PDM-Charged Quantum Particles Moving in Magnetic Fields

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ABSTRACT

The classical and quantum mechanical correspondence for constant mass settings is used, along with some nonlocal point transformation (NPT), to find the position-dependent mass (PDM) classical and quantum Hamiltonians. Consequently, the PDM-momentum operator is constructed. The same recipe is followed to identify the form of the PDM-minimal coupling of electromagnetic interactions for the classical and quantum PDM-Hamiltonians.

Using azimuthally symmetrized cylindrical coordinates, some PDM-charged particles moving in magnetic (constant or position-dependent (PD)) and Aharonov-Bohm (AB) flux fields along with some interaction potentials are considered. Their separability and solvability under PDM-settings is also reported. Systems of PDM-charged particles moving in three fields: constant magnetic, AB-flux, and pseudoharmonic oscillator potential, or generalized Killingbeck potential fields are solved for different radial cylindrical PDM settings. Spectral signatures of the one-dimensional z-dependent Schrödinger part on the overall eigenvalues and eigenfunctions, are reported using two z-dependent potential models (infinite potential well and Morse-type potentials). PDM-charged particles moving in an inverse power-law-type radial PD-magnetic fields are considered. Under such settings, the exact solutions of *almost-quasi-free* PDM-charged particles (i.e., no interaction potential) endowed with two type of radial cylindrical PDM settings are obtained. Furthermore, a Yukawa-type PDM-charged particle with a specific PDM setting moving in PD-magnetic and AB-flux fields along with Yukawa plus Kratzer type potential force fields is analyzed (using the Nikiforov-Uvarov (NU) method) to come out with exact solutions of the

system. Exact or conditionally exact eigenvalues and eigenfunctions are analytically obtained.

Keywords: position-dependent mass Hamiltonian, point transformation, PDM - momentum operator, PDM minimal-coupling, cylindrical coordinates, constant magnetic and position-dependent magnetic fields, Aharonov-Bohm flux field, almost-quasi-free PDM-charged particles, pseudo-harmonic osillator and Killingbeck potentials, Yukawa-plus-Kratzer potential, Nikiforov-Uvarov exact solvability.

ÖZ

Klasik ve Kuantum Mekaniksel tekâbuliyet ve Yerel Olmayan Nokta Dönüşüm (YOND) yöntemi kullanarak Pozisyon Bağımlı Kütle (PBK) Hamilton ve momentum operatörleri tespit edilmiştir. Bu yöntemle minimal kuplajlı elektromagnetik etkileşmeler için Klasik ve Kuantum PBK-Hamilton fonksiyonları bulunmuştur.

Eksensel simetrik silindirik koordinatlar kullanarak şarjlı partiküllerin, sabit (veya pozisyon bağımlı (PB)), magnetik Aharanov–Bohm (AB) akısı ve etkileşim potansiyelleri içindeki hareketleri ele alınmıştır. PBK durumunda hareket denklemleri ayrım ve çözüm açısından incelenmiştir. Üç durum ele alınmıştır: Sabit magnetik alan, AB-akısı, sahte harmonik hareket potansiyeli veya genellenmiş Killingbeck potansiyel alanlarında radyal silindirik çözümler verilmiştir. Tek boyutlu, z–bağımlı Schrödinger denkleminin spektral işaretlerinden, iki farklı potansiyel (sonsuz kuyu ve Morse–gibi) için uygun değer ve fonksiyonları elde edilmiştir. Burada PBK partikülleri için ters–üstel etkileşim alanları durumunda radyal PB-magnetik alan ele alınmıştır. Yaklaşık, sahte serbest görünümlü, şarjlı PBK durumunda kesin silindirik çözümler verilmiştir. İlâveten Yukawa tipi şarjlı PBK, PB–magnetik, AB–akılı ve Kratzer tipi potansiyel katkılı alanlar için Nikiforov–Uvarov (NU) yöntemi sayesinde çözümler bulunmuştur. Kesin veya şartlı uygun değer fonksiyonları elde edilmiştir.

Anahtar Kelimeler: pozisyon bağımlı kütle Hamilton fonksiyonu, momentum operatörü, minimal kuplaj, silindirik koordinatlar, sabit ve pozisyon bağımlı magnetik alan, Aharanov – Bohm akı alanı, yaklaşık serbest pozisyon bağımlı şarj partikül,

sahte–harmonik titreşim, Killingbech, Yukawa – Kratzer potansiyelleri, Nikiforov – Uvarov kesin çözümlülüğü.

... Dedicated to my family

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LIST OF ABBREVIATIONS

1D	One Dimensional
AB	Aharonov-Bohm
CC	Cylindrical Coordinates
NPT	Nonlocal Point Transformation
NU	Nikiforov-Uvarov
PD	Position-Dependent
PDM	Position-Dependent Mass
PT	Point Transformation

Chapter 1

INTRODUCTION

In the non-relativistic Schrödinger equation, the position-dependent mass (PDM) concept have attracted much attention in the literature over the years [1–26]. Particles endowed with PDM are considered interesting and unavoidable in both quantum and classical mechanics [27–33]. Such PDM settings find their applications in condensed matter physics (see, e.g., [9, 15, 16]), in optical physics (see, e.g., [34, 35]), etc. They are not to be necessarily understood as particles with PDM literally. A position-dependent deformation in the coordinate system may very well render the mass position-dependent. Hereby, the most prominent non-relativistic PDM-Hamiltonian is the von Roos Hamiltonian [1]

$$\hat{H} = -\frac{1}{4}\left[M(\vec{x})^{a}\partial_{x_j}M(\vec{x})^{b}\partial_{x_j}M(\vec{x})^{c} + M(\vec{x})^{c}\partial_{x_j}M(\vec{x})^{b}\partial_{x_j}M(\vec{x})^{a}\right] + V(\vec{x}). \quad (1.1)$$

Where $M(\vec{x}) = m_\circ m(\vec{x})$, $m_\circ$ is the textbook constant mass, $m(\vec{x})$ is a position-dependent dimensionless scalar multiplier that forms the position-dependent mass $M(\vec{x})$, $\vec{x} = (x_1, x_2, x_3)$, $\partial_{x_j} = \partial/\partial x_j$, $j = 1, 2, 3$, $V(\vec{x})$ is the potential force field, and the summation runs over repeated indices, unless otherwise mentioned. The parameters a, b, c are called the ambiguity parameters that satisfy von Roos constraint $a + b + c = -1$. Yet, this Hamiltonian is known to be associated with an ordering ambiguity problem as a result of the non-unique representation of the kinetic energy operator. An obvious radical change in the profile of the kinetic energy term occurs when the values of the ambiguity parameters are changed (consequently, the profile of the effective potential will radically change). There exist an infinite number of ambiguity parametric settings that satisfy the von Roos constraint above. In the literature, however, one may find many suggestions on the ambiguity parametric values [2–14]. Yet, the only physically acceptable condition (along with the von Roos constraint) on the ambiguity parameters is that $a = c$ to ensure continuity at the abrupt heterojunction (e.g., Refs. [15, 16]). The rest are based on different eligibility proposals which are, at least, mathematically challenging and useful models that enrich the class of exactly solvable or conditionally exactly solvable quantum mechanical systems [17–26].

However, no attempts have ever been made to construct and identify the PDM-momentum operator. Only very recently, we have reported the detailed construction of the PDM-momentum operator [36], and fixed the ambiguity parameters at $a = c = -1/4$ and $b = -1/2$. Such parametric ordering was suggested, early on, by Mustafa and Mazharimousavi [13], as a consequence of some factorization method. Nevertheless, it should be noted that PDM-Hamiltonian (1.1) would, in a straightforward manner, imply a time-independent PDM Schrödinger equation of the form (in $\hbar = 2m_\circ = 1$ units)

$$\left\{ -\frac{1}{m(\vec{x})}\partial_{x_j}^2 + \left[\frac{\partial_{x_j} m(\vec{x})}{m(\vec{x})^2}\right]\partial_{x_j} - [a(a+b+1)+b+1]\left(\frac{\left[\partial_{x_j} m(\vec{x})\right]^2}{m(\vec{x})^3}\right) + \frac{1}{2}(1+b)\left[\frac{\partial_{x_j}^2 m(\vec{x})}{m(\vec{x})^2}\right] + V(\vec{x}) \right\}\phi(\vec{x}) = E\phi(\vec{x}). \quad (1.2)$$

This equation plays a critical role in the determination of the ambiguity parameters and consequently in the construction of the PDM-momentum operator as well as in the identification of the minimal coupling of electromagnetic interactions. This is done in chapter 2.

Moreover, quantum mechanical constant mass charged particles moving in a uniform magnetic and/or an Aharonov-Bohm (AB) flux fields have been a subject of research interest over the years (e.g., see the sample of references [37–44]). On the classical mechanical and mathematical side of the problem, it is crucial to know that the canonical momentum is no longer the mass times velocity but an extra term is added so that $p_i = m_\circ v_i + eA_i$ (where e is the charge of the particle and A_i is the ith component of the vector potential). The problem is readily of a delicate nature, especially when the magnetic field is no longer a constant but rather a position-dependent one. Only a handful number of attempts were made to treat PDM charged particles in uniform magnetic field [23, 25, 26, 44]. Hereby, Eshghi et al. [44] have used Ben Danial and Duke's parametric settings $a = c = 0$ and $b = -1$ (c.f., e.g., [2–4, 22–24]) and considered PDM-charged particles moving in both magnetic and AB-flux fields.

In this dissertation, our objective is to study the effect of the magnetic fields (constant and position-dependent) on PDM-charged particles with and without the confining potentials (including the AB-flux field). We also discuss the separability and exact solvability of these problems. Therefore, the organization of this dissertation is in the respective order.

In chapter 2, we start with the Lagrangian of a classical particle of mass $m_\circ$ moving in a scalar potential field $V(\vec{q})$, in the generalized coordinates $\vec{q} = (q_1, q_2, q_3)$, to build up the classical and consequently the quantum mechanical Hamiltonians. Based on the very recent work on the PDM-nonlocal point transformation by Mustafa [20], we detail out the mapping(s)/connection(s) between the PDM-Schrödinger equation (1.2) and the apparently standard textbook Schrödinger equation for constant mass $m_\circ$ in the generalized coordinates. Once the mapping is made clear, the ordering ambiguity in (1.2) disappears and the parametric setting becomes strictly determined. In section 2.2, we first find the so called PDM pseudo-momentum operator $\widehat{\pi}_j(\vec{q}(\vec{x}))$ and connect it with the PDM-momentum operator through $\widehat{P}_j(\vec{x}) = \sqrt{m(\vec{x})}\widehat{\pi}_j(\vec{q}(\vec{x}))$. Then, we test the eligibility of the commonly used vector potentials and single out $\overrightarrow{A}(\vec{x}) \sim (-x_2, x_1, 0)$ as the only eligible vector potential within our current methodical proposal settings, of course. Two illustrative examples are given which include magnetic and electric fields.

Next, we start, in chapter 3, using the PDM-minimal-coupling recipe by Mustafa [26], along with the PDM-momentum operator obtained in chapter 2, and discuss the separability and solvability of the problem (within the azimuthally symmetric cylindrical coordinates (ρ, φ, z) settings of course). A purely radial (i.e, ρ-coordinate dependent) and a simplistic one-dimensional (1D) z-dependent, Schrödinger equations are obtained. For the radial ρ-dependent part, we choose to follow two different ways. The first of which is to transform it into a 1D-radial Schrödinger form, and discuss its exact solvability using a pseudo-harmonic oscillator potential (often used for quantum dots and antidotes, e.g., [40–42]). In the same section, we report exact eigenvalues and eigenfunctions for two PDM models, $g(\rho) = \eta\rho^2$ and $g(\rho) = \eta/\rho^2$. The second way is to use, the biconfluent Heun differential form (c.f., e.g., [44–47]). However, the implementation the biconfluent Heun equation and its biconfluent Heun solutions necessarily means that these solutions belong to the set of *conditionally exact solutions* for Schrödinger equation. Consequently, we report (in the same section) some *conditionally exact* eigenvalues and eigenfunctions for two PDM models, $g(\rho) = \lambda\rho$ and $g(\rho) = \lambda/\rho^2$. Moreover, the spectral signatures of the eigenvalues of the one-dimensional z-dependent Schrödinger part, on the overall spectra, are also reported for each of the four models used.

In chapter 4, we consider a PDM-charged particle in PD-magnetic and AB-flux fields and discuss the separability and solvability of the PDM-Schrödinger equation. Therein, we use a general form of the vector potential $\overrightarrow{A}(\overrightarrow{r})$ so that a radial PD-magnetic field results in the process (i.e., $\overrightarrow{B} = B_\circ F(\rho)\hat{z}$, where $F(\rho)$ is a dimensionless radial scalar multiplier). Furthermore, we construct our PD-magnetic field in such a way that it is of a feasibly experimentally applicable nature (i.e., inverse power-law type $\overrightarrow{B} = B_\circ(\mu/\rho^\sigma)\hat{z}$) to be used along with a PDM $m(\overrightarrow{r}) = g(\rho) = \eta f(\rho)\exp(-\delta\rho)$ (i.e., the PDM is only radial-dependent). Then, we consider the what may be called *almost-quasi-free* PDM-charged particles (i.e., no other interaction potential than the interaction of the PDM-charged particles with the PD-magnetic and AB-flux fields, where the conventional confinement $V(\overrightarrow{r}) = 0$) endowed with two unavoidable exactly solvable PDM models $g(\rho) = \eta/\rho$ and $g(\rho) = \eta/\rho^2$. A PDM-charged particle, with $m(\overrightarrow{r}) = g(\rho) = \eta\exp(-\delta\rho)/\rho$; $f(\rho) = 1/\rho$, interacting with a PD-magnetic plus AB-flux fields and moving in a Yukawa plus a Kratzer type potential field $V(\rho) = -V_\circ\exp(-\delta\rho)/\rho - V_1/\rho + V_2/\rho^2$ is considered. Hereby, the Nikiforov-Uvarov (NU) method (e.g., [6, 48–50]) is used to obtain exact eigenvalue and eigenfunctions.

For the sample of examples mentioned above, in both chapters 3 and 4, we have studied the effects of all parametric settings involved in the PDM, constant and PD-magnetic fields, and/or interaction potential on the spectra. We have observed that energy levels crossings (that may very well be considered as *occasional degeneracies* at some specific parametric settings) are unavoidable in the process. Such energy levels crossings are the signatures of the PDM settings. We report our concluding remarks in chapter 5.

Chapter 2

PDM-MOMENTUM OPERATOR AND MINIMAL COUPLING: NPT AND ISOSPECTRALITY

In order to facilitate exact solvability for PDM charged particles in electromagnetic fields, we use, in this chapter, some nonlocal point transformation (NPT) that maps the PDM Hamiltonian into conventional constant mass setting. In so doing, the exact solutions for PDM systems are inferred from those for conventional constant mass systems.

2.1 NPT and Classical-Quantum Correspondence

In this section, we start with a system of classical particle of a constant mass $m_\circ$ moving in a potential field $V(\vec{q})$, where $\vec{q} = (q_1, q_2, q_3) = q_1\widehat{q}_1 + q_2\widehat{q}_2 + q_3\widehat{q}_3$ are the generalized coordinates. For such a system, the corresponding Lagrangian reads

$$L\left(q_j, \tilde{q}_j; \tau\right) = \frac{1}{2} m_\circ \tilde{q}_j^2 - V(\vec{q}) \ ; \ \ \tilde{q}_j = \frac{dq_j}{d\tau}; \ \ j = 1,2,3. \tag{2.1}$$

where τ is a re-scaled time [20] and $L\left(q_j, \tilde{q}_j; \tau\right) = L(q_1, q_2, q_3, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3; \tau)$ is to be used for economy of notations. Under such settings, the classical Hamiltonian reads

$$H\left(q_j, P_j; \tau\right) = \tilde{q}_j P_j - L\left(q_j, \tilde{q}_j; \tau\right) = \frac{1}{2} m_\circ \tilde{q}_j^2 + V(\vec{q}), \tag{2.2}$$

and represents a constant of motion where $dH\left(q_j, P_j; \tau\right)/d\tau = 0$ (c.f., e.g., Mustafa [20]). Here, the jth component of the canonical momentum (associated with the generalized coordinate q_j)

$$P_j = \frac{\partial}{\partial \tilde{q}_j} L\left(q_j, \tilde{q}_j; \tau\right) \Longrightarrow P_j = m_\circ \tilde{q}_j \,, \tag{2.3}$$

is used. The Hamiltonian, however, is often interpreted as a function of position q_j and canonical momentum P_j. It is more appropriate, therefore, to re-cast the classical Hamiltonian (2.2) as

$$H\left(q_j, P_j; \tau\right) = \frac{P_j^2}{2m_\circ} + V(\vec{q}). \tag{2.4}$$

Hence, the corresponding quantum mechanical Hamiltonian is obtained by the identification of the jth canonical momentum P_j with the operator $\widehat{P}_j = -i\partial/\partial q_j = -i\partial_{q_j}$, that satisfies the canonical commutation relations $\left[q_i, \widehat{P}_j\right] = -i\left[q_i, \partial_{q_j}\right] = i\delta_{ij}$ and consequently yields (with $\hbar = 2m_\circ = c = 1$)

$$\widehat{H}\left(q_j, P_j; \tau\right) = -\partial_{q_j}^2 + V(\vec{q}). \tag{2.5}$$

Then, the related time-independent Schrödinger equation is given by

$$\left\{-\partial_{q_j}^2 + V(\vec{q})\right\}\psi(\vec{q}) = \Lambda\psi(\vec{q}) \,, \tag{2.6}$$

where Λ denotes eigenvalues and $\psi(\vec{q})$ eignfunctions. At this very point, we would like to figure out the mapping(s)/connection(s) between the quantum mechanical PDM-Schrödinger equation(1.2) and the apparently standard textbook Schrödinger equation for constant mass in (2.6). To do that, we use the NPT suggested by Mustafa [20] and define

$$dq_i = \delta_{ij}\sqrt{g(\vec{x})}\,dx_j = \sqrt{g(\vec{x})}\,dx_i \Longrightarrow \frac{\partial q_i}{\partial x_j} = \delta_{ij}\sqrt{g(\vec{x})}$$
$$\Longrightarrow q_j = \int \sqrt{g(\vec{x})}\,dx_j, \;\; d\tau = f(\vec{x})\,dt. \tag{2.7}$$

