Counting Shortest Paths in Grids

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ABSTRACT

It is well known that Pascal's triangle represents the binomial coefficients in binomial expansion. These numbers can also be interpreted as the numbers of (shortest) paths from the given position to top element of the triangle allowed to use two types of steps in these paths (left-up and right-up steps). The binomial coefficients show, in fact, the number of shortest paths in the square grid if only grid paths, i.e., paths on the grid lines (paths with cityblock distance), are allowed, and they also give the number of shortest paths in the hexagonal grid. When diagonal steps are also allowed in the square grid (paths with chessboard distance), the number of shortest paths can be described by trinomial coefficients. They can be obtained also in a triangle form by summing up three neighbour elements in the previous row. We consider also further generalisations of such triangles and their elements, quadrinomial and *n*-nomial coefficients. In this context, *n*-nomial coefficients of *n*-nomial expansions represent the numbers of paths from the position of the coefficient up to the top element of the triangle allowed to use *n* different types of steps, such that for trinomial coefficients, we use three types of steps, and quadrinomial coefficients we use four types of steps. We also present formulae to calculate trinomial, quadrinomial and *n*-nomial coefficients based on trinomial, quadrinomial and *n*-nomial expansions, where the power of the sum of more than two items is computed, respectively. Multinomial expansions are also related. We give also a comparison of those values known as various ways of generalisations of the binomial coefficients.

The number of shortest paths between any point pairs of the square grid, based on weighted distances, is computed. We use an 8-adjacency square grid, that is, a first weight is associated to the horizontal and vertical movements, while a second weight (not necessarily different from the first) is assigned to the diagonal steps. The chamfer distance of two points depends on the numbers and weights of the steps in a 'shortest path'. In special cases, as we have already mentioned, the cityblock and the chessboard distances, the two most basic and widely used digital distances of the two-dimensional digital space occur. Although our combinatorial result is theoretical, it is closely connected to applications, such as communication networks, path counting in digital images, traces and trajectories in 2D digital grids. We consider all the seven cases with non-negative weights and also the case when negative weights are allowed.

Also, we will discuss the number of weighted shortest paths between any two pixels in the triangular grid, where the number of shortest paths depend on the values of α, β and γ weights. In the triangular grid for each pixel, we have three types of neighbourhood:1st, 2nd and 3rd neighbourhood, where we assign a weight for each neighbourhood type, and according to these weights, we use Chamfer distance to define these shortest paths, and we use combinations of absolute differences between pixels to define number of these paths.

Keywords: Binomial Coefficients; Trinomial Coefficients; Quadrinomial Coefficients; *n*-nomial Coefficients; Multinomial Coefficients; Trajectories; Weighted Distances; Digital Distances; Combinatorics; Triangular Grid, Neighbourhood Types, Chamfer Distance; Shortest Weighted Paths; Path Counting.

ÖZ

Pascal üçgeninin binom açılımındaki binom katsayılarını temsil ettiği iyi bilinmektedir. Bu sayılar, aynı zamanda, bu konumlarda iki tür adım (sol-yukarı ve sağ-yukarı adımlar) kullanılmasına izin verilen, verilen konumdan üçgenin üst elemanına kadar (en kısa) yolların sayısı olarak da yorumlanabilir. Binom katsayıları, aslında sadece ızgara yollarının, yani ızgara çizgilerindeki yollara (şehir bloğu uzaklığına sahip yollar) izin verilirse, kare ızgaradaki en kısa yolların sayısını gösterir ve ayrıca altıgen ızgaradaki en kısa yolların sayısını verir. Kare ızgarada (satranç tahtası uzaklıklı yollar) köşegen adımlara da izin verildiğinde, en kısa yolların sayısı üçlü katsayılarla tanımlana bilir. Bir önceki satırda üç komşu elemanı toplayarak üçgen şeklinde de elde edilebilirler.

Bu tür üçgenlerin ve elemanlarının, kuadrinomiyal ve n-nominal katsayıların daha fazla genelleştirilmesini de düşünüyoruz.

Bu bağlamda, n-nomal açılımların n-nomal katsayıları, katsayı pozisyonundan üçgenin üst elemanına kadar n farklı tipte adım kullanılmasına izin verilen yolların sayısını temsil eder, böylece üçlü katsayılar için üç adım türleri ve dört adım katsayısı dört tür adım kullanırız.

Ayrıca, ikiden fazla öğenin toplamının kuvvetinin hesaplandığı üçlü, dörtlü ve n-nomiyal açılımlara dayanan üçlü, dörtlü ve n-nomal katsayıları hesaplamak için formüller sunuyoruz. Çok terimli açılımlar da ilişkilidir. Ayrıca binom katsayılarının çeşitli genelleme yöntemleri olarak bilinen değerlerin bir karşılaştırmasını da veriyoruz.

Ağırlıklı mesafelere göre kare ızgaranın herhangi bir nokta çifti arasındaki en kısa yolların sayısı hesaplanır. 8-komşu kare ızgara kullanıyoruz, yani bir ilk ağırlık yatay ve dikey hareketlerle ilişkiliyken, ikinci bir ağırlık (birinciden farklı olması gerekmez) köşegen adımlara atanmıştır.

İki noktanın chamfer uzaklığı, 'en kısa yoldaki' adımların sayılarına ve ağırlıklarına bağlıdır. Özel durumlarda, daha önce de belirttiğimiz gibi, şehir bloğu ve satranç tahtası mesafeleri, iki boyutlu dijital alanın en temel ve yaygın olarak kullanılan iki dijital mesafesi meydana gelir.

Kombinatorik sonucumuz teorik olmasına rağmen, iletişim ağları, dijital görüntülerde yol sayma, 2D dijital ızgaralardaki izler ve yörüngeler gibi uygulamalarla yakından bağlantılıdır. Negatif olmayan ağırlıkları olan yedi durumu ve negatif ağırlığa izin verilen durumu da ele alıyoruz.

Ayrıca, α, β ve γ ağırlıklarının değerlerine bağlı olarak en kısa yol sayısının olduğu üçgen ızgaradaki herhangi iki piksel arasındaki ağırlıklı en kısa yolların sayısını tartışacağız. Her piksel için üçgen ızgarada, üç tip komsuluk var: her komsuluk tipi için bir ağırlık atadığımız 1., 2. ve 3. komsuluk ve bu ağırlıklara göre, bu en kısa yolları tanımlamak için Chamfer mesafesini kullanıyoruz ve biz bu yolların sayısını tanımlamak için pikseller arasındaki mutlak farklılıkların kombinasyonlarını kullandık.

Anahtar Kelimeler: Binom Katsayıları; Trinomiyal Katsayılar; Dörtlü Katsayıları; n-nominal Katsayıları; Çok Terimli Katsayılar; Yörüngeleri; Ağırlıklı Mesafeler;

Dijital Mesafeler; Bir Kombinasyon; Üçgen Izgara, Komşuluk Tipleri, Chamfer Mesafesi; En Kısa Ağırlıklı Yollar; Yol Sayımı.

DEDICATION

This thesis is dedicated: to the sake of Allah, my creator; to my master, my great teacher and messenger, Mohammed (May Allah bless and grant him), who taught us the purpose of life; My great parents, who never stop giving of themselves in countless ways, the symbol of love and giving; to my brother Ammar, and my sisters Lujain and Sewar; to all my family; to my future wife; to my friends who encourage and support me; to my homeland Jordan; to the EMU University; my second magnificent home, and all the people in my life who touch my heart, I dedicate this research.

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Chapter 1

INTRODUCTION

On the first hand, Pascal's triangle is one of the most famous patterns of numbers in mathematics. One of the well-known applications of the Pascal's triangle is to find and represent the binomial coefficients for binomial expansion. On one hand, trinomial coefficients for expansion in the form of n-th power of $(1 + x + x^{-1})$ and $(1 + x + x^2)$ were discussed in [1, 2], moreover their q-analogues, the q-trinomial coefficients were introduced and used in applications related to statistical physics in [3]: trinomial coefficients were used to find local densities on lattice gas generalization on hexagonal grid. In [4], the triangles of trinomial and quadrinomial coefficients were introduced by applying recurrence relations. The sum of terms along any rising diagonals in any such array, given by $(1 + x + \cdots + x^{r-1})^n$ using generalized form of Pascal's triangle is presented in [5], consequently trinomial, quadrinomial, pentanomial and hexanomial coefficients are studied. Some algebraic properties of these coefficients are studied in [6]. On the other hand, general multinomial and, specially, trinomial coefficients, the coefficients of the expansions of the form $(a + b + c)^n$, and their usage were discussed in [7, 8, 9, 10]. Specially, Chapter 6 of [8] and some earlier papers of the same authors discuss also Pascal's tetrahedron.

On the other hand, in communication networks the transmitters, receivers, etc. can be represented by nodes of a graph and their connections, the possible ways of communication, can be shown by the edges. Concepts such as paths, shortest paths and distances in these graphs are understood and give some important features of the communication network. The number of shortest paths also has importance in applications for transmitting messages over networks, since they refer to the width of the connection channel between the given points. Any shortest path can be useful and used to increase the performance, in this case the amount of information transmitted during a unit of time (the width of the network), speeding up communication [11]. These networks are usually artificial, meaning that graphs with special properties such as the square grid can be considered. In social networks, the various graph measures, such as eccentricity, are defined based on the number of some shortest paths [12]. In some physical simulations connected to random walks, percolations, trajectories and traces [13, 14, 15], it is also important to count the number of shortest paths. Several related applications have already been detailed in [16]. Path counting in discrete spaces are closely related to graph theory.

Digital grids and their applications in various fields play important roles in science and technology. Digital grids are used in applications such as image processing [17], computer graphics, communication networks, crystallography and physical simulations. The space, it in this case the considered grid, is discrete, so theoretic tools from discrete mathematics, graph theory, combinatorics and, especially, from digital geometry can be used. In most cases only coordinates with integer values are used to address points (nodes). The square grid (also called rectangular grid) is the most usual digital grid, as it is the most frequently applied grid in two dimensions (2D). It is essential in image processing, cellular automata and other fields of applied information technology as well as 2D physical simulations. In many grids instead of the original graph its Voronoi dual is used, that is, instead of the vertices, pixels/voxels are used. One of the benefits of working on the square grid is that it is self-dual: the square grid can be seen by connecting (the midpoints of) neighbour pixels to each other, and also by using the original grid, with points where the gridlines cross. On the other hand, the honeycomb grid is dual of the triangular grid, that is instead of having vertices of the hexagons in the honeycomb (or hexagonal) grid, we may use the triangle pixels of the triangular grid keeping both the coordinate system and neighborhood structure [18, 19].

Opposite to discrete space, Euclidean space is continuous space, and there is no neighbour relation. The natural, Euclidean distance has several well-known and beneficial properties. However, using Euclidean distance on a discrete space may not be the best option. For example, one topological paradox is that the grid points having an exact Euclidean distance of seven from the origin do not really form a circle in any usual sense; the determined four pixels are not even connected. When working with computers, one may prefer digital distances, i.e., distances based on paths through neighbour points. In most cases, it is easy to work with these distances which have integer values. In a discrete space, shortest paths between any two points are computed depending on the grid and on the allowed types of steps of these paths. For instance, there is only 1 type of widely used neighborhood on the hexagonal grid. There are two popular types of neighborhood relations in the square grid: the cityblock and the chessboard neighborhoods [20]. These two neighborhood types are shown in Figure 1.1. In the cityblock neighborhood there are four neighbors (left, right, up and down) for each point, while in chessboard neighborhood, there are eight neighbors for each point: left, right, up, down and four diagonal neighbors. Related to the neighborhood relations, there are two types of basic distances between any two points of the square grid: cityblock and chessboard distances [21]. The cityblock distance (also called Manhattan distance) can be computed as d(p,q) = w_1 + w_2, while the chessboard distance can be computed as d(p,q) = $max\{w_1,w_2\}$, where w_1 and w_2 are the absolute differences of the first and second coordinates of the points p and q, respectively.

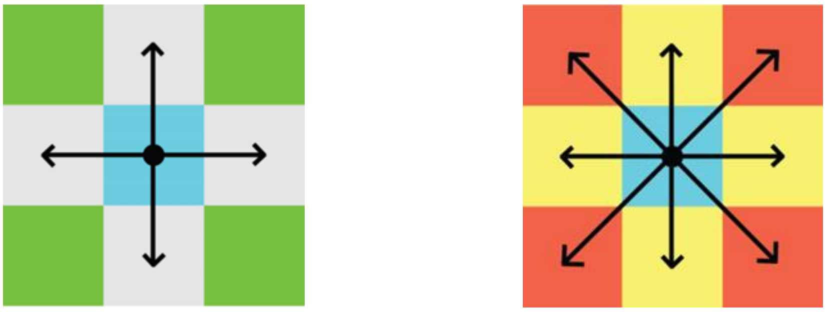


Figure 1.1: Cityblock neighborhood of the pixel marked by black point in the center (left, neighbors are in grey color). Chessboard neighborhood (right, neighbors are in yellow and red color).

These two digital distances give very rough approximations of the Euclidean distance, it was recommended to use them alternating along a path (the obtained distance is called octagonal distance). From the end of 1980's, extending this idea more formally, digital distances based on predefined neighbourhood sequences have been introduced and used in which both chessboard and cityblock neighbourhoods are combined in a sequence that can be periodic [22, 23, 24] or non-periodic [25]. Distances based on neighbourhood sequences on other grids have also been defined, see, e.g., [26, 27, 28]. Other digital distances, the weighted distances give another way to have distances on a grid with integer values [29]. They are also called chamfer distances. A reason to prefer weighted distances versus distances based on neighbourhood sequences is that the former ones are always metric, while there are plenty of neighbourhood sequences that do not provide metrics [25] (since the triangular inequality may fail).

In the triangular grid, the triangle pixels (also called trixels) are used as the elements of the grid (however, we may also refer them as the `points' of the grid). The hexagonal and the triangular grids have different symmetries and properties than the square grid, they behave in a different way and there are various advantages to apply them [30, 31, 32]. There are three types of neighbors widely used on the triangular grid [26, 33]. Thus, the triangular grid has the most complex neighborhood structure among the three regular two-dimensional grids. Digital distances based on these neighborhoods were described in [34], based on neighborhood sequences were studied in [19, 26, 35], chamfer distances in triangular grid were investigated in [36, 37, 38]. Because the three types of neighbors, chamfer distances are based on three weights on the triangular grid. Some path counting results were obtained for distances based on a given neighborhood in [39] and [40].

In the second chapter, we will discuss binomial, trinomial and quadrinomial coefficients in term of number of shortest paths, along with a connection between trinomial, quadrinomial polynomial coefficients and trinomial, quadrinomial multinomial coefficients. In the third chapter, we will study number of shortest paths in the square grid using different weights α and β for the cityblock and diagonal neighbours. In the fourth chapter, we will introduce number of shortest paths in triangular grid using different weights of movements for 1st, 2nd and 3rd neighbours.

Chapter 2

POLYNOMIAL AND MULTINOMIAL COEFFICIENTS IN TERMS OF NUMBER OF SHORTEST PATHS

In this part of the thesis, our aim is twofold. First, to make a clear differentiation of the two types of widely used generalisations of binomial coefficients, since in the literature trinomials, quadrinomials, etc., occur with two different meanings. On the other hand, we highlight the differences of the two types of generalisations of the binomial coefficients by counting lattice paths and their generalisations: In this thesis, we will express binomial and *n*-nomial coefficients in terms of number of shortest paths.

2.1 Binomial Coefficients as Numbers of Shortest Paths

The binomial coefficients build up the Pascal's triangle, see Figure 2.1. We start to write Pascal's triangle by writing the number 1 to the top. Then we write a new row with the number 1 twice. The remaining numbers in each row are calculated by adding the two numbers in the row above which lie above-left and above-right. For example, if we want to expand $(a+b)^3$ we select the coefficients from the row of the triangle beginning 1,3: these are 1,3,3,1 (see Figure 2.1). We can immediately write down the expansion $(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$. By the well-known Binomial theorem, the binomial expansion, when k is a positive integer, can be expressed as follows:

$$(a+b)^k = \binom{k}{0} a^k + \binom{k}{1} a^{k-1}b + \binom{k}{2} a^{k-2}b^2 + \cdots + \binom{k}{k} b^k$$

Alternatively, we may consider the following related expression as well, which also highlight the role of binomial coefficients:

$$(1+a)^k = \binom{k}{0} a^0 + \binom{k}{1} a^1 + \binom{k}{2} a^2 + \cdots + \binom{k}{k} a^k$$

Now we show a discrete geometric interpretation of the binomial coefficients, they can be viewed as the number of shortest paths in the traditional square/rectangular grid. We formulate this result as follows.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Figure 2.1: Pascal’s triangle with binomial coefficients.

Theorem 2.1 *Let k and j be nonnegative integers such that* $k \geq j \geq 0$. *Let us consider the points* $O(0,0)$ *and* $P(k-j, j)$ *of the square grid. The city-block distance of these points is k. The number of shortest paths between O and P is* $\binom{k}{j} = \frac{k!}{(k-j)!j!}$.

