applications were reported in many fields of science and engineering such as chaotic systems [6], fluid mechanics [8], viscoelasticity [9], optimal control problems [10, 11], chemical kinetics [12], electrochemistry [14], biology [15], physics [16], bioengineering [17], finance [18], social sciences [47], economics [48], optics [49], chemical reactions [50], rheology [51] and so on. Due to the importance of FDEs, the solutions of them are attracting widespread interest. On the other hand, due to reason that we mention before about the difficulty of analytical solutions, numerical techniques becomes more crucial for solving such equations.

In this thesis, we will focus to solve multi-term FDEs numerically, which are one of the most important type of FDEs, that is a system of mixed fractional and ordinary differential equations and involving more than one fractional differential operators. Nowadays, they are widely appearing for modeling of many important processes, especially for multirate systems. Their approximate solution is then a strong subject that deserves high interest.

1.2 Literature review on numerical methods for fractional differential equations

The extension of present numerical techniques for classical integer order differential equations to their corresponding FDEs is not an easy process. However, there are various numerical techniques have been developed for solving FDEs in literature. In this section, we give some examples of existing numerical methods in literature that used to solve FDEs. An Adams type predictor-corrector method is discussed in [19] by Diethelm et al. Laplace transforms for the solution of FDEs is introduced in [20] by Podlubny. In [21], Çenesiz et al. solved Bagley–Torvik equation by generalized Taylor collocation method. In [8, Chapter 6], Zheng and Zhang used variational iteration method and homotopy perturbation technique to solve FDEs. In [22], Ray and Bera applied Adomian decomposition method for the solution of a FDEs as an alternative method of Laplace transform. Tau method introduces for solving fractional partial differential equations in [23] by Vavani and Aminataei. In [54], Rani et al. applied the numerical inverse Laplace transform technique based on the Bernstein operational matrix to find the solution to FDEs. In [55], Khashan et al. introduced a collocation technique based on Haar wavelet to solve Riccati type differential equations with non-integer order numerically. In [56], Li and Sun applied block pulse operational matrices of differentiation to approximate FDEs. In [57], Saadamandi and Denghan presented a method based on the shifted Legendre-tau idea for solving a class of initial BVPs for the fractional diffusion equations with variable coefficients. In [58], Abuasad et al. applied fractional multi-step differential transformed technique to get numerical approximations to fractional stochastic SIS epidemic model with imperfect vaccination. In [59], Veeresha et al. used the q -homotopy analysis transform method to solve fractional Kolmogorov–Petrovskii–Piskunov (FKPP) equation. In [60], Silva et al. used the conformable Laplace transform to discuss solution of some fractional linear differential equations with constant coefficients. In [61], Pitolli applied a collocation method based on fractional B-splines for the solution of FDEs. In [62], finite difference method on Non-Uniform Meshes for Time-Fractional Advection–Diffusion Equations used by Fazio et al. In [24], Odibat et al. applied homotopy analysis technique to solve nonlinear FDEs and so on.

1.3 Spectral methods

Spectral methods are numerical techniques used to solve classical or fractional differential equations in applied mathematics. In 1938, spectral methods introduced by Lanczos [68] by showing the powerful role of Fourier series and Chebyshev polynomials for solutions of some problems. Applying spectral methods to solve many different types of integral and differential equations numerically, has received considerable interest in recent years, because of their easy applicability over finite and infinite intervals. Spectral methods are highly related to finite element methods and they depend on very similar ideas. The principal difference is that the finite element methods utilize basis functions that are nonzero only on trivial subdomain, while spectral methods utilize basis functions which are nonzero over the entire domain. That is to say, finite element methods utilize a local approach, whereas spectral methods take on a global approach. Therefore, when the solution is smooth, spectral methods have very good error properties, that is the so-called ”exponential convergence” being the fastest possible. These highly accurate methods are based on expressing the approximate solution of differential equation as a linear combination of a chosen set of orthogonal basis functions and choosing the coefficients in the sum in order to satisfy the solution of differential equation [69]. In general, there are three types of such a methods; collocation, Tau and Galerkin. We focus on collocation spectral method.

The collocation is based on interpolation. Similar to finite difference method, the collocation spectral method uses collocation points, namely a set of grid points in the domain. In our work, for discretization of multi-term FDEs, we use spectral collocation method with fractional Taylor basis which are easy to approximate the functions.

1.4 Structure of the thesis

In this thesis, motivated by the results reported in [27, 30] for solving a smaller class of problems where the highest order of derivative is an integer and involving at most one noninteger order derivative, we go further and establish a method for numerical solutions for higher order and arbitrary multi-term fractional FDEs which have a general form

$$D^{\alpha}y(t) = f\left(t, y(t), D^{\beta_0}y(t), D^{\beta_1}y(t), ..., D^{\beta_k}y(t)\right),\ t \in [0, R] \tag{1.1}$$

where D^{α} representing the Caputo fractional derivative of order $\alpha > 0$ and we assume that $0 < \beta_0 < \beta_1 < ... < \beta_k < \alpha$, $y^{(p)}(0) = Y_p$, $p = 0, 1, ...n$ where $n-1 < \alpha < n$.

In this work, our main purpose is to present an effective, reliable method to approximate IVP for the Eq.(1.1). Therefore, a numerical approach based on fractional Taylor vector is proposed to solve the initial value problem of general type of multi-term FDEs. The core idea of this method is to present and employ the operational matrix of fractional integration based on fractional Taylor vector to given problem and reduce it to a set of algebraic equations which can be efficiently solved.

The structure of the thesis is organized as follows. In Chapter 2, we briefly introduce some necessary definitions and preliminary ideas of fractional calculus. In Chapter 3, we give existence and uniqueness results for FDEs. Also, linear multi-term FDEs are introduced in Section 3.3 and some existing numerical techniques are given in Section 3.3.1. In Chapter 4, we introduce an algorithm based on fractional Taylor operational matrix of fractional integration to solve multi-term FDEs numerically. Also, given method has been applied to nine examples to demonstrate the efficiency and applicability. A final conclusion is presented in the last chapter.

Chapter 2

PRELIMINARIES

In this chapter, we introduce basic definitions of some special functions which have very important roles in fractional calculus. We also briefly give some necessary definitions of fractional derivatives and integrals and some properties that will be used later.

2.1 Basic Functions of Fractional Calculus

2.1.1 Gamma Function

The gamma function is a very useful and well-known function in mathematics, that is one commonly used generalisation of the factorial function to complex numbers. This function is introduced by Euler in the 18th century.

Definition 2.1.1. The Gamma function is given by the Euler integral of the second kind

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$$

where $Re(k) > 0$ and $t^{n-1} = e^{(n-1)\log t}$.

Gamma function is related to factorial by following relation:

$$\Gamma(k) = (k-1)!.$$

2.1.2 Beta Function

The beta function has a crucial role in calculus because of its close relation to the gamma function. It's also called as Euler's integral of the first kind.

Definition 2.1.2. The beta function or the Euler integral of the first kind is given as following

$$\beta(n,m) = \int_0^1 t^{n-1}(1-t)^{m-1}dt$$

for Re(n), Re(m)> 0.

The beta and gamma functions have relation as given in following equation

$$\beta(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}.$$

2.1.3 Mittag–Leffler Function

The ML function is a simple generalisation of the exponential function $exp(m)$. i.e. replacing $m! = \Gamma(m+1)$ by $(\alpha m)! = \Gamma(\alpha m+1)$ in the denominator of the power terms of the exponential series. The definition of ML function is given in following:

Definition 2.1.3. The ML function of one parameter is defined as

$$E_n(k) = \sum_{i=0}^{\infty} \frac{k^i}{\Gamma(ni+1)},$$

where $n > 0$.

The two parameter ML function is given as

$$E_{n,m}(k) = \sum_{i=0}^{\infty} \frac{k^i}{\Gamma(ni+m)}$$

where $n, m > 0$.

2.2 Fractional Derivative and Integral

2.2.1 Riemann–Liouville Fractional Integral

Definition 2.2.1. The RL fractional integral to order α of an integrable function $y(t)$ is defined to be

$$I^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s)\, \mathrm{d}s, & \alpha > 0 \\ y(t), & \alpha = 0 \end{cases} \tag{2.1}$$

When applied to a power function, it yields the following result:

$$I^{\alpha}(t)^c = \frac{\Gamma(c+1)}{\Gamma(c+\alpha+1)}(t)^{c+\alpha},\ \alpha \geq 0,\ c > -1 \tag{2.2}$$

The operator has a commutativity property, that is

$$I^{\alpha} I^{\beta} y(t) = I^{\beta} I^{\alpha} y(t),\ \alpha, \beta > 0$$

and it is linear, that is to say

$$I^{\alpha}(A_1 y_1(t) + A_2 y_2(t)) = A_1 I^{\alpha} y_1(t) + A_2 I^{\alpha} y_2(t)$$

for any two functions y_1,y_2 and constants A_1,A_2.

2.2.2 Caputo Fractional Derivative

Definition 2.2.2. The fractional derivative of $y(t)$ of the order α in the Caputo sense is given as

$$D^{\alpha}y(t) = I^{j-\alpha}\left(\frac{\mathrm{d}^j}{\mathrm{d}t^j}y(t)\right),\ j-1<\alpha\leq j,\ j\in\mathbb{N} \tag{2.3}$$

2.2.3 Some properties

1. The RL fractional integral and Caputo fractional derivative do not usually commute with each other. The Newton-Leibniz identity given below provides an important relation between them:

   $$I^{\alpha}(D^{\alpha}y(t)) = y(t) - \sum_{i=0}^{j-1} y^{(i)}(0)\frac{t^i}{i!} \tag{2.4}$$

   where $j-1<\alpha\leq j,\ j\in\mathbb{N}$.

2. The Caputo fractional derivative also has the following substitution identity. If we write $y_1(q) = y(qR)$ and $q = t/R$, then we have

   $$D^{\alpha}y(t) = \frac{1}{R^{\alpha}}D^{\alpha}y_1(q) \tag{2.5}$$

   where $j-1<\alpha\leq j,\ j\in\mathbb{N}$.

Chapter 3

FRACTIONAL DIFFERENTIAL EQUATIONS

3.1 Introduction

Fractional differential equations (FDEs), which are the generalisation of the ordinary differential equations to a arbitrary order, involve fractional derivatives of the form $(\mathrm{d}^{\alpha}/\mathrm{d}x^{\alpha})$, where $\alpha > 0$. Here, α is not necessarily to be an integer number.

In this part, existence and uniqueness theorems for FDEs are presented. Linear multi-term FDEs, which is one of the most important type of FDEs and some existing numerical methods for solving such equations are also briefly presented.

3.2 Existence and uniqueness theorems for FDEs

For any kind of differential equations, existence and uniqueness of the solution are too crucial. Therefore, in this part, we will discuss about the existence and uniqueness results of IVP for FDE in the following form

$$D^p y(t) = f(t, y(t)) \tag{3.1}$$

$$y^{(i)}(0) = y_0^i,\ i = 0, 1, 2, ...n-1 \tag{3.2}$$

where $D^p y(t)$ denotes the Caputo fractional derivative of order $p > 0$, with $n-1 < p < n$.

The existence and uniqueness results of the given IVP is presented in [31] that are a very close to the corresponding ordinary theorems known in the first-order equations case.

Theorem 1 (Existence). *[31] Let* $B := [0,R^*] \times [y_0^0 - \beta, y_0^0 + \beta]$ *with* $R^* > 0$ *and* $\beta > 0$ *and the* $f : B \to \mathbb{R}$ *be continous function. Moreover, let* $B := \min\left\{R^*, \left(\frac{\beta\Gamma(p+1)}{\|f\|_\infty}\right)\right\}$. *Then,* $\exists y : [0,R] \to \mathbb{R}$ *solving the IVP (3.1)-(3.2).*

Theorem 2 (Uniqueness). *[31] Let* $B := [0,R^*] \times [y_0^0 - \beta, y_0^0 + \beta]$ *with* $R^* > 0$ *and* $\beta > 0$. *Moreover, assume that the* $f : B \to \mathbb{R}$ *be bounded function on B and satisfy a Lipschitz condition with respect to the second variable, namely,*

$$|f(t,y) - f(t,z)| \leq L|y - z|$$

subject to a constant $L > 0$ *independent of* t,y,z. *Then, expressing R as given in Theorem 1, there exists mostly one function* $y : [0,R] \to \mathbb{R}$ *that solves the IVP (3.1)-(3.2).*

To prove these two theorems we need to use following results.

Lemma 3. *[31] If f is a continuous function, then IVP (3.1)-(3.2) is equivalent to the nonlinear second kind Volterra integral equation*

$$y(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(p)} \int_0^t (t-z)^{p-1} f(z,y(z))dz \tag{3.3}$$

with $n-1 < p \leq n$. *That is to say, each solution of Volterra equation (3.3) is also a solution of given IVP (3.1)-(3.2) and vice versa.*

The generalisation of Banach's fixed point theorem is used to get proof of the uniqueness theorem.

Theorem 4. *[31] Assume that V be a nonempty closed subset of a Banach space X, and let $\beta_n \geq 0$ $\forall n$ and so that $\sum_{n=0}^{\infty} \beta_n$ converges. Furthermore, assume that the mapping $M : V \to V$ satisfy the following inequality*

$$\|M^n v - M^n u\| \leq \beta_n \|v - u\|, \tag{3.4}$$

since $M^n v = M(M^{n-1} v)$ where $M^0 v = Mv$, $\forall n \in \mathbb{N}$ and for each $v, u \in V$. So, M has a unique fixed point v^*. Moreover, for any $v_0 \in V$, the sequence $(M^n v_0)_{n=1}^{\infty}$ converges to point v^*.

Proof of Theorem 2. [31] As defined previously, discussing the case $0 < p < 1$ only will be enough to prove the uniqueness. Therefore, the Volterra equation (3.3) brings to form

$$y(t) = y_0^0 + \frac{1}{\Gamma(p)} \int_0^t (t - z)^{p-1} f(z, y(z)) dz \tag{3.5}$$

Hence, $V = \{y \in C[0, R] : \|y - y_0^0\|_\infty \leq \beta\}$. Clearly, the set V is a closed subset of the Banach space of all continuous functions on $[0, R]$, equipped with the Chebyshev norm. It can be also seen that V is non-empty since $y \equiv y_0^0$ is in V. The operator M on V is defined by

$$(My)(t) = y_0^0 + \frac{1}{\Gamma(p)} \int_0^t (t - z)^{p-1} f(z, y(z)) dz. \tag{3.6}$$

By using this operator, we can rewrite the equation under consideration as following

$$y = My$$

and to prove uniqueness of the solution, we need to prove that M has a unique fixed point. Therefore, let us investigate the features of the operator M. Firstly, noting that, for $0 \le t_1 \le t_2 \le R$,

$$
\begin{aligned}
& |(My)(t_1) - (My)(t_2)| \\
= \; & \frac{1}{\Gamma(p)} \left| \int_0^{t_1} (t_1 - z)^{p-1} f(z, y(z)) dz - \int_0^{t_2} (t_2 - z)^{p-1} f(z, y(z)) dz \right| \qquad (3.7) \\
= \; & \frac{1}{\Gamma(p)} \left| \int_0^{t_1} \left((t_1 - z)^{p-1} - (t_2 - z)^{p-1} \right) f(z, y(z)) dz + \int_{t_1}^{t_2} (t_2 - z)^{p-1} f(z, y(z)) dz \right| \\
\le \; & \frac{\|f\|_\infty}{\Gamma(p)} \left[\int_0^{t_1} \left((t_1 - z)^{p-1} - (t_2 - z)^{p-1} \right) dz + \int_{t_1}^{t_2} (t_2 - z)^{p-1} dz \right] \\
= \; & \frac{\|f\|_\infty}{\Gamma(p)} \left(2(t_2 - t_1)^p + t_1^p - t_2^p \right). \qquad (3.8)
\end{aligned}
$$

shows that My is a continuous function. Furthermore, for $y \in V$ and $t \in [0, R]$ we get

$$
\begin{aligned}
\left| (My)(t) - y_0^0 \right| &= \frac{1}{\Gamma(p)} \left| \int_0^t (t - z)^{p-1} f(z, y(z)) dz \right| \le \frac{1}{\Gamma(p+1)} \|f\|_\infty t^p \\
&\le \frac{1}{\Gamma(p+1)} \|f\|_\infty R^p \le \frac{1}{\Gamma(p+1)} \|f\|_\infty \frac{\beta \Gamma(p+1)}{\|f\|_\infty} = \beta.
\end{aligned}
$$

Hence, we can see that if $y \in V$ then $My \in V$; namely, M maps the set V to itself.