No summation over repeated index holds in (2.7). Therefore, this type of transformation necessarily means that the differential change in q_j would result

$$dq_j = \sqrt{g(\vec{x})}\,dx_j \Longrightarrow \;\tilde{q}_j = \frac{\sqrt{g(\vec{x})}}{f(\vec{x})}\dot{x}_j\,;\; \dot{x}_j = \frac{dx_j}{dt},$$

Consequently, the unit vectors in the direction of q_i are obtained as

$$\widehat{q}_i = \sqrt{g(\vec{x})}\left[\left(\frac{\partial x_1}{\partial q_j}\right)\widehat{x}_1 + \left(\frac{\partial x_2}{\partial q_j}\right)\widehat{x}_2 + \left(\frac{\partial x_3}{\partial q_j}\right)\widehat{x}_3\right] \Longrightarrow \widehat{q}_i = \widehat{x}_i. \tag{2.8}$$

Then, with the substitution of

$$\psi(\vec{q}) = g(\vec{x})^{\upsilon}\,\phi(\vec{x}), \tag{2.9}$$

the corresponding time-independent Schrödinger equation (2.6) would yield

$$\left\{-\frac{1}{g(\vec{x})}\partial_{x_j}^2 - \left(2\upsilon - \frac{1}{2}\right)\left(\frac{\partial_{x_j} g(\vec{x})}{g(\vec{x})^2}\right)\partial_{x_j} - \upsilon\left(\upsilon - \frac{3}{2}\right)\left(\frac{\left[\partial_{x_j} g(\vec{x})\right]^2}{g(\vec{x})^3}\right)\right.$$
$$\left. -\upsilon\left(\frac{\partial_{x_j}^2 g(\vec{x})}{g(\vec{x})^2}\right) + V(\vec{q}(\vec{x}))\right\}\phi(\vec{x}) = \Lambda\phi(\vec{x}). \tag{2.10}$$

Here, we shall be interested in the quantum mechanical systems in (2.6) that are exactly solvable, conditionally exactly solvable, or quasi-exactly solvable to reflect on the solvability of a given PDM system as that in (2.10). Therefore, the eigenvalues Λ of (2.10) should be not only position-independent but also isospectral to E of (1.2), i.e., $E = \Lambda$. Under such conditions, one immediately concludes that $f(\vec{x}) = 1 \Longrightarrow$

$\tau = t$ and $g(\vec{x}) = m(\vec{x})$ to keep the total energy position-independent and ensures isospectrality between (1.2) and(2.6), Now, we compare the second term of (1.2) with second term of (2.10) to imply that

$$2\upsilon - \frac{1}{2} = -1 \Longrightarrow \upsilon = -\frac{1}{4} \Longrightarrow \psi(\vec{q}) = m(\vec{x})^{-1/4}\phi(\vec{x}). \tag{2.11}$$

Hence, equation (2.10) reduces to

$$\left\{ -\frac{1}{m(\vec{x})}\partial_{x_j}^2 + \left[\frac{\partial_{x_j} m(\vec{x})}{m(\vec{x})^2}\right]\partial_{x_j} - \frac{7}{16}\left(\frac{\left[\partial_{x_j} m(\vec{x})\right]^2}{m(\vec{x})^3}\right) + \frac{1}{4}\left[\frac{\partial_{x_j}^2 m(\vec{x})}{m(\vec{x})^2}\right] + V(\vec{q}(\vec{x}))\right\}\phi(\vec{x}) = E\phi(\vec{x}). \tag{2.12}$$

where $V(\vec{q}(\vec{x})) = V(\vec{x}(\overrightarrow{q}))$ of (1.2) (which would, in the process, determine the form of $V(\vec{x}(\overrightarrow{q}))$ for a given $V(\vec{q}(\vec{x}))$ and vice versa). Consequently, one obtains the identities

$$a(a+b+1)+b+1 = \frac{7}{16},\ \frac{1}{2}(1+b) = \frac{1}{4}. \tag{2.13}$$

Equations (2.6) and (2.12) are isospectral. Yet, the comparison clearly suggests that the ordering ambiguity parameters are strictly determined in (2.13) (along with the von Roos constraint $a+b+c=-1$) as $b=-1/2$, and $a=c=-1/4$. The result that $a=c=-1/4$ satisfy the continuity condition at the abrupt heterojunction between two crystals (c.f., e.g., Ref. [16]). Hereby, we may safely conclude that the PDM quantum mechanical correspondence of the PDM classical mechanical settings removes the ordering ambiguity in the von Roos PDM-Hamiltonian (1.1). We adopt these parametric results and proceed with the NPT settings used above.

2.2 Construction of the PDM-Momentum Operator

Having identified the correlation between the Schrödinger equation (in the generalized coordinates) and the PDM-Schrödinger equation (in the rectangular coordinates), through (2.6) - (2.13), we now need to address a question of delicate nature as to ”*what is the form of the position-dependent mass momentum operator?”*. The answer to this question very well lies in the very fundamentals of *”Quantum Mechanics”* by S. Gasiorowicz [51]. Therein, the one-dimensional quantum mechanical momentum operator $\widehat{p}_x = -i\partial/\partial x$ is determined through

$$\langle p_x \rangle = m_\circ \frac{d}{dt} \langle x \rangle = \int\limits_{-\infty}^{\infty} dx \Psi^* (x,t) \left(-i \frac{\partial}{\partial x} \right) \Psi (x,t) \Longrightarrow \widehat{p}_x = -i\partial/\partial x. \tag{2.14}$$

This would suggest that the one-dimensional (1D) quantum momentum operator in the generalized coordinate q for the 1D-quantum mechanical system is also obtainable through the same recipe as

$$\left\langle P_q \right\rangle = m_\circ \frac{d}{dt} \langle q \rangle = \int\limits_{-\infty}^{\infty} dq \Psi^* (q,t) \left(-i \frac{\partial}{\partial q} \right) \Psi (q,t) \Longrightarrow \widehat{P}_q = -i\partial/\partial q. \tag{2.15}$$

Which is, in fact, what we have readily used above. Next, if we use the corresponding 1D-point transformations

$$dq = \sqrt{m(x)} dx \, , \, \frac{\partial x}{\partial q} = \frac{1}{\sqrt{m(x)}}, \tag{2.16}$$

and $\Psi (q, \tau) = m(x)^{-1/4} \Phi (x,t)$ in (2.15), we immediately get

$$\left\langle P_q \right\rangle = \int\limits_{-\infty}^{\infty} dx \frac{\Phi^* (x,t)}{\sqrt{m(x)}} \left(-i \left[\frac{\partial}{\partial x} - \frac{1}{4} \left(\frac{\partial_x m(x)}{m(x)} \right) \right] \right) \Phi (x,t) \, . \tag{2.17}$$

Which clearly suggests that

$$\widehat{P}\left(q\left(x\right)\right)=\frac{-i}{\sqrt{m\left(x\right)}}\left[\frac{\partial}{\partial x}-\frac{1}{4}\left(\frac{\partial_{x}m\left(x\right)}{m\left(x\right)}\right)\right] \tag{2.18}$$

This is not yet the PDM-momentum operator and shall be called PDM pseudo-mementum operator (with the identity $\widehat{\pi}_x\left(q\left(x\right)\right)$ as reported in [13]. The generalization of which is straightforward and takes the form

$$\widehat{\pi}_{j}\left(\overrightarrow{q}\left(\vec{x}\right)\right)=\frac{-i}{\sqrt{m\left(\vec{x}\right)}}\left[\frac{\partial}{\partial x_{j}}-\frac{1}{4}\left(\frac{\partial_{x_j}m\left(\vec{x}\right)}{m\left(\vec{x}\right)}\right)\right] \tag{2.19}$$

We may rewrite it in general as

$$\widehat{\pi}\left(\overrightarrow{q}\left(\vec{x}\right)\right)=\frac{-i}{\sqrt{m\left(\vec{x}\right)}}\left[\overrightarrow{\nabla}-\frac{1}{4}\left(\frac{\overrightarrow{\nabla}m\left(\vec{x}\right)}{m\left(\vec{x}\right)}\right)\right], \tag{2.20}$$

where $\widehat{\pi}_j\left(\overrightarrow{q}\left(\vec{x}\right)\right)\rightarrow\widehat{p}_j=-i\partial/\partial x_j$ for constant mass settings (i.e., $m\left(\vec{x}\right)=1$). In fact, equation (2.20) gives the differential form of the Hamilton's canonical PDM pseudo-momentum operator $\widehat{\pi}\left(\overrightarrow{q}\left(\vec{x}\right)\right)$. Under such settings, our PDM Schrödinger equation (2.12) inherits the simplistic form

$$\left\{\widehat{\pi}_{j}^{2}\left(\overrightarrow{q}\left(\vec{x}\right)\right)+V\left(\overrightarrow{q}\left(\vec{x}\right)\right)\right\}\phi\left(\vec{x}\right)=E\phi\left(\vec{x}\right). \tag{2.21}$$

Furthermore, one should be aware that for $2m_{\circ}\neq 1$ the first term of equation (2.21) would result in $\widehat{\pi}_j^2\left(\overrightarrow{q}\left(\vec{x}\right)\right)/2m_{\circ}$ as the quantum PDM-kinetic energy operator (i.e., $\widehat{T}=\widehat{\pi}_j^2\left(\overrightarrow{q}\left(\vec{x}\right)\right)/2m_{\circ}$). Only under such transformation procedure's settings the quantum Hamiltonian implies the classical one, the other way around holds true as well. That is,

$$\begin{aligned} \widehat{H}_{quantum} &= \frac{\widehat{\pi}_j^2\left(\overrightarrow{q}\left(\vec{x}\right)\right)}{2m_\circ} + V\left(\overrightarrow{q}\left(\vec{x}\right)\right); \\ H_{classical.} &= \frac{1}{2} m_\circ m\left(\vec{x}\right)\dot{x}_j^2 + V\left(\overrightarrow{q}\left(\vec{x}\right)\right) = \frac{\pi_j^2\left(\overrightarrow{q}\left(\vec{x}\right)\right)}{2m_\circ} + V\left(\overrightarrow{q}\left(\vec{x}\right)\right) \end{aligned} \tag{2.22}$$

where $\pi_j\left(\overrightarrow{q}\left(\vec{x}\right)\right)$ is the jth-component of the classical PDM pseudo-momentum obtained through

$$\dot{q}_j\left(\vec{x}\right) = \sqrt{m\left(\vec{x}\right)}\dot{x}_j \Longrightarrow \pi_j\left(\overrightarrow{q}\left(\vec{x}\right)\right) = m_\circ\left[\sqrt{m\left(\vec{x}\right)}\dot{x}_j\right], \tag{2.23}$$

and $\widehat{\pi}_j\left(\overrightarrow{q}\left(\vec{x}\right)\right)$ is now the corresponding jth-component of the quantum PDM pseudo-momentum operator. At this very point, however, one recollects the classical PDM-Lagrangian $L = m_\circ m\left(\vec{x}\right)\dot{x}_j^2/2 - V\left(\vec{x}\right)$ to imply the classical PDM Hamiltonian

$$H = m_\circ m\left(\vec{x}\right)\dot{x}_j^2/2 + V\left(\vec{x}\right) = P_j^2/\left[2m_\circ m\left(\vec{x}\right)\right] + V\left(\vec{x}\right) \tag{2.24}$$

where $P_j\left(\vec{x}\right) = \partial L/\partial\dot{x}_j = m_\circ m\left(\vec{x}\right)\dot{x}_j$ is the canonical PDM-momentum. This would, in effect, imply that $P_j\left(\vec{x}\right) = \sqrt{m\left(\vec{x}\right)}\pi_j\left(\overrightarrow{q}\left(\vec{x}\right)\right)$ and consequently the PDM-momentum operator reads

$$\widehat{P}_j\left(\vec{x}\right) = -i\left[\frac{\partial}{\partial x_j} - \frac{1}{4}\left(\frac{\partial_{x_j} m\left(\vec{x}\right)}{m\left(\vec{x}\right)}\right)\right] \Longleftrightarrow \widehat{\pi}_j\left(\overrightarrow{q}\left(\vec{x}\right)\right) = \frac{\widehat{P}_j\left(\vec{x}\right)}{\sqrt{m\left(\vec{x}\right)}} \tag{2.25}$$

where $\widehat{\pi}_j\left(\overrightarrow{q}\left(\vec{x}\right)\right)$ is given in (2.20). This would necessarily mean that $\widehat{\pi}_j^2\left(\overrightarrow{q}\left(\vec{x}\right)\right)$ of (2.21) should be expressed as $\left(\widehat{P}_j\left(\vec{x}\right)/\sqrt{m\left(\vec{x}\right)}\right)^2$ and not as $\widehat{P}_j^2\left(\vec{x}\right)/m\left(\vec{x}\right)$.

2.3 Classical Electromagnetic Interaction and the PDM-Quantum Mechanical Correspondence

In this section, we begin with the motion of a classical particle of charge e and a constant rest mass $m_\circ$ moving in an electromagnetic interaction represented by the

4-vector potential $A_\mu = \left(\vec{A}, i\varphi\right)$ with the vector potential $\vec{A}(\vec{q})$ and a scalar potential $\varphi(\vec{q})$. The Lagrangian for such a system is given by

$$L\left(q_j, \dot{q}_j; t\right) = \frac{1}{2} m_\circ \dot{q}_j^2 + e\,\dot{q}_j A_j(\vec{q}) - \left[e\,\varphi(\vec{q}) + V(\vec{q})\right]. \tag{2.26}$$

Where $V(\vec{q})$ is any other potential energy than the electric and magnetic ones. Consequently, the classical Hamiltonian reads

$$H\left(q_j, P_j; t\right) = \dot{q}_j P_j - L\left(q_j, \dot{q}_j; t\right) = \frac{1}{2} m_\circ \dot{q}_j^2 + W(\vec{q});\; W(\vec{q}) = e\,\varphi(\vec{q}) + V(\vec{q}). \tag{2.27}$$

Here, the jth component of the canonical momentum (associated with the generalized coordinate q_j) is given by

$$P_j = \frac{\partial}{\partial \dot{q}_j} L\left(q_j, \dot{q}_j; t\right) \Longrightarrow P_j = m_\circ \dot{q}_j + eA_j(\vec{q}), \tag{2.28}$$

and the classical Hamiltonian (2.27) takes the form

$$H\left(q_j, P_j; t\right) = \frac{1}{2m_\circ}\left(P_j - eA_j(\vec{q})\right)^2 + W(\vec{q}). \tag{2.29}$$

Hence, the corresponding quantum mechanical Hamiltonian, with $2m_\circ = 1$ unit and $\widehat{P}_j = -i\partial_{q_j}$, consequently yields

$$\hat{H}\left(q_j, P_j; t\right) = -\partial_{q_j}^2 + ie\left[\partial_{q_j} A_j(\vec{q})\right] + 2ieA_j(\vec{q})\,\partial_{q_j} + e^2 A_j(\vec{q})^2 + W(\vec{q}). \tag{2.30}$$

Now, we follow our methodical proposal in section (2.1) and obtain the corresponding PDM-Schrödinger equation

$$
\begin{aligned}
\Bigg\{ -\frac{1}{m(\vec{x})}\partial_{x_j}^2 + \left[\frac{\partial_{x_j} m(\vec{x})}{m(\vec{x})^2}\right]\partial_{x_j} - \frac{7}{16}\left(\frac{\left[\partial_{x_j} m(\vec{x})\right]^2}{m(\vec{x})^3}\right) + \frac{1}{4}\left[\frac{\partial_{x_j}^2 m(\vec{x})}{m(\vec{x})^2}\right] \\
+ie\frac{\partial_{x_j} A_j(\vec{q}(\vec{x}))}{\sqrt{m(\vec{x})}} + 2ie\frac{A_j(\vec{q}(\vec{x}))}{\sqrt{m(\vec{x})}}\left[\partial_{x_j} - \frac{1}{4}\left(\frac{\partial_{x_j} m(\vec{x})}{m(\vec{x})}\right)\right] \\
+e^2 A_j(\vec{q}(\vec{x}))^2 + W(\vec{q}(\vec{x}))\Bigg\}\phi(\vec{x}) = E\phi(\vec{x}),
\end{aligned}
\tag{2.31}
$$

that reduces into

$$
\begin{aligned}
\left\{\left[\frac{-i}{\sqrt{m(\vec{x})}}\left[\frac{\partial}{\partial x_j} - \frac{1}{4}\left(\frac{\partial_{x_j} m(\vec{x})}{m(\vec{x})}\right)\right] - eA_j(\vec{q}(\vec{x}))\right]^2 + W(\vec{q}(\vec{x}))\right\} \\
\times\,\phi(\vec{x}) = E\phi(\vec{x}).
\end{aligned}
\tag{2.32}
$$

or in a more simplistic format

$$
\left\{\left[\frac{\widehat{P}_j(\vec{x})}{\sqrt{m(\vec{x})}} - eA_j(\vec{q}(\vec{x}))\right]^2 + W(\vec{q}(\vec{x}))\right\}\phi(\vec{x}) = E\phi(\vec{x}).
\tag{2.33}
$$

where the scalar potential $W(\vec{q}(\vec{x})) = W(\vec{x}(\vec{q}))$ and the vector potential $A_j(\vec{q}(\vec{x}))$ is yet to be correlated with $A_j(\vec{x})$ in the sequel. Classical mechanically, the PDM-Lagrangian and PDM-Hamiltonian with electromagnetic interaction are of the forms

$$
\begin{aligned}
L = \frac{1}{2}m_\circ m(\vec{x})\dot{x}_j^2 + e\dot{x}_j A_j(\vec{x}) - W(\vec{x}) \Longleftrightarrow H = \frac{1}{2}m_\circ m(\vec{x})\dot{x}_j^2 + W(\vec{x}); \\
W(\vec{x}) = e\varphi(\vec{x}) + V(\vec{x}),
\end{aligned}
\tag{2.34}
$$

where the PDM-canonical momentum reads

$$
P_j(\vec{x}) = \frac{\partial L}{\partial \dot{x}_j} = m_\circ m(\vec{x})\dot{x}_j + eA_j(\vec{x}) \Longleftrightarrow m_\circ\sqrt{m(\vec{x})}\dot{x}_j = \frac{P_j(\vec{x}) - eA_j(\vec{x})}{\sqrt{m(\vec{x})}}.
\tag{2.35}
$$

