Deterministic and Probabilistic Modeling of the Logistic Growth

Yves Yannick Yameni Noupoue

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Applied Mathematics and Computer Science

Eastern Mediterranean University
August 2019
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Ali Hakan Ulusoy
Acting Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy in Applied Mathematics and Computer Science.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy in Applied Mathematics and Computer Science.

Asst. Prof. Dr. Yücel Tandoğdu
Supervisor

Examining Committee

1. Prof. Dr. Çerkez Ağayeva
2. Prof. Dr. Hüseyin Aktuğlu
3. Prof. Dr. Agamirza Bashirov
4. Prof. Dr. Emine Mısırlı
5. Asst. Prof. Dr. Yücel Tandoğdu

ABSTRACT

The logistic growth concept investigated by many researchers has wide applications in different fields. An exact solution to the logistic growth problem can always be obtained using the first order differential equation. However, this is not always possible when the fractional order of derivative is used.

This work investigated the use of deterministic and probabilistic approaches for modeling the logistic growth models. The deterministic model was built using classical and fractional differential equations. Hadamard type fractional derivative and integral were used to prove the existence and uniqueness of the solution to the fractional logistic differential equation using theorems. Numerical methods were employed to approximate the solution in the fractional case since it has no analytic form. The probabilistic approach used by employing the Gaussian kernel smoothing. A comparison of deterministic and probabilistic methods performance in modeling the logistic growth concept, minimum error levels were achieved with the fractional method, and Gaussian kernel smoother method with bandwidth 22.

Keywords: Gaussian kernel, optimal bandwidth, fractional differential equation, Hadamard derivative, Caputo-Fabrizio, Grünwald-Letnikov, generalized Euler method, carrying capacity.

ÖZ

Birçok araştırmacının üzerinde çaliştığı lojistik büyüme kavramı pek çok farklı alanda uygulanabilir. Birinci dereceden diferansiyel denklemler kullanılarak, lojistik büyüme problemlerinin tam çözümü mümkündür. Ancak, kesirli diferansiyel denklemler kullanıldığında tam çözüm bulmak mümkün olmayabilir.

Bu çalışmada lojistik büyüme kavramının modellenmesi deterministik ve probabilistik yaklaşımlarla araştırıldı. Deterministik model, klasik ve kesirli diferansiyel denklemler kullanılarak tayin edildi. Hadamard türü kesirli differansiyel ve integral kullanılarak kesirli diferansiyel türü lojistik denklem için tek çözüm olacağı teoremlerle ispatlanmıştır. Analitik çözümün elde edilemediği kesirli durumlar için, nümerik yöntemlerle yaklaşık değerler bulunmuştıur. Probabilistik yaklaşımda Gauss kernel düzleştiricisi kullanılmıştır. Lojistik büyüme modellenmesi sürecinde kullanılan deterministik ve istatistiksel yöntemler, tahmin işleminde ortaya çıkan hatala gözönünde bulundurularak karşılaştırılmıştır. En düşük hatalar Kesirli üssel metod ve Gauss kernel düzleştirici metodunda band genişliği 22 iken elde edilmiştir.

Anahtar kelimeler: Gauss kernel, optimum band genişliği, kesirli diferansiyel denklem, Hadamard türevi, Caputo-Fabrizio, Grünwald-Letnikov, genelleştirilmiş Euler metodu, taşıma kapasitesi.

DEDICATION

To my Family

ACKNOWLEDGMENT

I would like to thank my supervisor Asst. Prof. Dr. Yücel Tandoğdu for his support and guidance throughout my postgraduate studies.

I would like to say thanks to everyone who in one way or another has contributed to my success during my PhD studies.

A special thanks to my wife Pauline Milaure Ngugnie Diffouo, who has always encouraged and provided me with all kind of supports.

I express my gratitude to Dr. Muath Awadalla and his wife Dr. Kanda Abuassba for their fraternity love and strong collaboration in scientific research.

My dear parents and siblings Kanyep Mathilde, Noupoue Jean, Seve Landry Nguematcha, Claude Martial Tanguep, Marie-Therese Ngamakoua, Justine Fideline Tcheumeni, Nadege Deumeni, You are forever in my heart, Thanks for your supports.

Dear grandmother Marceline, you left early, I will not be able to celebrate with you on my return. Thank you for your support and advices. You taught me that hard work always pays off. Your advices have paid off.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

${}_{RL}I_{0^+}^{\gamma}$	Riemann-Liouville fractional integral of order γ
${}_{RL}D_{0^+}^{\gamma}$	Riemann-Liouville fractional derivative of order γ
${}^{H}I^{\gamma}$	Hadamard fractional integral of order γ
${}^{H}D^{\gamma}$	Hadamard fractional derivative of order γ
$L^1[a,b]$	Set of integrable functions on $[a,b]$
${}_{C}D_{0^+}^{\gamma}$	Caputo fractional derivative of order γ
${}_{CF}D_{0}^{\gamma}$	Caputo-Fabrizio fractional derivative of order γ
$I_{0^+}^{\gamma,\psi}$	ψ -Caputo fractional integral of order γ
${}_{C}D_{0^+}^{\gamma,\psi}$	ψ -Caputo fractional derivative of order γ
${}_{GL}D_{a^+}^{\gamma}$	Grünwald–Letnikov fractional derivative of order γ
$H^1(a,b)$	Hilbert Space
$\partial\Omega$	Boundary of the set Ω
$\bar{\Omega}$	Closure set Ω
$\mathbb{N}$	The set of all integers
$\mathbb{R}$	The set of all real numbers
$\lVert\bullet\rVert$	Norm of $\bullet$
$C^n([a,T],\mathbb{R})$	Set of *n*-times continuously differentiable functions from $[a,T]$ to $\mathbb{R}$
AMISE	Asymptotic Mean Integrated Square Error
CF	Caputo-Fabrizio
FDE	Fractional Differential Equation

FLDE	Fractional Logistic Differential Equation
GDP	Gross Domestic Product
GEM	Generalized Euler Method
GL	Grünwald- Letnikov
MISE	Mean Integrated Square Error
MSE	Mean Square Error
ODE	Ordinary Differential Equations
OLS	Ordinary Least Squared estimator
PDE	Partial Differential Equations
PSE	Power Series Expansion

Chapter 1

INTRODUCTION

Applied mathematics focuses on mathematical modeling of various processes from all fields of life. Approach used in applied mathematics consists of design, analysis and simulation of models. Models are often built upon a problem from different fields such as biology, physics, chemistry, economics or any other field of study [10]. In general, models are defined with the aim to propose a general solution to particular type of problems. In this respect a well designed model with solid mathematical proof of their efficiency are often adopted unanimously by researchers worldwide. Hence, some scientific phenomena are identified by the model upon which they are built. For instance the Malthusian model [9], also known as population growth model was introduced by Thomas Malthus. Prior to a phenomenon modeling, it is always required to make some assumptions based to experimental observations. Failure to undertake the right assumptions might lead to poor model. Natural phenomenon modeling usually undergoes through an iterative process consisting of model validation followed by improvement and finally identification of model limitations [10].

Technically, models are classified into two subcategories, which are deterministic model and non-deterministic models that may be probabilistic or stochastic in nature. In the deterministic model, the assumption is that the initial state of the model is enough to determine all other states of the process [10].This means randomness is ignored, whereas in the probabilistic and stochastic models randomness is taken into account.

The use of differential equations in modeling was introduced more than a century ago. Kuang [72] studied population growth modeling using a large range of differential equation. Ordinary Differential Equations (ODE) and Partial Differential Equations (PDE) whether classical or fractional are used as main tools in building a deterministic model. ODE usually involved a single variable. Generally experimental data set involving a single variable is considered and the model is built upon it. PDE involves more than one variable. It is common to have time and space as variables in modeling. This is because many systems change over time or space. However, there are many other variables that can be involved in modeling such as temperature, strength of a material, height etc. Differential equations and their solutions are surrounded by some conditions. For instance it is fundamental to prove the existence and uniqueness of the solution to a problem to some extend that scientists refer to as '*principle of determinism*' [11].

Sometimes it is difficult or even impossible to provide exact and explicit solutions to ODE and PDE, in which case numerical approaches to solve those problems are suggested. This usually concerns non-linear differential equations, which analytic form of their solutions are difficult to establish. In many research works, the goal is to optimize the numerical approach in order to minimize the error term. Scientific computing is the field that focuses on writing algorithms for numerical computation. While using numerical approach to solve differential equations, it is necessary to study the stability and prove the convergence of the algorithm [12].

Probability theory studies the randomness in a process. Probability laws and models are often able to appraise and explain random variation in natural phenomena. Random variations occur in natural phenomena as a result of unknown or unpredictable cause. However, stochastic processes aim to quantify the dynamic of the relationship among sequences of random events [13]. Stochastic models are often driven by probability laws. In general, the role of probability laws in a stochastic model is to formally provide a framework within which the randomness of a variable can be appraised. In other words, stochastic model tries to appraise the uncertainty of an event. In general statistical modeling focuses on providing necessary tools for the modeling of data set while considering the random nature of the variable.

Modeling of some natural phenomena is possible through the combination of two approaches. Hence, it is required in some cases to combine both deterministic and stochastic methods to be able to model a problem. Stochastic Differential Equations (SDE) are used for this purpose. This category of differential equations contains a random element [14]. Fundamentally the difference between ODE and SDE is that, solution to ODE is functions whereas a solution to a SDE is given in term of probability density due to the randomness in the model. The main interest in SDE is to study the average behavior of the system variation.

During the past three decades, researchers focused on the Fractional Differential Equations (FDE) that was theoretically introduced back in 19th century. FDE differs from the ODE by its type of derivative. Existential theory of FDE is studied by many in literature [15]. In fact, derivative order is assumed to be a fractional number in the case of FDE. It has been proven by researchers that FDE sometimes is more efficient than ODE in modeling. A set of examples are given in [31] supporting the claim about the advantages of fractional differential over the ordinary differential equation in modeling.

In this work modeling of some commonly known phenomena using both deterministic and non-deterministic approaches is undertaken. In the deterministic case ODE, FDE are used, while in the non-deterministic case Smoothing approach is employed. Model comparisons are provided in some cases for performance evaluation. The work is divided into six chapters organized as follow. The current is the introductory chapter; it is followed by a review of existing work. Chapter 3 is a review of fractional calculus tools needed in the work. Chapters 4,5 and 6 are core case studies, which contents are summarized as follows.

Chapter 4 studies deterministic model known as the logistic growth model. Hadamard derivative and integral were used to prove the existence and uniqueness of the solution of the Fractional Logistic Differential Equation (FLDE). Previous works have shown that there isn't an analytic solution to the FLDE. Therefore, some numerical approaches such as power series expansion (PSE) method, generalized Euler's method (GEM) and Caputo-Fabrizio (CF) method were used to find an approximate solution. The classical solution obtained from the first order non-linear differential equation was also considered for error comparison.

Chapter 5 studies logistic growth model. The ψ-Caputo model is built using the Rayleigh function as kernel function. The study focuses on a special case when the population carrying capacity K tends to infinity. The existence and uniqueness of the solution to the defined problem using the ψ-Caputo method is proven. The Chinese population is taken as a case study, which has a carrying capacity K that tends to infinity. As a result, the proposed ψ-Caputo approach with Rayleigh kernel fits population with K that tends to infinity better than the usual logistic growth equation.

In application many data sets do not exhibit a certain pattern or model. Therefore, the use of nonparametric smoothing methods is employed to fit a function that best represents the variable of concern. Chapter 6 is a comparative study of the logistic growth model using both deterministic and non-deterministic approaches. The Logistic Growth Model (LGM) is fitted using a statistical approach known as kernel smoothing using the Nadaraya Watson estimator [73]. World population from 1910 to 2010 is used as a case study data [74]. The same data is modeled using ODE and FDE, which are commonly used deterministic methods in exponential and logistic growth modeling. A comparative study on their performances was undertaken and results show that the kernel smoothing gives better estimates compared with other methods used. The choice of optimum bandwidth is obviously essential for the success in kernel estimation.

Concluding remarks of the study are in Chapter 7.

Chapter 2

LITERATURE REVIEW ON MATHEMATICAL MODELING

There is a rich collection of research works in literature related with natural and physical phenomena modeling. Models are built using experimental data and process description. In some cases, existing models are used to fit a given phenomenon. Consistent and mathematically proven models are generally adopted as reference and used to solve problems similar to the problem upon which the model was built. In this review, attention is paid to logistic growth model and smoothing model among others.

Tashiro and Yoshimura [1] studied bacteria growth based on a modified logistic model they have introduced and called neo-logistic model. They first of all identified four phases of the bacteria growth, when a culture is enclosed in favorite liquid environment with a unique nutrient. These phases are the *lag phase*, within which the germs adapt to their new environment with no growth observed. The second phase is called *exponential phase*, during which the bacteria grow exponentially. The third phase is called *stationary phase*, during which the number of bacteria is saturated and the growth stops; and finally *death phase*, during which there is a continuous decline of the bacteria number in the culture. The authors highlighted the fact that, for the four phases, the model presents an S-shape. However, the fourth phase is often neglected and the growth is represents by either a logistic model or Gompertz model. Authors contributed in the study by considering the fourth phase which is often neglected. The iterative construction of the logistic growth equation was done based on the relations between the number of bacteria and the volume of substrate absorbed by bacteria. Hence the iterative variation of the number of bacteria and the number of substrate cubes are given respectively by following equations

$$\begin{aligned} BC_{k+1} &= \left(1-\lambda\frac{SC_k}{n^3}\right)BC_k + 2\lambda\frac{SC_k}{n^3}BC_k \\ &= BC_k + \lambda\frac{SC_k}{n^3}BC_k \end{aligned}, \tag{2.1}$$

$$SC_{k+1} = SC_k - \lambda\frac{BC_k}{n^3}SC_k\,, \tag{2.2}$$

where SC_k and BC_k are the number of substrate cubes and bacteria at the time *k*, respectively. The quantity n^3 is the number of space cubes into which the space is divided, and λ is a constant.

Differential equations of the model are built from Eq.(2.1) and Eq.(2.2) by dividing both by the time variation unit Δt, and taking the limit $\lim\limits_{\substack{n\to\infty \\ \Delta t\to 0}}\frac{1}{\Delta t\times n^3}=\frac{\gamma}{\lambda}$. At the limiting case, it appears that

$$\frac{dBC(t)}{dt} = \gamma SC(t)BC(t)\,, \tag{2.3}$$

and

$$\frac{dSC(t)}{dt} = -\gamma BC(t)SC(t)\,. \tag{2.4}$$

It was revealed that the total amount of substrate cubes and bacteria are converged toward infinity following the conservation equation

$$BC_{\infty} = SC(t) + BC(t). \tag{2.5}$$

Substitution of Eq.(2.5) into Eq.(2.3) leads to the bacteria growth logistic model defined by

$$\frac{dBC(t)}{dt} = \gamma\left(BC_{\infty} - BC(t)\right)BC(t). \tag{2.6}$$

Eq.(2.6) is known as the logistic model. The authors' proposal in [1] to build what they called neo-logistic is derived based as shown in what follows. It is worth mentioning terminology and meaning of the terms used. In what follows, '*rank*' is used to refer to the quantity in number of substrate cube the culture has absorbed from the initial time to the first cell division. Hence absorption of each additional substrate will increase the bacterium rank of one unit starting from its initial rank which 0. Moreover, bacteria with rank *n* are produced from bacteria with rank *n-1*, hence any bacteria of rank *k* doesn't decrease, which motivated to write a recursive formula similar to Eq.(2.1) as

$$\begin{aligned}
BC_{k+1}^{n} &= \left(1 - \lambda\frac{SC_k}{l^3}\right)BC_k^n + 2\lambda\frac{SC_k}{l^3}BC_k^n + \lambda\frac{SC_k}{l^3}BC_k^{n-1} \\
BC_{k+1}^{n} &= BC_k^n + \lambda\frac{SC_k}{l^3}BC_k^n + \lambda\frac{SC_k}{l^3}BC_k^{n-1}; \\
BC_{k+1}^{n-1} &= \left(1 - \lambda\frac{SC_k}{l^3}\right)BC_k^{n-1} + \lambda\frac{SC_k}{l^3}BC_k^{n-2}; \\
&\cdots \\
BC_{k+1}^{0} &= \left(1 - \lambda\frac{SC_k}{l^3}\right)BC_k^0
\end{aligned} \tag{2.7}$$

$$SC_{k+1} = SC_k - \lambda\sum_{j=0}^{n}\frac{BC_k^j}{t^3}SC_k \ , \tag{2.8}$$

where SC_k and BC_k are the number of substrate cubes and bacteria at the time *k*, respectively. l^3 is the number of spaces cubes into which the space is divided and λ is a constant.

The model differential equations is obtained from Eq.(2.7) and Eq.(2.8) by taking the limits values as $l \to \infty$ and $\Delta t \to 0$.

$$\begin{aligned}
&\frac{dBC^{n}(t)}{dt} = -\rho SC(t)BC^{n}(t) + \rho SC(t)BC^{n-1}(t); \\
&\frac{dBC^{n-1}(t)}{dt} = -\rho SC(t)BC^{n-1}(t) + \rho SC(t)BC^{n-2}(t); \\
&\dots \\
&\frac{dBC^{0}(t)}{dt} = -\rho SC(t)BC^{0}(t).
\end{aligned} \tag{2.9}$$

$$\frac{dSC(t)}{dt} = -\rho \sum_{j=0}^{n} BC^{j}(t)SC(t) \ . \tag{2.10}$$

Eq.(2.9) and Eq.(2.10) are called neo-logistic model. Moreover, the total number of bacteria in the bacterium at time *t* is

$$BC(t) = \sum_{j=0}^{n} BC^{j}(t) \ . \tag{2.11}$$

There are analogies between the classical logistic model and the neo-logistic model proposed here. For instance Eq.(2.3) and Eq.(2.4) appear as particular cases of Eq.(2.9) and Eq.(2.10) respectively when the bacteria rank $n=0$.

It is sometime difficult to fit data collected from a process using a single model. Due to some modifications that can happen during an experiment or a process, the shape of the collected data might change. Harris et al. [2] studied the US energy consumption data between 1949 and 2015. Interestingly, the data shows a double ‘S-shape’ in a form of two logistic equations put together, see Figure 2.1.

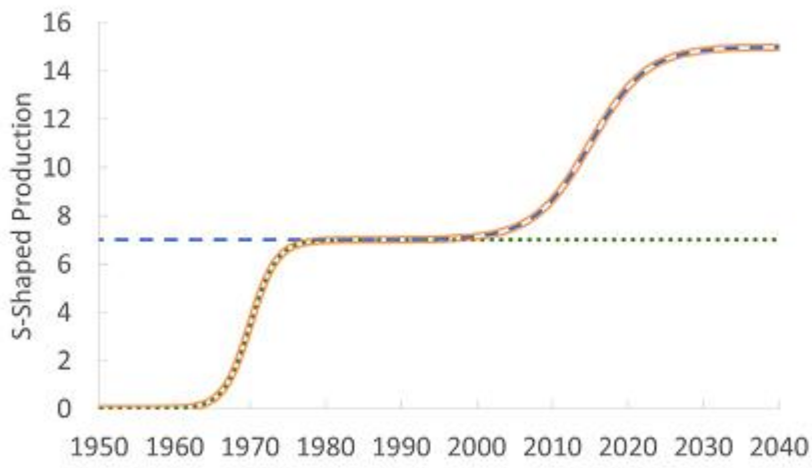


Figure 2.1: US energy from 1949 to 2015 and forecast to 2040 [2].

The authors proposed a four-parameter multi-cycle logistic growth model to fit and to forecast the US energy data shown above. The four-parameter logistic function proposed is given by

$$P(t) = \frac{(K_Z - H_Z)\exp\left(\frac{t-N}{h}\right)}{Q\left(1+\exp\left(\frac{t-N}{h}\right)\right)^2}, \tag{2.12}$$

where P is annual based production rate. H_Z represents the cumulative production low-plateau and K_Z represents the cumulative production high-plateau. The time *t* is the production year. *h* is the all in years width factor, N is the growth Midpoint.