Proof. One can consider the rotated square grid with axes *x* and *y* such that the first “column” of the Pascal’s triangle coincides to axis *x* (growing downward left) and the last column of the triangle coincides to axis *y*. Then, considering the point (x,y) with nonnegative integer coordinates, every shortest path has $x+y$ steps and among them *x* is to the direction of the *x* axis, and *y* is to the direction of the *y* axis. The order of these steps is arbitrary, and thus, there are $\binom{x+y}{x} = \binom{x+y}{y}$ shortest paths. Now letting $x=j$ and $x + y = k$, the statement follows. ■

We can give explanation also by Figure 2.1. Each coefficient of this triangle is representing the number of shortest paths from the position of the coefficient (i.e., point $P(j, k-j)$) to the top element (i.e., point $O(0,0)$). These paths are composed of two types of steps which are left steps and right steps (in addition the row position is also changing by 1), where the length of these shortest paths equals to k. Thus, the number of left steps in these paths equals to j. Hence the number of right steps in the shortest paths equals to $k-j$. Therefore, the number of different arrangements of these left and right steps in the shortest path to $O(0,0)$ are represented by the binomial coefficient $\binom{k}{j} = \binom{k}{k-j} = \frac{k!}{(k-j)!j!}$, where j and k-j are the row number and the column number of the coefficient, respectively.

Example 2.1 The number of shortest grid paths (i.e. cityblock paths) between (0,0) and (2,2) is the coefficient of term x^2y^2 in the expansion $(x+y)^4$. Therefore, $k=4$ and $j = 2$, then the binomial coefficient equals to

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = 6.$$

The digital distance is the number of steps in a shortest path, where a path built up by steps to neighbour pixels. The hexagonal grid is also a two-dimensional grid, where, using the usual 6-neighbourhood of the pixels, there could be steps in a shortest path in at most two directions. Consequently, the following result can be established in a very similar manner as Theorem 2.1.

Theorem 2.2 *Binomial coefficients appear as number of the shortest paths in the hexagonal grid as it is shown in Figure* 2.2. *For any hexagon with distance k from the origin there are at most two types of steps in a shortest path, and their number is* *determined by the position of the hexagons. However, the order of the steps is arbitrary, and, thus, the binomial coefficients give the number of shortest paths. One can observe the Pascal's triangle six times in the figure starting from the middle (hexagon with mark* 0*).*

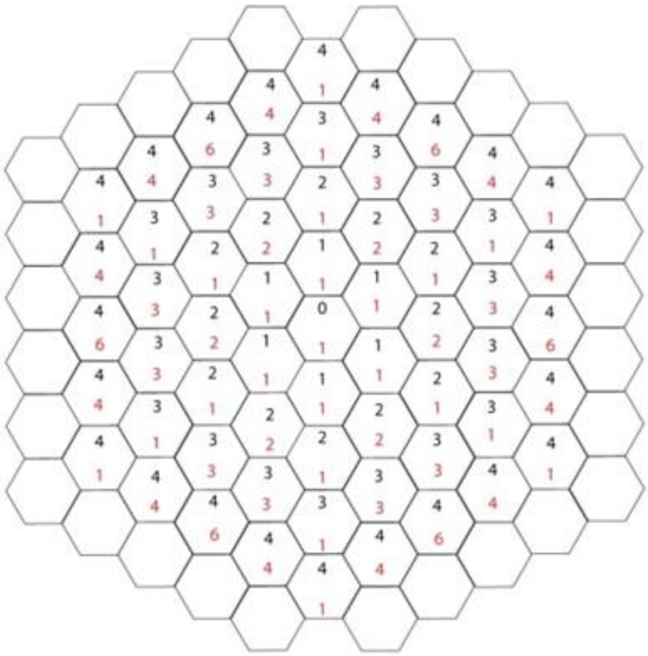


Figure 2.2: Binomial coefficients in the hexagonal grid in terms of number of shortest paths: black numbers show the distance of the given hexagon from the origin, i.e., the one marked by 0, red numbers give the number of shortest paths between the actual hexagon and the origin.

2.2 N-nomial Coefficients – Generalising Binomial Coefficients

As we have already mentioned, in this thesis we consider generalisations of the binomial coefficients. Binomial expansion is a sum of two terms raised to power k, while n-nomial expansion is a sum of n terms raised to power k. In this thesis, we recall two types of n-nomial coefficients and we differentiate them as multinomial and polynomial coefficients, depending on the way we generalise the binomial expansion. At the first type of extension, the number of variables is increased and the dimension of the triangle is increased to 3D (e.g., to have a "pyramid" or a tetrahedron), to 4D and, generally to nD. These coefficients will be used in Section 2.4 to count shortest paths in grids having steps in n independent directions. Consequently, number of shortest paths in higher dimensional grids can be given. On the other hand, having the other type of generalisation, the coefficients are represented by triangle with right or obtuse angle, such that as n increases the triangle will be wider, these n-nomial-polynomial coefficients form 2D triangles, and thus, these coefficients will represent number of the shortest paths between two points in the 2D grid, when, in these paths, steps of n different type are allowed based on an extended neighbourhood relation.

2.3 Multinomial N-nomial Coefficients

In this type of generalisation, the coefficients of n-nomial expansion

$$(x_1 + x_2 + x_3 + \cdots + x_n)^k$$

are used, where $x_1, x_2, x_3, \ldots, x_n$ are different variables. The binomial coefficients were represented by equilateral triangle in the two dimensional space (2D), as we have shown in Figure 2.1. The n-nomial coefficients, however, can be represented in the n dimensional space, i.e. trinomial coefficients are represented as a tetrahedron in the 3D space [8], quadrinomial coefficients are represented as a hypertetrahedron in the 4D space, etc. These *n*-nomial coefficients will be expressed as the number of shortest paths in higher dimensional grids in the next subsections.

2.3.1 Trinomial-Multinomial Coefficients

Among these multinomial coefficients, the trinomial coefficients can be computed as the coefficients of the trinomial expansion on the form: $(x_1 + x_2 + x_3)^k$, where x_1, x_2 and x_3 are distinct variables. (One may consider similarly the expansion $(a + b + c)^k$.) These coefficients were shown in the form of the Pascal's Pyramid in [8]. We show that they can be used to count shortest paths in the cubic grid with the closest neighbourhood, i.e., 6-neighbourhood (that is the 3D form of cityblock neighbourhood).

Theorem 2.3 *For* $r \geq 0, s \geq 0, t \geq 0$, *let* $k = r + s + t$. *The trinomial coefficient*

$$T_1(k,\ r, s, t) = \binom{k}{r}\binom{k-r}{s} = \frac{k!}{r!\,s!\,t!}$$

provides the number of shortest grid paths between point O(0,0,0) *and P*(*r*,*s*,*t*) *in the cubic grid.*

Proof. Each shortest path between *O* and *P* contains exactly *k* steps. Moreover, *r* steps are along the first axis, *s* steps along the second axis and *t* steps along the third axis. The number of steps in the different directions are fixed by the points, however the order of the steps is arbitrary, thus there is exactly $T_1(k, r, s, t)$ ways to order these steps, and each of these orderings is describing a shortest path in a bijective way. Hence the theorem. ■

Consequently, these coefficients can be represented in a cube in the 3D space as it is shown on Figure 2.3, where *k* is the distance from the position of point 0 (top back), *r*, *s* and *t* are the numbers of the steps needed along the gridlines of the three directions of the axis, respectively. We may call such cube as Pascal's cube.

All points which have the same distance from the position of point 0 (all points with the same value of *k*) can be represented in an equilateral triangle as shown in Figure 2.4. The coefficients of any such triangle of the Pascal cube represent the number of shortest paths between the position of point 0 and position of any coefficient on the k^{th} triangle.

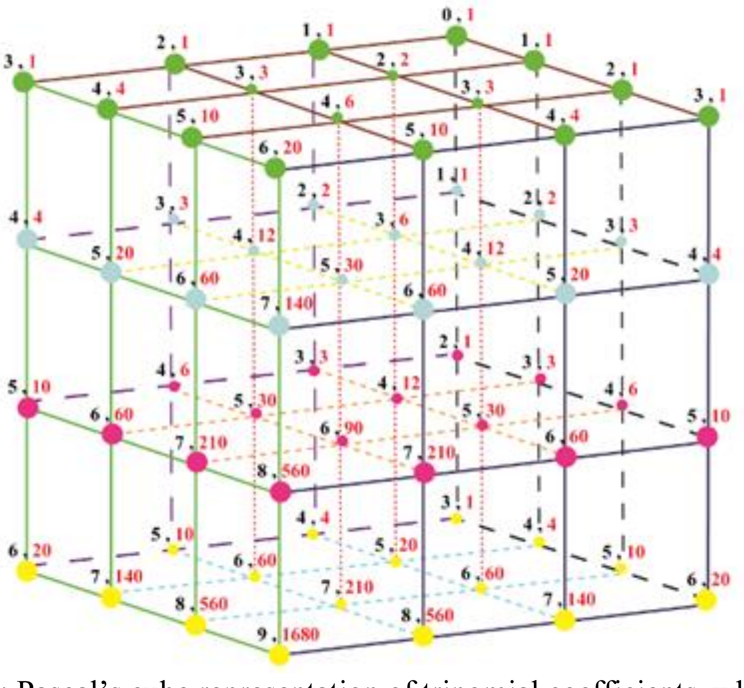


Figure 2.3: Pascal's cube representation of trinomial coefficients, where black numbers represent the discrete (step based) distance between the given point and point 0, and red numbers represent the number of shortest paths between the given points.

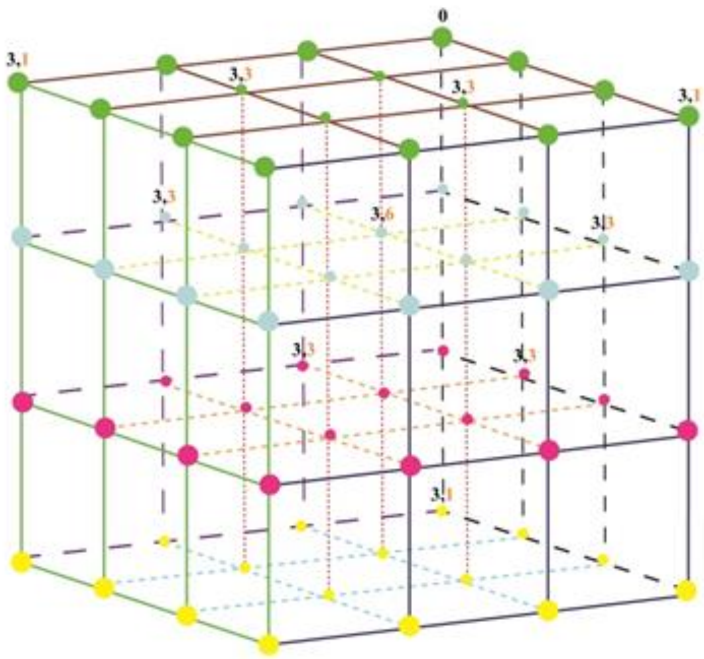


Figure 2.4: Points which have distance 3 from the position of point 0 form a triangle mesh in the cube.

Example 2.2 To find the number of shortest paths between $O(0,0,0)$ and $P(2,3,1)$ is the same as the coefficient of $x^2y^3z^1$ in the trinomial expansion $(x+y+z)^6$, $k=6$, $r=2$, $s=3$, and $t=1$. Therefore the coefficient *of* $x^2y^3z^1$ is*:*

$$T_1(k,r,s,t) = T_1(6,2,3,1) = \frac{6!}{2!\,3!\,1!} = 60$$

2.3.2 Generalization to Arbitrary Dimension: N-nomial-Multinomial Coefficients

Extent to *n*-nomial (multinomial) coefficients, the *n*-dimensional analogue of Pascal's triangle and Pascal's tetrahedron is called Pascal's *n*-dimensional simplex [8,9]. In Pascal's *n*-dimensional simplex, with nonnegative integers $k,\ k_1, k_2, k_3, \dots, k_n$ with $k = \sum_i k_i$, the *n*-nomial coefficients is given by:

$$\binom{k}{k_1, k_2, k_3, \dots, k_n} = \frac{k!}{k_1!\,k_2!\,k_3!\dots k_n!}$$

Where k_i is the power of the of x_i in the *n*-nomial-multinomial expansion:

$$(x_1 + x_2 + x_3 + \cdots + x_n)^k$$

In a similar manner as we have proven Theorem 2.3, its generalisation can also be established:

Theorem 2.4 *The n-nomial coefficient* $\binom{k}{k_1,k_2,k_3,\dots,k_n}$ *is the number of shortest paths in the n-dimensional grid between the points* (0,0,0,…,0) *and* $(k_1, k_2, k_3, \dots, k_n)$ *where* $k = \sum_i k_i$.

2.4 Polynomial N-nomial Coefficients

This type of *n*-nomial coefficients are represented by triangles with right and obtuse angles, such that as *n* increases the triangle will be wider as we are showing them in this section. These coefficients represent number of the shortest paths between element the origin at $k = 0$ and $j = 0$ and the given position of the coefficients, where in these paths we use *n* neighbourhood relations. Therefore, in order to find these coefficients, in the shortest paths we may use *n* different types of steps.

The coefficients of this type represent *n*-nomial expansion of the following form:

$$(1 + x + x^2 + \cdots + x^{n-1})^k$$

where *k* represents the k^{th} row of the *n*-nomial triangle, where $k \geq 0$. In this type of *n*-nomial expansions, we use only one variable, e.g. x, but it is used in various powers. To show relation to path counting we present these coefficients with relatively small value of *n* in the next subsections. For larger values the results are analogous to those we present here.

2.4.1 Trinomial-Polynomial Coefficients

These polynomial coefficients are used at powers of sums with three elements, similarly to the trinomial-multinomial coefficients. However, at this new type of coefficients, we understand the trinomial coefficients by the trinomial expansion of the form $(1 + x + x^2)^k$. The variables are not independent, but powers of the same variable. The coefficients are computed by a triangle such that three neighbours in the previous line of the triangle are summed up (close to the sides of the triangles the missing elements are substituted by 0's). The trinomial triangle is shown in Figure 2.5. Let $k \geq 0$, $j \geq 0$, and $j \leq k$, where k and j are the row number and the column number of the coefficient, respectively. The value of the corresponding trinomial-polynomial coefficient is

$$T_2(k,j) = \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \binom{k}{i} \binom{k-i}{j-2i} = \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{k!}{i!\,(j-2i)!\,(k-j+i)!}$$

where i is the number of x^2.

Theorem 2.5 *Each trinomial coefficient equals to the number of shortest paths between the first element at position* (0,0)*, and the position of the trinomial coefficient by chessboard distance.*

Proof. The proof goes by induction. The base of the induction contains the first coefficients up to distance 1. As Figure 2.5 and 2.6 show, the top (or the middle) element have exactly one shortest path to itself which contains no step. The next 3 points can be reached directly by 1 step, meaning exactly 1 shortest path for each case. Since there could occur three direction steps in shortest paths in the yellow part of the grid (namely diagonal steps to down and to right directions and cityblock steps in the direction between those directions), the number of shortest path to any further points can be computed as the number of the shortest paths to its three neighbours in the previous row. But, the sum of these values gives exactly the trinomial coefficients, thus the theorem is proven. ■

For $k \geq 0$, $j \geq 0$, and $j \leq k$, each coefficient in this case is representing number of shortest paths from position of that coefficient (which is represented by k^{th} row and j^{th} column) to the first coefficient (for which $k = 0$ and $j = 0$) of the triangle shown in Figure 2.5. As it was shown in Figure 2.6, these paths are composed by three types of steps which are: left, right and middle and the length of these paths equal to k. Considering the shortest paths from a point to the top, the number of different arrangements of these left, right and vertical steps are computed as $T_2(k, j)$.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1
\end{array}
$$

Figure 2.5: Trinomial triangle with three different neighbours in the preceding line: an extension of the Pascal's triangle to right angled triangle.

Example 2.3 For trinomial expansion $(1 + x + x^2)^3$, the coefficients are coloured with red colour in Figure 2.6 as follows:

$$(1 + x + x^2)^3 = 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + 1x^6$$

These coefficients represent the 4^{th} row in Figure 2.5 and the bottom yellow row in Figure 2.6, where each coefficient represents number of shortest paths between point 0 and each point of the given, 4^{th} row using three types of steps allowed in chessboard neighbourhood.

Figure 2.6 shows the trinomial-polynomial coefficients in term of number of shortest paths on the square grid using chessboard neighbourhood.