Therefore, in terms of the canonical momentum the PDM-Hamiltonian (2.34) takes the form

$$H=\frac{1}{2m_\circ}\left(\frac{P_j(\vec{x})-eA_j(\vec{x})}{\sqrt{m(\vec{x})}}\right)^2+W(\vec{x}),\tag{2.36}$$

and the quantum mechanical PDM-Hamiltonian hence reads

$$\widehat{H}=\left(\frac{\widehat{P}_j(\vec{x})-eA_j(\vec{x})}{\sqrt{m(\vec{x})}}\right)^2+W(\vec{x}).\tag{2.37}$$

Which immediately, when compared with the PDM-Hamiltonian of (2.33), suggests the correlation between the vector potentials $A_j(\vec{q}(\vec{x}))$ and $A_j(\vec{x})$ as

$$A_j(\vec{q}(\vec{x}))=\frac{A_j(\vec{x})}{\sqrt{m(\vec{x})}}.\tag{2.38}$$

Consequently equation (2.33) should look like

$$\left\{\left(\frac{\widehat{P}_j(\vec{x})-eA_j(\vec{x})}{\sqrt{m(\vec{x})}}\right)^2+W(\vec{x})\right\}\phi(\vec{x})=E\phi(\vec{x}).\tag{2.39}$$

It is now obvious, therefore, that the simplest way of coupling the electromagnetic interaction is to take the Hamilton's canonical pseudo-momentum $\pi_j(\vec{q}(\vec{x}))$ as the sum of the kinetic momentum $m_\circ\dot{q}_j=m_\circ\left(\sqrt{m(\vec{x})}\dot{x}_j\right)$ and $eA_j(\vec{q}(\vec{x}))$ (i.e., $\pi_j(\vec{q}(\vec{x}))=m_\circ\left(\sqrt{m(\vec{x})}\dot{x}_j\right)+eA_j(\vec{q}(\vec{x}))$). Hence, for the classical Hamiltonian in (2.22) one may simply use the minimal coupling

$$\begin{gathered}\pi_j(\vec{q}(\vec{x}))=\left(\frac{P_j(\vec{x})}{\sqrt{m(\vec{x})}}\right)\longrightarrow\pi_j(\vec{q}(\vec{x}))-eA_j(\vec{q}(\vec{x}))\\ \text{and }\; E=H_{classical}\longrightarrow E-e\varphi(\vec{q}(\vec{x})),\end{gathered}\tag{2.40}$$

or in terms of the canonical PDM-momentum, it precisely reads

$$\left(\frac{P_j(\vec{x})}{\sqrt{m(\vec{x})}}\right)\longrightarrow\left(\frac{P_j(\vec{x})-eA_j(\vec{x})}{\sqrt{m(\vec{x})}}\right)\;\text{and }\; E=H_{classical}\longrightarrow E-e\varphi(\vec{q}(\vec{x})),\tag{2.41}$$

to incorporate electromagnetic interactions. Consequently, in quantum mechanics, it is obvious that the electromagnetic interactions for PDM are integrated into the PDM-Schrödinger equation (2.21) through the the minimal coupling

$$\widehat{\pi}_j(\vec{q}(\vec{x})) = \frac{\widehat{P}_j(\vec{x})}{\sqrt{m(\vec{x})}} \longrightarrow \left(\frac{\widehat{P}_j(\vec{x}) - eA_j(\vec{x})}{\sqrt{m(\vec{x})}} \right) \text{ and } E \longrightarrow E - e\varphi(\vec{q}(\vec{x})). \tag{2.42}$$

Nevertheless, one should notice that a proper reverse engineering of (2.39), with $\phi(\vec{x}) = m(\vec{x})^{1/4}\psi(\vec{q})$, would immediately yield

$$\left\{ \left[\widehat{P}_j - eA_j(\vec{q}) \right]^2 + W(\vec{q}) \right\} \psi(\vec{q}) = E\psi(\vec{q}); \;\; \widehat{P}_j = -i\partial_{q_j} \tag{2.43}$$

Obviously, equation (2.43) represents a textbook example which is known to be exactly or conditionally exactly solvable model for some $W(\vec{q})$ forms. The solutions of which can be mapped into the PDM Schrödinger equation (2.39).

2.4 Eligibility of the Vector Potentials and PDM-Settings

In this section, we shall consider the two vector potentials that satisfy the Coulomb gauge $\partial_{q_j}A_j(\vec{q}) = 0$ and are often used in the literature as illustrative examples. They are,

$$\vec{A}(\vec{q}) = B_\circ(-q_2, 0, 0) = -B_\circ q_2 \widehat{q}_1 \tag{2.44}$$

and

$$\vec{A}(\vec{q}) = \frac{B_\circ}{2}(-q_2, q_1, 0) = \frac{B_\circ}{2}(-q_2\widehat{q}_1 + q_1\widehat{q}_2) \tag{2.45}$$

where $\widehat{q}_i$ is the unit vector for the generalized coordinate q_i. Consequently, they result a constant magnetic field

$$\vec{B}(\vec{q}) = \vec{\nabla}_q \times \vec{A}(\vec{q}) = \begin{cases} B_\circ \widehat{q}_3 \text{ for } \vec{A}(\vec{q}) = B_\circ(-q_2, 0, 0) \\ \\ \frac{B_\circ}{2} \widehat{q}_3 \text{ for } \vec{A}(\vec{q}) = \frac{B_\circ}{2}(-q_2, q_1, 0) \end{cases} \tag{2.46}$$

Hereby, we shall subject the two vector potentials $\vec{A}(\vec{q})$ in (2.44) and (2.45) to some eligibility test in order to be able to deal with Schrödinger equation (2.43) for different interaction potentials (be it the vector potentials $A_j(\vec{q})$ and/or scalar potentials $W(\vec{q}) = e\varphi(\vec{q}) + V(\vec{q})$) and hence to reflect on the corresponding PDM settings in (2.39).

Let us recollect the correlation of (2.38) and recast it as

$$A_j(\vec{q}(\vec{x})) = \frac{S(\vec{x})}{\sqrt{m(\vec{x})}} \tilde{A}_j(\vec{x}) \; ; \; \overrightarrow{\tilde{A}}(\vec{x}) = \begin{cases} B_\circ(-x_2, 0, 0) \\ \\ \frac{B_\circ}{2}(-x_2, x_1, 0) \end{cases} , \tag{2.47}$$

where the introduction of the scalar multiplier $S(\vec{x})$ in the assumption $A_j(\vec{x}) = S(\vec{x})\tilde{A}_j(\vec{x})$ absorbs any other position-dependent terms that may emerge from the construction of the vector potential $\overrightarrow{A}(\vec{x})$ (such as that of a long solenoid for example). Yet, the Coulomb gauge $\partial_{q_j} A_j(\vec{q}) = 0$ should be satisfied and remain invariant under our point transformation. That is, with $m(\vec{x}) = m(r)$, $S(\vec{x}) = S(r)$ and $S'(r) = dS(r)/dr$; $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, the condition

$$\partial_{q_j} A_j(\vec{q}(\vec{x})) = \frac{S(r)}{m(r)} \left[\partial_{x_j} \tilde{A}_j(\vec{x}) + \frac{x_j \tilde{A}_j(\vec{x})}{r} \left(\frac{S'(r)}{S(r)} - \frac{m'(r)}{2m(r)} \right) \right] = 0. \tag{2.48}$$

has to be satisfied. It is obvious that, whilst the first term $\partial_{x_j} \tilde{A}_j(\vec{x}) = 0$ for both forms of $\overrightarrow{\tilde{A}}(\vec{x})$ in (2.47), the second term $x_j \tilde{A}_j(\vec{x}) = 0$ if and only if $\overrightarrow{\tilde{A}}(\vec{x}) = \frac{B_\circ}{2}(-x_2, x_1, 0)$. We, therefore, consider the only eligible vector potential setting

$$\vec{A}(\vec{q}) = \frac{B_\circ}{2}(-q_2, q_1, 0) = \frac{S(r)}{\sqrt{m(r)}}\frac{B_\circ}{2}(-x_2, x_1, 0). \tag{2.49}$$

This would immediately imply that

$$q_j(\vec{x}) = \frac{S(r)}{\sqrt{m(r)}}x_j \Longrightarrow \frac{\partial q_j}{\partial x_j} = \frac{S(r)}{\sqrt{m(r)}}\left[1 + \frac{x_j^2}{r}\left(\frac{S'(r)}{S(r)} - \frac{m'(r)}{2m(r)}\right)\right], \tag{2.50}$$

where there is no summation in the above equation, but when summed over the repeated index yields

$$\frac{\partial q_j}{\partial x_j} = \frac{S(r)}{\sqrt{m(r)}}\left[N + r\left(\frac{S'(r)}{S(r)} - \frac{m'(r)}{2m(r)}\right)\right], \tag{2.51}$$

where $N \geq 2$ denotes the number of degrees of freedom involved in the problem at hand and in our case $N = 3$. If we now use equation (2.7) (with $g(\vec{x}) = m(\vec{x}) = m(r)$) summed up over the repeated index, we get

$$\frac{\partial q_j}{\partial x_j} = N\sqrt{m(\vec{x})} = N\sqrt{m(r)} \tag{2.52}$$

Hence, (2.51) and (2.52) suggest the relation

$$m(r) = S(r)\left[1 + \frac{r}{N}\left(\frac{S'(r)}{S(r)} - \frac{m'(r)}{2m(r)}\right)\right];$$
$$S(r) = N\sqrt{m(r)}r^{-N}\int r^{N-1}\sqrt{m(r)}dr. \tag{2.53}$$

Although $N = 3$ for the current methodical proposal, we choose to cast the above equation in terms of N to identify the number of degrees of freedom involved in the problem at hand. Moreover, for a given $m(r)$ one may find $S(r)$ using (2.53), the other way around works as well.

2.5 Illustrative Examples

2.5.1 PDM-Charged Particle in $W(\vec{q})=0$

A charged particle moving under the influence of the vector potential $\vec{A}(\vec{q}) = B_\circ(-q_2,q_1,0)/2 \Longrightarrow \vec{B}(\vec{q}) = B_\circ\widehat{q}_3/2 = B_\circ\widehat{x}_3/2$ would be described by the Schrödinger equation (2.43) as

$$\left\{\left[\widehat{P}_1+\frac{eB_\circ}{2}q_2\right]^2+\left[\widehat{P}_2-\frac{eB_\circ}{2}q_1\right]^2+\widehat{P}_3^2\right\}\psi(\vec{q})=E\psi(\vec{q}). \tag{2.54}$$

This would, in effect, suggest that the Hamiltonian

$$\widehat{H}(q,P)=\left[\widehat{P}_1+\frac{eB_\circ}{2}q_2\right]^2+\left[\widehat{P}_2-\frac{eB_\circ}{2}q_1\right]^2+\widehat{P}_3^2 \tag{2.55}$$

does not explicitly depend on q_3, and the commutation relations

$$\left[q_i,\widehat{P}_j\right]=i\delta_{ij}\;\;,\;\;\left[\widehat{P}_i,\widehat{P}_j\right]=0\;\;,\;\;\left[\widehat{P}_3,\widehat{H}(q,P)\right]=0 \tag{2.56}$$

are satisfied. Hence, $\widehat{P}_3$ is no longer an operator but rather a constants of motion (i.e., it can be replaced by a number, therefore). Consequently, the solution of (2.54) can be expressed as

$$\psi(\vec{q})=\exp\left[i\left(k_1q_1+k_3q_3+\frac{eB_\circ}{2}q_1q_2\right)\right]Y(q_2) \tag{2.57}$$

to result in a shifted harmonic oscillator like Schrödinger equation

$$\left\{-\frac{d^2}{dq_2^2}+e^2B_\circ^2\left[q_2+\frac{k_1}{eB_\circ}\right]^2+k_3^2\right\}Y_n(q_2)=E_nY_n(q_2). \tag{2.58}$$

Which admits exact energy eigenvalues and eigenfunctions, respectively, as

$$E_n = k_3^2 + (2n+1)\,|e|\,B_\circ, \tag{2.59}$$

$$\begin{gathered} Y_n(\zeta) \sim \exp\left[-\frac{|e|\,B_\circ}{2}\zeta^2\right] H_n\left(\sqrt{|e|\,B_\circ}\zeta\right); \\ \zeta = q_2 + \frac{k_1}{eB_\circ},\ n = 0,1,2,\cdots, \end{gathered} \tag{2.60}$$

where $H_n(x)$ are the Hermite polynomials.

2.5.2 PDM-Charged Particle in $W(\vec{q}) = -e\mathcal{E}_\circ q_2$

Here we take the same charged particle as above and subject it not only to a constant magnetic field but also to a constant electric field $\overrightarrow{E} = \mathcal{E}_\circ \widehat{q}_2$ (i.e., $\overrightarrow{E} = \mathcal{E}_\circ \widehat{x}_2$; $\widehat{q}_2 = \widehat{x}_2$). In this case, our Schrödinger equation (2.43) reads

$$\left\{\left[\widehat{P}_1 + \frac{eB_\circ}{2}q_2\right]^2 + \left[\widehat{P}_2 - \frac{eB_\circ}{2}q_1\right]^2 + \widehat{P}_3^2 - e\mathcal{E}_\circ q_2\right\}\psi(\vec{q}) = E\psi(\vec{q}). \tag{2.61}$$

with the substitution of $\psi(\vec{q})$ in (2.57) we obtain, again, a shifted harmonic oscillator like Schrödinger equation

$$\left\{-\frac{d^2}{dq_2^2} + e^2B_\circ^2\left[q_2 + \left(\frac{k_1}{eB_\circ} - \frac{\mathcal{E}_\circ}{2eB_\circ^2}\right)\right]^2 + k_3^2\right\}Y_n(q_2) = \tilde{E}_n Y_n(q_2). \tag{2.62}$$

Which admits exact solution similar to that of (2.58) where the exact eigenenergies are given by

$$\tilde{E}_n = k_3^2 + (2n+1)\,|e|\,B_\circ \Longrightarrow E_n = (2n+1)\,|e|\,B_\circ + k_3^2 + \frac{k_1\mathcal{E}_\circ}{B_\circ} - \frac{\mathcal{E}_\circ^2}{4B_\circ^2}, \tag{2.63}$$

and the exact eigenfunctions are

$$\begin{gathered} Y_n(\varsigma) \sim \exp\left[-\frac{|e|\,B_\circ}{2}\varsigma^2\right] H_n\left(\sqrt{|e|\,B_\circ}\varsigma\right); \\ \varsigma = q_2 + \left(\frac{k_1}{eB_\circ} - \frac{\mathcal{E}_\circ}{2eB_\circ^2}\right), n = 0,1,2,\cdots. \end{gathered} \tag{2.64}$$

For both illustrative examples above, one may recollect our coordinates' settings of (2.7)-(2.8) and (2.49), along with $g(\vec{x}) = m(\vec{x}) = m(r)$, to build up the wavefunctions in the rectangular coordinates using $\phi(\vec{x}) = m(r)^{1/4}\psi(\vec{q}(\vec{x}))$ of (2.11). Hereby, we notice that all such PDM functions satisfying (2.53) share the same eigenvalues and eigenfunctions of either (2.59) and (2.60) or (2.63) and (2.64), respectively. Isospectrality is an obvious consequence of the current methodical proposal, of course.

Chapter 3

PDM CHARGED PARTICLES IN MAGNETIC AND AB-FLUX FIELDS

In the current chapter, we use the PDM momentum operator (2.25) along with the PDM minimal coupling (2.41) and study PDM-charged particles moving in a magnetic and an Aharonov-Bohm (AB) flux fields.