An application of logistic modeling in Biology was studied by Banks et al.[3] . They studied logistic growth of the green algae. In their study, the usual logistic growth equation was considered, with application to data set collected from a population of green algae. The authors focused on the residuals data observed during modeling. With regard to this they built a 95% confidence interval of the fitting parameters of a logistic growth model, which are: the initial population size, the growth rate and the carrying capacity. This confidence interval can be used in the estimation of the whole population growth. The confidence interval was developed admitting that there is a random error term between the collected data set and the estimated data. In order to evaluate the error due to measurement, the following statistical model is used

$$Y_i = g(x_i, \theta_0) + g(x_i, \theta_0)^{\alpha} \varepsilon_i \tag{2.13}$$

Where *i=1, 2,…, n* and *n* is the sample size; Y_i are the observation; α is a constant ; $g(x_i, \theta_0)$ is the fitting function of the data set; ε_i is the identically and independently distributed noise data causing deviation between observed values and model fitted value. Moreover ε_i is a zero mean random variable, that is $E(\varepsilon_i) = 0$ and variance $\operatorname{var}(\varepsilon_i) = \sigma_0^2 . \theta_0$ is simply an hypothesized 'nominal' or 'true' vector of parameter generator of the observations Y_i. When the constant $\alpha = 0$, Eq.(2.13) is a special case called Ordinary Least Squared estimator (OLS).

Logistic model also find its application in economy. A country's Gross Domestic Product (GDP), tends to grow under favorable economic conditions, and decline under adverse economic conditions. However, when it comes to GDP growth, it is important to mention that, although many countries present a continuous growth of their GDP over years, there is a saturation point. Researchers are not always in agreement about the modeling of GDP. Some claim that exponential model is suitable whereas other are convinced that logistic model is appropriated. Kwasnicki [4] proposed a framework in which he provided criteria that would help to decide when to use Exponential growth or logistic growth. In general, it is observed that the approach is more intuitive and explicative. It is not based on any direct mathematical formula manipulation. However error evaluation tools are proposed by the following formulae

$$E_1 = \sqrt{\frac{1}{x_{\max} - x_0 + 1} \sum_{x=x_0}^{x_{\max}} \left(y(x) - \tilde{y}(x) \right)^2} \tag{2.14}$$

$$E_2 = \sqrt{\frac{1}{x_{\max} - x_0 + 1} \sum_{x=x_0}^{x_{\max}} \left(\frac{y(x) - \tilde{y}(x)}{\tilde{y}(x)} \right)^2} \tag{2.15}$$

x_0 is the initial time and $x_{\max}$ is final time of the data observation. $y(x)$ is the true data value ; $\tilde{y}(x)$ is the fitted data value. Eq.(2.14) and Eq.(2.15) are respectively called Mean Squared Error and Relative Mean Squared Error.

Reliability test is important in every goods or services provider in industry. This means goods and services can sustainably be sold and purchased only if they have been proven reliable. For example in software industry, a reliability test is essential. Software testing is a tedious and time consuming task. After highlighting the fact that the existing testing approaches require heavy resources financial and/or human resources, Chin-Yu [72] has proposed a logistic based model to analyze performances of software reliability growth models. His model established the testing-consumption effort $C(t)$, in a cumulative form defined as

$$C(t) = N \left(\frac{(\mu + 1)/\theta}{1 + B\exp(-\lambda \mu t)} \right)^{1/\mu}, \tag{2.16}$$

where N is the amount of testing-efforts to be consumed, λ is the rate at which the testing-effort is consumed, B and θ are constants, and μ is the structuring index.

Gompertz model [5] and Logistic model are famous for their use in population growth modeling. They both have similar properties such as their S-shape. They also have property that differentiates both of them. For instance the Gompertz generating function is symmetric whereas the logistic generating function is asymmetric. It is clear that failing to select the right model in modeling a particular population growth will lead to a bad forecast. Nguimkeu [5], proposed a selection criteria that would help choosing the right model for a given population growth modeling. The criteria are derived as follow. Consider the logistic and Gompertz trend functions given respectively by

$$L(t)=\frac{\lambda_1}{\left(1+\lambda_2\exp\left(-\lambda_3 t\right)\right)}, \tag{2. 17}$$

$$G(t)=\alpha_1\exp\left(-\alpha_2\exp\left(-\alpha_3 t\right)\right), \tag{2. 18}$$

where $\alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2$,and λ_3 are all constants. Differentiating Eq.(2.17) and Eq.(2.18) following by rearrangement of their terms will respectively lead to Eq.(2.19) and Eq.(2.20) as follow

$$\frac{L'(t)}{L(t)}=\lambda_3\left(\ln\left(\lambda_1\right)-\ln L(t)\right) , \tag{2.19}$$

$$\frac{G'(t)}{G(t)}=\alpha_3\left(\alpha_1-G(t)\right) \tag{2.20}$$

It follows from Eq.(2.19) and Eq.(2.20) that a simple linear representation of the logistic Eq.(2.7) and Gompertz Eq.(2.18) are given respectively by

$$y_t=c_1+c_2\ln\left(Y_{t-1}\right)+v_{1t}, \tag{2.21}$$

and

$$y_t=d_1+d_2 Y_{t-1}+v_{2t} , \tag{2.22}$$

where Y_t is the variable of interest it can be *G(t)* or *L(t),* and the quantity $y_t=\frac{Y_t-Y_{t-1}}{Y_{t-1}}$ represents a relative increase in Y_t , the error term is v_t .

A comprehensive but artificial linear model that plays the role of both Eq.(2.21) and Eq.(2.22) is given by

$$y_t = b_1 + b_2 \ln\left(Y_{t-1}\right) + b_3 Y_{t-1} + v_t \quad . \tag{2.23}$$

Fractional Differential Equation is a widely used tool in modeling nowadays. The exact solution to fractional logistic equation defined as

$$D_t^{\gamma} f\left(t\right) = \lambda^{\gamma} f\left(t\right)\left(1 - f\left(t\right)\right), \tag{2. 24}$$

was first proposed by West B.[6] in the form of the following power series function

$$f\left(t\right) = \sum_{k=0}^{\infty} \left(\frac{f_0 - 1}{f_0}\right)^k E_{\gamma}\left(-k\lambda^{\gamma} t^{\gamma}\right) \ . \tag{2.25}$$

However Area et al.[7] have proven that the proposed solution, Eq.(2.25) fails to be correct for the fractional case, rendering West's claim incorrect. This has drawn the interest of many authors putting effort towards the solution of the fractional logistic growth problem. A set of numerical schemes to approach the solution of the fractional logistic equation is found in [8].

It is difficult to forecast the outcome of a random process. However, it is possible to appraise it using probability theory. Once the shape of the process is built based one experimental data, Kernel smoothing can be used to smooth the data set into a smooth curve. In practice, there exist many random phenomena. An example is the variation of wind speed. Bo et al. [16] proposed a review of existing smoothing method and proposed an application to wind speed modeling.

It is difficult to forecast future outcome in a random process. However, once the probabilistic law governing a process is defined, then the process can be monitor.

Monitoring consists of setting lower limit and upper limit of the possible outcome, in an optimal way. If a process is multivariate, then principal components analysis might be useful to identify the main axes of variation. Chen et al. [17] proposed a model for process monitoring combing principal components analysis and kernel smoothing.

Currency fluctuation, country's economy, portfolio investment are modeled by the meaning of stochastic process. Portfolio risk management and price derivative in finance require high precision in estimation. Hong et al. [18] proposed in their work a kernel smoothing based framework of portfolio risk management. They found out that although the kernel smoothing is a weak performer for a process with few risk factors, it performs very well for portfolio with high number of risk factors.

Chapter 3

ELEMENTS OF FRACTIONAL CALCULUS

The name fractional calculus comes from the fact that the integral and derivative orders are fractions or decimal numbers rather than the commonly used integers. Early work on fractional calculus dates back to early 19th century [19]. Researchers initially focused on proving existence and uniqueness of the solution to a fractional differential equation [20, 21, 22]. Discussion on the theory of fractional calculus can be found in [23, 24, 25].

In this chapter, some useful fractional calculus definitions and theory are discussed. These definitions, properties and theorems only represent a very small part of fractional calculus literature. However, the selected topics represent the set of useful tools needed in the course of this work. One might not start such discussion without mentioning the Mittag-Leffler function which is a generalization of the classical exponential function. This function plays a key role in fractional calculus. The Mittag-Leffler function usually appears in the representation of the solutions of fractional differential equations.

3.1 Basic Definitions and Theorems

Definition 3.1 [26] The one parameter Mittag-Leffler function is defined by

$$E_{\alpha}\left(t\right)=\sum_{i=0}^{\infty}\frac{t^{i}}{\Gamma\left(\alpha i+1\right)}, \tag{3.1}$$

where $\alpha > 0$ and $t \in \mathbb{C}$. The functional $\Gamma(.)$ is the well known gamma function. When $\alpha = 1$, then Eq.(3.1) coincides with the classical exponential function

$$E_1(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i+1)} = \exp(t) \ . \tag{3.2}$$

This shows that the exponential function is a special case of the Mittag-Leffler function, in other words, Mittag-Leffler function is a generalization of exponential function [26].

Definition 3.2 [26] The two parameter Mittag-Leffler function is given by

$$E_{\alpha,\beta}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + \beta)} , \tag{3.3}$$

where $\alpha, \beta > 0$ and $t \in \mathbb{R}$.

A similar reasoning to what leads to Eq.(3.2) can be adopted to derive classical exponential function from the two-parameters Mittag-Leffler function. In fact, for $\alpha = 1$ and $\beta = 1$, Eq.(3.3) becomes $E_{1,1}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i+1)} = \exp(t)$. Many others special functions can be derived for specific values the parameters α and β . These are for instance, the cosine function

$$E_{2,1}(-t^2) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{\Gamma(2k+1)} = \cos(t);$$

The cosine hyperbolic function

$$E_{2,1}(t^2) = \sum_{k=0}^{\infty} \frac{t^{2k}}{\Gamma(2k+1)} = \cosh(t),$$

and

$$E_{0.5,1}\left(\pm t^{0.5}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k t^{0.5k}}{\Gamma(0.5k+1)}=e^t\left(1+erf\left(\pm t^{0.5}\right)\right);$$

where $erf(t)=\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$, $\forall t\in\mathbb{C}$ is called the error function . The error function can also be defined in a complement form as $erfc(t)=1-erf(t)$.

The Mittag-Leffler function is a family of functions; hence it is worth studying some proprieties of its asymptotic behavior and its integral representation. The integral representation of the Mittag-Leffler function is useful in deriving some properties, as well as for its asymptotic behavior.

Definition 3.3 [26] Consider the two-parameter Mittag-Leffler function with $\beta=1$ and $0<\alpha<2$; or simply the one parameter Mittag-Leffler function, then the following integral representation are possible

$$E_{\alpha,\beta}(z)=\frac{1}{\alpha 2\pi i}\int_{c(\varepsilon;\gamma)}\frac{e^{x^{1/\alpha}}x^{(1-\beta)/\alpha}}{x-z}dx,. \quad ; \quad z\in G^{(-)}(\varepsilon;\gamma) \tag{3.4}$$

$$E_{\alpha,\beta}(z)=\frac{1}{\alpha}e^{z^{1/\alpha}}+\frac{1}{\alpha 2\pi i}\int_{c(\varepsilon;\gamma)}\frac{e^{x^{1/\alpha}}x^{(1-\beta)/\alpha}}{x-z}dx,\ z\in G^{(+)}(\varepsilon;\gamma) \tag{3.5}$$

where $c(\varepsilon,\gamma)$ is the integral contour defined such that $\frac{\pi\alpha}{2}<\gamma<\min(\pi\alpha,\pi)$.

Asymptotically, that is when $|z|\to\infty$, the Mittag-Leffler function can be represented by the mean of power series and residuals (*O* notation) [84]. Such representation is useful in the implementation of computational algorithm.

Definition 3.4 Consider the one-parameter Mittag-Leffler function with $0<\alpha<2$ and $\frac{\pi\alpha}{2}<\omega<\min(\pi\alpha,\pi)$. Then the following asymptotic formulas hold

$$E_\alpha(z)=\frac{1}{\alpha}e^{z^{1/\alpha}}-\sum_{n=1}^{q}\frac{z^{-n}}{\Gamma(1-n\alpha)}+O\left(|z|^{-1-q}\right),\ |\arg z|\le\omega,\quad |z|\to\infty \tag{3.6}$$

$$E_\alpha(z)=-\sum_{n=1}^{q}\frac{z^{-n}}{\Gamma(1-n\alpha)}+O\left(|z|^{-1-q}\right),\omega\le|\arg z|\le\pi,\ |z|\to\infty \tag{3.7}$$

with $q\in\mathbb{N}$, a randomly selected integer.

Under the conditions of proposition 3.4, if $\alpha\ge 2$, the asymptotic formula is given by

$$E_\alpha(z)=\frac{1}{\alpha}\sum_{\eta}\exp\left(z^{1/\alpha}e^{\frac{2\pi\eta i}{\alpha}}\right)-\sum_{n=1}^{q}\frac{z^{-n}}{\Gamma(1-n\alpha)}+O\left(|z|^{-1-q}\right); \tag{3.8}$$

where $|\arg z|<\pi$; $|z|\to\infty$, with $\eta\in P(z)=\left\{k:k\in\mathbb{Z},|2k\pi+\arg z|\le\frac{\pi\alpha}{2}\right\}$, and $-\pi\le\arg z\le+\pi$.

The Mellin-Barnes representation of a one-parameter Mittag-Leffler function, with $\alpha>0$, is the following

$$E_\alpha(z)=\frac{1}{2i\pi}\int_{\gamma_{ih}}\frac{\Gamma(t)\Gamma(1-t)}{\Gamma(1-\alpha t)}(-z)^{-t}\,dt,\ |\arg z|<\pi. \tag{3.9}$$

The contour over which the integration is carried γ_{ih}, is a straight line from $h-i\infty$ to $h+i\infty$; with the constant $0<h<1$.

In the section below, it is usually assumed that the variable $z\in\mathbb{C}$. Special properties of Mittag-Leffler function can be studied for real valued variables.

Definition 3.5 [26] Let a function g be a real-valued absolutely integrable on the open interval $(0,+\infty)$,its Laplace transform is defined as

$$(Lg)(u)=\int_0^\infty e^{-ux}g(x)\,dx. \tag{3.10}$$

The following Laplace integral relation exists with the one parameter Mittag-Leffler function

$$\int_0^\infty e^{-t}E_\alpha\left(t^\alpha z\right)dt=\frac{1}{1-z},\ \alpha\geq 0. \tag{3.11}$$

The Laplace transform of $E_\alpha\left(\pm x^\alpha\right)$ is obtained from Eq. (3.11) by setting $t^\alpha z=\pm x^\alpha$ as follows

$$L\left(E_\alpha\left(\pm x^\alpha\right)\right)(u)=\int_0^\infty e^{-ux}E_\alpha\left(\pm x^\alpha\right)dx=\frac{u^{\alpha-1}}{u^\alpha\pm 1}. \tag{3.12}$$

The main tools in modeling a dynamic process are derivative and integral. This section reviews the special type of derivatives and integrals, namely, fractional derivatives and integrals.

Definition 3.6 [27] The Riemann-Liouville fractional integral of order $q>0$ of a function $g:[0,+\infty)\to\mathbb{R}$ is defined as

$$\left({}_{RL}I_{0^+}^{q}g\right)(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)\,ds, \tag{3.13}$$

provided that the right hand side of the integral is point wise defined on $(0,+\infty)$ and Γ is the gamma function $\Gamma(\upsilon)=\int_0^\infty e^{-t}t^{\upsilon-1}dt, \forall\upsilon>0$.

Definition 3.7 [27] The Riemann-Liouville fractional derivative of order $q>0$ of a function $g:[0,+\infty)\to\mathbb{R}$ is defined as

$$\left({}_{RL}D_{0^+}^{q}g\right)(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d^n}{dt^n}\right)\int_0^t (t-s)^{n-q-1}g(s)ds, \tag{3.14}$$

where $n-1\le q<n,\ n\in\mathbb{N}.$

Definition 3.8 [27] The Caputo derivative of order $q>0$ for a function $g:[0,+\infty]\to\mathbb{R}$ is defined as

$$\left({}_{C}D_{0^+}^{q}g\right)(t)=\begin{cases}\displaystyle\int_0^t \frac{(t-s)^{n-q-1}g^{(n)}(s)}{\Gamma(n-q)}ds & ,n-1<q<n\in\mathbb{R},\\ g^{(n)}(t) & ,q\in\mathbb{N},\end{cases} \tag{3.15}$$

where $n=[q]+1,\ [q]$ is the integer part of q.

Definition 3.9 [23] Let g be a continuous function and $n=\frac{t-a}{h}$, then the Grünwald-Letnikov (GL) fractional derivative of g is given by

$$\left({}_{GL}D_{a^+}^{q}g\right)(t)=\lim_{h\to 0}\frac{1}{h^q}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{q}{j}g(t-jh), \tag{3.16}$$

where

$$\binom{q}{j}=\frac{q!}{j!(q-j)!}=\frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)},\ \text{and}\ \binom{q}{0}=1.$$

Definition 3.10 [28] Given an Hilbert space $H^1(a,b), b>a$, Let $g\in H^1(a,b)$, and $q\in[0,1]$ then the new Caputo version of fractional derivative of g is defined as

$$\left({}_{CF}D_0^q g\right)(t) = \frac{M(q)}{(1-q)} \int_a^t g'(s) \exp\left[-\frac{q}{1-q}(t-s)\right] ds\,, \tag{3.17}$$

where $M(q)$ is the normalization function with $M(0) = M(1) = 1$. If $g \notin H^1(a,b)$ then, a new fractional derivative can be built. It is called Caputo-Fabrizio fractional derivative [28], and it is defined as

$$\left({}_{CF}D_0^q g\right)(t) = \frac{M(q)}{(1-q)} \int_{-\infty}^t g'(s) \exp\left[-\frac{q}{1-q}(t-s)\right] ds\,. \tag{3.18}$$

The Caputo-Fabrizio fractional derivative is a derivative with non-singular kernel. This is the fundamental difference and advantage it has over other type of fractional derivatives [28].

Caputo-Fabrizio fractional derivative definition can be extended to a class of function that don't belong to $H^1(a,b)$. In fact given a function $g \in L^1(-\infty, d)$ and a fractional order derivative $0 < q < 1$, the Caputo-Fabrizio fractional derivative of g, is given by

$${}_{CF}D_x^q g(x) = \frac{qM(q)}{(1-q)} \int_{-\infty}^x \exp\left(-\frac{q}{1-q}(x-t)\right)\left(g(x) - g(t)\right) dt\ . \tag{3.19}$$

The Caputo-Fabrizio fractional derivative also has the advantage that it coincides with the classical derivative when the order the derivative q approaches the value 1; and it is a definite integral when the order of derivative q approaches the value 0.

Proposition 3.11 Considering the Caputo-Fabrizio fractional derivative given by Eq.(3.18) and Eq. (3.19), the following relations [28] are correct

i) $\underset{q\to 1}{Lim}\ {}_{CF}D_x^q g(x) = g'(x)$,

ii) $\underset{q\to 0}{Lim}\ {}_{CF}D_x^q g(x) = g(x) - g(a)$.

Proof of Proposition 3.11

Setting $\alpha = \frac{1-q}{q}$, leads to $q = \frac{1}{1+\alpha}$; with $0 \le q \le 1$, then it follows that $0 \le \alpha \le \infty$.

Based on these setting, Eq.(3.18) can be written as

$$ {}_{CF}D_x^q g\left(x\right) = \frac{N(\alpha)}{\alpha}\int_a^x \exp\left(-\frac{\left(x-t\right)}{\alpha}\right) g'\left(t\right) dt, \tag{3.20}$$

where the normalization function of the new form of derivative, $N(\alpha)$ is such that $N(0) = N(\infty) = 1$. Observe that when $q \to 1$ we have $\alpha \to 0$. Moreover, $\underset{\alpha\to 0}{Lim}\frac{1}{\alpha}\exp\left(-\frac{\left(x-t\right)}{\alpha}\right) = \lambda\left(x-t\right)$, it follows from Eq.(3.18) and Eq.(3.20) that

$$\begin{aligned}\underset{q\to 1}{Lim}\ {}_{CF}D_x^q g(x) &= \underset{q\to 1}{Lim}\frac{M(q)}{\left(1-q\right)}\int_a^x \exp\left(-\frac{q}{1-q}\left(x-t\right)\right) g'\left(t\right) dt \\ &= \underset{\alpha\to 0}{Lim}\frac{N(\alpha)}{\alpha}\int_a^x \exp\left(-\frac{\left(x-t\right)}{\alpha}\right) g'\left(t\right) dt \\ &= g'\left(x\right).\end{aligned}$$

This proves part *i)* of the proposition. It is also observed that when $q \to 0$ we have $\alpha \to +\infty$; $\underset{\alpha\to +\infty}{Lim}\frac{N(\alpha)}{\alpha} = \underset{q\to 0}{Lim}\frac{M(q)}{\left(1-q\right)} = 1$, and also $\exp_{\alpha\to\infty}\left(-\frac{\left(x-t\right)}{\alpha}\right) \to 1$, hence

$$\begin{aligned}\underset{q\to 0}{Lim}\ {}_{CF}D_x^q g(x) &= \underset{q\to 0}{Lim}\frac{M(q)}{\left(1-q\right)}\int_a^x \exp\left(-\frac{q}{1-q}\left(x-t\right)\right) g'\left(t\right) dt \\ &= \underset{\alpha\to +\infty}{Lim}\frac{N(\alpha)}{\alpha}\int_a^x \exp\left(-\frac{\left(x-t\right)}{\alpha}\right) g'\left(t\right) dt \\ &= g(x) - g(a).\end{aligned}$$

This proves part *ii)* of the proposition.