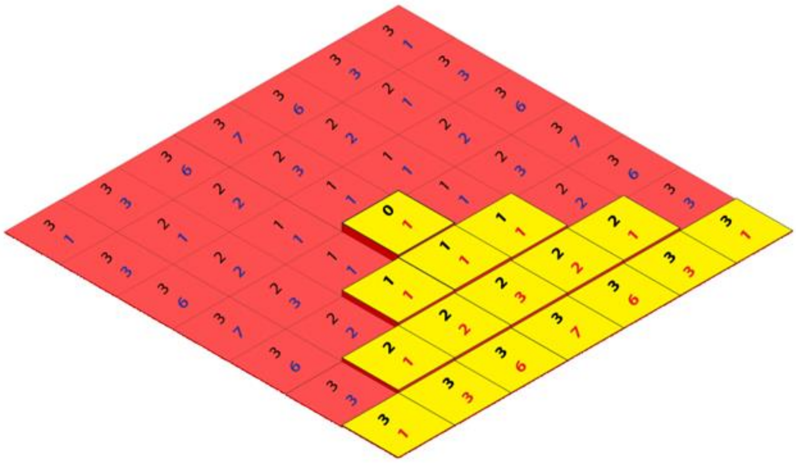


Figure 2.6: Trinomial-polynomial coefficients in square grid in terms of shortest paths using chessboard distance, black color: the distance from the pixel marked by 0, red color: the number of shortest paths between the given pixel and 0.

2.4.2 Quadrinomial-Polynomial Coefficients

The quadrinomial-polynomial coefficients are calculated as the coefficients of the quadrinomial expansion of the form: $(1 + x + x^2 + x^3)^k$. For $k \geq 0$, $j \geq 0$, and $j \leq \left\lfloor \frac{3k}{2} \right\rfloor$, where k and j are the row number and the column number of the coefficient, respectively. Therefore the value of any quadrinomial-polynomial coefficient is given by:

$$Q_2(k,j) = \sum_{i=0}^{\left\lfloor \frac{j}{3} \right\rfloor} \sum_{m=\max\{0,j-2i-k\}}^{\left\lfloor \frac{j-3i}{2} \right\rfloor} \binom{k}{i} \binom{k-i}{m} \binom{k-i-m}{j-3i-2m}$$

where i is the number of x^3, and m is the number of x^2.

Quadrinomial coefficients can be computed in a similar manner as Pascal's triangle, but in this case an obtuse angled triangle is obtained by summing up four of the previous elements. Regarding, our main topic, we can state the following result, which is somewhat analogous to Theorem 2.5.

Theorem 2.6 *The number of shortest paths between the top and any other elements of the triangle is exactly the quadrinomial-polynomial coefficient displayed at the corresponding element of the triangle when four different neighbourhood relation are used: by changing the row, the column can be changed to the closest two neighbours and also to their neighbours (see Figure* 2.7, *where arrows represent those relations).*

Proof. The way to obtain the coefficients in the triangle is exactly the same as the way to compute the number of shortest paths inductively from the top to any positions of the triangle. ■

$$
\begin{array}{c}
1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 1 \\
1 \quad 3 \quad 6 \quad 10 \quad 12 \quad 12 \quad 10 \quad 6 \quad 3 \quad 1 \\
1 \quad 4 \quad 10 \quad 20 \quad 31 \quad 40 \quad 44 \quad 40 \quad 31 \quad 20 \quad 10 \quad 4 \quad 1
\end{array}
$$

Figure 2.7: Quadrinomial triangle with four different types of neighbourhood relations.

Example 2.4 Quadrinomial-polynomial coefficient where k = 4 and j = 3 equals to the number of shortest paths from the top 1 to the 4th element of the last line: 1, 4, 10, 20 (in Figure 2.7) and it is computed as

$$Q_2(4,3) = \sum_{i=0}^{\left\lfloor\frac{3}{3}\right\rfloor} \sum_{m=\max\{0,3-2i-4\}}^{\left\lfloor\frac{3-3i}{2}\right\rfloor} \binom{4}{i}\binom{4-i}{m}\binom{4-i-m}{3-3i-2m} = 20$$

One can further generalise these coefficients to *n*-nomial coefficients for any positive integer *n*. Instead of having further subsections here, we present the general case in the next section.

2.5 Connection Between Multinominal and Polynominal Coefficients

Although both type of generalisations have the same names, e.g., trinomial, quadrinomial, etc. as we have seen they have very different meaning. However, they are not independent of each other, as one can see that by choosing $x_1 = 1$, $x_2 = x$, $x_3 = x^2$, ..., $x_n = x^{n-1}$, one can shift from the multinomial approach to the polynomial approach. Their names come from the fact that how many elements are summed up to obtain a given value, but these values may be neighbors from different independent directions (multinomial case, higher dimensions) or larger number of neighbors that are next to each other in a 2D triangle (polynomial case). At trinomials three numbers are summed up, generally, at *n*-nominals *n*.

Trinomial-polynomial coefficient ($T_2(k,j)$) can be calculated in terms of Trinomial-multinomial coefficients by the following formula:

In case $j \leq k$:

$$T_2(k,j) = \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} T_1(k, k-j+i, j-2i, i)$$

If $j \geq k$:

$$T_2(k,j) = T_2(k, 2k-j).$$

Similarly, quadrinomial-polynomial coefficients can also be calculated in terms of quadrinomial -multinomial coefficients by applying the following formula:

In case $j \leq \frac{3k}{2}$:

$$Q_2(k,j) = \sum_{i=0}^{\left\lfloor \frac{j}{3} \right\rfloor} \sum_{m=max\{0, j-2i-k\}}^{\left\lfloor \frac{j-3i}{2} \right\rfloor} Q_1(k, j-3i-2m, k-j+2i+m, i, m)$$

In case $j \geq \frac{3k}{2}$:

$$Q_2(k,j) = Q_2(k, 3k-j)$$

Generally, the *n*-nomial-polynomial coefficients can always be computed by summing up appropriate *n*-nomial-multiominal coefficients (with the same value of *n*).

Chapter 3

COUNTING THE NUMBER OF SHORTEST CHAMFER PATHS IN THE SQUARE GRID

Path counting (for cityblock and chessboard distances) in digital images (i.e., finite subgraphs of the square grid) is used to infer properties of images [21]. It was also considered in [41] based on matrix multiplication with various neighbourhood relations. In [42], the numbers of shortest paths are computed for the two above mentioned basic distances and also to the octagonal distance, which is a special neighborhood sequence based distance. For neighbourhood sequences in general, the problem was considered in [43, 44]. In this chapter, a similar combinatorial problem, the path counting for weighted distances considering the basic two types of steps is presented with enumerative combinatorial calculation. In most cases, we assume that both weights are non-negative. Moreover, we solve all the cases of the problem by providing the solutions by closed formulae. As we will see that there are five entirely different cases based on (the relation of) the used weights if both weights are positive; and there are two cases with 0 weights. Two of the cases, actually, provide the same result as the corresponding results for cityblock and chessboard distances, however, our proof technique is different than the technique used in [42]. We also present 3D charts to show how the number of shortest paths grows when the distance grows. Thus, the significance of the thesis is not only to consider and summarize all the possible cases, but also to give solutions for cases which were not analyzed before, e.g., the last three cases shown in this thesis.

3.1 Preliminaries

In weighted distance (or chamfer distance per the original terminology [29]) we associate a weight for each type of movement: we give weight α for cityblock movements and weight β for each diagonal movement, as shown in Figure 3.1. Formally, we can describe it as follows:

Let $p = (x_1,y_1)$ and $q = (x_2,y_2)$ be two points in the square grid; let $W = (w_1,w_2)$ be the absolute difference vector between the points, where $w_1 = |x_1 - x_2|$ and $w_2 = |y_1 - y_2|$. Then, as was previously computed [21].

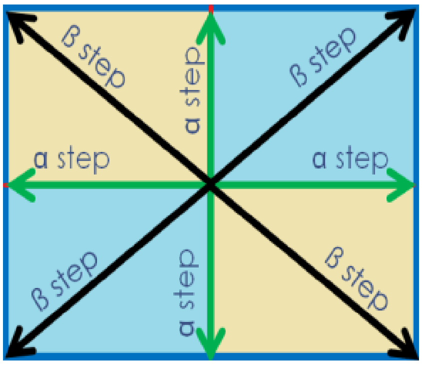


Figure 3.1: Weighted steps for chamfer distance (α steps, β steps) from the point in the centre.

The number of diagonal steps in a shortest path of the chessboard distance is $min\{w_1,w_2\}$, and the number of cityblock steps (i.e. the number of vertical or horizontal movements) in a shortest path with the chessboard distance is $max\{w_1,w_2\} - min\{w_1,w_2\}$. These values become important when calculating the number of shortest paths of chamfer distances.

When using both types of neighbours, but with different weights, in order to calculate the length of a shortest path (i.e. the chamfer distance between p and q), we must find how many α and β steps are in the given path. Their numbers in a shortest path depend on the respective values of α and β as well. According to their numbers and values, we will compute not only the length but also the number of shortest paths between any two points. The actual computation depends on the used weights. In this thesis, as usual both in graph theory and in digital geometry, we assume that both α and β are non-negative, and actually, in the first some cases we assume that they are positive. In some applications, there is also an assumption that $0 < \alpha \leq \beta$. We show Example 3.1 below, for this case. With this condition subcases are defined by the relation of 2α and β. However, in this thesis, we do not restrict our studies to these cases. We will also do computations when $0 < \beta < \alpha$ (see Subsection 3.2.5), and as we will see this case is the most interesting among all. For the sake of completeness we also present the cases, when one or both the weights have value 0.

Now, as an example, we show how to compute the distance, or the length of the shortest path if $0 < \alpha \leq \beta$ holds. Let N be the number of α steps and M be the number of β steps in a shortest path; then, the weighted distance between p and q is $d_w(p,q) = N\alpha + M\beta$.

Example 3.1 Let $p = (5,6)$, $q = (7,1)$ and $\alpha = 3, \beta = 4$. Then $w_1 = 2$ and $w_2 = 5$. Thus, the cityblock distance of these points is 7, their chessboard distance is 5 (note that in these distances unit weight is used). Now, computing the chamfer distance, since $\alpha < \beta < 2\alpha$, it is worth to use the path defined by the chessboard distance, i.e. with 2 diagonal and 3 cityblock steps: the chamfer distance equals to: $2{\cdot}4+3{\cdot}3=17$.

3.2 Results: Formulae for the Number of Shortest Paths

According to the values of α and β, we can compute weighted distances, and, consequently, we can compute the number of shortest paths. In this context, we have various cases depending on the respective ratio of the used weights, letting $f(w_1,w_2)$ be the function calculating the number of shortest paths between two points with an absolute difference vector (w_1,w_2). The cases are listed in the following subsections. The first two cases are equivalent to obvious discrete mathematical exercises (and have been proven also in [42] by a recursive method), and we explain them only for the sake of completeness using enumerative combinatorial techniques in our proofs. In the first five subsections cases with positive weights are studied, while in the last subsection we deal with the cases when one or both weights is/are zero.

3.2.1 Case of $\beta > 2\alpha > 0$

Theorem 3.1 *Let α and β be the weights for cityblock and diagonal movements, respectively, such that $\beta > 2\alpha > 0$. Let $p = (x_1,y_1)$ and $q = (x_2,y_2)$ be points of the square grid and $w_1 = |x_1 - x_2|$ and $w_2 = |y_1 - y_2|$ be the absolute differences between the corresponding coordinates of the points. Then, the number of shortest paths between p and q, denoted by $f(w_1,w_2)$, is given as $f(w_1,w_2) = \binom{w_1+w_2}{w_1}$.*

Proof. We have $\beta > 2\alpha$ such that the weights are positive, which means that in the shortest path between p and q, no diagonal steps occur since diagonal steps can be substituted by two consecutive cityblock (i.e. a vertical and a horizontal) steps to produce a path with smaller weight. Thus, all shortest paths contain only cityblock steps. The number of α steps between points p and q in the shortest weighted paths is computed in the same way as in the cityblock distance: $w_1 + w_2$. The distance between p and q is α times more, since each step has weight α. Moreover, in each shortest weighted path between p and q, the numbers of horizontal and vertical steps are w_1 and w_2, respectively. However, the order of these steps is arbitrary; thus, the number of shortest paths is given by the number of ways that we can arrange w_1 or w_2 steps among the total $w_1 + w_2$ steps; it is given by the binomial coefficient $\binom{w_1+w_2}{w_1}$. Actually, $\binom{w_1+w_2}{w_1}$ and $\binom{w_1+w_2}{w_2}$ give the same value. ■

Example 3.2 The number of shortest paths between the points $p(5,12)$ and $q(8,13)$ with $\alpha = 3$ and $\beta = 7$ is computed as follows: $w_1 = |\,8 - 5| = 3$ and $w_2 = |\,13 - 12\,| = 1$, $w_1 + w_2 = 4$. Thus, the result is $f(1,3) = \binom{4}{3} = \binom{4}{1} = 4$. See also Figure 3.2.

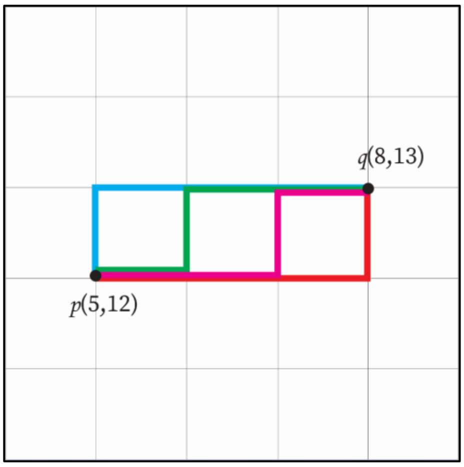


Figure 3.2: All shortest paths in case $\beta > 2\alpha > 0$; the four colors show the four shortest paths of Example 3.2.

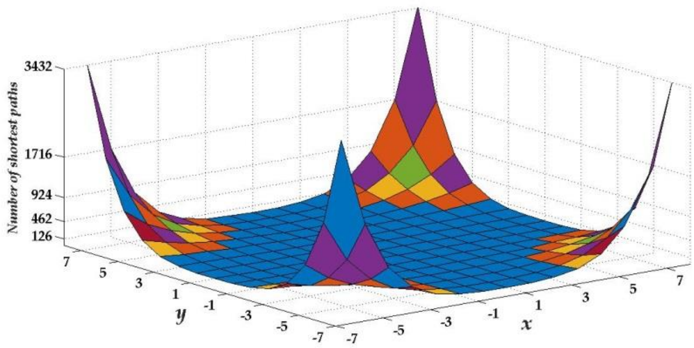


Figure 3.3: The number of shortest weighted paths from point (0,0) to other points in a 14 × 14 window with corners $(-7,-7)$, $(7,-7)$,$(7,7)$ and $(-7,7)$ in case $\beta > 2\alpha > 0$.

The paths of this case are also called grid-paths, since only the edges of the grid are used. Since these results are exactly the binomial coefficients (as they form the Pascal's triangle, as we have seen in section 2.1), we do not give them in a table form, only sketch them in Figure 3.3 as a 3D chart for the number of shortest weighted paths from point (0,0) to all points in the represented region. In the figure the origin is placed in the middle to show the symmetry distribution of the values. The formula grows rapidly at the corner directions when the coordinate differences are (almost) equal.

3.2.2 Case $0 < \alpha < \beta < 2\alpha$

Theorem 3.2 *Let α and β be the weights for cityblock and diagonal movements, respectively, with the condition $\alpha < \beta < 2\alpha$. Further, let $p = (x_1,y_1)$ and $q = (x_2,y_2)$ be points, and let $w_1 = |x_1 - x_2|$ and $w_2 = |y_1 - y_2|$. Then, the number of the shortest paths between p and q is given as $f(w_1,w_2)=\binom{max\{w_1,w_2\}}{min\{w_1,w_2\}}$.*

Proof. Both weights are positive and $\alpha < \beta < 2\alpha$. Thus, we may move in the shortest path from p to q using both α-steps and β-steps. We use β-steps as much as possible to get closer to the endpoint, which means that we will move diagonally by the minimum number of differences between the point coordinates, and the remaining steps are α-steps. According to this, the number of α-steps and β-steps in the shortest paths will be computed in the same way, as in a chessboard path from p to q (i.e. the number of steps is $max\{w_1,w_2\}$). Since the number of β-steps is $min\{w_1,w_2\}$, the number of α-steps is $(max\{w_1,w_2\} - min\{w_1,w_2\})$. The order of the steps are arbitrary; thus, the number of shortest weighted paths equals to the number of ways the β-steps can be arranged in the path with $max\{w_1,w_2\}$ steps altogether, which is exactly the binomial coefficient $\binom{max\{w_1,w_2\}}{min\{w_1,w_2\}}$. ■

Example 3.3 Let $p(-2,3)$, $q(2,0)$, and let α=3, β=4.Then w_1=4,w_2=3, further $min\{w_1,w_2\}$ = 3 and $max\{w_1,w_2\}$= 4. Applying the formula for this case, the number of shortest paths from p to q is$\binom{max\,\{4,3\}}{min\,\{4,3\}}$ =$\binom{4}{3}$= 4. Actually these four shortest weighted paths are illustrated in Figure 3.4 with various colors.