3.1 Construction of the Vector Potential: Cylindrical Coordinates and Separability

In this section, we start with the PDM momentum operator

$$\widehat{P}(\overrightarrow{r}) = -i\left[\overrightarrow{\nabla} - \frac{1}{4}\left(\frac{\overrightarrow{\nabla} m(\overrightarrow{r})}{m(\overrightarrow{r})}\right)\right], \tag{3.1}$$

in the PDM- Schrödinger equation

$$\left[\left(\frac{\widehat{P}(\overrightarrow{r}) - e\overrightarrow{A}(\overrightarrow{r})}{\sqrt{m(\overrightarrow{r})}}\right)^2 + W(\overrightarrow{r})\right]\psi(\overrightarrow{r}) = E\psi(\overrightarrow{r}), \tag{3.2}$$

where $W(\overrightarrow{r}) = e\varphi(\overrightarrow{r}) + V(\overrightarrow{r})$. Consequently, in a straightforward manner, equation (3.2) would read

$$\left[-\frac{1}{m(\overrightarrow{r})}\overrightarrow{\nabla}^2+\left(\frac{\overrightarrow{\nabla}m(\overrightarrow{r})}{m(\overrightarrow{r})^2}\right)\cdot\overrightarrow{\nabla}+\frac{1}{4}\left(\frac{\overrightarrow{\nabla}^2m(\overrightarrow{r})}{m(\overrightarrow{r})^2}\right)-\frac{7}{16}\left(\frac{\left[\overrightarrow{\nabla}m(\overrightarrow{r})\right]^2}{m(\overrightarrow{r})^3}\right)\right.$$
$$+\frac{2\,i\,e}{m(\overrightarrow{r})}\overrightarrow{A}(\overrightarrow{r})\cdot\overrightarrow{\nabla}+\frac{ie}{m(\overrightarrow{r})}\left(\overrightarrow{\nabla}\cdot\overrightarrow{A}(\overrightarrow{r})\right)-i\,\overrightarrow{A}(\overrightarrow{r})\cdot\left(\frac{\overrightarrow{\nabla}m(\overrightarrow{r})}{m(\overrightarrow{r})^2}\right)$$
$$\left.+\frac{e^2\overrightarrow{A}(\overrightarrow{r})^2}{m(\overrightarrow{r})}+W(\overrightarrow{r})\right]\psi(\overrightarrow{r})=E\psi(\overrightarrow{r}). \tag{3.3}$$

Here, we consider the interaction of a PDM particle of charge e moving in the vector potential

$$\overrightarrow{A}(\overrightarrow{r})=\overrightarrow{A}_1(\overrightarrow{r})+\overrightarrow{A}_2(\overrightarrow{r});\begin{cases}\overrightarrow{\nabla}\times\overrightarrow{A}_1(\overrightarrow{r})=\overrightarrow{B}=B_\circ\widehat{z}\\ \overrightarrow{\nabla}\times\overrightarrow{A}_2(\overrightarrow{r})=0\end{cases}, \tag{3.4}$$

where a uniform magnetic field $\overrightarrow{B}=B_\circ\widehat{z}$ is applied in the z-direction, $\overrightarrow{A}_1(\overrightarrow{r})=(0,B_\circ\rho/2,0)$ and $\overrightarrow{A}_2(\overrightarrow{r})=(0,\Phi_{AB}/2\pi\rho,0)$ are given in the cylindrical coordinates, with $\overrightarrow{A}_2(\overrightarrow{r})$ describing the so called Aharonov-Bohm flux field Φ_{AB} effect (c.f., e.g., [23, 40–42]). Hence, our PDM charged particle interacts with the total vector potential

$$\overrightarrow{A}(\overrightarrow{r})=\left(0,\frac{B_\circ}{2}\rho+\frac{\Phi_{AB}}{2\pi\rho},0\right), \tag{3.5}$$

that satisfies the Coulomb gauge $\overrightarrow{\nabla}\cdot\overrightarrow{A}=0$. Moreover, we shall use the assumptions that

$$m(\overrightarrow{r})=m(\rho,\varphi,z)=g(\rho)f(\varphi)k(z)=g(\rho);\,f(\varphi)=k(z)=1, \tag{3.6}$$

and

$$g(\rho)W(\rho,\varphi,z)=V(\rho)+V(\varphi)+V(z)=V(\rho)+V(z)\,;V(\varphi)=0, \tag{3.7}$$

where, $V(\varphi)=0$ assumes azimuthal symmetrization and our PDM scalar multiplier $m(\overrightarrow{r})=g(\rho)$ is only radially cylindrically symmetric. Under such assumptions, we may now follow the conventional textbook separation of variables and use the substitution

$$\psi(\overrightarrow{r})=\psi(\rho,\varphi,z)=R(\rho)Z(z)e^{im\varphi}, \tag{3.8}$$

(where $m=0,\pm1,\pm2,...,\pm\ell$ is the magnetic quantum number, and ℓ is angular momentum quantum number) in (3.3) to obtain

$$\left[\frac{R''(\rho)}{R(\rho)}-\left(\frac{g'(\rho)}{g(\rho)}-\frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}-\frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}+\frac{g'(\rho)}{\rho g(\rho)}\right)+\frac{7}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2-\frac{m^2}{\rho^2}\right.$$
$$\left.-\frac{e^2\Phi_{AB}^2}{4\pi^2\rho^2}+\frac{em\Phi_{AB}}{\pi\rho^2}-\frac{e^2B_\circ\Phi_{AB}}{2\pi}+eB_\circ m-\frac{e^2B_\circ^2\rho^2}{4}+g(\rho)E-V(\rho)\right]$$
$$+\left[\frac{Z''(z)}{Z(z)}-V(z)\right]=0. \tag{3.9}$$

It is obvious that this equation decouples into two parts, a purely z-dependent part

$$\left[-\partial_z^2+V(z)\right]Z(z)=k_z^2Z(z), \tag{3.10}$$

and a radial-dependent cylindrically-azimuthal part

$$\left[\frac{R''(\rho)}{R(\rho)}-\left(\frac{g'(\rho)}{g(\rho)}-\frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}-\frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}+\frac{g'(\rho)}{\rho g(\rho)}\right)+\frac{7}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2\right.$$
$$\left.-\frac{\tilde{m}^2}{\rho^2}+eB_\circ\tilde{m}-k_z^2-\frac{e^2B_\circ^2\rho^2}{4}+g(\rho)E-V(\rho)\right]=0, \tag{3.11}$$

where $\alpha=\Phi_{AB}/\Phi_{\circ}$, $\Phi_{\circ}=2\pi/e$ is the AB-flux quantum, and $\tilde{m}=m-\alpha$ is a new irrational magnetic quantum number that indulges within the Aharonov-Bohm quantum number α. Nevertheless, one may need to get rid of the first-order derivative and bring the radial part into the one-dimensional Schrödinger form. In so doing, one may use the substitution

$$R(\rho)=\sqrt{\frac{g(\rho)}{\rho}}U(\rho), \tag{3.12}$$

to obtain

$$\left\{-\frac{d^2}{d\rho^2}+\frac{\tilde{m}^2-1/4}{\rho^2}+V_{eff}(\rho)\right\}U(\rho)=\tilde{E}U(\rho). \tag{3.13}$$

Where,

$$\begin{gathered}V_{eff}(\rho)=V(\rho)+\frac{e^2B_{\circ}^2\rho^2}{4}-g(\rho)E+\\ \left[\frac{5}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2-\frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}\right)-\frac{1}{4}\left(\frac{g'(\rho)}{\rho g(\rho)}\right)\right],\end{gathered} \tag{3.14}$$

and

$$\tilde{E}=eB_{\circ}\tilde{m}-k_z^2 \tag{3.15}$$

represents the eigenvalues of (3.13) to be used to find the eigenvalues of the radial PDM problem $E_{n_\rho,m,\alpha}$ in (3.14).

We have, at our disposal, three types of Schrödinger differential equations to deal with. The z-dependent part of (3.10), the ρ-dependent part of (3.11) and the one-dimensional ρ-dependent part of (3.13). The ρ-dependent parts (3.11) and (3.13) are to be shown useful in their own skin and serve different PDM and/or interaction potential settings. Whereas, the z-dependent part of (3.10) will have its own spectral signatures on the overall spectra of the decoupled problem in (3.2).

3.2 Radial Cylindrical 1D-PDM-Schrödinger Form with a Pseudo-Harmonic Oscillator Potential

In this section, we consider our PDM-charged particle moving in the so called pseudo-harmonic oscillator potential [41]

$$V(\rho) = \mathcal{V}_1 \rho^2 + \frac{\mathcal{V}_2}{\rho^2} - 2\mathcal{V}_\circ \; ; \; \mathcal{V}_1 = \frac{\mathcal{V}_\circ}{\rho_\circ^2} \, , \; \mathcal{V}_2 = \mathcal{V}_\circ \rho_\circ^2 \tag{3.16}$$

in the presence of a uniform magnetic and an AB-flux fields of (3.5). Where, $\mathcal{V}_\circ$ is the chemical potential and $\rho_\circ$ is the zero point of the pseudo-harmonic oscillator potential. This potential includes both a harmonic quantum dot potential $\mathcal{V}_1 \rho^2$ and antidote potential $\mathcal{V}_2/\rho^2$ [40, 41]. Such a pseudo-harmonic oscillator potential is most suited for the 1D-PDM-Schrödinger form (3.13) and anticipated to be exactly solvable for a sample of PDM settings. Therefore, we treat, in what follows, some special PDM settings so that their exact solutions are inferred from some textbook models that are known to be exactly solvable.

3.2.1 Model-I: A Radial Cylindrical PDM $g(\rho) = \eta \rho^2$

Consider a charged particle with radial cylindrical PDM $g(\rho) = \eta \rho^2$ moving in the pseudo-harmonic oscillator potential (3.16), a uniform magnetic and an AB-flux fields of (3.5). Then the effective potential $V_{eff}(\rho)$ of (3.14) would read

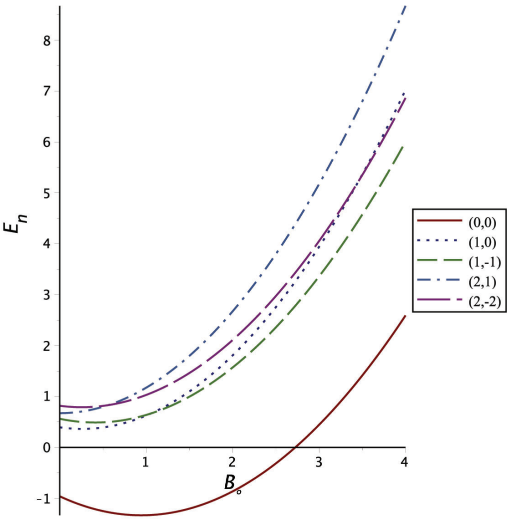
$$V_{eff}(\rho) = \mathcal{V}_1 \rho^2 + \frac{\mathcal{V}_2}{\rho^2} - 2\mathcal{V}_\circ + \frac{e^2 B_\circ^2 \rho^2}{4} - \eta E \rho^2 + \frac{1}{4\rho^2}. \tag{3.17}$$

Hence, equation (3.13) collapse into

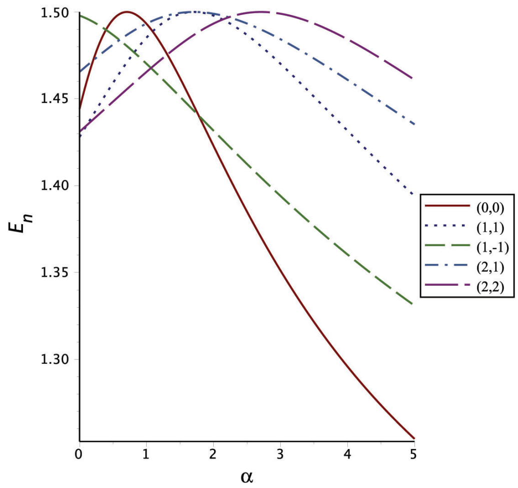
$$\left\{-\frac{d^2}{d\rho^2}+\frac{\tilde{\ell}_1^2-1/4}{\rho^2}+\frac{\left(4\mathcal{V}_1-4\eta E+e^2B_\circ^2\right)}{4}\rho^2\right\}U(\rho)=E_{eff}U(\rho), \tag{3.18}$$

where

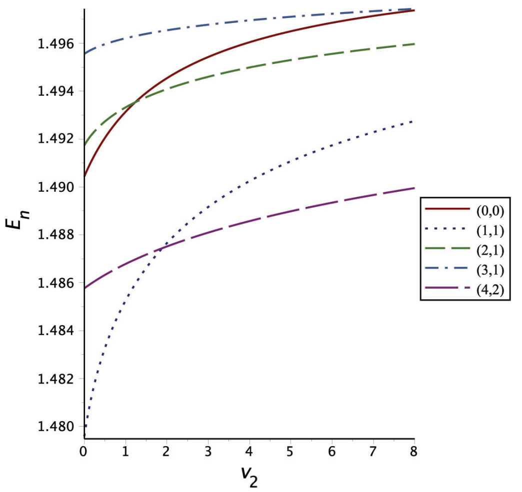
$$\tilde{\ell}_1^2-1/4=\tilde{m}^2+\mathcal{V}_2 \Longleftrightarrow \left|\tilde{\ell}_1\right|=\sqrt{(m-\alpha)^2+\mathcal{V}_2+1/4}. \tag{3.19}$$



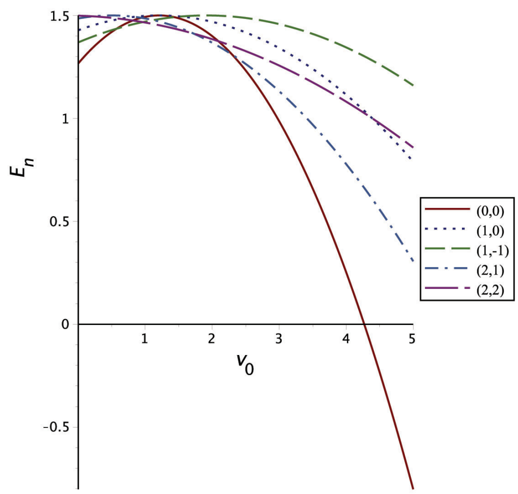
(a) Energy levels vs the magnetic field strenght $B_\circ$.



(b) Energy levels vs the AB-quantum number α.



(c) Energy levels vs the potential parameter $\mathcal{V}_2$.



(d) Energy levels vs the potential parameter $\mathcal{V}_\circ$.

Figure 3.1: Energy levels (n_ρ, m) of (3.21) for different values of the parameters $B_\circ$, α, $\mathcal{V}_2$, and $\mathcal{V}_\circ$ in (a), (b), (c) and (d) respectively.

Equation (3.18) is, in fact, the well know two-dimensional radial cylindrical harmonic oscillator problem (c.f., e.g., [43]) that admits exact solution in the form of

$$E_{eff}=\sqrt{4\mathcal{V}_1-4\eta E+e^2B_\circ^2}\left(2n_\rho+\left|\tilde{\ell}_1\right|+1\right)=2\mathcal{V}_\circ+eB_\circ\left(m-\alpha\right)-k_z^2. \quad (3.20)$$

Which would, in turn, imply that the eigenvalues are given by

$$E_{n_\rho,m,\alpha}=\frac{1}{4\eta}\left[4\mathcal{V}_1+e^2B_\circ^2-\left(\frac{2\mathcal{V}_\circ+eB_\circ\left(m-\alpha\right)-k_z^2}{2n_\rho+1+\sqrt{\left(m-\alpha\right)^2+\mathcal{V}_2+1/4}}\right)^2\right] \quad (3.21)$$

and radial wavefunctions are obtained in a similar manner to read

$$R_{n_\rho,m,\alpha}\left(\rho\right)\sim\rho^{1+\left|\tilde{\ell}_1\right|}\exp\left(-\frac{\sqrt{e^2B_\circ^2+4\mathcal{V}_2-4\eta E_{n_\rho,m,\alpha}}}{4}\rho^2\right)$$

$$\times{}_1F_1\left(-n_\rho;\left|\tilde{\ell}_1\right|+1;\frac{\sqrt{e^2B_\circ^2+4\mathcal{V}_2-4\eta E_{n_\rho,m,\alpha}}}{2}\rho^2\right) \quad (3.22)$$

3.2.2 Model-II: A Radial Cylindrical PDM $g\left(\rho\right)=\eta/\rho^2$

A charged particle with radial cylindrical PDM $g\left(\rho\right)=\eta/\rho^2$ moving in a pseudo-harmonic oscillator potential field (3.16), a uniform magnetic and an AB-flux fields of (3.5), would imply equation (3.13) be rewritten as

$$\left\{-\frac{d^2}{d\rho^2}+\frac{\tilde{\ell}_2^2-1/4}{\rho^2}+\frac{\left(4\mathcal{V}_1+e^2B_\circ^2\right)}{4}\rho^2\right\}U\left(\rho\right)=E_{eff}U\left(\rho\right), \quad (3.23)$$

where

$$\tilde{\ell}_2^2-1/4=\tilde{m}^2+\mathcal{V}_2-\Longleftrightarrow\left|\tilde{\ell}_2\right|=\sqrt{\left(m-\alpha\right)^2+\mathcal{V}_2-\eta E+1/4}. \quad (3.24)$$

Equation (3.23) is, again, in the form of the well known two-dimensional radial cylindrical harmonic oscillator and admits the exact solution

$$\begin{aligned} E_{eff} &= \sqrt{4\mathcal{V}_1 + e^2 B_\circ^2}\left(2n_\rho + \sqrt{(m-\alpha)^2 + \mathcal{V}_2 - \eta E + 1/4} + 1\right) \\ &= 2\mathcal{V}_\circ + eB_\circ (m-\alpha) - k_z^2, \end{aligned} \tag{3.25}$$

