A New Kantorovich Type *q*-analogue of the Balázs-Szabados Operators

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Eastern Mediterranean University
January 2021
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

This thesis consists of four chapters. In the first chapter, the introduction is offered. In the second chapter, we give the definitions, concepts and important theorems related with linear positive operators. We mention about the *q*-integers which are used to introduce *q*-analogue of the positive linear operators that have been intensive of research on approximation theory. After that, we mention about the definition of the operators which are introduced by Balázs and Szabados together. As well as, we shed light on various definitions of *q*-Balázs-Szabados operators, but we especially work on the new *q*-Balázs-Szabados operators which are defined by N. I. Mahmudov and denoted by $\Omega_{n,q}(f,x)$. We calculate the formulas of $\Omega_{n,q}(t^m,x)$ for $m=1, 2, 3, 4$ and we obtain the 1st, 2nd, 3rd and the 4th order moments of the new *q*-Balázs-Szabados operators. We also derive the recurrence formula of $\Omega_{n,q}(t^m,x)$ in terms of $\mathrm{B}_{n,q}\left(t^m,\frac{a_n x}{a_n x+1}\right)$ that represents a close connection between the new *q*-Balázs-Szabados operators and the *q*-Bernstein operators. As well as, we estimate the 2nd order and the 4th order central moments of the operators $\Omega_{n,q}(f,x)$, which have a great deal of importance of getting the results in approximation theory. Besides, we mention about the Kantorovich type *q*-analogue of the Balázs-Szabados operators (*q*-BSK operators) that have a nondecreasing restriction on $f(x)$ to maintain the positivity property. In the third chapter, we construct a new Kantorovich type *q*-analogue of the Balázs-Szabados operators, $\Omega^*_{n,q}(f,x)$. These new operators have an advantage compared to the previous ones, they maintain the positivity property without any restriction on $f(x)$. We give the recurrence formula for $\Omega^*_{n,q}(t^m,x)$, $m\in\mathrm{N}\cup\{0\}$ and

we calculate the formulas of $\Omega^{*}_{n,q}\left(t^{m},x\right)$ for $m=0,...,4$. Then, we give some significant auxiliary findings for the convergence properties of these operators $\Omega^{*}{}_{n,q}\left(f;x\right)$. In terms of the usual modulus of continuous functions, we investigate the local approximation properties and we give Korovkin type approximation theorem for the operators $\Omega^{*}{}_{n,q}\left(f;x\right)$. We prove Voronoskaja type theorem and we present the convergence rate in terms of the usual Lipschitz functions, $Lip_{M}\left(\alpha\right)$. In the fourth chapter, the conclusion is given.

Keywords: *q*-calculus; *q*-Bernstein basis function; *q*-Bernstein operators; *q*-analogue of the Balázs-Szabados operators; moments; Voronovskaja theorem.

ÖZ

Bu tez dört bölümden oluşmaktadır. Birinci bölüm giriş bölümüdür. Bu bölümde teze ilişkin önbilgiler, daha önce yapılan benzer çalışmalar ve daha önce yapılan çalışmaların zayıf yönlerinden bahsedilmiştir. Yine bu bölümde tezin amacı ve ilerleyen bölümlerde neler yapıldığından bahsedilmiştir. . İkinci bölümde lineer pozitif operatörler ve q-tamsayıları ile ilgili tanımlar, kavramlar, bağıntılar ve teoremlerden bahsedilmiştir. Yine bu bölümde Balázs-Szabados operatörlerinin tanımı verilip bu operatörlerin farklı q-analoglarından sözedilmiştir. Bu analoglar arasından özellikle N. Mahmudov tarafından önerilen q-Balázs-Szabados operatörü ele alınmış ve bu operatörün birinci, ikinci, üçüncü ve dördüncü mertebeden momentleri hesaplanmıştır. Ayrıca ikinci ve dördüncü mertebeden merkezi momentleri hesaplanmış ve bu operatörlerin çok popüler olan q-Bernstein operatörlerine bağlı rekürans formülü bulunmuştur. Bunun yanında E. Özkan tarafından önerilen Balázs-Szabados operatörlerinin Kantorovich tipli q-analoğundan ve bu analoğun zayıf yönlerinden bahsedilmiştir. Üçüncü bölümde Balázs-Szabados operatörlerinin yeni bir Kantorovich tipli q-analoğu önerilmiştir. Bu yeni operatörler daha önce önerilenlerle karşılaştırılıp, yeni operatörlerin avantajlarından bahsedilmiştir. Yine bu bölümde yeni önerilen operatörlere ilişkin rekürans formülü verilmiş ve bu formül yardımıyla birinci, ikinci, üçüncü ve dördüncü mertebeden momentleri ve birinci, ikinci ve dördüncü mertebeden merkezi momentleri hesaplanmıştır. Süreklilik modülü cinsinden yerel yaklaşım özellikleri incelenmiştir. Ayrıca bu bölümde yeni operatörlere ilişkin Korovkin tipli teorem ve Voronovskaya tipli teorem verilmiştir. Bunun yanında klasik Lipschitz fonksiyonu kullanılarak yakınsama oranı verilmiştir. Dördüncü bölümde ise sonuç ve ileride yapılması planlanan çalışmalar verilmiştir.

Anahtar Kelimeler: q-kalkülüs; q-Bernstein taban fonksiyonu; q-Bernstein operatörleri; Balázs-Szabados operatörlerinin q-analoğu; momentler; Voronovskaja tipli teorem.

DEDICATION

To my beloved father, who always strongly supports and encourages me from the beginning until this moment to fulfil my and his dream of obtaining a PhD in my beautiful and interesting study. Words are not enough to thank him for his continues support and his love and prayers. To my dear and lovely mother, to the one who was the reason for my presence in this world., the one who was a help and support for me in my difficult times to the source of tenderness and unlimited giving and prayers. Expressing words does not give her the right to thank and gratitude, I wish her happiness and longevity. To my love Mohammed Alshugaa for his prayers. brothers. To my dear Seraj for his help and kindness, I wish for him a prosperous future and happy marriage. To my lovely kids Abdullah, Ibrahim, Youssef and to my little angel Arrej, they were with me throughout the study journey and they shared the beautiful and difficult times and the challenges.

ACKNOWLEDGMENT

Praise to Allah, the number of atoms of the universe in the heavens and the earth and what is between them and beyond. Praise be to God who provided me with strength, patience and health to reach this moment.

I sincerely and deeply offer my thanks and gratitude to my supervisor Asst. Prof. Dr. Pembe Sabancigil who did not delay in supporting and granting me valuable advice for the success of this research. As I would like to thank her for her sympathetic and kind heart, for friendly conduct throughout the years of research. I wish her and her family a happy and healthy life of longevity.

I would like strongly to thank Prof. Dr Nazım Mahmudov who is very much better than holding his position as ahead of the department. I would like to thank you for your continuous support and the great effort for advance the educational process in the department since your door was open to any student who has a question or problem, who is also a model for an active and skilled scientist who does not stop working, studying and developing in his field of specialization. Thank you for everything you have done for me, the students, and the department's staff. Thank you for developing the department thanks to your effort and constant work. I wish you good health and longevity.

TABLE OF CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	The set of natural numbers including zero
$\mathbb{R}$	The set of real numbers
(a,b)	An open interval
$[a,b]$	A closed interval
$C[a,b]$	The space of all real-valued and continuous functions defined on the compact interval $[a,b]$, for $f \in C[a,b]$
$[0,\infty)$	The nonnegative real line
$\|\xi\|$	Norm of ξ
e_r	Denotes the r-th monomial with $e_r : [a,b] \in t \to t^r \in \mathbb{R},\ r \in \mathbb{N}_0$
C-S-I	Cauchy-Schwarz inequality
RHS	Right hand side
LHS	Left hand side
B-S	Balázs-Szabados Operators

Chapter 1

INTRODUCTION

Positive linear operators have a crucial effect on the theory of approximation and this theory has been stated as an essential subject of studies in the last three decades. Due to this, the solution of some issue in complex analysis, numerical analysis, solutions of some mathematical and physical equations and differential equations are affected particularly. The general approximation techniques for linear positive operators are intended to handle convergence conduct. The accuracy can be ascertained by applying various processes for a required degree. The approximation of sequences with the help of linear and positive operators of functions work on variable spaces such as normed space, complex space and other spaces. In the year 1885, Karl Weierstrass firstly proved his (fundamental) theorem on approximation by using techniques of linear algebra and trigonometric functions for polynomials, this method became very important in the improvement of approximation theory. The proof which he gave took a very long time and provoked by many well-known mathematicians to find simpler and more didactic proofs. As well as, the moments of linear positive operators have a great deal of importance on approximation theory. In [1], very important and effective results obtained on the convergence of sets of positive linear operators and linear contractions have motivated the development of Korovkin-type approximation theory. The famous Korovkin theorem considers that the convergence of operators can be studied by lots of researches. So lots of new operators were suggested and built by many studies after the famous Korovkin theorem due to Weierstrass and substantial

Korovkin type theorem. “A little while back, V. Gupta calculated the moments of some discrete and Kantorovich type operators by using the notion of moment generating functions in [2], also, in [3] the central moments of certain operators were estimated by using this approach”. In [4], the well-known sequence of operators for any $n \in \mathrm{N}$, $f(x)$ defined on an interval $0 \le x \le 1$ was introduced by S. N. Bernstein as follows:

$$B_n(f,x) = \sum_{j=0}^{n} p_{n,j}(x)\, f\left(\frac{j}{n}\right), \quad 0 \le x \le 1,$$

where the basis function of these polynomials is defined as

$$p_{n,j}(x) = \binom{n}{j} x^j \,(1-x)^{n-j}.$$

$\lim_{n\to\infty} B_n(f,x) = f(x)$ uniformly when $f(x)$ is continuous for $0 \le x \le 1$, thus supply with a deductive proof of Weierstrass's Theorem given in [5]. These polynomials are very important in the theory of approximation and also in some other fields of mathematics see [6-12]. In [13], Bernstein type rational functions were defined by Balázs as follows:

$$R_n(f;x) = \frac{1}{(1+a_n x)^n} \sum_{j=0}^{n} f\left(\frac{j}{b_n}\right)\binom{n}{j}(a_n x)^j \quad , n \in \mathrm{N} \tag{1.0.1}$$

where f is a real and single-valued function defined on the half open interval $[0,\infty)$, a_n and b_n are real numbers which are suitably chosen and do not depend on x. He studied the approximation properties of these operators. In [14], Balázs and Szabados studied together and they improved the estimation which is presented in [13]. They did this improvement by selecting convenient a_n and b_n under some conditions for $f(x)$.

On the other side, in the last three decades, q-calculus has obtained a considerable area of research on the approximation of functions by using positive linear operators. Many q-operators were introduced and studied by several researchers. The first q- Bernstein operators were defined by Lupaş [15] and examined for its approximating properties and shape-conserving properties. After that, the q-analogue of the very popular polynomials of Bernstein was introduced by George M. Phillips in [16]. The q-Bernstein polynomials have become very popular and some other authors introduced and investigated many operators which are based on the q-integers, examined their approximation and statistical approximation properties. We may mention here some of them as Durrmeyer variant of q-Bernstein-Schurer operators [17], q-Bernstein-Schurer-Kantorovich type operators [18], q-Bernstein-Durrmeyer polynomials in compact disks [19], q-Stancu-Beta operators [20] and Kantorovich type q-Bernstein operators [21]. Doğru ([22]) and Özkan ([23] and [24]) have studied on various q-analogues of Balázs-Szabados (shortly called B-S) operators. Approximation properties of the q-Balázs-Szabados (shortly called q-B-S) complex operators are examined by N. I. Mahmudov in [25] and by İspir and Özkan in [26]. The B-S operators based on the q-integers defined by N. I. Mahmudov in [25] is as follows:

$$\Omega_{n,q}(f,x)=\frac{1}{(1+a_n x)^n}\sum_{j=0}^{n} f\left(\frac{[j]_q}{b_n}\right)\begin{bmatrix} n \\ j \end{bmatrix}_q (a_n x)^j \prod_{s=0}^{n-j-1}\left(1+(1-q)[s]_q a_n x\right) \qquad (1.1.1)$$

where $q>0$, $[n]_q$ is the q-integer which is defined in section 2.2, f is a real-valued function which is defined on the nonnegative real line, $a_n=[n]_q^{\beta-1}, b_n=[n]_q^{\beta}$, $\beta>0$ and $\beta\le\frac{2}{3}$, $n\in\mathbb{N}$ and $x\neq-\frac{1}{a_n}$. In this thesis mainly we considered these operators to get a new Kantorovich type q-analogue of the B-S operator with its approximation properties.

The thesis consists of four chapters and is arranged as follows. In the second chapter we present some formulas and definitions about q-calculus and positive linear operators (more details on this topic can be found in the studies [6], [27-31]). We also present mth-order moments of the q-analogue of B-S operators for $m=\{0,1,2,3,4\}$, some of them can be found in [25]. We find an estimation for the 2nd and the 4th-order central moments of the operators defined by (1.1.1) and these moments are used to prove some important theorems which are given in Chapter 3. We also mention about the q-Balázs-Szabados operator (which is also called the q-BSK operator) introduced by Esma Yıldız Özkan in [24]. Here in the definition of the q-BSK operator, there is a restricted condition on the function $f(x)$ such that it must be a nondecreasing function to provide the positivity. On the other side the operators defined by the equation (1.1.1) cannot approximate integrable functions. We use this operator which is given by (1.1.1) as a building block to construct a new Kantorovich type q-analogue of the B-S operators. In the third chapter, we construct a new Kantorovich type q-analogue of the B-S operators, we examined the formula of these newly defined operators in the special case $q=1$. Also, we obtain a recurrence relation for $\Omega^{*}_{n,q}(t^m,x)$ and we calculate the moments and important estimations of the 2nd and the 4th-order central moments of these operators $\Omega^{*}_{n,q}(f,x)$. We prove the approximation property of Korovkin type in the special case when $q=q_n$ as a sequence of subset of the interval (0,1) and we study local approximation properties via modulus of continuity. We investigate the results of the rate of convergence of these new operators in items of the elements of the usual Lipschitz class and we give a proof of a Voronoskaja type theorem. In the fourth chapter, we give the conclusion part of the thesis and we mention about the future work.

Chapter 2

PRELIMINARY AND AUXILIARY RESULTS

2.1 Positive Linear Operators

Here in this chapter, we present some important definitions, initial concepts and some properties related with the positive linear operators that are used as tools to state and prove the theories in the next chapter. More information on this topic can be found in [31].