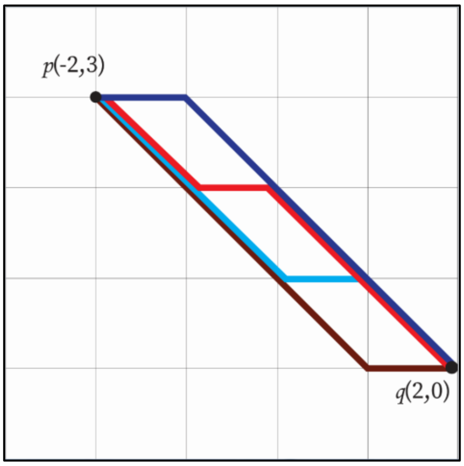


Figure 3.4: Example for all shortest paths of case $0 < \alpha < \beta < 2\alpha$ as in Example 3.3.

Again, the values of Pascal’s triangle appear, but in a different arrangement than in the previous case. Figure 3.5 gives a 3D chart for values for the number of shortest weighted paths from point (0,0) to all points in a 14 × 14 window. To show the symmetry of the distribution the origin is in the middle. This graph is already more interesting than the previous one, with more growing directions: the value grows fastest when one of the absolute coordinate differences is (approximately) half of the other one.

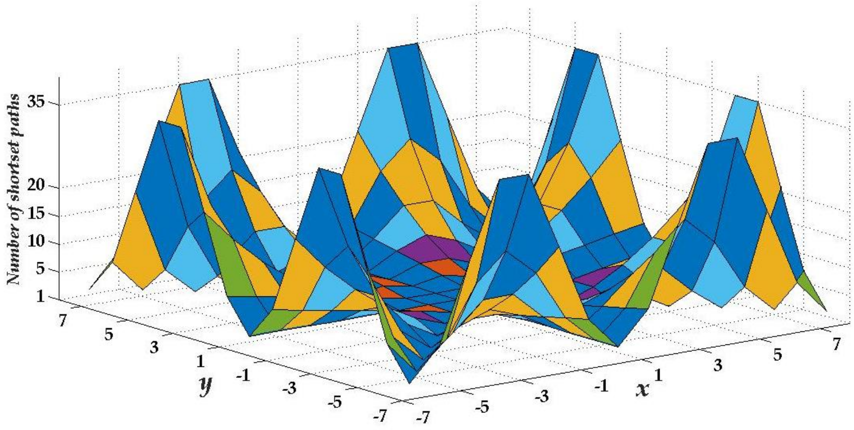


Figure 3.5: The number of shortest weighted paths in case $0 < \alpha < \beta < 2\alpha$ in a 14×14 window with corners $(-7, -7)$, $(7, -7)$, $(7, 7)$ and $(-7, 7)$. The minimums on the axes and on the diagonals can be seen.

3.2.3 Case of $\beta = 2\alpha > 0$

Theorem 3.3 *Let α and β be (positive) weights for cityblock and diagonal movements, respectively, such that $2\alpha = \beta$. Let $p = (x_1,y_1)$, $q = (x_2,y_2)$, $w_1 = |x_1 - x_2|$ and $w_2 = |y_1 - y_2|$. The number of shortest paths $f(w_1,w_2)$ between p and q is*

$$f(w_1, w_2) = \sum_{i=0}^{min\{w1,w2\}} \frac{(w_1 + w_2 - i)!}{i!\,(w_1 - i)!\,(w_2 - i)!} \tag{1}$$

Proof. In this case, a diagonal step has exactly the same weight as two consecutive movements to cityblock neighbours. The number of shortest weighted paths between p and q depends on the number of used diagonal steps (β-steps) between the two points, which is at most the minimum difference of the two coordinate values of p and q. Since each diagonal step can be substituted by two consecutive α-steps (a horizontal and a vertical one), the number of diagonal steps may be less, potentially equalling zero, meaning that the points are connected by only cityblock steps. (In special cases, when the two points p and q share a coordinate value, the shortest path cannot contain diagonal steps. Thus, the number of shortest weighted paths is exactly one in this case.) Let i be the number of diagonal steps in the shortest path (these steps can be replaced by α-steps); then, i has a range between $0 \leq i \leq min\{w_1,w_2\}$. Because each diagonal step can be replaced by two consecutive α-steps, we need to sum up the cases, such as the number of shortest weighted paths corresponding to various value of i. This can be done as follows:

i = 0, then all steps in the path are α-steps:$\binom{w_1+w_2}{w_1}$: vertical and horizontal steps in any order;

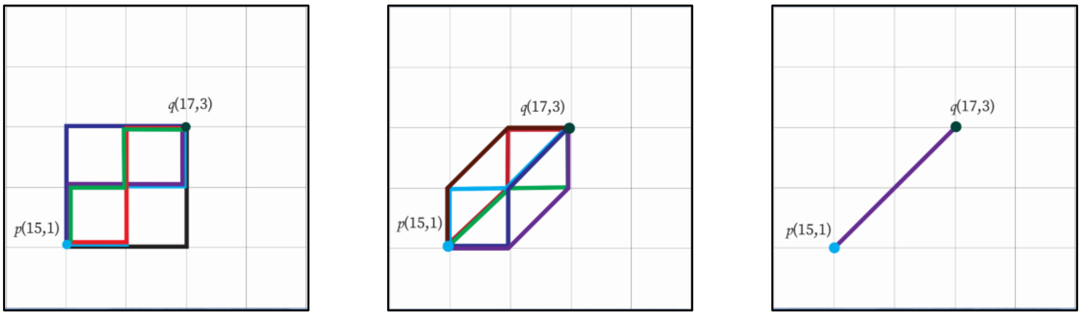
i = 1, then 1 diagonal step and remaining steps are α-steps(horizontal and vertical steps, accordingly): $\frac{(w_1+w_2-1)!}{1!(w_1-1)!(w_2-1)!}$;

i in general, the number of steps is $w_1 + w_2 - i$ from which i are diagonal, $w_1 - i$ and $w_2 - i$ are the number of horizontal and vertical steps. The number of such paths is $\frac{(w_1+w_2-i)!}{i!(w_1-i)!(w_2-i)!}$; $i = min\{w_1,w_2\}$, (the same formula applies for this special case, as we have used in the case $\alpha < \beta < 2\alpha$) : $\binom{max\{w_1,w_2\}}{min\{w_1,w_2\}}$.

To sum these numbers up, the number of shortest weighted paths is computed:

$$\sum_{i=0}^{min\{w_1,w_2\}} \frac{(w_1 + w_2 - i)!}{i!\,(w_1 - i)!\,(w_2 - i)!}$$

As is shown in the formula, each time we increment i by 1, the number of diagonal steps is increased by 1 and the number of α-steps is decreased by 2 (1 vertical step, 1 horizontal step); then, the overall number of steps in the shortest path is decreased by i for each i, where in this shortest path we have i diagonal steps, w_1-i horizontal steps and w_2-i vertical steps. Therefore, the number of shortest weighted paths according to the value of i of diagonal steps is given as $f(w_1,w_2,i) = \frac{(w_1+w_2-i)!}{i!(w_1-i)!(w_2-i)!}$ using the fact that the order of steps is arbitrary. ■



(a) (i = 0) (b) (i = 1) (c) (i = 2)

Figure 3.6 :The shortest paths between $p(15,1)$ and $q(17,3)$, with $\alpha = 3$ and $\beta = 6$ (case $2\alpha = \beta > 0$), when $i = 0$, 1 and 2 is the number of diagonal steps (starting from $i = 0$ to $i = \min\{w_1,w_2\}$ in the path from p to q).

Example 3.4 Let $q(17,3)$ and $p(15,1)$ and weights $\alpha = 3$, $\beta = 6$ be given. Then, $w_1 = 2$, $w_2 = 2$ and $min\{w_1,w_2\} = 2$, thus, the number of shortest paths is:

$$f(2,2) = \sum_{i=0}^{2} \frac{(2+2-i)!}{i!\,(2-i)!\,(2-i)!} = 13.$$

As we can see, the number of shortest weighted paths, in this case, can be computed by various numbers of diagonal steps with a maximum of $min\{w_1,w_2\}$. Figure 3.6 (a), (b) and (c) shows all the possible shortest paths between points p and q of Example 3.4 separated by the possible number of diagonal steps.

Summarizing the results of this case, Figure 3.7 shows the 3D chart for the values of the number of shortest weighted paths from point (0,0) to all points in a 14 × 14 window with the origin in the middle. The function grows most rapidly on the diagonal directions.

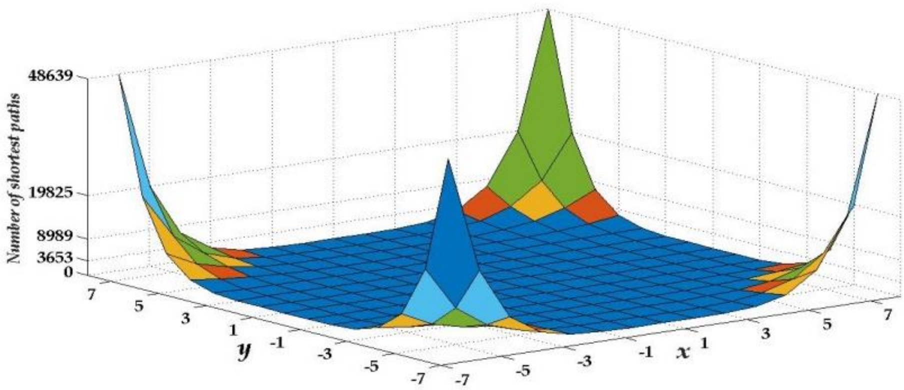


Figure 3.7: The number of shortest weighted paths from the point (0,0) to other points in a 14 × 14 window with corners $(-7, -7)$, $(7, -7)$,$(7, 7)$ and $(-7, 7)$, in case $\beta = 2\alpha > 0$.

3.2.4 Case of $\beta = \alpha > 0$

This case correspond to the chessboard distance, therefore the number of shortest paths of this case is already detailed. Actually, they are counted as the Trinomial-polynomial coefficients in section 2.4.1. In this section, we will only briefly recall these numbers and state the result in a more general way by counting the number of shortest chessboard paths between any two points of the square grid with an alternative proof.

Theorem 3.4. *Let α and β be* (*positive*) *weights for cityblock and diagonal movements, respectively, with $\alpha = \beta$. Let $p = (x_1,y_1)$, $q = (x_2,y_2)$, $w_1 = |x_1 - x_2|$ and $w_2 = |y_1 - y_2|$. The number $f(w_1,w_2)$ of the shortest paths between p and q is counted as*

$$f(w_1, w_2) = \sum_{i=0}^{\left\lfloor \frac{max\{w_1,w_2\} - min\{w_1,w_2\}}{2} \right\rfloor} \binom{max\{w_1, w_2\}}{i} \binom{max\{w_1, w_2\} - i}{min\{w_1, w_2\} + i} \qquad (2)$$

Proof. In this case, the weight of a diagonal step equals the weight of an α step (i.e. a vertical or horizontal step). The number of steps in a shortest path is clearly given by $max\{w_1,w_2\}$ (as in chessboard distance). Since one does not need to pay any extra for diagonal steps, it is possible, for example, that instead of having two consecutive α steps in the same direction, two diagonal steps are applied, reaching the same point after the two steps. In these paths, there are diagonal steps that are in an unnecessary direction (i.e. shortest paths can be obtained without any such direction steps). Let i denote the number of such unnecessary direction diagonal steps. Evidently, the minimum value of i is 0. For any such step, we need to have an extra diagonal step (in the other diagonal direction, to equalize its effect) instead of an α step. Originally, without any unnecessary diagonal steps (i = 0), there are exactly $max\{w_1,w_2\}$ – $min\{w_1,w_2\}$ number of α steps in a shortest weighted path. Thus, the number of α steps decreases by two when an unnecessary diagonal step is introduced. Thus, the maximum of i will be $\left\lfloor\frac{max\{w_1,w_2\}-min\{w_1,w_2\}}{2}\right\rfloor$, where the floor function is used. When i is fixed, we know the number of various steps in the shortest path(s): there are $max\{w_1,w_2\}$ steps, from which i are unnecessary diagonal steps, and we have also $min\{w_1,w_2\}+ i$ number of diagonal steps in the other diagonal direction. The remaining steps are α steps, and their number is $(max\{w_1,w_2\} - i) - (min\{w_1,w_2\} + i)$ $= max\{w_1,w_2\} - min\{w_1,w_2\} - 2i$.

Thus, the number of shortest paths with various values of i can be computed as follows:

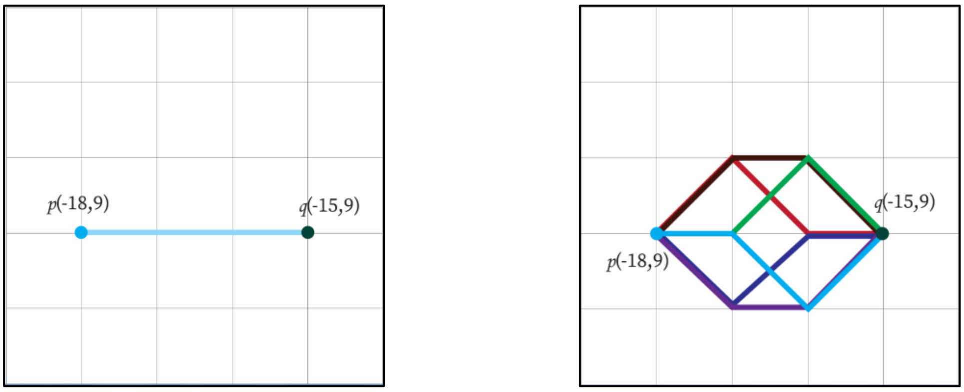
i = 0, then all steps in the path are in the right direction diagonal and α-steps, and their number is $\binom{max\{w_1,w_2\}}{min\{w_1,w_2\}}$;

for i in general:

$$\frac{max\{w_1,w_2\}!}{i!\,(min\{w_1,w_2\}+i)!\,(max\{w_1,w_2\}-min\{w_1,w_2\}-2i)!} \tag{3}$$

Where i=$\left\lfloor\frac{max\{w1,w2\}-min\{w1,w2\}}{2}\right\rfloor$is the maximum value for i. Thus, the total number of shortest paths is the sum of those:

$$f(w_1, w_2) = \sum_{i=0}^{\left\lfloor \frac{max\{w_1,w_2\} - min\{w_1,w_2\}}{2} \right\rfloor} \binom{max\{w_1, w_2\}}{i} \binom{max\{w_1, w_2\} - i}{min\{w_1, w_2\} + i}$$ ■



i = 0, $f(3, 0, i = 0) = 1$ (a) i = 1, $f(3, 0, i = 1) = 6$ (b)

Figure 3.8: Shortest paths between $p(-18,9)$ and $q(-15,9)$ with $\alpha = \beta$, when $i = 0$ and 1, respectively in (a) and (b), where i is the number possible diagonal steps to an unnecessary direction in the path from p to q.

Example 3.5 Let us use the points $p(-18,9)$ and $q(-15,9)$ with weight values α=1, β=1. Then w_1=3, w_2 = 0 and thus, $min\{w_1,w_2\}$ = 0, $max\{w_1,w_2\}$ =3. Further, the number of shortest weighted path (where the distance is 3) is:

$$f(3,0) = \sum_{i=0}^{1} \binom{max\{3,0\}}{i} \binom{max\{3,0\} - i}{min\{3,0\} + i} = 7$$

These paths are also illustrated in Figure 3.8 (a) and (b), with i = 0 and i = 1, respectively.

To show how these numbers are changing in the function of the coordinate differences, in Figure 3.9 we present a 3D chart for the number of shortest paths from the origin to other points in a 14 × 14 window when the diagonal and cityblock steps have the diagonals at the minimum places of this curve while it grows rapidly on the axes.

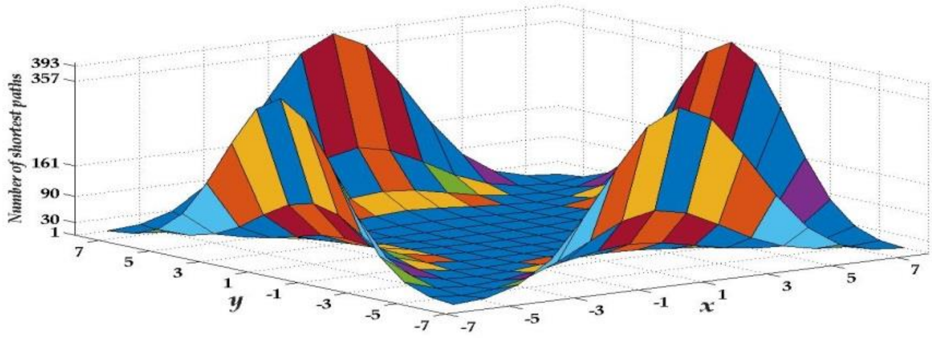


Figure 3.9: The number of shortest weighted paths from point (0,0) to other points in a 14 × 14 window with corners (−7, −7), (7, −7),(7, 7) and (−7, 7), in case $\beta = \alpha > 0$.