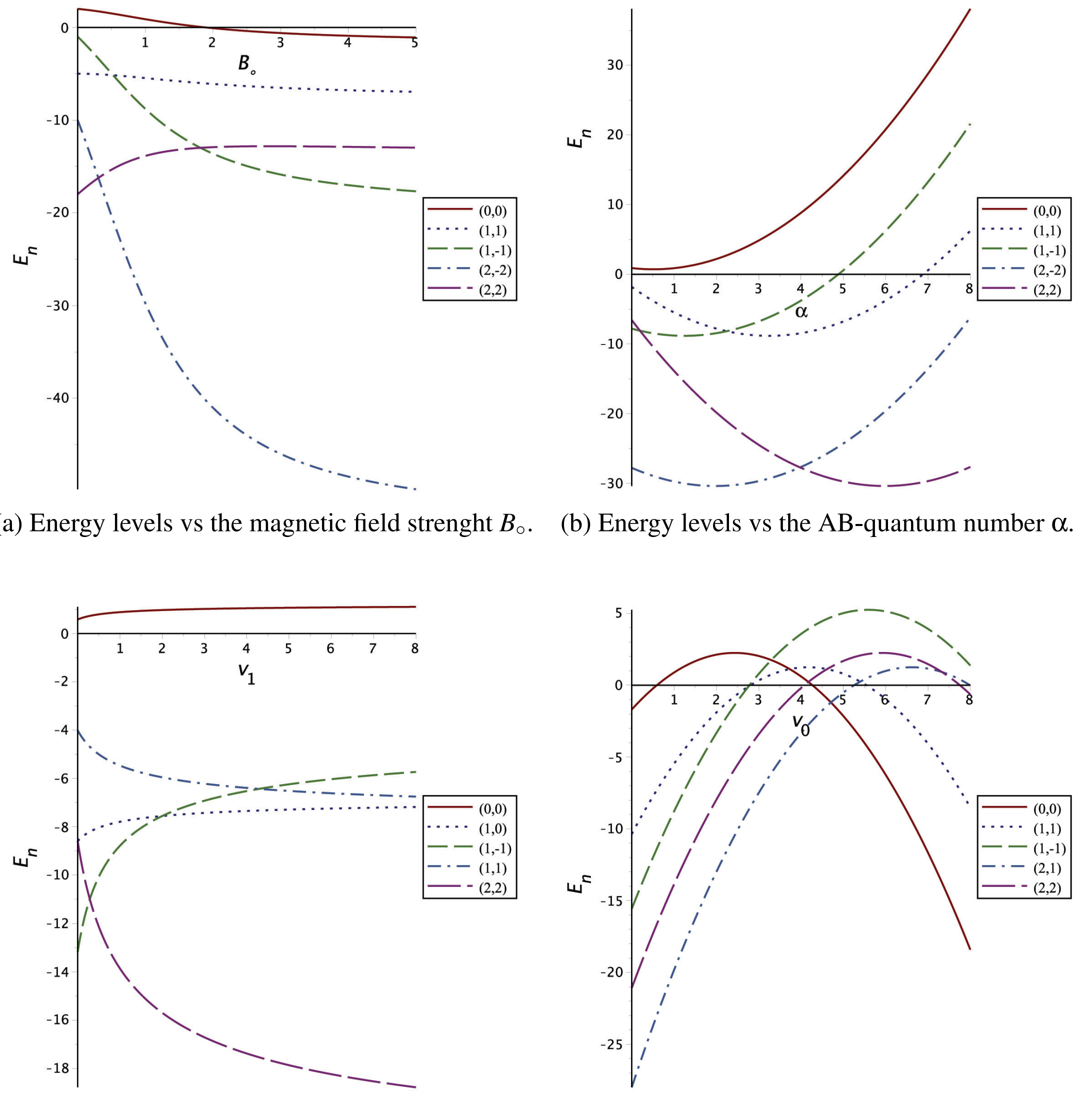
to yield the eigenvalues

$$E_{n_\rho,m,\alpha} = \frac{1}{\eta}\left\{(m-\alpha)^2 + \mathcal{V}_2 + 1/4 - \left[\frac{2\mathcal{V}_\circ + eB_\circ (m-\alpha) - k_z^2}{\sqrt{4\mathcal{V}_1 + e^2 B_\circ^2}} - \left(2n_\rho + 1\right)\right]^2\right\} \tag{3.26}$$

and the corresponding radial eigenfunctions

$$\begin{aligned} R_{n_\rho,m,\alpha}(\rho) &\sim \rho^{-1+|\widetilde{\ell}_2|} \exp\left(-\frac{\sqrt{e^2 B_\circ^2 + 4\mathcal{V}_2}}{4}\rho^2\right) \\ &\times {}_1F_1\left(-n_\rho; \left|\widetilde{\ell}_2\right| + 1; \frac{\sqrt{e^2 B_\circ^2 + 4\mathcal{V}_2}}{2}\rho^2\right) \end{aligned} \tag{3.27}$$

To understand the behavior of the energy levels of these two Models above, we plot the energy levels of different quantum states (n_ρ, m), in (3.21) and (3.26), as a function of the parameters of the magnetic and AB-flux fields $B_\circ$, α, and the parameters of the potential (3.16) (as shown in Figures 3.1 and 3.2, respectively). Energy levels crossings are observed in these figures, where there are multiple energy levels crossings for some specific quantum numbers. The energy levels crossings suggest that there could be more than one quantum state sharing the same energy at each crossing point. This would in turn indicate occasional degeneracies at some specific parametric settings.



(a) Energy levels vs the magnetic field strenght $B_{\circ}$. (b) Energy levels vs the AB-quantum number α.

(c) Energy levels vs the potential parameter $\mathcal{V}_1$. (d) Energy levels vs the potential parameter $\mathcal{V}_{\circ}$.

Figure 3.2: Energy levels (n_ρ, m) of (3.26) for different values of the parameters $B_{\circ}$, α, $\mathcal{V}_1$, and $\mathcal{V}_{\circ}$ in (a), (b), (c) and (d) respectively.

3.3 Radial Cylindrical 1D-PDM-Schrödinger Form with a Killingbeck-Type Potential

We now use the radial cylindrical PDM-Schrödinger form (3.11) and consider a generalized Killingbeck potential field (e.g., [49]) of the form

$$V(\rho) = V_0 + V_1 \rho + V_2 \rho^2 + \frac{V_3}{\rho} + \frac{V_4}{\rho^2}. \tag{3.28}$$

When such potential field is substituted in (3.11), one obtains

$$\left[\frac{R''(\rho)}{R(\rho)}-\left(\frac{g'(\rho)}{g(\rho)}-\frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}-\frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}+\frac{g'(\rho)}{\rho g(\rho)}\right)+\frac{7}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2 -\frac{\beta^2}{\rho^2}+\tilde{k}^2-\gamma^2\rho^2+g(\rho)E-V_1\rho-\frac{V_3}{\rho}\right]=0, \tag{3.29}$$

where

$$\beta^2=\tilde{m}^2+V_4, \qquad \tilde{k}^2=eB_\circ\tilde{m}-k_z^2-V_0,$$
$$\gamma^2=\tfrac{e^2B_\circ^2}{4}+V_2, \qquad \alpha=\Phi_{AB}/\Phi_\circ,\ \Phi_\circ=2\pi/e. \tag{3.30}$$

In the sample of illustrative examples below, we wish to benefit from the known solutions of the biconfluent Heun equation using two different PDM settings.

3.3.1 Model-III: A Radial Cylindrical PDM $g(\rho)=\lambda\rho$

A PDM-charged particle with radial cylindrical PDM $g(\rho)=\lambda\rho$ moving in the potential field (3.28), under the influence of both a uniform magnetic and an AB-flux fields of (3.5), would be described by the radial Schrödinger equation (3.29) as

$$\frac{R''(\rho)}{R(\rho)}-\frac{\beta^2-3/16}{\rho^2}-\gamma^2\rho^2-\frac{V_3}{\rho}+(\lambda E-V_1)\rho+\tilde{k}^2=0. \tag{3.31}$$

Which, in a straightforward manner, collapses into the standard 1D-Schrödinger form of the biconfluent Heun equation (c.f., e.g., [44])

$$R''(\rho)+\left[\frac{1-\tilde{\alpha}^2}{4\rho^2}-\frac{\tilde{\delta}}{2\rho}-\tilde{\beta}\rho-\rho^2+\tilde{\gamma}-\frac{\tilde{\beta}^2}{4}\right]R(\rho)=0, \tag{3.32}$$

where

$$\left\{\begin{array}{lll}\left(1-\tilde{\alpha}^{2}\right)/4=3/16-\beta^{2}, & -\tilde{\delta}/2=-V_{3}, & -\tilde{\beta}=\lambda E-V_{1} \\ \gamma^{2}=1=e^{2}B_{\circ}^{2}/4+V_{2}, & \tilde{\gamma}-\tilde{\beta}^{2}/4=\tilde{k}^{2}, & \end{array}\right\} \tag{3.33}$$

We now use the transformation

$$R(\rho)=\rho^{\left(1+\tilde{\alpha}^{2}\right)/2}\exp\left[-\frac{\tilde{\beta}\rho+\rho^{2}}{2}\right]U(\rho) \tag{3.34}$$

in (3.32) to obtain the biconfluent Heun-type equation

$$\rho U''(\rho)+\left[1+\tilde{\alpha}-\tilde{\beta}\rho-2\rho^{2}\right]U'(\rho)+\left\{(\tilde{\gamma}-2-\tilde{\alpha})\rho-\frac{1}{2}\left(\tilde{\delta}+[1+\tilde{\alpha}]\,\tilde{\beta}\right)\right\}U(\rho)=0. \tag{3.35}$$

Which is known to admit solutions in the form of biconfluent Heun functions

$$U(\rho)=H_{B}\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};\rho\right), \tag{3.36}$$

where,

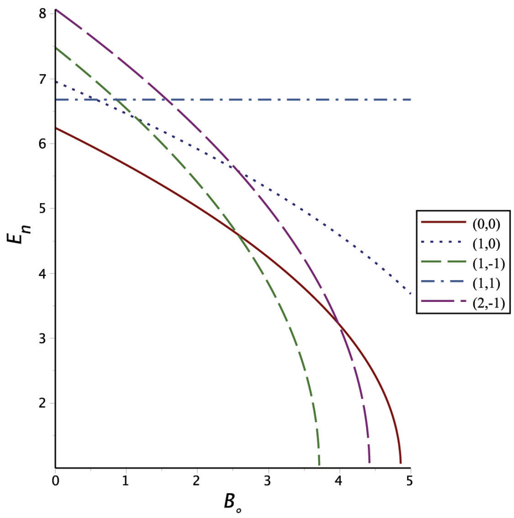
$$\tilde{\gamma}-2-\tilde{\alpha}\ =2n_{\rho}\ ;\ n_{\rho}=0,1,2\cdots, \tag{3.37}$$

provides the essential quantization, and

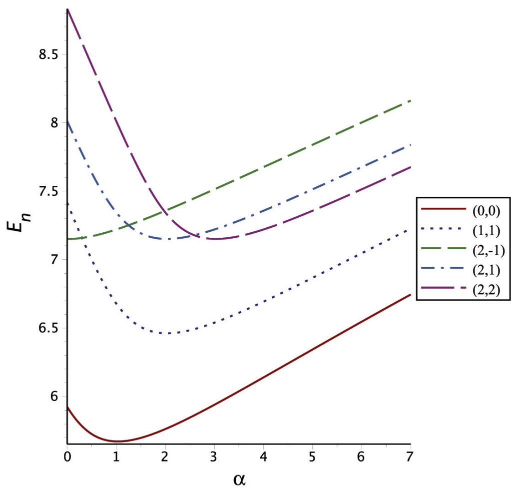
$$\tilde{\gamma}=\frac{\tilde{\beta}^{2}}{4}+\tilde{k}^{2}=\frac{(\lambda E-V_{1})^{2}}{4}+eB_{\circ}(m-\alpha)-k_{z}^{2}-V_{0}, \tag{3.38}$$

$$\tilde{\alpha}=2\sqrt{(m-\alpha)^{2}+V_{4}+\frac{1}{16}}. \tag{3.39}$$

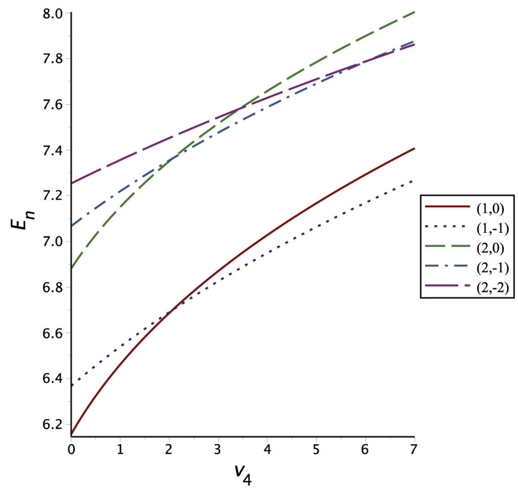
This would, in turn, imply that the eigenvalues are given as



(a) Energy levels vs the magnetic field strenght $B_{\circ}$.



(b) Energy levels vs the AB-quantum number α.



(c) Energy levels vs the potential parameter V_4.

Figure 3.3: Energy levels (n_{ρ}, m) of (3.40) for different values of the parameters $B_{\circ}$, α, and V_4 in (a), (b) and (c) respectively.

$$E_{n_{\rho},m,\alpha} = \frac{1}{\lambda}\left[V_1 + 2\left(2\left[n_{\rho} + 1 + \sqrt{(m-\alpha)^2 + V_4 + \frac{1}{16}}\right] - eB_{\circ}(m-\alpha) + k_z^2 + V_0\right)^{1/2}\right] \tag{3.40}$$

and the radial eigenfunctions are

$$R_{n_{\rho},m,\alpha}(\rho) \sim \rho^{(1+\tilde{\alpha}^2)/2} \exp\left(-\frac{\tilde{\beta}\rho + \rho^2}{2}\right) H_B\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho\right), \tag{3.41}$$

Where $\tilde{\alpha}$ and $\tilde{\beta}$ are defined, respectively, in(3.39) and (3.33). However, for more details on the biconfluent Heun the readers are advised to refer to the sample of references [44–49].

3.3.2 Model-IV: A Radial Cylindrical PDM $g(\rho) = \lambda/\rho^2$

For a PDM-charged particle with $g(\rho) = \lambda/\rho^2$ moving in the vicinity of the three fields above (i.e., the potential of (3.28), the uniform magnetic and the AB-flux fields of (3.5)), the radial Schrödinger equation (3.29) along with the substitution (3.12) would collapse into

$$\frac{U''(\rho)}{U(\rho)} - \frac{\xi^2}{\rho^2} - \gamma^2\rho^2 - V_1\rho - \frac{V_3}{\rho} + \tilde{k}^2 = 0\,;\;\; \xi^2 = \beta^2 - \lambda E. \tag{3.42}$$

Which, in a straight forward manner, reduces to

$$U''(\rho) + \left[\frac{1-\tilde{\alpha}^2}{4\rho^2} - \frac{\tilde{\delta}}{2\rho} - \tilde{\beta}\rho - \rho^2 + \tilde{\gamma} - \frac{\tilde{\beta}^2}{4}\right] U(\rho) = 0, \tag{3.43}$$

where

$$\left\{\begin{array}{ll} \left(1-\tilde{\alpha}^2\right)/4 = -\xi^2 = \lambda E - (m-\alpha)^2 - V_4, & -\tilde{\delta}/2 = -V_3,\;\tilde{\beta} = V_1, \\ \tilde{\gamma} - \tilde{\beta}^2/4 = \tilde{k}^2 = eB_\circ(m-\alpha) - k_z^2 - V_0 & \gamma^2 = 1 = e^2B_\circ^2/4 + V_2, \end{array}\right\} \tag{3.44}$$

Next, we use a transformation recipe similar to (3.34) and substitute

$$U(\rho) = \rho^{\left(1+\tilde{\alpha}^2\right)/2} \exp\left[-\left(\tilde{\beta}\rho + \rho^2\right)/2\right] Y(\rho) \tag{3.45}$$

in (3.43) to obtain a biconfluent Heun-type equation

$$\rho Y''(\rho)+\left[1+\tilde{\alpha}-\tilde{\beta}\rho-2\rho^{2}\right]Y'(\rho)+\left\{(\tilde{\gamma}-2-\tilde{\alpha})\rho-\frac{1}{2}\left(\tilde{\delta}+[1+\tilde{\alpha}]\,\tilde{\beta}\right)\right\}Y(\rho)=0. \tag{3.46}$$

Which admits solutions in the form of biconfluent Heun functions

$$Y(\rho)=H_{B}\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};\rho\right), \tag{3.47}$$

provided that

$$\tilde{\gamma}-2-\tilde{\alpha}\ =2n_{\rho}\ ;\ n_{\rho}=0,1,2\cdots, \tag{3.48}$$

gives again the essential quantization. Where, in this case,

$$\tilde{\gamma}=\frac{\tilde{\beta}^{2}}{4}+\tilde{k}^{2}=eB_{\circ}(m-\alpha)-k_{z}^{2}-V_{0}+\frac{V_{1}^{2}}{4}, \tag{3.49}$$

$$\tilde{\alpha}=\sqrt{1+4\left[(m-\alpha)^{2}+V_{4}-\lambda E\right]}. \tag{3.50}$$

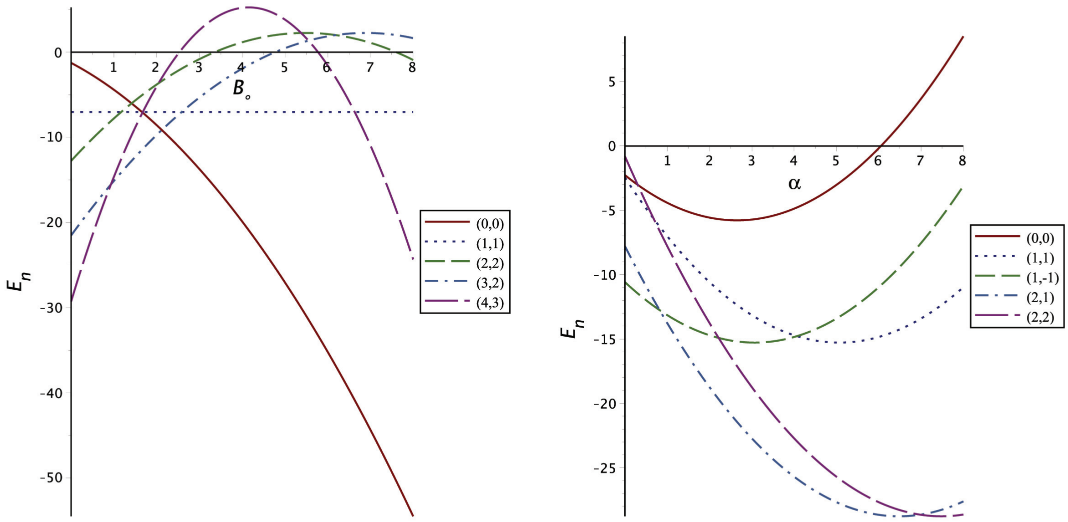
This would, in turn, imply that the eigenvalues are given by

$$E_{n_{\rho},m,\alpha}=\frac{1}{\lambda}\left\{(m-\alpha)^{2}+V_{4}+\frac{1}{4}-\frac{1}{4}\left[2\left(n_{\rho}+1\right)+k_{z}^{2}+V_{0}-\frac{V_{1}^{2}}{4}-eB_{\circ}(m-\alpha)\right]^{2}\right\}, \tag{3.51}$$

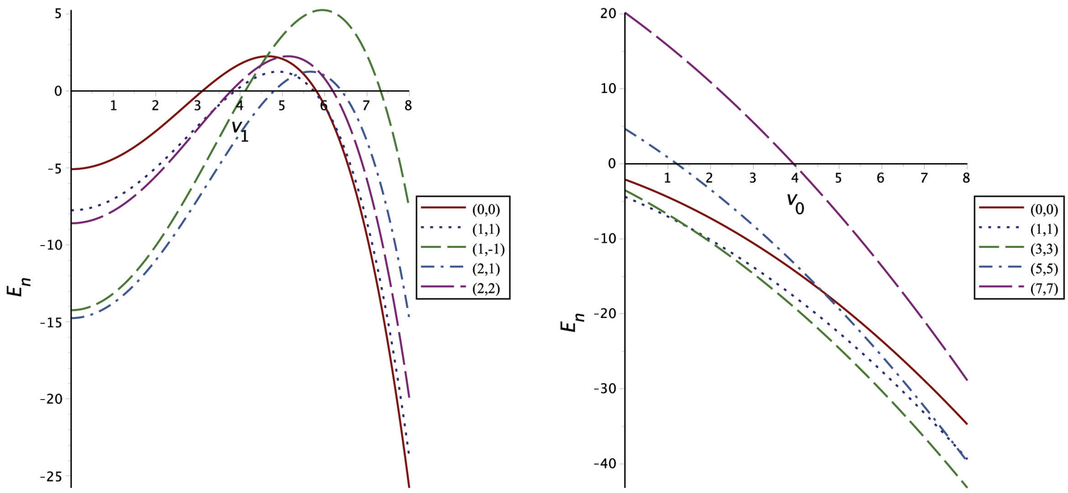
and the radial eigenfunctions are

$$R_{n_{\rho},m,\alpha}(\rho)\sim\rho^{(\tilde{\alpha}^{2}-2)/2}\exp\left(-\frac{\tilde{\beta}\rho+\rho^{2}}{2}\right)H_{B}\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};\rho\right). \tag{3.52}$$

Where $\tilde{\alpha}$ and $\tilde{\beta}$ are, respectively, defined in (3.50) and (3.44). In the two examples



(a) Energy levels vs the magnetic field strenght $B_\circ$. (b) Energy levels vs the AB-quantum number α.