Definition 3.12 [29] The Hadamard fractional integral of order q of a continuous function g is defined as

$$ {}^{H}I^{q}g(t)=\frac{1}{\Gamma(q)}\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}ds,\quad q>0, \tag{3.21}$$

provided that the integral exists.

Definition 3.13 [29] The Hadamard fractional derivative of order $q>0$ of a continuous function $g:[a,\infty)\to\mathbb{R}$ is defined as

$$ {}^{H}D^{q}g(t)=\delta^{n}\left({}^{H}I^{q}g\right)(t)=\left(t\frac{d}{dt}\right)^{n}\frac{1}{\Gamma(n-q)}\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}ds, \tag{3.22}$$

with $n-1<q<n,\ n=[q]+1;\ \delta=t\left(\frac{d}{dt}\right)$ and $[q]$ denotes the integer part of the real number q.

Definition 3.14 [29] Let $u,x\in C_{\delta}^{n}([a,T],\mathbb{R})$, where

$C_{\delta}^{n}[a,T]=\left\{u,x:[a,T]\to\mathbb{R}:\delta^{(n-1)}u\in C[a,T]\right\}$ then

$$ {}^{H}I^{q}\left({}^{H}D^{q}u\right)(t)=u(t)-\sum_{j=1}^{n}c_{j}\left(\ln\frac{t}{a}\right)^{q-j}. \tag{3.23}$$

Another useful fractional calculus tool is the fractional derivative of a function with respect to another function, which is similar to the chain from classical calculus.

Definition 3.15 [30] Let $\alpha>0$, $g\in L^{1}[a,b]$ and $\psi\in C^{1}[a,b]$ be an increasing function with $\psi'(x)\neq 0,\ \forall x\in[a,b]$ then $I_{0^{+}}^{\alpha,\psi}g(t)$ denotes the fractional integral of g with respect to ψ and it is given by

$$I_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} g(s)\,ds. \tag{3.24}$$

Definition 3.16 [30] Let $\alpha > 0$, $g,\psi \in C^n[a,b]$ with ψ being an increasing function and $\psi'(x) \neq 0,\ \forall x \in [a,b]$ then ${}_C D_{0^+}^{\alpha,\psi} g(t)$ denotes the fractional derivative of g with respect to ψ and it is given by

$${}_c D_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1} \left(\frac{1}{\psi'(s)}\frac{d}{dt}\right)^n g(s)\,ds. \tag{3.25}$$

Definition 3.17 [30] Let $\alpha > 0$, n a natural number such that $\alpha \in (n-1,n)$. If $g,\psi \in C^n[a,b]$ then

$$I_{0^+}^{\alpha,\psi}\left({}_C D_{0^+}^{\alpha.\psi} g\right)(t) = g(t) - \sum_{i=0}^{n-1} \frac{\left(\frac{1}{\psi'(s)}\frac{d}{dt}\right)^i g(0)}{i!}\left(\psi(t)-\psi(0)\right)^i. \tag{3.26}$$

Derivatives and integrals are tools used in differential equations. Some fractional types of derivatives and integrals are defined in the above section. In the following section these derivatives and integrals are used as tools in solving fractional differential equations.

3.2 Theory of Fractional differential Equation

Fractional derivative and integrals are useful tools in fractional differential equations. Similar to classical differential equation, it is possible to investigate the existence and uniqueness of solution of fractional differential equations.

The general form of a Riemann-Liouville non-linear fractional differential equation of the order $\gamma \in \mathbb{R}_+$, on a close interval $[b,c]$ of the real line $\mathbb{R}$ is given by

$$\left(D_{b+}^{\gamma} f\right)(t) = g\left(t, f(t)\right), \tag{3.27}$$

with $b < t$.

A Cauchy problem can be associated to Eq.(3.27), by the initial conditions

$$\left(D_{b+}^{\gamma-k} f\right)(b+) = t_k \ , \tag{3.28}$$

where $\left(t_k\right)_{1\le k\le m} \in \mathbb{R}$, and $m = [\gamma] + 1$.

A special case of the non-linear equations defined by Eqs.(3.27)-(3.28) is the linear fractional differential equation which is defined by

$$\left(D_{b+}^{\gamma} f\right)(t) - \kappa f(t) = g(t), \tag{3.29}$$

where $\gamma \in \mathbb{R}_+^*$, $\kappa \in \mathbb{R}$ and $t \in (b, c]$.

A Cauchy problem can be associated to the problem defined by Eq.(3.29) using an approach which is similar to what was used to obtain Eq.(3.28). Hence the Cauchy problem associated to the problem defined by Eq.(3.29) is also given by Eq.(3.28).

Proposition 3.18 [23] Consider $\gamma \in \mathbb{R}_+^*$, $m = [\gamma] + 1$; given also $0 \le \alpha < 1$ such that $\alpha \ge m - \gamma$, and $\eta \in \mathbb{R}$. If the function $g \in C_{\alpha}[b, c]$, it follows that the fractional differential equation defined with the initial condition (Cauchy problem), given by Eq.(3.29) has a solution $f(t)$, which is unique such that $f(t) \in C_{m-\gamma,\alpha}^{\gamma}[b, c]$. This solution has the following analytic form

$$f(t) = \sum_{i=1}^{m} u_i (t-b)^{\gamma-i} E_{\gamma,\gamma-i+1}\left[\eta (t-b)^{\gamma}\right] + \int_b^t (t-v)^{\gamma-1} E_{\gamma,\gamma}\left[\eta (t-b)^{\gamma}\right] f(v) dv. \tag{3.30}$$

Proposition 3.18 gives the analytic and explicit solution of fractional differential equations with Riemann-Liouville fractional derivative. Using a similar approach, the explicit solution to fractional differential equation with Caputo derivative is defined.

A linear fractional differential equation with Cauchy problem using Caputo derivative is defined as follow

$$\left({}^{C}D_{b+}^{\gamma} f\right)(t) - \kappa f(t) = g(t), \tag{3.31}$$

with $b \le t \le c$, $m-1 < \gamma < m$, $m \in \mathbb{N}$ and $\kappa \in \mathbb{R}$.

A Cauchy problem associated to Eq.(3.31) is given as follow

$$f^{(k)}(b) = t_k \ , \tag{3.32}$$

with $t_k \in \mathbb{R}$ and $0 \le k \le m-1$.

Lemma 3.19: [23]: Given $m-1 < \gamma < m$, $m \in \mathbb{N}$; $0 \le \lambda < 1$ such that $\lambda \le \gamma$,and also consider $\eta \in \mathbb{R}$. At this point if the function $g \in C_\lambda[b,c]$, then the fractional differential equation Eq.(3.31) with its associated Cauchy problem Eq.(3.32) has a unique solution $f(t)$ such that $f(t) \in C_\lambda^{\gamma,m-1}[b,c]$ which is defined by

$$f(t) = \sum_{i=1}^{m-1} u_i (t-b)^i E_{\gamma,i+1}\left[\eta(t-b)^\gamma\right] + \int_b^t (t-v)^{\gamma-1} E_{\gamma,\gamma}\left[\eta(t-b)^\gamma\right] f(v)\,dv \ . \tag{3.33}$$

The explicit solution given by Eq.(3.33) involves the two-parameter Mittag-Leffler function. In [27], an explicit solution to the Cauchy problem defined by Eqs. (3.31), (3.32) based on the one-parameter Mittag-Leffler function is proposed as follow

$$f\left(t\right)=\sum_{i=0}^{m-1} f^{(k)}\left(b\right)\times u_k\left(t\right)+\begin{cases} \dfrac{1}{(m-1)!}\int_b^x \left(x-t\right)^{m-1} g\left(t\right)dt & if\,\kappa=0 \\ \dfrac{1}{\kappa}\int_b^x g\left(x-t\right)u_0'\left(t\right)dt & if\,\kappa\neq 0 \end{cases}, \tag{3.34}$$

with $u_k\left(t\right)=\dfrac{1}{\left(k-1\right)!}\int_b^x E_\gamma\left(\kappa t^\gamma\right)dt.$

The above mentioned theory related with fractional derivative and integral is considered sufficient to handle the problems to be handled in subsequent chapters of the thesis.

Chapter 4

NUMERICAL SOLUTION OF THE FRACTIONAL LOGISTIC DIFFERENTIAL EQUATION

4.1 Introduction

In recent work, many researchers have focused on showing the advantages of fractional calculus methods over classical calculus approaches [22, 31, 32,33, 34]. Form many experiments and comparative studies, it has become evident that the fractional calculus model has provided better results in problem solving than their classical counterparts approaches used in solving the same problem. Usually the lower error levels obtained from modeling with fractional calculus is used to prove the strength of this approach [31, 32, 33, 35, 36].Computational methods are often used for implementation of an explicit equation or to iteratively approach the solution to a problem whose analytic form doesn’t exit. Some of the important work in which computational methods have been applied successfully are found in [37, 38, 39, 40, 41, 42].

Application of fractional calculus in modeling and solving real life problems have gradually been proven by research work in various branches of science, including but not limited in Economy[32]; Biology [31, 43, 34, 35, 44] and Physics[31, 33].

In the process of comparing classical calculus and fractional calculus approaches, it is common for researchers to evaluate the performance of both approaches in solving the same problem. It is however important to mention that the solutions obtained from both approaches might be different in some cases. For instance, a problem might have an explicit classical solution whereas its fractional counterpart might not. An evidence of this situation is the logistic differential equation [45] that has an exact solution. However an exact solution was proposed to Fractional Logistic Differential Equation (FLDE) by West [46]. Subsequently, West's proposal was proven limited by Area et.al.[47],since it is valid only when the derivative order is one. D'Ovidio et. al. [48] have also proven that West proposal could be fully valid only if the FLDE is modified into what they called 'Modified fractional logistic equation'. An Euler's based numerical approach to the FLDE was proposed in [49].

Up to date, an exact solution to the FLDE hasn't yet been established by researchers. In this regards, we follow the path of the authors of [49] in this chapter, by computing numerical solution of the FLDE using several numerical methods , namely the Caputo-Fabrizio (CF) method [52], the power series expansion (PSE) method also known as Letnikov method (LM) [51], and the generalized Euler method (GEM) [50]. Prior to the application of those numerical methods we have carefully proven the existence and the uniqueness of a solution to the FLDE.

4.2 Numerical Methods and Formula for Solving non-linear Differential Equations

In this section, some numerical methods and algorithms for solving non-linear differential equations are discussed.

4.2.1 The Generalized Euler's Method (GEM)

Here the GEM is discussed, and referred to the approach introduced by Odibat et al. [50]. GEM is inspired and defined from the Euler's method for solving differential equations. Given a fractional order non-linear differential equation defined by

$$\left({}_C D_{0^+}^{q} x\right)(t) = g\left(t, x(t)\right), x(0) = 0, \tag{4.1}$$

where $q \in (0,1]$ is the fractional order of derivative and $t > 0$. The functions $x(t)$, ${}_C D_{0^+}^{q} x(t)$ and ${}_C D_{0^+}^{2q} x(t)$ are assumed to be continuous on the closed interval $[0,T]$. Finding the numerical solution to the problem defined by Eq.(4.1)over the interval $[0,T]$ is only possible after a discretization of the closed interval $[0,T]$ into k sub-intervals $\left[t_j, t_{j+1}\right]$, of equal width $h = T/k$ is required. Beside this discretization, the corresponding set of points $\left\{t_j, x\left(t_j\right)\right\}$ are also required in the approximation process. Finally, the GEM algorithm to approximate the solution is defined as

$$x\left(t_{j+1}\right) = x\left(t_j\right) + \frac{h^q}{\Gamma(q+1)} g\left(t_j, x\left(t_j\right)\right), j = 0, 1, \ldots k-1, \tag{4.2}$$

with node $t_j = jh, j = 1, 2, \ldots, k$.

4.2.2 The Grünwald–Letnikov Method (GL) or Power Series Expansion (PSE)

A method for solving non-linear differential equation was established by Grünwald, more details on the method are found in [51]. This method is known in literature as Power Series Expansion (PSE) or Grünwald–Letnikov Method (GL).

Definition 4.1[51] The explicit formula for the fractional numerical approximation of qth derivative at the point kh, $(k = 1, 2, \ldots\ldots)$ in the Grünwald–Letnikov sense is defined as

$$_{(k - L_m/h)}\left({}_{GL} D_{t_k}^{q} g\right)(t) \approx \frac{1}{h^q} \sum_{j=0}^{k} (-1)^j \binom{q}{j} g\left(t_{k-j}\right), \tag{4.3}$$

where L_m is the memory length; $t_k = kh$, the time or space step of iteration is *h*, and $(-1)^j \binom{q}{j}$ are referred to as binomial coefficients (different from the commonly used in combinatory). The binomial coefficients in the computation process are denoted by $c_j^{(q)}, (j = 0,1,....)$ and computed using the formula $\begin{cases} c_0^{(q)} = 1, \\ c_j^{(q)} = \left(1 - \dfrac{1+q}{j}\right) c_{j-1}^{(q)} \end{cases}$.

Consider a non-linear fractional differential equation with initial conditions, with the derivative considered in the Grünwald–Letnikov sense, defined by

$$\left({}_{GL}D_{a^+}^{q} u \right)(t) = g\left(u(t), t\right). \tag{4.4}$$

Then a numerical solution to the problem defined by Eq.(4.4) is computed by the following formula

$$u(t_k) = g(u(t_k), t_k) h^q - \sum_{j=1}^{k} c_j^{(q)} u\left(t_{k-j}\right). \tag{4.5}$$

4.2.3 The Caputo-Fabrizio Method (CF)

Consider a non-linear fractional differential equation, with initial conditions and derivative taken in the Caputo-Fabrizio sense defined by

$$\left({}_{CF}D_0^{q} u \right)(t) = g\left(t, u(t)\right), u(0) = u_0. \tag{4.6}$$

A numerical solution to the problem stated by Eq.(4.6) is computed using the Adam-Basforth approach as (see [52,53])

$$u_{n+1} = u_n + \left(\frac{1-q}{M(q)} + \frac{3qh}{2M(q)} \right) g\left(t_n, u_n\right) + \left(\frac{1-q}{M(q)} + \frac{qh}{2M(q)} \right) g\left(t_{n-1}, u_{n-1}\right). \tag{4.7}$$

4.3 Fractional Logistic Differential Equation

In this section the logistic model is studied in both classical and fractional sense. Numerical methods mentioned in the previous section are used to establish and compute some numerical solution of the FLDE. As a reminder, the logistic model has long been used in modeling population growth. Consider a population that has a proliferation capability, and whose initial size is P_0. From a simplistic point of view, such a population would increase infinitely as the time increases toward infinity. Malthus, T. R(see[54]), has proven that a population growth tends to stabilize once it reach a certain size, hence doesn't grow infinitely when the time approaches infinity. Malthusian theory is what spans the logistic population growth model. The model is defined using integer order differential equation as follow

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right), \tag{4.8}$$

with the initial population size denoted as $N(0) = N_0$;*r* is called the growth rate and *K* is the carrying capacity. This capacity is the maximum size that the population can reach. $N(t)$ is the function representing the population size at any time *t*. The exact solution of the problem defined by Eq.(4.8) is given by

$$N_c(t) = \frac{K}{1 + \left(\frac{K - N_0}{N_0}\right)e^{-rt}}. \tag{4.9}$$

In order to investigate the FLDE, it is important to assume that Eq.(4.8) is built using a fractional order derivative. Moreover, let this derivative be the Hadamard fractional derivative without loss of generality. Then the FLDE is defined as

$$\left({}^{H}D_0^q N\right)(t) = rN(t)\left(1 - \frac{N(t)}{K}\right), N(a) = N_a < \infty, \tag{4.10}$$

where N_a is the initial population value.

4.3.1 Existence and Uniqueness of the Solution of Fractional Logistic Equation

Existence and uniqueness of the solution of FLDE are proven in this section. For conformity and without loss of generality, Hadamard type of fractional derivative and integral used.

Applying the Hadamard integral operator ${}^{H}I^{r}$ to Eq.(4.10)leads to the following equation

$$N(t)-c_1\left(\ln\frac{t}{a}\right)^{q-1}={}^{H}I^{q}\left(rN\left(1-\frac{N}{K}\right)\right). \tag{4.11}$$

From Eq.(4.11), the following notation is adopted for simplicity

$$Q\big(t,N(t)\big)=rN\left(1-\frac{N}{K}\right). \tag{4.12}$$

The initial value conditions is denoted $c_1 = N_a$.

Denote by $H=C\big([a,T],\mathbb{R}\big)$ the Banach space of all continuous functions defined from the closed interval $[a,T]$ to $\mathbb{R}$, and then build the operator $E:H\to H$ endowed with the norm $\|N\|=\sup\limits_{a\le t\le T}|N(t)|$, then using the operator *E*, it follows that

$$(EN)(t)=N_a\left(\ln\frac{t}{a}\right)^{q-1}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{Q\big(s,N(s)\big)}{s}ds. \tag{4.13}$$

4.3.1.1 Existence of solution

Based on the setting of Eq.(4.11) - Eq.(4.13), existence of the solution of FLDE is stated and proven in what follows.

Theorem 4.2 Let $Q:[a,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following assumptions hold

(A_1): $\exists N_Q > 0$ such that

$\left|Q(t,N_1)-Q(t,N_2)\right| \le N_Q \left|N_1 - N_2\right|, \forall t \in [a,T], \forall N_1, N_2 \in \mathbb{R}.$

(A_2): $\left|Q(t,N)\right| \le y(t), \forall (t,N) \in [0,T] \times \mathbb{R},\ y \in C([a,T],\mathbb{R}^+)$, where $C([a,T],\mathbb{R}^+)$ is endowed by the norm $\sup_{a \le t \le T} \left|y(t)\right| = \|y\|$.

In addition, it is assumed that $\dfrac{N_Q}{\Gamma(q+1)}\left(\ln\dfrac{T}{a}\right)^{q-1} < 1,$ then there exists at least one solution for the initial value problem defined by Eq.(4.10) .

Proof of Theorem 4.2: Consider the close set $B_\lambda = \{N \in H, \|N\| \le \lambda\}$ with

$\lambda \ge N_a \left(\ln\dfrac{T}{a}\right)^{q-1} + \dfrac{1}{\Gamma(q+1)}\left(\ln\dfrac{T}{a}\right)^{q} \|y\|.$

Define the following two operators E_1 and E_2 on B_λ by

$$(E_1 N)(t) = N_a \left(\ln\frac{t}{a}\right)^{q-1}, \tag{4.14}$$

$$(E_2 N)(t) = \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} \frac{Q(s,N(s))}{s} ds. \tag{4.15}$$

For all $N_1, N_2 \in B_\lambda$, the following holds from Eq.(4.14) and Eq.(4.15)

$$\|E_1 N_1 + E_2 N_2\| \le N_a \left(\ln\frac{T}{a}\right)^{q-1} + \frac{1}{\Gamma(q+1)}\left(\ln\frac{T}{a}\right)^{q} \|y\| \le \lambda, \tag{4.16}$$

leading to $E_1 N_1 + E_2 N_2 \in B_\lambda$.