Next, let us prove that $\forall n \in \mathbb{N}_0$ and for each $t \in [0, R]$, we have

$$
\|M^n y - M^n \hat{y}\|_{L_\infty[0,t]} \le \frac{(Lt^p)^n}{\Gamma(1 + pn)} \|y - \hat{y}\|_{L_\infty[0,t]}. \qquad (3.9)
$$

In order to prove this, the induction technique can be use. When $n = 0$, the statement is easily true. For $n-1 \to n$, we write

$$
\begin{aligned}
\|M^n y - M^n \hat{y}\|_{L_\infty[0,t]} &= \left\| \left\| M(M^{n-1}y) - M(M^{n-1}\hat{y}) \right\|_{L_\infty[0,t]} \right\| \\
&= \frac{1}{\Gamma(p)} \sup_{0\le q\le t} \left| \int_0^q (q-z)^{p-1} [f(z, M^{n-1}y(z)) - f(z, M^{n-1}\hat{y}(z))] dz \right|.
\end{aligned}
$$

Next, using the Lipschits assumption on f and induction, we get

$$
\begin{aligned}
\|M^n y - M^n \hat{y}\|_{L_\infty[0,t]} &\le \frac{L}{\Gamma(p)} \sup_{0\le q\le t} \int_0^q (q-z)^{p-1} \left| M^{n-1}y(z) - M^{n-1}\hat{y}(z) \right| dz \\
&\le \frac{L}{\Gamma(p)} \int_0^t (t-z)^{p-1} \sup_{0\le q\le t} \left| M^{n-1}y(q) - M^{n-1}\hat{y}(q) \right| dz \\
&\le \frac{L^n}{\Gamma(p)\Gamma(1+p(n-1))} \int_0^t (t-z)^{p-1} z^{p(p-1)} \sup_{0\le q\le t} |y(q) - \hat{y}(q)| \, dz \\
&\le \frac{L^n}{\Gamma(p)\Gamma(1+p(n-1))} \sup_{0\le q\le t} |y(q) - \hat{y}(q)| \int_0^t (t-z)^{p-1} z^{p(p-1)} dz \\
&= \frac{L^n}{\Gamma(p)\Gamma(1+p(n-1))} \|y - \hat{y}\|_{L_\infty[0,t]} \frac{\Gamma(p)\Gamma(1+p(n-1))}{\Gamma(1+pn)} t^{pn}.
\end{aligned}
$$

which is the desired result. Consequently, by taking Chebyshev norms on the interval $[0,t]$ we get

$$
\|M^n y - M^n \hat{y}\|_\infty \le \frac{(Lt^p)^n}{\Gamma(1+pn)} \|y - \hat{y}\|_\infty
$$

It's proved that the M satisfies the assumptions of Theorem 3 with $\beta_n = (Lt^p)/\Gamma(1+pn)$. To use Theorem 3, we need to verify that $\sum_{n=0}^{\infty} \beta_n$ converges. This is a well known result; the limit

$$
\sum_{n=0}^{\infty} \frac{(Lt^p)^n}{\Gamma(1+pn)} = E_p(Lt^p)
$$

is the ML function of one parameter p, evaluated at Lt^p. Then, applying the fixed point theorem will give the uniqueness result for the solution of FDE. □

Proof of Theorem 1. [31] Similarly, we utilize the same operator M defined in (3.6) and remember that it maps the convex, nonempty and closed set $V = \{y \in C[0,R] : \|y - y_0^0\|_\infty \leq \beta\}$ to itself.

Let us now show that M is a continuous operator. Given any $\gamma > 0$, we can find $\varphi > 0$ so that

$$|f(t,y) - f(t,z)| < \frac{\gamma}{t^p}\Gamma(p+1) \tag{3.10}$$

whenever $|y - z| < \varphi$

Next, assume that $y, \hat{y} \in V$ so that $\|y - \hat{y}\| < \varphi$. Then, from (3.10)

$$|f(t,y(t)) - f(t,\hat{y}(t))| < \frac{\gamma}{t^p}\Gamma(p+1) \tag{3.11}$$

$\forall t \in [0,R]$. Therefore,

$$\begin{aligned} |M^n y(t) - M^n \hat{y}(t)| &= \frac{1}{\Gamma(p)} \left| \int_0^t (t-z)^{p-1} (f(z,y(z)) - f(z,\hat{y}(z)))dz \right| \\ &\leq \frac{\Gamma(p+1)\gamma}{t^p \Gamma(p)} \int_0^t (t-z)^{p-1} dz = \frac{\gamma t^p}{t^p} \leq \gamma \end{aligned}$$

which shows that the operator M is continuous.

Next, let us consider the set of functions

$$M(V) := \{My : y \in V\}.$$

For $c \in M(V)$, we get $\forall x \in [0,R]$,

$$\begin{aligned} |c(x)| &= |(My)(t)| \leq |y_0^0| + \frac{1}{\Gamma(p)} \int_0^t (t-z)^{p-1} |f(z,y(z)| \, dz \\ &\leq |y_0^0| + \frac{1}{\Gamma(p+1)} \|f\|_\infty R^p. \end{aligned}$$

This result shows that $M(V)$ is bounded in a pointwise sense. Furthermore, from proof of Theorem 2 for $0 \leq t_1 \leq t_2 \leq R$, we get that

$$\begin{aligned} |(My)(t_1) - (My)(t_2)| &\leq \frac{\|f\|_\infty}{\Gamma(p+1)} \left(2(t_2-t_1)^p + t_1^p - t_2^p\right) \\ &\leq 2\frac{\|f\|_\infty}{\Gamma(p+1)} (t_2-t_1)^p. \end{aligned}$$

Hence, if $|t_2 - t_1| < \varphi$,

$$|(My)(t_1) - (My)(t_2)| \leq 2\frac{\|f\|_\infty}{\Gamma(p+1)} \varphi^p.$$

Here, we note that the right side of this expression is independent of y, and the set $M(V)$ is equicontinuous. Hence, by Arzelà-Ascoli theorem, each sequence of functions from $M(V)$ have a uniformly convergent subsequence. Hence, $M(V)$ is relatively compact. Then, by Schauder's fixed point theorem, M has a unique fixed point. By construction, a fixed point of M is a solution of given IVP (3.1)-(3.2). □

3.3 Linear multi-term FDEs

In this part, we rewrite and focus on the general type of multi-term FDE in Caputo sense given in Eq.(1.1) in the following linear form

$$D^{\alpha} y(t) = \sum_{i=0}^{k} u_i D^{\beta_i} y(t) + u_{k+1} y(t) + f(t), \ \ 0 \leq t \leq R, \tag{3.12}$$

with

$$y^{(p)}(0) = Y_p,\ p = 0,1,...,n-1 \text{ where } n-1 < \alpha < n, \tag{3.13}$$

u_i $(i = 0,1,...,k)$ are known coefficients and

$$0 < \beta_0 < \beta_1 < ... < \beta_k < \alpha$$

Here, it's also worth to mention that the highest order α need not to be an integer and $f(t)$ is a known function. This equation is important in applications due to the fact it can treat the problems with fractional force, therefore it is suitable for being treated within fractional operators of Caputo type.

Multi-term FDEs have very useful features and they can describe complex multi-rate physical processes in a numerous way and can be applied in many different kind of fields, see e.g. [2, 4, 20, 25]. Basset equation [28] and Bagley–Torvik [29] equations can be given as important examples for smaller class of multi-term FDEs. Existence, uniqueness and stability of solution for multi-term FDEs are discussed in [31–33, 46].

3.3.1 Some numerical techniques for solving Multi-term FDEs

In this subsection, we will briefly review some techniques that used to solve multi-term FDEs numerically. Due to difficulty of finding the exact solutions for multi-term FDEs, many new numerical techniques have been developed to investigate the numerical solutions for such equations. In [40], Diethelm et al. used a generalization of the classical one-step Adams–Bashforth–Moulton technique for first-order equations for solving nonlinear FDEs. Haar wavelets for the solution of fractional Volterra and Fredholm integral equations are considered in [41] by Lepik. Differential Transform Method (DTM) have been carried out for various types of problems, including the Bagley–Torvik, Ricatti and composite fractional oscillation equations for the application of the technique in [35] by Arıkoğlu and Özkol. In [26], Diethelm and Ford applied Adams–Bashforth–Moulton method to solve multi-order FDEs of the general form. In [34], Saw and Kumar introduced a scheme based on collocation technique and shifted Chebyshev polynomials (SCP) to solve multi-term fractional order IVP. A method based on using Boubaker polynomial operational matrix of fractional integration have been applied to solve multi-order FDEs in [38] by Bolandtalat et al. In [63], the solution of multi-term FDEs expressed in terms of ML functions evaluated at matrix arguments by Popolizio. In [64], the differential transformation is proposed as convenient for finding solution to the IVP involving multiple Caputo fractional derivatives of generally non-commensurate orders by Rebenda.

Chapter 4

NUMERICAL SOLUTIONS FOR MULTI-TERM FDEs WITH FRACTIONAL TAYLOR OPERATIONAL MATRIX OF FRACTIONAL INTEGRATION

4.1 Fractional Taylor Basis Vector

We shall make use of the fractional Taylor vector,

$$T_{m\delta}(t) = \left[1, t^{\delta}, t^{2\delta}, ..., t^{m\delta}\right] \tag{4.1}$$

for $m \in \mathbb{N}$ and $\delta > 0$ in the work of this thesis.

4.2 Approximation of function

Suppose that $T_{m\delta}(t) \subset H$, where H is the space of all square integrable functions on the interval $[0,1]$. For any $y \in H$, since $S = span\left\{1, t^{\delta}, t^{2\delta}, ..., t^{m\delta}\right\}$ is a finite dimensional vector space in H, then, y has a unique best approximation $y_* \in S$, so that

$$\forall \widehat{y} \in S, \;\; \|y - y_*\| \leq \|y - \widehat{y}\|$$

Therefore, the function y is approximated by fractional Taylor vector as following

$$y \simeq y_* = \sum_{i=0}^{m} c_i t^{i\delta} = C^T T_{m\delta}(t) \tag{4.2}$$

where $T_{m\delta}(t)$ denote the fractional Taylor vector and

$$C^T = [c_0, c_1, c_2, ..., c_m] \tag{4.3}$$

are the unique coefficients.

4.3 Construction of fractional Taylor operational matrix of fractional integration

By using the property of RL fractional integral given in Eq.(2.2) and Eq.(4.1), we construct the fractional Taylor operational matrix of fractional integration as following

$$\begin{aligned} I^{\alpha}(T_{m\delta}(t)) = & \\ & = \left[\frac{1}{\Gamma(\alpha+1)}t^{\alpha}, \frac{\Gamma(\delta+1)}{\Gamma(\delta+\alpha+1)}t^{\delta+\alpha}, \frac{\Gamma(2\delta+1)}{\Gamma(2\delta+\alpha+1)}t^{2\delta+\alpha}, ..., \frac{\Gamma(m\delta+1)}{\Gamma(m\delta+\alpha+1)}t^{m\delta+\alpha} \right] \\ & = t^{\alpha} M_{\alpha} T_{m\delta}(t) \end{aligned} \tag{4.4}$$

where

$$M_{\alpha} = diag\left[\frac{1}{\Gamma(\alpha+1)}, \frac{\Gamma(\delta+1)}{\Gamma(\delta+\alpha+1)}, \frac{\Gamma(2\delta+1)}{\Gamma(2\delta+\alpha+1)}, ..., \frac{\Gamma(m\delta+1)}{\Gamma(m\delta+\alpha+1)}\right]$$

denotes the operational matrix of integration.

If we define G_{α} as

$$G_{\alpha} = \left[\frac{1}{\Gamma(\alpha+1)}, \frac{\Gamma(\delta+1)}{\Gamma(\delta+\alpha+1)}, \frac{\Gamma(2\delta+1)}{\Gamma(2\delta+\alpha+1)}, ..., \frac{\Gamma(m\delta+1)}{\Gamma(m\delta+\alpha+1)}\right]$$

then, we can rewrite the Eq.(4.4) as

$$I^{\alpha}(T_{m\delta}(t)) = t^{\alpha} G_{\alpha} * T_{m\delta}(t) \tag{4.5}$$

where $*$ denotes the operation of multiplying matrices term by term.

4.4 The Numerical Algorithm

In this part, we give the numerical algorithm of fractional Taylor method to solve given multi-term IVP of fractional order [45].

Let us recall the linear multi-term FDE defined in Eq.(3.12) and Eq.(3.13),

$$D^{\alpha} y(t) = \sum_{i=0}^{k} u_i D^{\beta_i} y(t) + u_{k+1} y(t) + f(t), \;\; 0 \le t \le R,$$

subject to the

$$y^{(p)}(0) = Y_p, \; p = 0, 1, ..., n-1 \text{ where } n-1 < \alpha < n$$

$$u_i \; (i = 0, 1, ..., k) \text{ are known coefficients and}$$

$$0 < \beta_0 < \beta_1 < ... < \beta_k < \alpha$$

The procedure to solve given equation above is explained step by step as following.

As a first step, by using the transformation $q = t/R$, we replace the variable $t \in [0, R]$ with $q \in [0, 1]$. Now, by using Eq.(2.5) in Eq.(3.12), we get

$$\frac{1}{R^{\alpha}} D^{\alpha} y_1(q) = \sum_{i=0}^{k} \frac{1}{R^{\beta_i}} u_i D^{\beta_i} y_1(q) + u_{k+1} y_1(q) + f_1(q), \, 0 \le q \le 1 \tag{4.6}$$

where $f_1(q) = f(qR)$ and $y_1(q) = y(qR)$. Same as Eq.(4.2), we approximate the $y_1(q)$ as

$$y_1(q) = \sum_{i=0}^{m} c_i q^{i\delta} = C^T T_{m\delta}(q) \tag{4.7}$$

such that $T_{m\delta}(q) = [1, q^{\delta}, q^{2\delta}, ..., q^{m\delta}]^T$ is the fractional Taylor vector and the unique coefficients C^T is unknown vector which is defined in Eq.(4.3).

Next step, employing the RL fractional integral for the both sides of (4.6), we have

$$\frac{1}{R^{\alpha}}\left[y_1(q) - \sum_{j=0}^{n-1} y_1^{(j)}(0^+)\frac{t^j}{j!}\right] = \sum_{i=0}^{k} \frac{1}{R^{\beta_i}} u_i I^{\alpha-\beta_i}\left[y_1(q) - \sum_{j=0}^{n_i-1} y_1^{(j)}(0^+)\frac{t^j}{j!}\right]$$
$$+ u_{k+1} I^{\alpha} y_1(q) + I^{\alpha} f_1(q) \tag{4.8}$$

where $y^{(p)}(0) = V_p$, $p = 0, 1, ..., n-1$ where $n_i - 1 < \beta_i < n_i$.