Definition 2.1.1 ([31])

The mapping $\xi : Y \to Z$ where Y and Z are linear spaces of functions, is called a linear operator if

$$\xi(\alpha f + \beta g) = \alpha\xi(f) + \beta\xi(g)$$

$\forall\, f,\, g \in Y$ and $\forall\, \alpha, \beta \in \Re.$ If $f \geq 0,\, f \in Y$ implies that $\xi f \geq 0,$ then ξ is a positive linear operator.

For emphasizing the discussion of the function $\xi f \in Z,$ the notation $\xi(f;x)$ or in some cases $(\xi f)(x)$ is used.

Proposition 2.1.1. If $\xi : Y \to Z$ is a positive and linear operator, then

1. ξ is monotonic, that is, if f and g are in Y and g is greater than or equal to f, then ξg is greater than or equal to ξf.

2. For every f in Y the inequality $|\xi f| \leq \xi |f|$ is satisfied.

Definition 2.1.2. Suppose that Y and Z are two linear and normed spaces of real functions such that $Z \supseteq Y$ and let $\xi : Y \to Z$. For every linear operator ξ, one can appoint a norm $\|\xi\|$ defined by

$$\|\xi\| := \sup\ \left\{\ \|\xi f\|\ :\ f \in Y,\ \|f\| = 1\right\} = \sup\left\{\ \|\xi f\| :\ f \in Y,\ 0 < \|f\| \leq 1\right\}.$$

It is not difficult to verify that $\|.\|$ satisfies all the norm properties and thus it is called as the operator norm. If Y and Z are selected to be both equal to the space of continuous and real valued functions on the closed interval $[a,b]$, we will be able to state the next remark related to the continuity and the norm of an operator.

Remark 2.1.1 ([1]). Let $\xi : C[a,b] \to C[a,b]$ be linear and positive. Then ξ is continuous and $\|\xi\| = \|\xi e_0\|$ where $e_0(t) = t^0$.

A necessary and sufficient condition for the convergence of a positive linear operator towards the identity operator is provided in the next result. The current habitual finding of the theory of approximation is known very well as the Bohman Korovkin theorem. More details can be found in [1] and in [5].

Theorem 2.1.1 ([1]). Suppose that ξ_n is a sequence of positive linear operators from the space of real valued and continuous functions on $[a,b]$ to the space of real valued and continuous functions on $[a,b]$ and let $e_r(t) = t^r$. If $\xi_n e_r(t)$ converges uniformly to t^r for $r = 0,1,2$ for $t \in [a,b]$ then $\xi_n f$ converges uniformly to f for $t \in [a,b]$ for every function $f \in C[a,b]$.

The simple functions $e_r(t) = t^r$ for $r = 0,1,2,$ have a significant functionality in the approximation theory of positive linear operators on the continuous function spaces that depend on the result given above, and these are usually called as test functions. Lots of scientists in mathematics were charged up from this cute and basic finding, and they started to expand the final theorem by using various methods. This, in turn, generalizes the sequence notion and takes into consideration various spaces. One special part of the approximation theory that came in view with this way is said to be the Korovkin-type approximation theory. Rest of the details and wide explanation about this topic exists in [1]. For lots of estimations the C-S-I is employed:

$\left(\xi(gf)\right)^2 \le \xi\left(g^2\right)\xi\left(f^2\right),$ where f and g are in $C[a,b]$.

In the following theorem, an inequality of Hőlder-type for positive linear operators is given. If $q = p = 2$ it reduces to the C-S-I.

Theorem 2.1.2. Let $\xi : C[a,b] \to C[a,b]$ be a positive linear operator, $\xi e_0 = e_0$. If $q, p > 1,\ q^{-1} + p^{-1} = 1,$ f belongs to $C[a,b]$ and $x \in [a,b]$ one has

$$\xi\left(|gf|;x\right) \le \left(\xi\left(|g|^p\right)\right)^{\frac{1}{p}} \left(\xi\left(|f|^q\right)\right)^{\frac{1}{q}}.$$

The following are important quantities of linear positive operators.

The central moment of order r for the operator ξ_n is represented as follows

$\xi_n\left((e_1-e_0x)^r;x\right):=\xi_n\left((t-x)^r\right)\ (x),\ \ a\le x\le b,$

and for $r\ge 1$ also the absolute moments of odd order r, is represented as follows

$\xi_n\left(|e_1-e_0x|^r;x\right):=\xi_n\left(|e_1-e_0x|^r\right)\ (x),\ \ a\le x\le b.$

In particular, the second and the fourth-order central moments with their estimations and the first absolute moments have great importance. Calculating the first absolute moment is a hard task in many of the cases. Consequently, the C-S-I is used to estimate as below;

$\xi_n\left(|e_1-e_0x|;x\right)\le\sqrt{\xi_n\left(e_o^2;x\right)}\sqrt{\xi_n\left((e_1-e_0x)^2;x\right)}.$

2.2 The *q*-integers

In this section, we give some definitions and essential concepts of *q*-calculus, which are recently used to construct many various *q*-analogues of linear positive operators. More details can be found in [7], [26-27] and [29].

Definition 2.2.1 ([27]). For any nonnegative integer n, the *q*-integer of the number n is defined as

$$[n]_q=\begin{cases}\dfrac{1-q^n}{1-q} & \text{if} \quad q\ne 1\\ n & \text{if} \quad q=1\end{cases}\qquad \text{where } q \text{ is a positive real number.} \tag{2.2.1}$$

Let us define $\mathbb{N}_q$ as

$$\mathbb{N}_q=\left\{\ [n]_q,\ n\in\mathbb{N}_0\right\},\ \text{for any given } q>0 \tag{2.2.2}$$

and we can use from the definition 2.2.1 that

$$\mathbb{N}_q=\left\{\ 0,\ q+1,\ q^2+q+1,\ q^3+q^2+q+1,\ \ldots\right\} \tag{2.2.3}$$

It is clear that the set of q-integers $\mathbb{N}_q$ generalizes the set of non-negative integers $\mathbb{N}_0$, which we recover by putting $q=1$.

Definition 2.2.2 ([27]). Suppose that q is a positive number. We define the q-factorial, denoted by $[n]_q!$, where $n \in \mathbb{N}_0$, as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q [n-2]_q \dots [2]_q [1]_q & , \ n \geq 1 \\ 1 & , \ n=0 \end{cases}. \qquad (2.2.4)$$

Definition 2.2.3 ([28]). Suppose that k and n are two integers such that $0 \leq k \leq n$.

The q- binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{[n]_q!}{[k]_q![n-k]_q!} & \text{if} \quad q \neq 1 \\ \\ \dbinom{n}{k} & \text{if} \quad q=1 \end{cases} \qquad (2.2.5)$$

The q-binomial coefficient satisfies the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q .$$

Definition 2.2.4 ([27]). The q-analogue of $(x-a)^n$ is denoted by $(x-a)_q^n$ and is defined by the polynomial

$$(x-a)_q^n = \begin{cases} (x-a)\ (x-qa)(x-q^2a)\dots(x-q^{n-1}a) & \text{if} \quad n=1,2,3,\dots \\ \\ 1 & \text{if} \quad n=0. \end{cases}$$

The q-analogue of the common Pochhammer symbol which is also called the q-shifted factorial is defined in [27] as follows:

$$(x;q)_0 = 1 \ , \quad (x;q)_n = \prod_{j=0}^{n-1}\left(1-q^j x\right) \ , \quad (x;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j x\right).$$

Definition 2.2.5 ([27]). For $n \in \mathbb{N}_0$, the Gauss's binomial formula is

$$(x+a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{j(j-1)}{2}} a^j \, x^{n-j}, \tag{2.2.6}$$

and the Euler identity is given by

$$(1+x)_q^n = \sum_{j=0}^{n} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j.$$

Definition 2.2.6 ([27]). For $n \in \mathbb{N}_0$, the binomial formula found by Heine is

$$\frac{1}{(1-x)_q^n} = \sum_{j=0}^{n} \begin{bmatrix} n+j-1 \\ j \end{bmatrix} x^j . \tag{2.2.7}$$

We also present the following useful result:

$$x^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (x-1)_q^j .$$

Maybe a reader asks about the above formulas given in (2.2.6) and in (2.2.7), what could be the change in these formulas when n approaches to infinity. In the case $q=1$ which is the ordinary calculus, the change is not very interesting. Depending on x, it is either infinitely large or infinitely small. On the other hand, it is different in quantum calculus, because, for instance, the polynomial $(1+x)_q^\infty = (1+x)(1+qx)\left(1+q^2x\right)\ldots$

converges to a finite limit, for $|q|<1$. Likewise, as it is observed in [28], if we suppose $|q|<1$, then we can see that

$$\lim_{n\to\infty}[n]_q=\lim_{n\to\infty}(1-q^n)(1-q)^{-1}=(1-q)^{-1}. \tag{2.2.8}$$

and

$$\lim_{n\to\infty}\begin{bmatrix} n \\ j \end{bmatrix}_q=\lim_{n\to\infty}\frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^j)}.$$

Thus

$$\lim_{n\to\infty}\begin{bmatrix} n \\ j \end{bmatrix}_q=\lim_{n\to\infty}\frac{1}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^j)}. \tag{2.2.9}$$

So, the behavior of the q-analogue of integers and binomial coefficients is changing variously when n is so large if we compare with their ordinary counterparts.

If we apply the equations (2.2.8) and (2.2.9) to the binomial formulas given by Gauss and Heine, we get the two Euler's identities, whenever $n\to\infty$, which are power series in x for $-1<q<1$:

$$\prod_{j=0}^{\infty}(1+q^jx)=(1+x)_q^{\infty}=\sum_{j=0}^{\infty}q^{\frac{j(j-1)}{2}}\frac{x^j}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^j)}, \tag{2.2.10}$$

$$\prod_{j=0}^{\infty}\frac{1}{(1-q^jx)_q}=\frac{1}{(1-x)_q^{\infty}}=\sum_{j=0}^{\infty}\frac{x^j}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^j)}. \tag{2.2.11}$$

Definition 2.2.7 ([28]). For $|q|<1$, the $e_q(x)$ is in the following form

$$e_q(x)=\sum_{j=0}^{\infty}\frac{x^j}{[j]_q!}. \tag{2.2.12}$$

which is similar to the Taylor's expansion of the normal exponential function.

Using (2.2.11) we see that

$$\frac{1}{(1-x)_q^\infty} = \sum_{j=0}^{\infty} \frac{x^j}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^j)},$$

$$= \sum_{j=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^j}{\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^3}{1-q}\right)\ldots\left(\frac{1-q^j}{1-q}\right)},$$

$$= \sum_{j=0}^{\infty} \frac{(x/(1-q))^j}{[j]_q!}. \qquad (2.2.13)$$

Therefore, from (2.2.11) and (2.2.13), we directly have

$$e_q\left(\frac{x}{1-q}\right) = \frac{1}{(1-x)_q^\infty},$$

$$e_q(x) = \frac{1}{(1-(1-q)x)_q^\infty}.$$

Now we consider a different q-analogue of the exponential function, which is in the following form

$$E_q(x) = \sum_{j=0}^{n} q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]_q!}. \qquad (2.2.14)$$

Now, by using (2.2.10) we see that

$$(1+x)_q^\infty = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^j)},$$

$$= \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{\left(\frac{x}{1-q}\right)^j}{\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^3}{1-q}\right)\ldots\left(\frac{1-q^j}{1-q}\right)},$$

$$= \sum_{j=0}^{n} q^{\frac{j(j-1)}{2}} \frac{(x/1-q)^j}{[j]_q!}. \qquad (2.2.15)$$

Then, as it is mentioned in [28], from (2.2.14) and (2.2.15) we directly get

$E_q(x/1-q) = (1+x)_q^{\infty}$ and $E_q(x) = (1+(1-q)x)_q^{\infty}$.

Definition 2.2.8 ([28]). The q-integral (the Jackson integral) of the function f is defined by

$$\int_0^b f(t)\, d_q t = b(1-q)\sum_{j=0}^{\infty} f\left(bq^j\right) q^j, \quad 0 < q < 1, \;\; b > 0$$

and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \quad 0 < a < b.$$

As an example the q-integral of the function $f(t) = t^c$ on the interval [0,1] is

$$\int_0^1 t^c\, d_q t = \frac{1}{[c+1]_q}.$$

2.3 Balázs-Szabados Operators (B-S Operators)

As it is mentioned before, for $f \in C[0,1]$, Bernstein polynomials are introduced as follows (see [4])

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x)\; f\left(\frac{k}{n}\right), \qquad x \in [0,1] \qquad (2.3.1)$$

where $p_{n,k}(x)$ is the Bernstein basis function and is given by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \qquad (2.3.2)$$

It is a recognized fact that when $f(x)$ is continuous on the interval [0,1] then the above polynomials given in (2.3.1) converges uniformly to $f(x)$.

Obviously it can be seen that the sum of the values of $p_{n,k}(x)$ for $k=0,...,n$ is 1, i.e.

$$\sum_{k=0}^{n} p_{n,k}(x) = \sum_{k=0}^{n-1} p_{n-1,k}(x) = \sum_{k=0}^{n-2} p_{n-2,k}(x) = ...\sum_{k=0}^{1} p_{1,k}(x) = (x+1-x) = 1. \qquad (2.3.3)$$

In [6], it is mentioned about an important and efficient property of the Bernstein basis function, which is obtained by taking the derivative of (2.3.2) as it is given in the following:

$$p'_{n,k}(x) = \frac{d}{dx}\left(\binom{n}{k} x^k (1-x)^{n-k}\right).$$

Then, we can obtain

$$x(1-x)\, p'_{n,k}(x) = (k-nx)\, p_{n,k}(x), \qquad (2.3.4)$$

which helps in deducing the recurrence relation of $B_n(t^{m+1};x),\ m \in \{0\} \cup \mathbb{N}$.

The following lemma is given in [6].

Lemma 2.3.1 ([6]). Let $n \in \mathbb{N},\ x \in [0,1],$ then

$$B_n(t^{m+1};x) = \frac{x(1-x)}{n} B'_n(t^m;x) + xB_n(t^m;x).$$

Proof. The proof is done by writing explicitly

$$B_n(t^{m+1};x) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^{m+1}}{n^{m+1}} x^k (1-x)^{n-k} ,$$

since $\frac{k^{m+1}}{n^{m+1}} = \frac{k^m}{n^m}\frac{k}{n}$, we write

$$B_n(t^{m+1};x)=\sum_{k=0}^{n}\frac{k^{m+1}}{n^{m+1}}p_{n,k}(x)=\sum_{k=0}^{n}\left(\frac{k^m}{n^m}\right)\left(\frac{k}{n}p_{n,k}(x)\right).$$

Then, by using property (2.3.3) one get

$$B_n(t^{m+1};x)=\sum_{k=0}^{n}\left(\frac{k^m}{n^m}\right)\left(\frac{x(1-x)}{n}\right)p'_{n,k}(x)+\sum_{k=0}^{n}\left(\frac{k^m}{n^m}\right)\left(xp_{n,k}(x)\right)$$

$$=\frac{x(1-x)}{n}B'_n(t^m;x)+xB_n(t^m;x), \qquad (2.3.5)$$

which is the required result.