(c) Energy levels vs the potential parameter V_1. (d) Energy levels vs the potential parameter V_0.

Figure 3.4: Energy levels (n_ρ, m) of (3.51) for different values of the parameters $B_\circ$, α, V_1, and V_0 in (a), (b), (c) and (d) respectively.

reported above, Models-III and IV, it is obvious that the exact analytical solutions offered by the biconfluent Heun-type equations belong to the set of PDM-Schrödinger equations that are *conditionally exactly solvable*. This is mandated by the condition $\gamma^2 = 1 = e^2 B_\circ^2/4 + V_2$ in (3.33) and again in (3.44). This would, effectively, imply that $V_2 = 1 - e^2 B_\circ^2/4$ is a condition imposed by the exact solvability of the biconfluent Heun-type equation that renders our radial PDM-Schrödinger equation (3.11) *conditionally exactly solvable*. Whereas, in Model-II of the preceding section, we have used the same mass setting but not the same condition imposed upon Model-IV above. That is why the results for the two models are not the same as should be expected.

In Figures 3.3 and 3.4, we plot the energy levels (n_ρ, m) in(3.40) and (3.51) as functions of the parameters of the magnetic and AB-flux fields $B_\circ$, α, and the parameters of the potential (3.28) respectively. As shown in these figures, a similar pattern of the energy levels crossings is also observed.

3.4 Spectral Signatures of the 1D Z-dependent Schrödinger Part on the Overall Spectra

In this section, we shall include the z-dependent part (3.10) of the PDM Schrödinger equation in (3.9)

$$\left[-\partial_z^2 + V(z)\right] Z(z) = k_z^2 Z(z),$$

and explore its contribution on the overall spectra of the four examples discussed above. We may very well consider any of the conventional textbook exactly-solvable 1D-Schrödinger equations. Therefore, there exist a large number of feasible 1D-potentials that may contribute to the problem at hand. However, for the sake of clarification and illustration of the current methodical proposal, we only choose two 1D-potentials, an infinite potential well and a Morse-type oscillator potential.

3.4.1 Case 1: Infinite Potential Well

Let us assume that our charged PDM particle is also bound to move within an impenetrable potential well of width L on the z-axis, i.e.,

$$V(z)=\begin{cases}0 & ;0<z<L\\ \infty & ;\text{elsewhere}\end{cases}. \tag{3.53}$$

This would, by the textbook boundary conditions, manifest an exact solution in the form of

$$Z(z)\sim\sin(k_z z)\Longrightarrow k_z L=(n_z+1)\pi\Longrightarrow k_z^2=\frac{(n_z+1)^2\pi^2}{L^2}\ ;\ n_z=0,1,2,\cdots. \tag{3.54}$$

Under such settings, the total eigenenergies and eigenfunctions of the four examples above are, respectively, in order. For the two *exactly solvable* models, I and II, we get

$$E_{n_\rho,m,\alpha,n_z}=\frac{1}{4\eta}\left[4\mathcal{V}_1+e^2B_\circ^2-\left(\frac{2\mathcal{V}_\circ+eB_\circ(m-\alpha)-(n_z+1)^2\pi^2/L^2}{2n_\rho+1+\sqrt{(m-\alpha)^2+\mathcal{V}_2+1/4}}\right)^2\right], \tag{3.55}$$

$$\begin{aligned}\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z)&=\mathcal{N}\,e^{im\varphi}\rho^{1+|\tilde{\ell}_1|}\sin\left(\frac{(n_z+1)\pi}{L}z\right)\\&\times\exp\left(-\frac{\sqrt{e^2B_\circ^2+4\mathcal{V}_2-4\eta E_{n_\rho,m,\alpha,n_z}}}{4}\rho^2\right)\\&\times\,_1F_1\left(-n_\rho;|\tilde{\ell}_1|+1;\frac{\sqrt{e^2B_\circ^2+4\mathcal{V}_2-4\eta E_{n_\rho,m,\alpha,n_z}}}{2}\rho^2\right)\end{aligned} \tag{3.56}$$

for Model-I, and

$$\begin{aligned}E_{n_\rho,m,\alpha,n_z}&=\frac{1}{\eta}\left\{(m-\alpha)^2+\mathcal{V}_2+\frac{1}{4}\right.\\&\left.-\left[\frac{2\mathcal{V}_\circ+eB_\circ(m-\alpha)-(n_z+1)^2\pi^2/L^2}{\sqrt{4\mathcal{V}_1+e^2B_\circ^2}}-(2n_\rho+1)\right]^2\right\},\end{aligned} \tag{3.57}$$

$$\begin{aligned}\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z)&=\mathcal{N}\,e^{im\varphi}\rho^{-1+|\tilde{\ell}_2|}\sin\left(\frac{(n_z+1)\pi}{L}z\right)\\&\times\exp\left(-\frac{\sqrt{e^2B_\circ^2+4\mathcal{V}_2}}{4}\rho^2\right)\,_1F_1\left(-n_\rho;|\tilde{\ell}_2|+1;\frac{\sqrt{e^2B_\circ^2+4\mathcal{V}_2}}{2}\rho^2\right)\end{aligned} \tag{3.58}$$

for Model-II.

For the two *conditionally exactly solvable* models, III and IV, we obtain

$$E_{n_\rho,m,\alpha,n_z} = \frac{1}{\lambda}\left\{V_1 + 2\left(2\left[n_\rho + 1 + \sqrt{(m-\alpha)^2 + V_4 + \frac{1}{16}}\right]\right.\right.$$
$$\left.\left. -eB_\circ(m-\alpha) + \frac{(n_z+1)^2\pi^2}{L^2} + V_0\right)\right\}, \tag{3.59}$$

$$\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N}\, e^{im\varphi}\rho^{\left(1+\tilde{\alpha}^2\right)/2}\sin\left(\frac{(n_z+1)\pi}{L}z\right)$$
$$\times\exp\left(-\frac{\tilde{\beta}\rho+\rho^2}{2}\right) H_B\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};\rho\right) \tag{3.60}$$

for Model-III, and

$$E_{n_\rho,m,\alpha,n_z} = \frac{1}{\lambda}\left\{(m-\alpha)^2 + V_4 + \frac{1}{4}\right.$$
$$\left. -\left[2\left(n_\rho+1\right) + \frac{(n_z+1)^2\pi^2}{L^2} + V_0 - \frac{V_1^2}{4} - eB_\circ(m-\alpha)\right]^2\right\}, \tag{3.61}$$

$$\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N}\, e^{im\varphi}\rho^{\left(\tilde{\alpha}^2-2\right)/2}\sin\left(\frac{(n_z+1)\pi}{L}z\right)$$
$$\times\exp\left(-\frac{\tilde{\beta}\rho+\rho^2}{2}\right) H_B\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};\rho\right). \tag{3.62}$$

for Model-IV.

3.4.2 Case 2: A Morse-Type Potential

If our charged PDM-particle is also influenced by a Morse-type potential (c.f., e.g., [50, 52])

$$V(z) = D\left[\exp(-2\sigma z) - 2\exp(-\sigma z)\right] \tag{3.63}$$

in the z-direction, would result in the exact eigenvalues and eigenfunctions given, respectively, as

$$k_z^2 = \left(\frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2}\right)^2, \tag{3.64}$$

$$Z(z) \sim z^{k_z} e^{-z/2} L_{n_z}^{2k_z}(z), \tag{3.65}$$

where $L_{n_z}^{2k_z}(z)$ are the Laguerre polynomials. In this case, the total eigenenergies and eigenfunctions of the four examples at hand are in order. Starting with the two *exactly solvable* models, I and II, we obtain

$$E_{n_\rho,m,\alpha,n_z} = \frac{1}{4\eta}\left[4\mathcal{V}_1 + e^2B_\circ^2 - \left(\frac{2\mathcal{V}_\circ + eB_\circ(m-\alpha) - \left(\frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2}\right)^2}{2n_\rho + 1 + \sqrt{(m-\alpha)^2 + \mathcal{V}_2 + 1/4}}\right)^2\right], \tag{3.66}$$

$$\begin{aligned}\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z) &= \mathcal{N} z^{k_z} e^{-z/2} e^{im\varphi} L_{n_z}^{2k_z}(z)\, \rho^{1+|\tilde{\ell}_1|} \\ &\times \exp\left(-\frac{\sqrt{e^2B_\circ^2 + 4\mathcal{V}_2 - 4\eta E_{n_\rho,m,\alpha,n_z}}}{4}\rho^2\right) \\ &\times {}_1F_1\left(-n_\rho; |\tilde{\ell}_1| + 1; \frac{\sqrt{e^2B_\circ^2 + 4\mathcal{V}_2 - 4\eta E_{n_\rho,m,\alpha,n_z}}}{2}\rho^2\right).\end{aligned} \tag{3.67}$$

for Model-I, and

$$\begin{aligned}E_{n_\rho,m,\alpha,n_z} = \frac{1}{\eta}\Bigg\{&(m-\alpha)^2 + \mathcal{V}_2 + \frac{1}{4} \\ &- \left[\frac{2\mathcal{V}_\circ + eB_\circ(m-\alpha) - \left(\frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2}\right)^2}{\sqrt{4\mathcal{V}_1 + e^2B_\circ^2}} - (2n_\rho + 1)\right]^2\Bigg\}\end{aligned} \tag{3.68}$$

$$\begin{aligned}\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z) &= \mathcal{N} z^{k_z} e^{-z/2} e^{im\varphi} L_{n_z}^{2k_z}(z)\, \rho^{-1+|\tilde{\ell}_2|} \exp\left(-\frac{\sqrt{e^2B_\circ^2 + 4\mathcal{V}_2}}{4}\rho^2\right) \\ &\times {}_1F_1\left(-n_\rho; |\tilde{\ell}_2| + 1; \frac{\sqrt{e^2B_\circ^2 + 4\mathcal{V}_2}}{2}\rho^2\right).\end{aligned} \tag{3.69}$$

for Model-II.

Likewise, for the two *conditionally exactly solvable* models, III and IV, we obtain

$$E_{n_\rho,m,\alpha,n_z} = \frac{1}{\lambda}\left\{V_1 + 2\left(2\left[n_\rho + 1 + \sqrt{(m-\alpha)^2 + V_4 + \frac{1}{16}}\right]\right.\right.$$
$$\left.\left. -eB_\circ(m-\alpha) + \left(\frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2}\right)^2 + V_0\right)\right\}, \qquad (3.70)$$

$$\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N}\, z^{k_z} e^{-z/2} e^{im\varphi} L_{n_z}^{2k_z}(z)\, \rho^{\left(1+\tilde{\alpha}^2\right)/2}$$
$$\times \exp\left(-\frac{\tilde{\beta}\rho + \rho^2}{2}\right) H_B\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho\right) \qquad (3.71)$$

for Model-III, and

$$E_{n_\rho,m,\alpha,n_z} = \frac{1}{\lambda}\left\{(m-\alpha)^2 + V_4 + \frac{1}{4}\right.$$
$$\left. - \left[2\left(n_\rho + 1\right) + \left(\frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2}\right)^2 + V_0 - \frac{V_1^2}{4} - eB_\circ(m-\alpha)\right]^2\right\}, \qquad (3.72)$$

$$\psi_{n_\rho,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N}\, z^{k_z} e^{-z/2} . e^{im\varphi} L_{n_z}^{2k_z}(z)\, \rho^{\left(\tilde{\alpha}^2-2\right)/2}$$
$$\times \exp\left(-\frac{\tilde{\beta}\rho + \rho^2}{2}\right) H_B\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho\right) \qquad (3.73)$$

for Model-IV, where $\mathcal{N}$ is the corresponding normalization constant.

Chapter 4

PDM CHARGED PARTICLES IN PD-MAGNETIC AND AB-FLUX FIELDS

It would be interesting to consider a PDM-charged particle moving not only in a PD-magnetic plus an AB-flux fields but also in a Yukawa-type plus a Kratzer-type molecular interaction force fields. Hereby, we need to use the Nikiforov-Uvarov (NU) method (see e.g. [53, 54]) and explore its exact solvability. In this chapter, we start with the PDM-momentum operator of (3.1) and Schrödinger equation of (3.3). Where, the vector potential takes a conventional form that satisfies the Coulomb gauge $\overrightarrow{\nabla}\cdot\overrightarrow{A}\left(\overrightarrow{r}\right)=0$ and results in a uniform constant magnetic field through the traditional textbook recipe $\overrightarrow{\nabla}\times\overrightarrow{A}\left(\overrightarrow{r}\right)=\overrightarrow{B}=B_{\circ}\widehat{z}$. However, in the construction of the vector potential $\overrightarrow{A}\left(\overrightarrow{r}\right)$, the magnetic field may turn out to be a PD-magnetic field (see e.g., [36, 55]). Therefore, the current methodical proposal, we focus our attention on PDM-charged particles in PD-magnetic and AB-flux fields, with and without the confinement potential (i.e., $V\left(\overrightarrow{r}\right)\neq 0$ and $V\left(\overrightarrow{r}\right)=0$, respectively).

4.1 Construction of the Vector Potential and PD-magnetic Fields

Let us start with PDM-charged particles interacting with the vector potential

$$\overrightarrow{A}\left(\overrightarrow{r}\right)=\overrightarrow{A}_1\left(\overrightarrow{r}\right)+\overrightarrow{A}_2\left(\overrightarrow{r}\right)\Rightarrow\begin{cases}\overrightarrow{A}_1\left(\overrightarrow{r}\right)=\left(0,B_{\circ}\rho\,S\left(\rho\right)/2,0\right)\\ \\ \overrightarrow{A}_2\left(\overrightarrow{r}\right)=\left(0,\Phi_{AB}/2\pi\rho,0\right)\end{cases},\tag{4.1}$$

where a PD-magnetic field is manifested by the vector potential $\overrightarrow{A}_1\left(\overrightarrow{r}\right)$ so that

$$\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}_1(\overrightarrow{r}) = B_\circ \left[S(\rho) + \frac{\rho}{2} S'(\rho)\right] \widehat{z}\,; \;\; S'(\rho) = \frac{dS(\rho)}{d\rho} \tag{4.2}$$

Here, $\overrightarrow{\nabla} \times \overrightarrow{A}_2(\overrightarrow{r}) = 0$ with $\overrightarrow{A}_2(\overrightarrow{r})$ describing the AB-flux field Φ_{AB} effect (see, e.g., [42, 54, 56]), and $S(\rho)$ is a dimensionless scalar multiplier and is a byproduct of the construction process of the vector potential $\overrightarrow{A}_1(\overrightarrow{r})$. Consequently, our PDM-charged particle interacts with the total vector potential.

$$\overrightarrow{A}(\overrightarrow{r}) = \left(0, \frac{B_\circ}{2} \rho S(\rho) + \frac{\Phi_{AB}}{2\pi\rho}, 0\right) = (0, A_\varphi, 0)\,. \tag{4.3}$$

At this point, we use the assumptions that the PDM function is only radially dependent, i.e.,

$$m(\overrightarrow{r}) = m(\rho, \varphi, z) = g(\rho)\,, \tag{4.4}$$

and $V(\varphi) = 0$ to secure azimuthal symmetrization so that

$$g(\rho) W(\rho, \varphi, z) = V(\rho) + V(z)\,. \tag{4.5}$$

This would, in turn, facilitate separability of the PDM-Schrödinger equation (3.3) at hand and allow the substitution of the wavefunction

$$\psi(\overrightarrow{r}) = \psi(\rho, \varphi, z) = R(\rho) Z(z) e^{im\varphi}, \tag{4.6}$$

to obtain

$$\frac{R''(\rho)}{R(\rho)}-\left(\frac{g'(\rho)}{g(\rho)}-\frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}-\frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}+\frac{g'(\rho)}{\rho g(\rho)}\right)+\frac{7}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2$$
$$-\frac{m^2}{\rho^2}+\frac{2em}{\rho}A_\varphi-e^2A_\varphi^2+g(\rho)E-V(\rho)-k_z^2=0 \tag{4.7}$$

and

$$\frac{Z''(z)}{Z(z)}-V(z)=k_z^2. \tag{4.8}$$

Consequently, the radially-dependent part along with (4.3) reads

$$\left[\frac{R''(\rho)}{R(\rho)}-\left(\frac{g'(\rho)}{g(\rho)}-\frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}-\frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}+\frac{g'(\rho)}{\rho g(\rho)}\right)+\frac{7}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2\right.$$
$$\left.-\frac{\tilde{m}^2}{\rho^2}+e\tilde{m}B_\circ S(\rho)-k_z^2-\frac{e^2B_\circ^2}{4}\left[\rho S(\rho)\right]^2+g(\rho)E-V(\rho)\right]=0 \tag{4.9}$$