From this place, by substituting initial conditions (3.13), we have

$$\frac{1}{R^{\alpha}}[y_1(q)] = \sum_{i=0}^{k} \frac{1}{R^{\beta_i}} u_i I^{\alpha-\beta_i}[y_1(q)] + u_{k+1} I^{\alpha} y_1(q) + h_1(q) \tag{4.9}$$

so that $h_1(q) = I^{\alpha} f_1(q) + \frac{1}{R^{\alpha}}\left(\sum_{j=0}^{n-1} V_j \frac{t^j}{j!}\right) + \sum_{i=0}^{k} \frac{1}{R^{\beta_i}} u_i I^{\alpha-\beta_i}\left(\sum_{j=0}^{n_i-1} V_j \frac{t^j}{j!}\right)$.

Now, by using the Eq.(4.5), we approximate the fractional integrals in above equation and we get

$$\frac{1}{R^{\alpha}}\left[C^T T_{m\delta}(q)\right] = \sum_{i=0}^{k} \frac{1}{R^{\beta_i}} u_i C^T q^{\alpha-\beta_i}\left(G_{\alpha-\beta_i} * T_{m\delta}(q)\right)$$
$$+ u_{k+1} q^{\alpha} C^T \left(G_{\alpha} * T_{m\delta}(q)\right) + h_1(q) \tag{4.10}$$

As a final step, by taking the collocation points $q_j = j/m$ ($j = 0, 1, ..., m$) in Eq.(4.10), we get $m+1$ linear algebraic equations. This linear system can be solved for the unknown vector C^T. Consequently, $y_1(q)$ can be approximated by Eq.(4.7).

4.5 Error Estimation

In this part, an error estimation based on residual error function for the proposed method will be presented. The residual error estimation was used in [71, 72] and from these results, we can conclude that, this error estimation is very effective. Let $y_{m,\delta}(t)$ and $y(t)$ be numerical and exact solutions of given IVP (3.11)-(3.12).

Substituting $y_{m,\delta}(t)$ into IVP (3.11)-(3.12) we get,

$$D^{\alpha} y_{m,\delta}(t) - \sum_{i=0}^{k} u_i D^{\beta_i} y_{m,\delta} - u_{k+1} y_{m,\delta} - f(t) = R_m(t)$$

where $R_m(t)$ is the residual function. By using the above equation and Eq.(3.11) we have

$$D^{\alpha}(y(t) - y_{m,\delta}(t)) - \sum_{i=0}^{k} u_i D^{\beta_i}(y(t) - y_{m,\delta}(t)) - u_{k+1}(y(t) - y_{m,\delta}(t)) = R_m(t)$$

Now, we define the error function as

$$e_{m,\delta} = y(t) - y_{m,\delta}(t).$$

Next, using this error function we get

$$D^{\alpha} e_{m,\delta} - \sum_{i=0}^{k} u_i D^{\beta_i} e_{m,\delta} - u_{k+1} e_{m,\delta} = R_m(t)$$

with initial conditions $e_{m,\delta}(0) = 0$ and $e'_{m,\delta}(0) = 0$. Solving this equation in the same way presented in Section 4.4, we get the approximate error estimation $e_{m,\delta}(t)$ of proposed method. Consequently, the approximation of maximum absolute error can be estimated by

$$E_{m,\delta} = \max\left\{\left|e_{m,\delta}\right|, 0 \le t \le R\right\}.$$

In the case that the exact solution of the given problem is unknown, this presented error estimation can be used to show the accuracy of the obtained results.

4.6 Illustrative Examples

To show the applicability and effectiveness of the given method, we give nine examples in this section. To approximate the solution of given problems, the presented fractional Taylor method applied to each example. Approximate solutions obtained by use of presented method have been compared with analytical solutions and also with results of some other techniques in literature. From this comparisons, we can conclude that the presented technique is providing very good results and very effective for approximating the solution of multi-term FDEs. To compute the numerical results, MATLAB version R2015a has been used.

For choosing δ, we usually take either $\delta = 1$ or $\delta = \alpha - \lfloor\alpha\rfloor$, the fractional part of α.

4.6.1 Example 1

For the first example, let us focus on multi-order FDE in the form given below [37]

$$D^{\alpha}y(t) = u_0 D^{\beta_0}y(t) + u_1 D^{\beta_1}y(t) + u_2 D^{\beta_2}y(t) + u_3 D^{\beta_3}y(t) + f(t), \ \ 0 \le t \le R, \quad (4.11)$$

$$y(0) = V_0, \ \ y'(0) = V_1$$

We let $\alpha = 2, V_0 = V_1 = 0, R = 1$, the coefficients $u_0 = u_2 = -1, u_1 = 2, u_3 = 0$ and $\beta_0 = 0, \beta_1 = 1, \beta_2 = \frac{1}{2}$ and the function $f(t)$ is

$$f(t) = t^7 + \frac{2048}{429\sqrt{\pi}} t^{6.5} - 14t^6 + 42t^5 - t^2 - \frac{8}{3\sqrt{\pi}} t^{1.5} + 4t - 2.$$

The exact solution is $y(t) = t^7 - t^2$.

To solve Eq.(4.11), let us apply the given procedure step by step which is implemented in previous section.

As a first step, replace variable $t \in [0,R]$ to $q \in [0,1]$ by taking $q = t/R$.

Next, we use the Eq.(2.5) and get

$$\frac{1}{R^\alpha} D^\alpha y_1(q) = \frac{u_0}{R^{\beta_0}} D^{\beta_0} y_1(q) + \frac{u_1}{R^{\beta 1}} D^{\beta_1} y_1(q) + \frac{u_2}{R^{\beta_2}} D^{\beta_2} y_1(q) + \frac{u_3}{R^{\beta_3}} D^{\beta_3} y_1(q) + f_1(q) \tag{4.12}$$

where $0 \leq q \leq 1$.

Now, using Eq.(2.4) we have

$$\begin{aligned}\frac{1}{R^\alpha}(y_1(q) - y_1(0) - qy_1\prime(0)) = {} & \frac{u_0}{R^{\beta_0}} I^{\alpha-\beta_0}(y_1(q) - y_1(0) - qy_1\prime(0)) \\ & + \frac{u_1}{R^{\beta 1}} I^{\alpha-\beta_1}(y_1(q) - y_1(0) - qy_1\prime(0)) \\ & + \frac{u_2}{R^{\beta_2}} I^{\alpha-\beta_2}(y_1(q) - y_1(0) - qy_1\prime(0)) \\ & + \frac{u_3}{R^{\beta_3}} I^{\alpha-\beta_3}(y_1(q) - y_1(0) - qy_1\prime(0)) \\ & + I^\alpha f_1(q). \end{aligned} \tag{4.13}$$

Next, using Eq.(4.13) and putting initial conditions $y(0) = V_0,\ y'(0) = V_1$ into equation

$$\begin{aligned}\frac{1}{R^{\alpha}}(C^T T_{m\delta}(q) - V_0 - RqV_1) &= \frac{u_0}{R^{\beta_0}} I^{\alpha-\beta_0}(C^T T_{m\delta}(q) - V_0 - RqV_1) \\ &+ \frac{u_1}{R^{\beta 1}} I^{\alpha-\beta_1}(C^T T_{m\delta}(q) - V_0 - RqV_1) \\ &+ \frac{u_2}{R^{\beta_2}} I^{\alpha-\beta_2}(C^T T_{m\delta}(q) - V_0 - RqV_1) \\ &+ \frac{u_3}{R^{\beta_3}} I^{\alpha-\beta_3}(C^T T_{m\delta}(q) - V_0 - RqV_1) \\ &+ I^{\alpha} f_1(q). \end{aligned} \tag{4.14}$$

From Eq.(4.5), we have

$$\begin{aligned} &\frac{1}{R^{\alpha}}(C^T T_{m\delta}(q) - V_0 - RqV_1) \\ &= \frac{u_0}{R^{\beta_0}} q^{\alpha-\beta_0} C^T (G_{\alpha-\beta_0} * T_{m\delta}(q)) - \frac{u_0 q^{\alpha-\beta_0}}{R^{\beta_0}\Gamma(\alpha-\beta_0+1)} V_0 - \frac{u_0 q^{\alpha-\beta_0+1}}{R^{\beta_0}\Gamma(\alpha-\beta_0+2)} V_1 \\ &+ \frac{u_1}{R^{\beta 1}} q^{\alpha-\beta_1} C^T (G_{\alpha-\beta_1} * T_{m\delta}(q)) - \frac{u_1 q^{\alpha-\beta_1}}{R^{\beta_1}\Gamma(\alpha-\beta_1+1)} V_0 - \frac{u_1 q^{\alpha-\beta_1+1}}{R^{\beta 1}\Gamma(\alpha-\beta_1+2)} V_1 \\ &+ \frac{u_2}{R^{\beta_2}} q^{\alpha-\beta_2} C^T (G_{\alpha-\beta_2} * T_{m\delta}(q)) - \frac{u_2 q^{\alpha-\beta_2}}{R^{\beta_2}\Gamma(\alpha-\beta_2+1)} V_0 - \frac{u_2 q^{\alpha-\beta_2+1}}{R^{\beta_2}\Gamma(\alpha-\beta_2+2)} V_1 \\ &+ \frac{u_3}{R^{\beta_3}} q^{\alpha-\beta_3} C^T (G_{\alpha-\beta_3} * T_{m\delta}(q)) - \frac{u_3 q^{\alpha-\beta_3}}{R^{\beta_3}\Gamma(\alpha-\beta_3+1)} V_0 - \frac{u_3 s^{\alpha-\beta_3+1}}{R^{\beta_3}\Gamma(\alpha-\beta_3+2)} V_1 \\ &+ I^{\alpha} f_1(q). \end{aligned} \tag{4.15}$$

Now, taking $R = 1$ in Eq.(4.15) and putting the given values for V_0, V_1, u_i, β_i where $i = 0,1,2,3$ into this equation, we get

$$C^T T_{m\delta} = 2q^1 C^T (G_1 * T_{m\delta}(q)) - q^{3/2} C^T (G_{3/2} * T_{m\delta}(q)) - q^2 C^T (G_1 * T_{m\delta}(q)) + I^2 f_1(q) \tag{4.16}$$

Finally, taking the collocation points $q_j = j/m$ $(j = 0,1,...,m)$ generates a linear algebraic system of dimension $m+1$ with unknown vector C^T. In order to solve this system by using presented method and comparing the results, we choose $\delta = 1$ and different values of m.

To show the efficiency, we compared the numerical results with the method given in [37].

Table 4.1, compares the obtained results for absolute error with $m = 4, 6, 7$. We observe from Table 1 that, the absolute errors for presented method are smaller and the numerical solution is more accurate for the same size of m.

Table 4.1: The comparison for absolute errors of the proposed scheme and method given in [37] with $m = 4, 6, 7$

t	Present method $m = 4$	Method in [37] $m = 4$	Present method $m = 6$	Method in [37] $m = 6$	Present method $m = 7$	Method in [37] $m = 7$
0.2	0.0116	0.0844	6.81430698097618e-07	0.0044	1.040834086e-16	2.81025203108243e–15
0.4	0.0032	0.3501	1.01100805164899e-04	0.0079	2.498001805e-16	6.63358257213531e–15
0.6	0.0108	0.6734	1.2907314422994e-05	0.0143	1.665334537e-16	3.27515792264421e–15
0.8	0.0037	1.0234	1.16246682382747e-04	0.0214	3.330669074e-16	4.25770529943748e–14
1.0	0.0026	1.6700	1.11299947542775e-05	0.0280	1.110223025e-16	2.43819897540083e–13

In Fig. 4.1–Fig. 4.3, we present the graphical representation of comparison between exact solution and the numerical solutions obtained by given method and the method of [37] for the problem (4.11) with $m = 4, 6, 7$ respectively. From these results, we can conclude that $m = 4$ and $m = 6$ give larger absolute error, while $m = 7$ gives smaller absolute error (10^{-16}) and more precise numerical solution. These comparisons also shows that the results obtained by given method is closer to the exact solution than the results of [37].

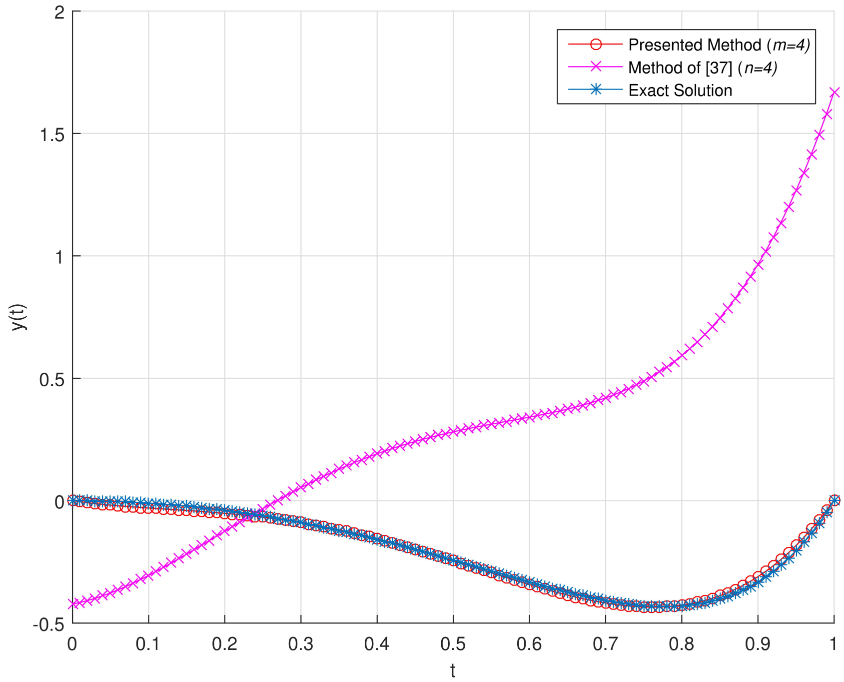

Figure 4.1: Graphical representation of exact solution and the numerical solutions obtained by presented method and the method of [37] with $m, n = 4$

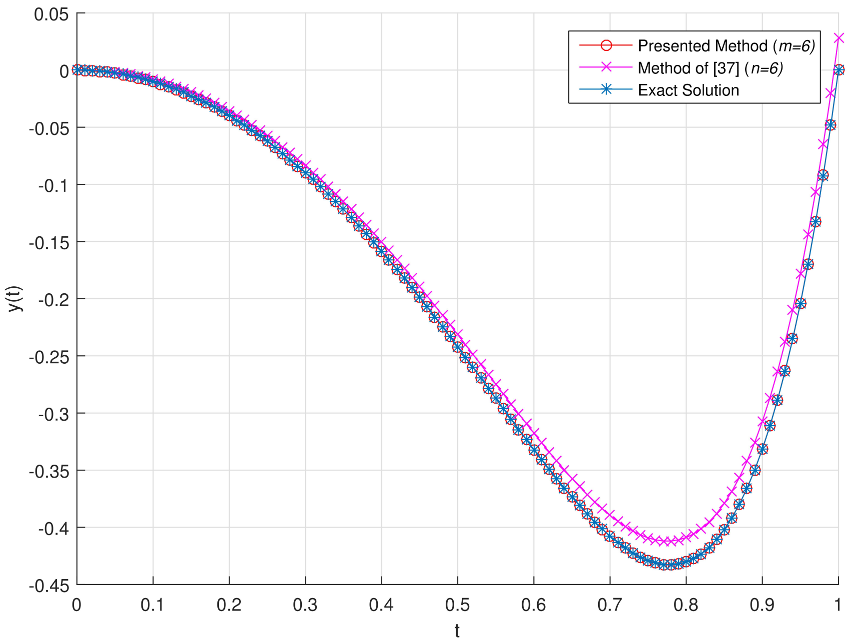

Figure 4.2: Graphical representation of exact solution and the numerical solutions obtained by presented method and the method of [37] with $m, n = 6$

