On the ω-Multiple Charlier Polynomials

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ABSTRACT

Appell polynomials are certain family having wide range of application areas from numerical analysis to analytic function theory. They were defined by Paul Emile Appell in 1880. The most famous Appell polynomials are Hermite, Bernoulli and Euler polynomials.

This thesis consists of five chapters. Chapter 1 is devoted to Introduction. In Chapter 2, we give basic definitions and properties that is used throughout the thesis. Chapters 3, 4 and 5 are original.

In Chapter 3, we introduce 3D-ω-Hermite Appell polynomials $\mathcal{A}_n(x,y,z;\omega)$ using ω-Hermite polynomials $\mathfrak{H}_n^{\omega}(x,y,z)$. We obtain their explicit forms, determinantal representations, recurrence relations, lowering and raising operators, difference equations, integro-difference equations, and partial difference equations.

In Chapter 4, we introduce the Δ_ω-multiple Appell polynomials and we give an explicit representation and recurrence relations for them.

In the last chapter, we define ω-multiple Charlier polynomials then we give raising operator, Rodrigues formula, explicit representation and generating function. Also an $(r+1)th$ order difference equation is given. As an example we consider the case $\omega = \frac{3}{2}$ and define $\frac{3}{2}-$multiple Charlier polynomials.

Keywords: Δ_ω-Appell polynomials, determinant, recurrence equation, Multiple orthogonal polynomials, ω- multiple Charlier polynomials, Appell polynomials,

Hypergeometric function, Rodrigues formula, Generating function, Difference equation.

ÖZ

Appell polinomları, sayısal analizden analitik fonksiyon teorisine kadar geniş uygulama alanlarına sahip belirli bir ailedir. 1880 yılında Paul Emile Appell tarafından tanımlanmıştır. En ünlü Appell polinomları, Hermite, Bernoulli ve Euler polinomlarıdır.

Bu tez beş bölümden oluşmaktadır. Bölüm 1'de giriş kısmı verilmiştir. Bölüm 2'de ω-ileri fark operatörü, ω-geri fark operatörü ve ω-kısmi toplam formülü tanımlanmış ve ω-hipergeometrik türde fark denklemi elde edilmiştir. 3., 4. ve 5. bölümler orijinaldir.

Bölüm 3'de ω-Hermite polinomlarını kullanarak 3D-ω-Hermite Appell polinomları tanımlanmış ve bu polinomlar için açık formlar, determinant, rekürans bağıntıları, indirgeme ve artırma operatörleri, fark denklemleri, tam fark denklemleri verilmiştir. Ayrıca bunlar tarafından sağlanan kısmi fark denklemleri elde edilmiştir.

Bölüm 4'de Δ_ω-Hermite Appell polinomları tanımlanmış, açık form ve rekürans bağıntıları verilmiştir.

Son bölümde ω katlı Charlier polinomları tanımlanmış ve bu polinomlar için yükseltme operatörü, Rodrigues formülü, açık form, üretici fonksiyon elde edilmiştir. Ek olarak, bu tezde incelenen teoriler örneklerle gösterilmiştir.

Anahtar Kelimeler: Δ_ω-Appell polinomları, determinant, yineleme denklemi, Çoklu ortogonal polinomlar, ω-çoklu Charlier polinomları, Appell polinomları, Hipergeometrik fonksiyonu, Rodrigues formülü, Üretici fonksiyonu, Fark denklemi

To My Lovely Family

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LIST OF SYMBOLS

$\mathcal{A}_n(x,y,z;\omega)$	3D-ω-Hermite Appell Polynomials
$\mathcal{A}_{n_1,n_2}(x,y,\omega)$	Δ_ω-multiple Appell Polynomials
$\mathscr{B}(x,y,z;\omega)$	3D-ω-Hermite Bernoulli Polynomials
$\mathcal{B}_n(x,y,z;\omega)$	3D-ω-Hermite Carlitz Bernoulli Polynomials
$\mathscr{B}l_n(x,y,z;\lambda;\omega)$	3D-ω-Hermite Boole Polynomials
$\mathcal{D}_n(x)$	Appell Polynomials
$\mathcal{E}_{n_1,n_2}(x,y;\omega)$	Δ_ω-Hermite Carlitz Euler Polynomials
$\mathcal{E}_n(x)$	Euler Polynomials
$\mathcal{E}_n(x,y,z;\omega)$	3D-ωHermite Carlitz Euler Polynomials
$\mathscr{G}_n(x,y,z;\omega)$	3D-ω-Hermite Genocchi Polynomials
$\mathfrak{G}_n^{\omega}(x,y,z)$	3D-ω-Hermite Polynomials
$\mathfrak{G}_{n_1,n_2}^{\omega}(x,y)$	ω-Gould Hopper Polynomials
$\mathcal{H}_n^{(2)}(x,y)$	Gould-Hopper Polynomials
$\mathcal{K}_n(x)$	Bernoulli Polynomials
$\mathscr{R}_n^{(j)}(x,y)$	Bivariate Appell Polynomials
$Bl_{n_1,n_2}(x,y;\lambda,\omega)$	Δ_ω-Hermite Boole Polynomials
$C_n^a(x,y,z;\omega)$	3D-ω-Hermite Charlier Polynomials
$C_{\vec{n}}^{\vec{a}}(x)$	ω-Multiple Charlier Polynomials
$C_{n_1,n_2}^{a_1,a_2}(x,y;\omega)$	Δ_ω-Hermite Charlier Polynomials
$H_n(x,y;\lambda)$	Degenerate Hermite Polynomials

Chapter 1

INTRODUCTION

Appell polynomials have been the subject of intense investigation of recent years. Many authors have studied Appell polynomials in pure and applied mathematics because of their applications.

There are many equivalent definitions for Appell polynomials. Their definition through the generating relation is given by the following formula

$$\mathsf{d}(t)\, e^{xt} = \sum_{\mathsf{n}=0}^{\infty} \mathcal{D}_{\mathsf{n}}(x) \frac{t^n}{n!}$$

where

$$d(t) = \sum_{n=0}^{\infty} \mathcal{D}_n \frac{t^n}{n!}, \quad \mathcal{D}_0 \neq 0.$$

As immediate examples to Appell polynomials, we have Bernoulli polynomials $\mathcal{K}_n$

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{K}_n(x) \frac{t^n}{n!}$$

and Euler polynomials $\mathcal{E}_n$

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n(x) \frac{t^n}{n!}.$$

There are many types of Appell polynomials that are under investigation nowadays.

- Multiple Appell polynomials were defined in [20] by

$$\mathcal{D}(t_1, t_2)\, e^{x(t_1+t_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mathcal{U}_{n_1,n_2}(x) \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!}$$

where $\mathcal{D}(t_1,t_2)$ has a series expansion

$$\mathcal{D}(t_1,t_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}d_{n_1,n_2}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!},$$

with $d_{0,0}\neq 0$.

- Degenerate Appell polynomials were introduced in [27] by,

$$d(t)(1+ht)^{\frac{x}{h}}=\sum_{n=0}^{\infty}\mathcal{D}_n(x,h)\frac{t^n}{n!}$$

where $d(t)$ is given by

$$d(t)=\sum_{k=0}^{\infty}\alpha_{k,h}\frac{t^k}{k!},\quad \alpha_{0,h}\neq 0.$$

- Bivariate Appell polynomials were defined in [4] by

$$\mathsf{a}(t)e^{xt+yt^j}=\sum_{n=0}^{\infty}\mathscr{R}_n^{(j)}(x,y)\frac{t^n}{n!},$$

with $a_0\neq 0$.

- The multiple $\Delta_\omega-$Appell polynomials were defined [29] by

$$\mathcal{A}(t_1,t_2,\cdots,t_n)(1+\omega(t_1+t_2+\cdots+t_r))^{\frac{x}{\omega}}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\mathcal{P}_{\vec{n}}(x)\frac{t_1^{n_1}t_2^{n_2}\cdots t_r^{n_r}}{n_1!n_2!\cdots n_r!}. \tag{1.1}$$

with $\vec{n}=(n_1,\cdots,n_r)$ and $a_{0,\cdots,0}\neq 0$.

It should be note that Appell polynomial sequence $\{\mathcal{A}_n(x)\}_{n\in\mathbb{N}}$ has an equivalent definition as

$$\mathfrak{D}_x(\mathcal{A}_n(x))=n\mathcal{A}_{n-1}(x),\quad n=1,2,\cdots,$$

where $\mathfrak{D}_x$ represents the derivative operator. The $\mathcal{A}_n(x)$ that results from the replacement of $\mathfrak{D}_x$ with Δ_ω, that is

$$\Delta_\omega(\mathcal{A}_n(x))=n\omega\mathcal{A}_{n-1}(x),\quad n=1,2,\cdots,$$

which was defined in [9] and named as Δ_ω-Appell sequences, where

$$\Delta_\omega \mathfrak{f}(x) = \mathfrak{f}(x+\omega) - \mathfrak{f}(x).$$

In [8], the authors introduced the δ_ω-Appell polynomial sets which are defined by

$$\delta_\omega \mathcal{P}_{n+1}(x) = (n+1)\mathcal{P}_n(x), \quad n \geq 0,$$

where

$$\delta_\omega(\mathfrak{f}(x)) := \frac{\Delta_\omega(\mathfrak{f}(x))}{\omega} := \frac{\mathfrak{f}(x+\omega) - \mathfrak{f}(x)}{\omega}, \quad \omega \neq 0.$$

They proved an equivalent definition in terms of the generating function given below

$$\mathcal{A}(t)(1+\omega t)^{\frac{x}{\omega}} = \sum_{n=0}^{\infty} \frac{\mathcal{P}_n(\omega;x)}{n!} t^n,$$

where

$$\mathcal{A}(t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_0 \neq 0.$$

It has also been shown in [8] that among all the δ_ω- Appell polynomials, d-orthogonal polynomial sets should have the generating function of the form

$$\mathfrak{G}(x,t) = exp(\mathcal{H}_d(t))(1+\omega t)^{\frac{x}{\omega}} = \sum_{n=0}^{\infty} \frac{\mathcal{P}_n(x)}{n!} t^n,$$

where $\mathcal{H}_d$ is a polynomial of degree d. In the special case

$$\mathcal{H}_d(t) = -at, \quad a \neq 0,$$

we have the polynomials generated as follows:

$$exp(-at)(1+\omega t)^{\frac{x}{\omega}} = \sum_{n=0}^{\infty} \frac{\mathcal{P}_n(x)}{n!} t^n, \quad a \neq 0.$$

These polynomials can be called $\omega-$Charlier polynomials, since the case $\omega = 1$ gives the usual Charlier polynomials.

Based on these motivation, in chapter 3 we introduced 3D-ω-Hermite Appell polynomials. Some of their properties such as explicit representation, determinantal representations, recurrence relations, lowering and raising operators are obtained. In Chapter 4 we introduced Δ_ω-Hermite Appell polynomials and we give an explicit representation and recurrence relations for them. Last chapter is devoted to the $\omega-$multiple Charlier polynomials. We start their investigation with multiple orthogonality relations with respect to the weight function of the form

$$w_i(x) = \frac{a_i^x}{\Gamma_\omega(x+\omega)}, \quad x \in \mathbb{R}^+, \quad i = 1, \cdots r,$$

and investigate, among other properties, the raising operator, Rodrigues formula, explicit representation and generating function. We also obtain an $(r+1)th$ order difference equation and give some special examples for certain choices of ω. So it can easily be observed from the generating function of $\omega-$multiple Charlier polynomials that these polynomials are examples of $\Delta_\omega-$multiple Appell polynomials.

Chapter 2

HYPERGEOMETRIC TYPE THE ω-DIFFERENCE EQUATION

The main aim of this chapter is to provide some properties for the ω-forward difference operator and ω-backward difference operator, such as product rule, ω-summation by parts formula which is used throughout the thesis.

The ω-forward difference operator is defined as,

$$\Delta_\omega(\mathfrak{f}(x)) = \mathfrak{f}(x+\omega) - \mathfrak{f}(x),$$

and ω- backward difference operator is defined as,

$$\nabla_\omega(\mathfrak{f}(x)) = \mathfrak{f}(x) - \mathfrak{f}(x-\omega).$$

where $\omega > 0$.

Theorem 2.1: The operators Δ_ω and ∇_ω have the following properties:

$$\Delta_\omega \mathfrak{f}(x) = \nabla_\omega \mathfrak{f}(x+\omega), \tag{2.1}$$

$$\Delta_\omega \nabla_\omega \mathfrak{f}(x) = \mathfrak{f}(x+\omega) - 2\mathfrak{f}(x) + \mathfrak{f}(x-\omega), \tag{2.2}$$

and

$$\Delta_\omega[\mathfrak{f}(x)\,\mathfrak{g}(x)] = \mathfrak{f}(x)\,\Delta_\omega \mathfrak{g}(x) + \mathfrak{g}(x+\omega)\,\Delta_\omega \mathfrak{f}(x). \tag{2.3}$$

Proof. We start with the proof of equation (2.1),

$$\nabla_{\omega}\mathfrak{f}(x+\omega)=\mathfrak{f}(x+\omega)-\mathfrak{f}(x)$$

$$=\Delta_{\omega}\mathfrak{f}(x).$$

The left hand side of (2.2) will be

$$\Delta_{\omega}\nabla_{\omega}\mathfrak{f}(x)=\Delta_{\omega}[\mathfrak{f}(x)-\mathfrak{f}(x-\omega)]$$

$$=\mathfrak{f}(x+\omega)-2\mathfrak{f}(x)+\mathfrak{f}(x-\omega).$$

For the proof of equation (2.3),

$$\Delta_{\omega}[\mathfrak{f}(x)\mathfrak{g}(x)]=\mathfrak{f}(x+\omega)\mathfrak{g}(x+\omega)-\mathfrak{f}(x)\mathfrak{g}(x)+\mathfrak{f}(x)\mathfrak{g}(x+\omega)-\mathfrak{f}(x)\mathfrak{g}(x-\omega)$$

$$=\mathfrak{g}(x+\omega)[\mathfrak{f}(x+\omega)-\mathfrak{f}(x)]+\mathfrak{f}(x)[\mathfrak{g}(x+\omega)-\mathfrak{g}(x)]$$

$$=\Delta_{\omega}\mathfrak{f}(x)\mathfrak{g}(x+\omega)+\mathfrak{f}(x)\Delta_{\omega}\mathfrak{g}(x).$$

Whence the result. □

Theorem 2.2: The ω-summation by parts formula is given by

$$\sum_{k=0}^{\infty}[\Delta_{\omega}\mathfrak{f}(\omega k)]\mathfrak{g}(\omega k)=-\sum_{k=0}^{\infty}[\nabla_{\omega}\mathfrak{g}(\omega k)]\mathfrak{f}(\omega k), \tag{2.4}$$

where $\mathfrak{g}(-\omega)=0$.

Proof. The left hand side of (2.4) can be written as,

$$\sum_{k=m}^{l}[[\Delta_{\omega}\mathfrak{f}(\omega k)]\mathfrak{g}(\omega k)+\mathfrak{f}(\omega k)\nabla_{\omega}[\mathfrak{g}(\omega k)]]$$

$$=\sum_{k=m}^{l}[\mathfrak{f}[\omega(k+1)]\mathfrak{g}(\omega k)-\mathfrak{f}(\omega k)\mathfrak{g}[\omega(k-1)]]$$

$$=[\mathfrak{f}[\omega(m+1)]\mathfrak{g}(\omega m)-\mathfrak{f}(\omega m)\mathfrak{g}[\omega(m-1)]]$$

$$+[\mathfrak{f}[\omega(m+2)]\mathfrak{g}(\omega(m+1))-\mathfrak{f}(\omega(m+1))\mathfrak{g}[\omega(m)]]+\cdots+$$

$$[\mathfrak{f}[\omega(l+1)]\mathfrak{g}(\omega(l))-\mathfrak{f}(\omega(l))\mathfrak{g}[\omega(l-1)]]$$

$$=\mathfrak{f}[\omega(l+1)]\mathfrak{g}(\omega l)-\mathfrak{f}(\omega m)\mathfrak{g}[\omega(m-1)].$$

Letting m tend to zero, $l\to\infty$ and considering that $\mathfrak{g}(-\omega)=0$, we get

$$\sum_{k=0}^{\infty}\left[\Delta_{\omega}\mathfrak{f}\left(\omega k\right)\right]\mathfrak{g}\left(\omega k\right)+\sum_{k=0}^{\infty}\left[\nabla_{\omega}\mathfrak{g}\left(\omega k\right)\right]\mathfrak{f}\left(\omega k\right)=0.$$

□

Chapter 3

3D-ω-APPELL POLYNOMIALS

In this chapter, 3D-ω-Hermite polynomials and 3D-ω-Hermite Appell polynomials are defined. Then, explicit and determinantal forms of 3D-ω-Appell polynomials are given. On the other hand, recurrence relation, difference equation, integro-difference equation and partial differential equation are obtained for such polynomials.

3D-ω-Hermite polynomials are defined as

$$(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}. \tag{3.1}$$

By means of the eq (3.1), we define 3D-ω-Hermite Appell polynomials as,

$$\mathsf{a}(t)(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!} \tag{3.2}$$

where

$$\mathsf{a}(t)=\sum_{k=0}^{\infty}\mathsf{a}_{k,\omega}\frac{t^k}{k!} \tag{3.3}$$

is the determining function which generates the corresponding degenerate numbers.

This generalized family leads to many potentially valuable new 3D-polynomials, some of which are listed below:

3D-ω-Hermite Charlier polynomials are defined by

$$exp\left(-a^{\omega}t\right)(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}C_n^a(x,y,z;\omega)\frac{t^n}{n!}.$$

The first kind 3D-Hermite Carlitz Bernoulli polynomials are introduced as

$$\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}B_n(x,y,z;\omega)\frac{t^n}{n!}.$$

3D-Hermite Carlitz Euler polynomials by

$$\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}\varepsilon_n(x,y,z;\omega)\frac{t^n}{n!}.$$

3D-ω-Hermite Genocchi polynomials by

$$\frac{2t}{(1+\omega t)^{\frac{1}{\omega}}+1}(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}\mathscr{G}_n(x,y,z;\omega)\frac{t^n}{n!}.$$

3D-ω-Hermite Boole polynomials are defined as

$$\frac{1}{1+(1+\omega t)^{\frac{\lambda}{\omega}}}(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}\mathscr{B}l_n(x,y,z;\lambda;\omega)\frac{t^n}{n!}.$$

The second kind 3D-ω-Hermite Bernoulli polynomials are defined as

$$\frac{\omega t}{ln(1+\omega t)}(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}\mathscr{B}_n^{II}(x,y,z;\omega)\frac{t^n}{n!}.$$

We should note that when $z=0$, each of the above definitions reduce to the bivariate versions given in [25].