Further simplification of the radial equation can be carried out by using the substitution in (3.12) to obtain the one-dimensional form of the PDM-Schrödinger equation (4.9)

$$\left\{-\frac{d^2}{d\rho^2}+\frac{\tilde{m}^2-1/4}{\rho^2}+V_{eff}(\rho)+k_z^2\right\}U(\rho)=0, \tag{4.10}$$

where, now,

$$V_{eff}(\rho)=V(\rho)-e\tilde{m}B_\circ S(\rho)+\frac{e^2B_\circ^2}{4}\rho^2S(\rho)^2-g(\rho)E$$
$$+\left[\frac{5}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2-\frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}\right)-\frac{1}{4}\left(\frac{g'(\rho)}{\rho g(\rho)}\right)\right]. \tag{4.11}$$

Equation (4.10) is to be solved for different PDM functions and PD-magnetic fields. Yet, should one be interested in the two-dimensional flat-land polar coordinates (ρ,φ), then the substitutions $Z(z)=1$, and $V(z)=k_z^2=0$ could perfectly get the job done. To construct the PD-magnetic fields, we observe that the choice of $S(\rho)$, in (4.2), is not a random one at all. It is very much related to the feasibly experimentally applicable nature of the PD-magnetic fields. The choice that

$$\overrightarrow{B} = B_\circ \left[\frac{\mu}{\rho^\sigma}\right]\widehat{z} \Longleftrightarrow S(\rho) = \left(\frac{2\mu}{2-\sigma}\right)\rho^{-\sigma} + \frac{\acute{\beta}}{\rho^2}; \sigma \neq 2, \tag{4.12}$$

looks viable and interesting. Where $\mu \neq 0$, otherwise the magnetic field is switched off. Therefore, $S(\rho)$ works as a generating function for the PD-magnetic fields, where for $\mu = 1$ and $\sigma = 0$ we recover the constant magnetic field settings. Nevertheless, in the current methodical proposal we wish to work with the most simplistic PD-magnetic field where $\sigma = 1$, so that

$$\overrightarrow{B} = B_\circ \left[\frac{\mu}{\rho}\right]\widehat{z} \Longleftrightarrow S(\rho) = \frac{2\mu}{\rho} + \frac{\acute{\beta}}{\rho^2} \tag{4.13}$$

This would, in turn, imply that equation(4.10) be rewritten as

$$\left\{-\frac{d^2}{d\rho^2} + \frac{\tilde{m}^2 - 1/4 - e\tilde{m}B_\circ\acute{\beta} + e^2B_\circ^2\acute{\beta}^2/4}{\rho^2} - \frac{\left(2e\tilde{m}B_\circ\mu - e^2B_\circ^2\mu\acute{\beta}\right)}{\rho} - g(\rho)E + V(\rho) \right.$$
$$\left. + \left[\frac{5}{16}\left(\frac{g'(\rho)}{g(\rho)}\right)^2 - \frac{1}{4}\left(\frac{g''(\rho)}{g(\rho)}\right) - \frac{1}{4}\left(\frac{g'(\rho)}{\rho\, g(\rho)}\right)\right]\right\} U(\rho) = \tilde{E}U(\rho), \tag{4.14}$$

where

$$\tilde{E} = -\left(k_z^2 + e^2B_\circ^2\mu^2\right) \tag{4.15}$$

Next, we shall be interested in a PDM function in the form of

$$g(\rho) = \eta\, f(\rho)\exp(-\delta\rho) \tag{4.16}$$

where $f(\rho)=1$, $\delta=0$, and $\eta=1$ allow the problem to recover constant mass settings. Yet, we shall choose some specific values for these parameters in such a way that serves and clarifies the current methodical proposal.

4.2 Almost Quasi-Free Case: $V(\rho)=0$

Equation (4.14) suggests two exactly solvable textbook-models that constitute two *almost-quasi-free* PDM-charged particles of fundamental Coulombic nature. The two examples are in order.

4.2.1 An Almost Quasi-Free PDM-Charged Particle of $g(\rho)=\eta/\rho$

Let us consider a PDM-charged particle with $g(\rho)=\eta/\rho$ (i.e., $f(\rho)=1/\rho$ and $\delta=0$ in (4.16)) moving in the vector potential (4.3) that yields the PD-magnetic field of (4.13). Hence, equation (4.14) reads

$$\left\{-\frac{d^2}{d\rho^2}+\frac{\tilde{\ell}^2-1/4}{\rho^2}-\frac{\acute{\alpha}}{\rho}\right\}U(\rho)=\tilde{E}U(\rho), \tag{4.17}$$

where

$$\acute{\alpha}=2e\tilde{m}B_\circ\mu-e^2B_\circ^2\mu\acute{\beta}+\eta E, \tag{4.18}$$

and

$$\tilde{\ell}^2=\tilde{m}^2+\frac{1}{16}-e\tilde{m}B_\circ\acute{\beta}+\frac{e^2B_\circ^2\acute{\beta}^2}{4}\Longleftrightarrow|\tilde{\ell}|=\sqrt{\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\frac{1}{16}} \tag{4.19}$$

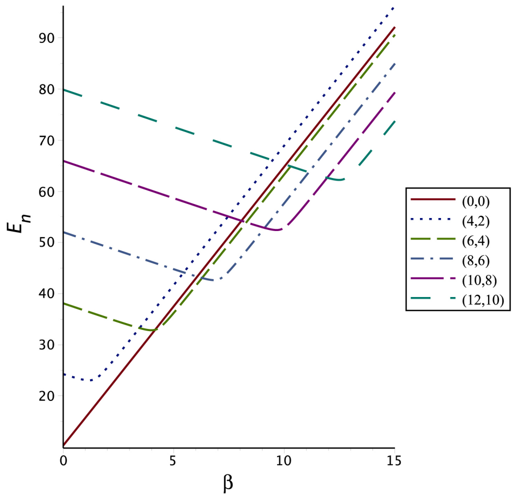
Equation (4.17) is similar to the radial Schrödinger equation of the two-dimensional Coulombic problem and admits exact eigenvalues

$$\tilde{E}=-\frac{\acute{\alpha}^2}{\left[2\left(n_\rho+|\tilde{\ell}|+1/2\right)\right]^2}\Longleftrightarrow\left(k_z^2+e^2B_\circ^2\mu^2\right)=\frac{\acute{\alpha}^2}{\left[2\left(n_\rho+|\tilde{\ell}|+1/2\right)\right]^2}, \tag{4.20}$$

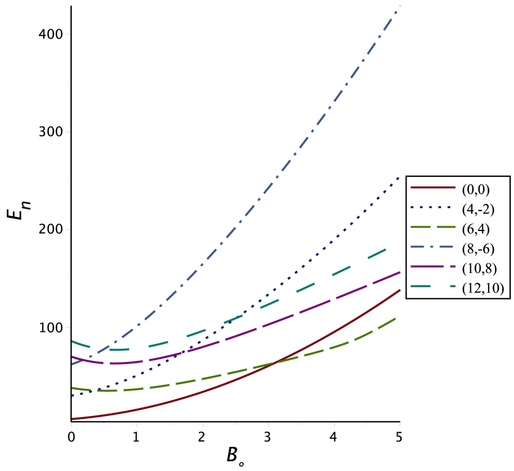
which would in turn lead to

$$E_{n_\rho,m,\alpha} = \frac{1}{\eta}\left[\acute{\beta}\mu e^2 B_\circ^2 - 2e\,(m-\alpha)\,B_\circ\mu \right.$$
$$\left. +2\sqrt{k_z^2+e^2B_\circ^2\mu^2}\left(n_\rho+\frac{1}{2}\sqrt{\left(m-\alpha-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\frac{1}{16}}\right)\right]. \quad (4.21)$$

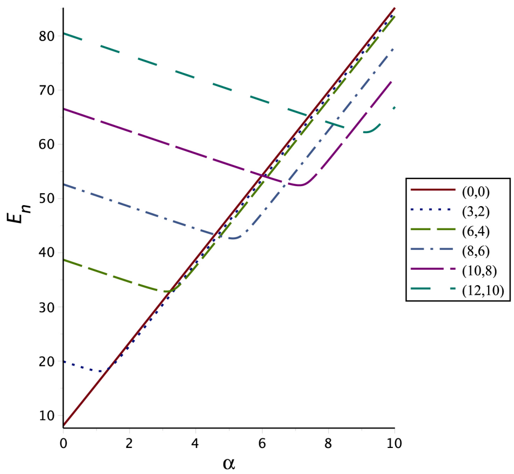
where $n_\rho = 0, 1, 2, \cdots$.



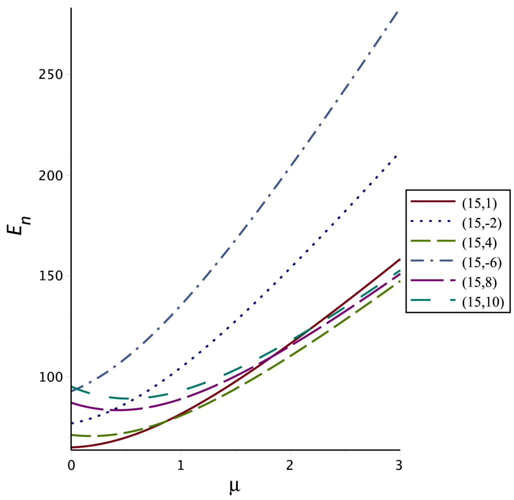
(a) Energy levels vs the parameter $\acute{\beta}$.



(b) Energy levels vs the magnetic field strenght $B_\circ$.



(c) Energy levels vs the AB-quantum number α.



(d) Energy levels vs the magnetic field parameter μ.

Figure 4.1: Energy levels (n_ρ, m) of (4.21) as a function of the parameters $\acute{\beta}$, $B_\circ$, α, and μ in (a), (b), (c) and (d), respectively.

The radial eigenfunctions are

$$R_{n_\rho,m,\alpha}(\rho)=\mathcal{N}\rho^{|\tilde{\ell}|-1/2}\exp\left(-\sqrt{k_z^2+e^2B_\circ^2\mu^2}\rho\right)L_{n_\rho}^{2|\tilde{\ell}|}\left(2\sqrt{k_z^2+e^2B_\circ^2\mu^2}\rho\right), \quad (4.22)$$

where $L_{n_\rho}^{2|\tilde{\ell}|}\left(2\sqrt{k_z^2+e^2B_\circ^2\mu^2}\rho\right)$ are the Laguerre polynomials, and n_ρ is the radial quantum number.

In Figures 4.1 we plot the energy levels $(n_\rho.m)$ of (4.21) as a functions of the parameters β, $B_\circ$, α, and μ in 4.1a, 4.1b, 4.1c, and 4.1d, respectively. The quantum numbers of a given state $(n_\rho.m)$ are chosen at random so that the phenomenon of energy levels crossings is made clear. Such energy levels crossing points indicate occasional degeneracies of the energy spectra.

4.2.2 An Almost Quasi-Free PDM-Charged Particle of $g(\rho)=\eta/\rho^2$

A PDM-charged particle with $g(\rho)=\eta/\rho^2$ (i.e., $f(\rho)=1/\rho^2$ and $\delta=0$ in (4.16)) moving under the influence of only the vector potential (4.3) would result in presenting (4.14) as

$$\left\{-\frac{d^2}{d\rho^2}+\frac{\ell^2-1/4}{\rho^2}-\frac{\acute{\gamma}}{\rho}\right\}U(\rho)=\tilde{E}\,U(\rho), \quad (4.23)$$

where,

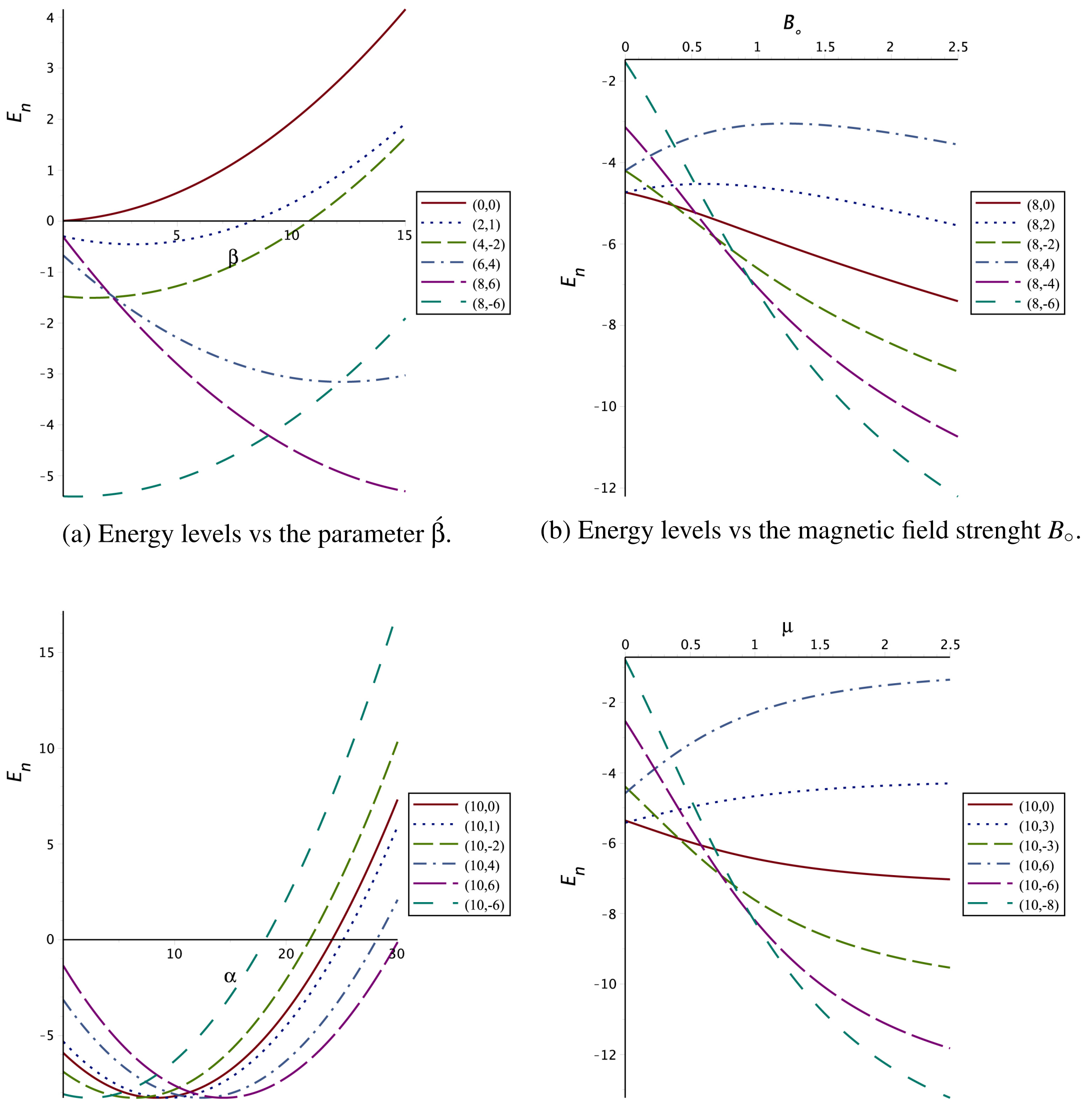
$$\acute{\gamma}=2e\tilde{m}B_\circ\mu-e^2B_\circ^2\mu\acute{\beta}, \quad (4.24)$$

and

$$\ell^2=\tilde{m}^2+\frac{1}{4}-e\tilde{m}B_\circ\acute{\beta}+\frac{e^2B_\circ^2\acute{\beta}^2}{4}-\eta E\Longleftrightarrow\left|\ell\right|=\sqrt{\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\frac{1}{4}-\eta E}. \quad (4.25)$$

We have again a similar two-dimensional radial Schrödinger equation of Coulombic nature. One may, in a straightforward manner, show that it admits the exact eigenvalues

$$E_{n_\rho,m,\alpha}=\frac{1}{\eta}\left[\left(m-\alpha-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\frac{1}{4}-\left(\frac{2e\,(m-\alpha)\,B_\circ\mu-e^2B_\circ^2\mu\acute{\beta}}{2\sqrt{k_z^2+e^2B_\circ^2\mu^2}}-n_\rho-\frac{1}{2}\right)^2\right], \tag{4.26}$$



(a) Energy levels vs the parameter $\acute{\beta}$.

(b) Energy levels vs the magnetic field strenght $B_\circ$.

(c) Energy levels vs the AB-quantum number α.

(d) Energy levels vs the magnetic field parameter μ.

Figure 4.2: Energy levels (n_ρ, m) of (4.26) as a function of the parameters $\acute{\beta}$, $B_\circ$, α, and μ in (a), (b), (c) and (d), respectively.

The exact radial wavefunctions

$$R_{n_\rho,m,\alpha}(\rho)=\mathcal{N}\rho^{-1+|\ell|}\exp\left(-\sqrt{k_z^2+e^2B_\circ^2\mu^2}\rho\right)L_{n_\rho}^{2|\ell|}\left(2\sqrt{k_z^2+e^2B_\circ^2\mu^2}\rho\right) \quad (4.27)$$

In Figures 4.2, we plot the energy levels $\left(n_\rho.m\right)$ in (4.26) versus different values of the parameters involved. The the quantum states $\left(n_\rho.m\right)$ are also chosen at random so that the phenomenon of energy levels crossings is made clear in these cases as well.