3.2.5 Case of $0 < \beta < \alpha$

In this case, β steps (diagonal steps) have less weight than α steps (i.e. horizontal and vertical steps); therefore, it will be more convenient and shorter to move from one point to another by diagonal steps, the shortest path between two points relying on the parity of the sum S of the absolute differences of the coordinates of the points. Therefore, we discuss two sub-cases in the two subsections below.

3.2.5.1 Sub-Case of $0 < \beta < \alpha$ for Points with Even Sum of Differences

Theorem 3.5 *Let α and β be the weights for cityblock and diagonal movements, respectively, with $\alpha > \beta$. Let $p = (x_1,y_1)$, $q = (x_2,y_2)$, $w_1 = |x_1 - x_2|$ and $w_2 = |y_1 - y_2|$. If $S = w_1 + w_2$ is an even number, then the number of the shortest paths between p and q, denoted by $f(w_1,w_2)$, is computed as*

$$f(w_1, w_2) = \binom{max\{w_1, w_2\}}{\frac{min\{w_1, w_2\} + max\{w_1, w_2\}}{2}} \tag{4}$$

Proof. The number of steps between two points is given as $max\{w_1,w_2\}$; moreover, all of them can be diagonal steps. As we showed previously in Subsection 3.2.2 (case $\alpha < \beta < 2\alpha$), $min\{w_1,w_2\}$ is the number of original diagonal steps in a shortest path. The remaining number of steps, $max\{w_1,w_2\}-min\{w_1,w_2\}$, can also be expressed by diagonal steps in this case; we call these diagonal steps 'added' diagonal steps. These added diagonal steps are used instead of the α-steps of the case $\alpha < \beta < 2\alpha$. These added diagonal steps are of two directions. One of them is the one we have called 'unnecessary' direction. We must have them in this case if $w_1 \neq w_2$. (In case of equality, the shortest path is built up by original diagonal steps to the same direction.) We need to add the same number of unnecessary direction diagonal steps and other (original) direction steps. Thus, the number of unnecessary direction diagonal steps is $\frac{max\{w_1,w_2\}-min\{w_1,w_2\}}{2}$, and the same number of added diagonal steps is needed. Therefore, the number of diagonal steps in a shortest path is $(min\{w_1,w_2\} + \frac{max\{w_1,w_2\}-min\{w_1,w_2\}}{2}) + \frac{max\{w_1,w_2\}-min\{w_1,w_2\}}{2}$. The first term gives the number of original direction diagonal steps (both the original and the added ones), while the second term gives the unnecessary direction diagonal steps. The sum equals $max\{w_1,w_2\}$.

Since the order of these steps is arbitrary, the number of shortest weighted paths between points p and q is the number of possible arrangements of these diagonal steps in the shortest path. Consequently, their number can be expressed by the following equation:

$$f(w_1, w_2) = \binom{max\{w_1, w_2\}}{\frac{max\{w_1, w_2\} + min\{w_1, w_2\}}{2}}$$

Equivalently, it can be written as the following binomial coefficient:

$$f(w_1, w_2) = \binom{max\{w_1,w_2\}}{min\{w_1,w_2\}+\frac{max\{w_1,w_2\}-min\{w_1,w_2\}}{2}} \qquad (5) \qquad \blacksquare$$

Let us analyse a special case. When w_1 or w_2 equals zero, the number of original diagonal steps is $min\{w_1, w_2\} = 0$, and the shortest path contains only added diagonal steps: one (any) of the directions is then unnecessary, and we have the same number of other added diagonal steps. In this case, the previous formula, the number of shortest weighted paths is simplified as follows:

$$f(w_1,w_2) = \binom{max\{w_1,w_2\}}{\frac{max\{w_1,w_2\}}{2}} \qquad (6)$$

Example 3.6 Let $p(0,0)$, $q(3,1)$, $\alpha = 2$ and $\beta = 1$ be given. Then, $w_1 = |3-0| = 3$ and $w_2 = |1-0| = 1$, then the number of shortest paths between p and q is:

$$f(3,1) = \binom{max\{3,1\}}{\frac{max\{3,1\}+min\{3,1\}}{2}} = \binom{3}{2} = 3$$

These paths are illustrated in Figure 3.10.

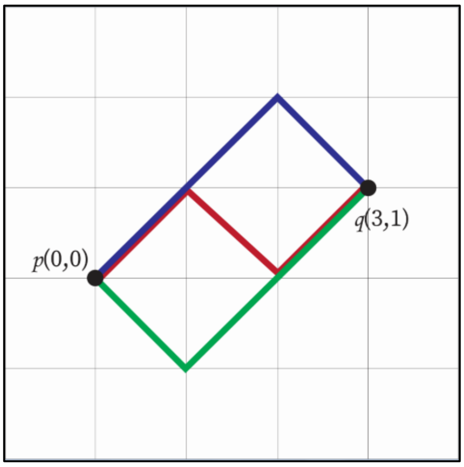


Figure 3.10: The shortest paths between $p(0,0)$ and $q(3,1)$ in case $0 < \beta < \alpha$ with even sum of differences.

3.2.5.2 Sub-Case of $0 < \beta < \alpha$ for Points with Odd Sum of Differences

Theorem 3.6 *Let α and β be the weights for cityblock and diagonal movements, respectively, with $\alpha > \beta$. Let $p = (x_1,y_1)$ and $q = (x_2,y_2)$ be two points in the square grid and $w_1 = |x_1 - x_2|$ and $w_2 = |y_1 - y_2|$. If $S=w_1+w_2$ is an odd number, then the number $f(w_1,w_2)$ of the shortest paths between the points p and q is determined as*

$$f(w_1,w_2) = \binom{max\{w_1,w_2\}-1}{\frac{min\{w_1,w_2\}+max\{w_1,w_2\}-1}{2}} \cdot max\{w_1,w_2\} \qquad (7)$$

Proof. In this case, we must have a cityblock step (α step) in the path, because there is no way to have shortest path with only diagonal steps (with β steps, two coordinates are always modified by ±1, and thus odd difference cannot be eliminated). The number of diagonal steps, thus, is $max\{w_1,w_2\}-1$, since the number of steps in this shortest path is $max\{w_1,w_2\}$. Let q' be the cityblock neighbour of q that is the closest to p (i.e. a shortest path between p and q' can be obtained by $max\{w_1,w_2\}-1$ diagonal steps). Actually, a shortest path from p to q contains exactly the same number of various direction steps as the shortest path from p to q' plus an additional cityblock step in the direction that is the same as from q' to q. The number of shortest paths is counted as the number of possible arrangements of the diagonal steps and the cityblock steps. Applying Theorem 3.5, the number of ways to have the diagonal steps between p and q is as follows:

$$\binom{max\{w_1,w_2\}-1}{\frac{min\{w_1,w_2\}+max\{w_1,w_2\}-1}{2}} \qquad (8)$$

Then, the number of ways to locate one cityblock step (which may not necessarily be the last step of the shortest path, but can be anywhere) is as follows:

$$\binom{max\{w_1,w_2\}}{1} = max\{w_1,w_2\}$$

From these, the number of shortest weighted paths is given by the following equation:

$$f(w_1,w_2) = \binom{max\{w_1,w_2\}-1}{\frac{min\{w_1,w_2\}+max\{w_1,w_2\}-1}{2}} \cdot max\{w_1,w_2\} \qquad \blacksquare$$

Example 3.7 Let the points $p(2,0)$ and $q(4,3)$, and the weights $\alpha = 3$ and $\beta = 2$ be given. Then $w_1 = |4-2| = 2$ and $w_2 = |3-0| = 3$, therefore $w_1 + w_2 = 5$ (which is odd number), then the number of shortest paths between p and q is:

$$f(2,3) = \binom{max\{2,3\} - 1}{\frac{max\{2,3\} + min\{2,3\} - 1}{2}} \cdot max\{2,3\} = \binom{2}{2} \cdot 3 = 3$$

Figure 3.11 shows all these shortest paths.

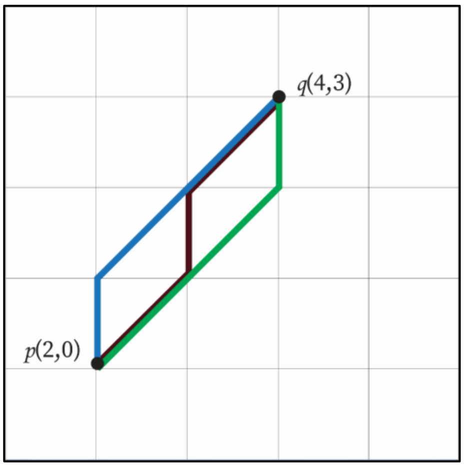


Figure 3.11: The shortest paths between $p(2,0)$ and $q(4,3)$, in case $0 < \beta < \alpha$ with odd sum of differences.

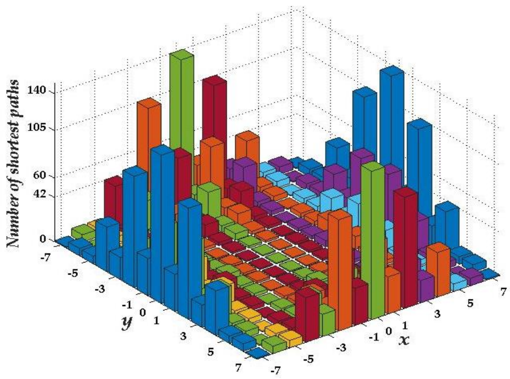


Figure 3.12: The number of shortest weighted paths from the origin (0,0) to other points in a 14 × 14 window with corners (−7, −7), (7, −7),(7, 7) and (−7, 7), in case $0 < \beta < \alpha$.

Finally, we summarize the case when diagonal steps have lower weights than cityblock steps. Figure 3.12 shows the number of shortest paths between (0,0) and other points in a 14 × 14 window. For the subcases, we also separately show the values: Figure 3.13 represents the cases ($\beta < \alpha$ for points with even coordinate sum S) and ($\beta < \alpha$ for points with odd coordinate sum S) for the number of shortest weighted paths from point (0,0) in a 14 × 14 window. One can observe that minimal values are given on the diagonals, while the function is growing with different speeds for the points with odd and even coordinate sums. For odd coordinate sums, it grows more rapidly. The largest growth values are on the axes with a growing coordinate difference.

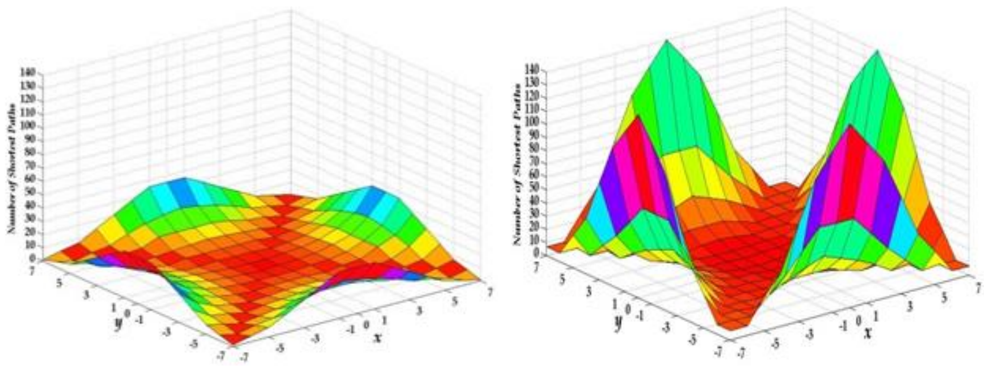


Figure 3.13: The number of shortest weighted paths from the origin (0,0) to other points in a 14 × 14 window with corners (−7, −7), (7, −7), (7, 7) and (−7, 7), in case $\beta < \alpha$ for points with even coordinate sum S shown on the left and with odd coordinate sum S shown on the right.

3.2.6 Cases with Zero Weights

In this subsection we consider the special scenarios when one or both of the weights is/are 0. In subsection 3.2.6.1 we consider the cases when $\alpha = 0$, while in 3.2.6.2 we consider the case when α is positive, but β has zero value. Up to our knowledge, these cases were never considered before.

3.2.6.1 Case of $\alpha = 0$

Theorem 3.7 *Let $\alpha = 0$ and $\beta \geq 0$ be the weights for cityblock and diagonal movements, respectively. The distance of points $p = (x_1,y_1)$ and $q = (x_2,y_2)$ is* 0*, since there are paths between any two points built up only by cityblock steps. Moreover, there are infinitely paths between p and q with sum of the weights* 0.

Proof. Consider the path built up by cityblock steps along the line from $p = (x_1,y_1)$ to $r = (x_1,y_2)$ with fixed first coordinate concatenated with the path from r to $q = (x_2,y_2)$ on the line with fixed second coordinate. (One or both of these paths could be empty, i.e., with 0 steps, depending on the fact if the points $p = (x_1,y_1)$ and $p = (x_1,y_1)$ share one or two or no coordinates.) The cost of this path is 0 and thus, the distance of the points with the condition $\alpha = 0$ is also zero.

Now, w.l.o.g., assume that $x_1 \leq x_2$. Let us consider the paths defined as follows: cityblock steps along the lines from $p = (x_1,y_1)$ to $p' = (x_1-n,y_1)$ (for any positive integer n), then from p' to $r' = (x_1-n,y_2)$, and from r' to q. Since the sum of the weights of this path is 0 for any value of n, all of these paths are considered as shortest paths between the two mentioned points, thus, there are infinitely many of them. ■

Theorem 3.8 *Let $\alpha = 0$ and $\beta \geq 0$ be the weights for cityblock and diagonal movements, respectively. The weighted distance defined by these weights is not metrical, but it is a pseudometric.*

Proof. A pseudometric is a distance function which has non-negative values, it is symmetric, it fulfils the triangular inequality, moreover the distance from any point to itself is 0. All of these properties are easily to check, since all distance values are 0. A distance is metric if it is a pseudometric, moreover if the distance of two points is 0, then the points coincide. This additional property is dropped by the considered distance function, thus it is not a metric. ■

Actually, the given pseudometric is the trivial pseudometric, since all the distance values are zero.

3.2.6.2 Case of $\alpha > \beta = 0$

Theorem 3.9 *Let $\beta = 0$ and $\alpha > 0$ be the weights for cityblock and diagonal movements, respectively. The distance of points $p = (x_1,y_1)$ and $q = (x_2,y_2)$ is 0 if and only if the sum of coordinate differences, $w_1 + w_2 = |x_1 - x_2| + |y_1 - y_2|$ is even. On the other hand, the distance of points p and q is α if and only if the sum of coordinate differences, $w_1 + w_2$ is odd. The number of paths between the points with the given length is infinite in both cases.*

Proof. Consider, first, the case, when the sum of the coordinate differences is even. There are paths between any two points built up only by diagonal steps. For instance, consider the diagonal line with slope 1 containing point *p* and the "antidiagonal" line, the line with slope –1 going through on *q*. These two lines will intersect each other at a point *r* with coordinates $x_1 + n = x_2 - m$ and $y_1 + n = y_2 + m$ for a pair of integers *n* and *m*, where these integers give the number (and the direction) of the diagonal steps from *p* to *r* and from *r* to *q*, respectively. Thus, it is clear that the distance of the points becomes zero. Furthermore, the given path can be easily modified to contain more and more diagonal steps (in a similar manner as we have shown in the proof of 3.7), thus the number of paths with length 0 becomes infinite.

Now, let us consider the case, when the sum of the coordinate differences is an odd number. Since in every diagonal step, both of the coordinates change by ±1, we cannot reach from one (of the points *p* and *q*) the other point only by diagonal steps. However, we can reach any of its cityblock neighbours by only diagonal zero-weight steps, thus, we need one extra cityblock step in the path resulting the distance of the points be α in this case. As the number of zero length paths between p and a given cityblock neighbour of q is (according to the first part of the proof) is infinite, each of them produces a shortest, i.e., α length path between p and q by adding the last cityblock step, hence the proof. ■

By a similar proof as the proof of Theorem 3.8, one can also establish the following result.

Theorem 3.10 *Let $\alpha > 0$ and $\beta = 0$ be the weights for cityblock and diagonal movements, respectively. The weighted distance defined by these weights is not metrical, but it is a pseudometric.*

Chapter 4

ON THE NUMBER OF SHORTEST WEIGHTED PATHS IN TRIANGULAR GRID

In this chapter of the thesis, we continue our work in terms of path counting, by counting the number of shortest paths for various cases based on weighted distances on the triangular grid.

4.1 Preliminaries

Each pixel in the triangular grid is addressed uniquely by a triplet of coordinates having axes with directions *x*,*y* and *z* to reflect the symmetry of the grid structure. In this grid, to the sum of the coordinate values reflects the orientation of the trixels (pixels or points of the triangular grid), thus we differentiate two types of trixels: even (zero sum trixels has orientation Δ) and odd trixels (pixel having orientation ∇ are addressed by triplets with 1-sum). Figure 4.1 shows the origin (trixel with coordinates (0,0,0)), the axes of the coordinate system and also a part of the grid with the assigned coordinate values.