3.1 3D-ω-Hermite Polynomials

Theorem 3.1: The following equations are satisfied by 3D-ω-Hermite polynomials

$$_x\Delta_\omega\left(\mathfrak{G}_n^{\omega}(x,y,z)\right)=\omega n\mathfrak{G}_{n-1}^{\omega}(x,y,z), \tag{3.4}$$

$$_y\Delta_\omega\left(\mathfrak{G}_n^{\omega}(x,y,z)\right)=\omega n(n-1)\mathfrak{G}_{n-2}^{\omega}(x,y,z) \tag{3.5}$$

and

$$_z\Delta_\omega\left(\mathfrak{G}_n^{\omega}(x,y,z)\right)=\omega n(n-1)(n-2)\mathfrak{G}_{n-3}^{\omega}(x,y,z). \tag{3.6}$$

Proof. Using (3.1) and applying the difference operator $_x\Delta_\omega$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}{}_x\Delta_\omega\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}&={}_x\Delta_\omega\left[\sum_{n=0}^{\infty}\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}\right]\\&={}_x\Delta_\omega\left[(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}\right].\end{aligned}$$

Using the Cauchy Product rule,

$$
\begin{aligned}
&{}_x\Delta_\omega\left[(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}\right]\\
&=\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}(1+\omega t)^{\frac{x}{\omega}}[1+\omega t-1]\\
&=\omega t\sum_{n=0}^{\infty}\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}\\
&=\omega\sum_{n=0}^{\infty}\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^{n+1}}{n!}\\
&=\omega\sum_{n=1}^{\infty}\mathfrak{G}_{n-1}^{\omega}(x,y,z)\frac{t^n}{(n-1)!}\\
&=\omega\sum_{n=0}^{\infty}n\mathfrak{G}_{n-1}^{\omega}(x,y,z)\frac{t^n}{n!}.
\end{aligned}
$$

Hence

$$\sum_{n=0}^{\infty}{}_x\Delta_\omega\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}=\omega\sum_{n=0}^{\infty}n\mathfrak{G}_{n-1}^{\omega}(x,y,z)\frac{t^n}{n!}.$$

Comparing the coefficients $\frac{t^n}{n!}$, we get

$${}_x\Delta_\omega\left(\mathfrak{G}_n^{\omega}(x,y,z)\right)=\omega n\mathfrak{G}_{n-1}^{\omega}(x,y,z).$$

Proofs of (3.5) and (3.6), follow in a similar manner. □

Corollary 3.1: The following properties are satisfied by 3D-ω-Hermite polynomials

$$\omega\,{}_y\Delta_\omega\left(\mathfrak{G}_n^{\omega}(x,y,z)\right)={}_x\Delta_\omega^2\left(\mathfrak{G}_n^{\omega}(x,y,z)\right) \tag{3.7}$$

and

$$\omega^2\,{}_z\Delta_\omega\left(\mathfrak{G}_n^{\omega}(x,y,z)\right)={}_x\Delta_\omega^3\left(\mathfrak{G}_n^{\omega}(x,y,z)\right). \tag{3.8}$$

Proof. We will give proof for (3.8). It follows from (3.4) and (3.6) that

$$
\begin{aligned}
\omega^2\,{}_z\Delta_\omega\left(\mathfrak{G}_n^{\omega}(x,y,z)\right)&=\omega^3 n(n-1)(n-2)\,\mathfrak{G}_{n-3}^{\omega}(x,y,z)\\
&={}_x\Delta_\omega^3\left(\mathfrak{G}_n^{\omega}(x,y,z)\right).
\end{aligned}
$$

□

Theorem 3.2: The polynomial $\mathfrak{G}_n^{\omega}(x,y,z)$ has the following explicit representation

$$\mathfrak{G}_n^{\omega}(x,y,z)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{l=0}^{\left[\frac{k}{3}\right]}(x)_{n-2k}^{\omega}(y)_{k-3l}^{\omega}(z)_l^{\omega}\frac{n!}{(n-2k)!\,(k-3l)!\,l!} \tag{3.9}$$

where

$$(x)_n^{\omega}=\left(-\frac{x}{\omega}\right)_n(-\omega)^n.$$

Proof. Using the series expansion of (3.1), we get

$$\begin{aligned}
&(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}\\
&=\sum_{n=0}^{\infty}\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}(x)_n^{\omega}\frac{t^n}{n!}\sum_{k=0}^{\infty}(y)_k^{\omega}\frac{t^{2k}}{k!}\sum_{l=0}^{\infty}(z)_l^{\omega}\frac{t^{3l}}{l!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{l=0}^{\left[\frac{k}{3}\right]}\frac{(x)_{n-2k}^{\omega}(y)_{k-3l}^{\omega}(z)_l^{\omega}}{(n-2k)!\,(k-3l)!\,l!}\frac{t^n}{n!}.
\end{aligned} \tag{3.10}$$

From (3.1) and (3.10), we get

$$\mathfrak{G}_n^{\omega}(x,y,z)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{l=0}^{\left[\frac{k}{3}\right]}(\mathsf{x})_{n-2k}^{\omega}(\mathsf{y})_{k-3l}^{\omega}(\mathsf{z})_l^{\omega}\frac{n!}{(n-2k)!\,(k-3l)!\,l!}.$$

The proof is completed. □

3.2 An Explicit and Determinantal Forms of 3D-ω-Hermite Appell Polynomials

Definition 3.1: 3D-ω-Hermite Appell polynomials are defined by

$$\mathsf{a}(t)(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}=\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!}, \tag{3.11}$$

where $\mathcal{A}_n(0,0,0;\omega)=\alpha_{n,\omega}\ \ (n=0,1,2,\cdots)$ are the degenerate numbers given by the series

$$\mathsf{a}(t)=\sum_{k=0}^{\infty}\alpha_{k,\omega}\frac{t^k}{k!},\ \ \alpha_{0,\omega}\neq 0. \tag{3.12}$$

Theorem 3.3: 3D-ω-Hermite Appell polynomials satisfy the following properties:

$$ {}_{x}\Delta_{\omega}\left(\mathcal{A}_{n}\left(x,y,z;\omega\right)\right)=n\omega\mathcal{A}_{n-1}\left(x,y,z;\omega\right), \tag{3.13} $$

$$ {}_{y}\Delta_{\omega}\left(\mathcal{A}_{n}\left(x,y,z;\omega\right)\right)=n\left(n-1\right)\omega\mathcal{A}_{n-2}\left(x,y,z;\omega\right) \tag{3.14} $$

and

$$ {}_{z}\Delta_{\omega}\left(\mathcal{A}_{n}\left(x,y,z;\omega\right)\right)=n\left(n-1\right)\left(n-2\right)\omega\mathcal{A}_{n-3}\left(x,y,z;\omega\right). \tag{3.15} $$

Proof. For the proof of (3.13), we use (3.11) and apply the difference operator ${}_{x}\Delta_{\omega}$ to get

$$ \begin{aligned} \sum_{n=0}^{\infty}{}_{x}\Delta_{\omega}\mathcal{A}_{n}\left(x,y,z;\omega\right)\frac{t^{n}}{n!} &= {}_{x}\Delta_{\omega}\left[\sum_{n=0}^{\infty}\mathcal{A}_{n}\left(x,y,z;\omega\right)\frac{t^{n}}{n!}\right] \\ &= {}_{x}\Delta_{\omega}\left[\left(1+\omega t\right)^{\frac{x}{\omega}}\mathsf{a}\left(t\right)\left(1+\omega t^{2}\right)^{\frac{y}{\omega}}\left(1+\omega t^{3}\right)^{\frac{z}{\omega}}\right]. \end{aligned} $$

Applying the Cauchy-Product rule, we have

$$ \begin{aligned} &{}_{x}\Delta_{\omega}\left[\left(1+\omega t\right)^{\frac{x}{\omega}}\mathsf{a}\left(t\right)\left(1+\omega t^{2}\right)^{\frac{y}{\omega}}\left(1+\omega t^{3}\right)^{\frac{z}{\omega}}\right] \\ &= {}_{x}\Delta_{\omega}\left[\mathsf{a}\left(t\right)\left(1+\omega t^{2}\right)^{\frac{y}{\omega}}\left(1+\omega t^{3}\right)^{\frac{z}{\omega}}\right]\left(1+\omega t\right)^{\frac{x}{\omega}} \\ &+ \mathsf{a}\left(t\right)\left(1+\omega t^{2}\right)^{\frac{y}{\omega}}\left(1+\omega t^{3}\right)^{\frac{z}{\omega}}{}_{x}\Delta_{\omega}\left(1+\omega t\right)^{\frac{x}{\omega}} \\ &= \mathsf{a}\left(t\right)\left(1+\omega t^{2}\right)^{\frac{y}{\omega}}\left(1+\omega t^{3}\right)^{\frac{z}{\omega}}\left(1+\omega t\right)^{\frac{x}{\omega}}\left[1+\omega t-1\right] \\ &= \omega t\sum_{n=0}^{\infty}\mathcal{A}_{n}\left(x,y,z;\omega\right)\frac{t^{n}}{n!} \\ &= \omega\sum_{n=0}^{\infty}\mathcal{A}_{n}\left(x,y,z;\omega\right)\frac{t^{n+1}}{n!} \\ &= \omega\sum_{n=0}^{\infty}n\mathcal{A}_{n-1}\left(x,y,z;\omega\right)\frac{t^{n}}{n!}. \end{aligned} $$

Comparing the coefficients on both sides,

$$ {}_{x}\Delta_{\omega}\left(\mathcal{A}_{n}\left(x,y,z;\omega\right)\right)=n\omega\mathcal{A}_{n-1}\left(x,y,z;\omega\right). $$

For the proof of (3.14) and (3.15), we can use the similar arguments. □

Corollary 3.2: The following equations are satisfied by 3D-ω-Hermite Appell polynomials,

$$\omega\, {}_{y}\Delta_{\omega}\left(\mathcal{A}_n\left(x,y,z;\omega\right)\right) = {}_{x}\Delta_{\omega}^{2}\left(\mathcal{A}_n\left(x,y,z;\omega\right)\right)$$

and

$$\omega^2\, {}_{z}\Delta_{\omega}\left(\mathcal{A}_n\left(x,y,z;\omega\right)\right) = {}_{x}\Delta_{\omega}^{3}\left(\mathcal{A}_n\left(x,y,z;\omega\right)\right).$$

Theorem 3.4: The polynomials $\{\mathcal{A}_n(x,y,z;\omega)\}_{n\in\mathbb{N}}$ have explicit representation,

$$\begin{aligned}&\mathcal{A}_n\left(x,y,z;\omega\right)\\&=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}\mathsf{a}_{n-k,\omega}\binom{k}{2m}\binom{m}{3l}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_{l}^{\omega}\frac{(2m)!}{m!}\frac{(3l)!}{l!}\end{aligned}\tag{3.16}$$

where

$$\mathsf{a}(t) = \sum_{n=0}^{\infty}\alpha_{n,\omega}\frac{t^n}{n!}.$$

Proof. Applying the Cauchy product and multiplying both sides of (3.1) by $\mathsf{a}(t)$, we have

$$\mathsf{a}(t)\left(1+\omega t\right)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}} = \mathsf{a}(t)\sum_{k=0}^{\infty}\mathfrak{G}_k^{\omega}(x,y,z)\frac{t^k}{k!}.$$

Therefore

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{A}_n\left(x,y,z;\omega\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\mathsf{a}_{n,\omega}\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathfrak{G}_k^{\omega}(x,y,z)\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathsf{a}_{n,\omega}\mathfrak{G}_k^{\omega}(x,y,z)\frac{t^{n+k}}{n!k!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathsf{a}_{n-k,\omega}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_{l}^{\omega}\frac{k!}{(k-2m)!(m-3l)!l!}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathsf{a}_{n-k,\omega}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}\binom{k}{2m}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_{l}^{\omega}\frac{(2m)!(3l)!}{(3l)!(m-3l)!l!}\frac{t^n}{n!}\end{aligned}\tag{3.17}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathsf{a}_{n-k,\omega}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}\binom{k}{2m}\binom{m}{3l}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_{l}^{\omega}\frac{(2m)!}{m!}\frac{(3l)!}{l!}\frac{t^n}{n!}.\tag{3.18}$$

From (3.17) and (3.18), we have

$$\begin{aligned}&\mathcal{A}_n(x,y,z;\omega)\\&=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}\mathsf{a}_{n-k,\omega}\binom{k}{2m}\binom{m}{3l}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_{l}^{\omega}\frac{(2m)!}{m!}\frac{(3l)!}{l!}.\end{aligned}$$

Whence the result. □

Theorem 3.5: For each $n=0,1,2,\cdots$, 3D-ω-Hermite Appell polynomials are represented by

$$\mathcal{A}_n(x,y,z;\omega)$$
$$=\frac{(-1)^n}{(\beta_{0,\omega})^{n+1}}\begin{vmatrix}1 & \mathfrak{G}_1^{\omega}(x,y,z) & \mathfrak{G}_2^{\omega}(x,y,z) & \cdots & \mathfrak{G}_{n-1}^{\omega}(x,y,z) & \mathfrak{G}_n^{\omega}(x,y,z)\\ \beta_{0,\omega} & \beta_{1,\omega} & \beta_{2,\omega} & \cdots & \beta_{n-1,\omega} & \beta_{n,\omega}\\ 0 & \beta_{0,\omega} & \binom{2}{1}\beta_{1,\omega} & \cdots & \binom{n-1}{1}\beta_{n-2,\omega} & \binom{n}{1}\beta_{n-1,\omega}\\ 0 & 0 & \beta_{0,\omega} & \cdots & \binom{n-1}{2}\beta_{n-3,\omega} & \binom{n}{2}\beta_{n-2,\omega}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_{0,\omega} & \binom{n}{n-1}\beta_{1,\omega}\end{vmatrix} \tag{3.19}$$

where the numbers $\beta_{k,\omega}$ $k=0,1,2,\cdots$ are the coefficients of the Maclaurin series of $\frac{1}{\mathsf{a}(t)}$.

Proof. Since

$$\mathsf{a}(t)\sum_{n=0}^{\infty}\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!} \tag{3.20}$$

we have

$$\sum_{n=0}^{\infty}\mathfrak{G}_n^{\omega}(x,y,z)\frac{t^n}{n!}=\frac{1}{\mathsf{a}(t)}\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!}$$

with

$$\frac{1}{\mathsf{a}(t)}=\sum_{k=0}^{\infty}\beta_{k,\omega}\frac{t^k}{k!}. \tag{3.21}$$

Using (3.21) in (3.20), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{G}_{n}^{\omega}(x,y,z)\frac{t^{n}}{n!} &= \sum_{k=0}^{\infty}\beta_{k,\omega}\frac{t^{k}}{k!}\sum_{n=0}^{\infty}\mathcal{A}_{n}(x,y,z;\omega)\frac{t^{n}}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\beta_{k,\omega}\mathcal{A}_{n}(x,y,z;\omega)\frac{t^{n+k}}{n!k!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\beta_{k,\omega}\mathcal{A}_{n-k}(x,y,z;\omega)\frac{t^{n}}{(n-k)!k!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\omega}\mathcal{A}_{n-k}(x,y,z;\omega)\frac{t^{n}}{n!}.\end{aligned}$$

Comparing the coefficients $\frac{t^n}{n!}$, we get

$$\mathfrak{G}_{n}^{\omega}(x,y,z)=\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\omega}\mathcal{A}_{n-k}(x,y,z;\omega). \tag{3.22}$$

This equality leads to the system of n-equations with unknown $\mathcal{A}_n(x,y,z;\omega)$, $n=0,1,2,\cdots$.

Solving this system using Cramer's rule and considering the assumption that the denominator is the determinant of a lower triangular matrix with determinant $(\beta_{0,\omega})^{n+1}$, and then taking transpose of the numerator of the resultant matrix and finally replacing the $i-th$ row with the $(i+1)-th$ position for $i=1,2,\cdots,n-1$, we get the result. □

3.3 Some Properties for 3D-ω-Hermite Appell Polynomials

Theorem 3.6: Recurrence relation satisfied by 3D-ω-Hermite Appell polynomials are given as

$$\mathcal{A}_{-k}(x,y,z;\omega)=0 \quad (k=1,2,\cdots)$$

and

$$
\begin{aligned}
&\mathcal{A}_{n+1}(x,y,z;\omega) \\
&= (x+\gamma_{0,\omega})\mathcal{A}_n(x,y,z;\omega) + \sum_{k=0}^{n-1}\binom{n}{k}\gamma_{n-k,\omega}\mathcal{A}_k(x,y,z;\omega) \\
&+ xn!\sum_{k=1}^{n}(-\omega)^k\frac{\mathcal{A}_{n-k}(x,y,z;\omega)}{(n-k)!} + 2ny\mathcal{A}_{n-1}(x,y,z;\omega) \\
&+ 2yn!\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\frac{\mathcal{A}_{n-2k-1}(x,y,z;\omega)}{(n-2k-1)!} + 3n(n-1)z\mathcal{A}_{n-2}(x,y,z;\omega) \\
&+ 3zn!\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\frac{\mathcal{A}_{n-3k-2}(x,y,z;\omega)}{(n-3k-2)!},
\end{aligned} \tag{3.23}
$$

where

$$
\frac{\mathsf{a}'(t)}{\mathsf{a}(t)} = \sum_{k=0}^{\infty}\gamma_{k,\omega}\frac{t^k}{k!}. \tag{3.24}
$$

Proof. Taking derivative with respect to t on both sides of (3.1), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{A}_{n+1}(x,y,z;\omega)\frac{t^n}{n!} \\
&= \frac{\mathsf{a}'(t)}{\mathsf{a}(t)}\mathsf{a}(t)(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}} \\
&+ \frac{1}{1+\omega t}x\mathsf{a}(t)(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}} \\
&+ 2y\frac{t}{1+\omega t^2}\mathsf{a}(t)(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}} \\
&+ 3z\frac{t^2}{1+\omega t^3}\mathsf{a}(t)(1+\omega t)^{\frac{x}{\omega}}\left(1+\omega t^2\right)^{\frac{y}{\omega}}\left(1+\omega t^3\right)^{\frac{z}{\omega}}.
\end{aligned} \tag{3.25}
$$

Now consider the expansions

$$
\begin{aligned}
\frac{1}{1+\omega t} &= \sum_{k=0}^{\infty}(-\omega t)^k \quad |\omega t| < 1 \\
\frac{1}{1+\omega t^2} &= \sum_{k=0}^{\infty}(-1)^k\omega^k t^{2k} \quad \left|\omega t^2\right| < 1 \\
\frac{1}{1+\omega t^3} &= \sum_{k=0}^{\infty}(-1)^k\omega^k t^{3k} \quad \left|\omega t^3\right| < 1.
\end{aligned} \tag{3.26}
$$

Inserting (3.26) into (3.25), we get

$$\sum_{n=0}^{\infty}\mathcal{A}_{n+1}(x,y,z;\omega)\frac{t^n}{n!}$$
$$=\sum_{k=0}^{\infty}\gamma_{k,\omega}\frac{t^k}{k!}\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!}+x\sum_{k=0}^{\infty}(-\omega t)^k\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!}$$
$$+2y\sum_{k=0}^{\infty}(-1)^k\omega^k t^{2k}\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^{n+1}}{n!}$$
$$+3z\sum_{k=0}^{\infty}(-1)^k\omega^k t^{3k}\sum_{n=0}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^{n+2}}{n!},$$

and

$$\mathcal{A}_1(x,y,z;\omega)+\sum_{n=1}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!}$$
$$=\gamma_{0,\omega}\mathcal{A}_0(x,y,z;\omega)+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\gamma_{k,\omega}\mathcal{A}_{n-k}(x,y,z;\omega)\frac{t^n}{n!}+x\mathcal{A}_0(x,y,z;\omega)$$
$$+x\sum_{n=1}^{\infty}\sum_{k=0}^{n}(-\omega)^k\frac{n!}{(n-k)!}\mathcal{A}_{n-k}(x,y,z;\omega)\frac{t^n}{n!}$$
$$+2y\sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\frac{n!}{(n-2k-1)!}\mathcal{A}_{n-2k-1}(x,y,z;\omega)\frac{t^n}{n!}$$
$$+3z\sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\frac{n!}{(n-3k-2)!}\mathcal{A}_{n-3k-2}(x,y,z;\omega)\frac{t^n}{n!}.$$

Finally we have,

$$\mathcal{A}_1(x,y,z;\omega)+\sum_{n=1}^{\infty}\mathcal{A}_n(x,y,z;\omega)\frac{t^n}{n!}$$
$$=(x+\gamma_{0,\omega})\mathcal{A}_0(x,y,z;\omega)+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\gamma_{k,\omega}\mathcal{A}_{n-k}(x,y,z;\omega)\frac{t^n}{n!}$$
$$+x\sum_{n=1}^{\infty}\sum_{k=0}^{n}(-\omega)^k\frac{n!}{(n-k)!}\mathcal{A}_{n-k}(x,y,z;\omega)\frac{t^n}{n!}$$
$$+2y\sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\frac{n!}{(n-2k-1)!}\mathcal{A}_{n-2k-1}(x,y,z;\omega)\frac{t^n}{n!}$$
$$+3z\sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\frac{n!}{(n-3k-2)!}\mathcal{A}_{n-3k-2}(x,y,z;\omega)\frac{t^n}{n!}. \tag{3.27}$$

On the other hand, by (3.16), we have $\mathcal{A}_0(x,y,z;\omega)=\alpha_0$ and $\mathcal{A}_1(x,y,z;\omega)=\alpha_1+\alpha_0 x$. Inserting (3.11) into (3.24), then making some calculations and finally using the

Cauchy product, we have $\gamma_0\alpha_0 = \alpha_1$. Hence, we can write that

$$\left(x+\gamma_{0,h}\right)\mathcal{A}_0\left(x,y,z;\omega\right) = \mathcal{A}_1\left(x,y,z;\omega\right). \tag{3.28}$$

Using (3.28) in (3.27), equating the coefficients of $\frac{t^n}{n!}$, and extending specific terms from the summations by shifting the series, we obtain the recurrence relation given by 3D-ω-Hermite Appell polynomials. □

Theorem 3.7: For the 3D-ω-Hermite Appell polynomials, we have the lowering operator as,

$${}_xL_n^- = \frac{1}{n\omega}\,{}_x\Delta_\omega, \tag{3.29}$$

and the raising operator as,

$$\begin{aligned} {}_xL_n^+ &= x+\gamma_{0,\omega}+\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!\,\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}+x\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k \\ &+2y\frac{{}_x\Delta_\omega}{\omega}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+1}}\,{}_x\Delta_\omega^{2k+1}+3z\frac{{}_x\Delta_\omega^2}{\omega^2}+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+2}}\,{}_x\Delta_\omega^{3k+2}. \end{aligned} \tag{3.30}$$