Lemma 2.3.2. ([6]). For $n\in\mathbb{N},\ x\in[0,1]$, we have

$$B_n(1;x)=1,\ B_n(t;x)=x,$$

$$B_n(t^2;x)=\frac{x(1-x)}{n}+x^2,$$

$$B_n(t^3;x)=\frac{x(1-x)}{n}\left\{\frac{1-2x}{n}+3x\right\}+x^3.$$

Proof. By using (2.3.3) it is obvious that $B_n(1,x)=1$ and $B_n(t;x)=x$.

For the evaluation of $B_n(t^2;x)$ and $B_n(t^3;x)$, we use the recurrence relation that is given in (2.3.5). We can see that

$$B_n(t^2;x)=\frac{x(1-x)}{n}B'_n(t;x)+xB_n(t;x)=\frac{x(1-x)}{n}+x^2,$$

$$B_n(t^3;x)=\frac{x(1-x)}{n}B'_n(t^2;x)+xB_n(t^2;x),$$

$$B_n\left(t^3;x\right)=\frac{x(1-x)}{n}\left(\frac{1-2x}{n}+2x\right)+\frac{x^2(1-x)}{n}+x^3,$$

$$B_n\left(t^3;x\right)=\frac{x(1-x)}{n}\left(\frac{1-2x}{n}+3x\right)+x^3.$$

In [13], Balázs defined and studied approximation properties of Bernstein type rational functions as in the following:

$$R_n(f;x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_nx)^k, \qquad n=1,2,3,... \tag{2.3.6}$$

where $f(x)$ is a real and single-valued function defined on the interval $[0,\infty)$, a_n and b_n are real numbers which are suitably chosen and don't depend on x. In the particular case where $a_n=n^{\beta-1}$, $b_n=n^{\beta}$, $n\in\mathbb{N}$, $0<\beta<1$, the operators (2.3.6) are denoted by $R_n^{[\beta]}$. In [13], Balázs stated and proved Voronoskaja type theorem under the assumption that $a_n=\frac{b_n}{n}\to 0$ and $\frac{\sqrt{n}}{b_n}\to 0$ as $n\to\infty$. Also, in the same paper Balázs gave the convergence theorems for the operators $R_n^{\left[\frac{2}{3}\right]}$ and the convergence of their derivatives to the derivatives of the function. Later in [14] Balázs and Szabados together improved the estimate in [13] by choosing suitable sequences a_n and b_n under some restrictions for $f(x)$. Besides, Balázs and Szabados together presented the weighted estimates of $R_n^{[\beta]}$ where $0<\beta\le\frac{2}{3}$ and developed certain questions of uniform convergence of $R_n^{[\beta]}$ where $0<\beta\le\frac{2}{3}$. As well as in [31], Gal introduced the rational complex B-S operators, he investigated and studied the approximation properties on complex disks.

2.4 *q*-Balázs-Szabados Operators

Recently, the studies on the q-operators have been one of the very attractive and effective subjects of research in approximation theory. Many researchers defined lots of different q-operators and examined their approximation properties. In [16], firstly, Philips defined and presented q-Bernstein operators in the rational form as in the following

$$B_{n,q}(f;x)=\sum_{k=0}^{n} p_{n,k}(q,x)\, f\left(\frac{[k]_q}{[n]_q}\right) \tag{2.4.1}$$

where f is contained in the space of all continuous functions defined on the interval $0 \le x \le 1$, $0 < q < 1$ and the q-Bernstein basis function is defined as follows

$$p_{n,k}(q,x)=\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}\left(1-q^s x\right),\ x\in[0,1].$$

Ostrovska in [33-34] and Mahmudov in [35-36] accomplished praiseworthy and appreciable work on these operators defined by (2.4.1) and they provided many valuable and interesting results. Besides, in [37], Mahmudov and Sabancigil proposed Voronovskaja type theorem for the Lupaş q-analogue of the Bernstein operators.

Different q-analogue of the Balázs-Szabados operators have recently been studied by several researchers. In [22], the q-analogue of B-S operators introduced by Doğru as in the following:

$$R_{n,q}(f;x)=\frac{1}{\prod_{s=0}^{n-1}\left(1+q^s a_n x\right)}\sum_{j=0}^{n} q^{j(j-1)/2}\ f\left(\frac{[j]_q}{b_n}\right)\begin{bmatrix} n \\ j \end{bmatrix}_q \left(a_n x\right), \tag{2.4.2}$$

where $x\in[0,\infty), a_n=[n]_q^{\beta-1}, b_n=[n]_q^{\beta}$ for all $n\in\mathbb{N}, q\in(0,1]$ and $0<\beta\le 2/3$.

In [23], Özkan introduced the q-B-S Stancu operators as follows

$$R_{n,q}^{(\alpha,\gamma)}(f;q,x)=\sum_{j=0}^{n} f\left(\frac{[j]_q+[\alpha]_q}{b_n+[\gamma]_q}\right)p_{n,j}(x;q), \tag{2.4.3}$$

where f is a real-valued function defined on $[0,\infty)$, $a_n=[n]_q^{\beta-1}, b_n=[n]_q^{\beta}$, for all $n\in\mathbb{N}$, $q\in(0,1]$, $0<\beta\leq 2/3$ and $0\leq\alpha\leq\gamma$,

$$p_{n,j}(x;q)=\frac{1}{(1+a_nx)_q^n}q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}_q (a_nx)^j.$$

She investigated and studied the statistical approximation properties of these operators that are given in (2.4.3). On the other hand, the newly defined q-B-S operators are given by N.I. Mahmudov as follows

$$\Omega_{n,q}(f,x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{[k]_q}{b_n}\right)\begin{bmatrix} n \\ k \end{bmatrix}_q (a_nx)^k \prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_n x\right) \tag{2.4.4}$$

where $q>0$ and f is a real-valued function defined on $[0,\infty)$, $a_n=[n]_q^{\beta-1}, b_n=[n]_q^{\beta}$, $n\in\mathbb{N}$, $\beta\in\left(0,\frac{2}{3}\right]$ and $x\neq\frac{1}{a_n}$. These operators can be called new q-analogue of B-S operators. They have an important role in the construction of the main operator of this thesis. Now for these new operators $\Omega_{n,q}$, we will evaluate the 1st, 2nd, 3rd and the 4th order moments in the following lemma.

Lemma 2.4.1. For $0<q<1, x\in[0,\infty)$ and for all $n\in\mathbb{N}$ we have

$$\Omega_{n,q}(1;x)=1,\quad \Omega_{n,q}(t;x)=\frac{x}{1+a_nx},$$

$$\Omega_{n,q}\left(t^2;x\right)=\frac{x}{b_n(1+a_nx)}+\frac{x^2}{(1+a_nx)^2},$$

$$\Omega_{n,q}\left(t^3;x\right)=\frac{q^3[n-1]_q[n-2]_q}{a_nb_n^2}\left(\frac{a_nx}{1+a_nx}\right)^3+\frac{q[n-1]_q(2+q)}{a_nb_n^2}\left(\frac{a_nx}{1+a_nx}\right)^2$$
$$+\frac{1}{a_nb_n^2}\frac{a_nx}{1+a_nx},$$

$$\Omega_{n,q}\left(t^4;x\right)=\frac{q^6[n-1]_q[n-2]_q[n-3]_q}{b_n^4}\left(\frac{a_nx}{1+a_nx}\right)^4$$

$$+\frac{\left(q^5+2q^4+3q^3\right)[n]_q[n-1]_q[n-2]_q}{b_n^4}\left(\frac{a_nx}{1+a_nx}\right)^3$$

$$+\frac{\left(q^3+3q^2+3q\right)[n]_q[n-1]_q}{b_n^4}\left(\frac{a_nx}{1+a_nx}\right)^2+\frac{[n]_q}{b_n^4}\left(\frac{a_nx}{1+a_nx}\right).$$

Proof. The formulas for $\Omega_{n,q}(1;x)$, $\Omega_{n,q}(t;x)$ and $\Omega_{n,q}\left(t^2;x\right)$ can be found in [24] without proofs. The proofs are as follows:

$$\Omega_{n,q}(1;x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(a_nx)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_qa_nx\right)$$

$$=\frac{1}{(1+a_nx)^{n-k+k}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(a_nx)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_qa_nx\right)$$

$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\frac{a_nx}{1+a_nx}\right)^k\frac{1}{(1+a_nx)^{n-k}}\times\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_qa_nx\right)$$

$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\frac{a_nx}{1+a_nx}\right)^k\left(1-\frac{a_nx}{1+a_nx}\right)_q^{n-k}=1.$$

$$\Omega_{n,q}(t;x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[k]_q}{b_n}(a_nx)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_qa_nx\right),$$

$$=\left(\frac{a_n x}{1+a_n x}\right)\times\frac{1}{\left(1+a_n x\right)^{n-1}}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(a_n x\right)^k \prod_{s=0}^{n-k-2}\left(1+\left(1-q\right)\left[s\right]_q a_n x\right)$$

$$=\frac{a_n x}{1+a_n x}.$$

Now to evaluate $\Omega_{n,q}\left(t^2;x\right)$, we write

$$\Omega_{n,q}\left(t^2;x\right)=\frac{1}{\left(1+a_n x\right)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q^2}{b_n^2}\left(a_n x\right)^k \prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)[s]_q a_n x\right)$$

$$\Omega_{n,q}\left(t^2;x\right)=\frac{1}{\left(1+a_n x\right)^n}\sum_{k=1}^{n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{[k]_q}{a_n b_n}\left(a_n x\right)^k \prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)[s]_q a_n x\right)$$

Using the fact that $[k]_q = q[k-1]_q + 1$, we get

$$\Omega_{n,q}\left(t^2;x\right)=\frac{1}{a_n b_n}\frac{1}{\left(1+a_n x\right)^n}\sum_{k=1}^{n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(q[k-1]_q + 1\right)\left(a_n x\right)^k \prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)[s]_q a_n x\right)$$

$$\Omega_{n,q}\left(t^2;x\right)=\frac{q[n-1]_q}{[n]_q}\left(\frac{x}{1+a_n x}\right)^2\frac{1}{\left(1+a_n x\right)^{n-2}}\sum_{k=0}^{n-2}\begin{bmatrix} n-2 \\ k \end{bmatrix}_q \left(a_n x\right)^k \prod_{s=0}^{n-k-3}\left(1+\left(1-q\right)[s]_q a_n x\right)$$

$$+\frac{x}{b_n\left(1+a_n x\right)}\frac{1}{\left(1+a_n x\right)^{n-1}}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(a_n x\right)^k \prod_{s=0}^{n-k-2}\left(1+\left(1-q\right)[s]_q a_n x\right)$$

$$\Omega_{n,q}\left(t^2;x\right)=\frac{q[n-1]_q}{[n]_q}\left(\frac{x}{1+a_n x}\right)^2+\frac{x}{b_n\left(1+a_n x\right)}$$

$$\Omega_{n,q}\left(t^2;x\right)=\frac{x}{b_n\left(1+a_n x\right)}+\frac{x^2}{\left(1+a_n x\right)^2}.$$

Now, $\Omega_{n,q}\left(t^3;x\right)$ is calculated as follows:

$$\Omega_{n,q}\left(t^3;x\right)=\frac{1}{\left(1+a_n x\right)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q^3}{b_n^3}\left(a_n x\right)^k \prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)[s]_q a_n x\right),$$

since $[k]_q\begin{bmatrix} n \\ k \end{bmatrix}_q = [n]_q\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ and $b_n = a_n [n]_q$, we have

$$\Omega_{n,q}\left(t^3;x\right)=\frac{1}{a_n^3}\frac{1}{\left(1+a_nx\right)^n}\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\frac{[k]_q^2}{b_n^2}\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right).$$

Now by using the simple facts that

$$[k]_q=q[k-1]_q+1 \quad \text{and} \quad [k-1]_q\begin{bmatrix}n-1\\k-1\end{bmatrix}_q=[n-1]_q\begin{bmatrix}n-2\\k-2\end{bmatrix}_q,$$

we get

$$\Omega_{n,q}\left(t^3;x\right)=\frac{1}{a_n^3}\frac{1}{\left(1+a_nx\right)^n}\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\frac{\left(q[k-1]_q+1\right)^2}{b_n^2}\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

$$\Omega_{n,q}\left(t^3;x\right)=\frac{q^2[n-1]_q}{a_nb_n^2\left(1+a_nx\right)^n}\sum_{k=2}^{n}\begin{bmatrix}n-2\\k-2\end{bmatrix}_q\left(q[k-2]_q+1\right)\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

$$+\frac{2q[n-1]_q}{a_nb_n^2\left(1+a_nx\right)^n}\sum_{k=2}^{n}\begin{bmatrix}n-2\\k-2\end{bmatrix}_q\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

$$+\frac{1}{a_nb_n^2\left(1+a_nx\right)^n}\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

$$\Omega_{n,q}\left(t^3;x\right)=\frac{q^3[n-1]_q[n-2]_q}{a_nb_n^2}\left(\frac{a_nx}{1+a_nx}\right)^3\Omega_{n-3,q}(1;x)$$

$$+\frac{q(2+q)[n-1]_q}{a_nb_n^2}\left(\frac{a_nx}{1+a_nx}\right)^2\Omega_{n-2,q}(1;x)$$

$$+\frac{1}{a_nb_n^2}\left(\frac{a_nx}{1+a_nx}\right)\Omega_{n-1,q}(1;x)$$

By the above calculation, we obtain

$$\Omega_{n,q}\left(t^3;x\right)=\frac{q^3[n-1]_q[n-2]_q}{a_nb_n^2}\left(\frac{a_nx}{1+a_nx}\right)^3+\frac{q[n-1]_q(2+q)}{a_nb_n^2}\left(\frac{a_nx}{1+a_nx}\right)^2$$

$$+\frac{1}{a_nb_n^2}\frac{a_nx}{1+a_nx}.$$

Similarly, $\Omega_{n,q}\left(t^4;x\right)$ is calculated in the following way:

Using the fact that $[k]_q = q[k-1]_q + 1$ we get

$$\Omega_{n,q}\left(t^4;x\right)=\frac{1}{a_n b_n^3\left(1+a_n x\right)^n}\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\left(q\left[k-1\right]+1\right)^3\left(a_n x\right)^k\prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)\left[s\right]_q a_n x\right).$$

$$\Omega_{n,q}\left(t^4;x\right)=\frac{q^3\left[n-1\right]_q}{a_n b_n^3\left(1+a_n x\right)^n}\sum_{k=2}^{n}\begin{bmatrix}n-2\\k-2\end{bmatrix}_q\left(q\left[k-2\right]+1\right)^2\left(a_n x\right)^k\prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)\left[s\right]_q a_n x\right)$$

$$+\frac{3q^2\left[n-1\right]_q}{a_n b_n^3\left(1+a_n x\right)^n}\sum_{k=2}^{n}\begin{bmatrix}n-2\\k-2\end{bmatrix}_q\left(q\left[k-2\right]+1\right)\left(a_n x\right)^k\prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)\left[s\right]_q a_n x\right)$$

$$+\frac{3q\left[n-1\right]_q}{a_n b_n^3\left(1+a_n x\right)^n}\sum_{k=2}^{n}\begin{bmatrix}n-2\\k-2\end{bmatrix}_q\left(a_n x\right)^k\prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)\left[s\right]_q a_n x\right)$$

$$+\frac{1}{a_n b_n^3\left(1+a_n x\right)^n}\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\left(a_n x\right)^k\prod_{s=0}^{n-k-1}\left(1+\left(1-q\right)\left[s\right]_q a_n x\right)$$

By simple calculations we get

$$\Omega_{n,q}\left(t^4;x\right)=\frac{q^6\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q}{a_n b_n^3\left(1+a_n x\right)^n}\left(\frac{a_n x}{1+a_n x}\right)^4\Omega_{n-4,q}\left(1;x\right)$$

$$+\frac{\left(q^5+2q^4+3q^3\right)\left[n-1\right]_q\left[n-2\right]_q}{a_n b_n^3\left(1+a_n x\right)^n}\left(\frac{a_n x}{1+a_n x}\right)^3\Omega_{n-3,q}\left(1;x\right)$$

$$+\frac{\left(q^3+3q^2+3q\right)\left[n-1\right]_q}{a_n b_n^3\left(1+a_n x\right)^n}\left(\frac{a_n x}{1+a_n x}\right)^2\Omega_{n-2,q}\left(1;x\right)$$

$$+\frac{1}{a_n b_n^3\left(1+a_n x\right)^n}\left(\frac{a_n x}{1+a_n x}\right)\Omega_{n-1,q}\left(1;x\right).$$

After that, we get

$$\Omega_{n,q}\left(t^4;x\right)=\frac{q^6\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q}{b_n^4}\left(\frac{a_n x}{1+a_n x}\right)^4$$

$$+\frac{\left(q^5+2q^4+3q^3\right)\left[n\right]_q\left[n-1\right]_q\left[n-2\right]_q}{b_n^4}\left(\frac{a_n x}{1+a_n x}\right)^3$$

$$+\frac{\left(q^3+3q^2+3q\right)[n]_q[n-1]_q}{b_n^4}\left(\frac{a_nx}{1+a_nx}\right)^2+\frac{[n]_q}{b_n^4}\left(\frac{a_nx}{1+a_nx}\right).$$

Notice. From the definition of the new *q*-analogue of B-S operators $\Omega_{n,q}$, we can observe that they have a close relationship with the *q*-Bernstein polynomials as it is given in (2.4.1).

For all $n\in\mathbb{N}$, $x\in[0,\infty)$ and $q\in(0,1)$ we have

$$\Omega_{n,q}(f;x)=B_{n,q}\left(F_n;\frac{a_nx}{1+a_nx}\right) \tag{2.4.5}$$

where $F_n(x)=f\left(\frac{[n]_q}{b_n}\right)$ and $0<\frac{a_nx}{1+a_nx}<1$. (2.4.6)

Besides, this relationship (2.4.5) is very important and effective for deducing the recurrence relation of $\Omega_{n,q}\left(t^m;x\right)$, which is given in the next lemma.

Lemma 2.4.2 ([25]). For all $n\in\mathbb{N}$, $x\in[0,\infty)$, $q\in(0,1)$ we have

$$\Omega_{n,q}\left(t^m;x\right)=\frac{1}{a_n^m}B_{n,q}\left(t^m;\frac{a_nx}{1+a_nx}\right),$$

where $0<\frac{a_nx}{1+a_nx}<1$.

Proof. With the help of the simple calculations, we obtain immediately

$$\Omega_{n,q}\left(t^m;x\right)=\frac{1}{\left(1+a_nx\right)^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[k]_q^m}{b_n^m}\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_qa_nx\right)$$

$$\Omega_{n,q}\left(t^m;x\right)=\frac{1}{\left(1+a_nx\right)^{n-k+k}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[k]_q^m}{b_n^m}\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

$$=\frac{1}{\left(1+a_nx\right)^{n}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[k]_q^m}{a_n^m[n]_q^m}\left(a_nx\right)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

$$=\frac{1}{a_n^m}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[k]_q^m}{[n]_n^m}\left(\frac{a_nx}{1+a_nx}\right)^k\frac{1}{\left(1+a_nx\right)^{n-k}}\times\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

$$=\frac{1}{a_n^m}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[k]_q^m}{[n]_n^m}\left(\frac{a_nx}{1+a_nx}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{a_nx}{1+a_nx}\right).$$

Therefore, by the above calculations and by (2.4.5) we get the explicit formula for

$x\in[0,\infty),\ q\in(0,1)$

$$\Omega_{n,q}\left(t^m;x\right)=\frac{1}{a_n^m}B_{n,q}\left(t^m;\frac{a_nx}{1+a_nx}\right),\quad 0<\frac{a_nx}{1+a_nx}<1,\forall n\in\mathrm{N}.$$

In the next lemma, we present the evaluation of the 2nd and 4th order central moments of the new q-analogue of B-S operators $\Omega_{n,q}$, which are needed to evaluate 2nd and 4th order central moments of $\Omega^*_{n,q}$ that will be given in the next chapter.

Lemma 2.4.3. $\forall n\in\mathbb{N},\ x\in[0,\infty)$ and $q\in(0,1)$ we have

$$\Omega_{n,q}\left((t-x)^2;x\right)\le\frac{\lambda_n(q)}{b_n}\left(x+x^2\right),\quad \lambda_n(q)=\max\{1,b_n\}$$

$$\Omega_{n,q}\left((t-x)^4;x\right)\le\frac{C(q)}{b_n}\left(x+x^2\right),\quad C(q)>0$$

Proof. By using Lemma 2.4.1 and the linearity property of the operator $\Omega_{n,q}$ directly, we have

$$\Omega_{n,q}\left((t-x)^2;x\right)=\Omega_{n,q}\left(t^2;x\right)-2x\Omega_{n,q}\left(t;x\right)+x^2\Omega_{n,q}\left(1;x\right),$$

$$\Omega_{n,q}\left((t-x)^2;x\right)=\frac{x}{b_n\left(1+a_nx\right)^2}+\frac{x^2}{\left(1+a_nx\right)^2}-\frac{2x^2}{\left(1+a_nx\right)^2}+x^2$$

$$\Omega_{n,q}\left((t-x)^2;x\right)=\frac{b_nx^2+x+a_nb_nx^3-b_nx^2+a_n^2b_nx^4-a_nb_nx^3}{b_n\left(1+a_nx\right)^2}$$

$$\Omega_{n,q}\left((t-x)^2;x\right)=\frac{x+a_n^2b_nx^4}{b_n\left(1+a_nx\right)^2}.$$

By using the facts that,

$$\frac{(a_nx)^2}{(1+a_nx)^2}<1 \text{ and } \frac{1}{(1+a_nx)^2}<1, \text{ for } x\in[0,\infty) \tag{2.4.7}$$

$$\Omega_{n,q}\left((t-x)^2;x\right)=\frac{x+a_n^2b_nx^4}{b_n\left(1+a_nx\right)^2}=\frac{x}{b_n\left(1+a_nx\right)^2}+\frac{a_n^2x^2\times b_nx^2}{b_n\left(1+a_nx\right)^2},$$

now by substituting the inequalities given in (2.4.7) into the above equation we obtain

$$\Omega_{n,q}\left((t-x)^2;x\right)\le\frac{x}{b_n}+\frac{b_nx^2}{b_n}\le\frac{1}{b_n}\left(x+b_nx\right)$$

$$\Omega_{n,q}\left((t-x)^2;x\right)\le\frac{\lambda_n(q)}{b_n}\left(x+x^2\right),\qquad \lambda_n(q)=\max\{1,b_n\}.$$

For the estimation of $\Omega_{n,q}\left((t-x)^4;x\right)$, we write the formula explicitly as follows:

$$\Omega_{n,q}\left((t-x)^4;x\right)$$

$$=\Omega_{n,q}\left(t^4;x\right)-4x\Omega_{n,q}\left(t^3;x\right)+6x^2\Omega_{n,q}\left(t^2;x\right)-4x^3\Omega_{n,q}\left(t;x\right)+x^4\Omega_{n,q}\left(1;x\right)$$

$$=\left\{q^6\frac{[n-1][n-2]_q[n-3]_q}{[n]_q^3(1+a_nx)^4}-4q^3\frac{[n-1]_q[n-2]_q}{[n]_q^2(1+a_nx)^3}+\frac{6}{(1+a_nx)^2}-\frac{4}{(1+a_nx)}+1\right\}x^4$$

$$+\left\{\left(q^5+2q^4+3q^3\right)\frac{[n-1]_q[n-2]_q}{a_n[n]_q^3(1+a_nx)^3}-\frac{4q(2+q)[n-1]_q}{a_n[n]_q(1+a_nx)^2}+\frac{6}{a_n[n]_q(1+a_nx)^2}\right\}x^3$$

$$+\left\{\left(q^3+3q^2+3q\right)\frac{[n-1]_q}{a_n^2[n]_q^3(1+a_nx)^2}-\frac{4}{a_n^2[n]_q^2(1+a_nx)}\right\}x^2$$

$$+\frac{1}{a_n^3[n]_q^3(1+a_nx)}x\,.$$

Now if we consider the following facts given below $q[n-1]_q=[n]_q-1$, $q^2[n-2]_q=[n]_q-[2]_q$ and $q^3[n-3]_q=[n]_q-[3]_q$, we may reorganize the last equality by assigning some variables to the terms in the numerators as it is shown below

$$\Omega_{n,q}\left((t-x)^4;x\right)=\frac{\sum_1}{a_n^3[n]_q^3(1+a_nx)}x+\frac{\sum_2}{a_n^2[n]_q^3(1+a_nx)^2}x^2$$

$$+\frac{\sum_3}{a_n[n]_q^3(1+a_nx)^3}x^3+\frac{\sum_4}{[n]_q^3(1+a_nx)^3}x^4$$

where

$$\sum\nolimits_1=1,\quad \sum\nolimits_2=\left(q^3+3q^2+3q\right)\left([n]_q-1\right)-4[n]_q(1+a_nx),$$

$$\sum\nolimits_3=\left(q^5+2q^4+3q^3\right)\left([n]_q-1\right)\left([n]_q-2\right)-4q\left(1+[2]_q\right)[n]_q\left([n]_q-1\right)(1+a_nx)+6[n]_q^2(1+a_nx)$$

and

$$\sum\nolimits_4=\Big\{\left([n]_q-1\right)\left([n]_q-[2]_q\right)\left([n]_q-[3]_q\right)-4q^3[n]_q\left([n]_q-1\right)\left([n]_q-[2]_q\right)(1+a_nx)$$

$$+6[n]_q^3(1+a_nx)^2-4[n]_q^3(1+a_nx)^3+[n]_q^3(1+a_nx)^4\Big\},$$

and now if we consider the powers of $[n]_q$ in $\sum_1, \sum_2, \sum_3, \sum_4$ and the facts that

$\frac{1}{(1+a_nx)}\leq 1$ and $\frac{a_nx}{1+a_nx}\leq 1,$ we see that for $[0,\infty)$

$$\Omega_{n,q}\left((t-x)^4;x\right)\leq\frac{1}{b_n^2}x+\frac{\left(q^3+3q^2+3q-4\right)}{b_n^2}x^2,$$

$\Omega_{n,q}\left((t-x)^4;x\right) \leq \frac{1}{b_n^2}C(q)\left(x+x^2\right)$. For the restricted interval $[0,a]$, we have

$$\Omega_{n,q}\left((t-x)^4;x\right) \leq \frac{1}{b_n^2}C(q,a).$$

In the following lemma we give a formula for the *m*-th order central moments of the *q*-Balázs-Szabados operators $\Omega_{n,q}(f;x)$.

Lemma 2.4.4 ([27]). For all $n\in\mathbb{N}$, $x\in[0,\infty)$ we have

$$\Omega_{n,q}\left((t-x)^m;x\right) = \frac{1}{a_n^m}\sum_{j==0}^{m}\binom{m}{j}(-a_nx)^{m-j}\,B_{n,q}\left(t^j,\frac{a_nx}{1+a_nx}\right)$$

where $0<\frac{a_nx}{1+a_nx}<1$.