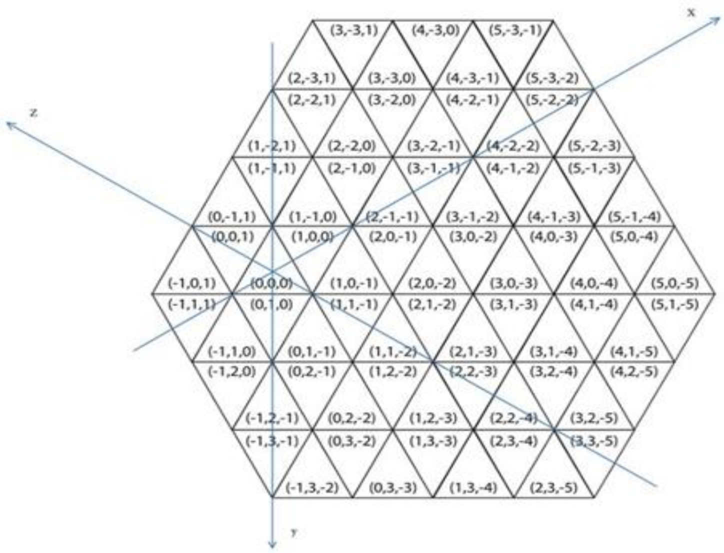


Figure 4.1 Coordinate system for the triangular grid with the Origin and the axes.

Each pixel in the triangular grid has three types of neighbors: there are three 1-neighbors, each of them shares a side with the original trixel, there are six more 2-neighbors and, further, there are three more 3-neighbor trixels. All twelve neighbors share at least one point on the boundary with the original trixel (see Figure 4.2). One can also define formally the neighborhood relations based on the coordinates of the trixels:

❖ The trixels $p(p(1),p(2),p(3))$ and $q(q(1),q(2),q(3))$ are in m-neighbor relation ($m \in \{1,2,3\}$) if:

- $|p(k)-q(k)| \leq 1$ for every $k \in \{1,2,3\}$ and
- $\sum_{k=1}^{3}|p(k) - q(k)| = m.$

We note here that, when working with a given neighborhood and also at neighborhood sequences, in the second condition the sign $\leq$ is used and the neighborhood relation is having the extensive property, that is all m-neighbors are also $(m - 1)$-neighbors (for $m > 1$). In case of equality of the last condition, the trixels are usually referred as strict m-neighbors in the literature. However, for chamfer distances, this strict neighborhood is more adequate, thus we use the definition as we have formally described above. Notice that 1- and 3-neighbors have different orientation than the original pixel (i.e. if the original pixel is even, then these neighbors are odd and vice versa), while 2-neighbors have the same orientation as the original pixel [36]. Figure 4.2 shows these three neighbor types for an even trixel.

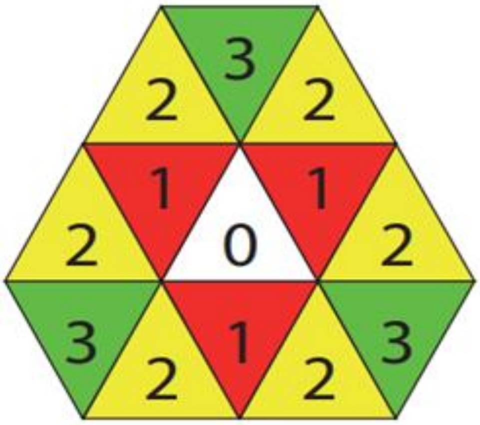


Figure 4.2: Three types of neighbors of trixel 0. 1-neighbors are red, 2-neighbors are yellow and 3-neighbors are green.

4.2 Number of Shortest Weighted Paths in Triangular Grid

Let $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ be two trixels in triangular grid, and let w_1, w_2 and w_3 be the absolute differences between the coordinates of p and q such that $w_1 = |x_1-x_2|$, $w_2 = |y_1-y_2|$ and $w_3 = |z_1-z_2|$. Let $S = w_1+w_2+w_3$, and $min(w_1, w_2, w_3)$, $mid(w_1, w_2, w_3)$ and $max(w_1, w_2, w_3)$ are minimum, middle (median) and maximum of w_1, w_2 and w_3 respectively. The number of shortest weighted paths between $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ depends on the values of the weights α, β and γ. According to this fact, we analyze the various cases in the next subsections.

4.2.1 The Case: $2\alpha < \beta$ and $3\alpha < \gamma$

Theorem 4.1 *Let α, β and γ be the weights of steps to a* 1-, 2- *and a* 3-*neighbor, respectively, such that $2\alpha < \beta$ and $3\alpha < \gamma$. Let $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ be two points of the triangular grid. Further, let $w_1=|x_1 - x_2|$, $w_2 = |y_1 - y_2|$ and $w_3 = |z_1 - z_2|$ be the absolute differences between the corresponding coordinates of the points. Then, the number of the shortest paths between p and q, denoted by $f(w_1,w_2,w_3)$, is computed as*

$$f(w_1, w_2, w_3) = \binom{mid(w_1,w_2,w_3) + min(w_1,w_2,w_3)}{min(w_1,w_2,w_3)} \tag{9}$$

Proof. By the given conditions on the weights, every shortest path is built up only by 1-steps since each 2-step can be substituted by two consecutive 1-steps and each 3-step can be substituted by three consecutive 1-steps with less sum of weights. Let us consider two different cases:

Case 1 If the two points are in the same lane:

The number of steps, i.e., the number of α-steps in a shortest weighted path between the two points is $mid(w_1, w_2, w_3)$ + $max(w_1, w_2, w_3)$, since $min(w_1, w_2, w_3)$=0. There is only 1 shortest path between any two points on the same lane, through the points `between' the two points in the same lane. Thus, there is only one path, any by (4.1) we will also get $\binom{mid(w_1,w_2,w_3)}{0}$= 1.

Case 2 If the two points are not in the same lane:

For simplicity we will take the two points to be (0,0,0) and (i,j,k), and let us assume, that the sector of triangular grid that we are interested in having values, $i,j > 0$ and $k < 0$, or j, $k < 0$ and $i > 0$. As we have already mentioned, based on the transformations detailed in [45], by mirroring of these sectors, one may obtain the whole triangular grid.

Firstly, case i, $j > 0$ and $k < 0$ is considered. We prove the formula (9) by induction. The base case of induction is $min(w_1, w_2, w_3) = 0$, which means that the two points p and q are in the same lane. It is already proven that formula (9) is satisfied, i.e., it gives 1 for these cases. Now, as the induction hypothesis, let us assume that formula (9) holds for every point with $|i|+|j|+|k| = i+j-k < M$, with a positive integer M. Further, let us consider a point with coordinates $|i|+|j|+|k| = i+j-k = M$. We may also assume that $min(w_1, w_2, w_3) > 0$. Since every trixel has either 0-sum or 1-sum triplet, this condition also means that in this region of the grid, one of i and j has the value $min(w_1, w_2, w_3)$ and the other has the value $mid(w_1, w_2, w_3)$. Then, let us analyze, first, the case when q is an odd pixel. In this case, all shortest paths from (0,0,0) to $(i,j-k)$ must contain, as the last step, a step either from $(i-1,j,k)$ or from $(i,j\text{-}1,k)$ the target trixel (i,j,k). However, both $(i-1,j,k)$ and $(i,j-1,k)$ are even pixels such that the sum of the absolute values of their coordinates is less than M. Thus, the number of the shortest paths to the trixels $(i\text{-}1,j,k)$ and $(i,j-1,k)$ are given by the formula (9) by our hypothesis, i.e., $\binom{i-1+j}{i-1}$ and $\binom{i+j-1}{i}$, respectively, not depending on which of i or j (or both) have the minimal value, since, e.g., $\binom{i-1+j}{i-1} = \binom{i-1+j}{j}$. Moreover, the number of shortest paths to (i,j,k) is, then, exactly the sum of those two values, that is,

$$f(w_1, w_2, w_3) = \binom{w_1 - 1 + w_2}{w_1 - 1} + \binom{w_1 + w_2 - 1}{w_1} = \binom{w_1 + w_2}{w_1} = \binom{i+j}{i} = \binom{i+j}{j},$$

which was to be proven.

Now, let us analyze the case when q is an even trixel. In this case all the shortest paths from (0,0,0) to (i,j,k) has the last step from the trixel $(i,j,k+1) = (i,j,-(|k|-1))$ to the trixel (i,j,k). Thus, the number of shortest path to the even trixel $q(i,j,k)$ is exactly the same as the number of shortest paths to the odd trixel $q'(i,j,k+1)$. However, for q' the sum of absolute coordinate values is $|i|+|j|+|k+1| = i+j+|k|-1=M-1$. Therefore, based on the hypothesis, the number of shortest path is $\binom{w_1+w_2}{w_1} = \binom{i+j}{i}$. Observing that q and q' shares the coordinates i and j, which are, in fact, $min(w_1, w_2, w_3)$ and $mid(w_1, w_2, w_3)$, the formula also holds for the trixel q.

Secondly, let us consider the case j, $k < 0$ and $i > 0$. Here, $max(w_1, w_2, w_3) = w_1 = |i|$. Again we use induction, on the base cases, where $min(w_1, w_2, w_3) = 0$, for which cases it is already proven that formula (9) is satisfied. Now, as the induction hypothesis, let us assume that formula (9) holds for every point with $|i|+|j|+|k| = i-j-k < M$, with a positive integer M. Further, let us consider a point with coordinates $|i|+|j|+|k| = i-j-k = M$. The number of shortest paths from pixel (0,0,0) to an even pixel (i,j,k) equals to the sum of the number of shortest paths to points $(i,j+1,k) = (i, -(|j|-1), -k)$ and $(i,j,k+1) = (i, -|j|, -(|k|-1))$, since each the shortest path from (0,0,0) to (i,j,k) is passing through exactly one of these two points having the last step from there to (i,j,k). However, by the induction hypothesis, formula (9) is correct for pixels $(i,j+1,k)$ and $(i,j,k+1)$ since their absolute coordinate sum is M-1. Therefore, for trixel (i,j,k) we have

$$f(w_1, w_2, w_3) = f(w_1, w_2 - 1, w_3) + f(w_1, w_2, w_3 - 1)$$

$$= \binom{w_3 + w_2 - 1}{w_2 - 1} + \binom{w_3 - 1 + w_2}{w_2}$$

$$f(w_1, w_2, w_3) = \binom{w_3 + w_2}{w_2} = \binom{-j - k}{-j}$$

$$= \binom{mid(w_1, w_2, w_3) + min(w_1, w_2, w_3)}{min(w_1, w_2, w_3)}.$$

For odd pixel (i,j,k), with $j,k < 0$ and $i > 0$, the number of shortest weighted paths equals to the number of shortest weighted paths for even pixel $(i-1,j,k)$, since in each shortest path the last step is from the even trixel $(i-1,j,k)$ to the odd trixel (i,j,k). Here $mid(w_1, w_2, w_3)$ and $min(w_1, w_2, w_3)$ have the values $|j|$ and $|k|$ (in an order) both for the trixels $(i-1,j,k)$ and (i,j,k). Therefore, the number of shortest paths to both of them is given by:

$$f(w_1, w_2, w_3) = f(w_1 - 1, w_2, w_3) = \binom{w_3 + w_2}{w_2} = \binom{-j - k}{-j}$$

$$= \binom{mid(w_1, w_2, w_3) + min(w_1, w_2, w_3)}{min(w_1, w_2, w_3)}.$$

The proof has been finished. ■

As one may also observe in the next example, the binomial coefficients appear in Figure 4.3, in fact, the space is cut to six parts, and in each part one can observe the Pascal's triangle. We also note here that in [39], based on a different approach, but, in fact, very similar results were presented (as the case of path counting for 1-neighborhood).

Example 4.1. Figure 4.3 illustrates the number of weighted shortest paths from point (0,0,0) to all other points in case $2\alpha < \beta$ and $3\alpha < \gamma$.

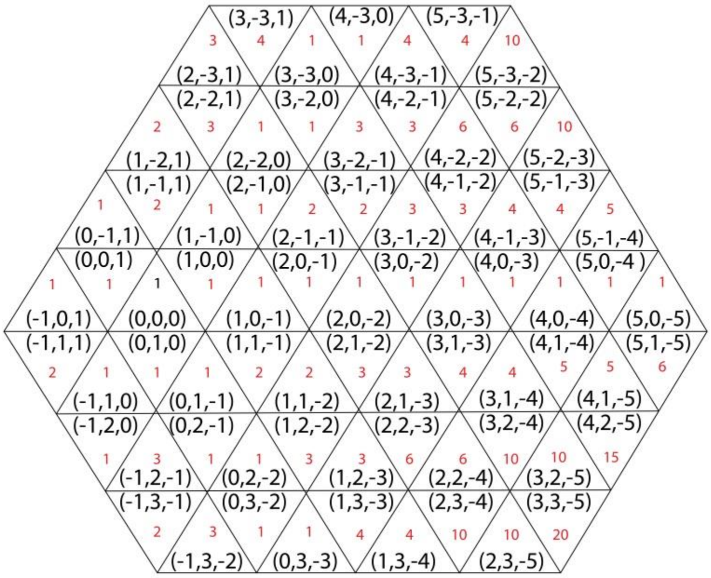


Figure 4.3: The number of shortest weighted paths from point (0,0,0) to other points of the grid with the condition $2\alpha < \beta$ and $3\alpha < \gamma$.

4.2.2 Case of $2\alpha > \beta$ and $3\alpha < \gamma$

In this case the shortest path between $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ contains a number of β-steps (plus one α-step in case p and q have different parities).The number of β-steps is equal to $\left\lfloor \frac{S}{2} \right\rfloor$. Furthermore, the number of shortest weighted paths, $f(w_1,w_2,w_3)$, between $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ is computed based on two sub-cases which are given by the following subsections.

4.2.2.1 Sub-Case ($2\alpha > \beta$ and $3\alpha < \gamma$) and S is an Even Number

Theorem 4.2 *Let α, β and γ be the weights for movements to* 1-, 2- *and* 3-*neighbor trixels in the triangular grid, respectively, such that $2\alpha > \beta$ and $3\alpha < \gamma$. Let $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ be two points of the triangular grid, and $w_1 = |x_1 - x_2|$, $w_2 = |y_1 - y_2|$ and $w_3 = |z_1 - z_2|$ be the absolute differences between the corresponding* *coordinates of the points such that* $S = w_1 + w_2 + w_3$ *is an even number. Then, the number of the shortest paths between p and q, denoted by* $f(w_1,w_2,w_3)$, *is computed as*

$$f(w_1,w_2,w_3) = \binom{mid(w_1,w_2,w_3) + min(w_1,w_2,w_3)}{min(w_1,w_2,w_3)}. \qquad (10)$$

Proof. By the given values of the weights, it is clear that the number of β-steps equals to $\left\lfloor \frac{S}{2} \right\rfloor$ in any of the shortest paths and there no other steps are considered (any 3-step can be broken to three consecutive 1-steps, and any 2 consecutive 1-steps can be joined to a β-step such that the total weight of the paths is decreasing if the path has other type of steps originally). First we deal with the case when the two trixels are on the same lane, i.e., they share one of the coordinates values. In this case, clearly, there is exactly one shortest path between them, built up by 2-steps on the given common lane; and since $min(w_1,w_2,w_3) = 0$, formula (10) also provides this result.

Now, without loss of generality, assume that *p* is the origin and *q* does not share any lane with *p*. By the symmetry of the grid, there are various, but equivalent cases. Let us consider the case that $q(i,j,k)$ has coordinates with the properties $i,j > 0$ and $k < 0$. The base of the induction consists the value 1 for the cases when $min(w_1,w_2,w_3) = 0$. We use induction on the sum of the coordinate difference, that is, in this case, $i+j-k$. By the induction hypothesis let us assume that equation (4.2) holds also for each even trixel (i,j,k) with $i+j-k < M$ for any given positive integer M. Then, let us consider an even trixel (i,j,k) with $i+j-k = M$ and count the number of the shortest paths from the origin to (i,j,k). It is clear that since only 2-steps are used, each shortest path goes through only on even trixels. On the other hand, to reach (i,j,k) in a shortest path the last step could be from exactly one of the two trixels (*i*-

1,j,k+1) = (i-1,j,-(|k|-1)) and (i,j-1,-(|k|-1)). However, for these two trixels, the condition that their absolute coordinate sum is less than M holds (it is actually, M-2 for any of these two trixels). Therefore, by the hypothesis, the number of shortest paths to them can be computed by formula (10), that is, actually, $\binom{i-1+j}{i-1} = \binom{i-1+j}{j}$ and $\binom{i+j-1}{i} = \binom{i+j-1}{j-1}$, since the first two coordinates correspond to the minimum and to the middle coordinate differences (in one of the others). Then the number of shortest paths to the pixel (i,j,k) is exactly the sum of the previous two values, i.e., $\binom{i-1+j}{i-1} + \binom{i+j-1}{i} = \binom{i+j}{i} = \binom{i+j}{j}$, which is the value we wanted to prove. ■

Now, let us give a comment on the previous case, since the result is exactly the same as in the first case, when only 1-steps were used.