The difference equation satisfied by 3D-ω-Hermite Appell polynomials is given by

$$\begin{aligned} &\Bigg[\left(\frac{x}{\omega}+1+\frac{\gamma_{0,\omega}}{\omega}\right){}_x\Delta_\omega+\sum_{k=1}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!\,\omega^{n-k+1}}\,{}_x\Delta_\omega^{n-k+1}+\left(\frac{x}{\omega}+1\right)\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^{k+1} \\ &+\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k+2y\frac{{}_x\Delta_\omega^2}{\omega^2}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+2}}\,{}_x\Delta_\omega^{2k+2}+3z\frac{{}_x\Delta_\omega^3}{\omega^3} \\ &+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+3}}\,{}_x\Delta_\omega^{3k+3}-n\Bigg]A_n\left(x,y,z;\omega\right)=0. \end{aligned} \tag{3.31}$$

Proof. Starting from (3.29) and using (3.12), we have

$$\begin{aligned} {}_x\Delta_\omega\left(\mathcal{A}_n\left(x,y,z;\omega\right)\right) &= n\omega\mathcal{A}_{n-1}\left(x,y,z;\omega\right) \\ \frac{1}{n\omega}\,{}_x\Delta_\omega\left(\mathcal{A}_n\left(x,y,z;\omega\right)\right) &= \mathcal{A}_{n-1}\left(x,y,z;\omega\right). \end{aligned}$$

Therefore, we immediately get

$${}_xL_n^- = \frac{1}{n\omega}\,{}_x\Delta_\omega$$

For raising operator, first, we write $\mathcal{A}_k(x,y,z;\omega)$, $\mathcal{A}_{n-k}(x,y,z;\omega)$, $\mathcal{A}_{n-1}(x,y,z;\omega)$, $\mathcal{A}_{n-2k-1}(x,y,z;\omega)$, $\mathcal{A}_{n-2}(x,y,z;\omega)$ and $\mathcal{A}_{n-3k-2}(x,y,z;\omega)$ in terms of the lowering operator as follows:

$$\begin{aligned}
\mathcal{A}_k(x,y,z;\omega) &= \left[L_{k+1}^{-}L_{k+2}^{-}\cdots L_n^{-}\right]\mathcal{A}_n(x,y,z;\omega)\\
&= \left[\frac{1}{(k+1)\,\omega}\,{}_x\Delta_\omega \frac{1}{(k+2)\,\omega}\,{}_x\Delta_\omega \cdots \frac{1}{n\omega}\,{}_x\Delta_\omega\right]\mathcal{A}_n(x,y,z;\omega)\\
&= \left[\frac{1}{(n)^{n-k}\,\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}\right]\mathcal{A}_n(x,y,z;\omega)\\
\mathcal{A}_{n-k}(x,y,z;\omega) &= \frac{(n-k)!}{n!\omega^k}\,{}_x\Delta_\omega^{k}\left(\mathcal{A}_n(x,y,z;\omega)\right),\\
\mathcal{A}_{n-1}(x,y,z;\omega) &= \frac{1}{n\omega}\,{}_x\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right),\\
\mathcal{A}_{n-2k-1}(x,y,z;\omega) &= \frac{(n-2k-1)!}{n!\omega^{2k+1}}\,{}_x\Delta_\omega^{2k+1}\left(\mathcal{A}_n(x,y,z;\omega)\right),\\
\mathcal{A}_{n-2}(x,y,z;\omega) &= \frac{1}{n(n-1)\,\omega^2}\,{}_x\Delta_\omega^{2}\left(\mathcal{A}_n(x,y,z;\omega)\right),\\
\mathcal{A}_{n-3k-2}(x,y,z;\omega) &= \frac{(n-3k-2)!}{n!\omega^{3k+2}}\,{}_x\Delta_\omega^{3k+2}\left(\mathcal{A}_n(x,y,z;\omega)\right).
\end{aligned}$$

Using the recurrence relation (3.23), we get

$$\begin{aligned}
&\mathcal{A}_{n+1}(x,y,z;\omega)\\
&= (x+\gamma_{0,\omega})\,\mathcal{A}_n(x,y,z;\omega) + \sum_{k=0}^{n-1}\frac{n!}{(n-k)!k!}\gamma_{n-k,\omega}\frac{k!}{n!\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+ xn!\sum_{k=1}^{n}\frac{(-1)^k\,\omega^k}{(n-k)!}\frac{(n-k)!}{n!\omega^k}\,{}_x\Delta_\omega^{k}\left(\mathcal{A}_n(x,y,z;\omega)\right) + 2ny\frac{1}{n\omega}\,{}_x\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+ 2yn!\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k\,\omega^k}{(n-2k-1)!}\frac{(n-2k-1)!}{n!\omega^{2k+1}}\,{}_x\Delta_\omega^{2k+1}\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+ 3n(n-1)\,z\frac{1}{n(n-1)\,\omega^2}\,{}_x\Delta_\omega^{2}\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+ 3zn!\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k\,\omega^k}{(n-3k-2)!}\frac{(n-3k-2)!}{n!\omega^{3k+2}}\,{}_x\Delta_\omega^{3k+2}\left(\mathcal{A}_n(x,y,z;\omega)\right).
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\mathcal{A}_{n+1}(x,y,z;\omega)\\
&=(x+\gamma_{0,\omega})\mathcal{A}_n(x,y,z;\omega)+\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+x\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k\left(\mathcal{A}_n(x,y,z;\omega)\right)+2y\frac{{}_x\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right)}{\omega}\\
&+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+1}}\,{}_x\Delta_\omega^{2k+1}\left(\mathcal{A}_n(x,y,z;\omega)\right)+3z\frac{{}_x\Delta_\omega^2\left(\mathcal{A}_n(x,y,z;\omega)\right)}{\omega^2}\\
&+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+2}}\,{}_x\Delta_\omega^{3k+2}\left(\mathcal{A}_n(x,y,z;\omega)\right).
\end{aligned}
\tag{3.32}
$$

We can write the raising operator as,

$$
\begin{aligned}
{}_xL_n^+=x+\gamma_{0,\omega}+\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}+x\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k+2y\frac{{}_x\Delta_\omega}{\omega}\\
+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+1}}\,{}_x\Delta_\omega^{2k+1}+3z\frac{{}_x\Delta_\omega^2}{\omega^2}+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+2}}\,{}_x\Delta_\omega^{3k+2}.
\end{aligned}
$$

For the difference equation applying the ${}_x\Delta_\omega$ operator on both sides of (3.32) and using

$$
{}_x\Delta_\omega\left(\mathfrak{f}(x)\,\mathfrak{g}(x,y,z)\right)=\mathfrak{f}(x+\omega)\,{}_x\Delta_\omega\mathfrak{g}(x,y,z)+\mathfrak{g}(x,y,z)\,{}_x\Delta_\omega\mathfrak{f}(x),
$$

we get,

$$
\begin{aligned}
&{}_x\Delta_\omega\left(\mathcal{A}_{n+1}(x,y,z;\omega)\right)\\
&=\ {}_x\Delta_\omega\left[(x+\gamma_{0,\omega})\mathcal{A}_n(x,y,z;\omega)\right]+{}_x\Delta_\omega\left[\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}\mathcal{A}_n(x,y,z;\omega)\right]\\
&+\ {}_x\Delta_\omega\left[x\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k\left(\mathcal{A}_n(x,y,z;\omega)\right)\right]+{}_x\Delta_\omega\left[\frac{2y\,{}_x\Delta_\omega\mathcal{A}_n(x,y,z;\omega)}{\omega}\right]\\
&+\ {}_x\Delta_\omega\left[2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+1}}\,{}_x\Delta_\omega^{2k+1}\mathcal{A}_n(x,y,z;\omega)\right]+{}_x\Delta_\omega\left[\frac{3z\,{}_x\Delta_\omega^2\mathcal{A}_n(x,y,z;\omega)}{\omega^2}\right]\\
&+\ {}_x\Delta_\omega\left[3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+2}}\,{}_x\Delta_\omega^{3k+2}\mathcal{A}_n(x,y,z;\omega)\right].
\end{aligned}
$$

Dividing both sides of the equation by ω, we can write

$$\left(\frac{x}{\omega}+1+\frac{\gamma_{0,\omega}}{\omega}\right){}_x\Delta_\omega \mathcal{A}_n(x,y,z;\omega)+\sum_{k=1}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!\,\omega^{n-k+1}}\,{}_x\Delta_\omega^{n-k+1}\mathcal{A}_n(x,y,z;\omega)$$

$$+\left(\frac{x}{\omega}+1\right)\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^{k+1}\mathcal{A}_n(x,y,z;\omega)+\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^{k}\mathcal{A}_n(x,y,z;\omega)$$

$$+2y\frac{{}_x\Delta_\omega^2\mathcal{A}_n(x,y,z;\omega)}{\omega^2}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+2}}\,{}_x\Delta_\omega^{2k+2}\mathcal{A}_n(x,y,z;\omega)+3z\frac{{}_x\Delta_\omega^3\mathcal{A}_n(x,y,z;\omega)}{\omega^3}$$

$$+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+3}}\,{}_x\Delta_\omega^{3k+3}\mathcal{A}_n(x,y,z;\omega)-n\mathcal{A}_n(x,y,z;\omega)=0.$$

Whence the result. □

Theorem 3.8: 3D-ω-Hermite Appell polynomials satisfy the following integro-lowering, integro-raising, and integro-difference equations

$$_x\mathscr{L}_n^- = \frac{1}{n}\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega, \tag{3.33}$$

$$\begin{aligned}
_x\mathscr{L}_n^+ &= x+\gamma_{0,\omega}+\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k}+x\sum_{k=1}^{n}(-\omega)^k\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k}\\
&+2y\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(2k+1)}\,{}_y\Delta_\omega^{2k+1}+3z\,{}_x\Delta_\omega^{-2}\,{}_y\Delta_\omega^2\\
&+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(3k+2)}\,{}_y\Delta_\omega^{3k+2},
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&\Bigg[(x+\gamma_{0,\omega})\frac{{}_y\Delta_\omega}{\omega}+\frac{1}{\omega}\sum_{k=1}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k+1}+x\sum_{k=1}^{n}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k+1}\\
&+2\left(\frac{y}{\omega}+1\right){}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega^{2}+2\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega+2\left(\frac{y}{\omega}+1\right)\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(2k+1)}\,{}_y\Delta_\omega^{2k+2}\\
&+2\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(2k+1)}\,{}_y\Delta_\omega^{2k+1}+\frac{3z}{\omega}\,{}_x\Delta_\omega^{-2}\,{}_y\Delta_\omega^{3}\\
&+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-(3k+2)}\,{}_y\Delta_\omega^{3k+3}-(n+1)\frac{{}_x\Delta_\omega}{\omega}\Bigg]\mathcal{A}_n(x,y,z;\omega)=0,
\end{aligned} \tag{3.35}$$

respectively.

Proof. Since

$$_x\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right)=n\omega\mathcal{A}_{n-1}(x,y,z;\omega),$$

$$
\begin{aligned}
{}_x\Delta_\omega\left(\mathcal{A}_{n-1}(x,y,z;\omega)\right) &= (n-1)\,\omega\mathcal{A}_{n-2}(x,y,z;\omega), \\
\mathcal{A}_{n-2}(x,y,z;\omega) &= \frac{1}{(n-1)\,\omega}\,{}_x\Delta_\omega\left(\mathcal{A}_{n-1}(x,y,z;\omega)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_y\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right) &= n(n-1)\,\omega\mathcal{A}_{n-2}(x,y,z;\omega) \\
&= n(n-1)\,\omega\frac{1}{(n-1)\,\omega}\,{}_x\Delta_\omega\left(\mathcal{A}_{n-1}(x,y,z;\omega)\right) \\
\mathcal{A}_{n-1}(x,y,z;\omega) &= \frac{1}{n}\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right).
\end{aligned}
$$

We can write the integro-lowering operator as,

$$
{}_x\mathscr{L}_n^- = \frac{1}{n}\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega.
$$

The inverse forward difference operators are introduced as

$$
\begin{aligned}
{}_x\Delta_\omega^{-1}\left(\mathcal{A}_{n-1}(x,y,z;\omega)\right) &= \frac{1}{n\omega}\mathcal{A}_n(x,y,z;\omega) \\
{}_x\Delta_\omega^{-k}\left(\mathcal{A}_{n-1}(x,y,z;\omega)\right) &= \frac{1}{n(n+1)\cdots(n+k-1)\,\omega^k}\mathcal{A}_{n+k-1}(x,y,z;\omega).
\end{aligned}
$$

Inserting terms into (3.33), we get

$$
\begin{aligned}
\mathcal{A}_k(x,y,z;\omega) &= \left[\mathscr{L}_{k+1}^-\mathscr{L}_{k+2}^-\cdots\mathscr{L}_n^-\right]\mathcal{A}_n(x,y,z;\omega) \\
&= \left[\frac{1}{(k+1)}\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega\frac{1}{k+2}\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega\cdots\frac{1}{n}\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega\right]\mathcal{A}_n(x,y,z;\omega) \\
&= \frac{1}{(n)^{n-k}}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k}\mathcal{A}_n(x,y,z;\omega) \\
&= \frac{k!}{n!}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k}\left(\mathcal{A}_n(x,y,z;\omega)\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathcal{A}_{n-k}(x,y,z;\omega) &= \frac{(n-k)!}{n!}\,{}_x\Delta_\omega^{-(n-n+k)}\,{}_y\Delta_\omega^{n-n+k}\left(\mathcal{A}_n(x,y,z;\omega)\right) \\
&= \frac{(n-k)!}{n!}\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k}\left(\mathcal{A}_n(x,y,z;\omega)\right), \\
\mathcal{A}_{n-1}(x,y,z;\omega) &= \frac{(n-1)!}{n!}\,{}_x\Delta_\omega^{-(n-n+1)}\,{}_y\Delta_\omega^{n-n+1}\left(\mathcal{A}_n(x,y,z;\omega)\right) \\
&= \frac{1}{n}\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right),
\end{aligned}
$$

$$\mathcal{A}_{n-2k-1}(x,y,z;\omega)=\frac{(n-2k-1)!}{n!}\,{}_x\Delta_\omega^{-(2k+1)}\,{}_y\Delta_\omega^{2k+1}\left(\mathcal{A}_n(x,y,z;\omega)\right),$$
$$\mathcal{A}_{n-2}(x,y,z;\omega)=\frac{1}{n(n-1)}\,{}_x\Delta_\omega^{-2}\,{}_y\Delta_\omega^{2}\left(\mathcal{A}_n(x,y,z;\omega)\right),$$
$$\mathcal{A}_{n-3k-2}(x,y,z;\omega)=\frac{(n-3k-2)!}{n!}\,{}_x\Delta_\omega^{-(3k+2)}\,{}_y\Delta_\omega^{3k+2}\left(\mathcal{A}_n(x,y,z;\omega)\right),$$

by writing the above equalities in (3.23) we get,

$$\begin{aligned}
&\mathcal{A}_{n+1}(x,y,z;\omega)\\
&=(x+\gamma_{0,\omega})\mathcal{A}_n(x,y,z;\omega)+\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k}\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+x\sum_{k=1}^{n}(-\omega)^k\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k}\left(\mathcal{A}_n(x,y,z;\omega)\right)+2y\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(2k+1)}\,{}_y\Delta_\omega^{2k+1}\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+3z\,{}_x\Delta_\omega^{-2}\,{}_y\Delta_\omega^{2}\left(\mathcal{A}_n(x,y,z;\omega)\right)\\
&+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(3k+2)}\,{}_y\Delta_\omega^{3k+2}\left(\mathcal{A}_n(x,y,z;\omega)\right).
\end{aligned}\tag{3.36}$$

Therefore the integro raising operator is given by

$$\begin{aligned}
{}_x\mathscr{L}_n^{+}&=x+\gamma_{0,\omega}+\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k}+x\sum_{k=1}^{n}(-\omega)^k\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k}\\
&+2y\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(2k+1)}\,{}_y\Delta_\omega^{2k+1}+3z\,{}_x\Delta_\omega^{-2}\,{}_y\Delta_\omega^{2}\\
&+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(3k+2)}\,{}_y\Delta_\omega^{3k+2}.
\end{aligned}$$

For integro-difference equation, we use (3.36). Applying ${}_y\Delta_\omega$ on both sides of (3.36), we get

$$\begin{aligned}
&{}_y\Delta_\omega\left[(x+\gamma_{0,\omega})\mathcal{A}_n(x,y,z;\omega)\right]+{}_y\Delta_\omega\left[\sum_{k=0}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k}\left(\mathcal{A}_n(x,y,z;\omega)\right)\right]\\
&+{}_y\Delta_\omega\left[x\sum_{k=1}^{n}(-\omega)^k\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k}\left(\mathcal{A}_n(x,y,z;\omega)\right)\right]
\end{aligned}$$

$$+ {}_y\Delta_\omega \left[2y\, {}_x\Delta_\omega^{-1}\, {}_y\Delta_\omega \left(\mathcal{A}_n(x,y,z;\omega)\right)\right]$$

$$+ {}_y\Delta_\omega \left[2y \sum_{k=1}^{\left[\frac{n-1}{2}\right]} (-\omega)^k\, {}_x\Delta_\omega^{-(2k+1)}\, {}_y\Delta_\omega^{2k+1} \left(\mathcal{A}_n(x,y,z;\omega)\right)\right]$$

$$+ {}_y\Delta_\omega \left[3z\, {}_x\Delta_\omega^{-2}\, {}_y\Delta_\omega^{2} \left(\mathcal{A}_n(x,y,z;\omega)\right)\right]$$

$$+ {}_y\Delta_\omega \left[3z \sum_{k=1}^{\left[\frac{n-2}{3}\right]} (-\omega)^k\, {}_x\Delta_\omega^{-(3k+2)}\, {}_y\Delta_\omega^{3k+2} \left(\mathcal{A}_n(x,y,z;\omega)\right)\right]$$

$$= {}_y\Delta_\omega \left(\mathcal{A}_{n+1}(x,y,z;\omega)\right).$$

Here using the product rule

$${}_y\Delta_\omega \left(\mathfrak{f}(y)\,\mathfrak{g}(x,y)\right) = \mathfrak{f}(y+\omega)\, {}_y\Delta_\omega \mathfrak{g}(x,y) + \mathfrak{g}(x,y)\, {}_y\Delta_\omega \mathfrak{f}(y),$$

we obtain,

$$(x+\gamma_{0,\omega})\, {}_y\Delta_\omega \mathcal{A}_n(x,y,z;\omega) + \sum_{k=0}^{n-1} \frac{\gamma_{n-k,\omega}}{(n-k)!}\, {}_x\Delta_\omega^{-(n-k)}\, {}_y\Delta_\omega^{n-k+1} \left(\mathcal{A}_n(x,y,z;\omega)\right)$$

$$+ x\sum_{k=1}^{n} (-\omega)^k\, {}_x\Delta_\omega^{-k}\, {}_y\Delta_\omega^{k+1} \left(\mathcal{A}_n(x,y,z;\omega)\right) + 2(y+\omega)\, {}_x\Delta_\omega^{-1}\, {}_y\Delta_\omega^{2} \left(\mathcal{A}_n(x,y,z;\omega)\right)$$

$$+ 2\omega\, {}_x\Delta_\omega^{-1}\, {}_y\Delta_\omega \left(\mathcal{A}_n(x,y,z;\omega)\right) + 2(y+\omega) \sum_{k=1}^{\left[\frac{n-1}{2}\right]} (-\omega)^k\, {}_x\Delta_\omega^{-(2k+1)}\, {}_y\Delta_\omega^{2k+2} \left(\mathcal{A}_n(x,y,z;\omega)\right)$$

$$+ 2\omega \sum_{k=1}^{\left[\frac{n-1}{2}\right]} (-\omega)^k\, {}_x\Delta_\omega^{-(2k+1)}\, {}_y\Delta_\omega^{2k+1} \left(\mathcal{A}_n(x,y,z;\omega)\right) + 3z\, {}_x\Delta_\omega^{-2}\, {}_y\Delta_\omega^{3} \left(\mathcal{A}_n(x,y,z;\omega)\right)$$

$$+ 3z \sum_{k=1}^{\left[\frac{n-2}{3}\right]} (-\omega)^k\, {}_x\Delta_\omega^{-(3k+2)}\, {}_y\Delta_\omega^{3k+3} \left(\mathcal{A}_n(x,y,z;\omega)\right) = (n+1)A_{n+1}(x,y,z;\omega).$$

We can express the integro-difference equation by dividing both sides of the equation by ω, to get the following,

$$\Bigg[(x+\gamma_{0,\omega})\frac{{}_y\Delta_\omega}{\omega} + \frac{1}{\omega}\sum_{k=1}^{n-1} \frac{\gamma_{n-k,\omega}}{(n-k)!}\, {}_x\Delta_\omega^{-(n-k)}\, {}_y\Delta_\omega^{n-k+1} + x\sum_{k=1}^{n} (-1)^k \omega^{k-1}\, {}_x\Delta_\omega^{-k}\, {}_y\Delta_\omega^{k+1}$$

$$+ 2\left(\frac{y}{\omega}+1\right) {}_x\Delta_\omega^{-1}\, {}_y\Delta_\omega^{2} + 2\, {}_x\Delta_\omega^{-1}\, {}_y\Delta_\omega + 2\left(\frac{y}{\omega}+1\right) \sum_{k=1}^{\left[\frac{n-1}{2}\right]} (-\omega)^k\, {}_x\Delta_\omega^{-(2k+1)}\, {}_y\Delta_\omega^{2k+2}$$

$$+2\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-(2k+1)}\,{}_y\Delta_\omega^{2k+1}+\frac{3z}{\omega}\,{}_x\Delta_\omega^{-2}\,{}_y\Delta_\omega^{3}$$

$$\left.+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-(3k+2)}\,{}_y\Delta_\omega^{3k+3}-(n+1)\frac{{}_x\Delta_\omega}{\omega}\right]\mathcal{A}_n(x,y,z;\omega)=0.$$