Proof. By writing $\Omega_{n,q}\left((t-x)^m;x\right)$ explicitly, we obtain

$$\Omega_{n,q}\left((t-x)^m;x\right) = \frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\frac{[k]_q}{b_n}-x\right)^m (a_nx)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q\,a_nx\right).$$

By using the facts that

$$a_n[n]_q = b_n \qquad \text{and} \qquad \left(\frac{[k]_q}{b_n}-x\right)^m = \sum_{j=0}^{m}\binom{m}{j}\left(\frac{[k]_q}{b_n}\right)^j(-x)^{m-j},$$

we have,

$$\Omega_{n,q}\left((t-x)^m;x\right) = \frac{1}{a_n^m}\sum_{j=0}^{m}\binom{m}{j}(-a_nx)^{m-j}\sum_{k=0}^{n}\left(\frac{[k]_q}{[n]_q}\right)^j\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\frac{a_nx}{1+a_nx}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{a_nx}{1+a_nx}\right)$$

If we use (2.4.1) we get

$$\Omega_{n,q}\left((t-x)^m;x\right)=\frac{1}{a_n^m}\sum_{j=0}^{m}\binom{m}{j}(-a_nx)^{m-j}\,B_{n,q}\left(t^j;\frac{a_nx}{1+a_nx}\right).$$

On the other hand, Kantorovich type q-analogue of B-S operators (q-BSK operators) introduced by E. Özkan in [22] is as follows:

$$\widetilde{R}_n(f;q,x)=\frac{b_n}{\prod_{s=0}^{n-1}(1+q^na_nx)}\sum_{j=0}^{n}q^{j(j-1)/2}\begin{bmatrix}n\\j\end{bmatrix}_q(a_nx)^j\int_{\frac{q[j]_q}{b_n}}^{\frac{q[j+1]_q}{b_n}}f(t)d_qt, \tag{2.4.8}$$

where f is a nondecreasing and continuous function on $[0,\infty)$, $a_n=[n]_q^{\beta-1}$ and $b_n=[n]_q^{\beta}$, for all $n\in\mathbb{N}$, $0<q<1$ and $\beta\in\left(0,\frac{2}{3}\right]$. Since f is nondecreasing and from the definition of q-integral , q-BSK operator given in (2.4.8) is a positive operator. The operator, which is given in (2.4.4) is a summation type operator, which is not capable to approximate integrable functions. So to maintain the positivity of the q-BSK operators defined by (2.4.8), f must be a nondecreasing function. The main motivation of this thesis is to construct a new Kantorovich type q-analogue of the B-S operators that approximate also the integrable functions on the interval $[0,\infty)$ and maintain the positivity of the operators without nondecreasing restriction on f.

In the next chapter we define these new operators and give the recurrence formula that helps us to evaluate the moments for these new operators. We obtain the local approximation property and establish a Voronoskaja type theorem.

Chapter 3

NEW KANTOROVICH TYPE *q*-ANALOGUE OF BALÁZS-SZABADOS OPERATORS

In this section, we give the definition of a new Kantorovich type q-analogue of B-S operators. We construct a recurrence formula and we calculate the moments up to the fourth order with the help of the recurrence formula for these new operators. As well as, we calculate the first, second and the fourth-order central moments of these new operators and at the same time we give their estimations. Estimations of these central moments play an important role in obtaining quantitative results for convergence rate in various cases. In addition, we study Korovkin's type approximation property and the local approximation theorem of these new operators. Also, we investigate the convergence rate in terms of the elements of the usual Lipschitz class and we prove Voronoskaja type theorem.

3.1 Construction of The Operators

Definition 3.1.1 Let $0<q<1$. For $f:[0,\infty)\to\mathbb{R}$, the new Kantorovich type q-analogue of the B-S operators is defined as follows:

$$\Omega^{*}_{n,q}(f,x)=\sum_{k=0}^{n}\omega_{n,k}(q,x)\int f\left(\frac{[k]_q+q^k t}{b_n}\right)d_q t,$$

where

$$\omega_{n,k}(q,x)=\frac{1}{(1+a_n x)^n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a_n x)^k \prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q\, a_n x\right),$$

$a_n=[n]_q^{\beta-1}, b_n=[n]_q^{\beta}, 0<\beta\le\frac{2}{3}, n\in\mathbb{N},\ x\ge 0$ and f is a real-valued continuous function defined on the interval $[0,\infty)$. In the case $q=1$ these polynomials reduce to

$$\Omega_{n,q}^{*}(f,x)=\sum_{k=0}^{n}\omega_{n,k}(x)\int_0^1 f\left(\frac{k+t}{b_n}\right)dt,$$

where

$$\omega_{n,k}(x)=\frac{1}{(1+a_n x)^n}\binom{n}{k}(a_n x)^k,\ a_n=n^{\beta-1}, b_n=n^{\beta}, 0<\beta\le\frac{2}{3}, n\in\mathbb{N}, x\in[0,\infty),$$ and

in this case the results coincide with the results for q-BSK operators defined in [24] by E. Özkan.

3.2 The Recurrence Formula of the Operators and their Moments

In the next lemma, we give the recurrence formula that is needed for the evaluation of the moments of new Kantorovich type q-analogue of the B-S operators.

Lemma 3.2.1 For all $n\in\mathbb{N}$, $x\in[0,\infty)$, $m\in\mathbb{Z}^{+}\cup\{0\}$ and $0<q<1$ we have

$$\Omega_{n,q}^{*}(t^m;x)=\sum_{j=0}^{m}\binom{m}{j}\frac{1}{b_n^{m-j}[m-j+1]_q}\sum_{i=0}^{m-j}\binom{m-j}{i}(a_n)^i\,(q^n-1)^i\,\Omega_{n,q}(t^{i+j};x)$$

Proof. By direct calculation, the recurrence formula is obtained as follows:

$$\Omega_{n,q}^{*}(t^m;x)=\sum_{k=0}^{n}\omega_{n,k}(q;x)\int_0^1\left(\frac{[k]_q+q^k t}{b_n}\right)^m d_q t,$$

using the binomial formula for $\left([k]_q+q^k t\right)^m$ and evaluating the q-integral as below

$$\left([k]_q + q^k t\right)^m = \sum_{j=0}^{m} \left([k]_q\right)^j \left(q^k t\right)^{m-j},$$

$$\int_0^1 t^{m-j}\, d_q t = (1-q)\sum_{k=0}^{\infty} q^k \cdot q^{k(m-j)} = \frac{1}{[m-j+1]_q},$$

we get,

$$\Omega^*_{n,q}\left(t^m; x\right) = \sum_{k=0}^{n} \omega_{n,k}(q,x) \sum_{j=0}^{m} \binom{m}{j} \frac{q^{(m-j)k}[k]_q}{b_n^m\, [m-j+1]_q}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[m-j+1]_q} \sum_{k=0}^{n} q^{(m-j)k} \frac{[k]_q}{b_n^m} \omega_{n,k}(q,x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[m-j+1]_q} \sum_{k=0}^{n} \left(q^k + 1 - 1\right)^{m-j} \frac{[k]_q^j}{b_n^m} \omega_{n,k}(q,x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}[m-j+1]_q} \times \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} \left(q^k - 1\right)^i \frac{[k]_q^j}{b_n^j} \omega_{n,k}(q,x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}[m-j+1]_q} \times \sum_{i=0}^{m-j} \binom{m-j}{i} \frac{\left(q^n - 1\right)^i}{[n]_q^i} \sum_{k=0}^{n} \frac{[k]_q^{i+j}}{(b_n)^j} \omega_{n,k}(q,x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}\, [m-j+1]_q} \times \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i \left(q^n - 1\right)^i \sum_{k=0}^{n} \frac{[k]_q^{i+j}}{(b_n)^{i+j}} \omega_{n,k}(q,x),$$

Since the last summation is $\Omega_{n,q}\left(t^{i+j}; x\right)$ then we obtain

$$\Omega^*_{n,q} = \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}\, [m-j+1]_q} \times \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i \left(q^n - 1\right)^i \Omega_{n,q}\left(t^{i+j}; x\right).$$

In the following lemma, we calculate $\Omega^*_{n,q}(f;x)$ for the monomials $f(t) = t^m$ for $m = \{0,1,2,3,4\}$.

Lemma 3.2.2. For all $n\in\mathbb{N}$, $0\le x<\infty$ and $0<q<1$, we have the following equalities:

$$\Omega^*_{n,q}(1;x)=1,$$

$$\Omega^*_{n,q}(t;x)=\frac{2q}{[2]_q}\frac{x}{1+a_nx}+\frac{1}{[2]_qb_n},$$

$$\Omega^*_{n,q}\left(t^2;x\right)=\frac{q[n-1]_q}{[n]_q}\frac{4q^3+q^2+q}{[2]_q[3]_q}\left(\frac{x}{1+a_nx}\right)^2+\frac{4q^3+5q^2+3q}{[2]_q[3]_qb_n}\left(\frac{x}{1+a_nx}\right)$$
$$+\frac{1}{[3]_qb_n^2},$$

$$\Omega^*_{n,q}\left(t^3;x\right)=\left(\frac{(q-1)^3}{[4]_q}+\frac{3(q-1)^2}{[3]_q}+\frac{3(q-1)}{[2]_q}+1\right)\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2}\left(\frac{x}{1+a_nx}\right)^3$$

$$+\left\{\frac{(q-1)^2\left(q^2+q+1\right)}{[4]_qb_n}+\frac{3q\left(q^2-1\right)}{[3]_qb_n}\right.$$
$$\left.+\frac{3\left(q^2+q-1\right)}{[2]_qb_n}+\frac{2+q}{b_n}\right\}\frac{q[n-1]_q}{[n]_q}\left(\frac{x}{1+a_nx}\right)^2+\left\{\frac{q^3-1}{[4]_qb_n^2}+\frac{3q^2}{[3]_qb_n^2}\right.$$
$$\left.+\frac{3q}{[4]_qb_n^2}+\frac{1}{b_n^2}\right\}\frac{x}{1+a_nx}+\frac{1}{[4]_qb_n^3},$$

$$\Omega^*_{n,q}\left(t^4;x\right)=\frac{q^6[n-1]_q[n-2]_q[n-3]_q}{[n]_q^3}\left\{\frac{(q-1)^4}{[5]_q}+\frac{4(q-1)^3}{[4]_q}+\frac{6(q-1)^2}{[3]_q}\right.$$
$$\left.+\frac{4(q-1)}{[2]_q}+1\right\}\left(\frac{x}{1+a_nx}\right)^4$$

$$+\left\{\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2 b_n}\left(\left([3]_q+[2]_q+1\right)+\frac{4\left([4]_q-3\right)}{[2]_q}+\frac{6(q-1)\left([4]_q-2\right)}{[3]_q}\right)\right.$$

$$\left.+\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2 b_n}\left(\frac{4[3]_q\,q(q-1)^2}{[4]_q}+\frac{[4]_q(q-1)^3}{[5]_q}\right)\right\}\left(\frac{x}{1+a_n x}\right)^3$$

$$+\left\{\frac{q[n-1]_q\left(3+3q+q^2\right)}{[n]_q b_n^2}+\frac{4q[n-1]_q\left(q^3+2q^2+q+1\right)}{[2]_q[n]_q b_n^2}\right.$$

$$+\frac{6q^2[n-1]_q\left((q-1)^3+2(q-1)+q^3\right)}{[3]_q[n]_q b_n^2}+\frac{4q^3[n-1]_q\left(q^3-1\right)}{[4]_q[n]_q b_n^2}$$

$$\left.+\frac{q[n-1]_q(q-1)^2\left([5]_q+q^2\right)}{[5]_q[n]_q b_n^2}\right\}\left(\frac{x}{1+a_n x}\right)^2$$

$$+\left\{\frac{q^4+4q^2-4q-1}{[5]_q b_n^3}+\frac{4q^3}{[4]_q b_n^3}+\frac{6q^2}{[3]_q b_n^3}+\frac{4q}{[2]_q b_n^3}+\frac{1}{b_n^3}\right\}\left(\frac{x}{1+a_n x}\right)+\frac{1}{[5]_q b_n^4}.$$

Proof. The proof is done by using the recurrence formula given in Lemma 3.2.1,

It can be easily seen that $\Omega^*_{n,q}(1;x)=1$.

Now, by using the recurrence formula we have

$$\Omega^*_{n,q}(t;x)=\sum_{j=0}^{1}\binom{1}{j}\frac{1}{b_n^{1-j}[2-j]_q}\times\sum_{i=0}^{1-j}\binom{1-j}{i}(a_n)^i\left(q^n-1\right)^i\,\Omega_{n,q}\left(t^{i+j};x\right),$$

$$=\frac{1}{[2]_q b_n}\left(\Omega_{n,q}(1;x)+a_n\left(q^n-1\right)\Omega_{n,q}(t;x)\right)+\Omega_{n,q}(t;x)$$

$$=\frac{1}{[2]_q b_n}+\left(\frac{q-1}{q+1}+1\right)\Omega_{n,q}(t;x)$$

$$=\frac{1}{[2]_q b_n}+\frac{2q}{[2]_q}\Omega_{n,q}(t;x),$$

now by using the equality for $\Omega_{n,q}(t;x)$ which is given in Lemma 2.4.1 we get

$$\Omega^*_{n,q}(t,x)=\frac{1}{[2]_q b_n}+\frac{2q}{[2]_q}\frac{x}{1+a_n x}.$$

In a similar way,

$$\Omega^*_{n,q}(t^2;x)$$

$$=\sum_{j=0}^{2}\binom{2}{j}\frac{1}{b_n^{2-j}[3-j]_q}\times\sum_{i=0}^{2-j}\binom{2-j}{i}(a_n)^i\,(q^n-1)^i\,\Omega_{n,q}(t^{i+j};x)$$

$$=\frac{1}{[3]_q b_n^2}\left(1+2a_n(q^n-1)\Omega_{n,q}(t;x)+a_n^2(q^n-1)^2\Omega_{n,q}(t^2;x)\right)$$

$$+\frac{2}{[2]_q b_n}\left(\Omega_{n,q}(t;x)+a_n(q^n-1)\Omega_{n,q}(t^2;x)\right)+\Omega_{n,q}(t^2;x)$$

$$=\frac{1}{[3]_q b_n^2}+\left(\frac{2(q-1)}{[3]_q b_n}+\frac{2}{[2]_q b_n}\right)\Omega_{n,q}(t;x)+\left(\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}+1\right)\Omega_{n,q}(t^2;x)$$

$$=\frac{1}{[3]_q b_n^2}+\left(\frac{4q^2+2q}{[2]_q[3]_q b_n}\right)\Omega_{n,q}(t;x)+\left(\frac{4q^3+q^2+q}{[2]_q[3]_q}\right)\Omega_{n,q}(t^2;x)$$

$$=\frac{1}{[3]_q b_n^2}+\left(\frac{4q^2+2q}{[2]_q[3]_q b_n}\right)\frac{x}{1+a_n x}+\frac{4q^3+q^2+q}{[2]_q[3]_q}\frac{x}{b_n(1+a_n x)}$$

$$+\frac{4q^3+q^2+q}{[2]_q[3]_q}\frac{q[n-1]_q}{[n]_q}\left(\frac{x}{1+a_n x}\right)^2$$

$$=\frac{1}{[3]_q b_n^2}+\frac{4q^3+5q^2+3q}{[2]_q[3]_q b_n}\left(\frac{x}{1+a_n x}\right)+\frac{q[n-1]_q}{[n]_q}\,\frac{4q^3+q^2+q}{[2]_q[3]_q}\left(\frac{x}{1+a_n x}\right)^2.$$

$$\Omega^*_{n,q}\left(t^3;x\right)$$

$$=\sum_{j=0}^{3}\binom{3}{j}\frac{1}{b_n^{3-j}\left[4-j\right]_q}\times\sum_{i=0}^{3-j}\binom{3-j}{i}\left(a_n\right)^i\left(q^n-1\right)^i\Omega_{n,q}\left(t^{i+j};x\right)$$