The next step is to show that the mapping E_2 is a contraction. The said proof is as follow, $\forall t \in [a,T], \forall N_1, N_2 \in B_\lambda$, we have the following relation

$$\begin{aligned}\left|E_2N_1(t)-E_2N_2(t)\right| &= \left|\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{Q(s,N_1(s))}{s}ds-\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{Q(s,N_2(s))}{s}ds\right| \\ &\le \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\left|Q(s,N_1(s))-Q(s,N_2(s))\right|\frac{ds}{s},\end{aligned} \tag{4.17}$$

it then follows that

$$\begin{aligned}\left\|E_2N_1+E_2N_2\right\| &\le \frac{1}{\Gamma(q+1)}N_Q\left(\ln\frac{T}{a}\right)^{q-1}\left\|N_1-N_2\right\| \\ &\le \left\|N_1-N_2\right\|.\end{aligned} \tag{4.18}$$

From Eq.(4.18), it is observable that E_2 is a contraction. In addition, to the contraction of E_2, E_1 it is a continuous operator as a result of the continuity of N. Furthermore, E_1 is uniformly bounded as

$$\left\|E_1N\right\| \le N_a\left(\ln\frac{T}{a}\right)^{q-1}. \tag{4.19}$$

The following step is to show the compactness of the operator E_1. For all $t_1, t_2 \in [a, T]$, $(t_1 < t_2)$, the following holds

$$\left\|(E_1N)(t_2)-(E_1N)(t_1)\right\| \le N_a\left(\left|\left(\ln\frac{t_2}{a}\right)^{q-1}-\left(\ln\frac{t_1}{a}\right)^{q-1}\right|\right). \tag{4.20}$$

The right hand side of Eq.(4.20) approaches zero as $t_1 \to t_2$. Note that $\left\|(E_1N)(t_2)-(E_1N)(t_1)\right\|$ doesn't dependent on N which implies that E_1 is relatively compact. By Arzela-Ascoli theorem (See appendix A for theorem and its proof) we conclude that E_1 is compact on B_λ. Hence, the existence of the solution of the initial value problem defined by Eq.(4.10) holds by Krasnoselskii's fixed point theorem (See appendix B for theorem and its proof).

4.3.1.2 Uniqueness of solution

Existence of a solution to the FLDE, Eq.(4.10) is proven by theorem 4.2. In this section the uniqueness of the solution is proven.

Theorem 4.3 Let $Q:[a,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function that satisfies (A_1) and assume that $\left(\frac{1}{\Gamma(q+1)}\left(\ln\frac{T}{a}\right)^q\right)N_Q<1,$ then the initial value problem given by Eq.(4.10) has a unique solution.

Proof of Theorem 4.3: Consider the close set $B_\lambda=\{N\in H,\|N\|\le\lambda\}$ where

$$\lambda\ge\frac{N_a\left(\ln\frac{T}{a}\right)^{q-1}+M\left(\frac{1}{\Gamma(q+1)}\left(\ln\frac{T}{a}\right)^q\right)}{1-N_Q\left(\frac{1}{\Gamma(q+1)}\left(\ln\frac{T}{a}\right)^q\right)},\ \text{and } M=\sup_{a\le t\le T}\left|Q(t,0)\right|. \tag{4.21}$$

The first step is to show that $EB_\lambda\subset B_\lambda$. Hence, $\forall N\in B_\lambda,\forall t\in[a,T]$, the following inequality holds

$\left|(EN)(t)\right|\le N_a\left(\ln\frac{t}{a}\right)^{q-1}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{\left|Q(s,N(s))\right|}{s}ds$. On the other hand, the following relation is derived from $Q(t,N(t))$,

$$\begin{aligned}\left|Q(t,N(t))\right|&=\left|Q(t,N(t))-Q(t,0)+Q(t,0)\right|\\&\le\left|Q(t,N(t))-Q(t,0)\right|+\left|Q(t,0)\right|\\&\le N_Q\|N\|+M\\&\le N_Q\lambda+M.\end{aligned} \tag{4.22}$$

It then follows from Eqs.(4.19) to Eq.(4.22) that

$$\begin{aligned}\|EN\|&\le N_a\left(\ln\frac{T}{a}\right)^{q-1}+\left(\frac{1}{\Gamma(q+1)}\left(\ln\frac{T}{a}\right)^q\right)\left(N_Q\lambda+M\right)\\&\le\lambda.\end{aligned} \tag{4.23}$$

Eq. (4.23) implies that $EN \in B_\lambda, \forall N \in B_\lambda$, which means that $EB_\lambda \subset B_\lambda$.

The next step is to show the contraction of the mapping operator E. For all $N_1, N_2 \in H$, the following relation holds

$$\left|EN_1(t) - EN_2(t)\right| \le \frac{1}{\Gamma(q)} \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} \left|Q\left(s, N_1(s)\right) - Q\left(s, N_2(s)\right)\right| \frac{ds}{s}. \tag{4.24}$$

Eq.(4.24) implies that

$$\begin{aligned} \left\|EN_1 - EN_2\right\| &\le \left(\frac{1}{\Gamma(q+1)}\left(\ln\frac{T}{a}\right)^q\right) N_Q \left\|N_1 - N_2\right\| \\ &\le \left\|N_1 - N_2\right\|. \end{aligned} \tag{4.25}$$

From Eq.(4.25), the mapping E is a contraction. From the Banach contraction mapping theorem (See appendix C for theorem and its proof) the initial value problem defined by Eq. (4.10) has a unique solution on the close interval $[a,T]$. *Q.E.D.*

4.4 Simulation Studies

In previous sections, we highlighted the fact that so far none has proposed an acceptable and valid analytic solution to the FLDE. Moreover, we have proven the existence and uniqueness of the solution of the FLDE using Hadamard's fractional derivatives and integral. In this section, three numerical methods namely GEM, CF and PSE are used to compute the numerical solution of the FLDE.

Consider a fractional order of derivative $q \in (0,1) \cup (1,2)$, then the numerical solution of the FLDE using GEM, PSE and CF are defined respectively as

$$N_{GEM}\left(t_{j+1}\right) = N\left(t_j\right) + \frac{h^q}{\Gamma(q+1)} rN\left(t_j\right)\left(1 - \frac{N\left(t_j\right)}{K}\right), j = 0,1,\ldots k-1. \tag{4.26}$$

$$N_{PSE}(t_k) = rN(t_k)\left(1 - \frac{N(t_k)}{K}\right)h^q - \sum_{j=1}^{k} c_j^{(q)} N(t_{k-j}). \quad (4.27)$$

$$\begin{aligned} N_{CF}(t_{n+1}) = N(t_n) &+ \left(\frac{1-q}{M(q)} + \frac{3qh}{2M(q)}\right) N(t_n)\left(1 - \frac{N(t_n)}{K}\right) \\ &+ \left(\frac{1-q}{M(q)} + \frac{qh}{2M(q)}\right) N(t_{n-1})\left(1 - \frac{N(t_{n-1})}{K}\right). \end{aligned} \quad (4.28)$$

For performance evaluation purpose of the methods defined by Eq.(4.26)-Eq.(4.28), a mean squared error [56], whose formula is given in Eq. (4.29) is used to compute the deviation rate between the true values and the values obtained through numerical approximations,

$$MSE\% = \sqrt{\frac{\sum_{i=0}^{n}\left(y_i(t) - \hat{y}_i(t)\right)^2}{n}} \Bigg/ \sqrt{\frac{\sum_{i=0}^{n}\left(y_i(t)\right)^2}{n}}, \quad (4.29)$$

where $y_i(t)$ is the true value at time t; $\hat{y}_i(t)$ is the approximated value computed at time t and finally n is the data size.

The data used for the numerical simulation in this section is the annual growth rate of the helianthus plant. The data set was retrieved from [55] and it is a record of the helianthus plant height measured at a constant rate of one measurement after every 7 days. The plants' heights are measured in centimeter. The plants on which the experiment was carried were considered from their 7$^{\text{th}}$ day of age up to their 84$^{\text{th}}$ day of age, representing 12 measurements. For uniformity a measurement is the average value of all the plants heights on the record day.

Recalling Eq.(4.9), which is the solution to the logistic differential equation using the classical approach, the Matlab optimization routine of non-linear model '*lsqcurvefit*' is used to find the carrying capacity $K=267.5301$and the growth rate $r=0.0760$ of the data set. Using these parameters, the error rate computed using Eq.(4.29) is $MSE\%=0.0320$ i.e 3.2%. Figure 4.1 shows the plots of the true data and the data computed using Eq.(4.9).

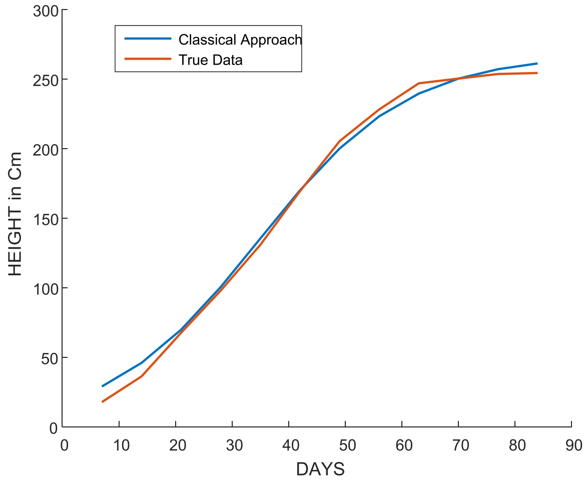


Figure 4.1: Classical Method

Consider Eq.(4.26) and Eq.(4.27) for the GEM and PSE numerical solution respectively. The value of fractional derivative that would minimize the error rate is investigated iteratively. A step or increment value of 10^{-3} was used to iteratively cover the interval [0.9, 1.2] representing the possible q –values and the corresponding error rate $MSE\%$ of each q-value was computed. As result, it appears that both GEM and PSE would produce a minimum error $MSE\%$, if the fractional order derivative is $q=1$ as shown on figure 4.2.

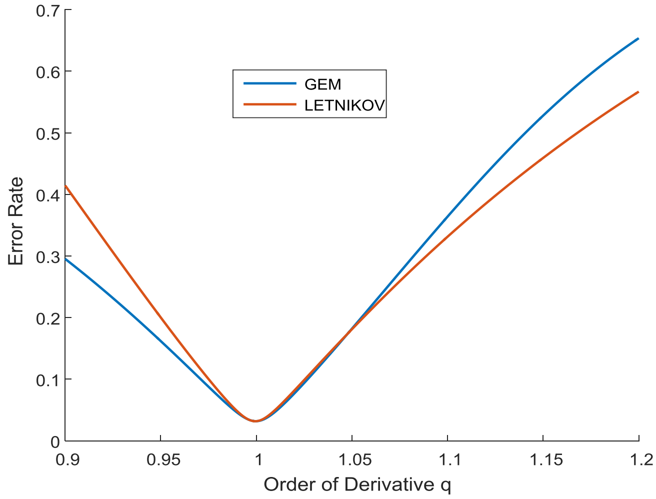
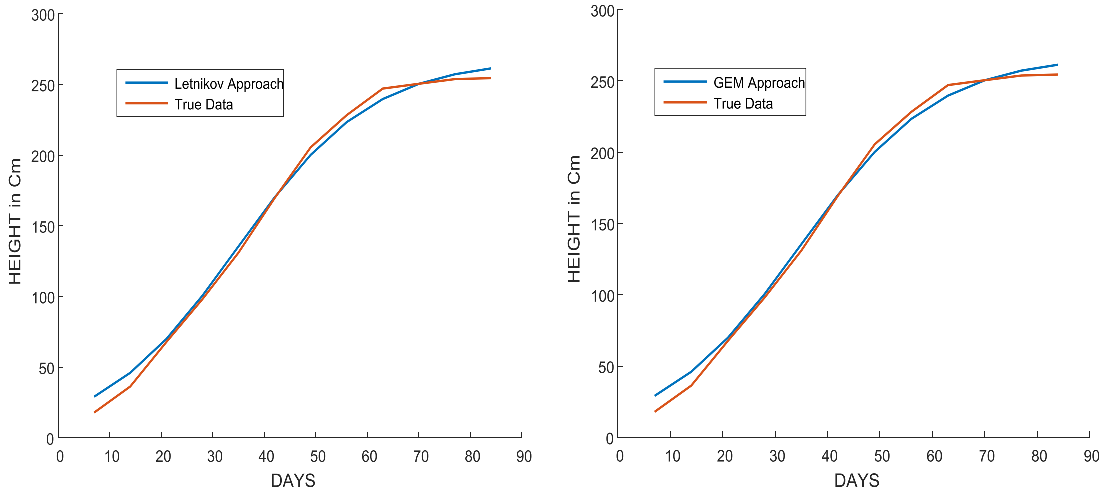


Figure 4.2: PSE and GEM error rate (MSE%) versus possible *q* values.

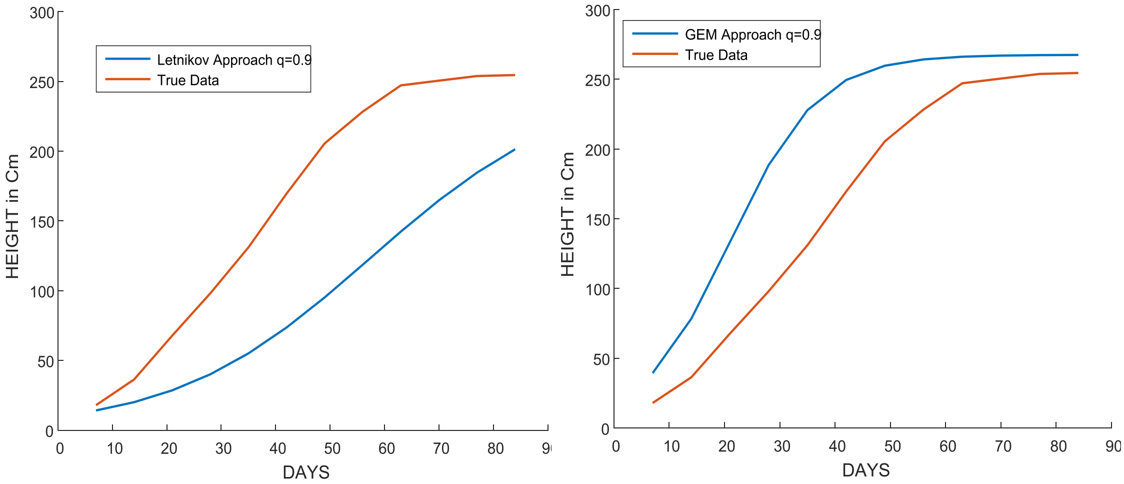
PSE and GEM would produce a minimum error only if their fractional order derivative is $q=1$, which means when both methods coincide with the classical approach. In which case, the common error rate value of GEM , PSE and the classical approach all coincide and equals to $MSE\%=0.0320$, i.e3.2%. Figure 4.3 shows the plots of the true data versus the computed data using GEM and PSE with $q=1$.



(a) (b)

Figure 4.3: Graphs of (a) PSE with $q=1$ and (b) GEM for $q=1$.

Different values of the fractional derivative order have significant effect on the estimation. In order to illustrate such effect, a *q=0.9*, was considered in simulation with PSE and GEM. The PSE produced *MSE%=0.4146* that is*41.46%* whereas GEM has produced *MSE%=0.2954*, i.e*29.54%*. Figure 4.4 shows the plots of both GEM and PSE methods for *q=0.9* alongside the true data values. Looking at Figure 4.3 and Figure 4.4, the behavior the difference is clearly perceptible.



(a) (b)

Figure 4.4: Graphs of (a) PSE with *q=0.9* and (b) GEM for *q=0.9*.

Consider Eq.(4.28) for the CF numerical solution. The value of fractional derivative that would minimize the error rate is investigated iteratively. Here a step value of 10^{-3} was used to iteratively cover the interval [0.9, 1.2] representing the possible *q* – values and the corresponding error rate *MSE%* of each *q*-value was computed. It is observed that CF would produce a minimum error *MSE%=3.21%*, if the fractional order of derivative is *q=1.005* as shown on Figure 4.5.

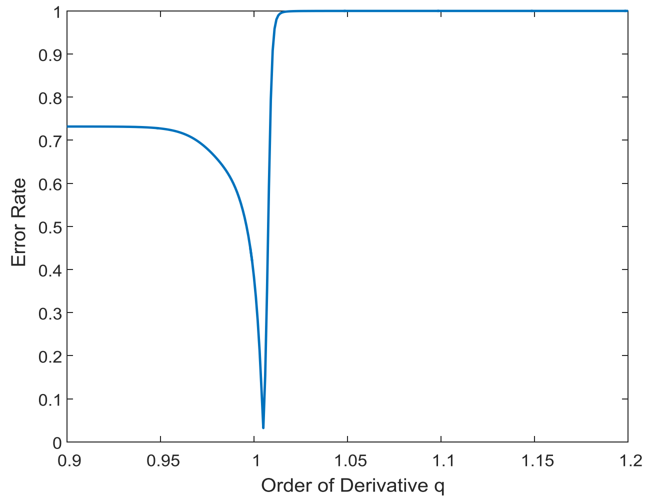
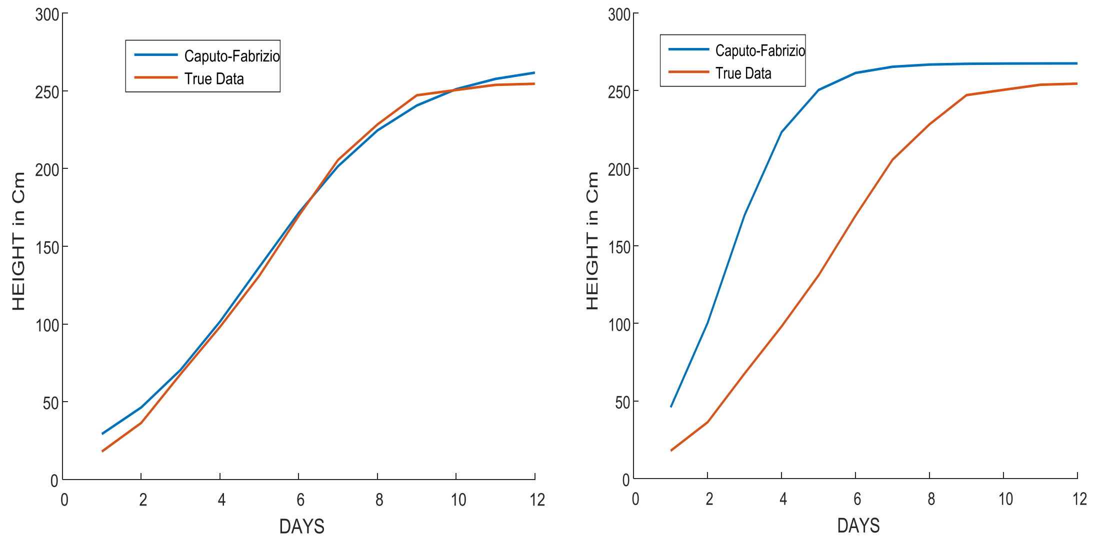


Figure 4.5: CF's error rate versus q values.

Under some approximation circumstances, $q=1.005$ could be approximated by $q=1$. However, it is important to highlight that such approximation would incorrect in this study. For a clear illustration, the CF was simulated for $q=1$ and has produced $ER=0.3824$ i.e. 38.24%. Figure 6.6 shows plots of the data obtained using CF with $q=1.005$and $q=1$, respectively.



(a) (b)

Figure 4.6: CF 's graph (a) for $q=1.005$ and (b) for $q=1$.

4.5 Analysis of Results

In this chapter the FLDE was studied. The model was derived from its counterpart logistic differential equation with integer order of derivative. Hadarmard fractional integral and derivatives were used to prove the existence and uniqueness of a solution for the FLDE. Moreover, CF, PSE and GEM numerical algorithms were used to build the numerical solution of the FLDE, since an exact analytic form doesn't exist yet.

In application with experimental data, when $q=1$, GEM and PSE both produced minimum error rate $MSE\%=3.2\ \%$, that coincides with the error rate of the classical approach. On the other hand, CF has produced a minimum $MSE\%=3.2\ \%$ for $q=1.005$.

As summary, the chapter's aim which was to build the FLDE and investigate its solution was achieved. It appears that the solution of the FLDE can only be computed numerically up today. The error rate of the FLDE for the best value of q is not different from error rate of the classical approach.

Chapter 5

FRACTIONAL LOGISTIC GROWTH MODELING WITH RAYLEIGH KERNEL FUNCTION

5.1 General Concepts

Mathematical modeling is of great importance in science since it finds its application in various fields of different disciplines. Physics [61, 62], Biology [58, 59, 60], Health Science [90], Economy [91], are just some of the long list of fields where mathematical modeling is used.