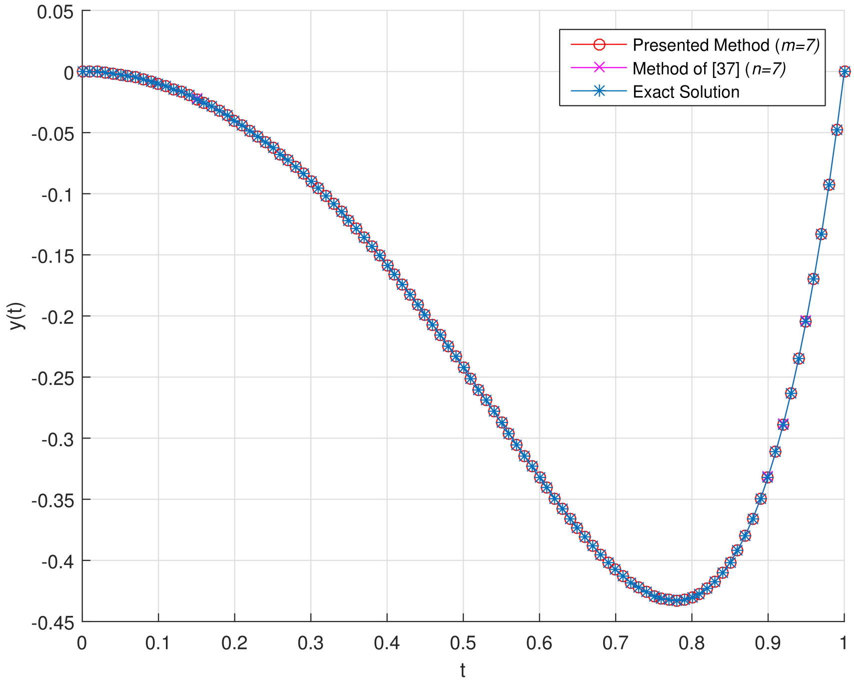

Figure 4.3: Graphical representation of exact solution and the numerical solutions obtained by presented method and the method of [37] with $m, n = 7$

In Fig. 4.4, we show the graphical representation of absolute errors obtained by using presented method and the method of [37] with $m, n = 6$.

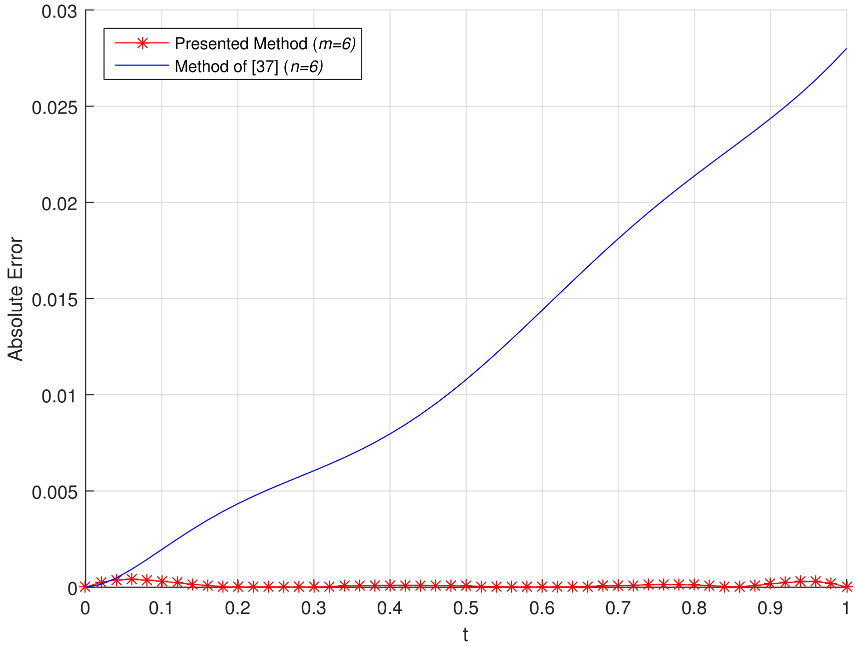

Figure 4.4: The behaviour of absolute errors obtained by using given technique and the method of [37] with $m, n = 6$.

From Fig. 4.4, we can conclude that the absolute error obtained by our method is remaining smaller and stable while the absolute error of other method is increasing in the interval $[0, 1]$.

In Fig. 4.5–Fig. 4.6, we give the graphical representation of absolute errors obtained by using proposed method with $m = 4, 7$ respectively.

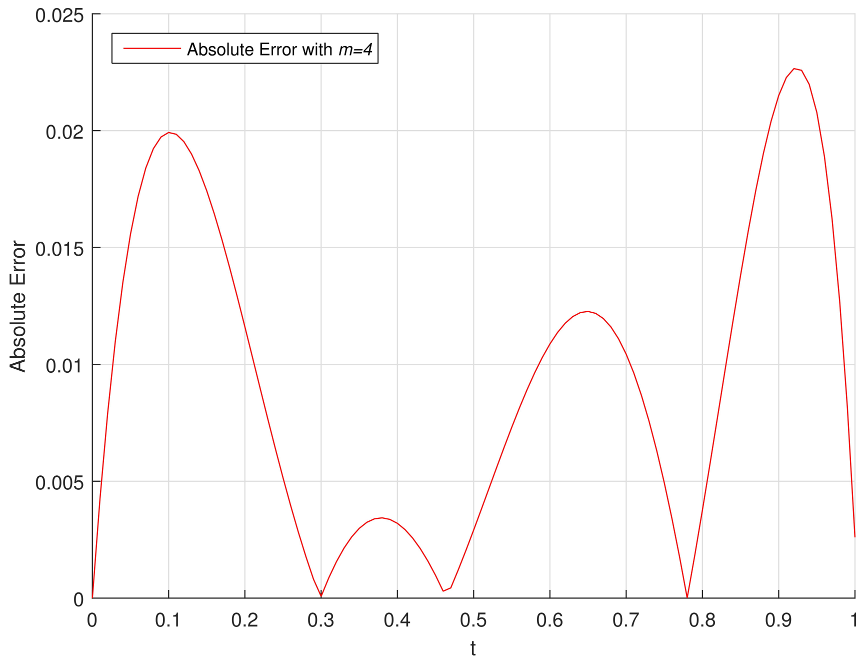

Figure 4.5: The absolute error result with $m = 4$.

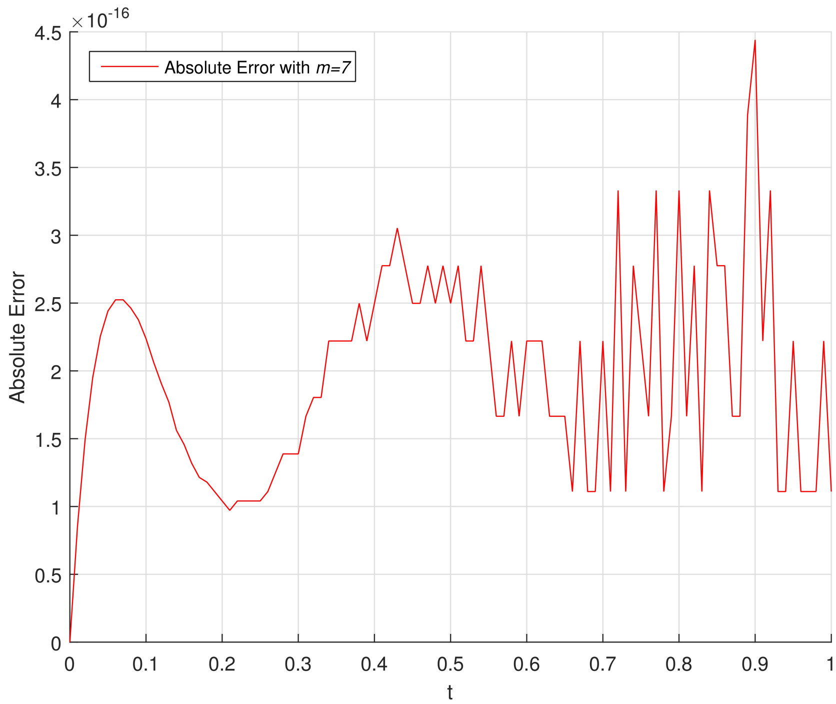

Figure 4.6: The absolute error result with $m = 7$.

A pseudo-code for MATLAB implementation of Example 1 is given below.

Algorithm 1: fractionalTaylormethod.m

```
alpha = 2;

beta = [1, 1/2, 0];

Uk = [2, -1, -1];

func =@(t) t^7 + 2048/(429*sqrt(pi))*t^6.5 - 14*t^6 + 42*t^5 - t^2 - ...
        8/(3*sqrt(pi))*t^1.5 + 4*t - 2;

t0 = 0 ; R = 1;

y0 = [0;0];

m = 4;

delta = 1;

[A,b] = fractionalTaylor(alpha, beta, Uk, func, t0, R, y0, m, delta)

C = linsolve(A,b)

[s,y] = approxSoln(C)
```

4.6.2 Example 2

In this example, we focus on Eq.(4.11) with $\alpha = 2, V_0 = V_1 = 0$, the coefficients $u_0 = u_2 = -1, u_1 = 0, u_3 = 2$ and $\beta_0 = 0, \beta_2 = \frac{2}{3}, \beta_3 = \frac{5}{3}$ and the function is

$$f(t) = t^3 + 6t - \frac{12}{\Gamma(\frac{7}{3})} t^{\frac{4}{3}} + \frac{6}{\Gamma(\frac{10}{3})} t^{\frac{7}{3}}.$$

The exact solution of this equation is $y(t) = t^3$. [37]

Applying the same procedure to given problem as presented in Example 1, we get the following equation

$$C^T T_{m\delta} = 2q^{1/3} C^T (G_{1/3} * T_{m\delta}(q)) - q^{4/3} C^T (G_{4/3} * T_{m\delta}(q)) - q^2 C^T (G_2 * T_{m\delta}(q)) + I^2 f_1(q) \tag{4.17}$$

As we stated in previous example, collocating this equation at the nodes $q_j = j/m$ ($j = 0, 1, ..., m$) generates a set of algebraic equations. In this example, to solve this sysem for C^T, we choose $\delta = 1, 1.5$ and different values of m.

Table 4.2 shows the results for obtained absolute errors by using presented method with $m = 2, 3$. From these results, we can see that, there is satisfactory agreement between the exact solution and numerical solutions. The absolute error is achieved about 10^{-15}. We also note that, the proposed method gives better results for $m = 2$ by taking $\delta = 1.5$.

In Fig. 4.7.(a), we show the graphical representation of obtained numerical solution

Table 4.2: The absolute errors with m = 2,3

t	$\delta = 1$, $m = 2$	$\delta = 1.5$, $m = 2$	$\delta = 1$, $m = 3$
0	0	0	0
0.1	0.010209105	1.3e-17	7.42e-17
0.2	0.008778787	4.68e-17	1.232e-16
0.3	0.001709047	1.11e-16	1.769e-16
0.4	0.005000117	2.082e-16	2.637e-16
0.5	0.005348703	3.608e-16	4.163e-16
0.6	0.006663287	5.829e-16	6.661e-16
0.7	0.037035855	8.882e-16	9.992e-16
0.8	0.091769001	1.2212e-15	1.5543e-15
0.9	0.176862723	1.6653e-15	1.9984e-15
1.0	0.2983170221	2.2204e-15	2.8866e-15

and the exact solution of the given problem. Fig. 4.7.(b) presents the obtained absolute error by using proposed method with $m = 3$.

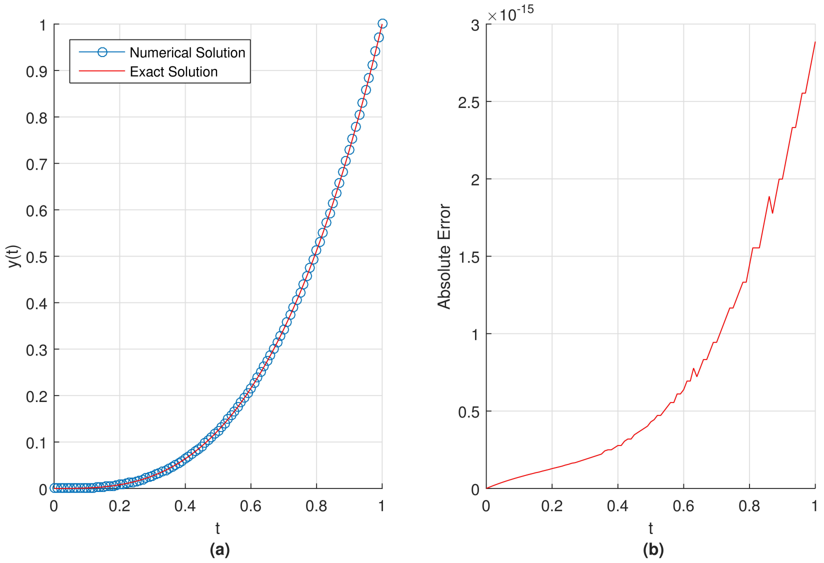

Figure 4.7: (**a**) The numerical and the exact solutions with $m = 3$. (**b**) The absolute error with $m = 3$.

4.6.3 Example 3

Consider the multi-term fractional order IVP [34]

$$D^{(2.2)}y(t) + 1.3D^{(1.5)}y(t) + 2.6y(t) = \sin(2t), \tag{4.18}$$

with initial conditions

$$y(0) = y'(0) = y''(0) = 0,$$

where the equation have the series solution given by [35]

$$\begin{aligned} y_s(t) = &\frac{28561}{3600000}t^6 + \frac{2}{\Gamma(4.2)}t^{3.2} - \frac{13}{5\Gamma(4.9)}t^{3.9} + \frac{169}{50\Gamma(5.6)}t^{4.6} \\ &- \frac{8}{\Gamma(6.2)}t^{5.2} - \frac{2197}{500\Gamma(6.3)}t^{5.3} - \frac{26}{5\Gamma(6.4)}t^{5.4} + \frac{52}{5\Gamma(6.9)}t^{5.9}. \end{aligned} \tag{4.19}$$

To solve this problem, we choose $\delta = 1$, and $m = 10$.

We give the comparison of series solution and the approximate solution obtained by given technique in Table 4.3. Table 4.4 compares the obtained absolute errors by using presented method with the results of [34]. From this compared results, it can be seen that the obtained approximate solution by use of given method is very close to series solution for a small number of m.