$$=\frac{1}{\left[4\right]_q b_n^3}\left\{1+3a_n\left(q^n-1\right)\Omega_{n,q}\left(t;x\right)+3a_n^2\left(q^n-1\right)^2\Omega_{n,q}\left(t^2;x\right)\right.$$

$$\left.+a_n^3\left(q^n-1\right)^3\Omega_{n,q}\left(t^3;x\right)\right\}$$

$$+\frac{3}{\left[3\right]_q b_n^2}\left\{\Omega_{n,q}\left(t;x\right)+2a_n\left(q^n-1\right)\Omega_{n,q}\left(t^2;x\right)+a_n^2\left(q^n-1\right)^2\Omega_{n,q}\left(t^3;x\right)\right\}$$

$$+\frac{3}{\left[2\right]_q b_n}\left\{\Omega_{n,q}\left(t^2;x\right)+a_n\left(q^n-1\right)\Omega_{n,q}\left(t^3;x\right)\right\}+\Omega_{n,q}\left(t^3;x\right)$$

$$=\frac{1}{\left[4\right]_q b_n^3}+\left(\frac{3a_n\left(q^n-1\right)}{\left[4\right]_q b_n^3}+\frac{3}{\left[3\right]_q b_n^2}\right)\Omega_{n,q}\left(t,x\right)$$

$$+\left\{\frac{3a_n^2\left(q^n-1\right)^2}{\left[4\right]_q b_n^3}+\frac{6a_n\left(q^n-1\right)}{\left[3\right]_q b_n^2}+\frac{3}{\left[2\right]_q b_n}\right\}\Omega_{n,q}\left(t^2;x\right)$$

$$+\left\{\frac{a_n^3\left(q^n-1\right)^3}{\left[4\right]_q b_n^3}+\frac{3a_n^2\left(q^n-1\right)^2}{\left[3\right]_q b_n^2}+\frac{3a_n\left(q^n-1\right)}{\left[2\right]_q b_n}+1\right\}\Omega_{n,q}\left(t^3;x\right)$$

by substituting the formulas of $\Omega_{n,q}\left(t;x\right)$, $\Omega_{n,q}\left(t^2;x\right)$ and $\Omega_{n,q}\left(t^3;x\right)$ we obtain

$$=\frac{q^3\left[n-1\right]_q\left[n-2\right]_q}{\left[n\right]_q^2}\left\{\frac{(q-1)^3}{\left[4\right]_q}+\frac{3(q-1)^2}{\left[3\right]_q}+\frac{3(q-1)}{\left[2\right]_q}+1\right\}\left(\frac{x}{1+a_nx}\right)^3$$

$$+\frac{q\left[n-1\right]_q}{\left[n\right]_q}\left\{\frac{(q-1)^2\left(q^2+q+1\right)}{\left[4\right]_q b_n}+\frac{3q\left(q^2-1\right)}{\left[3\right]_q b_n}+\frac{3\left(q^2+q-1\right)}{\left[2\right]_q b_n}+\frac{1+\left[2\right]_q}{b_n}\right\}\left(\frac{x}{1+a_nx}\right)^2$$

$$+\left\{\frac{q^3-1}{\left[4\right]_q b_n^2}+\frac{3q^2}{\left[3\right]_q b_n^2}+\frac{3q}{\left[2\right]_q b_n^2}+\frac{1}{b_n^2}\right\}\frac{x}{1+a_nx}+\frac{1}{\left[4\right]_q b_n^3}.$$

$$\Omega^*_{n,q}\left(t^4;x\right)$$

$$=\sum_{j-0}^{4}\binom{4}{j}\frac{1}{b_n^{4-j}\left[5-j\right]_q}\times\sum_{i-0}^{4-j}\binom{4-j}{i}\left(a_n\right)^i\left(q^n-1\right)^i\ \Omega_{n,q}\left(t^{i+j};x\right)$$

$$=\frac{1}{[5]_q b_n^4}\left\{1+4a_n\left(q^n-1\right)\Omega_{n,q}(t;x)+6a_n^2\left(q^n-1\right)^2\Omega_{n,q}\left(t^2;x\right)\right.$$

$$\left.+4a_n^3\left(q^n-1\right)^3\Omega_{n,q}\left(t^3;x\right)+a_n^4\left(q^n-1\right)^4\Omega_{n,q}\left(t^4;x\right)\right\}+\frac{4}{[4]_q b_n^3}\left\{\Omega_{n,q}(t;x)\right.$$

$$\left.+3a_n\left(q^n-1\right)\Omega_{n,q}\left(t^2;x\right)+3a_n^2\left(q^n-1\right)^2\Omega_{n,q}\left(t^3;x\right)+a_n^3\left(q^n-1\right)^3\Omega_{n,q}\left(t^4;x\right)\right\}$$

$$+\frac{6}{[3]_q b_n^2}\left\{\Omega_{n,q}\left(t^2;x\right)+2a_n\left(q^n-1\right)\Omega_{n,q}\left(t^3;x\right)+a_n^2\left(q^n-1\right)^2\Omega_{n,q}\left(t^4;x\right)\right\}$$

$$+\frac{4}{[2]_q b_n}\left\{\Omega_{n,q}\left(t^3;x\right)+a_n\left(q^n-1\right)\Omega_{n,q}\left(t^4;x\right)\right\}+\Omega_{n,q}\left(t^4;x\right)$$

$$\frac{1}{[5]_q b_n^4}+\left\{\frac{4a_n\left(q^n-1\right)}{[5]_q b_n^4}+\frac{4}{[4]_q b_n^3}\right\}\Omega_{n,q}(t;x)$$

$$+\left\{\frac{6a_n^2\left(q^n-1\right)^2}{[5]_q b_n^4}+\frac{12a_n\left(q^n-1\right)}{[4]_q b_n^3}+\frac{6}{[3]_q b_n^2}\right\}\Omega_{n,q}\left(t^2;x\right)$$

$$+\left\{\frac{4a_n^3}{[5]_q b_n^4}+\frac{12a_n^2\left(q^n-1\right)^2}{[4]_q b_n^3}+\frac{12a_n\left(q^n-1\right)}{[3]_q b_n^2}+\frac{4}{[2]_q b_n}\right\}\Omega_{n,q}\left(t^3;x\right)$$

$$+\left\{\frac{a_n^4\left(q^n-1\right)^4}{[5]_q b_n^4}+\frac{4a_n^3\left(q^n-1\right)^3}{[4]_q b_n^3}+\frac{6a_n^2\left(q^n-1\right)^2}{[3]_q b_n^2}+\frac{a_n\left(q^n-1\right)}{[2]_q b_n}+1\right\}\Omega_{n,q}\left(t^4;x\right)$$

now by substituting the formulas of $\Omega_{n,q}(t;x), \Omega_{n,q}\left(t^2;x\right), \Omega_{n,q}\left(t^3;x\right)$ and $\Omega_{n,q}\left(t^4;x\right)$,

as are calculated in Lemma 2.4.1 we obtain

$$\Omega_{n,q}^*\left(t^4;x\right)=\frac{q^6\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q}{\left[n\right]_q^3}\left\{\frac{\left(q-1\right)^4}{[5]_q}+\frac{4\left(q-1\right)^3}{[4]_q}+\frac{6\left(q-1\right)^2}{[3]_q}\right.$$

$$\left.+\ \frac{4\left(q-1\right)}{[2]_q}+1\right\}\left(\frac{x}{1+a_n x}\right)^4$$

$$+\left\{\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2 b_n}\left(([3]_q+[2]_q+1)+\frac{4([4]_q-3)}{[2]_q}+\frac{6(q-1)([4]_q-2)}{[3]_q}\right)\right.$$

$$\left.+\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2 b_n}\left(\frac{4[3]_q q(q-1)^2}{[4]_q}+\frac{[4]_q(q-1)^3}{[5]_q}\right)\right\}\left(\frac{x}{1+a_n x}\right)^3$$

$$+\left\{\frac{q[n-1]_q\left(3+3q+q^2\right)}{[n]_q b_n^2}+\frac{4q[n-1]_q\left(q^3+2q^2+q+1\right)}{[2]_q[n]_q b_n^2}\right.$$

$$+\frac{6q^2[n-1]_q\left((q-1)^3+2(q-1)+q^3\right)}{[3]_q[n]_q b_n^2}+\frac{4q^3[n-1]_q\left(q^3-1\right)}{[4]_q[n]_q b_n^2}$$

$$\left.+\frac{q[n-1]_q(q-1)^2\left([5]_q+q^2\right)}{[5]_q[n]_q b_n^2}\right\}\left(\frac{x}{1+a_n x}\right)$$

$$+\left\{\frac{q^4+4q^2-4q-1}{[5]_q b_n^3}+\frac{4q^3}{[4]_q b_n^3}+\frac{6q^2}{[3]_q b_n^3}+\frac{4q}{[2]_q b_n^3}+\frac{1}{b_n^3}\right\}\frac{x}{1+a_n x}+\frac{1}{[5]_q b_n^4}.$$

3.3 Auxiliary Results for Convergence Properties

Lemma 3.3.1. For all $n\in\mathbb{N}$ and $0<q<1$, we have the following estimations

$$\left(\Omega_{n,q}^*\left((t-x);x\right)\right)^2\leq\frac{2}{b_n}\left\{\frac{\left(1-q^n\right)^2}{b_n}\left(\frac{1}{1+q}+\frac{a_n x}{1-q}\right)^2+\frac{1}{b_n}\right\},\ x\in[0,\infty)$$

$$\Omega_{n,q}^*\left((t-x)^2;x\right)\ \leq\ \frac{2}{b_n}\left\{\frac{1}{[3]_q b_n}+\frac{x\left(1+a_n^2 b_n x^3\right)}{\left(1+a_n x\right)^2}\right\},\ x\in[0,\infty)$$

$$\Omega_{n,q}^*\left((t-x)^4;x\right)\leq\frac{1}{b_n^2}C_1(q,a)\ \text{for}\ x\in[0,a],$$

where $C_1(q,a)$ is a positive constant which depends on q and a.

Proof. First of all, we will give the estimation of $\left(\Omega_{n,q}^*\left((t-x);x\right)\right)^2$. For $x\in[0,\infty)$,

$$\left(\Omega_{n,q}^{*}\left(\left(t-x\right);x\right)\right)^{2}=\left(\frac{2q}{\left[2\right]_{q}}\frac{x}{1+a_{n}x}-x+\frac{1}{\left[2\right]_{q}b_{n}}\right)^{2}$$

$$\leq 2\left(\frac{\left(1-q\right)x}{\left[2\right]_{q}\left(1+a_{n}x\right)}+\frac{a_{n}x^{2}}{1+a_{n}x}\right)^{2}+2\left(\frac{1}{\left[2\right]_{q}b_{n}}\right)^{2}$$

$$\leq 2\left(\frac{\left(1-q\right)x}{\left[2\right]_{q}\left(1+a_{n}x\right)}+\frac{b_{n}}{b_{n}}\times\frac{a_{n}x^{2}}{1+a_{n}x}\right)^{2}+2\left(\frac{1}{\left[2\right]_{q}b_{n}}\right)^{2}$$

$$\leq\frac{2}{b_{n}^{2}}\left(\frac{1-q^{n}}{1+q}\times\frac{a_{n}x}{1+a_{n}x}+\frac{1-q^{n}}{1-q}\left(a_{n}x\right)\frac{a_{n}x}{1+a_{n}x}\right)^{2}+2\left(\frac{1}{\left[2\right]_{q}b_{n}}\right)^{2}$$

$$\leq\frac{2}{b_{n}^{2}}\left(\frac{1-q^{n}}{1+q}+\frac{1-q^{n}}{1-q}\left(a_{n}x\right)\right)^{2}+2\left(\frac{1}{\left[2\right]_{q}b_{n}}\right)^{2}$$

$$\leq\frac{2}{b_{n}^{2}}\left\{\left(1-q^{n}\right)^{2}\left(\left(\frac{1}{1+q}+\frac{a_{n}x}{1-q}\right)^{2}+1\right)\right\}$$

$$=\frac{2}{b_{n}}\left\{\frac{\left(1-q^{n}\right)^{2}}{b_{n}}\left(\frac{1}{1+q}+\frac{a_{n}x}{1-q}\right)^{2}+\frac{1}{b_{n}}\right\}.$$

Now for the estimation of, $\Omega_{n,q}^{*}\left(\left(t-x\right)^{2};x\right)$ we use the formula of $\Omega_{n,q}\left(\left(t-x\right)^{2};x\right)$ which is calculated in Lemma 2.4.3. For $x\in\left[0,\infty\right)$,

$$\Omega_{n,q}^{*}\left(\left(t-x\right)^{2};x\right)=\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\int_{0}^{1}\left(\frac{\left[k\right]_{q}+q^{k}t}{b_{n}}-x\right)^{2}d_{q}t$$

$$=\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\int_{0}^{1}\left(\frac{q^{k}t}{b_{n}}+\frac{\left[k\right]_{q}}{b_{n}}-x\right)^{2}d_{q}t$$

$$\leq 2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\int_{0}^{1}\left(\frac{q^{k}t}{b_{n}}\right)^{2}d_{q}t+2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\int_{0}^{1}\left(\frac{\left[k\right]_{q}}{b_{n}}-x\right)^{2}d_{q}t$$

$$\leq 2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\frac{q^{2k}}{b_n^2}\int_0^1\left(\frac{q^k}{b_n}\right)d_qt+2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\int_0^1\left(\frac{[k]_q}{b_n}-x\right)^2d_qt$$

$$\leq 2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\frac{q^{2k}}{[3]_q\,b_n^2}+2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)+2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\left(\frac{[k]_q}{b_n}-x\right)^2$$

$$=2\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\frac{q^{2k}}{[3]_q\,b_n^2}+2\Omega_{n,q}\left(\left(t-x\right)^2;x\right),$$

so we obtain

$$\Omega_{n,q}^*\left(\left(t-x\right)^2;x\right)\leq\frac{2}{[3]_q\,b_n^2}+2\left(\frac{x+a_n^2b_nx^4}{b_n\left(1+a_nx\right)^2}\right),$$

if we reorganize the quantity on the right side of the inequality we can write

$$\Omega_{n,q}^*\left(\left(t-x\right)^2;x\right)\leq\frac{2}{b_n}\left(\frac{1}{[3]_q\,b_n}+\frac{x\left(1+a_n^2b_nx^3\right)}{\left(1+a_nx\right)^2}\right).$$