Various approaches are used in modeling, namely deterministic and stochastic approaches. We mentioned in earlier chapters of this work that both approaches will be explored. When it comes to the deterministic approach, differential equations are known as the ultimate tool to be used. In recent decades, researchers have focused on a type of differential equations, known as fractional differential equations [57]; with the main goals to support the mathematical theory behind it [25, 63] and investigate how efficient it is in application [64, 65]. In some cases it is proven that fractional differential equations perform better than their classical counterpart in modeling [31, 33]. Derivatives and integrals are the main tools used in differential equations, whether it be fractional or classical. With reference to the chain rule derivative method [66], Imelda et Al. [30], introduced a similar fractional derivative, which they referred to as '*the* ψ *-Caputo derivative*'. That is the fractional derivative taken in Caputo sense, of a function with respect to another function. The function with respect to which the derivative is taken is called the kernel function.

There exists a probability distribution function called Rayleigh function [67], which is derived from Weibull family type of distribution. In this work a logistic model is built using the ψ *-Caputo derivative*' with the Rayleigh function as kernel function. The use of the Rayleigh function is motivated by its belt shape, which tends to stabilize the rate in a logistic growth model. A special attention is paid to logistic model with very large carrying capacity *K*. Preliminary work was done on the model to prove existence and uniqueness of solution and finally, the Chinese population growth was used in application since it has a very large carrying capacity.

5.2 Preliminaries Definitions and Rayleigh Distribution

The Rayleigh distribution is continuous density function that belongs to the Weibull and exponential families of distribution. It is defined only for positive valued random variable [68].The distribution was named after Lord Rayleigh [69], who did the initial work on its development. The application fields of this distribution include but not limited, wind trajectory, queuing system, life span of an object, magnetic resonance and biomedical image processing. In [85, page 111], authors proposed a generalized form of distribution called 'Gamma family' from which they derived the Rayleigh, Weibull and many others distributions as special cases. The one parameter gamma distribution has a density function given by

$$f\left(t\middle|\alpha\right)=\frac{t^{\alpha-1}e^{-t}}{\Gamma\left(\alpha\right)}, \tag{5.1}$$

with $t \geq 0$ and $\alpha > 0$. The distribution parameter α is known as shape. When $\alpha = 1$, Eq.(5.1) becomes the exponential distribution. A scale parameter $\beta > 0$ can be introduced into Eq.(5.1) to produce a two-parameters gamma distribution defined

$$f\left(t \middle| \alpha, \beta\right) = \frac{t^{\alpha-1} e^{-t/\beta}}{\beta^{\alpha} \Gamma\left(\alpha\right)}. \tag{5.2}$$

A location parameter $l \in \mathbb{R}$ can be added to Eq.(5.2) with the purpose of centralizing the formula leading to a three-parameters gamma distribution defined by

$$f\left(t \middle| \alpha, \beta, l\right) = \frac{\left(t-l\right)^{\alpha-1} e^{\left(-\frac{t-l}{\beta}\right)}}{\beta^{\alpha} \Gamma\left(\alpha\right)}. \tag{5.3}$$

Stacy et al. [86] introduced another shape parameter $\lambda > 0$, to the model given by Eq.(5.3) to produce a four-parameters distribution; which is also referred to as generalized gamma distribution

$$f\left(t \middle| \alpha, \beta, l, \lambda\right) = \frac{\lambda\left(t-l\right)^{\lambda\alpha-1} e^{\left(-\left(\frac{t-l}{\beta}\right)^{\lambda}\right)}}{\beta^{\lambda\alpha} \Gamma\left(\alpha\right)}. \tag{5.4}$$

Different distributions can be generated using Eq.(5.4) by simply changing the values of the parameters α, β, l and λ. The table below contains some of such distributions.

Table 5.1: Distribution derived from generalized gamma distribution.

$f\left(t \middle| \alpha, \beta, l, \lambda\right)$ ***from Eq.(5.4)***	***Name of the derived Distribution***
$f\left(t \middle| 1, 1, 0, \lambda\right)$	Reduced Weibull distribution
$f\left(t \middle| 1, \beta, l, \lambda\right)$	Three-parameters Weibull distribution
$f\left(t \middle| 1, \beta, l, 2\right)$	Rayleigh Distribution
$f\left(t \middle| 0.5, \sqrt{2}, 0, 2\right)$	Half-Normal distribution
$f\left(t \middle| 1, \sqrt{2}, 0, 2\right)$	Circular Normal distribution

$f\left(t \middle| 1,1,0,1\right)$	Reduced Exponential distribution

From table 1, one can observe that the Rayleigh distribution is a special case of the three-parameters Weibull distribution with $\lambda = 2$. Several similar formulae which might sometime differ only by constants appearing in notations are used to represent the Rayleigh distribution. For instance a definition of Rayleigh distribution function is given in [68] as follow.

Definition 5.1 A random variable *X*, is said to have a Rayleigh distribution with shape parameter σ, if the following holds

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & , x \geq 0 \\ 0 & \text{otherwise} \end{cases} . \tag{5.5}$$

It is important to observe that the distribution is a positive, bell shaped distribution. The cumulative Rayleigh distribution is easily derived as follow

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}}\, dt = -e^{-\frac{t^2}{2\sigma^2}} \Big|_0^x = 1 - e^{-\left(\frac{x^2}{2\sigma^2}\right)}. \tag{5.6}$$

The mean and variance of a random variable *X* having a Rayleigh distribution with a shape parameter σ are proven to respectively be given by $E(X) = \sigma\sqrt{\frac{\pi}{2}}$ and $Var(X) = \frac{4-\pi}{2}\sigma^2$.

Notation:

- In all what will follow, the notation $X \sim Ray(\alpha)$ will refers to a Rayleigh random variable with parameter α .
- For simplification purpose, from Eq.(5.5) let us perform the following change of variable $\alpha = 2\sigma^2$, then it follows a simplified Rayleigh distribution function given by

$$f_X(x) = \begin{cases} \dfrac{2x}{\alpha} e^{-\frac{x^2}{\alpha}} & , x \geq 0 \\ 0 & \text{otherwise} \end{cases} . \tag{5.7}$$

Theorem 5.2 The scaling property is verified by Rayleigh distribution. That is given a random variable X such that $X \sim Ray(\alpha)$.Then $\forall \lambda > 0$, it follows that $\lambda X \sim Ray(\lambda^2 \alpha)$.

Proof of Theorem 5.2 [68] Since the distribution is defined for *X>0,* and also $\forall \lambda > 0$, define the following bijection

$$\begin{cases} \mathbb{R}^+ \to \mathbb{R}^+ \\ X \to h(X) = \lambda X = Y \end{cases} . \tag{5.8}$$

The inverse of h, is then defined as $h^{-1}(Y) = \dfrac{Y}{\lambda} = X$. The Jacobian of the linear transformation, Eq.(5.8) is $\dfrac{dX}{dY} = \dfrac{1}{\lambda}$. Using the variable transformation technique, it follows that

$$\begin{aligned} \forall y \in \mathbb{R}^+ ; f_Y(y) &= f_X\left(h^{-1}(y)\right)\left|\frac{dX}{dY}\right| \\ &= f_X\left(\frac{y}{\lambda}\right)\left|\frac{1}{\lambda}\right| \end{aligned}$$

$$
\begin{aligned}
&= \frac{y/\lambda}{\sigma^2} e^{-\frac{(y/\lambda)^2}{2\sigma^2}} \left|\frac{1}{\lambda}\right| \\
&= \frac{y}{(\lambda\sigma)^2} e^{-\frac{y^2}{2(\lambda\sigma)^2}}.
\end{aligned} \tag{5.9}
$$

Hence $y \sim Ray\left(\lambda^2\alpha\right)$. *Q.E.D.*

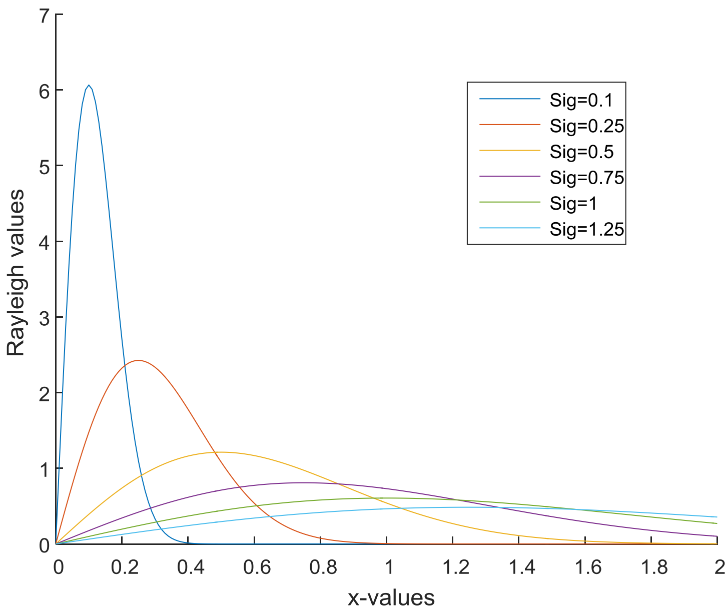


Figure 5.1: Rayleigh distribution

5.3 Logistic Population Growth Modeling with ψ -Caputo derivative

The theory of the population growth modeling using logistic equation was introduced by an economist named Malthus T.R [9, 54, 70]. The size of a population with growing capacity would theoretically approach infinity when the time approaches infinity. However, such idea was proven incorrect by Malthus, who claimed that growing populations always reach a saturation point. Modeling of a population growth with FLDE is discussed. Furthermore, the solution to the FLDE was built and computed numerically. In this chapter, the FLDE is considered with the 'ψ *-Caputo derivative*'.

Consider the classical logistic growth model defined by

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right), \tag{5.10}$$

where r is the growth rate and K is the carrying capacity which represents the maximum value that the population size may reach. Hence at that size the population growth stabilizes. A general solution for the classical logistic model defined by Eq.(5.10) is

$$N_c(t) = \frac{K}{1 + \left(\frac{K - N_0}{N_0}\right)e^{-rt}}, \tag{5.11}$$

where $N_0 = N(0)$ is the initial size of the population at $t=0$.

Considering problem from fractional point of view, the Fractional equivalent to Eq.(5.10) using the *psi-Caputo* derivative is defined as

$${}_cD^{\alpha,\psi}N(t) = rN(t)\left(1 - \frac{N(t)}{K}\right), \tag{5.12}$$

where $\alpha \in (0,1),\ N(0) = 0$.

The right hand side of Eq.(5.12) is written as a function of t and $N(t)$ as

$${}_cD^{\alpha,\psi}N(t) = f(t, N(t)), \tag{5.13}$$

with $\alpha \in (0,1),\ N(0) = 0$.

The notation provided in Eq.(5.13) is adopted for simplification and conformity in proving Lemmas and theorems that follow in the course of the work.

Lemma 5.3 Assume f is an integrable function defined on $[0,T]$, then the general solution of the fractional differential equation given by Eq.(5.13) is equivalent to the following integral equation

$$N(t) = N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x)\left(\psi(t)-\psi(x)\right)^{\alpha-1} f\left(x, N(x)\right) dx. \qquad (5.14)$$

Applying the 'ψ -*Caputo*' fractional integral operator $I_{0^+}^{\alpha,\psi}$ to both sides of Eq.(5.13) leads to

$$N(t) - N_0 = I_{0^+}^{\alpha,\psi} f\left(t, N(t)\right). \qquad (5.15)$$

At this point it is worth building the framework under which the method is applicable. After building the framework, existence and uniqueness of the problem defined by Eq.(5.12) and Eq.(513) are proven.

Consider $C[0,T] = \{N \in C[0,T]\}$, let $\Phi = \left(C[0,T], \|\cdot\|\right)$ representing the Banach space of all continuous functions from the close interval $[0,T]$ to $\mathbb{R}$, endowed with the norm defined by $\|N\| = \sup_{0 \le t \le T} \left|N(t)\right|$.

An operator $\mathcal{F} : \Phi \to \Phi$ can be associated with the problem defined by Eq.(513) and defined as

$$\left(\mathcal{F}N\right)(t) = N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x)\left(\psi(t)-\psi(x)\right)^{\alpha-1} f\left(x, N(x)\right) dx \quad . \qquad (5.16)$$

Prior to the statement and proof of the main results, the existence and uniqueness of solution for the problem given by Eq.(5.14) is discussed. Consider the following hypotheses

(A1): $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous.

(A2): There exists $L_N>0$ such that $\left|f(t,N_1)-f(t,N_2)\right|\le L_N\left|N_1-N_2\right|, \forall t\in[0,T]$.

(A3): There exists a function $g\in C([0,T],\mathbb{R}^+)$ and a non-decreasing function $\chi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\left|f(t,u)\right|\le g(t)\chi(|u|), \forall (t,u)\in[0,T]\times\mathbb{R}$, $\forall u,v\in\mathbb{R}$.

(A4): There exists a constant $W>0$ such that

$$\frac{W}{N_0+\|g\|\chi(\lambda)\left(\frac{1}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha}\right)}>1.$$

Theorem 5.4 Assume that (A1), (A3) and (A4) hold. Then the problem defined by Eq.(5.13) has at least one solution on the close interval $[0,T]$.

Proof of Theorem 5.4: This proof will be split into several steps. The first step consists of showing that the operator $\mathcal{F}:\Phi\to\Phi$ maps bounded sets into bounded sets of Φ. In this regard, let $B_\lambda=\{N\in\Phi:\|N\|\le\lambda\}$ be a bounded set in Φ, then

$$\begin{aligned}\left|(\mathcal{F}N)(t)\right|&\le N_0+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(x)\left(\psi(t)-\psi(x)\right)^{\alpha-1}\left|f(x,N(x))\right|dx\\&\le N_0+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(x)\left(\psi(t)-\psi(x)\right)^{\alpha-1}g(x)\chi(\|N\|)dx\ .\end{aligned}\tag{5.17}$$

Taking the norm $\|\mathcal{F}N\|=\sup\limits_{0\le t\le T}\left|\mathcal{F}N(t)\right|$, implies that

$$\begin{aligned}\|\mathcal{F}N\|&\le N_0+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(x)\left(\psi(x)-\psi(x)\right)^{\alpha-1}\left|f(x,N(x))\right|dx\\&\le N_0+\frac{1}{\Gamma(\alpha+1)}\|g\|\chi(\lambda)\left(\psi(T)-\psi(0)\right)^{\alpha}.\end{aligned}\tag{5.18}$$

The second step is to show that the operator $\mathcal{F}:\Phi\to\Phi$ maps bounded sets into equicontinuous sets of Φ.

Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $N\in B_\lambda$. Moreover Denote by

$k_1(t_1)=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(x)\left[\left(\psi(t_2)-\psi(x)\right)^{\alpha-1}-\left(\psi(t_1)-\psi(x)\right)^{\alpha-1}\right]g(x)\chi(\|N\|)dx$ and

$k_2(t_2)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(x)\left[\left(\psi(t_2)-\psi(x)\right)^{\alpha-1}\right]g(x)\chi(\|N\|)dx$ then it follows that

$$\left|(\mathcal{F}N)(t_2)-(\mathcal{F}N)(t_1)\right|\le\left|k_1(t_1)+k_2(t_2)\right|. \tag{5.19}$$

The right hand side of inequality defined by Eq.(5.19) tends to zero as $t_1\to t_2$, implying that $\left|(\mathcal{F}N)(t_2)-(\mathcal{F}N)(t_1)\right|\to 0$ as $t_1\to t_2$. Note that the right hand side of Eq.(5.19) is independent of $N\in B_\lambda$ hence, by Arzela-Ascoli theorem (See appendix A for theorem and its proof) we conclude that $\mathcal{F}$ is completely continuous.

The last step to complete the assumptions of Leray-Schauder nonlinear alternative theorem (See appendix D for theorem and its proof) is to show the boundedness of the set of all solutions to the following equation

$$N=\delta\mathcal{F}N. \tag{5.20}$$

Assume that N is a solution of Eq.(5.20), then

$$\begin{aligned}|N(t)|&=|\delta(\mathcal{F}N)(t)|\\&\le\delta\left(N_0+\frac{1}{\Gamma(\alpha+1)}\|g\|\chi(\lambda)\left(\psi(T)-\psi(0)\right)^{\alpha}\right)\\&\le N_0+\frac{1}{\Gamma(\alpha+1)}\|g\|\chi(\lambda)\left(\psi(T)-\psi(0)\right)^{\alpha}.\end{aligned} \tag{5.21}$$

Eq.(5.21) implies that

$$\frac{\|N\|}{N_0+\frac{1}{\Gamma(\alpha+1)}\|g\|\chi(\lambda)\left(\psi(T)-\psi(0)\right)^{\alpha}}\leq 1. \tag{5.22}$$

Recalling (A4), there exists a constant $W>0$ such that $W\neq N$. At this point, considering the set $\Omega=\{N\in\Phi:N<W\}$, it follows that the operator $\mathcal{F}:\overline{\Omega}\to\Phi$ is continuous and completely continuous, by the constructed Ω, $\nexists N\in\partial\Omega$ such that $N=\delta\mathcal{F}N$ for some $\delta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we conclude that $\mathcal{F}$ has a fixed point $N\in\overline{\Omega}$ which is a solution of the problem given by Eq.(5.14).

Theorem 5.5 Assume that (A1), (A2) hold. If $\frac{L_N}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha}<1$, then the problem given by Eq.(5.13) has a unique solution on $[0,T]$.

Proof of Theorem 5.5: Consider the operator $\mathcal{F}$ defined by Eq.(5.16) and define a ball $B_\varepsilon=\{N\in C[0,T]:\|N\|\leq\varepsilon\}$ with $\varepsilon\geq\frac{N_0+M_N\left(\frac{1}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha}\right)}{1-L_N\frac{1}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha}}$,

where $M_N=\sup\limits_{0\leq t\leq T}\left|f(t,0)\right|$.

First show that $\mathcal{F}B_\varepsilon\subset B_\varepsilon$. For any $N\in B_\varepsilon, t\in[0,T]$, we have

$$\left|(\mathcal{F}N)(t)\right|\leq N_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left|\psi'(x)\right|\left|\psi(t)-\psi(x)^{\alpha-1}\right|\left|f\left(x,N(0)\right)\right|dx. \tag{5.23}$$

Moreover, the following inequality can be built from a functions' norm

$$
\begin{aligned}
\left|f\left(x,N(x)\right)\right| &= \left|f\left(x,N(x)\right)-f\left(x,0\right)+f\left(x,0\right)\right| \\
&\le \left|f\left(x,N(x)\right)-f\left(x,0\right)\right|+\left|f\left(x,0\right)\right| \\
&\le L_N\left\|N\right\|+M_N \\
&\le L_N\varepsilon+M_N.
\end{aligned} \tag{5.24}
$$

Using Eq.(5.23) and Eq.(5.24) , it follows that

$$
\begin{aligned}
\left\|\mathcal{F}N\right\| &\le N_0+\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(x)\left(\psi(t)-\psi(x)\right)^{\alpha-1}dx\right)\left(L_N\varepsilon+M_N\right) \\
&\le N_0+\left(\frac{1}{\Gamma(\alpha)}\left(\psi(T)-\psi(0)\right)^{\alpha}\right)\left(L_N\varepsilon+M_N\right) \\
&\le \varepsilon.
\end{aligned} \tag{5.25}
$$

Eq.(5.25) implies that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$.

The second step is to prove that the operator is a contraction. $\forall N_1, N_2 \in \Phi$, we have

$$
\begin{aligned}
\left|(\mathcal{F}N_1)(t)-(\mathcal{F}N_2)(t)\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(x)\left(\psi(t)-\psi(x)\right)^{\alpha-1}\left|f\left(x,N_1(x)\right)-f\left(x,N_2(x)\right)\right|dx \\
&\le \left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(x)\left(\psi(t)-\psi(x)\right)^{\alpha-1}dx\right)L_N\left\|N_1-N_2\right\| \\
&\le L_N\left(\frac{1}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha}\right)\left\|N_1-N_2\right\| \\
&\le \left\|N_1-N_2\right\|.
\end{aligned} \tag{5.26}
$$

From Eq.(5.26), $\mathcal{F}N$ is a contraction. Hence, by the Banach contraction mapping theorem (See Appendix C for theorem and proof), the fractional differential equation given by Eq.(5.13) has a unique solution over the interval $[0,T]$.