4.3 PDM-Charged Particles in PD-Magnetic and AB Flux Fields: NU Exact Solvability

In this section, we shall be interested in a PDM-charged particle endowed with a Yukawa-type mass function

$$g(\rho)=\eta\left(\frac{\exp(-\delta\rho)}{\rho}\right) \quad (4.28)$$

(i.e., $f(\rho)=1/\rho$ and $\delta\neq 0$) moving in the vector potential (4.3) that yields the PD-magnetic field in (4.13). Moreover, we would like to subject this PDM-charged particle to radial confining potential of the form

$$V(\rho)=-\frac{\widetilde{V}_\circ\exp(-\delta\rho)}{\rho}-\frac{\widetilde{V}_1}{\rho}+\frac{\widetilde{V}_2}{\rho^2}, \quad (4.29)$$

which indulges within, a Yukawa-type (i.e., the first term) plus a Kratzer-type (the last two terms) potentials. A confinement potential type commonly used in the spectroscopy of the diatomic molecules, where the Greene-Aldrich approximation

$$\frac{1}{\rho}\simeq\frac{\delta}{1-\exp(-\delta\rho)}\Longleftrightarrow\frac{1}{\rho^2}\simeq\frac{\delta^2}{\left[1-\exp(-\delta\rho)\right]^2} \quad (4.30)$$

is valid for $\rho\ll 1$. Hence, equation (4.14) reads

$$\left\{-\frac{d^2}{d\rho^2}+\frac{a_1}{\rho^2}+\frac{a_2}{\rho}-a_3\left(\frac{\exp(-\delta\rho)}{\rho}\right)+a_4\right\}U(\rho)=0, \tag{4.31}$$

where

$$\begin{aligned}
a_1 &= \tilde{m}^2-3/16-e\tilde{m}B_\circ\acute{\beta}+e^2B_\circ^2\acute{\beta}^2/4+\widetilde{V}_2,\\
a_2 &= e^2B_\circ^2\mu\acute{\beta}-2e\tilde{m}B_\circ\mu+3\delta/8-\widetilde{V}_1\\
a_3 &= \widetilde{V}_\circ+\eta E \qquad (4.32)\\
a_4 &= k_z^2+e^2B_\circ^2\mu^2+\delta^2/16
\end{aligned}$$

Next, the use of Greene-Aldrich approximation (4.30) in (4.31) would allow us to rewrite it as

$$\left\{-\frac{d^2}{d\rho^2}+\frac{a_1\delta^2}{[1-\exp(-\delta\rho)]^2}+\frac{a_2\delta}{1-\exp(-\delta\rho)}-a_3\left(\frac{\delta\exp(-\delta\rho)}{1-\exp(-\delta\rho)}\right)+a_4\right\}U(\rho)=0. \tag{4.33}$$

Let us now use the substitution $\xi=\exp(-\delta\rho)$ and convert this equation into a Nikiforov-Uvarov type (see e.g. [53, 54, 57]) to obtain

$$\begin{aligned}
&\frac{d^2U(\xi)}{d\xi^2}+\frac{(1-\xi)}{\xi(1-\xi)}\frac{dU(\xi)}{d\xi}+\frac{1}{[\xi(1-\xi)]^2}\\
&\times\left[-(\tilde{a}_1-\tilde{a}_2+\tilde{a}_4)+(-\tilde{a}_2+\tilde{a}_3+2\tilde{a}_4)\,\xi-(\tilde{a}_3+\tilde{a}_4)\,\xi^2\right]U(\xi)=0 \qquad (4.34)
\end{aligned}$$

where

$$\tilde{a}_1=a_1\;,\;\tilde{a}_2=-a_2/\delta,\;\tilde{a}_3=a_3/\delta\;,\;\tilde{a}_4=a_4/\delta^2. \tag{4.35}$$

We may, therefore, express this equation in the Nikiforov-Uvarov form

$$U''(\xi)+\frac{\tilde{\tau}(\xi)}{\sigma(\xi)}U'(\xi)+\frac{\tilde{\sigma}(\xi)}{\sigma(\xi)^2}U(\xi)=0, \tag{4.36}$$

where

$$\begin{aligned}&\tilde{\tau}(\xi)=1-\xi\,,\quad \sigma(\xi)=\xi(1-\xi)\\ &\tilde{\sigma}(\xi)=-(\tilde{a}_1-\tilde{a}_2+\tilde{a}_4)+(-\tilde{a}_2+\tilde{a}_3+2\tilde{a}_4)\,\xi-(\tilde{a}_3+\tilde{a}_4)\,\xi^2\end{aligned}\;. \tag{4.37}$$

Which obviously satisfies the requirements of NU-method, where $\sigma(\xi)$, $\tilde{\sigma}(\xi)$ are polynomials of at most second degree, and $\tilde{\tau}(\xi)$ is at most a first degree polynomial. The NU-method is a well known approach. We, therefore, closely follow Mustafa and Algadhi's [58] Appendix (namely, equations (A.1) to (A.20), where instructive and informative details on NU-method are available), with $\tilde{a}_3=\left(\eta E+\widetilde{V}_\circ\right)/\delta$ in (4.35) and (4.32), we obtain

$$\tilde{a}_3=\left(n_\rho^2+n_\rho+1/2\right)+\left(2n_\rho+1\right)\varepsilon_1+\varepsilon_2 \tag{4.38}$$

where ε_1 and ε_2 are given through the relations $\varepsilon_1=\tilde{\varepsilon}_1/\delta$ and $\varepsilon_2=\tilde{\varepsilon}_2/\delta$ so that

$$\begin{aligned}\tilde{\varepsilon}_1=&\left[\delta^2\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\delta^2\widetilde{V}_2+\frac{\delta^2}{4}-2eB_\circ\mu\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)\delta\right.\\ &\left.-\delta\widetilde{V}_1+e^2B_\circ^2\mu^2+k_z^2\right]^{1/2}+\delta\left[\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\widetilde{V}_2+\frac{1}{16}\right]^{1/2}\end{aligned} \tag{4.39}$$

and

$$\tilde{\varepsilon}_2 = 2\left\{\left[\delta^2\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\delta^2\widetilde{V}_2+\frac{\delta^2}{4}-2eB_\circ\mu\right.\right.$$
$$\left.\left.\times\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)\delta-\delta\widetilde{V}_1+e^2B_\circ^2\mu^2+k_z^2\right]\left[\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\widetilde{V}_2+\frac{1}{16}\right]\right\}^{1/2}$$
$$+2\left[\delta\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)^2+\delta\widetilde{V}_2-eB_\circ\mu\left(\tilde{m}-\frac{eB_\circ\acute{\beta}}{2}\right)\right]-\widetilde{V}_1. \tag{4.40}$$

This would eventually imply

$$E_{n_\rho,m,\alpha} = \frac{1}{\eta}\left\{\left(n_\rho^2+n_\rho+1/2\right)\delta+\left(2n_\rho+1\right)\tilde{\varepsilon}_1+\tilde{\varepsilon}_2-\widetilde{V}_\circ\right\} \tag{4.41}$$

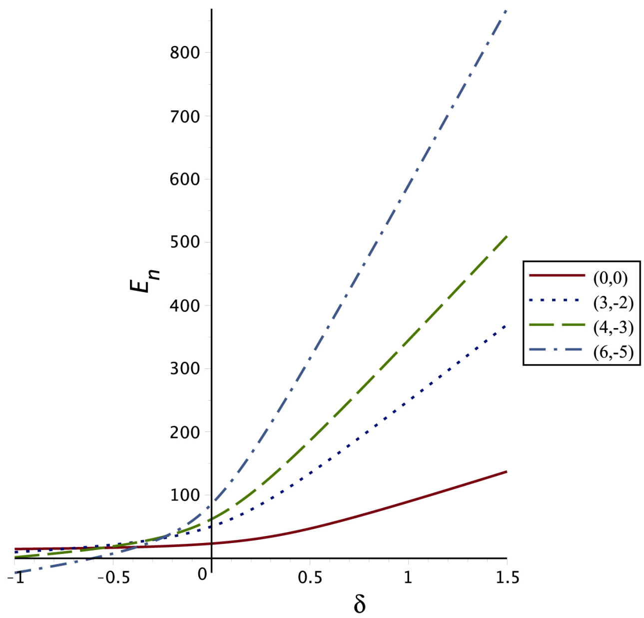


Figure 4.3: Energy levels $(n_\rho.m)$ crossings of (4.41) for different values δ in (4.28).

One should notice that the result in (4.41) recovers that of the almost quasi-free PDM-charged particle in (4.21) by setting $\delta=0$ and $\widetilde{V}_\circ=\widetilde{V}_1=\widetilde{V}_2=0$ in (4.28) and (4.29). This should be the typical tendency (as well as a double check) of the exact analytical solution of the more general problem discussed here, of course.

In Figure 4.3, we plot the energies of (4.37) against the PDM parameter δ of (4.28). We observe a direct effect of the PDM on the energy levels crossings indicating again occasional degeneracies.

Furthermore, the radial wave functions are given by (3.12), and ((A.1), and (A.23)) in the Appendix of [58] to yield

$$R_{n_\rho,m,\alpha}(\rho) = \mathcal{N}\rho^{-(1-\upsilon)/2}\exp\left(-\delta\rho\left(1+\kappa\right)/2\right)P_{n_\rho}^{(\kappa,\upsilon)}\left(1-2e^{-\delta\rho}\right) \tag{4.42}$$

where $\mathcal{N}$ are the corresponding normalization constants.

Chapter 5

CONCLUSIONS

We have investigated/studied PDM-quantum particles under the influence of electromagnetic and/or interaction potential fields. In the process, however, we have also included the AB-flux field effect to provide a more general treatment for the problems at hand. We were able to extract some exact or conditionally exact solutions for the corresponding PDM-Schrödinger equations. The strategy we have followed is in a sequential order.

We have first introduced some basic concepts of fundamental importance for PDM-quantum mechanics. We have introduced and build-up the PDM-momentum operator and consequently the PDM-minimal coupling of electromagnetic interactions. In so doing, we had to return back to the fundamentals of classical and/or quantum mechanical constant mass settings. Using some non-local point transformation of some generalized coordinates, we were able to map our constant mass settings into PDM-settings. As such we were able to elaborate on the structures of the PDM-momentum operator and PDM-minimal coupling. This approach allowed us to fix the ordering-ambiguity of the von Ross Hamiltonian (1.1), through the strictly determined ordering-ambiguity parametric settings, at $a = c = -1/4$ and $b = -1/2$ (known in the literature as MM-ordering, i.e, Mustafa and Mazharimousavi's [13]).

However, we have also reported/argued-out that while in PDM-classical mechanics it is safe to use the textbook minimal coupling,

$$\pi_j\left(\vec{q}\left(\vec{x}\right)\right) \longrightarrow \pi_j\left(\vec{q}\left(\vec{x}\right)\right) - eA_j\left(\vec{q}\left(\vec{x}\right)\right) \Longrightarrow P_j\left(\vec{x}\right) \longrightarrow P_j\left(\vec{x}\right) - eA_j\left(\vec{x}\right), \tag{5.1}$$

it is necessary and vital to use

$$\left(\frac{\widehat{P}_j\left(\vec{x}\right)}{\sqrt{m\left(\vec{x}\right)}}\right) \longrightarrow \left(\frac{\left(\widehat{P}_j\left(\vec{x}\right) - eA_j\left(\vec{x}\right)\right)}{\sqrt{m\left(\vec{x}\right)}}\right) \tag{5.2}$$

for PDM-quantum mechanics. This would, in turn, suggest that the PDM-kinetic energy operator must be expressed as $\widehat{T} = \left(\widehat{P}_j\left(\vec{x}\right)/\sqrt{m\left(\vec{x}\right)}\right)^2$ and not $\left(\widehat{P}_j\left(\vec{x}\right)\right)^2/m\left(\vec{x}\right)$. Yet, we have found that among the two commonly used vector potentials $\vec{A}\left(\vec{q}\left(\vec{x}\right)\right) = B_\circ(-q_2\left(\vec{x}\right),0,0)$ and $\vec{A}\left(\vec{q}\left(\vec{x}\right)\right) = B_\circ(-q_2\left(\vec{x}\right),q_1\left(\vec{x}\right),0)/2)$, only the later

$$\vec{A}\left(\vec{q}\left(\vec{x}\right)\right) = \frac{B_\circ}{2}(-q_2\left(\vec{x}\right),q_1\left(\vec{x}\right),0) = \frac{S\left(r\right)}{\sqrt{m\left(r\right)}}\frac{B_\circ}{2}(-x_2,x_1,0), \tag{5.3}$$

satisfies the Coulomb gauge $\partial_{q_j}A_j\left(\vec{q}\right) = 0$ (within our PDM-point transformation settings, of course). This is done in chapter 2 along with illustrative examples.

Next, using our findings above, we have considered (in chapter 3) PDM-charged particles in a uniform magnetic plus AB-flux fields and some interaction potentials (including some pseudo-harmonic oscillator, and Killingbeck-type interaction potentials). Hereby, we have explored the separability of the corresponding PDM-Schrödinger equation under radial cylindrical and azimuthal symmetrization settings. A simple one-dimensional textbook, a purely z-dependent (3.10), and a purely radial ρ-dependent (3.11) Schrödinger equations are obtained. In the radial ρ-dependent (3.11) part, we have transformed it into a radial one-dimensional Schrödinger form (3.13) and used two PDM settings, $g(\rho) = \eta\rho^2$ and $g(\rho) = \eta/\rho^2$. We have reported on the *exact solvability* (both eigenvalues and eigenfunctions) of our PDM charged particles moving in three fields: a uniform magnetic, an AB-flux, and the pseudo-harmonic oscillator potential (i.e., usual settings for charged particles in quantum dots and antidotes, e.g., [40–42], but here we have PDM-charged particles). This is documented in section 3.2. Moreover, we have used the radial ρ-dependent part (3.11) as is and used the biconfluent Heun differential forms for two PDM settings, $g(\rho) = \lambda\rho$ (3.35) and $g(\rho) = \lambda/\rho^2$ (3.46). We have reported on their *conditionally exact solvability* (for both eigenvalues and eigenfunctions) in section 3.3. Yet, the spectral signatures of the one-dimensional z-dependent Schrödinger part (3.10) on the overall eigenvalues and eigenfunctions are reported, in section 3.4. Where two z-dependent potential models (infinite potential well (3.53) and Morse type potentials (3.63)) wre used for each of the four examples in section 3.2 and 3.3. In Figures 3.1 - 3.4, moreover, we have plotted the energy levels against the uniform magnetic field $B_\circ$, AB-flux quantum number α, and some interaction potential parameters (one at a time, of course). Energy levels crossings are observed at some specific parametric values. This necessarily means that there could be more than one quantum state sharing the same energy at each crossing point. Therefore, such energy levels crossings may very well be classified as "*occasional degeneracies*" that have erupted as a result of PDM setting.

As to the last part of our dissertation, we have considered a more general assumption for the vector potential. That is, we suggested that

$$\vec{A}_1(\vec{r}) = S(\rho)\overrightarrow{\tilde{A}_1}(\vec{r}) = \left(0, \frac{B_\circ}{2}\rho S(\rho), 0\right), \tag{5.4}$$

where $S(\rho)$ is a scalar multiplier that may absorb any position-dependent terms that may emerge in the construction process of the vector potential $\vec{A}_1(\vec{r})$ (c.f, e.g., [36]). This is the focal point of chapter 4, where we have considered some PDM-charged particles in PD-magnetic and AB-flux fields. Two *almost-quasi-free* PDM-charged particles, with $g(\rho) = \eta/\rho$ and $g(\rho) = \eta/\rho^2$, turned out to imply exactly solvable radial Schrödinger equations of a Coulombic nature (documented in (4.21) and (4.22)). Their exact solutions were inferred from the textbook solutions. Moreover, a Yukawa-type PDM-charged particles with $g(\rho) = \eta\exp(-\delta\rho)/\rho$ moving in a PD-magnetic plus AB-flux fields and a Yukawa-Kratzer type confining potential $V(\rho) = -\widetilde{V}_\circ\exp(-\delta\rho)/\rho - \widetilde{V}_1/\rho + \widetilde{V}_2/\rho^2$ were considered. In this case, we have used the NU-method to obtain exact analytical eigenvalues and eigenfunctions (reported in (4.41) and (4.42), respectively). Moreover, the phenomenon of energ levels crossings repeats itself again in Figures 4.1-4.3. Therein, PDM-settings were observed to manifest " *occasional degeneracies*" of the energy spectral properties.

In the light of our experience above, our concluding remarks are in order.

Our methodical proposal is not restricted to analytically exact or analytically conditionally exact solvabilities reported in this dissertation. It is also applicable to Schrödinger-like models that admit numerically exact or numerically conditionally exact solvabilities (c.f. e.g., [59]). It may very well be applied to quasi-exactly solvable models (c.f. e.g., Quesne [60]), or even to non-Hermitian and pseudo-Hermitian Hamiltonian settings (c.f. e.g., [61–64]). Likewise, this would hold true for the z-dependent Schrödinger-like equation (3.10).

Having had the eigenenergies exactly or conditionally exactly obtained, one may use them to calculate the partition functions and discuss some thermodynamical properties (e.g., [50,65,66]) of such PDM systems in PD-magnetic and AB-fields along with any confining potential.

Although our methodical proposal above is introduced to deal with a three-dimensional PDM-Schrödinger equation, it is also feasibly applicable to a more commonly used two-dimensional problems (c.f., e.g. Dutra and Oliveira [25] or Correa et al. [2]). However, the three-dimensional case is a more general and instructive one.

Finally, we may also report that our methodical proposal above is used to study the Landau quantization for an electric quadrupole moment of PDM-neutral particles interacting with electromagnetic fields. The details of which are comprehensively and instructively reported by Algadhi and Mustafa [67]. Moreover, the reader is advised to seek more details on chapter 2, 3, and 4 in [36], [56], and [58] by Mustafa and Algadhi, respectively, where comprehensive discussions are provided.

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