□

Theorem 3.9: 3D-ω-Hermite Appell polynomials satisfy the partial difference equation

$$\left[\left(\frac{x}{\omega^n}+\frac{\gamma_{0,\omega}}{\omega^n}\right){}_x\Delta_\omega^{n-1}\,{}_y\Delta_\omega+\frac{1}{\omega^n}\sum_{k=1}^{n}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{k-1}\,{}_y\Delta_\omega^{n-k+1}\right]\mathcal{A}_n(x,y,z;\omega)$$

$$+\frac{1}{\omega^{n-1}}\sum_{j=0}^{n-1}(-1)^{n-1-j}\binom{n-1}{j}(x+j\omega)$$

$$\times\sum_{k=1}^{n}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k+1}\mathcal{A}_n(x+j\omega,y,z;\omega)$$

$$+\frac{1}{\omega^{n-1}}\left[2\left(\frac{y}{\omega}+1\right){}_x\Delta_\omega^{n-2}\,{}_y\Delta_\omega^{2}+2\,{}_x\Delta_\omega^{n-2}\,{}_y\Delta_\omega\right.$$

$$\left.+2\left(\frac{y}{\omega}+1\right)\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-2k+n-2}\,{}_y\Delta_\omega^{2k+2}\right]\mathcal{A}_n(x,y,z;\omega)$$

$$+\frac{1}{\omega^{n-1}}\left[2\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-2k+n-2}\,{}_y\Delta_\omega^{2k+1}+\frac{3z}{\omega}\,{}_x\Delta_\omega^{n-3}\,{}_y\Delta_\omega^{3}\right.$$

$$\left.+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-3k+n-3}\,{}_y\Delta_\omega^{3k+3}-(n+1)\frac{{}_x\Delta_\omega^{n}}{\omega}\right]\mathcal{A}_n(x,y,z;\omega)=0. \quad (3.37)$$

Proof. We use integro-difference equation and apply the forward difference operator with respect to x $n-1$-times, to have

$$\left[{}_x\Delta_\omega^{n-1}\left[(x+\gamma_{0,\omega})\frac{{}_y\Delta_\omega}{\omega}\right]+{}_x\Delta_\omega^{n-1}\left[\frac{1}{\omega}\sum_{k=1}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{-(n-k)}\,{}_y\Delta_\omega^{n-k+1}\right]\right]\mathcal{A}_n(x,y,z;\omega)$$

$$+\,{}_x\Delta_\omega^{n-1}\left[x\sum_{k=1}^{n}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k+1}\mathcal{A}_n(x,y,z;\omega)\right]$$

$$+\left[{}_x\Delta_\omega^{n-1}\left(2\left(\frac{y}{\omega}+1\right){}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega^{2}\right)+{}_x\Delta_\omega^{n-1}\left[2\,{}_x\Delta_\omega^{-1}\,{}_y\Delta_\omega\right]\right.$$

$$
\begin{aligned}
&+{}_x\Delta_\omega^{n-1}\left[2\left(\frac{y}{\omega}+1\right)\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-2k-1}\,{}_y\Delta_\omega^{2k+2}\right]\Bigg]\mathcal{A}_n(x,y,z;\omega)\\
&+\Bigg[{}_x\Delta_\omega^{n-1}\left[2\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-2k-1}\,{}_y\Delta_\omega^{2k+1}\right]+{}_x\Delta_\omega^{n-1}\left[\frac{3z}{\omega}\,{}_x\Delta_\omega^{-2}\,{}_y\Delta_\omega^{3}\right]\\
&+{}_x\Delta_\omega^{n-1}\left[3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-3k-2}\,{}_y\Delta_\omega^{3k+3}\right]\\
&-{}_x\Delta_\omega^{n-1}\left[(n+1)\frac{{}_x\Delta_\omega}{\omega}\right]\Bigg]\mathcal{A}_n(x,y,z;\omega)=0.
\end{aligned}
$$

Using the ω-summation by parts formula

$$
{}_x\Delta_\omega^{n-1}[\mathfrak{f}\mathfrak{g}](x)=\sum_{j=0}^{n-1}(-1)^{n-1-j}\binom{n-1}{j}\mathfrak{f}(x+j\omega)\,\mathfrak{g}(x+j\omega)
$$

then we have,

$$
\begin{aligned}
&\left[(x+\gamma_{0,\omega})_x\Delta_\omega^{n-1}\frac{{}_y\Delta_\omega}{\omega}+\frac{1}{\omega}\sum_{k=1}^{n-1}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{k-1}\,{}_y\Delta_\omega^{n-k+1}\right]\mathcal{A}_n(x,y,z;\omega)\\
&+\sum_{j=0}^{n-1}(-1)^{n-1-j}\binom{n-1}{j}(x+j\omega)\sum_{k=1}^{n}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k+1}\mathcal{A}_n(x+j\omega,y,z;\omega)\\
&+\Bigg[2\left(\frac{y}{\omega}+1\right){}_x\Delta_\omega^{n-2}\,{}_y\Delta_\omega^{2}+2\,{}_x\Delta_\omega^{n-2}\,{}_y\Delta_\omega\\
&+2\left(\frac{y}{\omega}+1\right)\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-2k+n-2}\,{}_y\Delta_\omega^{2k+2}\Bigg]\mathcal{A}_n(x,y,z;\omega)\\
&+\Bigg[2\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\,{}_x\Delta_\omega^{-2k+n-2}\,{}_y\Delta_\omega^{2k+1}+\frac{3z}{\omega}\,{}_x\Delta_\omega^{n-3}\,{}_y\Delta_\omega^{3}\\
&+3z\sum_{k=1}^{\frac{n-2}{3}}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-3k+n-3}\,{}_y\Delta_\omega^{3k+3}-(n+1)\frac{{}_x\Delta_\omega^{n}}{\omega}\Bigg]\mathcal{A}_n(x,y,z;\omega)=0.
\end{aligned}
$$

Finally, dividing both sides by ω^{n-1}, we get the partial difference equation,

$$
\begin{aligned}
&\left[\left(\frac{x}{\omega^n}+\frac{\gamma_{0,\omega}}{\omega^n}\right){}_x\Delta_\omega^{n-1}\,{}_y\Delta_\omega+\frac{1}{\omega^n}\sum_{k=1}^{n}\frac{\gamma_{n-k,\omega}}{(n-k)!}\,{}_x\Delta_\omega^{k-1}\,{}_y\Delta_\omega^{n-k+1}\right]\mathcal{A}_n(x,y,z;\omega)\\
&+\frac{1}{\omega^{n-1}}\sum_{j=0}^{n-1}(-1)^{n-1-j}\binom{n-1}{j}(x+j\omega)\\
&\times\sum_{k=1}^{n}(-1)^k\omega^{k-1}\,{}_x\Delta_\omega^{-k}\,{}_y\Delta_\omega^{k+1}\mathcal{A}_n(x+j\omega,y,z;\omega)
\end{aligned}
$$

$$+\frac{1}{\omega^{n-1}}\left[2\left(\frac{y}{\omega}+1\right){}_x\Delta_\omega^{n-2}{}_y\Delta_\omega^{2}+2{}_x\Delta_\omega^{n-2}{}_y\Delta_\omega\right.$$

$$\left.+2\left(\frac{y}{\omega}+1\right)\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k{}_x\Delta_\omega^{-2k+n-2}{}_y\Delta_\omega^{2k+2}\right]\mathcal{A}_n(x,y,z;\omega)$$

$$+\frac{1}{\omega^{n-1}}\left[2\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k{}_x\Delta_\omega^{-2k+n-2}{}_y\Delta_\omega^{2k+1}+\frac{3z}{\omega}{}_x\Delta_\omega^{n-3}{}_y\Delta_\omega^{3}\right.$$

$$\left.+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-1)^k\omega^{k-1}{}_x\Delta_\omega^{-3k+n-3}{}_y\Delta_\omega^{3k+3}-(n+1)\frac{{}_x\Delta_\omega^{n}}{\omega}\right]\mathcal{A}_n(x,y,z;\omega)=0.$$

□

3.4 Special Cases of 3D-ω-Hermite Appell Polynomials

As a particular case of main theorems we introduce special cases of 3D-ω-Hermite Appell polynomials and provide explicit forms, determinants, recurrence relation, lowering operator, raising operator, and difference equations for them. The special cases that we propose are 3D-ω-Hermite Charlier polynomials, the first kind 3D-Hermite Carlitz Bernoulli polynomials , 3D-Hermite Carlitz Euler polynomials, and 3D-ω-Hermite Boole polynomials.

3.4.1 3D-ω-Hermite Charlier Polynomials

In this subsection, we exhibit the explicit form, determinants, recurrence relation, lowering operator, raising operator and difference equation satisfied by 3D-ω-Hermite Charlier polynomials.

Corollary 3.3: 3D-ω-Hermite Charlier polynomials sequence has an explicit representation

$$C_n^a(x,y,z;\omega)=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}C_{n-k,\omega}^a\binom{k}{2m}\binom{m}{3l}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_l^{\omega}\frac{(2m)!}{m!}\frac{(3l)!}{l!} \tag{3.38}$$

where 3D-ω-Hermite Charlier numbers $C_n^a(0,0,0;\omega)=C_{n,\omega}^a$ are given by the series

$$a(t)=exp(-a^{\omega}t)=\sum_{n=0}^{\infty}C_{n,\omega}^a\frac{t^n}{n!}. \tag{3.39}$$

Some 3D-ω-Hermite Charlier numbers are given by

$$C_0^a(\omega) = 1,\ C_1^a(\omega) = -a^\omega,\ C_2^a(\omega) = \frac{a^{2\omega}}{2}$$
$$C_3^a(\omega) = -\frac{a^{3\omega}}{6},\ C_4^a(\omega) = \frac{a^{4\omega}}{24}.$$

Some 3D-ω-Hermite Charlier polynomials are given by

$$C_0^a(x,y,z;\omega) = 1,$$

$$C_1^a(x,y,z;\omega) = x - a^\omega,$$

and

$$C_2^a(x,y,z;\omega) = \frac{a^{2\omega}}{2} - 2a^\omega - x(x-\omega) + 2y.$$

In the case $\omega \to 0$, $y = 0$ and $z = 0$, 3D-ω-Hermite Charlier polynomials reduce to Charlier polynomials.

Corollary 3.4: Determinant satisfied by 3D-ω-Hermite Charlier polynomials is given by

$$C_n^a(x,y,z;\omega)$$
$$= (-1)^n \begin{vmatrix} 1 & \mathfrak{G}_1^\omega(x,y,z) & \mathfrak{G}_2^\omega(x,y,z) & \cdots & \mathfrak{G}_{n-1}^\omega(x,y,z) & \mathfrak{G}_n^\omega(x,y,z) \\ 1 & a^\omega & \frac{a^{2\omega}}{2} & \cdots & \frac{a^{\omega(n-1)}}{(n-1)!} & \frac{a^{\omega n}}{n!} \\ 0 & 1 & \binom{2}{1}a^\omega & \cdots & \binom{n-1}{1}\frac{a^{\omega(n-2)}}{(n-2)!} & \binom{n}{1}\frac{a^{\omega(n-1)}}{(n-1)!} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{a^{\omega(n-3)}}{(n-3)!} & \binom{n}{2}\frac{a^{\omega(n-2)}}{(n-2)!} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}a^\omega \end{vmatrix}. \quad (3.40)$$

Corollary 3.5: Recurrence relation for 3D-ω-Hermite Charlier polynomials is given by

$$\left(x+C_{0,\omega}^{a}\right)C_{n}^{a}(x,y,z;\omega)+\sum_{k=0}^{n-1}\binom{n}{k}C_{n-k,\omega}^{a}C_{k}^{a}(x,y,z;\omega)$$
$$+xn!\sum_{k=1}^{n}(-\omega)^{k}\frac{C_{n-k}^{a}(x,y,z;\omega)}{(n-k)!}+2nyC_{n-1}^{a}(x,y,z;\omega)$$
$$+2yn!\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^{k}\frac{C_{n-2k-1}^{a}(x,y,z;\omega)}{(n-2k-1)!}+3n(n-1)zC_{n-2}^{a}(x,y,z;\omega)$$
$$+3zn!\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^{k}\frac{C_{n-3k-2}^{a}(x,y,z;\omega)}{(n-3k-2)!}=C_{n+1}^{a}(x,y,z;\omega) \tag{3.41}$$

Corollary 3.6: The 3D-ω–Hermite Charlier polynomials satisfy the following difference equation, lowering and raising operators

$$_{x}L_{n}^{-}=\frac{1}{n\omega}\,{}_{x}\Delta_{\omega} \tag{3.42}$$
$$_{x}L_{n}^{+}=x+C_{0,\omega}^{a}+\sum_{k=0}^{n-1}\frac{C_{n-k,\omega}^{a}}{(n-k)!\omega^{n-k}}\,{}_{x}\Delta_{\omega}^{n-k}+x\sum_{k=1}^{n}(-1)^{k}\,{}_{x}\Delta_{\omega}^{k}$$
$$+2y\frac{_{x}\Delta_{\omega}}{\omega}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^{k}}{\omega^{k+1}}\,{}_{x}\Delta_{\omega}^{2k+1}+3z\frac{_{x}\Delta_{\omega}^{2}}{\omega^{2}}+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^{k}}{\omega^{2k+2}}\,{}_{x}\Delta_{\omega}^{3k+2} \tag{3.43}$$

$$\Bigg[\left(\frac{x}{\omega}+1+\frac{C_{0,\omega}^{a}}{\omega}\right){}_{x}\Delta_{\omega}+\sum_{k=1}^{n-1}\frac{C_{n-k,\omega}^{a}}{(n-k)!\omega^{n-k+1}}\,{}_{x}\Delta_{\omega}^{n-k+1}+\left(\frac{x}{\omega}+1\right)\sum_{k=1}^{n}(-1)^{k}\,{}_{x}\Delta_{\omega}^{k+1}$$
$$+\sum_{k=1}^{n}(-1)^{k}\,{}_{x}\Delta_{\omega}^{k}+2y\frac{_{x}\Delta_{\omega}^{2}}{\omega^{2}}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^{k}}{\omega^{k+2}}\,{}_{x}\Delta_{\omega}^{2k+2}+3z\frac{_{x}\Delta_{\omega}^{3}}{\omega^{3}}$$
$$+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^{k}}{\omega^{2k+3}}\,{}_{x}\Delta_{\omega}^{3k+3}-n\Bigg]C_{n}^{a}(x,y,z;\omega)=0. \tag{3.44}$$

3.4.2 First Kind 3D-ω-Hermite Carlitz Bernoulli Polynomials

This subsection contains explicit form, determinants, recurrence relation, lowering operator, raising operator, and difference equations fulfilled by 3D-ω-Hermite Carlitz Bernoulli polynomials of the first kind.

Corollary 3.7: The $\{\mathscr{B}_{n}(x,y,z;\omega)\}_{n\in\mathbb{N}}$ has an explicit representation,

$$\mathscr{B}_n(x,y,z;\omega)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}\mathscr{B}_{n-k,\omega}\binom{k}{2m}\binom{m}{3l}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_{l}^{\omega}\frac{(2m)!}{m!}\frac{(3l)!}{l!} \tag{3.45}$$

where 3D-ω-Hermite Carlitz Bernoulli numbers $\mathscr{B}_n(0,0,0;\omega)=\mathscr{B}_{n,\omega}$ are given by the series

$$\mathsf{a}(t)=\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}=\sum_{n=0}^{\infty}\mathscr{B}_{n,\omega}\frac{t^n}{n!}. \tag{3.46}$$

The first kind of 3D-ω-Hermite Carlitz Bernoulli numbers are provided by

$$\mathscr{B}_0(\omega)=1,\ \mathscr{B}_1(\omega)=\frac{\omega-1}{2},\ \mathscr{B}_2(\omega)=\frac{(1-\omega)(1+\omega)}{6},$$
$$\mathscr{B}_3(\omega)=\frac{(\omega-1)\,\omega\,(\omega+1)}{4},\ \mathscr{B}_4(\omega)=\frac{(1-\omega)\left(19\omega^3+19\omega^2-\omega-1\right)}{30}.$$

In the case $\omega\to 0$, the first kind 3D-ω-Hermite Carlitz Bernoulli numbers reduce to the first kind Bernoulli numbers.

Some first kind of 3D-ω-Hermite Carlitz Bernoulli polynomials are given by

$$\mathscr{B}_0(x,y,z;\omega)=1,$$
$$\mathscr{B}_1(x,y,z;\omega)=x+\frac{\omega-1}{2},$$
$$\mathscr{B}_2(x,y,z;\omega)=\frac{(1-\omega)(1+\omega)}{6}+x(\omega-1)+x(x-\omega)+2y.$$

In the case $\omega\to 0$ and $y=0$, $z=0$, first kind 3D-ω-Hermite Carlitz Bernoulli polynomials reduce to first kind Bernoulli polynomials.

Corollary 3.8: Determinant satisfied by first kind of 3D-ω-Hermite Carlitz Bernoulli polynomials is given by

$$\mathscr{B}_n(x,y,z;\omega)$$

$$= (-1)^n \begin{vmatrix} 1 & \mathfrak{G}_1^{\omega}(x,y,z) & \mathfrak{G}_2^{\omega}(x,y,z) & \cdots & \mathfrak{G}_{n-1}^{\omega}(x,y,z) & \mathfrak{G}_n^{\omega}(x,y,z) \\ 1 & \frac{(1)_2^{\omega}}{2} & \frac{(1)_3^{\omega}}{3} & \cdots & \frac{(1)_n^{\omega}}{n} & \frac{(1)_{n+1}^{\omega}}{n+1} \\ 0 & 1 & \binom{2}{1}\frac{(1)_2^{\omega}}{2} & \cdots & \binom{n-1}{1}\frac{(1)_{n-1}^{\omega}}{n-1} & \binom{n}{1}\frac{(1)_n^{\omega}}{n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{(1)_{n-2}^{\omega}}{n-2} & \binom{n}{2}\frac{(1)_{n-1}^{\omega}}{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{(1)_2^{\omega}}{2} \end{vmatrix}. \tag{3.47}$$

Corollary 3.9: First-kind 3D-ω-Hermite Carlitz Bernoulli polynomials satisfy a recurrence relation, which is given by

$$\begin{aligned}
&(x+\mathscr{B}_{0,\omega})\mathscr{B}_n(x,y,z;\omega)+\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{B}_{n-k,\omega}\mathscr{B}_k(x,y,z;\omega) \\
&+xn!\sum_{k=1}^{n}(-\omega)^k\frac{\mathscr{B}_{n-k}(x,y,z;\omega)}{(n-k)!}+2ny\mathscr{B}_{n-1}(x,y,z;\omega) \\
&+2yn!\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\frac{\mathscr{B}_{n-2k-1}(x,y,z;\omega)}{(n-2k-1)!}+3n(n-1)z\mathscr{B}_{n-2}(x,y,z;\omega) \\
&+3zn!\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\frac{\mathscr{B}_{n-2k-1}(x,y,z;\omega)}{(n-2k-1)!}=\mathscr{B}_{n+1}(x,y,z;\omega).
\end{aligned} \tag{3.48}$$

Corollary 3.10: Lowering operator, raising operator and difference equation of the first kind 3D-ω-Hermite Carlitz Bernoulli polynomials are given by

$$ {}_xL_n^- = \frac{1}{n\omega}\,{}_x\Delta_\omega \tag{3.49}$$

$$\begin{aligned}
{}_xL_n^+ &= x+\mathscr{B}_{0,\omega}+\sum_{k=0}^{n-1}\frac{\mathscr{B}_{n-k,\omega}}{(n-k)!\,\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}+x\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k \\
&+2y\frac{{}_x\Delta_\omega}{\omega}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+1}}\,{}_x\Delta_\omega^{2k+1}+3z\frac{{}_x\Delta_\omega^2}{\omega^2}+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+2}}\,{}_x\Delta_\omega^{3k+2}
\end{aligned} \tag{3.50}$$

$$\left[\left(\frac{x}{\omega}+1+\frac{\mathscr{B}_{0,\omega}}{\omega}\right){}_x\Delta_\omega+\sum_{k=1}^{n-1}\frac{\mathscr{B}_{n-k,\omega}}{(n-k)!\,\omega^{n-k+1}}\,{}_x\Delta_\omega^{n-k+1}+\left(\frac{x}{\omega}+1\right)\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^{k+1}\right.$$
$$+\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k+2y\frac{{}_x\Delta_\omega^2}{\omega^2}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+2}}\,{}_x\Delta_\omega^{2k+2}+3z\frac{{}_x\Delta_\omega^3}{\omega^3}$$
$$\left.+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+3}}\,{}_x\Delta_\omega^{3k+3}-n\right]\mathscr{B}_n(x,y,z;\omega)=0, \tag{3.51}$$

respectively.