Table 4.3: Comparison of numerical solution with series solution for Example 3

t	Series Solution [35]	Present Method $m = 10$
0.0	0	0
0.1	0.000147766	0.000147731
0.2	0.001274983	0.001275552
0.3	0.00439917	0.00440567
0.4	0.010405758	0.010441315
0.5	0.019962077	0.020094648
0.6	0.033452511	0.033841301
0.7	0.050923716	0.051890573
0.8	0.0720381	0.074169634
0.9	0.096035415	0.100321388

Table 4.4: Comparison for absolute errors of Example 3

t	Present Method $m = 10$	Method in [34] $m = 20$
0.0	0	0
0.1	3.47449e-08	5.2560e-7
0.2	5.69366e-07	1.7150e-6
0.3	6.49968e-06	8.2260e-6
0.4	3.55576e-05	3.7820e-5
0.5	0.000132571	0.0001353
0.6	0.00038879	0.000392
0.7	0.000966858	0.0009704
0.8	0.002131534	0.002135
0.9	0.004285973	0.00429

The compared results of Table 4.4 conclude that the proposed technique has better approach to series solution with a smaller m.

The graphical representation of comparison between series solution and approximate solutions obtained by presented method and the method of [34] in the interval $[0,1]$ is illustrated in Fig. 4.8.

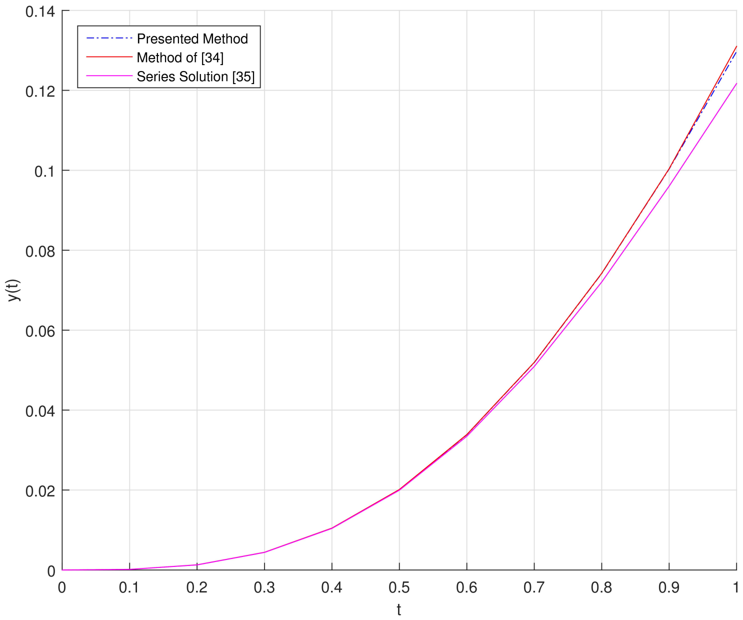

Figure 4.8: The comparison between series solution and numerical solutions obtained by presented method and the method of [34] with $m = 10$.

In Fig. 4.9, we show present graphical representation of absolute errors obtained by using given technique and the method of [34] with $m = 10$.

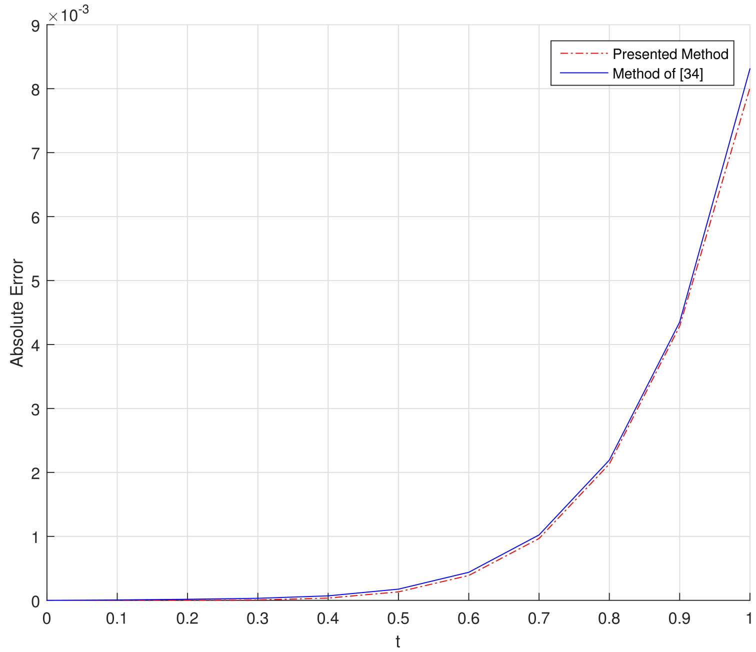

Figure 4.9: The behaviour of absolute errors obtained by using given technique and the method of [34].

In Fig. 4.10, we show the graphical representation for series solution and the numerical results of presented method for the interval $[0, 10]$. The results plotted in Fig. 4.10 are in a very good and satisfactory agreement with the series solution given in [35] and the results of [36].

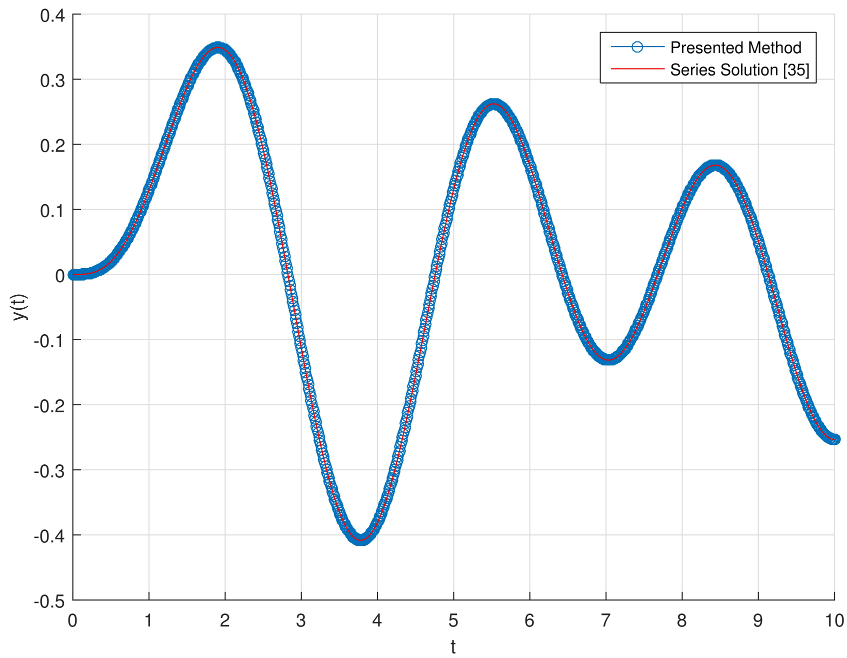

Figure 4.10: The behaviour of series solution and the approximate solution obtained by proposed method for the interval $[0,10]$.

4.6.4 Example 4

Motivated by [40], we consider the following form of FDE,

$$D^{\alpha}y(t)+y(t)=\begin{cases}\dfrac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+t^{2}-t, & \alpha>1\\ \dfrac{2}{\Gamma(3-\alpha)}t^{2-\alpha}-\dfrac{1}{\Gamma(2-\alpha)}t^{1-\alpha}+t^{2}-t, & \alpha\leq 1\end{cases} \tag{4.20}$$

with

$$y(0)=0,\ y'(0)=-1$$

whose exact solution is $y(t)=t^{2}-t$.

To apply the presented method to Eq.(4.20) and compare the results with methods of [44], [40] and [67], we solve this problem with $\alpha=0.3,0.5,0.7,1.25,1.5,1.85$, and various values for δ and m. The obtained results are presented as below.

In Table 4.5, we list the results of obtained absolute errors for $\alpha = 0.3, 0.5, 0.7$ by use of presented method. Also, the results for $\alpha = 1.25, 1.5, 1.85$ are given in Table 4.6.

Table 4.5: The Absolute Errors with *m=3* and $\alpha < 1$ for Example 4

t	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
0	0	0	0
0.1	4.16e-17	8.33e-17	1.94e-16
0.2	8.33e-17	5.55e-17	2.78e-16
0.3	1.11e-16	2.78e-17	2.50e-16
0.4	1.67e-16	1.39e-16	2.50e-16
0.5	1.67e-16	1.11e-16	1.67e-16
0.6	1.67e-16	5.55e-17	2.78e-17
0.7	1.67e-16	8.33e-17	8.33e-17
0.8	3.05e-16	5.55e-17	1.11e-16
0.9	2.08e-16	1.25e-16	1.39e-16
1.0	1.91e-16	1.26e-16	8.91e-17

Table 4.6: The Absolute Errors with *m=3* and $\alpha > 1$ for Example 4.

t	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.85$
0.0	0	0	0
0.1	1.39e-17	2.78e-17	1.25e-16
0.2	5.55e-17	5.55e-17	1.94e-16
0.3	5.55e-17	5.55e-17	2.22e-16
0.4	5.55e-17	2.78e-17	2.50e-16
0.5	1.11e-16	0	2.22e-16
0.6	1.67e-16	5.55e-17	1.67e-16
0.7	1.94e-16	5.55e-17	5.55e-17
0.8	3.05e-16	1.39e-16	5.55e-17
0.9	1.11e-16	8.33e-17	1.39e-17
1.0	8.21e-17	1.97e-16	1.06e-16

In Fig. 4.11.(a) and Fig. 4.11.(b), we give the graphical representation of obtained results for numerical and exact solution of the given problem and absolute error for $\alpha = 1.5$ in the interval $[0, 1]$

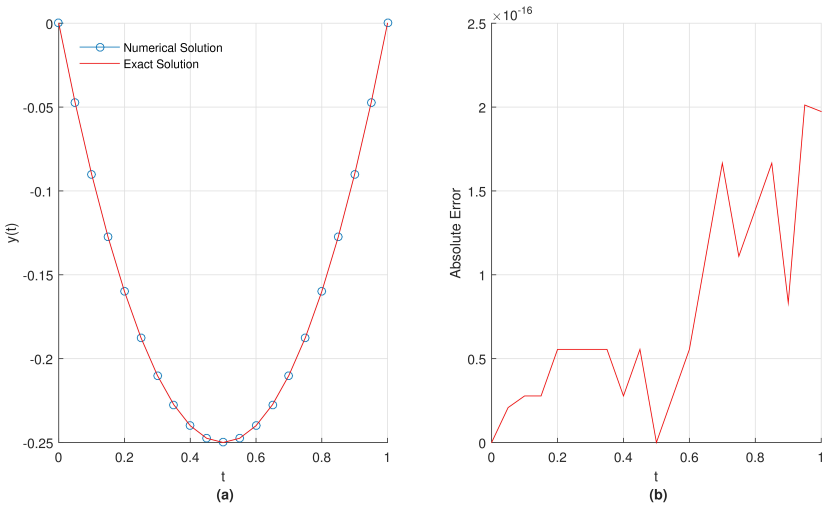

Figure 4.11: (**a**) The numerical and exact solutions for $\alpha = 1.5$. (**b**) The absolute error for $\alpha = 1.5$.

In Fig. 4.12, we plot the graphical representation for behavior of the obtained approximate solution by use of the given method and the exact solution of the given problem for $\alpha = 1.5$ in the interval $[0, 15]$.

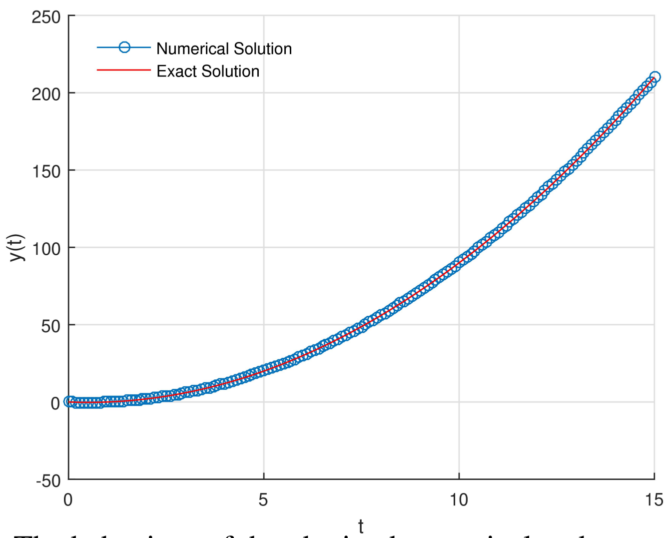

Figure 4.12: The behaviour of the obtained numerical and exact solutions with $\alpha = 1.5$ for the interval $t \in [0, 15]$

Table 4.7 lists the obtained absolute errors for the given problem (4.20) at $t = 1, 5, 10, 50$ and $\alpha = 1.5$ by use of presented method and some other methods in literature [40, 44, 67]. From this compared results, we can say that the numerical solution obtained by use of given technique is in better agreement with the exact one and obtained absolute error is smaller.

Table 4.7: Comparison for absolute errors of proposed method and some other numerical methods in literature at $t = 1, 5, 10, 50$ for $\alpha = 1.5$

t	Presented method	Method of [44]	Method of [40]	Method of [67]
	$\delta = 1/2, m = 4$	$n = 20$	$h = 1/320$	$p = 1, T = 1$
1	7.99361e-14	9.10e-5	3.42e-3	-
5	2.55795e-13	2.42e-3	-	-
10	1.42109e-13	5.50e-3	-	-
50	3.63798e-12	3.74e-2	-	1.2

In Fig. 4.13, the behaviour of absolute error for $\alpha = 1.5$ with $m = 4$ and $\delta = 1/2, 1$ at $t \in [0, 50]$ is presented. From this graph, it can be seen that we get better results by taking $\delta = 1/2$ for this example and the approximate solution is very close to exact solution for a small number of m.

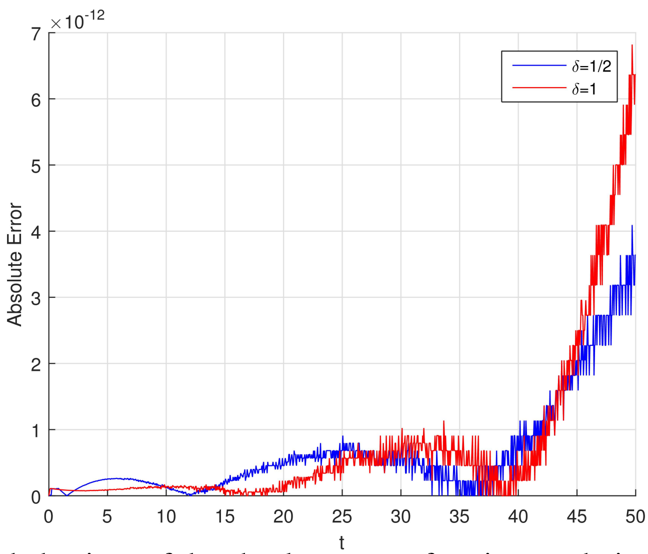

Figure 4.13: The behaviour of the absolute errors for given technique where $\alpha = 1.5$, $t \in [0, 50]$ with $m = 4$ and $\delta = 1/2, 1$.

4.6.5 Example 5

This example considers the following form of linear multi-term FDE with variable coefficients [65]

$$aD^2y(t)+b(t)D^{\beta_1}y(t)+c(t)Dy(t)+e(t)D^{\beta_2}y(t)+k(t)y(t)=f(t), \tag{4.21}$$

with,

$$y(0)=2,\ y'(0)=0$$

where $0<\beta_2<1,\ 1<\beta_1<2$ and

$$f(t)=-a-\frac{b(t)}{\Gamma(3-\beta_1)}t^{2-\beta_1}-c(t)t-\frac{e(t)}{\Gamma(3-\beta_2)}t^{2-\beta_2}+k(t)\left(2-\frac{t^2}{2}\right)$$

whose the exact solution is $y(t)=2-\frac{t^2}{2}$.