Remark 1. Every two consecutive 1-steps can be joined to a 2-step and any 2-step can be broken to two consecutive 1-steps, in fact there is a bijection between the set of shortest paths used in Theorem 4.1 and the set of shortest paths used in Theorem 4.2 between the same pixels (since Theorem 4.2, only same parity trixels are considered here). By the used sixth of the grid one of the directions of any two consecutive 1-steps is a necessary direction step in a shortest path, while there could be two choices in the other (if the actual point is not in the same lane as the target point). Thus, by describing every second steps of a shortest path with only 1-steps (case of Theorem 4.1), one can still uniquely defined the whole path, and in fact, this description gives a shortest path between the same two points in case of only 2-steps are used (i.e., case of Theorem 4.2).

4.2.2.2 Sub-Case (2α > β and 3α < γ) and S is an Odd Number

Theorem 4.3 *Let α, β and γ be the weights of the* 1-, 2- *and* 3-*steps, respectively, with the conditions* $2\alpha > \beta$ *and* $3\alpha < \gamma$. *Let $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ be two trixels* *of the triangular grid, and let* $w_1 = |x_1 - x_2|$, $w_2 = |y_1 - y_2|$ *and* $w_3 = |z_1 - z_2|$ *be the absolute differences between the corresponding coordinates of the points, such that* $S = w_1 + w_2 + w_3$ *is odd. Then, the number of shortest paths between p and q, denoted by* $f(w_1,w_2,w_3)$*, is computed as*

$$f(w_1, w_2, w_3) = \binom{mid(w_1, w_2, w_3) + min(w_1, w_2, w_3)}{min(w_1, w_2, w_3)} \cdot \frac{S+1}{2}. \qquad (11)$$

Proof. In this case the shortest path composed of $\frac{S-1}{2}$ β-steps, and one α-step. Thus, the total number of steps in any shortest path is $\frac{S+1}{2}$. By Remark 1, we know that each shortest path in this case correspond to a shortest path with only 1-steps (in the sense that only those pixels are used during the path of the actual case which are included in that shortest path with only 1-steps). However, the mapping is not a bijection in this case. There could be many actual shortest paths that correspond to the same shortest path with only 1-steps: in fact, any one of the $\frac{S+1}{2}$ steps can be the 1-step, and then, all others are 2-steps. This gives us the possibility to use multiplication rule to count the number of shortest paths, first we can fix a shortest path with only 1-steps in $\binom{mid(w_1,w_2,w_3) + min(w_1,w_2,w_3)}{min(w_1,w_2,w_3)}$ many ways, and then, we can choose $\frac{S+1}{2}$ different ways the place of the 1-step in the path. Thus, the formula of (11) has been proven.■

Example 4.2. Figure 4.4 shows the number of shortest weighted paths from (0,0,0) to the displayed points of the grid in case $2\alpha > \beta$ and $3\alpha < \gamma$.

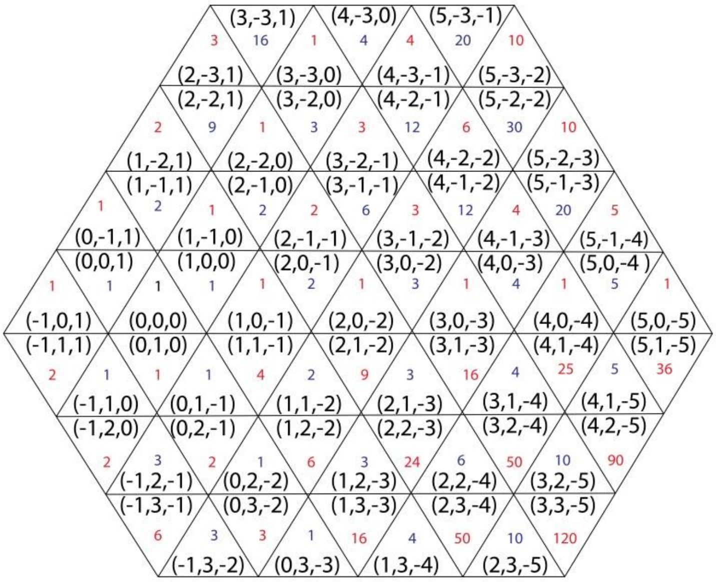


Figure 4.4: The number of shortest weighted paths between point (0,0,0) and some other trixels in the case $2\alpha > \beta$ and $3\alpha < \gamma$.

4.2.3 Case of $2\alpha < \beta < \gamma < 3\alpha$

In this case 3-steps have the smallest relative weight. Moreover, two consecutive 1-steps give less sum of weights than a 2-step, thus, in this case 2-steps are not used in any shortest paths.

Theorem 4.4 *Let α, β and γ be the weights of* 1-, 2- *and* 3-*steps, respectively, such that the weights satisfy the conditions* $2\alpha < \beta < \gamma < 3\alpha$. *Let $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ be pixels of the triangular grid, further, let $w_1=|x_1 - x_2|$, $w_2 = |y_1 - y_2|$ and $w_3 = |z_1 - z_2|$ be the absolute differences between the corresponding coordinates of the points. Then, the number of the shortest paths between p and q, denoted by $f(w_1,w_2,w_3)$, is computed as*

$$f(w_1, w_2, w_3) = \binom{mid(w_1, w_2, w_3)}{min(w_1, w_2, w_3)}. \qquad (12)$$

Proof. The proof consists of various cases. We start it with the case when the two points are in the same lane. Then the number of γ-steps (and also the number of β-steps) is zero. The number of shortest paths becomes one, having exactly one path through 1-neighbors between the two points in their common lane. Since $min(w_1, w_2, w_3) = 0$, the formula (12) leads also to this result:

$$f(w_1, w_2, w_3) = \binom{mid(w_1, w_2, w_3)}{min(w_1, w_2, w_3)} = \binom{mid(w_1, w_2, w_3)}{0} = 1.$$

Let us consider the cases when the two trixels are not lying on a common lane. Because of the symmetry of the grid, further we need to differentiate two cases, i.e., we do the proof for two of the sixths of the grid. The sixths of the triangular grid that we are interested in is x, $y > 0$ and $z < 0$ (2 positive coordinates and 1 negative), and $x > 0$ and y, $z < 0$ (2 negative and 1 positive coordinates). By mirroring these sixths one gets the whole triangular grid. Further, without loss of generality, we assume that $p(0,0,0)$ is the origin and $q(x,y,z)$ with the above properties. Let us consider the possible cases one by one.

Case a. If the two points $p(0,0,0)$ and $q(x,y,z)$ are not in the same lane and $x,y > 0$ and $z < 0$. Further, let us assume, first that q is an even trixel and let us see how a shortest path is built up from (0,0,0) to q. The shortest path contains the possible maximum number of "$\gamma\alpha$-combo" steps (any of those is a γ-step followed by an α-step, such that both of the first 2 coordinates are increased by 1 and the third coordinate is decreased by 2 during such a "combo" step). Thus, the number of these "combo" steps equals to $min(w_1, w_2, w_3)$. Notice that both the order and the direction of these steps is fixed by the coordinate values of q. In case $x = y$, one can reach q in this way, otherwise, "double" α-steps (those are two consecutive α-steps by increasing one of the first two coordinates, the one that had the value $mid(w_1, w_2, w_3)$, and by also decreasing the value of the third coordinate). The direction and the order of these α-steps is also fixed in a "double" step. Now, any of the shortest paths built up altogether by $mid(w_1, w_2, w_3)$ many "$\gamma\alpha$-combo" and "double" α-steps. The order of these combined steps, however, can be arbitrary, and $min(w_1, w_2, w_3)$ of them are "$\gamma\alpha$-combo". This leads to the formula that we wanted to prove. Observe that in each of the shortest paths in this case the last step, in fact, is an α-step, which decreases the third coordinate. This leads us the solution of the next case.

The next case includes the same sixth of the grid, but q is an odd point (i.e., $x + y + z = 1$). Instead of this odd trixel, let us consider the even trixel q'(x,y,z-1). The number of shortest paths from (0,0,0) to q is the same as the number of shortest paths to q', in fact, there is a bijection between these sets of paths, such that to any paths to q the last α-step from q to q' is concatenated. However, in this sixth of the grid x and y are playing the role of $min(w_1, w_2, w_3)$ and $mid(w_1, w_2, w_3)$ (in some order), thus the formula (12) also holds for this case.

Case b. Let us consider the other sixth of the grid, thus let p(0,0,0) and q(x,y,z) be given such that $x > 0$ and y, $z < 0$ (two negative coordinates and one positive coordinate). First, let q be even. A shortest path contains the possible maximum number of "$\alpha\gamma$-combo" steps (their number is $min(w_1, w_2, w_3)$, each of them is increasing the first coordinate by 2 and decreasing each of the other two by 1) and "double" α-steps (two consecutive α-steps, their directions are also fixed by q). Altogether the path contains $mid(w_1, w_2, w_3)$ number of those combined steps from which $min(w_1, w_2, w_3)$ is "$\alpha\gamma$-combo" steps. Since the order of these combined steps is arbitrary, the number of the shortest paths is

$$f(w_1, w_2, w_3) = \binom{mid(w_1, \mathrm{w}_2, w_3)}{min(w_1, w_2, w_3)},$$

which was to be proven.

Finally, let us consider the case in the same sixth of the grid when $q(x,y,z)$ is odd ($x + y + z$ =1). In this case every shortest path from (0,0,0) will have the last step an α-step from the even trixel q'(x-1,y,z) to q. Therefore, the number of shortest paths from the origin to q coincides to the number of shortest paths to q'. However, in this case y and z have the values $-min(w_1, w_2, w_3)$ and $-mid(w_1, w_2, w_3)$ (in some order) both for q and q', and therefore, the formula (12) gives also the number of the shortest paths to q. The theorem is proven. ■

Example 4.3. In Figure 4.5, one can observe the number of shortest weighted paths from the pixel (0,0,0) to some other pixels in case $2\alpha < \beta < \gamma < 3\alpha$.

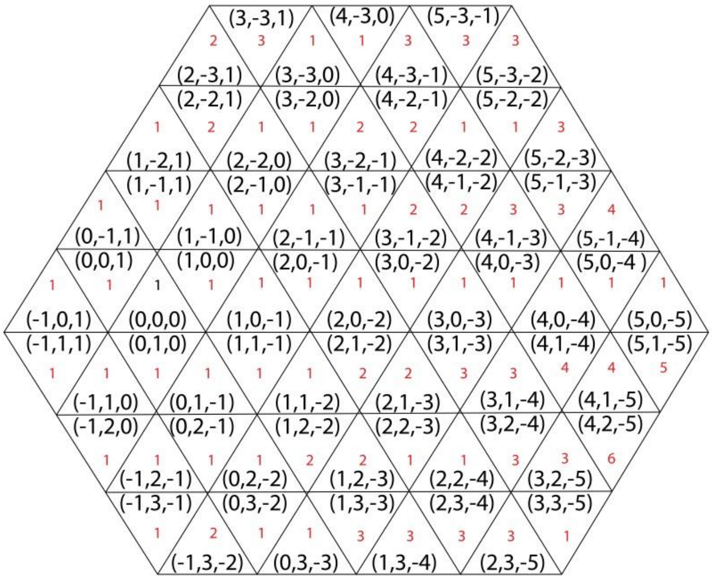


Figure 4.5: The number of shortest weighted paths from point (0,0,0) to some other points with the condition $2\alpha < \beta < \gamma < 3\alpha$.

4.2.4 Case of $2\alpha = \beta$ and $3\alpha < \gamma$

In this case every shortest path is built up by 1-steps and 2-steps, they are equally preferred, since their relative weight for changing a coordinate value is the same. Since 3-steps has a larger respective weight, they are not used in any shortest path. Moreover, every two consecutive 1-steps can be changed to a 2-step and vice versa without changing the sum of the weights in the path.

Theorem 4.5 *Let α, β and γ be the weights of 1-, 2- and 3-steps, respectively, such that the weights satisfy the conditions $2\alpha = \beta$ and $3\alpha < \gamma$. Let $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ be two points of the triangular grid, further, let $w_1=|x_1 - x_2|$, $w_2 = |y_1 - y_2|$ and $w_3 = |z_1 - z_2|$ be the absolute differences between the corresponding coordinates of the points, and S be the sum of these absolute differences. Let F_S*

denote the S-th element of the Fibonacci sequence (starting the sequence by $F_0 = F_1 = 1$*). Then, the number of the shortest paths between p and q, denoted by* $f(w_1,w_2,w_3)$, *is computed as*

$$f(w_1, w_2, w_3) = \binom{mid(w_1,w_2,w_3)+min(w_1,w_2,w_3)}{min(w_1,w_2,w_3)} F_S. \qquad (13)$$

Proof. In this case a shortest weighted path may contain only α-steps, or only β-steps or both of them. Each β-step can be substituted by two α-steps and vice versa. The number of shortest weighted paths with only α-steps equals to $\binom{mid(w_1,w_2,w_3)+min(w_1,w_2,w_3)}{min(w_1,w_2,w_3)}$, as we have seen in formula (9). In each of these paths there are exactly S α-steps. In each path, we can always substitute two consecutive α-steps by a β-step, such that the original path only 1-steps is clearly identifiable. (The obtained β-step can be broken to two consecutive α-steps in a unique way). Therefore, we may apply multiplication rule, first by counting the number of base paths with only 1-steps, and then, to count the number of paths when various number of 2-steps are used in various places. That is actually, the number of the possible orders of 1's and 2's such that their sum is $S = w_1 + w_2 + w_3$. Let, then i be the number of β-steps ($0 \le i \le \left\lfloor \frac{S}{2} \right\rfloor$), and each time we increase the number of β-steps by 1, we decrease the number of α-steps by 2. Thus, there is i 2-steps, and the path contains totally $S - i$ steps. Therefore, we need to sum up the values $\binom{S-i}{i}$ to get the number of possible ways. Actually, $\sum_{i=0}^{\left\lfloor \frac{S}{2} \right\rfloor} \binom{S-i}{i} = F_S$, that is the S-th Fibonacci number: One can see it as follows. When $p = q$, or they are 1-neighbors, there is exactly 1 shortest path, without any steps (any number) or with a 1-step (one number 1), respectively. Also F_0 and F_1 have the value 1, as the initial values of the sequence. Now, as an induction hypothesis, let us assume that the number of possible orders of 1's and 2's such that their sum is S is exactly F_S when $S < M$ for any fixed

M (where *M* is at least 2). Now there are exactly two ways to have such a sequence of 1's and 2's such that their sum is *M*: either the last element is a 1 or a 2 (1-step and 2-step, respectively, considering paths). However, by the assumption, the number of those sequences (paths) with sum *M* that have a 1 as their last element is exactly F_{M-1} while the number of those that have a 2 as their last element is exactly F_{M-2}. Using the addition rule, we get that $F_M = F_{M-1} + F_{M-2}$, which is exactly the recursive formula for the Fibonacci sequence, thus F_M is exactly the *M*-th element of this sequence. Summarizing it, we have the formula what we wanted to prove:

$$f(w_1, w_2, w_3) = \binom{mid(w_1, w_2, w_3) + min(w_1, w_2, w_3)}{min(w_1, w_2, w_3)} \sum_{i=0}^{\left\lfloor \frac{S}{2} \right\rfloor} \binom{S-i}{i}$$

$$= \binom{mid(w_1, w_2, w_3) + min(w_1, w_2, w_3)}{min(w_1, w_2, w_3)} F_S.$$

Example 4.4. Figure 4.6 shows the number of shortest paths from point (0,0,0) to the displayed points in the case $2\alpha = \beta$ and $3\alpha < \gamma$.

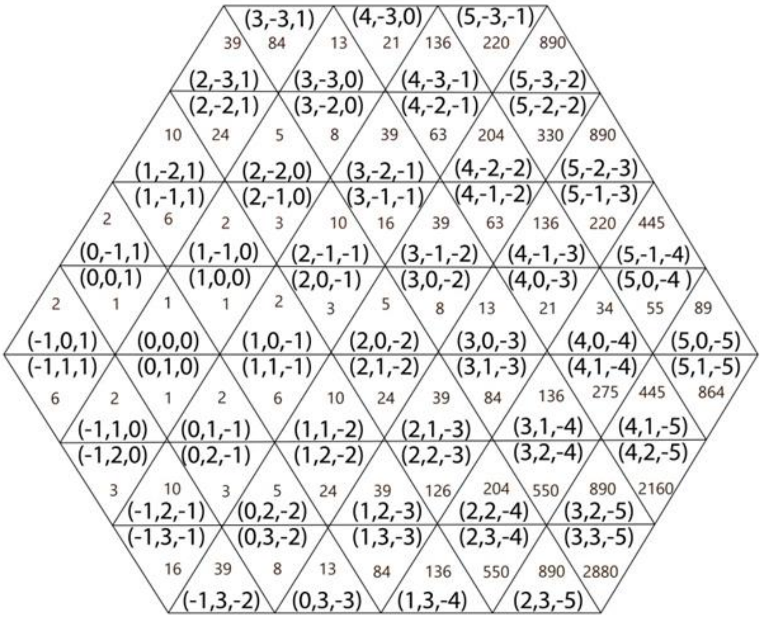


Figure 4.6: The number of the shortest weighted paths from the Origin to some other points with the condition $2\alpha = \beta$ and $3\alpha < \gamma$.