Now if we take $x\in[0,a]$, we can estimate the 4th-order central moment by using similar calculations.

$$\Omega_{n,q}^*\left(\left(t-x\right)^4;x\right)=\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\int_0^1\left(\frac{[k]_q+q^kt}{b_n}-x\right)^4d_qt$$

$$=\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\int_0^1\left(\frac{q^kt}{b_n}+\frac{[k]_q}{b_n}-x\right)^4d_qt$$

$$\leq 4\sum_{k=0}^{n}\omega_{n,k}(q,x)\int_0^1\left(\frac{q^kt}{b_n}\right)^4d_qt+4\sum_{k=0}^{n}\omega_{n,k}(q,x)\int_0^1\left(\frac{[k]_q}{b_n}-x\right)^4d_qt$$

$$\leq 4\sum_{k=0}^{n}\omega_{n,k}(q,x)\frac{q^{4k}}{b_n^4}\int_0^1t^4d_qt+4\sum_{k=0}^{n}\omega_{n,k}(q,x)\left(\frac{[k]_q}{b_n}-x\right)^4$$

$$=4\sum_{k=0}^{n}\omega_{n,k}\left(q,x\right)\frac{q^{4k}}{[5]_q\,b_n^4}+4\Omega_{n,q}\left(\left(t-x\right)^4;x\right)$$

$$\leq \frac{4}{[5]_q b_n^4} + \frac{4}{b_n^2} C(q,a)$$

$$\leq \frac{1}{b_n^2} C_1(q,a).$$

The next lemma is very important to state and prove the Voronoskaja-type theorem for the new Kantorovich type q-analogue of B-S operators, but before that, we make an observation for the limit of the quantity $a_n^2 [n]_q$ when $0 < \beta \leq \frac{2}{3}$. We can easily reach that $a_n^2 [n]_q$ approaches to infinity on the interval $1/2 < \beta \leq 2/3$ when $n \to \infty$, and $\lim\limits_{n\to\infty} a_n^2 [n]_q = \lim\limits_{n\to\infty} [n]_q^{2\beta-1}$ exists on the interval $0 < \beta \leq 1/2$. This observation shows that the convergence behavior of the new Kantorovich type q-analogue of the B-S operators can only be defined when $0 < \beta < \frac{1}{2}$, otherwise can not be defined.

Lemma 3.3.2. Assume that $0 < q_n < 1$, $q_n \to 1$, $q_n^n \to \mu$ as $n \to \infty$ and $0 < \beta < \frac{1}{2}$. Then we have the following limits

$$i) \lim_{n\to\infty} b_{n,q_n} \Omega^*_{n,q_n}\left((t-x);x\right) = \frac{1}{2},$$

$$ii) \lim_{n\to\infty} b_{n,q_n} \Omega^*_{n,q_n}\left((t-x)^2;x\right) = x,$$

Proof.

$$\lim_{n\to\infty} b_{n,q_n} \Omega^*_{n,q_n}\left((t-x);x\right)$$

$$= \lim_{n\to\infty} b_{n,q_n} \left(\Omega^*_{n,q_n}(t;x) - x\,\Omega^*_{n,q_n}(1;x)\right)$$

$$=\lim_{n\to\infty} b_{n,q_n}\left(\frac{1}{[2]_q b_{n,q_n}}+\frac{2q_n}{[2]_{q_n}}\frac{x}{1+a_{n,q_n}x}-x\right)$$

$$=\lim_{n\to\infty} b_{n,q_n}\left(\frac{1}{(1+q_n)b_{n,q_n}}+\frac{2q_n}{(1+q_n)}\frac{x}{1+a_{n,q_n}x}-x\right)$$

$$=\lim_{n\to\infty} b_{n,q_n}\left(\frac{q_n-1}{q_n+1}\frac{x}{1+a_{n,q_n}x}-\frac{a_{n,q_n}x^2}{1+a_{n,q_n}x}+\frac{1}{(1+q_n)b_{n,q_n}}\right)$$

$$=\lim_{n\to\infty}\frac{q_n^n-1}{q_n+1}\left(\frac{a_{n,q_n}x}{1+a_{n,q_n}x}\right)-\lim_{n\to\infty}\frac{[n]_{q_n}^{2\beta-1}x^2}{1+a_{n,q_n}x}+\lim_{n\to\infty}\frac{1}{q_n+1}=\frac{1}{2}.$$

For the second statement we write

$$\lim_{n\to\infty} b_{n,q_n}\Omega^*_{n,q_n}\left((t-x)^2;x\right)$$

$$=\lim_{n\to\infty} b_{n,q_n}\left\{\Omega^*_{n,q_n}\left(t^2;x\right)-x^2-2x\Omega^*_{n,q_n}\left((t-x);x\right)\right\}$$

$$=\lim_{n\to\infty} b_{n,q_n}\left\{\frac{1}{[3]_q b_{n,q_n}^2}+\frac{4q_n^3+5q_n^2+3q_n}{[2]_q[3]_q b_{n,q_n}}\frac{x}{1+a_n x}-x^2\right.$$

$$\left.+\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}}\left(\frac{x}{1+a_{n,q_n}x}\right)^2-2x\Omega^*_{n,q_n}(D_1;x)\right\},$$

$$=\lim_{n\to\infty} b_{n,q_n}\frac{1}{[3]_q b_{n,q_n}^2}+\lim_{n\to\infty} b_{n,q_n}\frac{4q_n^3+5q_n^2+3q_n}{[2]_q[3]_q b_{n,q_n}}\frac{x}{1+a_{n,q_n}x}$$

$$+\lim_{n\to\infty} b_{n,q_n}\left(1-\frac{1}{[n]_{q_n}}\right)\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}}\left(\frac{x}{1+a_{n,q_n}x}\right)^2-\lim_{n\to\infty} b_{n,q_n}x^2$$

$$-2x\lim_{n\to\infty} b_{n,q_n}\Omega^*_{n,q_n}(D_1;x),$$

$$=\lim_{n\to\infty}\frac{1}{[3]_q b_{n,q_n}}+\lim_{n\to\infty}\frac{4q_n^3+5q_n^2+3q_n}{[2]_q[3]_q}\times\lim_{n\to\infty}\frac{x}{1+a_{n,q_n}x}$$

$$+\lim_{n\to\infty} b_{n,q_n}\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}}\times\lim_{n\to\infty}\left(\frac{x}{1+a_{n,q_n}x}\right)^2-\lim_{n\to\infty} a_{n,q_n}\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}}$$

$$\times \lim_{n\to\infty}\left(\frac{x}{1+a_{n,q_n}x}\right)^2 - \lim_{n\to\infty} b_{n,q_n} x^2 - x,$$

$$= 2x - \lim_{n\to\infty} b_{n,q_n}\left(\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}} - 1\right)x^2 - x$$

$$= x + \lim_{n\to\infty} a_{n,q_n}\left(1-q_n^n\right)\frac{3q_n^3-q_n^2-q_n-1}{q_n^4-q_n^3+q_n+1} = x$$

which proves the lemma.

3.4 Convergence Properties

In this section, we will present the approximation property of Korovkin type for the new Kantorovich type q-analogue of B-S operators. We establish and investigate the local approximation theorem for the new Kantorovich type q-analogue of B-S operators.

Remark 3.4.1. We can see that $\lim_{n\to\infty}[n]_q = \begin{cases} \dfrac{1}{1-q} & , \quad q\in(0,1) \\ \infty & , \quad q=q_n\in(0,1) \end{cases}$

For example, if we choose $q_n = \dfrac{n-1}{n}$ such that $0<q_n<1$, then it is obvious that

$\lim_{n\to\infty} q_n = \lim_{n\to\infty}\dfrac{n-1}{n} = 1$, $\lim_{n\to\infty} q_n^n = \lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n = e^{-1}$, then $\lim_{n\to\infty}[n]_{q_n} = \lim_{n\to\infty}\dfrac{1-q_n^n}{1-q_n} = \infty$.

The study of the convergence rate of the operators $\Omega^*_{n,q}(f,x)$ in a particular case by considering $q=q_n\in(0,1)$ as a sequence is the main idea in the next theorem for investigation approximation property of Korovkin type of the new Kantorovich type q-analogue of B-S operators.

Theorem 3.4.2. Let $0 < q_n < 1$. The sequence $\left\{\Omega^*_{n,q_n}\left(f,x\right)\right\}$ converges to *f* uniformly on the interval $0 \le x \le a, a > 0$ if and only if $q_n \to 1$ as $n \to \infty$. For each $f \in C[0,\infty)$, with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$.

Proof. Assume that $q_n \to 1$, as $n \to \infty$. Now we need to show that $\left\{\Omega^*_{n,q_n}\left(f,x\right)\right\}$ converges to *f* uniformly on the interval $0 \le x \le a, a > 0$. we consider the lattice homomorphism $S_a : C[0,\infty) \to C[0,a]$ defined by

$$S_a(f) = f_{|[0,a]}$$

according to the well –known property of Korovkin type theorem in [1] and [6], it is sufficient to prove the following:

$\lim_{n\to\infty}\left\|\Omega^*_{n,q}\left(t^m;x\right) - x^m\right\| = 0$, $m = 0,1,2,$ uniformly on the interval $0 \le x \le a, a > 0$.

For $m = 0$ it is obvious since $\Omega^*_{n,q}(1;x) = 1$. Using the formula of $\Omega^*_{n,q}(t,x)$ in Lemma 3.1.2 we get

$$\begin{aligned}
\left\|\Omega^*_{n,q_n}(t;x) - x\right\|_{0\le x\le a} &= \sup_{0\le x\le a}\left|\Omega^*_{n,q_n}(t;x) - x\right| \\
&= \sup_{0\le x\le a}\left|\frac{2q_n}{[2]_{q_n}}\,\frac{x}{1+a_n x} - x + \frac{1}{[2]_{q_n} b_{n,q_n}}\right| \\
&\le \sup_{0\le x\le a}\left(\frac{2q_n}{[2]_{q_n}\left(1+a_{n,q_n}x\right)} - 1\right)x + \frac{1}{[2]_{q_n} b_{n,q_n}}
\end{aligned}$$

which implies that $\lim_{n\to\infty}\left\|\Omega^*_{n,q_n}(t;x) - x\right\| = 0$. By similar calculations, we can check that for $m = 2$, $\lim_{n\to\infty}\left\|\Omega^*_{n,q_n}\left(t^2;x\right) - x^2\right\|_{[0,a]} = 0.$ Therefore, with respect to the interval $0 \le x \le a$, we have

$\lim\limits_{n\to\infty}\left\|\Omega^*_{n,q_n}(f,x)-f(x)\right\|=0, \quad \forall f\in[0,\infty)$, which is the required result.

Now, we prove the converse result by contradiction. Assume that the sequence $\{q_n\}$ does not converge to 1. Then the sequence $\{q_n\}$ must contain a subsequence $\{q_{n_k}\}$ which is contained in the interval $(0,1)$ such that $q_{n_k}\to c\in[0,1)$ as $k\to\infty$. Consequently,

$$\lim_{n\to\infty}[n_k]_{q_{n_k}}=\lim_{n\to\infty}\frac{1-q_{n_k}^n}{1-q_{n_k}}=\frac{1}{1-c},$$

then we can see that

$\Omega^*_{n,q_{n_k}}(t;x)-x=\dfrac{1}{[2]_{q_{n_k}}b_{n_k}}+\dfrac{2q_{n_k}}{[2]_{q_{n_k}}}\dfrac{x}{1+a_{n_k}x}-x\neq 0,$ since $a_{n_k}\to(1-c)^{1-\beta}$ and $b_{n_k}\to(1-c)^{\beta}$ as $k\to\infty$, which implies that $\Omega^*_{n,q_{n_k}}(f;x)$ does not converge to $f(x)$, it is a contradiction. Then $\{q_n\}$ must converge to 1.

Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions f on the interval $[0,\infty)$, the norm of each function f denoted by $\|f\|=\sup\limits_{0\le x<\infty}|f(x)|$. We consider Peetre's K-functional: $K_2(f,\delta):=\inf\left\{\|f-g\|+\delta\|g''\|:g\in C_B^2[0,\infty)\right\},\ \delta\ge 0.$

where

$$C_B^2[0,\infty):=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}.$$

Then from the known result in [41], there exists an absolute constant $C_0>0$ such that

$$K_2(f,\delta)\le C_0\,\omega_2\left(f,\sqrt{\delta}\right) \tag{3.4.1}$$

where

$$\omega_2\left(f,\sqrt{\delta}\right):=\sup_{0<h\le\delta}\ \sup_{x\pm h\in[0,\infty)}\left|f\left(x-h-2f(x)+f(x+h)\right)\right|$$

is the 2nd modulus of smoothness of $f\in C_B[0,\infty)$. Also, the usual modulus of continuity is defined as $\omega(f,\delta)=\sup_{0\le h\le\delta}\ \sup_{x\in[0,\infty)}\left|f(x+h)-f(x)\right|$.

The first main result for the local approximation property of $\Omega_{n,q}^{*}(f;x)$ is stated in the following theorem.