We give the numerical solution for the given problem by proposed method for $a=1, b(t)=\sqrt{t}, c(t)=t^{\frac{1}{3}}, e(t)=t^{\frac{1}{4}}, k(t)=t^{\frac{1}{5}}, \beta_2=0.333, \beta_1=1.234$ with $\delta=1$.

In Table 4.8, we give the results for maximum errors obtained by use of proposed method and comparison with the results of [65, 66]. From this compared results, we can see that the numerical solution obtained by use of given technique is closer to the exact solution.

Table 4.8: Maximum Errors of Example 5 for $t = 1$ with $m = 3, 4, 5, 6, 10, 20, 40$.

m	Present Method	Method given in [66]	Method given in [65]
3	4.44089e-16	4.4409e-16	-
4	6.66134e-16	1.4633e-13	-
5	4.44089e-16	3.2743e-12	6.88384e-5
6	4.44089e-16	1.0725e-13	-
10	2.22045e-15	-	3.00351e-6
20	3.47278e-13	-	1.67837e-7
40	1.46549e-13	-	1.02241e-8

Fig. 4.14 presents the graphical representation for behaviour of numerical and exact solutions with $m = 6$. From this representation, we can see that the obtained approximate solution is in a very good agreement with exact solution.

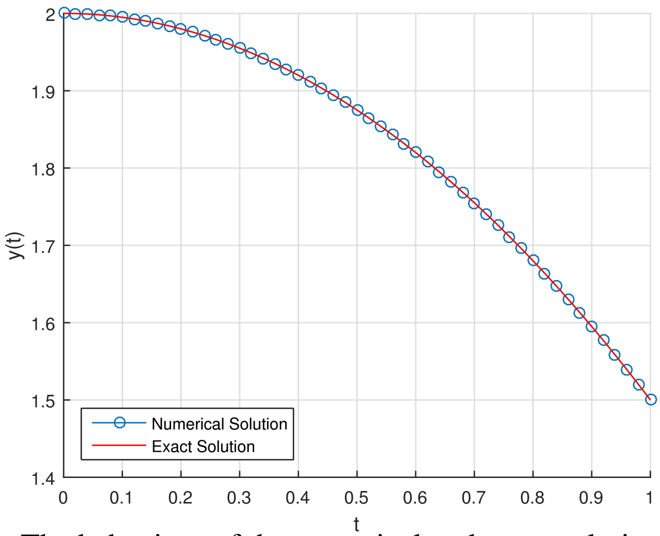

Figure 4.14: The behaviour of the numerical and exact solutions with $m = 6$.

4.6.6 Example 6

Now, we consider the below FDE [44]

$$y'(t) + D^{1/2}y(t) - 2y(t) = 0,\ t \in (0, R], \tag{4.22}$$

$$y(0) = 1$$

which arises, for example, in the study of generalized Basset force occuring when a spherical object sinks in a (relatively dense) incompressible viscous fluid; see [28,53]. By use of Laplace Transformation of Caputo derivatives, we get the analytical solution as following

$$y(t) = \frac{2}{3\sqrt{t}} E_{1/2,1/2}(\sqrt{t}) - \frac{1}{6\sqrt{t}} E_{1/2,1/2}(-2\sqrt{t}) - \frac{1}{2\sqrt{\pi t}},$$

where the ML function $E_{\lambda,\mu}(t)$ with parameters $\lambda, \mu > 0$ is given as

$$E_{\lambda,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\lambda k + \mu)}.$$

This ML function and its variations are very significant in fractional calculus and FDEs [43].

In order to solve given problem by use of given method and compare the results, we take $t \in (0,5]$ and use different values of δ and m.

Table 4.9 lists the exact and obtained numerical solutions by use of presented method and method of [44] for the given problem for $m = 5, 10, 15, 20$. Comparison of this results shows that, even for small values of m, the numerical solution obtained by use of given technique is in a better agreement with exact one.

Table 4.9: The resulting values for Example 6, with $R = 5$ in some values of t.

t	Exact	Proposed Method $m=5$	Method given in [44] $m=5$	Proposed Method $m=10$	Method given in [44] $m=10$	Proposed Method $m=15$	Method given in [44] $m=15$	Proposed Method $m=20$	Method given in [44] $m=20$
1	3.42445	3.42415	2.714336	3.425121	3.426525	3.42376044	3.42496	3.424563	3.424807
2	9.69088	9.670891	8.922571	9.692732	9.696794	9.68896761	9.692754	9.691185	9.691706
3	26.6414	26.60757	24.59981	26.64646	26.65929	26.6362145	26.64683	26.64225	26.64381
4	72.6729	72.53849	65.78029	72.68665	72.72038	72.6587861	72.68787	72.6752	72.67936
5	197.77	197.5757	180.1481	197.8077	197.8994	197.731934	197.8112	197.7766	197.7879

In Fig. 4.15.(a)–Fig. 4.17.(a), we present the graphical representation of comparison between exact solution and the numerical solutions obtained by using given method and the method of [44] with taking $m = 5, 10, 20$ respectively. Also in Fig. 4.15.(b)–Fig. 4.17.(b) we show the behaviour of absolute errors obtained by given method and the method of [44] in the interval $[0,1]$ with $m = 5, 10, 20$.

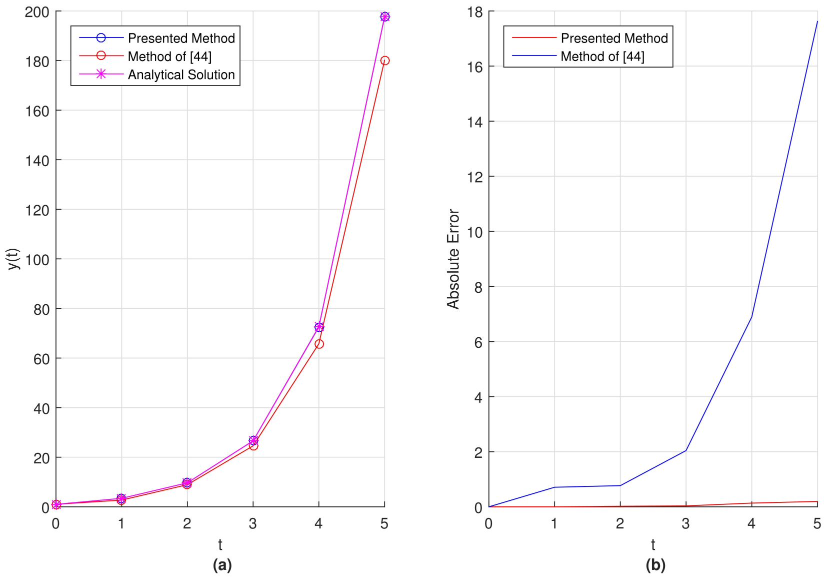

Figure 4.15: (**a**) The comparison of analytical solution and approximate solutions obtained by the given technique and the method of [44] with $m = 5$. (**b**) The behaviour of the absolute errors between the exact solution and numerical solutions obtained by our method and the method given in [44] with $m = 5$.

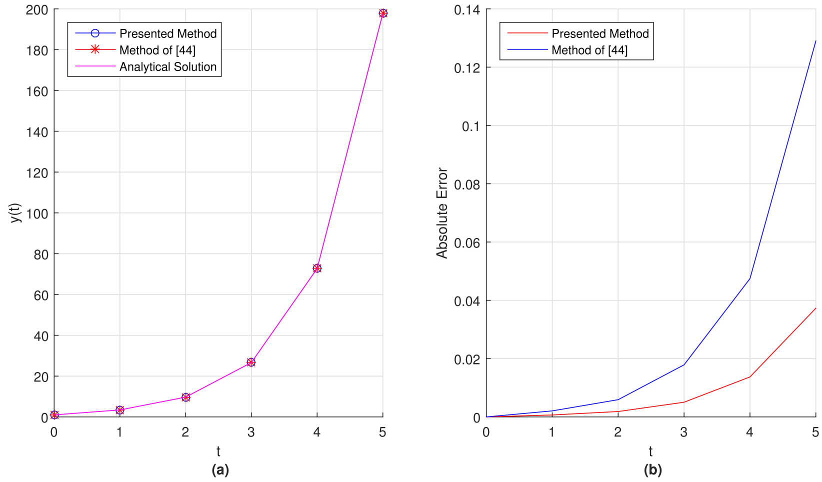

Figure 4.16: (**a**) The comparison of analytical solution and approximate solutions obtained by the given technique and the method of [44] with $m = 10$. (**b**) The behaviour of the absolute errors between the exact solution and numerical solutions obtained by our method and the method given in [44] with $m = 10$.

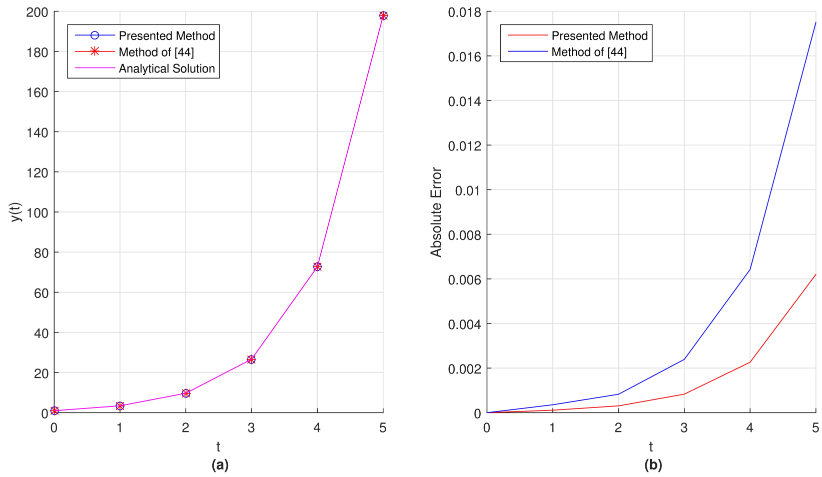

Figure 4.17: (**a**) The comparison of analytical solution and approximate solutions obtained by the given technique and the method of [44] with $m = 20$. (**b**) The behaviour of the absolute errors between the exact solution and numerical solutions obtained by our method and the method given in [44] with $m = 20$.

From these graphical results represented in Fig. 15–Fig. 17, we can conclude that the absolute error obtained by our method is remaining smaller when compared the absolute error of method given in Ref. [44].

4.6.7 Example 7

In this example, we consider a fractional linear differential equation involving two fractional derivative operator with non-homogeneous initial condition [70]

$$\begin{aligned} D_{0,t}^{2\alpha}y(t)+\frac{3}{2}D_{0,t}^{\alpha}y(t) &= -\frac{1}{2}y(t),\ t\in(0,R],\ R>0 \\ y(0) &= 1 \end{aligned} \tag{4.23}$$

The analytical solution is given by

$$y(t)=2E_{\alpha}(-t^{\alpha}/2)-E_{\alpha}(-t^{\alpha})$$

Here, $E_{\alpha}(t)$ denotes the ML function with one parameter.

We solve this problem for $\alpha=0.5$ and $R=1,10$. The approximate results obtained by using presented technique with $\delta=1/2$, $R=1$ and step size $m=2,3,5,15,25$ are presented in Table 4.10. Table 4.11 shows the relative error (%) results in percentage values. In Figure 4.18, we show the graphical comparison of numerical and exact solutions for $t\in(0,10]$. The graphical representation of obtained absolute errors for $m=2,5,15,25$ and $t\in(0,1]$ are presented in Figures 4.19-4.22 respectively. Also in Figures 4.23-4.26, we give graphical comparison of numerical solution and exact solution for $m=2,5,15,25$ and $t\in(0,1]$ respectively.

Table 4.10: Absolute Errors of Example 7 for $R = 1$ with $m = 2, 3, 5, 15, 25$.

t	$m = 2$	$m = 3$	$m = 5$	$m = 15$	$m = 25$
0	0	0	0	0	0
0.1	0.0089091	0.0017007	0.0000187	9.396e-12	3.973e-12
0.2	0.0048821	0.0002407	0.0000193	4.884e-12	2.138e-12
0.3	0.0014756	0.0004188	0.0000142	3.294e-12	1.439e-12
0.4	0.0009667	0.0005774	0.0000072	2.452e-12	1.068e-12
0.5	0.0024974	0.0004673	0.0000048	1.927e-12	8.41e-13
0.6	0.0032202	0.0002495	0.000005	1.573e-12	6.84e-13
0.7	0.0032345	0.0000381	0.0000048	1.315e-12	5.72e-13
0.8	0.0026266	0.0000842	0.0000032	1.121e-12	4.87e-13
0.9	0.0014694	0.000056	0.0000013	9.71e-13	4.22e-13
1.0	0.0001758	0.0001694	0.0000031	8.51e-13	3.67e-13

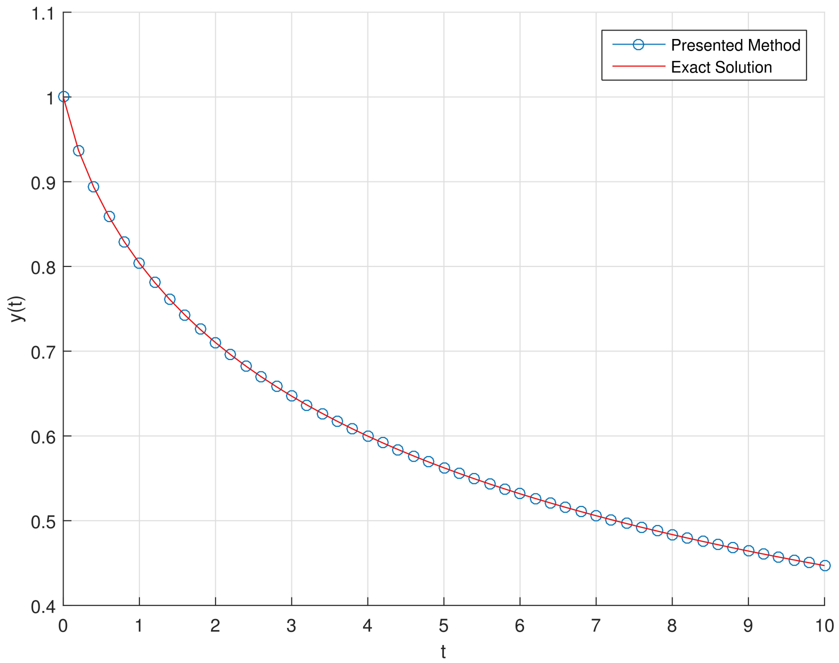

Figure 4.18: The behaviour of the numerical solution and exact solution for Example 7 with $R = 10$.