5.4 Population Growth with Carrying Capacity K Approaching infinity

In section 5.3 the framework of FLDE using the ψ *-Caputo derivative*' was built. In this section the FLDE is built with the assumption that the population carrying capacity is very large. The size of the carrying capacity highly influences the model variations. Recalling Eq.(5.10), if the carrying capacity is too large with respect to the current population size, then the following mathematical relations can be derived

$$N(t) \ll K, \tag{5.27}$$

equivalently

$$\frac{N(t)}{K} \to 0. \tag{5.28}$$

Under the conditions defined by Eq.(5.27) and Eq.(5.28) , the logistic model defined by Eq.(5.10) becomes an exponential growth model

$$\frac{dN(t)}{dt} = rN(t). \tag{5.29}$$

A general solution of the exponential growth model defined by Eq.(5.29) is obtained as

$$\frac{dN(t)}{N(t)} = rdt \to \int \frac{dN(t)}{N(t)} = \int rdt \to \ln(N(t)) = rt + c \to e^{\ln(N(t))} = e^{rt+c} \to N(t) = N_0 e^{rt}.$$

Then

$$N(t) = N_0 e^{rt}, \tag{5.30}$$

where N_0 is the initial population size. Eq.(5.30) shows that the population size will increases infinitely as $t \to \infty$. However, in many application problems this does not hold true. For example in the Chinese population case from Figure 5.2 it can be seen that exponential growth leaves its place to linear growth after mid 1970s.

In what follows, an alternative approach is proposed for the modeling of large size population with carrying capacity $K \to +\infty$. Applying the 'ψ *-Caputo derivative*' to the model defined by Eq.(5.29) the following is obtained

$$\begin{cases} {}_C D_{0^+}^{\alpha,\psi} N(t) = rN(t) \\ N(0) = N_0. \end{cases} \tag{5.31}$$

The solution of the model defined by Eq. (5.31) is defined as

$$N(t) = N_0 E_\alpha \left[r\left(\psi(t) - \psi(0)\right)^\alpha \right]. \tag{5.32}$$

Choosing the Rayleigh cumulative density function $\psi(t) = 1 - e^{-\left(t^2/2\sigma^2\right)}$ as kernel, Eq.(5.32) becomes

$$N(t) = N_0 E_\alpha \left[r\left(1 - e^{-\left(t^2/2\sigma^2\right)}\right)^\alpha \right]. \tag{5.33}$$

In simulations, a choice of the Kernel is made based on their shape in a way to best fit the data.

5.5 Numerical Simulation Using Chinese Population Data

Annual population of China between 1900 and 2020 is taken from the World Bank web site [71] as input data to test the accuracy of the logistic, exponential and Rayleigh models. It is observed that the Rayleigh model produced better fit to the population data with large carrying capacity, than the others in terms of lower Percent Mean Square Error. Using the Matlab non-linear optimization routine '*lsqcurvefit*' the parameter that fits best to the classical logistic model Eq.(5.16), the exponential model Eq.(5.29) and the Psi-Caputo fractional model Eq.(5.33) are determined. Referring to Eq.(4.29), Percent Mean Square Error can be opened up as, Mean Square Error (MSE)

$$MSE = \sum_{i=1}^{n} \frac{(x_i - \widehat{x}_i)^2}{n}. \tag{5.34}$$

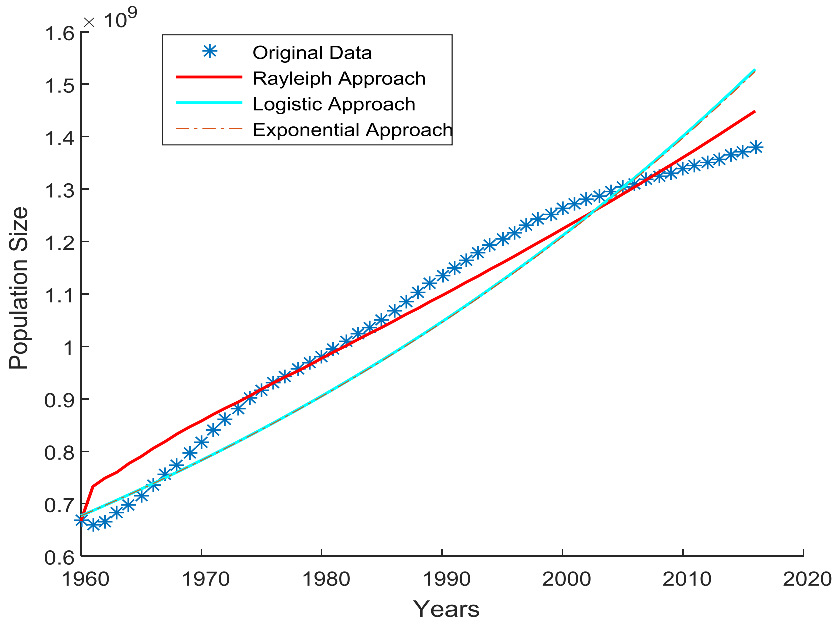
A fair comparison of MSE values from different methods or models can be obtained by expressing the MSE value as a percentage of the average of the squares of the variable under study, given as $AS = \frac{\sum_{i=1}^{n} x_i^2}{n}$. Then

$$MSE\% = \left(\frac{MSE}{AS}\right)100. \tag{5.35}$$

Computation of MSE% values for each methodology lead to the following results

- The classical logistic approach with a growth rate of $r=0.0145$ and a carrying capacity of $K=3.3198\text{x}10^{23}$would best fit the data by producing a total error rate of *MSE%=6.27%.*
- The classical exponential approach produced the same result as the classical logistic approach. In fact, $r=0.0145$, minimize the error rate to *MSE%=6.27%.*
- The Rayleigh Kernel fractional approach with a fractional order of derivative $\alpha = 0.2867$ and a rate of $3.050\text{x}10^3$, both obtained through non linear optimization routine '*lsqcurvefit*' produced a minimum error rate to *MSE%=3.67%.*

Based on the MSE% values the Rayleigh Kernel fractional approach produced the lowest error, meaning its estimates are better than the other two methods.



-

Figure 5.2: Chinese population growth modeling

Note: The difference in the error rate value from 3.67% to 6.27% seems to be small. However, this difference is highly significant as the population size is too large.

A close examination of the raw data in Figure 5.2 led to the idea of splitting the data into 3 sub sections. Sub group 1 is between 1960 to 1975 the graph exhibits an exponential pattern. Sub group 2 is from 1976 to 1995 a linear behavior is evident. Finally sub group 3 from 1996 onwards a downward curvature is visible. Hence, for sub group 1 an exponential model with growth rate estimated as r=0.0186, for sub group 2 a linear model with slope 0.1476×10^8 and y intercept 6.737×10^8, and for sub group 3 a ψ *-Caputo* model with Rayleigh kernel with parameters $\alpha = 0.1527,\ r = 31.9236$ are used. Obtained models and raw data are given in Figures 5.3, 5.4, and 5.5 respectively. Respective error levels are 2.97%, 0.38% and 0.38%.

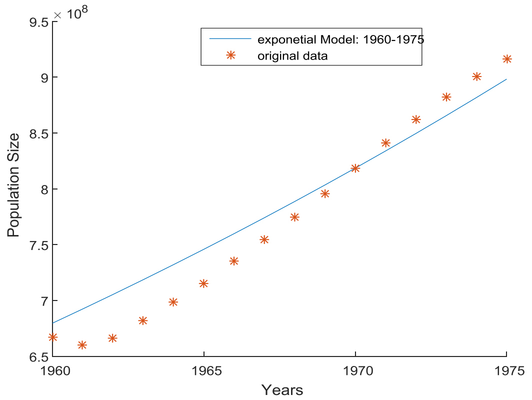


Figure 5.3: Population from 1960-1975 by exponential model

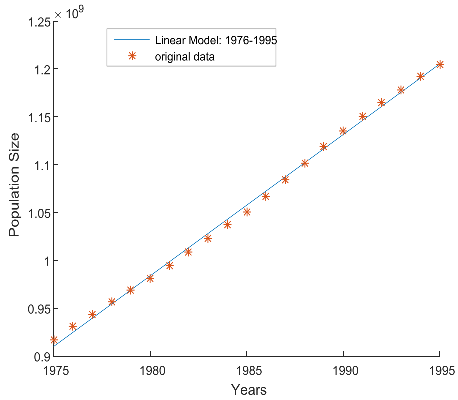


Figure 5.4: Population from 1976-1995 by Linear model

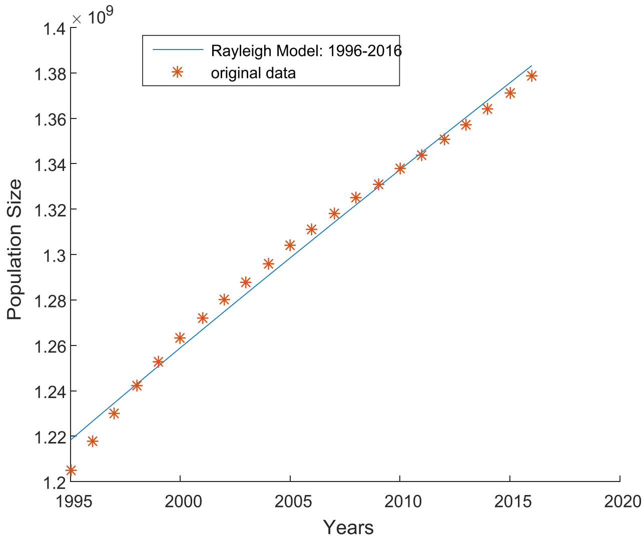


Figure 5.5: Population from 1996-2016 by Rayleigh kernel model

Chapter 6

DETERMINISTIC AND PROBABILISTIC MODELING OF THE WORLD POPULATION GROWTH

6.1 Kernel Smoothing

This method is used for fitting a non-parametric model to a data set. Kernel smoothing is the most widely used method for nonparametric model fitting to a data set with no visible pattern. Without loss of generality, we consider the univariate Kernel density function in this section.

Let X be a random variable. Consider the vector (X_1, X_2,…,X_n) representing n observation of the random variable X, generated by an unknown density function $f(x)$. If the vector (X_1, X_2,…, X_n)doesn't show any standard parametric trend, such as linear or quadratic shape, then the unknown density function $f(x)$ can be estimate using the kernel density estimation method given by the formula [75, 76]

$$\hat{f}(x,h) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right). \tag{6.1}$$

In (6.1), h is called the bandwidth, K is the kernel density, it is usually a symmetric non-negative and continuous function having the properties [75], $\int_{-\infty}^{\infty} K(x)\,dx = 1$; $\int_{-\infty}^{\infty} xK(x)\,dx = 0$ and $0 < \int_{-\infty}^{\infty} x^2 K(x)\,dx < \infty$. The most widely used kernel density functions are known to be the Gaussian kernel function [76] $K(x) = \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{1}{2}x^2\right)}$

Epanechnikov kernel function $K\left(x\right)=\frac{3}{4}\left(1-x^2\right)I_{\left(|x|\le1\right)}$, and the Uniform kernel function $K\left(x\right)=\frac{1}{2}I_{\left(|x|\le1\right)}$.

Let $f(x)$ be the unknown function and X be a variable that is one of p variables that governs some process. Let the kernel density estimator of $f(x,h)$ be $\hat{f}\left(x,h\right)$, h being the band width used by the kernel estimator. The goal is to select the optimal h value that will minimize the error level. That is to minimize the absolute value of the quantity $f(x)-\hat{f}(x,h)$. The use of the sample data (X_1, $X_2,\ldots,X_n$) to derive the optimal bandwidth h is known as bandwidth selector technique [75]. Different approaches exist for computing the optimal bandwidth value. One way of estimating the optimal bandwidth is based on the minimization of the mean squared error (MSE) or the mean integrated squared error (MISE).

Definition 6.1 Consider a statistical parameter Ψ, which is estimated by $\hat{\Psi}$. The mean square error (MSE) between the parameter and its estimated value is the expectation of the squared deviation between the true value of the parameter Ψ and its estimated value $\hat{\Psi}$. Its given by

$$MSE\left(\hat{\Psi}\right)=E\left(\Psi-\hat{\Psi}\right)^2. \tag{6.2}$$

In practice, Eq.(6.2) is split into summation of a bias and variance terms as shows below

$$MSE\left(\Psi\right)=Var\left(\hat{\Psi}\right)+\left(E\hat{\Psi}-\Psi\right)^2. \tag{6.3}$$

Details on how Eq.(6.3) is derived is found in [92].

The bias-variance form of the MSE is used to derive the optimal bandwidth value *h*.

Lemma 6.2 Consider *n* observations $(X_1, X_2, \ldots, X_n)$, of a univariate random variable *X* from a non-parametric distribution. An optimal bandwidth of the kernel density function defined by Eq.(6.1) is computed based on the asymptotic mean integrated squared error (AMISE), by the formula[75]

$$Opt_{AMISE}(h) = \sqrt[5]{\left(\frac{R(K)}{\mu_2(K)^2 R(f'')n}\right)}, \tag{6.4}$$

with $\mu_2(K) = \int t^2 K(t)\,dt$ and $R(f) = \int f(t)^2 dt$.

The unknown density function *f* is assume to be defined such that $f \in C^2$.

Proof of Lemma 6.2: Recalling definition 6.1 and formula (6.2), $\hat{f}(x,h)$ which is used to fit the original *f(x)* has a MSE defined by

$$MSE\left(\hat{f}(x,h)\right) = E\left(f - \hat{f}(x,h)\right)^2. \tag{6.5}$$

Using the idea of Eq.(6.3), Eq.(6.5) becomes

$$MSE\left(\hat{f}(x,h)\right) = Var\left(\hat{f}(x,h)\right) + \left(E\hat{f}(x,h) - f\right)^2. \tag{6.6}$$

At this point, it is important to explicitly find the variance term $Var\left(\hat{f}(x,h)\right)$ and the bias term $\left(E\hat{f}(x,h) - f\right)^2$ that appear in Eq.(6.6) in order to complete the proof. Expected value of $\hat{f}(x,h)$ is computed as follows

$$E\left(\hat{f}(x,h)\right) = E\left(\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)\right),$$

$$=\frac{1}{nh}\sum_{i=1}^{n}E\left(K\left(\frac{x-X_i}{h}\right)\right)=\frac{1}{h}E\left(K\left(\frac{x-X_i}{h}\right)\right). \tag{6.7}$$

Since the kernel density is a continuous function and the unknown function $f\in C^2$, which means is also continuous, the integral form of summation is used to expand Eq.(6.7) as follow

$$E\left(\hat{f}(x,h)\right)=\frac{1}{h}E\left(K\left(\frac{x-X_i}{h}\right)\right)=\frac{1}{h}\int K\left(\frac{x-t}{h}\right)f(t)\,dt\,. \tag{6.8}$$

Consider the following variable change, $\frac{x-t}{h}=u\Rightarrow t=x-hu$, it follows from Eq.(6.8) that

$$E\left(\hat{f}(x,h)\right)=\frac{1}{h}\int K\left(\frac{x-t}{h}\right)f(t)\,dt=\int K(u)f(x-hu)\,du\,. \tag{6.9}$$

The Taylor expansion of $f(x-hu)$ is needed in the next steps. Hence, the second order Taylor expansion of $f(x-hu)$ is given by the relation

$$f(x-hu)=f(x)-\frac{hu}{1!}f'(x)+\frac{(hu)^2}{2!}f''(x)+o\left(h^2\right). \tag{6.10}$$

Substituting Eq.(6.10) into Eq.(6.9), it follows that

$$\begin{aligned}E\left(\hat{f}(x,h)\right)&=\int K(u)f(x-hu)\,du\\&=\int K(u)\left[f(x)-\frac{hu}{1!}f'(x)+\frac{(hu)^2}{2!}f''(x)\right]du+o\left(h^2\right),\\&=f(x)\int K(u)du-\frac{h}{1!}f'(x)\int uK(u)du\\&\quad+\frac{h^2}{2!}f''(x)\int u^2K(u)du+o\left(h^2\right).\end{aligned} \tag{6.11}$$

Given that the kernel density K(.) has the properties [75,76] $\int_{-\infty}^{\infty} K(x)\,dx = 1$; $\int_{-\infty}^{\infty} xK(x)\,dx = 0$, Eq.(6.11) becomes

$$E\left(\hat{f}(x,h)\right) = f(x) + \frac{h^2}{2!} f''(x) \int u^2 K(u) du + o\left(h^2\right). \qquad (6.12)$$

The squared bias term appearing in Eq.(6.6) as $\left(E\hat{f}(x,h) - f\right)^2$, is derived from Eq.(6.12) as follow

$$\left(E\left(\hat{f}(x,h)\right) - f(x)\right)^2 = \left(\frac{h^2}{2!} f''(x) \int u^2 K(u) du\right)^2 + o\left(h^4\right). \qquad (6.13)$$

Recalling from Eq.(6.4) that $\mu_2(K) = \int t^2 K(t)\,dt$, it follows from Eq.(6.13) that ,

$$\left(E\left(\hat{f}(x,h)\right) - f(x)\right)^2 = \frac{h^4}{4}\left(f''(x)\right)^2 \mu_2^2(K) + o\left(h^4\right). \qquad (6.14)$$

Eq. (6.14) is the bias.

The variance of $\hat{f}(x,h)$ is computed below based on similar assumptions used for the bias computation [77].

$$Var\left(\hat{f}(x,h)\right) = Var\left(\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)\right) = \frac{1}{n^2} Var\left(\frac{1}{h}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)\right). \quad (6.15)$$

Placing the summation before the variance symbol in Eq.(6.15) leads to

$$\frac{1}{n^2} Var\left(\frac{1}{h}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var\left(K\left(\frac{x - X_i}{h}\right)\right) = \frac{1}{n} Var\left(K\left(\frac{x - X_i}{h}\right)\right). \quad (6.16)$$

Using the fact that the variance of a random variable *X* is the difference between the square expectation of *X* and its expectation squared, Eq.(6.16) becomes

$$\frac{1}{n} Var\left(K\left(\frac{x - X_i}{h}\right)\right) = \frac{1}{n}\left\{E\left(K^2\left(\frac{x - X_i}{h}\right)\right) - \left(E\left(K\left(\frac{x - X_i}{h}\right)\right)\right)^2\right\}. \quad (6.17)$$

Since the kernel density is a continuous function and the unknown function $f \in C^2$; also using the relation and transform given by Eq.(6.8), Eq.(6.9)into the Eq.(6.17) leads to

$$\frac{1}{n}\left(\frac{1}{h^2}\int K^2\left(\frac{x-t}{h}\right)f(t)dt-\left(\frac{1}{h}\int K\left(\frac{x-t}{h}\right)f(t)dt\right)^2\right). \tag{6.18}$$

The variable change $\frac{x-t}{h}=u \Rightarrow t=x-hu$ and the first order Taylor expansion $f(x-hu)=f(x)+o(h)$ of Eq. (6.18) leads to

$$\begin{aligned}&\frac{1}{n}\left(\frac{1}{h}\int K^2(u)f(x-hu)du-\left(f(x)+o(h)\right)^2\right)\\&=\frac{1}{n}\left(\frac{1}{h}\int K^2(u)f(x)du+o(h)-\left(f(x)+o(h)\right)^2\right).\end{aligned} \tag{6.19}$$

Recalling from Eq.(6.4) the notation $R(f)=\int f(t)^2 dt$, it follows from Eq.(6.19) that

$$Var\left(\hat{f}(x,h)\right)=\frac{1}{nh}R(K)f(x)+O\left((nh)^{-1}\right). \tag{6.20}$$

Recalling Eq.(6.6) in which the MSE is expressed as summation of variance and squared bias , and using the variance terms and squared bias term derived by Eq.(6.20) and Eq.(6.14)respectively, Eq.(6.6) is written as

$$MSE\left(\hat{f}(x,h)\right)=\frac{1}{nh}R(K)f(x)+\frac{h^4}{4}\left(f''(x)\right)^2\mu_2^2(K)+o\left(h^4\right)+O\left((nh)^{-1}\right). \tag{6.21}$$

Neglecting the small o and big O terms in Eq.(6.21) it follows that

$$MSE\left(\hat{f}(x,h)\right)\approx\frac{1}{nh}R(K)f(x)+\frac{h^4}{4}\left(f''(x)\right)^2\mu_2^2(K). \tag{6.22}$$

Eq.(6.22) is the mean square error. Integrating the MSE over the real line gives the mean integrated square error (MISE) which is

$$MISE\left(\hat{f}(x,h)\right)=\int MSE\left(\hat{f}(x,h)\right)dx$$
$$=\frac{1}{nh}R(K)+\frac{h^4}{4}\left(f''(x)\right)^2\mu_2^2(K) \quad (6.23)$$

The optimal value of the bandwidth is obtained by finding the critical point of Eq.(6.23), which obviously is a local maximum. The partial derivative with respect to *h* is found and set to 0.

$$\frac{\partial MISE\left(\hat{f}(x,h)\right)}{\partial h}=-\frac{1}{nh^{-2}}R(K)+h^3\left(f''(x)\right)^2\mu_2^2(K)=0\,. \quad (6.24)$$

$$Opt_{AMISE}(h)=\sqrt[5]{\left(\frac{R(K)}{\mu_2(K)^2R(f'')n}\right)}\,. \quad (6.25)$$

The curvature of *f* given by its second derivative f'' determines the nature of the bandwidth. The data will be over smoothed or under smoothed, if *f* is assumed to have a small curvature or large curvature respectively. The normality assumption of the unknown *f* is required when Eq.(6.25) is used in application.