3.4.3 3D-ω-Hermite Carlitz Euler Polynomials

The explicit form, determinants, recurrence relation, lowering operator, raising operator, and difference equation defined by 3D-ω-Hermite Carlitz Euler polynomials are studied in this subsection.

Corollary 3.11: 3D-ω-Hermite Carlitz Euler polynomial sequence has the following explicit representation

$$\mathcal{E}_n(x,y,z;\omega)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}e_{n-k,\omega}\binom{k}{2m}\binom{m}{3l}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_l^{\omega}\frac{(2m)!}{m!}\frac{(3l)!}{l!}, \tag{3.52}$$

where 3D-ω-Hermite Carlitz Euler numbers $\mathcal{E}_n(0,0,0;\omega)=\mathcal{E}_{n,\omega}$ are given by the formula

$$a(t)=\frac{2}{(1+\omega t)^{\frac{1}{\omega}}+1}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\omega}\frac{t^n}{n!}. \tag{3.53}$$

Some first few 3D-ω-Hermite Carlitz Euler numbers are given by

$$\mathcal{E}_{0,\omega}=1,\ \mathcal{E}_{1,\omega}=-\frac{1}{2},\ \mathcal{E}_{2,\omega}=\frac{\omega}{2}$$
$$\mathcal{E}_{3,\omega}=\frac{1}{4}-\omega^2,\ \mathcal{E}_{4,\omega}=3\omega^3+\frac{\omega^2}{2}-2\omega.$$

Some 3D-ω-Hermite Carlitz Euler polynomials are given by

$$\mathcal{E}_0(x,y,z;\omega) = 1,$$

$$\mathcal{E}_1(x,y,z;\omega) = x - \frac{1}{2},$$

$$\mathcal{E}_2(x,y,z;\omega) = \frac{\omega}{2} - x + x(x-\omega) + 2y.$$

Corollary 3.12: 3D-ω-Hermite Carlitz Euler polynomials has the following determinantal representation,

$$\mathcal{E}_n(x,y,z;\omega)$$
$$= (-1)^n \begin{vmatrix} 1 & \mathfrak{G}_1^{\omega}(x,y,z) & \mathfrak{G}_2^{\omega}(x,y,z) & \cdots & \mathfrak{G}_{n-1}^{\omega}(x,y,z) & \mathfrak{G}_n^{\omega}(x,y,z) \\ 1 & \frac{1}{2} & \frac{1}{2}(1)_2 & \cdots & \frac{1}{2}(1)_{n-1} & \frac{1}{2}(1)_n \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2}(1)_{n-2} & \binom{n}{1}\frac{1}{2}(1)_{n-1} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2}(1)_{n-3} & \binom{n}{2}\frac{1}{2}(1)_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}. \tag{3.54}$$

Corollary 3.13: The following recurrence relation is satisfied by 3D-ω-Hermite Carlitz Euler polynomials,

$$\begin{aligned} &(x+e_{0,\omega})\mathcal{E}_n(x,y,z;\omega) + \sum_{k=0}^{n-1}\binom{n}{k} e_{n-k,\omega}\mathcal{E}_k(x,y,z;\omega) \\ &+ xn!\sum_{k=1}^{n}(-\omega)^k \frac{\mathcal{E}_{n-k}(x,y,z;\omega)}{(n-k)!} + 2ny\mathcal{E}_{n-1}(x,y,z;\omega) \\ &+ 2yn!\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k \frac{\mathcal{E}_{n-2k-1}(x,y,z;\omega)}{(n-2k-1)!} + 3n(n-1)z\mathcal{E}_{n-2}(x,y,z;\omega) \\ &+ 3zn!\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^k \frac{\mathcal{E}_{n-3k-2}(x,y,z;\omega)}{(n-3k-2)!} = \mathcal{E}_{n+1}(x,y,z;\omega) \end{aligned} \tag{3.55}$$

Corollary 3.14: Lowering operator, raising operator and difference equation satisfied by 3D-ω-Hermite Carlitz Euler polynomials are given by

$$
\begin{aligned}
{}_xL_n^- &= \frac{1}{n\omega}\,{}_x\Delta_\omega && (3.56)\\
{}_xL_n^+ &= x + e_{0,\omega} + \sum_{k=0}^{n-1} \frac{e_{n-k,\omega}}{(n-k)!\,\omega^{n-k}}\,{}_x\Delta_\omega^{n-k} + x\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k + 2y\frac{{}_x\Delta_\omega}{\omega}\\
&\quad + 2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k}{\omega^{k+1}}\,{}_x\Delta_\omega^{2k+1} + 3z\frac{{}_x\Delta_\omega^2}{\omega^2} + 3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]} \frac{(-1)^k}{\omega^{2k+2}}\,{}_x\Delta_\omega^{3k+2} && (3.57)
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg[\left(\frac{x}{\omega}+1+\frac{e_{0,\omega}}{\omega}\right){}_x\Delta_\omega + \sum_{k=1}^{n-1}\frac{e_{n-k,\omega}}{(n-k)!\,\omega^{n-k+1}}\,{}_x\Delta_\omega^{n-k+1} + \left(\frac{x}{\omega}+1\right)\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^{k+1}\\
&+\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k + 2y\frac{{}_x\Delta_\omega^2}{\omega^2} + 2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+2}}\,{}_x\Delta_\omega^{2k+2} + 3z\frac{{}_x\Delta_\omega^3}{\omega^3}\\
&+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+3}}\,{}_x\Delta_\omega^{3k+3} - n\Bigg]\,\mathcal{E}_n(x,y,z;\omega) = 0. && (3.58)
\end{aligned}
$$

3.4.4 3D-ω-Hermite Boole Polynomials

The explicit form, determinants, recurrence relation, lowering operator, raising operator, and difference equation satisfied by 3D-ω-Hermite Boole polynomials are presented in this subsection.

Corollary 3.15: 3D-ω-Hermite Boole polynomials sequence has the explicit form

$$
\begin{aligned}
&\mathscr{B}l_n(x,y,z;\lambda;\omega)\\
&= \sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{\left[\frac{m}{3}\right]}\mathscr{B}l_{n-k,\omega}\binom{k}{2m}\binom{m}{3l}(x)_{k-2m}^{\omega}(y)_{m-3l}^{\omega}(z)_l^{\omega}\frac{(2m)!}{m!}\frac{(3l)!}{l!}, \qquad (3.59)
\end{aligned}
$$

where 3D-ω-Hermite Boole numbers $\mathscr{B}l_n(0,0,0;\omega) = \mathscr{B}l_n(\lambda;\omega) = \mathscr{B}l_{n,\omega}$ are given by

$$
a(t) = \frac{1}{1+(1+\omega t)^{\frac{\lambda}{\omega}}} = \sum_{n=0}^{\infty}\mathscr{B}l_{n,\omega}\frac{t^n}{n!}. \qquad (3.60)
$$

Some 3D-ω-Hermite Boole numbers are given by

$$
\begin{aligned}
&\mathscr{B}l_0(\lambda;\omega) = \frac{1}{2},\ \mathscr{B}l_1(\lambda;\omega) = -\frac{\lambda}{4},\ \mathscr{B}l_2(\lambda;\omega) = \frac{\lambda\omega}{4}\\
&\mathscr{B}l_3(\lambda;\omega) = \frac{\lambda^3-4\lambda\omega^2}{8},\ \mathscr{B}l_4(\lambda;\omega) = \frac{3}{4}\lambda^3\omega.
\end{aligned}
$$

In the case $\omega = 1$, $y = 0$ and $z = 0$, 3D-ω-Hermite Boole numbers reduce to the Boole numbers.

Some 3D-ω-Hermite Boole polynomials are given by

$$\begin{aligned}
\mathscr{B}l_0(x,y,z;\lambda;\omega) &= \frac{1}{2} \\
\mathscr{B}l_1(x,y,z;\lambda;\omega) &= \frac{x}{2} - \frac{\lambda}{4} \\
\mathscr{B}l_2(x,y,z;\lambda;\omega) &= \frac{\lambda\omega - 2x\lambda + 2x(x-\omega) + 4y}{4}.
\end{aligned}$$

In the case $\omega = 1$, $y = 0$ and $z = 0$, 3D-Hermite Boole polynomials reduce to the Boole polynomials.

Corollary 3.16: 3D-ω-Hermite Boole polynomials satisfy the determinant given by

$$\mathscr{B}l_n(x,y,z;\lambda;\omega)$$
$$= (-1)^n \begin{vmatrix}
1 & \mathfrak{G}_1^{\omega}(x,y,z) & \mathfrak{G}_2^{\omega}(x,y,z) & \cdots & \mathfrak{G}_{n-1}^{\omega}(x,y,z) & \mathfrak{G}_n^{\omega}(x,y,z) \\
1 & (\lambda)_1^{\omega} & (\lambda)_2^{\omega} & \cdots & (\lambda)_{n-1}^{\omega} & (\lambda)_n^{\omega} \\
0 & 1 & \binom{2}{1}(\lambda)_1^{\omega} & \cdots & \binom{n-1}{1}(\lambda)_{n-2}^{\omega} & \binom{n}{1}(\lambda)_{n-1}^{\omega} \\
0 & 0 & 1 & \cdots & \binom{n-1}{2}(\lambda)_{n-3}^{\omega} & \binom{n}{2}(\lambda)_{n-2}^{\omega} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}(\lambda)_1^{\omega}
\end{vmatrix}. \tag{3.61}$$

Corollary 3.17: Recurrence relation satisfied by 3D-ω-Hermite Boole polynomials is given by

$$
\begin{aligned}
&(x+\mathscr{B}l_{0,\omega}(\lambda;\omega))\,\mathscr{B}l_n(x,y,z;\lambda;\omega)+\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{B}l_{n-k,\omega}(\lambda;\omega)\,\mathscr{B}l_k(x,y,z;\lambda;\omega)\\
&+xn!\sum_{k=1}^{n}(-\omega)^k\frac{\mathscr{B}l_{n-k}(x,y,z;\lambda;\omega)}{(n-k)!}+2ny\mathscr{B}l_{n-1}(x,y,z;\lambda;\omega)\\
&+2yn!\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-\omega)^k\frac{\mathscr{B}l_{n-2k-1}(x,y,z;\lambda;\omega)}{(n-2k-1)!}+3n(n-1)z\mathscr{B}l_{n-2}(x,y,z;\lambda;\omega)\\
&+3zn!\sum_{k=1}^{\left[\frac{n-2}{3}\right]}(-\omega)^k\frac{\mathscr{B}l_{n-3k-2}(x,y,z;\lambda;\omega)}{(n-3k-2)!}=\mathscr{B}l_{n+1}(x,y,z;\lambda;\omega).
\end{aligned}
\tag{3.62}
$$

Corollary 3.18: 3D-ω-Hermite Boole polynomials satisfy has the following lowering and raising operators and they satisfy the difference equation given in (3.65),

$$
{}_xL_n^-=\frac{1}{n\omega}\,{}_x\Delta_\omega \tag{3.63}
$$

$$
\begin{aligned}
{}_xL_n^+&=x+\mathscr{B}l_{0,\omega}(\lambda;\omega)+\sum_{k=0}^{n-1}\frac{\mathscr{B}l_{n-k,\omega}(\lambda;\omega)}{(n-k)!\,\omega^{n-k}}\,{}_x\Delta_\omega^{n-k}+x\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k\\
&+2y\frac{{}_x\Delta_\omega}{\omega}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+1}}\,{}_x\Delta_\omega^{2k+1}+3z\frac{{}_x\Delta_\omega^2}{\omega^2}+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+2}}\,{}_x\Delta_\omega^{3k+2}
\end{aligned}
\tag{3.64}
$$

$$
\begin{aligned}
&\Bigg[\left(\frac{x}{\omega}+1+\frac{\mathscr{B}l_{0,\omega}(\lambda;\omega)}{\omega}\right){}_x\Delta_\omega+\sum_{k=1}^{n-1}\frac{\mathscr{B}l_{n-k}(\lambda;\omega)}{(n-k)!\,\omega^{n-k+1}}\,{}_x\Delta_\omega^{n-k+1}\\
&+\left(\frac{x}{\omega}+1\right)\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^{k+1}+\sum_{k=1}^{n}(-1)^k\,{}_x\Delta_\omega^k\\
&+2y\frac{{}_x\Delta_\omega^2}{\omega^2}+2y\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{\omega^{k+2}}\,{}_x\Delta_\omega^{2k+2}+3z\frac{{}_x\Delta_\omega^3}{\omega^3}\\
&+3z\sum_{k=1}^{\left[\frac{n-2}{3}\right]}\frac{(-1)^k}{\omega^{2k+3}}\,{}_x\Delta_\omega^{3k+3}-n\Bigg]\mathscr{B}l_n(x,y,z;\lambda;\omega)=0.
\end{aligned}
\tag{3.65}
$$

Chapter 4

BIVARIATE Δ_ω-MULTIPLE APPELL POLYNOMIALS

In chapter 4, we intoduce Δ_ω-Hermite Appell polynomials and we give an explicit form and recurrence relations for them. First, we define Δ_ω-Hermite Appell polynomials. First of all, we start with defining the bivariate multiple ω-Hermite polynomials.

The bivariate multiple ω-Hermite polynomials are defined by

$$(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathsf{G}_{n_1,n_2}^{\omega}(x,y)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}. \tag{4.1}$$

Theorem 4.1: The polynomial $\mathsf{G}_{n_1,n_2}^{\omega}(x,y)$ has the following explicit form,

$$\begin{aligned}
&\mathsf{G}_{n_1,n_2}^{\omega}(x,y)\\
&=\sum_{m=0}^{n_1}\sum_{r=0}^{n_2}\sum_{s=0}^{k_1}\sum_{l=0}^{k_2}\sum_{k_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_2=0}^{\left[\frac{n_2}{2}\right]}\binom{n_1}{2k_1}\binom{n_2}{2k_2}\binom{n_1-2k_1}{m}\binom{n_2-2k_2}{r}\\
&\times\binom{k_1}{s}\binom{k_2}{l}(x)_{n_1-2k_1}^{\omega}(x)_{n_2-2k_2}^{\omega}(y)_{k_1}^{\omega}(y)_{k_2}^{\omega}\frac{(2k_1)!}{k_1!}\frac{(2k_2)!}{k_2!}
\end{aligned}$$

Proof. Starting from the left hand side of (4.1), we have

$$\begin{aligned}
&(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(x)_{n_1}^{\omega}(x)_{n_2}^{\omega}\frac{(t_1+t_2)^{n_1}(t_1+t_2)^{n_2}}{n_1!n_2!}\\
&\times\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(y)_{k_1}^{\omega}(y)_{k_2}^{\omega}\frac{\left(t_1^2+t_2^2\right)^{k_1}\left(t_1^2+t_2^2\right)^{k_2}}{k_1!k_2!}
\end{aligned}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{(x)_{n_1}^{\omega}(x)_{n_2}^{\omega}}{n_1!n_2!} \sum_{m=0}^{n_1}\sum_{r=0}^{n_2} \binom{n_1}{m}\binom{n_2}{r} t_1^{n_1} t_2^{n_2}$$

$$\times \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(y)_{k_1}^{\omega}(y)_{k_2}^{\omega}}{k_1!k_2!} \sum_{s=0}^{k_1}\sum_{l=0}^{k_2} \binom{k_1}{s}\binom{k_2}{l} t_1^{2k_1} t_2^{2k_2}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{m=0}^{n_1}\sum_{r=0}^{n_2}\sum_{s=0}^{k_1}\sum_{l=0}^{k_2} \binom{n_1}{m}\binom{n_2}{r}\binom{k_1}{s}\binom{k_2}{l}$$

$$\times (x)_{n_1}^{\omega}(x)_{n_2}^{\omega}(y)_{k_1}^{\omega}(y)_{k_2}^{\omega} \frac{t_1^{n_1+2k_1} t_2^{n_2+2k_2}}{n_1!n_2!k_1!k_2!}.$$

Applying the Cauchy product rule, we get

$$\left(1+\omega\left(t_1+t_2\right)\right)^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{m=0}^{n_1}\sum_{r=0}^{n_2}\sum_{s=0}^{k_1}\sum_{l=0}^{k_2}\sum_{k_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_2=0}^{\left[\frac{n_2}{2}\right]} \binom{n_1}{2k_1}\binom{n_2}{2k_2}\binom{n_1-2k_1}{m}\binom{n_2-2k_2}{r}$$

$$\times \binom{k_1}{s}\binom{k_2}{l}(x)_{n_1-2k_1}^{\omega}(x)_{n_2-2k_2}^{\omega}(y)_{k_1}^{\omega}(y)_{k_2}^{\omega} \frac{(2k_1)!}{k_1!}\frac{(2k_2)!}{k_2!}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Comparing the coefficient's of $\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$, we have

$$G_{n_1,n_2}^{\omega}(x,y) = \sum_{m=0}^{n_1}\sum_{r=0}^{n_2}\sum_{s=0}^{k_1}\sum_{l=0}^{k_2}\sum_{k_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_2=0}^{\left[\frac{n_2}{2}\right]} \binom{n_1}{2k_1}\binom{n_2}{2k_2}\binom{n_1-2k_1}{m}\binom{n_2-2k_2}{r}$$

$$\times \binom{k_1}{s}\binom{k_2}{l}(x)_{n_1-2k_1}^{\omega}(x)_{n_2-2k_2}^{\omega}(y)_{k_1}^{\omega}(y)_{k_2}^{\omega} \frac{(2k_1)!}{k_1!}\frac{(2k_2)!}{k_2!}.$$

Whence the result. □

Definition 4.1: Δ_ω-multiple Appell polynomials are defined by

$$A\left(t_1,t_2\right)\left(1+\omega\left(t_1+t_2\right)\right)^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A_{n_1,n_2}\left(x,y,\omega\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$

where $A_{n_1,n_2}(0,0,\omega) = \alpha_{n_1,n_2,\omega}$ are given by the series

$$A\left(t_1,t_2\right) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \alpha_{k_1,k_2,\omega}\frac{t_1^{k_1}}{k_1!}\frac{t_2^{k_2}}{k_2!}, \quad \alpha_{0,0,\omega} \neq 0.$$

Remark 4.1: Taking limit as $\omega \to 0$ in the above definition, we get

$$A(t_1,t_2)\,e^{x(t_1+t_2)+y\left(t_1^2+t_2^2\right)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A_{n_1,n_2}(x,y)\,\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$

where we can call these polynomials as bivariate multiple Appell polynomials, since the case $y = 0$ reduces to

$$A(t_1,t_2)\,e^{x(t_1+t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A_{n_1,n_2}(x)\,\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$

which gives the definition of multiple Appell polynomials defined and investigated in [20].

Theorem 4.2: The following properties are satisfied by Δ_ω-multiple Appell polynomials

$$_{x}\Delta_\omega A_{n_1,n_2}(x,y,\omega) = n_1\omega A_{n_1-1,n_2}(x,y,\omega) + n_2\omega \mathbb{A}_{n_1,n_2-1}(x,y,\omega) \tag{4.2}$$

and

$$_{y}\Delta_\omega A_{n_1,n_2}(x,y,\omega) = n_1(n_1-1)\,\omega A_{n_1-2,n_2}(x,y,\omega) + n_2(n_2-1)\,\omega A_{n_1,n_2-2}(x,y,\omega). \tag{4.3}$$

Proof. We will give the proof for (4.2). The proof of (4.3) is similar.