Table 4.11: Relative Errors (%) of Example 7 for $R = 1$ with $m = 2,3,5,15,25$.

t	m=2	m=3	m=5	m=15	m=25
0	0	0	0	0	0
0.1	0.92396	0.17638	0.00194	9.74494e-10	4.12036e-10
0.2	0.52105	0.02569	0.00206	5.21303e-10	2.28214e-10
0.3	0.16147	0.04583	0.00156	3.60427e-10	1.57459e-10
0.4	0.10818	0.06462	0.00081	2.74378e-10	1.19539e-10
0.5	0.28532	0.05338	0.00055	2.20175e-10	9.5977e-11
0.6	0.375	0.02905	0.00058	1.83136e-10	7.9693e-11
0.7	0.38348	0.00452	0.00057	1.55872e-10	6.7761e-11
0.8	0.31672	0.01015	0.00038	1.35142e-10	5.8715e-11
0.9	0.18005	0.00686	0.00016	1.18905e-10	5.1665e-11
1.0	0.02187	0.02107	0.00038	1.05871e-10	4.5663e-11

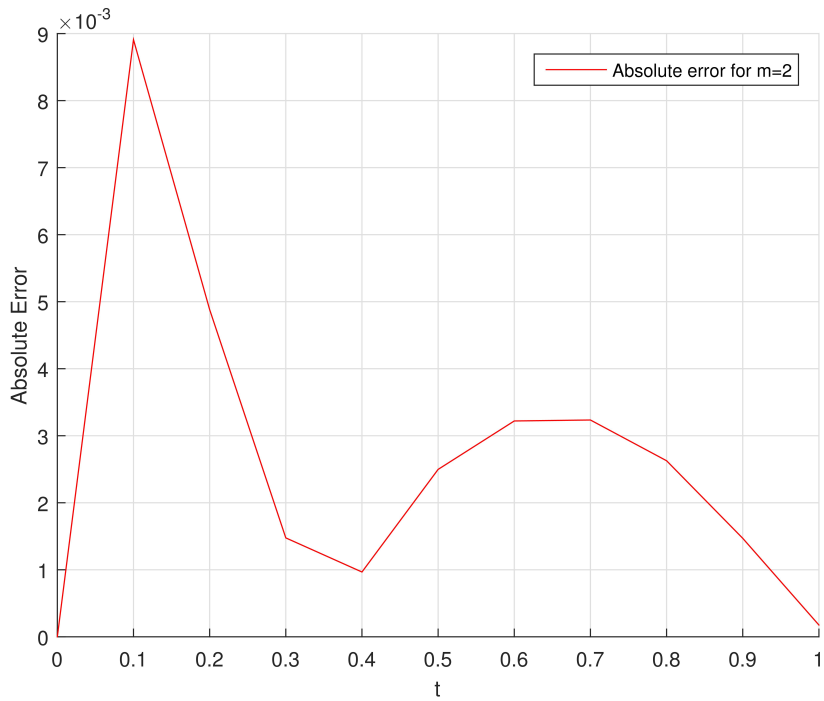

Figure 4.19: Graphical results of absolute errors for Example 7 with $m = 2$.

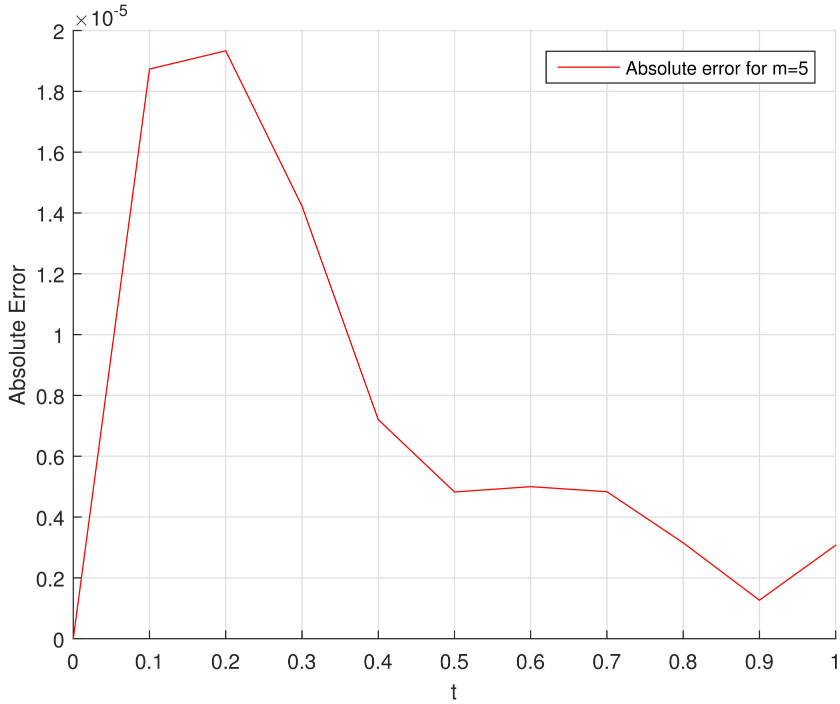

Figure 4.20: Graphical results of absolute errors for Example 7 with $m = 5$.

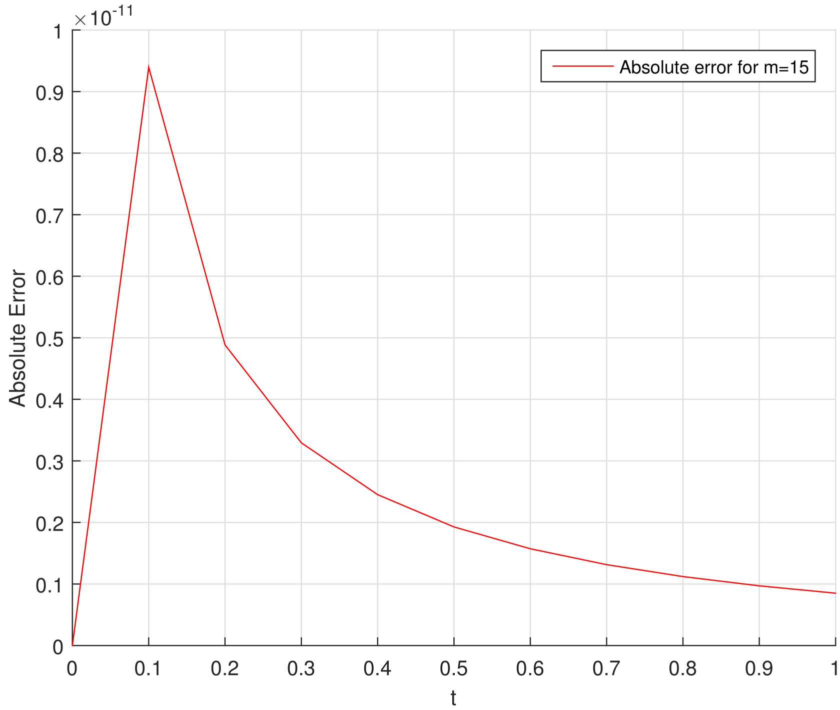

Figure 4.21: Graphical results of absolute errors for Example 7 with $m = 15$.

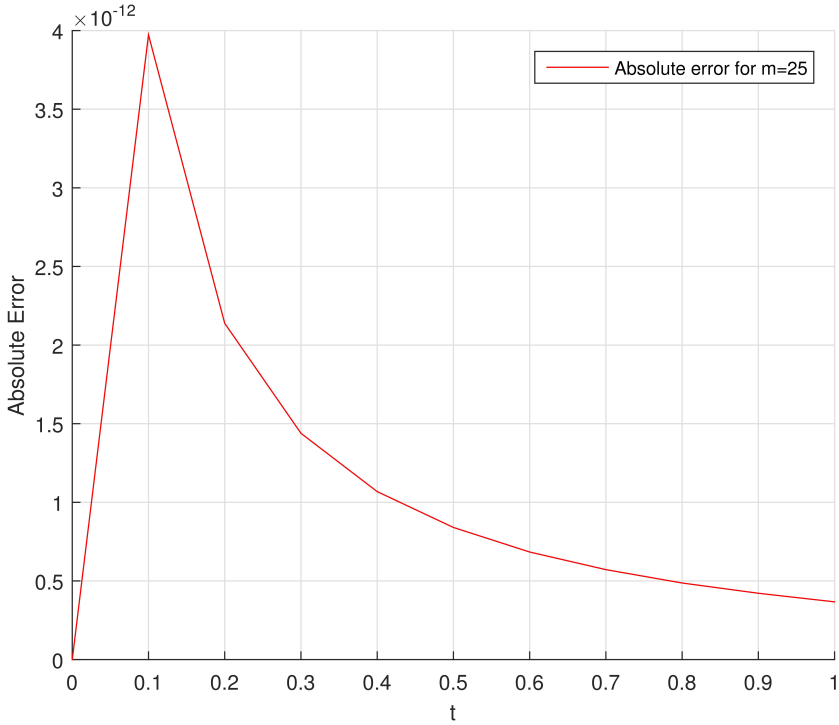

Figure 4.22: Graphical results of absolute errors for Example 7 with $m = 25$.

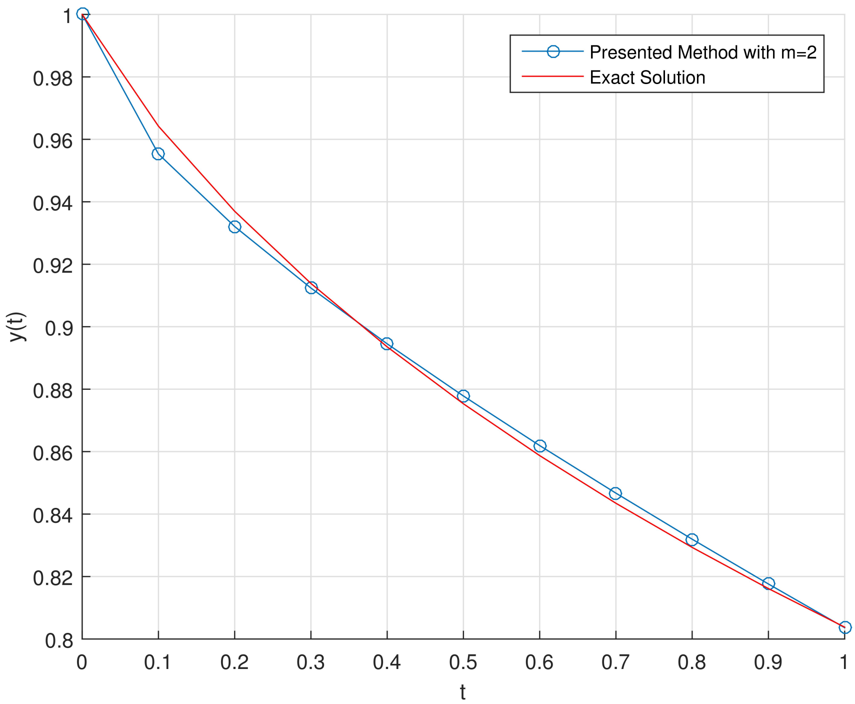

Figure 4.23: The behaviour of numerical solution and exact solution for Example 7 with $m = 2$.

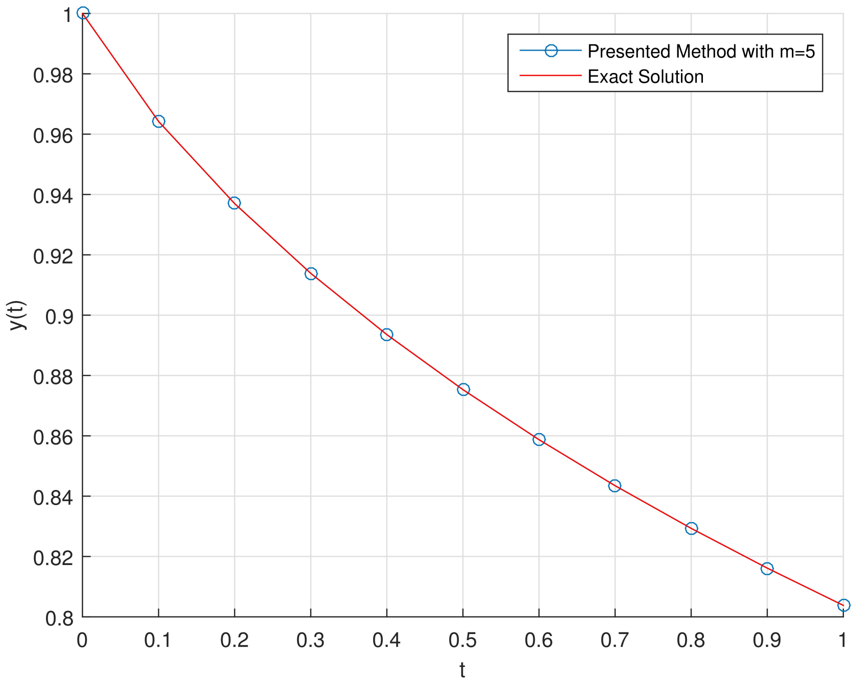

Figure 4.24: The behaviour of numerical solution and exact solution for Example 7 with $m = 5$.

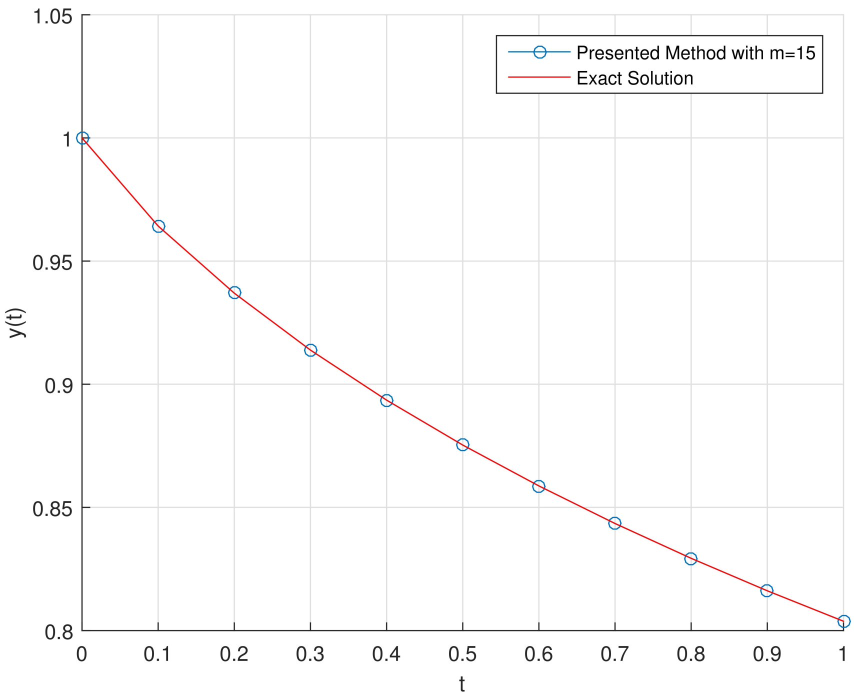

Figure 4.25: The behaviour of numerical solution and exact solution for Example 7 with $m = 15$.

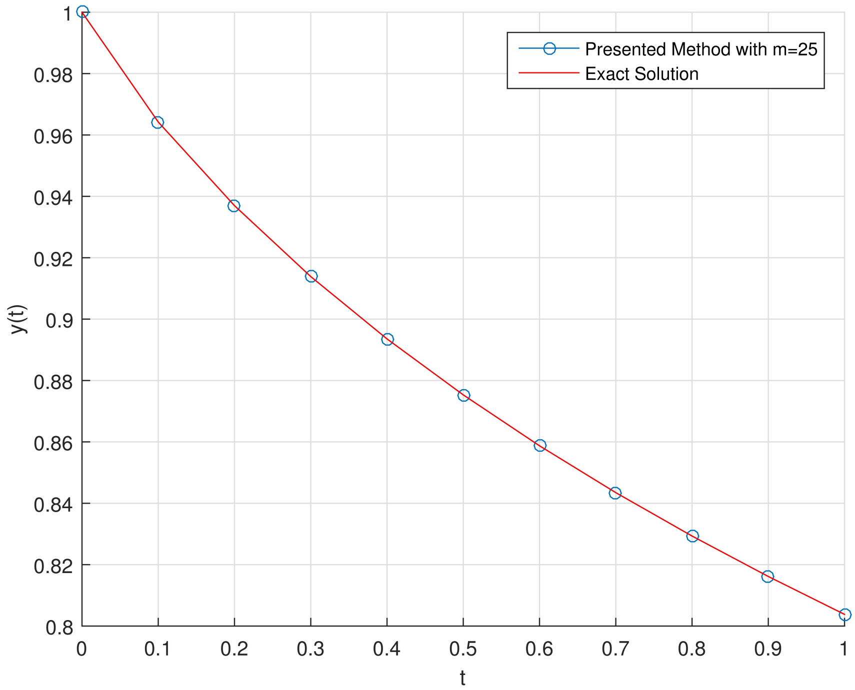

Figure 4.26: The behaviour of numerical solution and exact solution for Example 7 with $m = 25$.