Lemma 6.3 Assume that *f* is a Gaussian (normally distributed) function with variance σ^2 and mean μ. Then a robust estimation of the optimal bandwidth value *h* is given by [77, 78]

$$Opt_{AMISE}(h)=1.06\times n^{-1/5}\times\min\left(\frac{R}{1.34},\sigma\right). \quad (6.26)$$

R and σ are the inter-quartile range and the standard deviation of the distribution, their respective approximated values $\hat{R}$ and $\hat{\sigma}$ can be computed from the sample data. Moreover, the approximations are in practice used in computation.

Definition 6.4 Bias-variance tradeoff, there exists a tradeoff between the bias and the variance terms found in Eq.(6.3) and Eq.(6.6) respectively. For an over smoothed data set, the variance is small and the bias term is large; whereas for under smoothed data, the variance is large and the bias term is small.

Theorem 6.5 The optimal value of the bandwidth *h* can be computed using iterative approximation. From the bias-variance tradeoff, let us define the following functions from $\mathbb{R}_+ \to \mathbb{R}_+ : g : h \mapsto g(h) = Var\left(\hat{f}(x,h)\right)$ and $u : h \mapsto u(h) = \left(E\hat{f}(x,h) - f\right)^2$. The solution to the equation $g(h) = u(h)$ is the optimal bandwidth value h_{opt}.

Proof of Theorem 6.5: The function $g(h)$ is a strictly increasing function with $\lim_{h\to 0} g(h) = 0$ and $\lim_{h\to\infty} g(h) = \infty$. On the other hand, $u(h)$ is a strictly decreasing function with $\lim_{h\to 0} u(h) = \infty$ and $\lim_{h\to\infty} u(h) = 0$. These imply the existence and uniqueness of a solution to $g(h) = u(h)$. A bias-variance tradeoff graph can also support this result graphically.

Lemma 6.6 Given (T,Y) a 2-dimmensional random variable, let $(t_1, y_1), (t_2, y_2), \ldots, (t_m, y_m)$ be an empirical sample data representing *m* observations of (T,Y). The Nadaraya-Watson regression function of Y on T is defined by

$$\hat{Y}_m(t) = \frac{\sum_{k=1}^{m} y_k K\left(\frac{t-t_k}{h_n}\right)}{\sum_{k=1}^{m} K\left(\frac{t-t_k}{h_n}\right)}. \tag{6.27}$$

In Eq.(6.27), h_n and K are the bandwidth and kernel function respectively [73].

6.2 Logistic and exponential Growth Models

Since the aim is to carry out deterministic and probabilistic estimations and compare the results, it is considered useful to introduce some of deterministic growth models. Logistic and exponential growth models are built using both classical and fractional approaches of differential equations.

6.2.1 Exponential Growth and Fractional Exponential Growth Models

The general idea that leads to the building of an Exponential growth equation is the following [79]. The population size at the time t of a certain species is a function of time and it is defined by

$$P = \varphi(t). \tag{6.28}$$

Usually, it assumes that the rate of change of a population size is proportional to the size at the current state. This means

$$\frac{dP}{dt} = \lambda P \ . \tag{6.29}$$

The parameter λ is often considered as the difference between the population's birth rate B and its mortality rate M. That is $\lambda = B - M$.

Note: The constant λ, is known as the rate of decline or the rate of growth, meaning that λ is negative or positive respectively. $\lambda > 0$ leads to exponential growth and $\lambda < 0$ leads to exponential decay.

Eq.(6.29) is solved below using two approaches. The first is the ordinary differential equation with integer order derivative and the second is the fractional differential approach.

1st approach: First order ordinary differential equation.

From Eq.(6.29) it follows that $\frac{dP}{dt} = \lambda P \Leftrightarrow P' = \lambda P \Rightarrow \frac{P'}{P} = \lambda$, integrating both sides with respect to the variable *t* as $\int \frac{P'}{P} dt = \int \lambda dt \Rightarrow \ln(P) = \lambda t + c$ leads to the general form of the model

$$P = \exp(\lambda t + c) \Rightarrow P = \exp(c)\exp(\lambda t). \tag{6.30}$$

The constant $\exp(c)$ represents the population size at the initial time *t=0*. The initial population size is usually denoted by $P(0) = P_0$; hence, Eq.(6.30) takes the common form of

$$P = P_0 \exp(\lambda t). \tag{6.31}$$

2nd approach: Fractional differential equations

Using similar assumptions to the 1st approach, the fractional rate of change using the Caputo fractional derivative is

$${}_0^c D_u^{\gamma} P(t) = \lambda P(t). \tag{6.32}$$

A general form of the solution to Eq.(6.32) is given by

$$P(t) = P_0 E_{\gamma}\left(\lambda t^{\gamma}\right). \tag{6.33}$$

6.2.2 Logistic Growth and Fractional Logistic Growth Model.

Similar assumptions used for building the exponential growth model are used in building logistic growth model. However the only difference is that the population rate of change is not constant but it is a function of the current population size. The classical and the fractional approaches to the solution will be presented respectively.

Classical approach

$$\frac{dP}{dt} = f(P)P, \tag{6.34}$$

with *f (P)* referring to a function of *P*.

Note: The function $f\left(P\right)$ in Eq.(6.34) is chosen such that, when *P* is sufficiently large $f\left(P\right)<0$, when *P* grows larger $f\left(P\right)$ decreases and finally for sufficiently small value of *P*, $f\left(P\right)$ is almost constant. For this purpose a suitable choice is a linear function with the form

$$f\left(P\right)=-\lambda P+\kappa. \tag{6.35}$$

Solution to Eq.(6.34) is built as below. Substituting Eq.(6.35) into Eq.(6.34) leads to

$$\frac{dP}{dt}=\left(-\lambda P+\kappa\right)P \Leftrightarrow \frac{dP}{dt}=\kappa\left(-\frac{P}{\gamma}+1\right)P, \tag{6.36}$$

with $\lambda=\frac{\kappa}{\gamma}$, where $\lambda,\ \kappa,\ \gamma$ are constants.

Note: In the absence of any possible limiting factor due to the population size, the growth rate defined above by Eq.(6.35) is represented only by the constant κ. In this case, κ is known as the intrinsic growth rate [79]. The quantity γ is known as the carrying capacity.

Using the variable separation technique, Eq.(6.36) is written as

$$\frac{dP}{\left(-\frac{P}{\gamma}+1\right)P}=\kappa dt \Rightarrow \int\frac{dP}{\left(-\frac{P}{\gamma}+1\right)P}=\int\kappa dt. \tag{6.37}$$

Few steps of algebraic manipulation leads to the analytic solution of Eq.(6.37), given by

$$P=\frac{P_0\gamma}{P_0+\left(\gamma-P_0\right)e^{-\kappa t}}. \tag{6.38}$$

Setting $\beta = \frac{\gamma - P_0}{P_0}$, the general solution is given in a more compact form as

$$P = \frac{\gamma}{1 + \beta e^{-\kappa t}}. \tag{6.39}$$

Fractional model approach

Assuming that the rate of change is fractional, Eq.(6.34) can be written in the form

$$D^{\gamma} P(t) = \lambda P(t)\big(1 - P(t)\big). \tag{6.40}$$

Several methods have been proposed to solve the fractional logistic differential equation [6, 7, 80, 81, 82, 83] , most of which used numerical approach. In chapter 4 several numerical approaches were proposed to solve Eq.(6.40).

6.3 Numerical Simulation

In this section a numerical simulation is undertaken aiming to illustrate and compare the results of data modeling using a probabilistic approach or a deterministic approach. A similar comparative study in the fields of Differential equation was proposed by [31] in which the fractional differential equation has been proven more efficient than integer differential equations in solving problem. The mean square error (MSE) term is used for the evaluation of the performance of used models. Let $(P_i)_{1 \le i \le n}$ represent *n* observations of random variable and $(\hat{P}_i)_{1 \le i \le n}$ be the estimation of $(P_i)_{1 \le i \le n}$ through modeling approach. Then MSE term is computed by

$$MSE = \sum_{i=1}^{n} \left(P_i - \hat{P}_i\right)^2. \tag{6.41}$$

The data set used for the simulation purpose comes from [74]. It gives the world's population in billions from 1910 up to 2010 and forecasted values up to 2050.

Case 1: we consider a realistic part of the data from [74], which gives the true population figures of the world from 1910 to 2010. Then the integer, the fractional order differential equation and kernel smoothing are used for exponential growth modeling. From Figure 6.1, it is evident that the data exhibits an exponential shape.

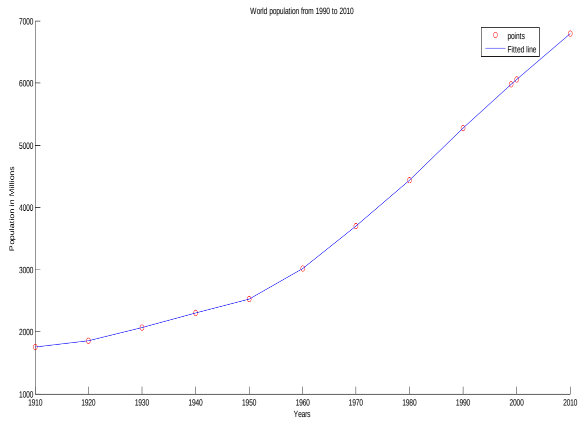


Figure 6.1: World population from 1910 to 2010

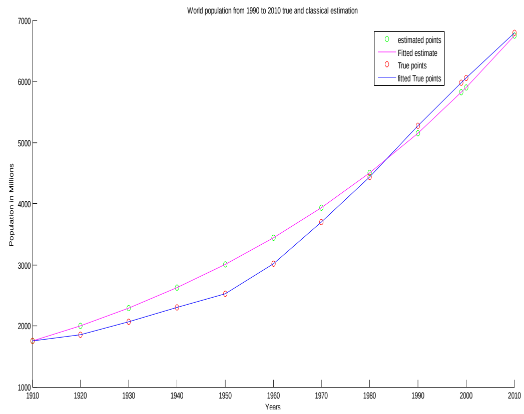


Figure 6.2: World population from 1910 to 2010 estimated by exponential equation.

Recall Eq.(6.31), representing the exponential growth using the classical approach, the estimation of the world population from 1910 to 2010 using the classical approach of differential equation is shown in Figure 6.2 together with the true data values from 1910 to 2010. The total MSE=0.9132 billion.

Consider Eq.(6.33), that gives the general solution to the fractional exponential growth differential equation. The constant $\gamma = 1.3933$ is given in [31] and also confirmed using MATLAB to be correct, that gives the best estimation of the fractional derivative. Figure 6.3shows the estimated curve together with the curve obtained from true data. In this case MSE=0.2051 billion.

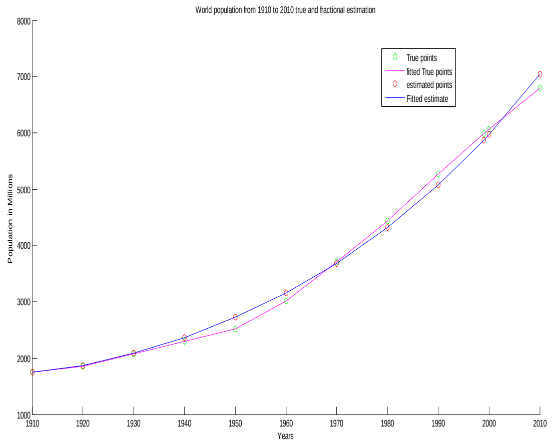


Figure 6.3World population from 1910 to 2010 estimated by fractional differential equation.

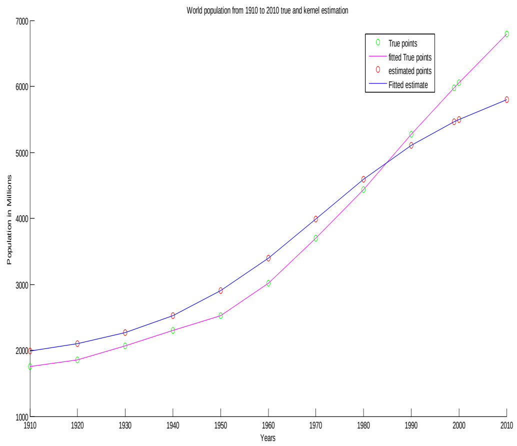


Figure 6.4 World population from 1910 to 2010 estimated by kernel smoothing with *h=21*.

The Gaussian kernel smoothing technique is used and obtained results are given in Figure 6.4. Optimal bandwidth to be used is computed using Eq. (6.26), resulting in h_{opt} =21. MSE= 2.197 billion is obtained.

Knowing that theoretically computed bandwidth (*h*) tend to result in large error levels, errors obtained from ordinary and fractional exponential models were taken as reference. Using an iterative algorithm error levels were computed for different *h* values using the Gaussian kernel model. It was observed that the error level obtained from ordinary exponential model (MSE=0.9132) was obtained at around *h=15.5*. Similarly the error level of fractional exponential model (MSE=0.2051) was reached at *h*=8.5. Corresponding estimated values and true values are shown in Figures 6.5 and 6.6.

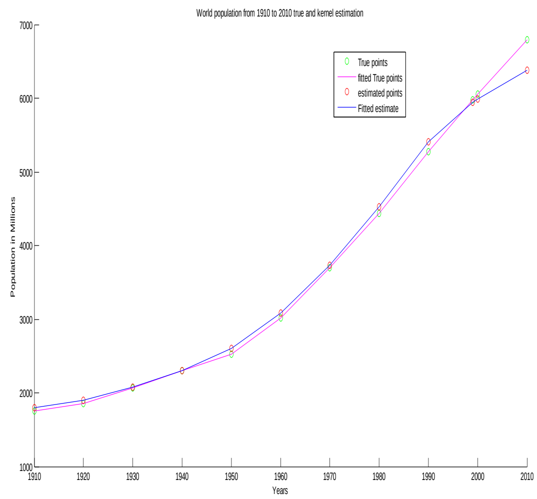


Figure 6.5: World population from 1910 to 2010 estimated by kernel smoothing with *h=8.5*

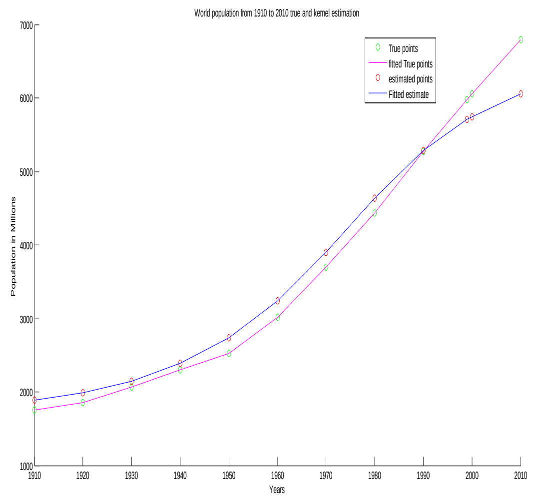


Figure 6.6: World population from 1910 to 2010 estimated by kernel smoothing with *h=15.5*.

Case 2: The data obtained from [74], included population annual projections until the year 2050. The graph of this data (1900 to 2050) exhibited a logistic model type trend. The data set of interest is from year 1900 to 2050. Using the Eq. (6.39) estimated values were computed using parameter values $\gamma = 21.9050$ and $\kappa = 0.0148$. Obtained MSE=2.2731, and the same MSE obtained using fractional logistic method when the derivative order is 1. For order values different than 1, level of error were larger. Figure 6.7 shows the logistic shape of the data.

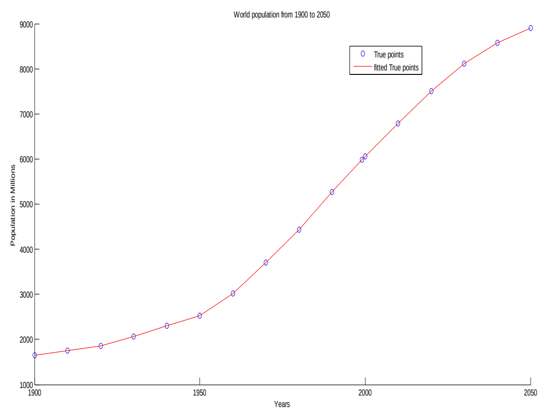


Figure 6.7: World population from 1900 to 2050

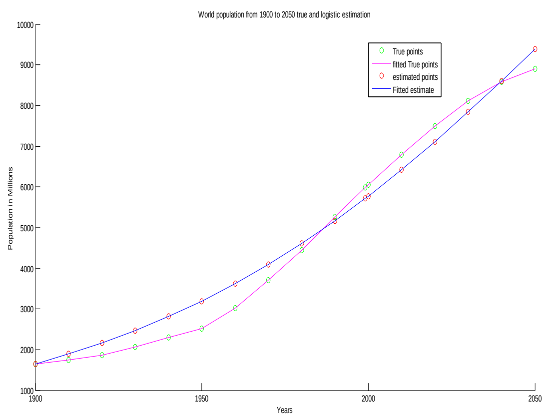


Figure 6.8: World population from 1900 to 2050 estimated by logistic approach.

Using the Gaussian kernel method the where the bandwidth was computed as *h=27.95* from Eq. (6.26), resulted in an MSE=5.122. Using the iterative method, an error level around that of the Logistic method (2.2731) was achieve for *h=22*.

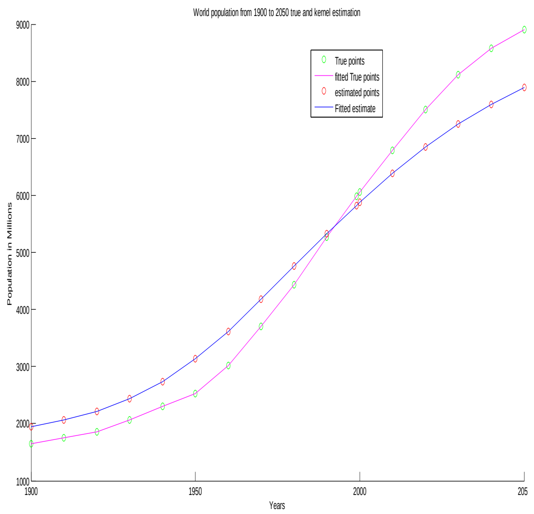


Figure 6.9: World population from 1900 to 2050 estimated by kernel approach with *h=27.9454*.

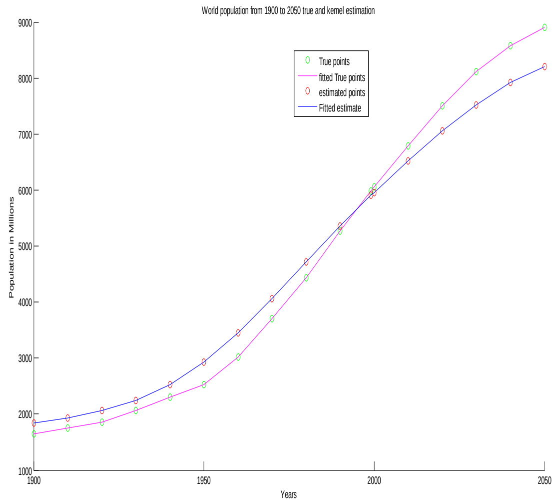


Figure 6.10: World population from 1900 to 2050 estimated by kernel approach with *h=22*.