Applying the difference operator $_{x}\Delta_\omega$, we get

$$\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} {}_{x}\Delta_\omega A_{n_1,n_2}(x,y,\omega)\,\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&={}_{x}\Delta_\omega\left[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A_{n_1,n_2}(x,y,\omega)\,\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\right]\\
&={}_{x}\Delta_\omega\left[A(t_1,t_2)\,(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}\right].
\end{aligned}$$

Using the Cauchy product rule,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} {}_x\Delta_\omega A_{n_1,n_2}(x,y,\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$

$$={}_x\Delta_\omega\left[A(t_1,t_2)\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}\right](1+\omega(t_1+t_2))^{\frac{x}{\omega}}$$

$$+A(t_1,t_2)\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}\ {}_x\Delta_\omega(1+\omega(t_1+t_2))^{\frac{x}{\omega}}$$

$$=\omega(t_1+t_2)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y,\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$

$$=\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y,\omega)\frac{t_1^{n_1+1}}{n_1!}\frac{t_2^{n_2}}{n_2!}+\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y,\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2+1}}{n_2!}$$

$$=\omega n_1\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1-1,n_2}(x,y,\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}+\omega n_2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2-1}(x,y,\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$

$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\omega n_1 A_{n_1-1,n_2}(x,y,\omega)+\omega n_2 A_{n_1,n_2-1}(x,y,\omega)\right]\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Comparing the coefficients $\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$, we get

$${}_x\Delta_\omega A_{n_1,n_2}(x,y,\omega)=n_1\omega A_{n_1-1,n_2}(x,y,\omega)+n_2\omega A_{n_1,n_2-1}(x,y,\omega).$$

□

Theorem 4.3: The double sequence of polynomials $\{A_{n_1,n_2}(x,y,\omega)\}_{n_1,n_2\in\mathbb{N}}$ has the explicit form

$$A_{n_1,n_2}(x,y,\omega)$$
$$=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{m=0}^{k_1}\sum_{r=0}^{k_2}\sum_{s=0}^{j_1}\sum_{l=0}^{j_2}\sum_{j_1=0}^{\left[\frac{k_1}{2}\right]}\sum_{j_2=0}^{\left[\frac{k_2}{2}\right]}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{k_1}{2j_1}\binom{k_2}{2j_2}\binom{k_1-2j_1}{m}$$
$$\times\binom{k_2-2j_2}{r}\binom{j_1}{s}\binom{j_2}{l}a_{n_1-k_1,n_2-k_2}(x)_{k_1-2j_1}^{\omega}(x)_{k_2-2j_2}^{\omega}(y)_{j_1}^{\omega}(y)_{j_2}^{\omega}$$
$$\times\frac{(2j_1)!}{j_1!}\frac{(2j_2)!}{j_2!}\tag{4.4}$$

, where the coefficients $\{a_{n_1,n_2}\}_{n_1,n_2\in\mathbb{N}}$ are given by

$$A(t_1,t_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}a_{n_1,n_2}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Proof. Multiplying both sides of (4.1) by $A(t_1,t_2)$, we have

$$A(t_1,t_2)(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}$$
$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}A(t_1,t_2)\,\mathsf{G}_{k_1,k_2}^{\omega}(x,y)\frac{t_1^{k_1}}{k_1!}\frac{t_2^{k_2}}{k_2!}$$
$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y,\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}a_{n_1,n_2}\mathsf{G}_{k_1,k_2}^{\omega}(x,y)\frac{t_1^{n_1+k_1}}{n_1!k_1!}\frac{t_2^{n_2+k_2}}{n_2!k_2!}.$$

Applying the Cauchy product, we get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y,\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}a_{n_1-k_1,n_2-k_2}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$\times\sum_{m=0}^{k_1}\sum_{r=0}^{k_2}\sum_{s=0}^{j_1}\sum_{l=0}^{j_2}\sum_{j_1=0}^{\left[\frac{k_1}{2}\right]}\sum_{j_2=0}^{\left[\frac{k_2}{2}\right]}\binom{k_1}{2j_1}\binom{k_2}{2j_2}\binom{k_1-2j_1}{m}\binom{k_2-2j_2}{r}\binom{j_1}{s}\binom{j_2}{l}$$
$$\times(x)_{k_1-2j_1}^{\omega}(x)_{k_2-2j_2}^{\omega}(y)_{j_1}^{\omega}(y)_{j_2}^{\omega}\frac{(2j_1)!}{j_1!}\frac{(2j_2)!}{j_2!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{m=0}^{k_1}\sum_{r=0}^{k_2}\sum_{s=0}^{j_1}\sum_{l=0}^{j_2}\sum_{j_1=0}^{\left[\frac{k_1}{2}\right]}\sum_{j_2=0}^{\left[\frac{k_2}{2}\right]}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{k_1}{2j_1}\binom{k_2}{2j_2}$$
$$\times\binom{k_1-2j_1}{m}\binom{k_2-2j_2}{r}\binom{j_1}{s}\binom{j_2}{l}a_{n_1-k_1,n_2-k_2}(x)_{k_1-2j_1}^{\omega}(x)_{k_2-2j_2}^{\omega}$$
$$\times\frac{(2j_1)!}{j_1!}\frac{(2j_2)!}{j_2!}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Comparing the coefficients $\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$, we get

$$A_{n_1,n_2}(x,y,\omega)$$
$$=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{m=0}^{k_1}\sum_{r=0}^{k_2}\sum_{s=0}^{j_1}\sum_{l=0}^{j_2}\sum_{j_1=0}^{\left[\frac{k_1}{2}\right]}\sum_{j_2=0}^{\left[\frac{k_2}{2}\right]}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{k_1}{2j_1}\binom{k_2}{2j_2}$$
$$\times\binom{k_1-2j_1}{m}\binom{k_2-2j_2}{r}\binom{j_1}{s}\binom{j_2}{l}a_{n_1-k_1,n_2-k_2}(x)_{k_1-2j_1}^{\omega}(x)_{k_2-2j_2}^{\omega}(y)_{j_1}^{\omega}(y)_{j_2}^{\omega}$$
$$\times\frac{(2j_1)!}{j_1!}\frac{(2j_2)!}{j_2!},$$

which completes the proof. □

Theorem 4.4: Δ_ω-multiple Appell polynomials satisfy the recurrence relations

$$\begin{aligned}
&A_{n_1+1,n_2}(x,y;\omega)\\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}\gamma_{n_1-k_1,n_2-k_2,\omega}A_{k_1,k_2}(x,y;\omega)\\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{A_{n_1-k_1,n_2-k_2(x,y;\omega)}(x,y;\omega)}{(n_1-k_1)!(n_2-k_2)!}\\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{k_2=0}^{n_2}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{A_{n_1-2k_1-1,n_2-k_2}(x,y;\omega)}{(n_1-2k_1-1)!(n_2-k_2)!} \qquad (4.5)
\end{aligned}$$

and

$$\begin{aligned}
&A_{n_1,n_2+1}(x,y;\omega)\\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}\gamma_{n_1-k_1,n_2-k_2,\omega}A_{k_1,k_2}(x,y;\omega)\\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{A_{n_1-k_1,n_2-k_2}(x,y;\omega)}{(n_1-k_1)!(n_2-k_2)!}\\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_2=0}^{n_2-1}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{A_{n_1-2k_1,n_2-k_2-1}(x,y;\omega)}{(n_1-2k_1)!(n_2-k_2-1)!} \qquad (4.6)
\end{aligned}$$

where

$$\frac{A'(t_1,t_2)}{A(t_1,t_2)}=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\gamma_{k_1,k_2,\omega}\frac{t_1^{k_1}}{k_1!}\frac{t_2^{k_2}}{k_2!}.$$

Proof. We will give the proof for (4.5),

$$A(t_1,t_2)(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!} \qquad (4.7)$$

Lets take the derivative both sides of (4.7) with respect to t_1 to get

$$\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1+1,n_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&=\frac{A'(t_1,t_2)}{A(t_1,t_2)}A(t_1,t_2)(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}\\
&+(1+\omega(t_1+t_2))^{-1}xA(t_1,t_2)(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}}\\
&+\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{-1}2yt_1A(t_1,t_2)(1+\omega(t_1+t_2))^{\frac{x}{\omega}}\left(1+\omega\left(t_1^2+t_2^2\right)\right)^{\frac{y}{\omega}} \qquad (4.8)
\end{aligned}$$

, where

$$\left(1+\omega\left(t_1+t_2\right)\right)^{-1}=\sum_{k_1=0}^{\infty}(-\omega)^{k_1}\left(t_1+t_2\right)^{k_1}$$
$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}t_1^{k_1}t_2^{k_2}, \tag{4.9}$$

$$\frac{1}{1+\omega\left(t_1^2+t_2^2\right)}=\sum_{k_1=0}^{\infty}(-\omega)^{k_1}\left(t_1+t_2\right)^{2k_1}$$
$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}t_1^{2k_1}t_2^{k_2}. \tag{4.10}$$

Using (4.9) and (4.10) in (4.8), we get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1+1,n_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\gamma_{k_1,k_2,\omega}\frac{t_1^{k_1}}{k_1!}\frac{t_2^{k_2}}{k_2!}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$+x\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}t_1^{k_1}t_2^{k_2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$+2y\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}t_1^{2k_1}t_2^{k_2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1,n_2}(x,y;\omega)\frac{t_1^{n_1+1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1+1,n_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\gamma_{k_1,k_2,\omega}A_{n_1,n_2}(x,y;\omega)\frac{t_1^{n_1+k_1}t_2^{n_2+k_2}}{k_1!k_2!n_1!n_2!}$$
$$+x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}A_{n_1,n_2}(x,y;\omega)\frac{t_1^{n_1+k_1}t_2^{n_2+k_2}}{n_1!n_2!}$$
$$+2y\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}A_{n_1,n_2}(x,y;\omega)\frac{t_1^{n_1+2k_1+1}t_2^{n_2+k_2}}{n_1!n_2!}$$
$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}A_{n_1+1,n_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\gamma_{k_1,k_2,\omega}A_{n_1-k_1,n_2-k_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$+x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{n_1!}{(n_1-k_1)!}\frac{n_2!}{(n_2-k_2)!}$$

$$\times A_{n_1-k_1,n_2-k_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}+2y\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{k_2=0}^{n_2}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}$$

$$\times\frac{n_1!}{(n_1-2k_1-1)!}\frac{n_2!}{(n_2-k_2)!}A_{n_1-2k_1-1,n_2-k_2}(x,y;\omega)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Comparing the coefficients of $\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$,

$$\begin{aligned}&A_{n_1+1,n_2}(x,y;\omega)\\&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}\gamma_{n_1-k_1,n_2-k_2,\omega}A_{k_1,k_2}(x,y;\omega)\\&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{A_{n_1-k_1,n_2-k_2}(x,y;\omega)}{(n_1-k_1)!(n_2-k_2)!}\\&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{k_2=0}^{n_2}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{A_{n_1-2k_1-1,n_2-k_2}(x,y;\omega)}{(n_1-2k_1-1)!(n_2-k_2)!}.\end{aligned}$$

The relation (4.6) can be proved in a similar manner. □

4.1 Explicit Forms and Recurrence Relations for Δ_ω-multiple Carlitz Euler Polynomials

Corollary 4.1: Δ_ω-multiple Carlitz Euler polynomial sequence has the explicit form,

$$\begin{aligned}&\mathcal{E}_{n_1,n_2}(x,y;\omega)\\&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{m=0}^{k_1}\sum_{r=0}^{k_2}\sum_{s=0}^{j_1}\sum_{l=0}^{j_2}\sum_{j_1=0}^{\left[\frac{k_1}{2}\right]}\sum_{j_2=0}^{\left[\frac{k_2}{2}\right]}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{k_1}{2j_1}\binom{k_2}{2j_2}\binom{k_1-2j_1}{m}\binom{k_2-2j_2}{r}\\&\times\binom{j_1}{s}\binom{j_2}{l}\mathcal{E}_{n_1-k_1,n_2-k_2}(x)_{k_1-2j_1}^{\omega}(x)_{k_2-2j_2}^{\omega}(y)_{j_1}^{\omega}(y)_{j_2}^{\omega}\frac{(2j_1)!}{j_1!}\frac{(2j_2)!}{j_2!},\end{aligned}$$

where

$$A(t_1,t_2)=\frac{2}{(1+\omega(t_1+t_2))^{\frac{1}{\omega}}+1}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathcal{E}_{n_1,n_2,\omega}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Corollary 4.2: Δ_ω-multiple Carlitz Euler polynomials that satisfy the following recurrence relations,

$$\begin{aligned}
&\mathcal{E}_{n_1+1,n_2}(x,y;\omega) \\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}e_{n_1-k_1,n_2-k_2,\omega}\mathcal{E}_{k_1,k_2}(x,y;\omega) \\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{\mathcal{E}_{n_1-k_1,n_2-k_2}(x,y;\omega)}{(n_1-k_1)!(n_2-k_2)!} \\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{k_2=0}^{n_2}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{\mathcal{E}_{n_1-2k_1-1,n_2-k_2}(x,y;\omega)}{(n_1-2k_1-1)!(n_2-k_2)!}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{E}_{n_1,n_2+1}(x,y;\omega) \\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}e_{n_1-k_1,n_2-k_2,\omega}\mathcal{E}_{k_1,k_2}(x,y;\omega) \\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{\mathcal{E}_{n_1-k_1,n_2-k_2}(x,y;\omega)}{(n_1-k_1)!(n_2-k_2)!} \\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_2=0}^{n_2-1}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{\mathcal{E}_{n_1-2k_1,n_2-k_2-1}(x,y;\omega)}{(n_1-2k_1)!(n_2-k_2-1)!}.
\end{aligned}$$

4.2 Explicit Forms and Recurrence Relations for Δ_ω-multiple Boole Polynomials

Corollary 4.3: Δ_ω-multiple Boole polynomial sequence has explicit form

$$\begin{aligned}
&Bl_{n_1,n_2}(x,y,\lambda;\omega) \\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{m=0}^{k_1}\sum_{r=0}^{k_2}\sum_{s=0}^{j_1}\sum_{l=0}^{j_2}\sum_{j_1=0}^{\left[\frac{k_1}{2}\right]}\sum_{j_2=0}^{\left[\frac{k_2}{2}\right]}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{k_1}{2j_1}\binom{k_2}{2j_2}\binom{k_1-2j_1}{m}\binom{k_2-2j_2}{r} \\
&\binom{j_1}{s}\binom{j_2}{l}Bl_{n_1-k_1,n_2-k_2}(\lambda)(x)_{k_1-2j_1}^{\omega}(x)_{k_2-2j_2}^{\omega}(y)_{j_1}^{\omega}(y)_{j_2}^{\omega}\frac{(2j_1)!}{j_1!}\frac{(2j_2)!}{j_2!},
\end{aligned}$$

where

$$A(t_1,t_2)=\frac{1}{1+(1+\omega(t_1+t_2))^{\frac{\lambda}{\omega}}}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}Bl_{n_1,n_2,\omega}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Corollary 4.4: Recurrence relations satisfied by Δ_ω-multiple Boole polynomials are given by

$$
\begin{aligned}
&Bl_{n_1+1,n_2}(x,y,\lambda;\omega)\\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}bl_{n_1-k_1,n_2-k_2,\omega}Bl_{k_1,k_2}(x,y,\lambda;\omega)\\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{Bl_{n_1-k_1,n_2-k_2}(x,y,\lambda;\omega)}{(n_1-k_1)!(n_2-k_2)!}\\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{k_2=0}^{n_2}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{Bl_{n_1-2k_1-1,n_2-k_2}(x,y,\lambda;\omega)}{(n_1-2k_1-1)!(n_2-k_2)!}
\end{aligned}
$$

and

$$
\begin{aligned}
&Bl_{n_1,n_2+1}(x,y,\lambda;\omega)\\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}bl_{n_1-k_1,n_2-k_2,\omega}Bl_{k_1,k_2}(x,y,\lambda;\omega)\\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{Bl_{n_1-k_1,n_2-k_2}(x,y,\lambda;\omega)}{(n_1-k_1)!(n_2-k_2)!}\\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_2=0}^{n_2-1}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{Bl_{n_1-2k_1,n_2-k_2-1}(x,y,\lambda;\omega)}{(n_1-2k_1)!(n_2-k_2-1)!}.
\end{aligned}
$$

4.3 Explicit Forms and Recurrence Relations for Δ_ω-multiple Charlier Polynomials

Corollary 4.5: Δ_ω-multiple Charlier polynomial sequence has explicit form

$$
\begin{aligned}
&C_{n_1,n_2}^{a_1,a_2}(x,y;\omega)\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{m=0}^{k_1}\sum_{r=0}^{k_2}\sum_{s=0}^{j_1}\sum_{l=0}^{j_2}\sum_{j_1=0}^{\left[\frac{k_1}{2}\right]}\sum_{j_2=0}^{\left[\frac{k_2}{2}\right]}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{k_1}{2j_1}\binom{k_2}{2j_2}\binom{k_1-2j_1}{m}\binom{k_2-2j_2}{r}\\
&\times\binom{j_1}{s}\binom{j_2}{l}c_{n_1-k_1,n_2-k_2}^{a_1,a_2}(x)_{k_1-2j_1}^{\omega}(x)_{k_2-2j_2}^{\omega}(y)_{j_1}^{\omega}(y)_{j_2}^{\omega}\frac{(2j_1)!}{j_1!}\frac{(2j_2)!}{j_2!}
\end{aligned}
$$

where

$$A(t_1,t_2)=exp(-a_1^{\omega}t_1-a_2^{\omega}t_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Corollary 4.6: Recurrence relations satisfied by Δ_ω-multiple Charlier polynomials are given by

$$
\begin{aligned}
&C^{a_1,a_2}_{n_1+1,n_2}(x,y;\omega)\\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}c_{n_1-k_1,n_2-k_2,\omega}C^{a_1,a_2}_{n_1,n_2}(x,y;\omega)\\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{C^{a_1,a_2}_{n_1-k_1,n_2-k_2}(x,y;\omega)}{(n_1-k_1)!(n_2-k_2)!}\\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1-1}{2}\right]}\sum_{k_2=0}^{n_2}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{C^{a_1,a_2}_{n_1-2k_1-1,n_2-k_2}(x,y;\omega)}{(n_1-2k_1-1)!(n_2-k_2)!}
\end{aligned}
$$

and

$$
\begin{aligned}
&C^{a_1,a_2}_{n_1,n_2+1}(x,y;\omega)\\
&=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2}{k_2}c_{n_1-k_1,n_2-k_2,\omega}C^{a_1,a_2}_{k_1,k_2}(x,y;\omega)\\
&+xn_1!n_2!\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{k_1+k_2}{k_2}(-\omega)^{k_1+k_2}\frac{C^{a_1,a_2}_{n_1-k_1,n_2-k_2}(x,y;\omega)}{(n_1-k_1)!(n_2-k_2)!}\\
&+2yn_1!n_2!\sum_{k_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_2=0}^{n_2-1}\binom{2k_1+k_2}{k_2}(-\omega)^{\frac{2k_1+k_2}{2}}\frac{C^{a_1,a_2}_{n_1-2k_1,n_2-k_2-1}(x,y;\omega)}{(n_1-2k_1)!(n_2-k_2-1)!}.
\end{aligned}
$$

Chapter 5

DISCRETE ω-MULTIPLE CHARLIER POLYNOMIALS

In chapter 5, we give ω-multiple Charlier polynomials. We start by defining the discrete multiple orthogonality on the linear lattice $\omega\mathbb{N} = \{0, \omega, 2\omega, \cdots\}$ $(\omega > 0)$ and call them ω-multiple orthogonal polynomials. We present the raising operator, the Rodrigues formula and explicit representation for them. We use the Rodrigues type formula (5.6) to give the explicit representation of the multiple ω-Charlier polynomials. Furthermore we obtain the generating function and some recurrence relations for these polynomials. Throughout this section for recurrence relations, we concentrate on the case $r = 2$, since the proof techniques for the general r will be similar. Finally, we obtain the $(r+1)th$ de(difference equation) for $\omega-$multiple Charlier polynomials. As a corollary, we give the third order de for the case $r = 2$.In this section, as an illustrative example of our new definition and its main results, we consider the case $\omega = \frac{3}{2}$ and define $\frac{3}{2}-$multiple Charlier polynomials. The corresponding consequences of our main results for $\frac{3}{2}-$multiple Charlier polynomials are also given. We will start by recalling some basic knowledge about the discrete orthogonal and discrete multiple orthogonal polynomials.

The nth degree monic orthogonal polynomial p_n is defined by

$$\int p_n(\mathrm{x})\, x^{\mathrm{k}} d\mu(x) = 0, \quad k = 0, 1, 2, \ldots, n-1,$$

where μ is a positive measure on the real line. In general, in the case of discrete orthogonal polynomials, the term x^k is replaced by $(-x)_k$, since $\Delta(-x)_k = -k(-x)_{k-1}$, where

$$(a)_k = a(a+1)\dots(a+k-1)$$

is the Pochhammer symbol and

$$\Delta\mathfrak{f}(x) = \mathfrak{f}(x+1) - \mathfrak{f}(x),$$

is the forward difference operator.

Hahn, Meixner, Kravchuk, and Charlier polynomials are the classical orthogonal polynomials of a discrete variable (on a linear lattice). The Charlier polynomials are the focus of this thesis.

The orthogonality measure (Poisson distribution) for Charlier polynomials is

$$\mu = \sum_{k=0}^{\infty} \frac{a^k}{k!} \delta_k,$$

with $k \in \mathbb{N}$ ($\mathbb{N} := \{0,1,2,\dots\}$) and $a > 0$.

The type *II* multiple orthogonal polynomial $p_{\vec{n}}$ of degree $\leq |\vec{n}| := n_1 + \cdots + n_r$ ($r \geq 2$) with respect to r non-negative measures $\mu_1, \dots, \mu_r$ on $\mathbb{R}$, are defined by

$$\int_{I_i} p_{\vec{n}}(x) \mathrm{x}^k d\mu_i(x) = 0, \quad k = 0,1,\dots,n_i - 1, \quad (i = 1,\dots,r). \tag{5.1}$$

Here

$$\mathrm{supp}(\mu_i) = \{x \in \mathbb{R} : \mu_i((x-\varepsilon, x+\varepsilon)) > 0 \; for \; all \; \varepsilon > 0\}$$

and I_i $(i = 1,2,\dots,r)$ is the smallest interval containing supp(μ_i). For the $|\vec{n}| + 1$ unknown coefficients of $p_{\vec{n}}$, conditions (5.1) provide $|\vec{n}|$ linear equations. The $\vec{\mathrm{n}}$ is considered to be normal if $p_{\vec{\mathrm{n}}}$ is unique (up to a multiplicative factor) and has degree $|\vec{n}|$. In general, the monic polynomials are considered.