Theorem 3.4.3. $\exists\ C>0$ where C is an absolute constant such that

$$\left|\Omega_{n,q_n}^{*}(f;x)-f(x)\right|\le C\omega_2\left(f,\sqrt{\delta_n(x)}\right)+\omega\left(f,\left|\theta\frac{x}{1+a_n x}+\eta_n\right|\right)$$

where f belongs to space $C_B[0,\infty)$, $\theta=\dfrac{2q}{[2]_q}$, $\eta_n=\dfrac{1}{[2]_q b_n}$, $0\le x<\infty$, $0<q<1$, and

$$\delta_n(x)=\frac{2}{b_n}\left\{\left(\frac{1}{[3]_q b_n}+\frac{x\left(1+a_n^2 b_n x^3\right)}{(1+a_n x)^2}\right)+\left(\frac{\left(1-q^n\right)^2}{b_n}\left(\frac{1}{1+q}+\frac{a_n x}{1-q}\right)^2+\frac{1}{b_n}\right)\right\}.$$

Proof. Let

$$\tilde{\Omega}_{n,q}^{*}(f;x)=\Omega_{n,q}^{*}(f;x)+f(x)-f\left(\theta\frac{x}{1+a_n x}+\eta_n\right)$$

where $f\in C_B[0,\infty)$, $\theta=\dfrac{2q}{[2]_q}$, $\eta_n=\dfrac{1}{[2]_q b_n}$. By linearity of $\tilde{\Omega}_{n,q}^{*}$ and Lemma 3.2.2, we obtain

$$\tilde{\Omega}_{n,q}^{*}\left((t-x);x\right)=\Omega_{n,q}^{*}\left((t-x);x\right)+x-\left(\theta\frac{x}{1+a_n x}+\eta_n\right)$$

$$=\Omega_{n,q}^{*}\left(t;x\right)-x\Omega_{n,q}^{*}\left(1;x\right)+x-\left(\theta\frac{x}{1+a_{n}x}+\eta_{n}\right)=0. \qquad (3.4.2)$$

Now by the formula of Taylor, we write

$$h(t)=h(x)+h'(x)(t-x)+\int_{x}^{t}(t-s)h''(s)\,ds, \qquad h\in C_{B}^{2}[0,\infty),$$

applying the operator $\tilde{\Omega}_{n,q}^{*}$ to each side of the last equation we get

$$\tilde{\Omega}_{n,q}^{*}\left(h;x\right)=h\left(x\right)+h'\left(x\right)\tilde{\Omega}_{n,q}^{*}\left(\left(t-x\right);x\right)+\tilde{\Omega}_{n,q}^{*}\left(\int_{x}^{t}\left(t-s\right)h''\left(s\right)ds;x\right)$$

$$=h(x)+\Omega_{n,q}^{*}\left(\int_{x}^{t}(t-s)h''(s)\,ds;x\right)$$

$$-\int_{x}^{\theta\frac{x}{1+a_{n}x}+\eta_{n}}\left(\theta\frac{x}{1+a_{n}x}+\eta_{n}-s\right)h''(s)\,ds.$$

Now by (3.4.2), we have

$$\tilde{\Omega}_{n,q}^{*}\left(h;x\right)=h\left(x\right)+\Omega_{n,q}^{*}\left(\int_{x}^{t}\left(t-s\right)h''\left(s\right);x\right)$$

$$-\int_{x}^{\theta\frac{x}{1+a_{n}x}+\eta_{n}}\left(\theta\frac{x}{1+a_{n}x}+\eta_{n}-s\right)h''\left(s\right)ds.$$

On the other side, since

$$\left|\int_{x}^{t}(t-s)h''(x)\,ds\right|\leq\int_{x}^{t}|t-s||h''(x)|\,ds$$

$$\leq\left\|h''\left(s\right)\right\|\int_{x}^{t}|t-s|\,ds$$

$$\leq\left\|h''\left(s\right)\right\|\left(t-x\right)^{2}.$$

Also

$$\left|\int_{x}^{\theta\frac{x}{1+a_nx}+\eta_n}\left(\theta\frac{x}{1+a_nx}+\eta_n-s\right)h''(s)\,ds\right|$$

$$\leq\left\|h''(s)\right\|\int_{x}^{\theta\frac{x}{1+a_nx}+\eta_n}\left|\theta\frac{x}{1+a_nx}+\eta_n-s\right|ds$$

$$\leq\left\|h''(s)\right\|\left(\theta\frac{x}{1+a_nx}+\eta_n-s\right)^2$$

Hence, we may write

$$\left|\tilde{\Omega}^*_{n,q}(h,x)-h(x)\right|\leq\tilde{\Omega}^*_{n,q}\left(\left|\int_x^t|t-s|\,|h''(s)|\right|;x\right)$$

$$+\left|\int_{x}^{\theta\frac{x}{1+a_nx}+\eta_n}\left|\theta\frac{x}{1+a_nx}+\eta_n-s\right|\left|h''(s)\right|ds\right|$$

$$\leq\|h''\|\Omega^*_{n,q}\left((t-x)^2;x\right)+\|h''\|\left(\theta\frac{x}{1+a_nx}+\eta_n-x\right)^2$$

$$\leq\|h''\|\Omega^*_{n,q}\left((t-x)^2;x\right)+\|h''\|\left(\Omega^*_{n,q}\left((t-x);x\right)\right)^2$$

$$\leq\|h''\|\frac{2}{b_n}\left(\frac{1}{[3]_q b_n}+\frac{x\left(1+a_n^2b_nx^3\right)}{(1+a_nx)^2}\right)$$

$$+\|h''\|\frac{2}{b_n}\left\{\frac{(1-q^n)^2}{b_n}\left(\frac{1}{1+q}+\frac{a_nx}{1-q}\right)^2+\frac{1}{b_n}\right\}\qquad(3.4.3)$$

$$=\|h''\|\,\delta_n(x).\qquad(3.4.4)$$

Using (3.4.4) and the uniform boundedness of $\tilde{\Omega}^*_{n,q}$ we get

$$\left|\Omega_{n,q}^{*}(f;x)-f(x)\right|\le\left|\tilde{\Omega}_{n,q}^{*}(f-h);x\right|+\left|\tilde{\Omega}_{n,q}^{*}(h;x)-h(x)\right|$$

$$+\left|f(x)-h(x)\right|+\left|f\left(\theta\frac{x}{1+a_n x}+\eta_n\right)-f(x)\right|$$

$$\le 4\|f-h\|+\|h''\|\,\delta_n(x)+\omega\left(f,\left|\theta\frac{x}{1+a_n x}+\eta_n-x\right|\right).$$

On the right side, if we take the infimum overall $h\in C_B^2[0,\infty)$, we obtain

$$\left|\Omega_{n,q}^{*}(f;x)-f(x)\right|\le 4K_2(f;\delta_n)+\omega\left(f,\left|\theta\frac{x}{1+a_n x}+\eta_n-x\right|\right)$$

which together with (3.4.1) we obtain

$$\left|\Omega_{n,q}^{*}(f;x)-f(x)\right|\le C\omega_2(f;\delta_n)+\omega\left(f,\left|\theta\frac{x}{1+a_n x}+\eta_n-x\right|\right),$$

and this completes the proof of the theorem.

Now in the below theorem we investigate and present the convergence rate of the operators $\Omega_{n,q}^{*}$ in terms of items of the usual Lipschitz function $Lip_M(\alpha)$.

Theorem 3.4.4. Let $\alpha\in(0,1]$ and $E\subset[0,\infty)$. Then, if $f\in C_B[0,\infty)$ is locally $Lip_M(\alpha)$; i.e the condition

$$\left|f(y)-f(x)\right|\le\zeta\left|y-x\right|^{\alpha},\ y\in E \text{ and } x\in[0,\infty) \qquad (3.4.5)$$

holds, then, for each $x\in[0,\infty)$, we have

$$\left|\Omega_{n,q}^{*}(f,x)-f(x)\right|\le\zeta\left\{\lambda_n^{\frac{\alpha}{2}}+2\left(d(x,E)\right)^{\alpha}\right\},$$

where ζ is a constant depending on α and f; and $d(x,E)$ is the distance between x and E defined as

$$d(x,E)=\inf\left\{|t-x|:t\in E\right\}.$$

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists a point $x_0 \in \overline{E}$ such that $|x-x_0| = d(x,E)$. By the triangle inequality

$$|f(t)-f(x)| \le |f(t)-f(x_0)| + |f(x)-f(x_0)|$$

and by (3.4.5) we get

$$\left|\Omega_{n,q}^*(f,x)-f(x)\right| \le \Omega_{n,q}^*\left(|f(t)-f(x_0)|;x\right)+\Omega_{n,q}^*\left(|f(x)-f(x_0)|;x\right)$$

$$\le \zeta\left\{\Omega_{n,q}^*\left(|t-x_0|^\alpha;x\right)+|x-x_0|^\alpha\right\}$$

$$\le \zeta\left\{\Omega_{n,q}^*\left(|t-x|^\alpha+|x-x_0|^\alpha;x\right)+|x-x_0|^\alpha\right\}$$

$$\le \zeta\left\{\Omega_{n,q}^*\left(|t-x_0|^\alpha;x\right)+2|x-x_0|^\alpha\right\}.$$

Now if we use the Hölder inequality with the values $p=\dfrac{2}{\alpha}$ and $q=\dfrac{2}{(2-\alpha)}$, we get

$$\left|\Omega_{n,q}^*(f,x)-f(x)\right| \le \zeta\left\{\left[\Omega_{n,q}^*\left(|t-x|^{\alpha p}\right);x\right]^{\frac{1}{p}}\left[\Omega_{n,q}^*\left(1^q;x\right)\right]^{\frac{1}{q}}+2\left(d(x,E)\right)^\alpha\right\}$$

$$= L\left\{\left[\Omega_{n,q}^*\left(|t-x|^2\right);x\right]^{\frac{\alpha}{2}}+2\left(d(x,E)\right)^\alpha\right\}$$

$$\le \left\{\left[\frac{2}{b_n}\left(\frac{1}{[3]_q b_n}+\frac{x\left(1+a_n^2 b_n x^3\right)}{(1+a_n x)^2}\right)\right]^{\frac{\alpha}{2}}+2\left(d(x,E)\right)^\alpha\right\}$$

$$= L\left\{\lambda_n(x)^{\frac{\alpha}{2}}+2\left(d(x,E)\right)^\alpha\right\},$$

and the proof is completed.

3.5 Voronovskaja Type Result

In the following theorem, we give a Voronovskaja type result for the new Kantorovich type q-analogue of the B-S operators.

Theorem3.5.1. Assume that $q_n \to 1$ and $q_n^n \to \mu$ as $n \to \infty$ and let $0 < \beta < \frac{1}{2}$. For any $f \in C_B^2[0,\infty)$ the following equality holds

$$\lim_{n\to\infty} b_{n,q_n}\left(\Omega_{n,q}^*(f,x) - f(x)\right) = \frac{1}{2} f'(x) + \frac{1}{2} x f''(x)$$

uniformly on $[0,a]$.

Proof. Suppose the function f belongs to the space $C_B^2[0,\infty)$ and $0 \le x < \infty$ is fixed. By using the formula of Taylor we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \rho(t,x)(t-x)^2, \tag{3.5.1}$$

where the function $\rho(t,x)$ is the remainder in the Peano form, $\rho(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x}\rho(t,x) = 0$. Applying Ω_{n,q_n}^* to (3.5.1) we obtain

$$\Omega_{n,q}^*(f;x) - f(x) = f'(x)\Omega_{n,q}^*\left((t-x);x\right) + \frac{1}{2} f''(x)\Omega_{n,q}^*\left((t-x)^2;x\right)$$

$$+\Omega_{n,q}^*\left(\rho(t,x)D_2;x\right).$$

Now multiplying the left side and the right side of the above equation by b_{n,q_n} we get

$$b_{n,q_n}\left(\Omega_{n,q_n}^*(f;x) - f(x)\right)$$

$$= f'(x) b_{n,q_n}\Omega_{n,q_n}^*\left((t-x);x\right) + \frac{1}{2} f''(x) b_{n,q_n}\Omega_{n,q_n}^*\left((t-x)^2;x\right)$$

$$+ b_{n,q_n}\Omega_{n,q_n}^*\left(\rho(t,x)(t-x)^2;x\right).$$

Then by using C.S.I we can write the following inequality

$$\Omega_{n,q_n}^*\left(\rho(t,x)(t-x)^2,x\right) \le \sqrt{\Omega_{n,q_n}^*\left(\rho^2(t,x),x\right)} \times \sqrt{\Omega_{n,q_n}^*\left((t-x)^4,x\right)}. \tag{3.5.2}$$

We observe that $\rho^2(x,x) = 0$ and $\rho^2(.,x) \in C_B[0,\infty)$. Now from theorem (3.4.2) it follows that

$$\lim_{n\to\infty} \Omega^{*}_{n,q_n}\left(\rho^2(t,x),x\right)=\rho^2(x,x)=0 \tag{3.5.3}$$

uniformly w.r.t $x\in[0,a]$. Finally, from the inequality (3.5.2), from (3.5.3) and from Lemma 3.3.1, we immediately obtain

$$\lim_{n\to\infty} b_{n,q_n}\ \Omega^{*}_{n,q_n}\left(\rho(t,x)(t-x)^2,x\right)$$

$$\leq \lim_{n\to\infty}\sqrt{\Omega^{*}_{n,q_n}\left(\rho^2(t,x),x\right)}\times\lim_{n\to\infty} b_{n,q_n}\sqrt{\Omega^{*}_{n,q_n}\left((t-x)^4,x\right)}$$

$$\leq 0\times\lim_{n\to\infty} b_{n,q_n}\times\frac{1}{b^2_{n,q_n}}C(a,q)=0.$$

which completes the proof.

Chapter 4

CONCLUSION AND FUTURE WORK

In this thesis, we constructed a new Kantorovich type q-analogue of the Balázs-Szabados operators by using the concepts of the q-integers. These newly defined operators have some advantages when they are compared with the other q-analogues given in the other studies. First advantage is that they are positive for all continuous and real valued functions on the half open interval $[0,\infty)$. Second advantage is that they can be used to approximate also the integrable functions. If we choose the special case $q=1$, the operators coincide with the q-BSK operators which are defined in [24] by E. Özkan. For these new operators we gave a recurrence relation and then by using this recurrence relation we established the moments up to the fourth order and we estimated also the central moments. We studied a Korovkin type theorem, we investigated the local approximation properties of these operators in terms of modulus of continuity and we proved a Voronovskaja type theorem.

The prospective methodology will be in the next research points:

1- New Kantorovich type (p,q)- Balázs-Szabados operators will be presented by using the concepts of (p,q)- calculus for $q\in(0,1)$, $p\in(q,1)$ and they will be denoted by $\Omega^{*}_{n,p,q}(f;x)$. These operators $\Omega^{*}_{n,p,q}(f;x)$ will be examined in several cases. For example in the case $p=1$, the operators $\Omega^{*}_{n,p,q}(f;x)$ turn out to be those operators $\Omega^*_{n,q}(f;x)$ which is defined in [47]. In another case where $p=1,\ q=1$ the operators coincide with the ones defined in [24].

2- A recurrence relation formula for $\Omega^*_{n,p,q}(t^m;x),\ m\in\mathbb{N}\bigcup\{0\}$ will be derived and by using this formula moments of the operators up to the 4th order will be calculated. Besides this 2nd order and 4th order central moments will be calculated and estimated.

3- Convergence properties of the operators $\Omega^*_{n,p,q}(f;x)$ will be examined. In terms of the usual modulus of continuous functions, local approximation properties will be investigated. Korovkin type approximation theorem for the operators $\Omega^*_{n,p,q}(f;x)$ will be given and a Voronoskaja type theorem will be proved. Also, a theorem of approximation error in terms of the weighted modulus of continuity will be given.

4- Finally, an example for these new Kantorovich type (p,q)- analogue of the Balázs-Szabados-Operators $\Omega^*_{n,p,q}(f;x)$ of certain functions and for different values of q will be presented.

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