Table 6.1: error terms of estimations

	MSE		
	Integer derivative approach	Fractional derivative approach	Kernel smoothing approach
Exponential model	*0.9132*	*0.2051*	*2.197 for h=21* *0.2164 for h=8.5* *0.9430 for h=15.5*
Logistic Model	*2.2731*	*2.2731*	*5.122 for h=27.95* *2.3269 for h=22*

Chapter 7

CONCLUSION

The work focused on the mathematical modeling of natural phenomena. In particular, the logistic growth model was at the center of the study. Deterministic methods (classical and fractional differential equations) and non-deterministic methods (non-parametric, kernel smoothing) were used throughout the study. Deterministic approach does not cater for the random element in any process, and this may lead to considerable error in modeling. The non-deterministic modeling is designed to handle the random element in a process, therefore providing an idea about the possible magnitude of error involved in the model. Randomness of the non-deterministic model doesn’t mean a trendless output, but rather the level of chance for the possible outcome that can be expected.

Following the review of important concepts in fractional calculus, one main achievement was the proof of the existence and uniqueness of the FLDE. Since the FLDE doesn’t have analytic solution, numerical schemes, such as the CF method, the PSE or LM method and the GEM are used to compute approximate solutions to the FLDE.

Another important work in this thesis is the study of a logistic growth model with a large carrying capacity. It is known that when the carrying capacity approaches infinity, the logistic growth model coincides with the exponential growth model. In chapter 5 an application for the logistic growth model with large carrying capacity with an application to the Chinese population growth modeling is given.

In Chapter 5 it was determined that the Ψ-Caputo derivative with Rayleigh kernel function performed better compared with the classical logistic or exponential models, based on obtained MSE values.

It is difficult to give a practical meaning to a fractional order derivative. In this study a simulation to determine the error levels obtained using CF, PSE and GEM numerical algorithms to find the optimum solution of the FLDE, indicated that only the CF method resulted in minimum error for the fractional derivative order $q=1.005$. The PSE and GEM methods achieved minimum error levels at $q=1$ which is the same as the classical approach.

Non-parametric kernel smoothing was used as a non-deterministic model in modeling data set that would be fitted by the deterministic logistic growth model. However, it is generally observed that the optimal bandwidth determined using theoretical formulae, tends to result in large estimation errors. A trial an error method could be used in selecting optimal bandwidth, where the balance between bias and variance is observed, and error level minimized.

REFERENCES

[1] Tashiro, T., & Yoshimura, F. (2019). A neo-logistic model for the growth of bacteria. *Physica A: Statistical Mechanics and its Applications*.

[2] Harris, T. M., Devkota, J. P., Khanna, V., Eranki, P. L., & Landis, A. E. (2018). Logistic growth curve modeling of US energy production and consumption. *Renewable and Sustainable Energy Reviews*, *96*, 46-57.

[3] Banks, H. T., Collins, E., Flores, K., Pershad, P., Stemkovski, M., & Stephenson, L. (2017). Statistical error model comparison for logistic growth of green algae (Raphidocelis subcapitata). *Applied Mathematics Letters*, *64*, 213-222.

[4] Kwasnicki, W. (2013). Logistic growth of the global economy and competitiveness of nations. *Technological Forecasting and Social Change*, *80*(1), 50-76.

[5] Nguimkeu, P. (2014). A simple selection test between the Gompertz and Logistic growth models. *Technological Forecasting and Social Change*, *88*, 98-105.

[6] West, B. J. (2015). Exact solution to fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*, *429*, 103-108.

[7] Area, I., Losada, J., & Nieto, J. J. (2016). A note on the fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*, *444*, 182-187.

[8] Noupoue, Y. Y. Y., Tandoğdu, Y., & Awadalla, M. (2019). On numerical techniques for solving the fractional logistic differential equation. *Advances in Difference Equations*, *2019*(1), 108.

[9] Malthus, T. R. (1959). *Population: the first essay* (Vol. 31). University of Michigan Press.

[10] Cain, J. W. (2014). Mathematical models in the sciences. *Molecular Life Sciences: An Encyclopedic Reference*, 1-6.

[11] Chicone, C. (2006). *Ordinary differential equations with applications* (Vol. 34). Springer Science & Business Media.

[12] Shampine, L. F. (2018). *Numerical solution of ordinary differential equations*. Routledge.

[13] Pinsky, M., & Karlin, S. (2010). *An introduction to stochastic modeling*. Academic press.

[14] Arnold, L. (1974). Stochastic differential equations. *New York*.

[15] Yong, Z. H. O. U., Jinrong, W., & Lu, Z. (2016). *Basic theory of fractional differential equations*. World Scientific.

[16] Bo, H. U., Yudun, L. I., Hejun, Y. A. N. G., & He, W. A. N. G. (2017). Wind speed model based on kernel density estimation and its application in reliability

assessment of generating systems. *Journal of Modern Power Systems and Clean Energy*, *5*(2), 220-227.

[17] Chen, Q., Wynne, R. J., Goulding, P., & Sandoz, D. (2000). The application of principal component analysis and kernel density estimation to enhance process monitoring. *Control Engineering Practice*, *8*(5), 531-543.

[18] Hong, L. J., Juneja, S., & Liu, G. (2017). Kernel smoothing for nested estimation with application to portfolio risk measurement. *Operations Research*, *65*(3), 657-673.

[19] Debnath, L. (2004). A brief historical introduction to fractional calculus. *International Journal of Mathematical Education in Science and Technology*, *35*(4), 487-501.

[20] Mahmudov, N. I., Awadalla, M., &Abuassba, K. (2017). Nonlinear sequential fractional differential equations.

[21] Mahmudov, N. I., Awadalla, M., &Abuassba, K. (2017). Hadamard and Caputo-Hadamard FDE's with Three Point Integral Boundary Conditions.

[22] Inc, M., Yusuf, A., Aliyu, A. I., & Baleanu, D. (2018). Investigation of the logarithmic-KdV equation involving Mittag-Leffler type kernel with Atangana–Baleanu derivative. *Physica A: Statistical Mechanics and its Applications*, *506*, 520-531.

[23] Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (Vol. 198). Elsevier.

[24] Sabatier, J. A. T. M. J., Agrawal, O. P., & Machado, J. T. (2007). *Advances in fractional calculus* (Vol. 4, No. 9). Dordrecht: Springer.

[25] Oldham, K., & Spanier, J. (1974). *The fractional calculus theory and applications of differentiation and integration to arbitrary order* (Vol. 111). Elsevier.

[26] Gorenflo, R., Kilbas, A. A., Mainardi, F., & Rogosin, S. V. (2014). *Mittag-Leffler functions, related topics and applications* (Vol. 2). Berlin: Springer.

[27] Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. Springer Science & Business Media.

[28] Caputo, M., &Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl*, *1*(2), 1-13.

[29] Pooseh, S., Almeida, R., & Torres, D. F. (2012). Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative. *Numerical Functional Analysis and Optimization, 33*(3), 301-319.

[30] Almeida, R. (2017). A Caputo fractional derivative of a function with respect to another function. *Communications in Nonlinear Science and Numerical Simulation*, *44*, 460-481.

[31] Almeida, R., Bastos, N. R., & Monteiro, M. T. T. (2016). Modeling some real phenomena by fractional differential equations. *Mathematical Methods in the Applied Sciences*, *39*(16), 4846-4855.

[32] Almeida, R., Malinowska, A. B., & Monteiro, M. T. T. (2018). Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Mathematical Methods in the Applied Sciences*, *41*(1), 336-352.

[33] M. Awadalla, M., &Yameni Y. Y. (2018). Modeling Exponential Growth and Exponential Decay Real Phenomena by Ψ-Caputo Fractional Derivative, *Journal of Advances in Mathematics and Computer Science*,28(2), 2456-9968 , DOI : 10.9734/JAMCS/2018/43054.

[34] Yusuf, A., Qureshi, S., Inc, M., Aliyu, A. I., Baleanu, D., & Shaikh, A. A. (2018). Two-strain epidemic model involving fractional derivative with Mittag-Leffler kernel. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *28*(12), 123121

[35] Qureshi, S., Yusuf, A., Shaikh, A. A., Inc, M., & Baleanu, D. (2019). Fractional modeling of blood ethanol concentration system with real data

application. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *29*(1), 013143.

[36] Yusuf, A., Inc, M., Aliyu, A. I., & Baleanu, D. (2018). Efficiency of the new fractional derivative with nonsingular Mittag-Leffler kernel to some nonlinear partial differential equations. *Chaos, Solitons & Fractals*, *116*, 220-226.

[37] Ullah, A., & Shah, K. (2018). Numerical analysis of Lane Emden–Fowler equations. *Journal of Taibah University for Science*, *12*(2), 180-185.

[38] Shah, K., Ali, A., & Khan, R. A. (2015). Numerical solutions of Fractional Order system of Bagley-torvik Equation using Operational Matrices. *Sindh University Research Journal-SURJ (Science Series)*, *47*(4).

[39] Khalil, H., Shah, K., & Khan, R. A. (2017). Approximate solution of boundary value problems using shifted Legendre polynomials. *Applied and Computational Mathematics*, *16*(3), 269-285.

[40] Inc, M., Abdel-Gawad, H. I., Tantawy, M., & Yusuf, A. On multiple soliton similariton-pair solutions, conservation laws via multiplier and stability analysis for the Whitham–Broer–Kaup equations in weakly dispersive media. *Mathematical Methods in the Applied Sciences*.

[41] Abdel-Gawad, H. I., Tantawy, M., Inc, M., & Yusuf, A. (2018). On multi-fusion solitons induced by inelastic collision for quasi-periodic propagation with

nonlinear refractive index and stability analysis. *Modern Physics Letters B*, *32*(29), 1850353.

[42] Ghanbari, B., Yusuf, A., & Baleanu, D. (2019). The new exact solitary wave solutions and stability analysis for the (2+1)-dimensional Zakharov–Kuznetsov equation. *Advances in Difference Equations*, *2019*(1), 49.

[43] Bushnaq, S., Khan, S. A., Shah, K., & Zaman, G. (2018). existence theory of hiv-1 infection model by using arbitrary order derivative of without singular kernel type. *Journal of Mathematical Analysis*, *9*(1), 16-28.

[44] Aliyu, A. I., Inc, M., Yusuf, A., & Baleanu, D. (2018). A fractional model of vertical transmission and cure of vector-borne diseases pertaining to the Atangana–Baleanu fractional derivatives. *Chaos, Solitons & Fractals*, *116*, 268-277.

[45] Petropoulou, E. (2010). A discrete equivalent of the logistic equation. *Advances in Difference Equations*, *2010*(1), 457073.

[46] West, B. J. (2015). Exact solution to fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*, *429*, 103-108.

[47] Area, I., Losada, J., & Nieto, J. J. (2016). A note on the fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*, *444*, 182-187.

[48] D'Ovidio, M., Loreti, P., &Ahrabi, S. S. (2018). Modified fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*, *505,* 818-824.

[49] Vivek, D., Kanagarajan, K., &Harikrishnan, S. (2013). Numerical solution of fractional-order logistic equations by fractional Euler's method.

[50] Odibat, Z. M., &Momani, S. (2008). An algorithm for the numerical solution of differential equations of fractional order. *Journal of Applied Mathematics & Informatics*, *26*(1_2), 15-27.

[51] Petráš, I. (2011). *Fractional-order nonlinear systems: modeling, analysis and simulation*. Springer Science & Business Media.

[52] Atangana, A., &Owolabi, K. M. (2018). New numerical approach for fractional differential equations. *Mathematical Modelling of Natural Phenomena*, *13*(1), 3.

[53] Bastos, N. R. (2018). Calculus of variations involving Caputo-Fabrizio fractional differentiation. *Statistics*, *2310*, 5070.

[54] Malthus, T. R. (1888). *An essay on the principle of population: or, A view of its past and present effects on human happiness*. Reeves & Turner.

[55] Reed, H. S., & Holland, R. H. (1919). The growth rate of an annual plant Helianthus. *Proceedings of the National Academy of Sciences*, *5*(4), 135-144.

[56] Tandoğdu, Y., & Erbilen, M. (2018). Imputing Missing Values Using Support Variables with Application to Barley Grain Yield. *Journal of Agricultural Science and Technology*, *20*(4), 829-839.

[57] Kilbas, A. A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations* (Vol. 204). Elsevier Science Limited.

[58] Area, I., Batarfi, H., Losada, J., Nieto, J. J., Shammakh, W., & Torres, Á. (2015). On a fractional order Ebola epidemic model. *Advances in Difference Equations*, *2015*(1), 278.

[59] Sofuoglu, Y., & Ozalp, N. U. R. I. (2014). A fractional order model on bilingualism. Commun. Fac. Sci. Univ. Ank. Ser. A, 2014, 81-89.

[60] Sofuoglu, Y., & Ozalp, N. (2017). Fractional order bilingualism model without conversion from dominant unilingual group to bilingual group. *Differential Equations and Dynamical Systems*, *25*(1), 1-9.

[61] Rudolf, H. (Ed.). (2000). *Applications of fractional calculus in physics*. World Scientific.

[62] Mainardi, F. (1996). Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos, Solitons & Fractals*, *7*(9), 1461-1477.

[63] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). Fractional integrals and derivatives: theory and applications.

[64] Debnath, L. (2003). Recent applications of fractional calculus to science and engineering. *International Journal of Mathematics and Mathematical Sciences*, *2003*(54), 3413-3442.

[65] Koh, C. G., & Kelly, J. M. (1990). Application of fractional derivatives to seismic analysis of base-isolated models. *Earthquake engineering & structural dynamics*, *19*(2), 229-241.

[66] Courant, R., & John, F. (2012). *Introduction to calculus and analysis I*. Springer Science & Business Media.

[67] Kishore, N. (1963). The Rayleigh function. *Proceedings of the American Mathematical Society*, *14*(4), 527-533.

[68] Maymon, G. (2018). *Stochastic Crack Propagation: Essential Practical Aspects*. Academic Press.

[69] Rayleigh, L. (1878). On the instability of jets. *Proceedings of the London mathematical society*, *1*(1), 4-13.

[70] Malthus, T. R., & Pullen, J. (1989). *TR Malthus: Principles of Political Economy* (Vol. 2). Cambridge University Press.

[71] Population, total. (n.d.). Retrieved from https://data.worldbank.org/indicator/SP.POP.TOTL?locations=CN.

[72] Kuang, Y. (Ed.). (1993). *Delay differential equations: with applications in population dynamics* (Vol. 191). Academic Press.

[73] Nadaraya, E. A. (1964). On estimating regression. *Theory of Probability & Its Applications*, *9*(1), 141-142.

[74] Index of /esa/population/publications/sixbillion. (n.d.). Retrieved October 11, 2017, from http://www.un.org/esa/population/publications/sixbillion/.

[75] Wand, M. P., & Jones, M. C. (1994). *Kernel smoothing*. Crc Press.

[76] Zambom, A. Z., & Dias, R. (2012). A review of kernel density estimation with applications to econometrics. *arXiv preprint arXiv:1212.2812*.

[77] Härdle, W. (2012). *Smoothing techniques: with implementation in S*. Springer Science & Business Media.

[78] Silverman, B. W. (1986). *Density estimation for statistics and data analysis* (Vol. 26). CRC press.

[79] Boyce, W. E., DiPrima, R. C., & Haines, C. W. (1969). *Elementary differential equations and boundary value problems* (Vol. 9). New York: Wiley.

[80] Khader, M. M., & Babatin, M. M. (2013). On approximate solutions for fractional Logistic differential equation. *Mathematical Problems in Engineering*, *2013*.

[81] El Raheem, Z. F., & Salman, S. M. (2014). On a discretization process of fractional-order logistic differential equation. *Journal of the Egyptian Mathematical Society*, *22*(3), 407-412.

[82] El-Sayed, A. M. A., El-Mesiry, A. E. M., & El-Saka, H. A. A. (2007). On the fractional-order logistic equation. *Applied Mathematics Letters*, *20*(7), 817-823.

[83] Sweilam, N. H., Khader, M. M., & Mahdy, A. M. S. (2012). Numerical studies for solving fractional-order Logistic equation. *International Journal of Pure and Applied Mathematics*, *78*(8), 1199-1210.

[84] Avigad, J., & Donnelly, K. (2004, July). Formalizing O notation in Isabelle/HOL. In *International Joint Conference on Automated Reasoning* (pp. 357-371). Springer, Berlin, Heidelberg.

[85] Rinne, H. (2008). *The Weibull distribution: a handbook*. Chapman and Hall/CRC.

[86] Stacy, E. W., & Mihram, G. A. (1965). Parameter estimation for a generalized gamma distribution. *Technometrics*, *7*(3), 349-358.

[87] Liu, Y., & Li, Z. (2008). Krasnoselskii type fixed point theorems and applications. *Proceedings of the American Mathematical Society*, *136*(4), 1213-1220.

[88] Dutta, P. N., & Choudhury, B. S. (2008). A generalization of contraction principle in metric spaces. *Fixed Point Theory and Applications*, *2008*(1), 406368.

[89] Dhage, B. C. (2006). On some nonlinear alternatives of Leray-Schauder type and functional integral equations. *Arch. Math.(Brno)*, *42*(1), 11-23.

[90] Ackerman, E., & Gatewood, L. C. (1979). *Mathematical Models in the Health Sciences: A Computer Aided Approach*. U of Minnesota Press.

[91] Dunbar, S. R. (2019). *Mathematical Modeling in Economics and Finance: Probability, Stochastic Processes, and Differential Equations* (Vol. 49). American Mathematical Soc.

[92] Zhang, P., Song, D., Wang, J., & Hou, Y. (2014). Bias–variance analysis in estimating true query model for information retrieval. *Information Processing & Management*, *50*(1), 199-217.

[93] Nachbar, J. (2016). The Arzela-Ascoli Theorem.

APPENDICES

Appendix A: Arzela-Ascoli Theorem and proof

This theorem is usually stated in different forms. Below are given the two forms that are the commonly used. Denote by $C(X,\mathbb{C})$, the set of continuous functions from X to $\mathbb{C}$.

Form 1: A subset S of $C(X,\mathbb{C})$ is compact if and only if it is bounded, closed and equicontinuous.

Form 2: If a sequence $\{g_n\}_1^\infty$ in $C(X,\mathbb{C})$ is bounded and equicontinuous then it has a uniformly convergent subsequence.

Note: boundedness and equicontinuity are mathematically written respectively as

i) '$S \subset C(X,\mathbb{C})$ *is bounded*' means that there is a constant $0 < \lambda < \infty$, such that $|g(x)| < \lambda$, for $x \in X$ and $g \in S$.

ii) '$S \subset C(X,\mathbb{C})$ *is equicontinuous*' means that : $\forall \varepsilon > 0$, $\exists \delta(\varepsilon)$ such that $\forall x, y \in X : d(x,y) < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$, $\forall f \in S$.

Proof: see [93]

Appendix B: Krasnoselskii's fixed point theorem

Let S be a closed, bounded and convex nonempty subset of a Banach space $\left(Y,\|.\|\right)$.

Moreover assume that U and V are continuous application that map S into Y, such that [87]

i) $\left(I-U\right)(S)\subset V(S)$,

ii) $\left(I-U\right)(S)$ is contained in a compact subset of S.

iii) If $Vx_n \to y$ then there is a convergent subsequence $\left\{x_{nk}\right\}\subset\left\{x_n\right\}$,

iv) $\forall y\in Range(V)$, $D_y=\left\{x\in S: Vx=y\right\}$ isa convex set.

Then there is $y\in S$, $y=Uy+Vy$.

Proof: See [87]

Appendix C: Banach contraction mapping theorem

See [88] for details information about the theorem which is stated as follow:

Consider a complete metric space (X,d). A mapping $P:X \rightarrow X$ is said to be a contraction mapping if $\forall x_1, x_2 \in X$, $d(Px_1, Px_2) \leq \lambda d(x_1, x_2)$. With $0 < \lambda < 1$.

Proof: See [88].

Appendix D: Leray-Schauder nonlinear alternative theorem

See [89] for detailed information about the theorem which is stated as follow.

Denote by V and $\bar{V}$ respectively the open and the closed subset of a convex U of a normed and linear space Y such that $0 \in V$. Moreover let $P : \bar{V} \to U$ be a continuous and compact operator. Then either

a) Equation $Px = x$ has a solution in the closed set $\bar{V}$, or
b) There is a point $v \in \partial V$ such that $v = \rho P v$; with $\rho \in (0,1)$ and ∂V represents the boundary of V.

Proof: See [89].