In the case where r is a non-negative discrete measures on $\mathbb{R}$:

$$\mu_i = \sum_{m=0}^{N_i} \rho_{i,m} \delta_{x_{i,m}}, \quad \rho_{i,m} > 0, \ \ x_{i,m} \in \mathbb{R}, \ N_i \in \mathbb{N} \cup \{\infty\}, \ \ i = 1, \ldots, r,$$

where all $x_{i,m}$ is different for each $m = 0, 1, \ldots, N_i$ $(i = 1, 2, \ldots, r)$, we have the discrete multiple orthogonal polynomials (on the linear lattice), and the above orthogonality conditions can be written as

$$\sum_{j=0}^{\infty} p_{\vec{n}}(j)(-j)_k \rho_{i,j} = 0, \quad k = 0, 1, \ldots, n_i - 1, \quad i = 1, \ldots, r, \tag{5.2}$$

where $p_{\vec{n}}$ is a polynomial of degree $\leq |\vec{n}|$.

In this thesis, we pay attention to the AT system of r non-negative discrete measures, where we recall its definition below:

Definition 5.1: [2] An AT system of r non-negative discrete measures is a system of measures

$$\mu_i = \sum_{m=0}^{N} \rho_{i,m} \delta_{x_m}, \ \ \rho_{i,m} > 0, \ \ x_m \in \mathbb{R}, \ \ N \in \mathbb{N} \cup \{+\infty\}, \ \ i = 1, \ldots, r,$$

where $\operatorname{supp}(\mu_i)$ $(i = 1, \ldots, r)$ is the closure of x_m and the orthogonality intervals (5.2) are the same, namely I. It is also assumed that there exist r continuous functions $w_1, \ldots, w_r$ on I with $w_i(x_m) = \rho_{i,m}$ $(m = 1, \ldots, N, \ i = 1, \ldots, r)$ such that $|\vec{n}|$ functions

$$\left\{w_1, xw_1, \ldots, x^{n_1-1}w_1, w_2, xw_2, \ldots, x^{n_2-1}w_2, \ldots, w_r, xw_r, \ldots, x^{n_r-1}w_r\right\},$$

form a Chebyshev system on I for each multi-index $|\vec{n}| < N+1$. This means, all the linear combinations of the form

$$\sum_{i=1}^{r} Q_{n_i-1} w_i(x),$$

where Q_{n_i-1} is a polynomial of degree $\leq n_i - 1$, has at most $|\vec{n}| - 1$ zeros on I.

Remark 5.1: If we have r continuous functions $w_1, \ldots, w_r$ on I with $w_i(x_m) = \rho_{i,m}$, then the orthogonality conditions (5.2) can be written as

$$\sum_{j=0}^{\infty} p_{\vec{n}}(j)(-j)_k w_i(j) = 0, \ \ k = 0,1,\ldots,n_i - 1, \ \ i = 1,\ldots,r.$$

Every discrete orthogonal polynomials of type II, corresponding to the multi-index $\vec{n}$, has exact degree $|\vec{n}|$, and every multi-index $\vec{n}$ with $|\vec{n}| < N+1$ is normal in an AT system, as stated in [2].

Definition 5.2: The $\omega-$multiple orthogonal polynomials are defined as

$$\sum_{k=0}^{\infty} p_{\vec{n}}(\omega k)(-\omega k)_{j,\omega} w_i(\omega k) = 0, \quad j = 0,\ldots,n_i - 1, \quad i = 1,2,\ldots,r,$$

where ω is a fixed positive real number, $\vec{n} = (n_1,\ldots,n_r)$ and $p_{\vec{n}}$ is a polynomial of degree $|\vec{n}|$ and

$$\begin{aligned}(-\omega k)_{j,\omega} &= (-\omega k)(-\omega k+\omega)\ldots(-\omega k+\omega(j-1)) \\ &= \omega^j(-k)_j .\end{aligned}$$

Now we choose the orthogonality measures as

$$\mu_i = \sum_{k=0}^{+\infty} \frac{a_i^{\omega k}}{\Gamma_\omega(\omega k+\omega)} \delta_{\omega k}, \quad a_i > 0, \quad i = 1,...,r,$$

where $a_1,\ldots,a_r$ are different parameters and

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt, \quad x > 0$$

is the $k-$gamma function [23].

For each measure the weights form an extended Poisson distribution on $\omega\mathbb{N}$ ($\omega\mathbb{N} = \{0,\omega,2\omega,\ldots\}$). It is easily seen from Example 2.1 in [2] that, these r measures form a *Chebyshev* system on $\mathbb{R}^+$ for every $\vec{n} = (n_1,\ldots,n_r) \in \omega\mathbb{N}^r$ since the weight functions,

$$w_i(x) = \frac{a_i^x}{\Gamma_\omega(x+\omega)}, \quad x \in \mathbb{R}^+, \quad i = 1, ..., r,$$

are continuous and they have no zeros on $\mathbb{R}^+$. So every multi-index is normal and the monic solution is unique.

The corresponding multiple orthogonality conditions are given on $\omega\mathbb{N}$ as

$$\sum_{k=0}^{\infty} C_{\vec{n}}^{\vec{a}}(\omega k)(-\omega k)_{j,\omega} \frac{a_i^{\omega k}}{\Gamma_\omega(\omega k+\omega)} = 0, \quad j = 0, \ldots, n_i - 1, \ \ i = 1, 2, \ldots, r, \tag{5.3}$$

where $\vec{n} = (n_1, \ldots, n_r)$ and $\vec{a} = (a_1, \ldots, a_r)$. We represent these polynomials by $C_{\vec{n}}^{\vec{a}}$ and call them as $\omega-$ multiple Charlier orthogonal polynomials.

Theorem 5.1: The raising relation for the $\omega-$multiple Charlier polynomials is given as

$$\frac{a_i^\omega}{w_i(x)} \nabla_\omega \left[w_i(x) C_{\vec{n}}^{\vec{a}}(x) \right] = -C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x), \quad i = 1, ..., r. \tag{5.4}$$

where $\nabla_\omega \mathfrak{f}(\mathrm{x}) = \mathfrak{f}(x) - \mathfrak{f}(x-\omega)$ and $\vec{e}_i = (0, \ldots, 0, 1, \ldots, 0)$.

Proof. Applying the product rule $\nabla_\omega[\mathfrak{f}(x)\mathfrak{g}(x)] = \mathfrak{f}(x)\nabla_\omega \mathfrak{g}(x) + \mathfrak{g}(x-\omega)\nabla_\omega \mathfrak{f}(x)$, we have

$$\nabla_\omega \left[w_i(x) C_{\vec{n}}^{\vec{a}}(x) \right] = w_i(x) \nabla_\omega C_{\vec{n}}^{\vec{a}}(x) + C_{\vec{n}}^{\vec{a}}(x-\omega) \nabla_\omega w_i(x). \tag{5.5}$$

Since $\nabla_\omega w_i(x) = w_i(x)\left[1 - \frac{x}{a_i^\omega}\right]$, by using (4.3), we get

$$\begin{aligned} \nabla_\omega \left[w_i(x) C_{\vec{n}}^{\vec{a}}(x) \right] &= w_i(x) \nabla_\omega C_{\vec{n}}^{\vec{a}}(x) + w_i(x) C_{\vec{n}}^{\vec{a}}(x-\omega) \left[1 - \frac{x}{a_i^\omega} \right] \\ &= w_i(x) \left[\nabla_\omega C_{\vec{n}}^{\vec{a}}(x) + C_{\vec{n}}^{\vec{a}}(x-\omega) \left[1 - \frac{x}{a_i^\omega} \right] \right] \\ &= -\frac{w_i(x)}{a_i^\omega} P_{\vec{n}+\vec{e}_i}^{\vec{\alpha}}(x). \end{aligned} \tag{5.6}$$

Hence

$$\sum_{x=0}^{\infty} (-x)_{j,\omega} \nabla_\omega \left[w_i(x) C_{\vec{n}}^{\vec{a}}(x) \right] = -\frac{1}{a_i^\omega} \sum_{x=0}^{\infty} w_i(x)(-x)_{j,\omega} P_{\vec{n}+\vec{e}_i}^{\vec{a}}(x).$$

Applying the $\omega-$summation by parts formula, which is

$$\sum_{x=0}^{\infty}\Delta_{\omega}\left[f\left(\omega x\right)\right]g\left(\omega x\right)=-\sum_{x=0}^{\infty}\nabla_{\omega}\left[g\left(\omega x\right)\right]f\left(\omega x\right),\ \ g\left(-\omega\right)=0,$$

we obtain,

$$\sum_{x=0}^{\infty}\left(-\omega x\right)_{j,\omega}\nabla_{\omega}\left[w_{i}\left(\omega x\right)C_{\vec{n}}^{\vec{a}}\left(\omega x\right)\right]=-\sum_{x=0}^{\infty}\Delta_{\omega}\left[\left(-\omega x\right)_{j,\omega}\right]w_{i}\left(\omega x\right)C_{\vec{n}}^{\vec{a}}\left(\omega x\right).$$

Since $\Delta_{\omega}\left(-\omega x\right)_{j,\omega}=-\omega j\left(-\omega x\right)_{j-1,\omega}$, we have

$$\sum_{x=0}^{\infty}\omega j\left(-\omega x\right)_{j-1,\omega}w_{i}\left(\omega x\right)C_{\vec{n}}^{\vec{a}}\left(\omega x\right)=-\frac{1}{a_{i}^{\omega}}\sum_{x=0}^{\infty}w_{i}\left(\omega x\right)\left(-\omega x\right)_{j,\omega}P_{\vec{n}+\vec{e}_{i}}^{\vec{a}}\left(\omega x\right).$$

Then for $j=0,\ldots,n$, the summation on the left hand side will be zero from the $\omega-$multiple orthogonality conditions. Hence

$$-\frac{1}{a_{i}^{\omega}}\sum_{x=0}^{\infty}w_{i}\left(\omega x\right)\left(-\omega x\right)_{j,\omega}P_{\vec{n}+\vec{e}_{i}}^{\vec{a}}\left(\omega x\right)=0. \tag{5.7}$$

By the uniqueness of the $\omega-$multiple orthogonal polynomials, we have

$$P_{\vec{n}+\vec{e}_{i}}^{\vec{a}}\left(x\right)=C_{\vec{n}+\vec{e}_{i}}^{\vec{a}}\left(x\right).$$

Considering the above equality in (5.6), the proof is completed. □

Theorem 5.2: The Rodrigues formula for the $\omega-$multiple Charlier polynomials is given by

$$C_{\vec{n}}^{\vec{a}}\left(x\right)=\left[\prod_{j=1}^{r}\left(-a_{j}^{\omega}\right)^{n_{j}}\right]\Gamma_{\omega}\left(x+\omega\right)\left[\prod_{i=1}^{r}\left(\frac{1}{a_{i}^{x}}\nabla_{\omega}^{n_{i}}\left(a_{i}^{x}\right)\right)\right]\left(\frac{1}{\Gamma_{\omega}\left(x+\omega\right)}\right). \tag{5.8}$$

Proof. We will give the proof for the case $r=2$. The proof of the general case is similar. Repeatedly using the raising operators and using the fact that $C_{0,0}^{a_{1},a_{2}}\left(x\right)=1$, we have

$$\begin{aligned}C_{n_{1},n_{2}}^{a_{1},a_{2}}\left(x\right)&=\frac{\left(-a_{1}^{\omega}\right)^{n_{1}}\left(-a_{2}^{\omega}\right)^{n_{2}}}{a_{1}^{x}a_{2}^{x}}\Gamma_{\omega}\left(x+\omega\right)\nabla_{\omega}^{n_{1}}\left[\left(a_{1}^{x}\right)\nabla_{\omega}^{n_{2}}\left[\left(a_{2}^{x}\right)\frac{1}{\Gamma_{\omega}\left(x+\omega\right)}\right]\right]\\&=\left[\prod_{j=1}^{2}\left(-a_{j}^{\omega}\right)^{n_{j}}\right]\Gamma_{\omega}\left(x+\omega\right)\left[\prod_{i=1}^{2}\left(\frac{1}{a_{i}^{x}}\nabla_{\omega}^{n_{i}}\left(a_{i}^{x}\right)\right)\right]\left(\frac{1}{\Gamma_{\omega}\left(x+\omega\right)}\right).\end{aligned}$$

Hence, we get (5.8) for $r=2$. □

Theorem 5.3: The explicit representation for the $\omega-$multiple Charlier polynomials is given by

$$C_{\vec{n}}^{\vec{a}}(x) = (-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}\dots(-a_r^{\omega})^{n_r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}\frac{(-n_1)_{k_1}(-n_2)_{k_2}\dots(-n_r)_{k_r}}{k_1!k_2!\dots k_r!}$$
$$\times\left(-\frac{x}{\omega}\right)_{k_1+k_2+\cdots+k_r}\left(\left(-\frac{1}{a_1}\right)^{\omega}\omega\right)^{k_1}\cdots\left(\left(-\frac{1}{a_r}\right)^{\omega}\omega\right)^{k_r}. \quad (5.9)$$

Proof. We will give the proof for $r = 2$. The general case (5.9) can be proved in a similar manner. Using (5.8) for $r = 2$, we write

$$C_{n_1,n_2}^{a_1,a_2}(x) = (-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}\Gamma_{\omega}(x+\omega)\frac{1}{a_1^x}\nabla_{\omega}^{n_1}(a_1^x)\left(\frac{1}{a_2^x}\nabla_{\omega}^{n_2}\frac{a_2^x}{\Gamma_{\omega}(x+\omega)}\right).$$

Since $\nabla_{\omega}^{n}f(x) = \sum_{i=1}^{n}(-1)^i\binom{n}{i}f(x-i\omega)$, we have

$$C_{n_1,n_2}^{a_1,a_2}(x) = (-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}\Gamma_{\omega}(x+\omega)\sum_{k=0}^{n_2}\frac{1}{a_2^x}\binom{n_2}{k}(-1)^k a_2^{x-k\omega}$$
$$\times\left(\frac{1}{a_1^x}\nabla_{\omega}^{n_1}\left(\frac{a_1^x}{\Gamma_{\omega}(x+\omega-k\omega)}\right)\right)$$
$$= (-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}\Gamma_{\omega}(x+\omega)\sum_{k=0}^{n_2}\binom{n_2}{k}a_2^{-k\omega}\sum_{m=0}^{n_1}\binom{n_1}{m}(-1)^{k+m}$$
$$\times\frac{a_1^{-m\omega}}{\Gamma_{\omega}(x+\omega-k\omega-m\omega)}$$
$$= (-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}\sum_{m=0}^{n_1}\sum_{k=0}^{n_2}\binom{n_2}{k}\binom{n_1}{m}(-1)^{k+m}a_1^{-m\omega}a_2^{-k\omega}$$
$$\times\frac{\Gamma_{\omega}(x+\omega)}{\Gamma_{\omega}(x+\omega-k\omega-m\omega)}$$
$$= (-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}\sum_{m=0}^{n_1}\sum_{k=0}^{n_2}(-n_1)_m(-n_2)_k\frac{a_1^{-m\omega}}{m!}\frac{a_2^{-k\omega}}{k!}$$
$$\times\frac{\Gamma_{\omega}(x+\omega)}{\Gamma_{\omega}(x+\omega-k\omega-m\omega)} \quad (5.10)$$
$$= (-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}\sum_{m=0}^{n_1}\sum_{k=0}^{n_2}\frac{(-n_1)_m(-n_2)_k}{k!m!}(-1)^{-k\omega-m\omega}$$
$$\times(-x)_{k+m,\omega}a_1^{-m\omega}a_2^{-k\omega}$$

$$= \left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\sum_{m=0}^{n_1}\sum_{k=0}^{n_2}\left(-n_1\right)_m\left(-n_2\right)_k\left(-x\right)_{k+m,\omega}\frac{\left(-\frac{1}{a_1}\right)^{\omega m}}{m!}\frac{\left(-\frac{1}{a_2}\right)^{\omega k}}{k!}$$

$$= \left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\sum_{m=0}^{n_1}\sum_{k=0}^{n_2}\frac{\left(-n_1\right)_m\left(-n_2\right)_k\left(-\frac{x}{\omega}\right)_{k+m}}{m!k!}$$

$$\times\left(\left(-\frac{1}{a_1}\right)^{\omega}\omega\right)^{m}\left(\left(-\frac{1}{a_2}\right)^{\omega}\omega\right)^{k}.$$

Whence the result. □

Corollary 5.1: The equation (5.10) can be written as

$$C_{n_1,n_2}^{a_1,a_2}(x)$$
$$= \left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\lim_{\gamma\to+\infty}F_2\left(-\frac{x}{\omega},-n_1,-n_2;\gamma,\gamma;\left(-\frac{1}{a_1}\right)^{\omega}\gamma\omega,\left(-\frac{1}{a_2}\right)^{\omega}\gamma\omega\right),$$

where

$$F_2\left(\alpha,\beta,\beta';\gamma,\gamma';x,y\right) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}x^m y^n$$

is the second Appell's hypergeometric function of two variables [12].

5.1 Generating Function

Theorem 5.4: The $\omega-$multiple Charlier polynomials have the following generating function

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}$$
$$= \left(1+\omega t_1+\omega t_2+\cdots+\omega t_r\right)^{\frac{x}{\omega}}\exp\left(-a_1^{\omega}t_1-\cdots-a_r^{\omega}t_r\right). \tag{5.11}$$
$$\left(\sum_{i=1}^{r}|t_i| < \omega^{-r}\right).$$

Proof. Using the explicit representation of the polynomials given in Theorem 5.3, we can write that

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}$$
$$=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\frac{\left(-a_1^{\omega}\right)^{n_1}\ldots\left(-a_r^{\omega}\right)^{n_r}(-n_1)_{k_1}\ldots(-n_r)_{k_r}}{k_1!k_2!\ldots k_r!}\left(-\frac{x}{\omega}\right)_{|\vec{k}|}$$
$$\times\left(\left(-\frac{1}{a_1}\right)^{\omega}\omega\right)^{k_1}\left(\left(-\frac{1}{a_2}\right)^{\omega}\omega\right)^{k_2}\cdots\left(\left(-\frac{1}{a_r}\right)^{\omega}\omega\right)^{k_r}\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}$$

$$=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\frac{\left(-a_1^{\omega}\right)^{n_1}\cdots\left(-a_r^{\omega}\right)^{n_r}}{k_1!\cdots k_r!}\frac{(-1)^{k_1}n_1!\cdots(-1)^{k_r}n_r!}{(n_1-k_1)!\cdots(n_r-k_r)!}$$
$$\times\left(-\frac{x}{\omega}\right)_{|\vec{k}|}\left(-a_1^{\omega}\right)^{-k_1}\cdots\left(-a_r^{\omega}\right)^{-k_r}\omega^{k_1}\cdots\omega^{k_r}\frac{t_1^{n_1}\cdots t_r^{n_r}}{n_1!\cdots n_r!}$$
$$=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\frac{\left(-a_1^{\omega}\right)^{n_1-k_1}\cdots\left(-a_r^{\omega}\right)^{n_r-k_r}}{k_1!\cdots k_r!}\frac{(-\omega)^{k_1}\cdots(-\omega)^{k_r}}{(n_1-k_1)!\cdots(n_r-k_r)!}$$
$$\times\left(-\frac{x}{\omega}\right)_{|\vec{k}|}t_1^{n_1}\cdots t_r^{n_r}.$$

Changing the order of the summation,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}$$
$$=\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\sum_{n_1=k_1}^{\infty}\cdots\sum_{n_r=k_r}^{\infty}\frac{(-\omega)^{k_1}\cdots(-\omega)^{k_r}}{k_1!\cdots k_r!}\left(-\frac{x}{\omega}\right)_{|\vec{k}|}$$
$$\times\frac{\left(-a_1^{\omega}\right)^{n_1-k_1}\cdots\left(-a_r^{\omega}\right)^{n_r-k_r}}{(n_1-k_1)!\cdots(n_r-k_r)!}t_1^{n_1}\cdots t_r^{n_r}$$
$$=\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{(-\omega)^{k_1}\cdots(-\omega)^{k_r}t_1^{k_1}\cdots t_r^{k_r}}{k_1!\cdots k_r!}\left(-\frac{x}{\omega}\right)_{|\vec{k}|}$$
$$\times\sum_{l_1=0}^{\infty}\cdots\sum_{l_r=0}^{\infty}\frac{\left(-a_1^{\omega}\right)^{l_1}\cdots\left(-a_r^{\omega}\right)^{l_r}t_1^{l_1}\cdots t_r^{l_r}}{l_1!\cdots l_r!}$$
$$=(1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\exp\left(-a_1^{\omega}t_1-\cdots-a_r^{\omega}t_r\right).$$

Whence the result. □

Remark 5.2: It can be easily seen from (1.1) and (5.11) that, $\omega-$multiple Charlier polynomials are examples to the $\Delta_{\omega}-$multiple Appell polynomials.