4.6.8 Example 8

Consider the equation [34]

$$D^{\alpha}y(t) + y(t) = t^4 - \frac{1}{2}t^3 - \frac{3}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)}t^{4-\alpha}$$

with

$$y(0) = 0$$

The exact solution is given by

$$y(t) = t^4 - \frac{1}{2}t^3.$$

We take $\alpha = 0.5$ and applying presented method to this problem with a choose of $\delta = 1/2$ and $m = 11$.

In Table 4.12, the results for absolute errors obtained by using given technique and the method of [34] are presented. From this table, if we compare the given error values, we can conclude that the presented method gives better results for solving the given problem for the same step-size. In Figure 4.27, we give the graphical representation for the behaviour of numerical and exact solutions for the given problem with taking $m = 10$. From this figure, we can see that the numerical solutions and exact solutions are in a very good agreement.

Table 4.12: Results for absolute errors of Example 8 for $R = 1$ with $m = 10$.

t	Present Method $m = 10$	Method in [34] with $m = 10$
0	0	0
0.1	4.15e-14	4.78e-14
0.2	2.1e-14	9.78e-14
0.3	1.46e-14	1.19e-13
0.4	1.11e-14	1.26e-13
0.5	8.9e-15	1.46e-13
0.6	7.6e-15	1.68e-13
0.7	6.3e-15	1.66e-13
0.8	5.4e-15	1.52e-13
0.9	5.2e-15	1.84e-13

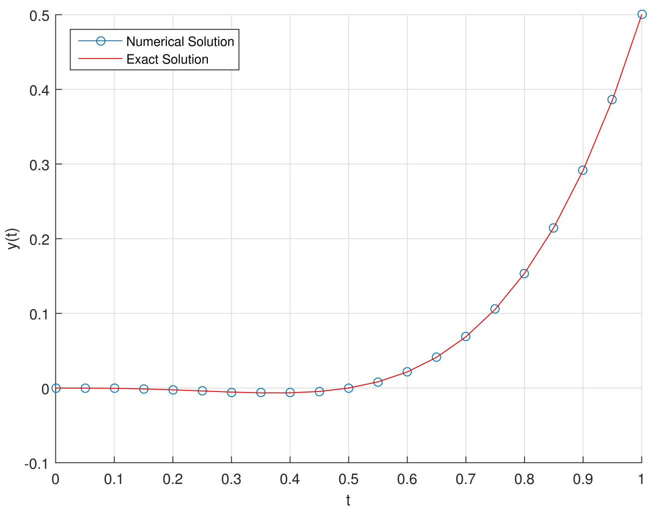

Figure 4.27: The behaviour of numerical and exact solutions for Example 8 with $m = 10$.

4.6.9 Example 9

Now, we consider the FDE with two fractional derivative operator as given in following [7]

$$D^{1.8}y(t)+0.5D^{0.5}y(t)+y(t)=p(t),\ t\in(0,R],\ R>0$$

with

$$y(0)=1, y'(0)=2.$$

Let $p(t)=2.1782t^{1/5}+1.1284t^{1/2}+0.75225t^{3/2}+(1+t)^2$. The exact solution is given by $y(t)=(1+t)^2$.

In order to solve this problem by presented technique, we choose $\delta=1$, $m=2$ and $R=10,20$.

In Figures 4.28 and 4.29, the graphical representation of absolute errors obtained by use of presented method for $t\in(0,10]$ and $t\in(0,20]$ are given respectively. From these graphs, we can conclude that the given technique gives very good results even for a small step-size $m=2$.

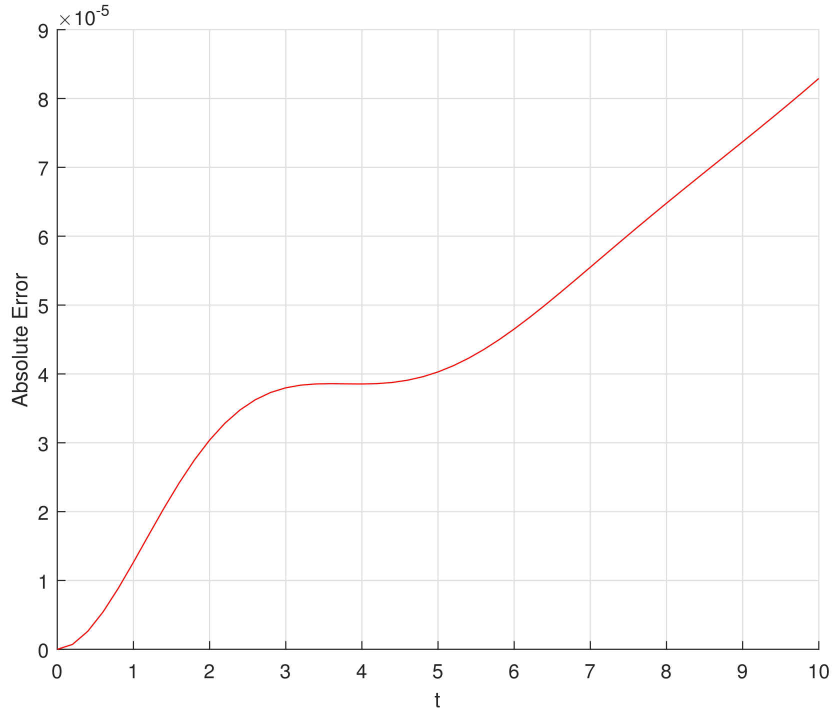

Figure 4.28: Absolute error for Example 9 in the interval $t \in (0, 10]$.

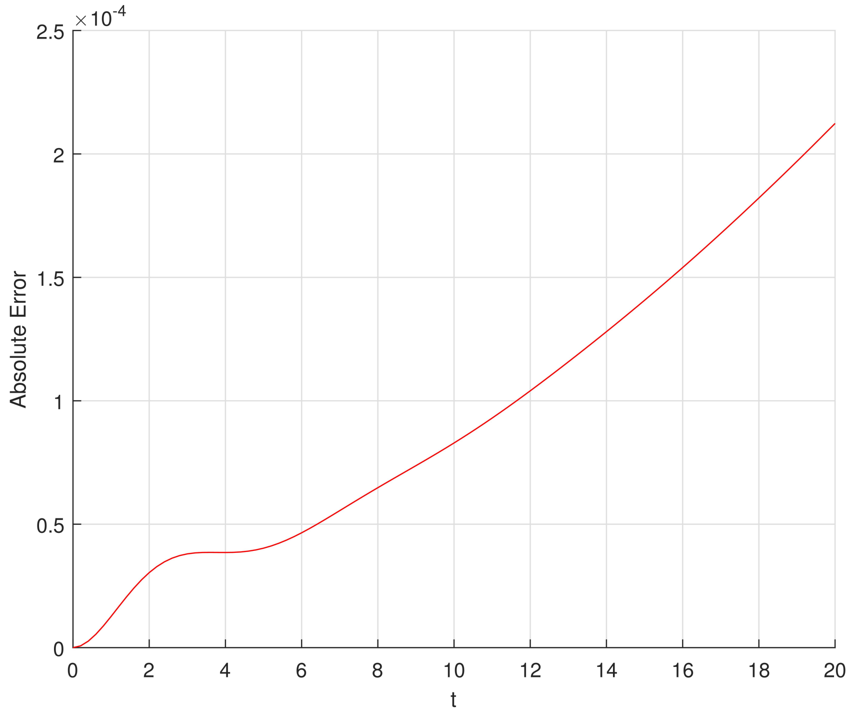

Figure 4.29: Absolute error for Example 9 in the interval $t \in (0, 20]$.

Figures 4.30 and 4.31 shows the graphical representation for the behaviour of exact solution and approximate solution obtained by use of given technique for given problem in the intervals $t \in (0, 10]$ and $t \in (0, 20]$ respectively. These graphical results shows that the approximate solution is remain stable for different values of R and in a very good agreement with exact solution.

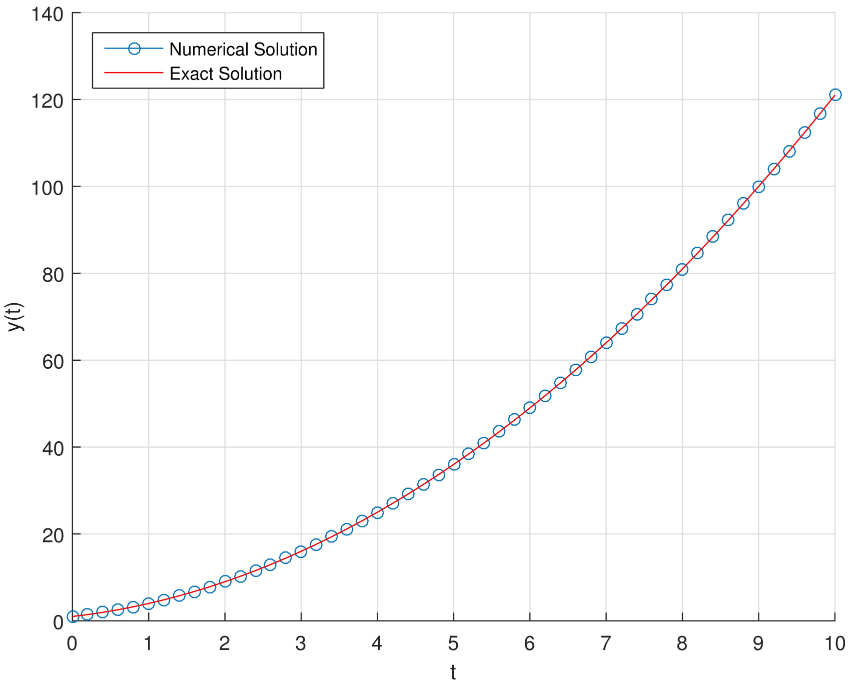

Figure 4.30: The behaviour of exact and approximate solutions for $t \in (0, 10]$.

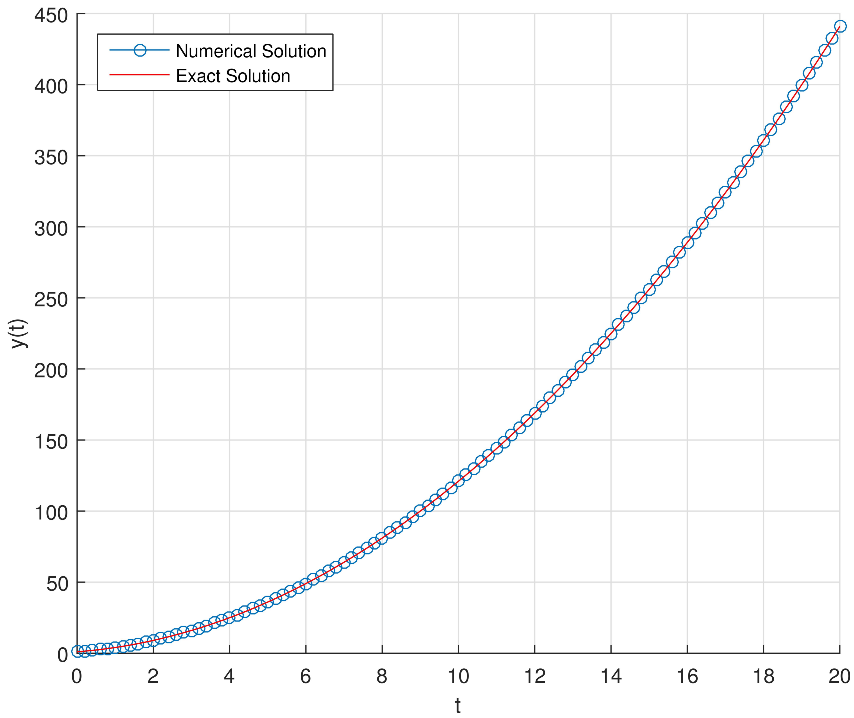

Figure 4.31: The behaviour of exact and approximate solutions for $t \in (0, 20]$

Chapter 5

CONCLUSION

During the past decades, multi-term FDEs has found many crucial application in many branches of applied science and engineering. Thus, their solutions becomes more and more important. In this study, we have focused on approximating the solution of such a equations.

In this thesis, an operational matrix based on the fractional Taylor vector is used to solve the multi-term FDEs numerically by reducing them to a set of linear algebraic equations, which simplifies the problem. From comparison of the obtained results with exact solutions and also with results of other techniques in the literature, we conclude that the given method provides the solution with high accuracy. The findings also show that, even for the small number of steps, we can get satisfactory results by using presented method. All computational results are obtained by using MATLAB.

In presented method, constructing the operational matrix without any approximation except the unknown function is an important benefit of using Taylor polynomials and this is a crucial reason for better results. Also, fractional derivative of Taylor polynomials can be evaluated easily and the use of these polynomials also provides ease to approximate the functions.

It is important to highlight that the MATLAB program that used to calculate the computational problems, was particularly designed for given problem. However, the general algorithm may be used for any problem of similar structure.

Concerning the difficulty of solving FDEs analytically, the presented method can have important contributions to the field of numerical techniques by having very efficient results and applicability even for a very less step size.

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APPENDIX

Pseudo Code for the Numerical Algorithm

The pseudo code given below allows us to use proposed method in MATLAB to get a numerical solution of a given problem. [52]

Algorithm 2: Fractional Taylor Method

$[A,b]$ = $fractionalTaylor(alpha, beta, Uk, func, t0, R, y0, m, delta)$

% Input variables
% $alpha$ is the highest order of fractional derivative of given equation
% $beta$ is the order of fractional derivatives other than alpha. $beta$ must be a vector with descending ordered values
% Uk is the vector of coefficients
% $func$ is defining the right hand side of given problem
% $t0$ and R denotes the left and right endpoints
% $y0$ is the initial conditions
% m denotes the number of steps
% $delta$ is a real number greater than zero. We usually take $delta = 1$ or $delta =$fractional part of $alpha$

% Output variables
% A is an $(m+1)$ x $(m+1)$ matrix
% b is an $(m+1)$ x 1 matrix
% using $fractionalTaylor.m$, where command $fractionalTaylor.m$ is defined by the Equation (4.10), gives us the linear system $AC = B$ which is $(m+1)$
% algebraic equations with unknown coefficients C^T
% Next step is to use matlab function $linsolve(A,b)$ to solve obtained algebraic equation for unknown coefficient vector C^T with dimension $(m+1)$.
$C = linsolve(A,b)$

% Output variables
% C is an $(m+1)$ x 1 matrix which is the solution of linear system $AC = B$
% Next step is substituting obtained coefficients to $approxSoln()$ as input, where the command $approxSoln()$ defined by Equation (4.7), we get the approximate solution of given problem
$[s,y]$ = $approxSoln(C)$

% Input variables
% C is the vector of coefficients obtained in previous step.

% Output variables
% s is the nodes on $[t0,R]$ in which the approximate solution calculated
% y is the numerical solution evaluated in the points of s.