Observe that if the two trixels are in a common lane (i.e., $min(w_1, w_2, w_3) = 0$), the number of the shortest paths between them is, in fact, the S-th (i.e., $(w_1 + w_2 + w_3)$-th) element of the Fibonacci sequence, starting with F_0=F_1=1. (Actually, as we have shown, see also, e.g., [40], the number of {1,2}-sequences having sum S is F_S, that is the S-th element of the Fibonacci sequence.) Based on that, we can see the results as a kind of two-dimensional extension of the Fibonacci sequence.

4.2.5 Case of 2α < β and 3α = γ

In this case the shortest weighted path between $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$ is composed from α-steps and γ-steps, and we will never use β-steps, where the number of γ-steps is between 0 and $min(w_1, w_2, w_3)$ in a shortest weighted path. Note that a γ-

step can always be substituted by three consecutive α-steps, but the converse does not hold. The number of shortest weighted paths between $p(x_1,y_1,z_1)$ and $q(x_2,y_2,z_2)$, $f(w_1,w_2, w_3)$, can be computed according to four sub-cases which are given in the next theorem.

Theorem 4.6 *Let α, β and γ be the weights of* 1-, 2- *and* 3-*steps, respectively, such that $2\alpha < \beta$ and $3\alpha = \gamma$ hold. Let $p(0,0,0)$ and $q(x,y,z)$ be two trixels of the triangular grid, further, let the absolute coordinate differences $w_1=|x|$, $w_2 = |y|$ and $w_3 = |z|$ and the sum of them $S = w_1 + w_2 + w_3$ be given. Then, there are the following cases for counting the number of the shortest paths, denoted by $f(w_1,w_2,w_3)$, between p and q:*

If the trixels are in a common lane, that is if $min(w_1, w_2, w_3) = 0$, then there is exactly 1 shortest path.

If the trixels are not in a common lane then, if S is even, then

$$f(w_1, w_2, w_3) = \sum_{i=0}^{min(w_1,w_2,w_3)} \binom{(S/2) - i}{min(w_1, w_2, w_3)} \binom{min(w_1, w_2, w_3)}{i}. \tag{14}$$

If S is odd and q has 2 *positive and a negative coordinate, then*

$$f(w_1, w_2, w_3) = \sum_{i=0}^{min(w_1,w_2,w_3)} \binom{((S+1)/2) - i}{min(w_1, w_2, w_3)} \binom{min(w_1, w_2, w_3)}{i}. \tag{15}$$

If S is odd and q has 2 *negative and* 1 *positive coordinate, then*

$$f(w_1, w_2, w_3) = \sum_{i=0}^{min(w_1,w_2,w_3)} \binom{((S-1)/2) - i}{min(w_1, w_2, w_3)} \binom{min(w_1, w_2, w_3)}{i}. \tag{16}$$

Proof. If the points are in the same lane, clearly, the shortest path built up by 1-steps including each trixel between them, and there is only 1 such path. Now, let us consider the shortest weighted paths from point $p(0,0,0)$ to $q(x,y,z)$ where none of the coordinates of q is zero, i.e., the two points are not in the same lane. A shortest path may contain α-steps only. It may contain "$\gamma\alpha$-combo" steps if q has 1 negative and 2 positive coordinates or it may contain "$\alpha\gamma$-combo" steps if q has 2 negative coordinates and 1 positive coordinate. A shortest path may also contain many α-steps and also many combo steps (based on the case as they were described above). Now, let us consider the remaining three cases one after the other.

When q is an even trixel, i.e., S is an even number, every shortest path is built up by combo steps and "double" α-steps (two consecutive α-steps). The type of the combo steps, i.e., either "$\alpha\gamma$-combo" or "$\gamma\alpha$-combo" depends on the number of negative and positive values among the coordinates x, y and z (as we have already described). We can partition the set of shortest paths to equivalent classes based on the number of combo steps used in them. Consequently, we will compute the number of shortest paths in each such blocks and we sum up those values. On the one hand, we may have only α-steps in a shortest path, that means that 0 combo steps are used. On the other hand, the maximal number of combo steps in a shortest path (since they change all the three coordinate values), is $min(w_1, w_2, w_3) = min(|x|, |y|, |z|)$. When only α-steps are used, the number of such shortest paths is $\binom{mid(w_1,w_2,w_3)+min(w_1,w_2,w_3)}{min(w_1,w_2,w_3)}$ (from Theorem 4.1). The number of shortest paths with the maximal number of combo steps is $\binom{mid(w_1,w_2,w_3)}{min(w_1,w_2,w_3)}$ (from Theorem 4.4). In one combo step the sum of the coordinate changes in absolute value is 4 (3 + 1 in $\gamma\alpha$ and 1 + 3 in $\alpha\gamma$-combo), while a double α-step changes 2 of the coordinates with sum 2 in absolute value, that implies that a combo step can be changed to two double α-steps (although the reverse may not hold). Let i be the number of the combo steps in a shortest path where $0 \leq i \leq min(w_1, w_2, w_3)$, then the number of combo and double steps in such shortest path is $\frac{S}{2} - i$. It is actually, $\frac{S}{2} = mid(w_1, w_2, w_3) + min(w_1, w_2, w_3)$ double α-steps, if combo steps are not used: and $min(w_1, w_2, w_3)$ of them in one direction, while $mid(w_1, w_2, w_3)$ of them in other direction (60 degree between the two directions). Moreover, each combo steps decrease the number of double steps by 2, i.e., by one and one both directions double α-steps, and therefore the sum of all types of combined steps by 1. In any path the used combo and double α-steps can be put in any order. Thus, the combination of the i combo, $mid(w_1, w_2, w_3) - i$ double α-steps in one direction and $min(w_1, w_2, w_3) - i$ double α-steps in the other direction, altogether $\frac{S}{2} - i$ steps, gives the number of shortest paths in this block. This number can be written as $\binom{(S/2)-i}{min(w_1,w_2,w_3)}\binom{min(w_1,w_2,w_3)}{i}$. By summing up these values for the possible values of i, one gets exactly the formula (14). The proof of this case is finished.

In the case S = |x|+|y|+|z| is odd and q(x,y,z) has 2 positive and one negative coordinate, by the symmetry of the triangular grid, we consider only x, y > 0 and z < 0. In this sixth of the grid, in the shortest paths "$\gamma\alpha$-combo" and "double" α-steps can be used (to any even point). In what follows, for any even point q' the shortest path finishes with an α-step into the opposite direction than axis z, i.e., by decreasing the third coordinate and not changing the other two. Therefore, the number of shortest paths from the trixel (0,0,0) to the odd trixel q(x,y,z) is exactly the same as the number of shortest paths from (0,0,0) to the even trixel q'(x,y,$z-1$). However, the number of shortest paths to q' is already computed in the previous case. Knowing that in this sixth of the grid, one of x and y plays the role of $mid(w_1, w_2, w_3)$ and the other plays the role of $min(w_1, w_2, w_3)$ and for q' the sum of the coordinate differences is one more than it is for q (it is S+1 for q'), the number of shortest paths is proven to be:

$$f(w_1, w_2, w_3) = \sum_{i=0}^{min(w_1,w_2,w_3)} \binom{((S+1)/2) - i}{min(w_1, w_2, w_3)} \binom{min(w_1, w_2, w_3)}{i}.$$

Finally, considering the last case, because of symmetry, we use $x > 0$ and $y,z < 0$. In this sixth of the grid, all shortest paths to an even point q' built up by "$\alpha\gamma$-combo" and "double" α-steps. To reach an odd trixel $q(x,y,z)$ every shortest path from (0,0,0) has the last step as an α-step from the trixel $q'(x\text{-}1,y,z)$ to $q(x,y,z)$. From this fact, it flows that the number of shortest path from (0,0,0) to q is the same as to q'. However, the latter one is already proven and it is computed by formula (14). In this sixth of the grid $-y$ and $-z$ play the role of $min(w_1, w_2, w_3)$ and $mid(w_1, w_2, w_3)$ in an order. Further, the sum of the absolute coordinate values is S for q, then it is S-1 for q'. Therefore, one needs to modify the formula (14) according to this and gets

$$f(w_1, w_2, w_3) = \sum_{i=0}^{min(w_1,w_2,w_3)} \binom{((S-1)/2) - i}{min(w_1, w_2, w_3)} \binom{min(w_1, w_2, w_3)}{i},$$

which was to be proven. Thus, each case of the theorem is proven. ■

Example 4.5. In Figure 4.7 the number of weighted shortest paths from the origin (0,0,0) to some points are presented in case of $2\alpha<\beta$ and $3\alpha = \gamma$.

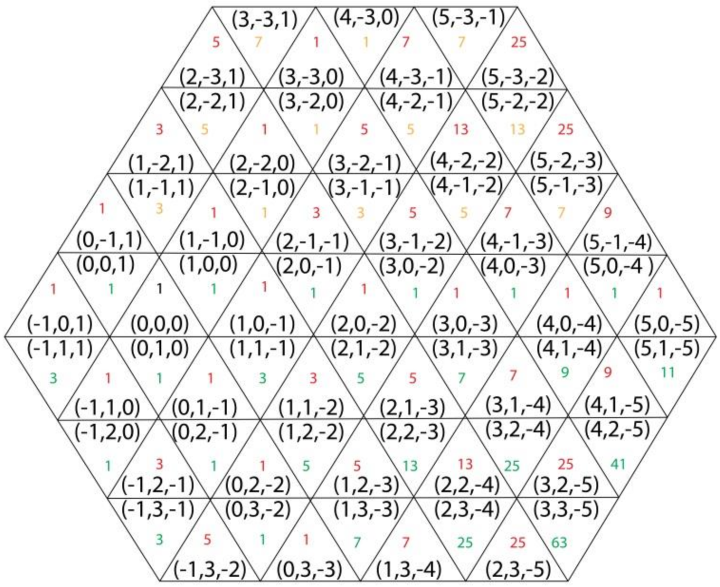


Figure 4.7: The number of shortest weighted paths from point (0,0,0) to all other displayed points in the case of $2\alpha < \beta$ and $3\alpha = \gamma$.

Chapter 5

CONCLUSION

In this work (second chapter), we have presented two types of *n*-nomial coefficients: to differentiate them, the terms multinomial and polynomial coefficients are used. We have interpreted both types of coefficients in terms of numbers of shortest paths using various neighbourhood relations in various grids. Connected to the first type, we have discussed *n*-nomial expansions which have *n* distinct variables, and as a special case trinomial-multinomial coefficients. We have shown also formulas to compute trinomial- and quadrinomial-polynomial coefficients which are found in trinomial- and quadrinomial-polynomial expansions (using various powers of only 1 variable), respectively. Connection to counting number of shortest paths has also been shown. While in the multinomial case, the dimension of the grid is changed, at the case of polynomial coefficients counting the shortest path in the 2D grid but with an extended neighbourhood was considered. We have also underlined the connection between the two types of coefficients, especially, e.g., by a formula for trinomial-polynomial coefficients in terms trinomial-multinomial coefficients: in both cases the number of shortest paths of previously computed three neighbours are summed, but while in the multinomial case the directions are independent, in the polynomial case we are still in 2D.

We should also mention that in [46], another kind of extension of the binomial coefficients was shown which allows also negative numbers, consequently, the

Pascal's triangle is expanded to the Pascal's hexagon. We also note that counting only paths satisfying a given restriction, as, e.g., determining the Catalan numbers, has also various extensions. To consider and compare these extended variants of Catalan numbers is a topic of a future work.

Then (in the third chapter, based on [47]), number of shortest paths in the square grid using chamfer distance has been discussed. These shortest paths can be represented by trajectories on the digital grid. A combinatorial problem, the number of shortest paths is computed in various scenarios. The numbers of shortest paths with the cityblock and chessboard metrics were already known [42]. However, we have presented results for a much larger class of digital distances, for chamfer distances, in this way our study can be seen as a generalisation of these previous results. Digital distances can be used in various ways in communication networks [31], and they are also related to combinatorial problems. For example, the number of shortest paths gives important features of a network. Results on the number of shortest paths for neighbourhood sequence distances were presented in [43, 44], in this sense, we have completed the picture by presenting here analogous results for the other type of widely used digital distance family. In this thesis, we have analysed rigorously all the cases to find the number of minimum weighted paths between any two points in a square grid. The cases depend on the value of weights given to cityblock steps (α steps) and diagonal steps (β steps). We have discussed five cases with positive weights and two cases when weight zero is allowed. We have seen that the results obtained in them are pairwise different. By Table 1 one can also be sure that there are no more cases, all the possibilities to have positive and/or zero weights for both cityblock and diagonal steps are discussed. Our results with positive weights are also displayed in 3D graphs, which show how the resulting functions grow. In most cases the functions have strong monotonic behaviour as one goes further from the origin.

Table 1: The discussed cases for the weights α and β.

Condition	Only positive weights						
	$2\alpha < \beta$	$2\alpha = \beta$	$\alpha < \beta < 2\alpha$	$\alpha = \beta$	$\beta < \alpha$	$\alpha = 0 \leq \beta$	$\alpha > \beta = 0$
Case /subsection	**3.1**	**3.3**	**3.2**	**3.4**	**3.5**	**3.6.1**	**3.6.2**

The cases of $\beta > 2\alpha$ and $\beta = 2\alpha$ show very similar behaviour (see Figure 3.3 and 3.7). The case of $\alpha < \beta < 2\alpha$ does not seem to relate to any other cases (Figure 3.5). Case of $\beta = \alpha$, displayed in Figure 3.9, shows some relation to the case of $\beta < \alpha$, however this latter is more complicated than the others, see Figure 3.12. We highlight the results of this case, i.e., when the diagonal steps have less weight than cityblock steps. As we have seen, the result is described by two different functions depending on the parity of the sum of the coordinate differences of the points, thus it does not behave in a monotonic way. We have shown also the cases when zero weight is allowed. If the cityblock step has zero weight, all the distances become 0. Contrary, if only the diagonal movements are without cost, but the cityblock steps have a positive weight, then somewhat similarly to the case of $\beta < \alpha$, the result is not monotonous, but given by two different values alternating for the points of the grid. Our results are useful in network analysis, in digital image processing and in shape analysis. We believe that it is important also for application point of view to consider all the possible cases depending on the possible values of the weights. The number of shortest weighted paths between points that contain a given point or a set of given points can be discussed in the future. For example, if we have path $s,\ldots,b,\ldots,t$, then it can be computed how many shortest paths between s and t contain b. Extensions to higher dimensional or other grids (architectures) can also be done in the future.

The fourth chapter of this thesis (based on [48]) discusses five of the most popular cases for the number of shortest weighted paths between any two pixels in the triangular grid. The number of these paths depends on the weights α, β and γ of the movements from the pixel to its various types of neighbors. In Subsection 4.2.1, α is preferred, it has the smallest relative weight for changing a coordinate value in a path, and thus, no shortest path contain any β- and γ-steps. In Subsection 4.2.2, β-steps are preferred, β has the smallest relative weight for changing a coordinate value in a path, and thus, no shortest path contain any γ-steps (and at most one α-step is used, since we may need to change the parity when we are looking for a shortest path between an odd and an even trixel). In contrast to this, in Subsection 4.2.3 β-steps are not used at all, in fact γ-steps are preferred (even by adding also many α-steps because of the parities of the trixels). In the case considered in Subsection 4.2.4, α-steps and β-steps are equally preferred, and no γ-step can occur in a shortest path. In our last studied case, α-steps and γ-steps are equally preferred, and no β-step can occur. While in some cases the computation results clearly well-known binomials, the structure of the grid give some more interesting cases. We have seen that based on the case $2\alpha = \beta$ and $3\alpha < \gamma$ one can define two dimensional extension of the Fibonacci numbers. We believe that the cases presented here are among the most basic and usual ones: we have studied the cases, when exactly one or two types of steps are not preferred, and therefore they have not appear any of the shortest paths. However, there are also some other interesting cases that can be discussed later on, e.g., when $2\alpha = \beta$ and $3\alpha = \gamma$, when all the three types of steps are equally preferred. Other possible future task is to consider “inhomogeneous” distribution of the weights, which causes to count the number of shortest weighted paths of one case concatenated by shortest weighted paths of other case. A somehow connected result was discussed in [37]: “digital disks” were defined and used to approximate the Euclidean disks, where the set of gridpoints having less (or equal) distance than a given radius from a given gridpoint defined the digital disk.

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