5.2 Recurrence Relations

Proposition 5.1: Let $G(x,t_1,t_2) = (1+\omega t_1+\omega t_2)^{\frac{x}{\omega}} e^{-\left(a_1^{\omega}t_1+a_2^{\omega}t_2\right)}$. We have the following properties,

$$\frac{\partial}{\partial t_1}G(x,t_1,t_2) - \frac{\partial}{\partial t_2}G(x,t_1,t_2) = (a_2^{\omega}-a_1^{\omega})\,G(x,t_1,t_2) \tag{5.12}$$

and

$$(1+\omega t_1+\omega t_2)\frac{\partial}{\partial t_1}G(x,t_1,t_2) = (x-a_1^{\omega}(1+\omega t_1+\omega t_2))\,G(x,t_1,t_2). \tag{5.13}$$

Proof. Proofs can be given by elementary calculations. □

Theorem 5.5: The following recurrence relations,

$$(a_2^{\omega}-a_1^{\omega})\,C_{n_1,n_2}^{a_1,a_2}(x) = C_{n_1+1,n_2}^{a_1,a_2}(x) - C_{n_1,n_2+1}^{a_1,a_2}(x), \tag{5.14}$$

$$\begin{aligned} xC_{n_1,n_2}^{a_1,a_2}(x) &= C_{n_1+1,n_2}^{a_1,a_2}(x) + (a_1^{\omega}+\omega n_1+\omega n_2)\,C_{n_1,n_2}^{a_1,a_2}(x) \\ &\quad + (\omega a_1^{\omega}n_1+\omega a_2^{\omega}n_2)\,C_{n_1,n_2-1}^{a_1,a_2}(x) + n_1 a_1^{\omega}\omega\,(a_1^{\omega}-a_2^{\omega})\,C_{n_1-1,n_2-1}^{a_1,a_2}(x) \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} xC_{n_1,n_2}^{a_1,a_2}(x) &= C_{n_1+1,n_2}^{a_1,a_2}(x) + (a_1^{\omega}+\omega n_1+\omega n_2)\,C_{n_1,n_2}^{a_1,a_2}(x) \\ &\quad + \omega a_2^{\omega}n_2C_{n_1,n_2-1}^{a_1,a_2}(x) + \omega a_1^{\omega}n_1C_{n_1-1,n_2}^{a_1,a_2}(x), \end{aligned} \tag{5.16}$$

hold for the $\omega-$multiple Charlier polynomials.

Proof. Using (5.12), we get

$$\begin{aligned} &(a_2^{\omega}-a_1^{\omega})\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!} \\ &= \frac{\partial}{\partial t_1}\left[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\right] - \frac{\partial}{\partial t_2}\left[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\right] \\ &= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1-1}}{(n_1-1)!}\frac{t_2^{n_2}}{n_2!} - \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}C_{n_1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2-1}}{(n_2-1)!} \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1+1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!} - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2+1}^{a_1,a_2}(x)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}. \end{aligned}$$

Hence

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\left(a_2^{\omega}-a_1^{\omega}\right)C_{n_1,n_2}^{a_1,a_2}\left(x\right)\right]\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[C_{n_1+1,n_2}^{a_1,a_2}\left(x\right)-C_{n_1,n_2+1}^{a_1,a_2}\left(x\right)\right]\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.$$

Comparing the coefficients of $\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$, (5.14) follows.

The left hand side of (5.13) can be written as

$$\begin{aligned}
&\left(1+\omega t_1+\omega t_2\right)\frac{\partial}{\partial t_1}G\left(x,t_1,t_2\right)\\
&=\left(1+\omega t_1+\omega t_2\right)\frac{\partial}{\partial t_1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&=\left(1+\omega t_1+\omega t_2\right)\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1-1}}{\left(n_1-1\right)!}\frac{t_2^{n_2}}{n_2!}\\
&=\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1-1}}{\left(n_1-1\right)!}\frac{t_2^{n_2}}{n_2!}+\omega\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{\left(n_1-1\right)!}\frac{t_2^{n_2}}{n_2!}\\
&+\omega\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1-1}}{\left(n_1-1\right)!}\frac{t_2^{n_2+1}}{n_2!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1+1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}+\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}n_1C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2}\\
&+\omega\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}C_{n_1+1,n_2-1}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{\left(n_2-1\right)!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1+1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\omega n_1C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\omega n_2C_{n_1+1,n_2-1}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[C_{n_1+1,n_2}^{a_1,a_2}\left(x\right)+\omega n_1C_{n_1,n_2}^{a_1,a_2}\left(x\right)+\omega n_2C_{n_1+1,n_2-1}^{a_1,a_2}\left(x\right)\right]\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}. \qquad (5.17)
\end{aligned}$$

The right hand side of (5.13) will be

$$
\begin{aligned}
&\left(x-a_1^{\omega}\left(1+\omega t_1+\omega t_2\right)\right)G\left(x,t_1,t_2\right)\\
&=\left(x-a_1^{\omega}\left(1+\omega t_1+\omega t_2\right)\right)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&=\left(x-a_1^{\omega}\right)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}-a_1^{\omega}\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1+1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&-a_1^{\omega}\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2-1}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2+1}}{n_2!}\\
&=\left(x-a_1^{\omega}\right)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}-a_1^{\omega}\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}n_1C_{n_1-1,n_2}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&-a_1^{\omega}\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}n_2C_{n_1,n_2-1}^{a_1,a_2}\left(x\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\left(x-a_1^{\omega}\right)C_{n_1,n_2}^{a_1,a_2}\left(x\right)-a_1^{\omega}\omega n_1C_{n_1-1,n_2}^{a_1,a_2}\left(x\right)-a_1^{\omega}\omega n_2C_{n_1,n_2-1}^{a_1,a_2}\left(x\right)\right]\\
&\times\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.
\end{aligned}
\tag{5.18}
$$

Combining (5.17) and (5.18), we get

$$
\begin{aligned}
&C_{n_1+1,n_2}^{a_1,a_2}\left(x\right)+\omega n_1C_{n_1,n_2}^{a_1,a_2}\left(x\right)+\omega n_2C_{n_1+1,n_2-1}^{a_1,a_2}\left(x\right)\\
&=\left(x-a_1^{\omega}\right)C_{n_1,n_2}^{a_1,a_2}\left(x\right)-a_1^{\omega}\omega n_1C_{n_1-1,n_2}^{a_1,a_2}\left(x\right)-a_1^{\omega}\omega n_2C_{n_1,n_2-1}^{a_1,a_2}\left(x\right).
\end{aligned}
\tag{5.19}
$$

Replacing n_2 by n_2-1 and n_1 by n_1-1 in (5.14), we have

$$
C_{n_1+1,n_2-1}^{a_1,a_2}\left(x\right)=C_{n_1,n_2}^{a_1,a_2}\left(x\right)+\left(a_2^{\omega}-a_1^{\omega}\right)C_{n_1,n_2-1}^{a_1,a_2}\left(x\right) \tag{5.20}
$$

and

$$
C_{n_1-1,n_2}^{a_1,a_2}\left(x\right)=C_{n_1,n_2-1}^{a_1,a_2}\left(x\right)-\left(a_2^{\omega}-a_1^{\omega}\right)C_{n_1-1,n_2-1}^{a_1,a_2}\left(x\right) \tag{5.21}
$$

respectively.

Using (5.20) and (5.21), we get (5.15).

Using (5.21), we have

$$
\left(a_1^{\omega}-a_2^{\omega}\right)C_{n_1-1,n_2-1}^{a_1,a_2}\left(x\right)=C_{n_1-1,n_2}^{a_1,a_2}\left(x\right)-C_{n_1,n_2-1}^{a_1,a_2}\left(x\right). \tag{5.22}
$$

Comparing (5.22) and (5.15), we get (5.16). □

5.3 Difference Equations for ω-Multiple Charlier Polynomials

Theorem 5.6: The raising operator can be rewritten as

$$L_{a_i}\left[C_{\vec{n}}^{\vec{a}}(x)\right] = -C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x), \quad i = 1,2,\dots,r, \tag{5.23}$$

where $\vec{e}_i = (0,\dots,0,1,\dots,0)$ and $L_{a_i}[.]$ is defined by

$$L_{a_i}[y] = x\nabla_\omega y + (a_i^\omega - x)\,y.$$

Proof. From the raising relation (5.4), we have

$$a_i^\omega \nabla_\omega\left[w_i(x)\,C_{\vec{n}}^{\vec{a}}(x)\right] = -w_i(x)\,C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x).$$

Applying the $\omega-$product rule, we can write that

$$a_i^\omega\left[C_{\vec{n}}^{\vec{a}}(x)\,\nabla_\omega w_i(x) + w_i(x-\omega)\,\nabla_\omega C_{\vec{n}}^{\vec{a}}(x)\right] = -w_i(x)\,C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x). \tag{5.24}$$

Since $\nabla_\omega w_i(x) = w_i(x)\left[1 - \frac{x}{a_i \omega}\right]$, we get by using (5.22) that

$$a_i^\omega\left[C_{\vec{n}}^{\vec{a}}(x)\,w_i(x)\left[1 - \frac{x}{a_i^\omega}\right] + \frac{a_i^{x-\omega}}{\Gamma_\omega(x)}\nabla_\omega C_{\vec{n}}^{\vec{a}}(x)\right] = -w_i(x)\,C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x)$$

$$a_i^\omega\left[C_{\vec{n}}^{\vec{a}}(x)\,w_i(x)\left[1 - \frac{x}{a_i^\omega}\right] + \frac{x a_i^x a^{-\omega}}{\Gamma_\omega(x+\omega)}\nabla_\omega C_{\vec{n}}^{\vec{a}}(x)\right] = -w_i(x)\,C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x)$$

$$a_i^\omega C_{\vec{n}}^{\vec{a}}(x) - xC_{\vec{n}}^{\vec{a}}(x) + x\nabla_\omega C_{\vec{n}}^{\vec{a}}(x) = -C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x).$$

Hence

$$x\nabla_\omega C_{\vec{n}}^{\vec{a}}(x) + (a_i^\omega - x)\,C_{\vec{n}}^{\vec{a}}(x) = -C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x),$$

and therefore

$$L_{a_i}\left[C_{\vec{n}}^{\vec{a}}(x)\right] = x\nabla_\omega C_{\vec{n}}^{\vec{a}}(x) + (a_i^\omega - x)\,C_{\vec{n}}^{\vec{a}}(x),$$

where

$$L_{a_i}\left[C_{\vec{n}}^{\vec{a}}(x)\right] = -C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x).$$

This completes the proof. □

Theorem 5.7: The lowering operator of the polynomials is determined from the following relation:

$$\Delta_\omega C_{\vec{n}}^{\vec{a}}(x) = \sum_{i=1}^{r} \omega n_i C_{\vec{n}-\vec{e}_i}^{\vec{a}}(x), \tag{5.25}$$

where $\vec{e}_i = (0,\ldots,1,\ldots,0)$.

Proof. Applying Δ_ω on both sides of (5.11), we get

$$\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} \Delta_\omega C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}\\
&= \Delta_\omega\left[(1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\exp\left(-a_1^\omega t_1-\cdots-a_r^\omega t_r\right)\right]\\
&= \exp\left(-a_1^\omega t_1-\cdots-a_r^\omega t_r\right)\Delta_\omega\left[(1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\right]\\
&+ (1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x+\omega}{\omega}}\Delta_\omega\left[\exp\left(-a_1^\omega t_1-\cdots-a_r^\omega t_r\right)\right]\\
&= \exp\left(-a_1^\omega t_1-\cdots-a_r^\omega t_r\right)\Delta_\omega\left[(1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\right]\\
&= (\omega t_1+\omega t_2+\cdots+\omega t_r)\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}\cdots t_r^{n_r}}{n_1!\cdots n_r!}\\
&= \omega\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1+1}\cdots t_r^{n_r}}{n_1!\cdots n_r!}+\cdots+\omega\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}\cdots t_r^{n_r+1}}{n_1!\cdots n_r!}\\
&= \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\left(\omega n_1 C_{n_1-1,\ldots,n_r}^{\vec{a}}(x)+\cdots+\omega n_r C_{n_1,\ldots,n_r-1}^{\vec{a}}(x)\right)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}
\end{aligned}$$

Comparing the coefficients of $\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}$, we get the result. □

Corollary 5.2: In particular, if $r=2$,

$$\Delta_\omega C_{n_1,n_2}^{a_1,a_2}(x) = \omega n_1 C_{n_1-1,n_2}^{a_1,a_2} + \omega n_2 C_{n_1,n_2-1}^{a_1,a_2}(x).$$

Theorem 5.8: The $\omega-$multiple Charlier polynomials $\left\{C_{\vec{n}}^{\vec{a}}(x)\right\}_{|n|=0}^{\infty}$ satisfy the following $(r+1)$ order difference equations

$$L_{a_1}L_{a_2}\cdots L_{a_r}\left[\Delta_\omega C_{\vec{n}}^{\vec{a}}(x)\right]+\sum_{i=1}^{r}\omega n_i L_{a_1}L_{a_2}\ldots L_{a_i-1}L_{a_i+1}\ldots L_{a_r}\left[C_{\vec{n}}^{\vec{a}}(x)\right]=0,$$

where $L_{a_i}[.]$ is the raising operator $(i=1,...,r)$ given in Theorem 5.6.

Proof. Applying $L_{a_1}\ldots L_{a_r}$ to both sides of (5.25), we get

$$L_{a_1}\dots L_{a_r}\left[\Delta_\omega C_{\vec{n}}^{\vec{a}}(x)\right]=\sum_{i=1}^{r}\omega n_i L_{a_1}\dots L_{a_r}C_{\vec{n}-\vec{e}_i}^{\vec{a}}(x).$$

Since $L_{a_j}L_{a_k}(y)=L_{a_k}L_{a_j}(y)$ for $a_j,a_k\in\mathbb{R}$, we obtain for $i=1,2,\dots,r$ that,

$$\begin{aligned}L_{a_1}\dots L_{a_r}&=L_{a_1}\dots L_{a_{i-1}}L_{a_i}L_{a_{i+1}}L_{a_{i+2}}\dots L_{a_r}\\&=L_{a_1}\dots L_{a_{i-1}}L_{a_{i+1}}L_{a_i}L_{a_{i+2}}\dots L_{a_r}\\&\vdots\\&=L_{a_1}\dots L_{a_{i-1}}L_{a_{i+1}}\dots L_{a_r}L_{a_i}.\end{aligned}$$

Hence

$$L_{a_1}\dots L_{a_r}\left[\Delta_\omega C_{\vec{n}}^{\vec{a}}(x)\right]=\sum_{i=1}^{r}\omega n_i L_{a_1}\dots L_{a_{i-1}}L_{a_{i+1}}\dots L_{a_r}L_{a_i}\left[C_{\vec{n}-\vec{e}_i}^{\vec{a}}(x)\right].$$

Using (5.23) with $\vec{n}$ replaced by $\vec{n}-\vec{e}_i$, we get the result. □

Corollary 5.3: The $\omega-$multiple Charlier polynomials $\left\{C_{n_1,n_2}^{a_1,a_2}(x)\right\}_{n_1+n_2=0}^{\infty}$ satisfy the difference equation,

$$\begin{aligned}&x(x-\omega)\Delta_\omega\nabla_\omega^2 y+x(2\omega+a_1^\omega+a_2^\omega-2x)\Delta_\omega\nabla_\omega y+\left[(a_1^\omega-x)(a_2^\omega-x)-x\omega\right]\Delta_\omega y\\&+(\omega n_1+\omega n_2)x\nabla_\omega y+(n_1(a_2^\omega-x)+n_2(a_1^\omega-x))\omega y=0.\end{aligned}\qquad(5.26)$$

5.4 Special Cases of the $\omega-$Multiple Charlier polynomials

Taking the weight function as

$$w_i(x)=\frac{a_i^x}{x\left(\frac{3}{2}\right)^{\frac{2x-3}{3}}\Gamma\left(\frac{2x}{3}\right)},$$

we can define the $\frac{3}{2}-$multiple Charlier polynomial by the following orthogonality conditions:

$$\sum_{k=0}^{\infty}C_{\vec{n}}^{\vec{a}}\left(\frac{3k}{2}\right)\left(\frac{3}{2}\right)^j(-k)_j\frac{a_i^{\frac{3k}{2}}}{\left(\frac{3k}{2}\right)\Gamma_{\frac{3}{2}}\left(\frac{3k}{2}\right)}=0,\quad j=0,\dots,n_i-1,\quad i=1,\dots r.$$

Their explicit representation can be written from Theorem 5.3 as

$$C_{\vec{n}}^{\vec{a}}(x) = (-a_1)^{\frac{3n_1}{2}} \dots (-a_r)^{\frac{3n_r}{2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \cdots \sum_{k_r=0}^{n_r} \frac{(-n_1)_{k_1} \dots (-n_r)_{k_r}}{k_1! \dots k_r!}$$
$$\times \left(-\frac{2x}{3}\right)_{k_1+\cdots+k_r} \left(-\frac{1}{a_1}\right)^{\frac{3k_1}{2}} \cdots \left(-\frac{1}{a_r}\right)^{\frac{3k_r}{2}} \left(\frac{3}{2}\right)^{k_1+k_2+\cdots+k_r}.$$

The genereting function of the $\frac{3}{2}-$multiple Charlier polynomial is written from Theorem 5.4 as

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} C_{\vec{n}}^{\vec{a}}(x) \frac{t_1^{n_1} t_2^{n_2} \dots t_r^{n_r}}{n_1! n_2! \dots n_r!}$$
$$= \left(1 + \frac{3}{2}(t_1 + t_2 + \cdots + t_r)\right)^{\frac{2x}{3}} \exp\left(-a_1^{\frac{3}{2}} t_1 - a_2^{\frac{3}{2}} t_2 - \cdots - a_r^{\frac{3}{2}} t_r\right).$$
$$\left(\sum_{i=1}^{r} |t_i| < \left(\frac{2}{3}\right)^r\right)$$

Their recurrence relations can be written from Theorem 5.5 as

$$\left(a_2^{\frac{3}{2}} - a_1^{\frac{3}{2}}\right) C_{n_1,n_2}^{a_1,a_2}(x) = C_{n_1+1,n_2}^{a_1,a_2}(x) - C_{n_1,n_2+1}^{a_1,a_2}(x),$$

$$xC_{n_1,n_2}^{a_1,a_2}(x) = C_{n_1+1,n_2}^{a_1,a_2}(x) + \left(a_1^{\frac{3}{2}} + \frac{3}{2}(n_1+n_2)\right) C_{n_1,n_2}^{a_1,a_2}(x)$$
$$+ \left(\frac{3}{2}\left(a_1^{\frac{3}{2}} n_1 + a_2^{\frac{3}{2}} n_2\right)\right) C_{n_1,n_2-1}^{a_1,a_2}(x) + \frac{3}{2} n_1 a_1^{\frac{3}{2}} \left(a_1^{\frac{3}{2}} - a_2^{\frac{3}{2}}\right) C_{n_1-1,n_2-1}^{a_1,a_2}(x),$$

and

$$xC_{n_1,n_2}^{a_1,a_2}(x) = C_{n_1+1,n_2}^{a_1,a_2}(x) + \left(a_1^{\frac{3}{2}} + \frac{3}{2}(n_1+n_2)\right) C_{n_1,n_2}^{a_1,a_2}(x)$$
$$+ \frac{3}{2} a_2^{\frac{3}{2}} n_2 C_{n_1,n_2-1}^{a_1,a_2}(x) + \frac{3}{2} a_1^{\frac{3}{2}} n_1 C_{n_1-1,n_2}^{a_1,a_2}(x).$$

The difference equation of the $\frac{3}{2}-$multiple Charlier polynomials for the case $r=2$,

$$x\left(x-\frac{3}{2}\right)\Delta_{\frac{3}{2}}\nabla_{\frac{3}{2}}^2 y + x\left(3 + a_1^{\frac{3}{2}} + a_2^{\frac{3}{2}} - 2x\right)\Delta_{\frac{3}{2}}\nabla_{\frac{3}{2}} y + \left[\left(a_1^{\frac{3}{2}} - x\right)\left(a_2^{\frac{3}{2}} - x\right) - \frac{3x}{2}\right]\Delta_{\frac{3}{2}} y$$
$$+ \frac{3}{2}(n_1+n_2) x \nabla_{\frac{3}{2}} y + \left(n_1\left(a_2^{\frac{3}{2}} - x\right) + n_2\left(a_1^{\frac{3}{2}} - x\right)\right)\frac{3y}{2} = 0.$$

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