q-Multiple Appell Polynomials

Merve Çil

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Approval of the Institute of Graduate Studies and Research

Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazım Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Mehmet Ali Özarslan
Supervisor

Examining Committee

1. Prof. Dr. Mehmet Ali Özarslan
2. Assoc. Prof. Dr. Suzan Cival Buranay
3. Asst. Prof. Dr. Ceren Ustaoğlu

ABSTRACT

In 1880, Paul Emile Appell introduced a certain kind of sequence which is named Appell polynomials in the literature. Besides the trivial examples, the most famous Appell polynomials are the Hermite, Bernoulli, and Euler polynomials. An interesting generalization of Appell polynomials, namely q-Appell polynomials were introduced by Walled A. Al-Salam in 1967. The multiple Appell polynomials have recently introduced and investigated in 2011 by D.W.Lee. Also, 2 iterated Appell polynomials defined by Subuhi Khan and Nusrat Raza in 2013.

The main purpose of this thesis is to define and investigate univariate q-multiple Appell polynomials, bivariate q-multiple Appell polynomials and 2 iterated q-multiple Appell polynomials.

This thesis consist of 5 chapters.

In Chapter 1, we recalled the main definitions and properties of the Appell polynomials, the 2 iterated Appell polynomials, the multiple Appell polynomials, the q-Calculus, and the q-Appell polynomials.

Chapters 2,3,4 and 5 are original.

In chapter 2 we define univariate q-multiple Appell polynomials and obtain equivalence theorem and recurrence relations for them.

In chapter 3, we introduce bivariate q-multiple Appellpolynomials via the concept of univariate q-multiple Appell polynomials and obtain explicit representation, equivalence theorem, and recurrence relations for them.

In chapter 4, we provide some examples for the polynomials that we define in chapters 2 and 3 such as q-multiple power polynomials, bivariate q-multiple Bernoulli polynomials, bivariate q-multiple Euler polynomials, bivariate q-multiple Bernoulli-Euler polynomials, and q-multiple Hermite polynomials.

In the last chapter, we define 2-iterated q-multiple Appell polynomials and we show how we can obtain q-analogue of multiple Hermite polynomials from this definition. We further obtain reccurencen relation for 2-iterated q-mutliple Appell polynomials.

Keywords: q-Calculus or Quantum Calculus, Appell Polynomials, q-Appell Polynomials, Two Iterated Multiple Appell Polynomials.

ÖZ

1880’de Paul Emile Appell, literatürde Appell polinomları olarak adlandırılan polinom dizisini tanımlamıştır. Aşikar örneklerin yanı sıra, en bilindik Appell polinomları Hermite, Bernoulli ve Euler polinomlarıdır. Appell polinomlarının ilginç bir genelleşmesi olan q-Appell polinomları 1967’de Walled A. Al-Salam tarafından tanımlanmıştır. Katlı Appell polinomları yakın zamanda 2011’de D.W. Lee tarafından tanımlanmış ve araştırılmıştır. Ayrıca Subuhi Kahn ve Nusrat Raza tarafından 2013 yılında 2 iterasyonlu Appell polinomu tanımlanmıştır.

Bu tezin temel amacı tek değişkenli katlı q-Appell polinomlarını, iki değişkenli katlı q-Appell polinomlarını ve 2 iterasyonlu katlı q-Appell polinomlarını tanımlamak ve araştırmaktır. Bu tez, 5 bölümden oluşmaktadır.

Bölüm 1’de Appell polinomlarının, 2 iterasyonlu Appell polinomlarının, katlı Appell polinomlarının, q-Kalkülüsün ve q-Appell polinomlarının tanımları verilmiş ve özellikleri özetlenmiştir.

2,3,4 ve 5. bölümler orjinaldir.

Bölüm 2’de tek değişkenli katlı q-Appell polinomları tanımlanmış ve bu tanımın denklik teoremi elde edilmiştir. Ayrıyeten bu tanımın rekürans bağıntıları da verilmiştir.

Bölüm 3’te, tek değişkenli katlı q-Appell polinomları kavramı aracılığıyla, iki değişkenli katlı q-Appell polinomları tanımlanmıştır. Bu tanım için denklik teoremi

ispatlanmış ve rekürans bağıntıları elde edilmiştir.

Bölüm 4'te 2. Ve 3. Bölümlerde tanımladığımız polinomlara örnek olarak katlı q-qüç polinomlarını, katlı q-Bernoullie polinomlarını, katlı q-Euler polinomlarını, katlı q-Bernoullie-Euler polinomlarını ve katlı q-Hermite polinomlarını sağlıyoruz.

Son bölümde 2 iterasyonlu katlı q-Appell polinomları tanımlanmıştır. Bu tanım yardımıyla katlı q-Hermite polinomlarının q-analoğu elde edilmiştir. Ayrıca 2 iterasyonlu katlı a-Appell polinomları için rekürans bağıntıları elde edilmiştir.

Anahtar Kelimeler: q-Kalkülüs veya Kuantum Kalkülüs, Appell Polinomları, q-Appell Polinomları, İki kez yinelenen çoklu Appell Polinomları.

To My Family

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TABLE OF CONTENTS

LIST OF SYMBOLS

$\mathscr{A}^{[2]}_{q,n_1,n_2}(x)$	2-Iterated q-Multiple Appell Polynomials
$\mathcal{B}^{[2]}_n(x)$	2- Iterated Bernoulli Polynomials
$\mathcal{B}_n(x)$	Bernoulli Polynomials
$\mathfrak{B}^{(\alpha)}_{n,q}(x,y)$	q-Bernoulli Polynomials
$\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)$	Bivariate q-Multiple Bernoulli Polynomials
$\mathcal{E}^{[2]}_n(x)$	2-Iterated Euler Polynomials
$\mathcal{E}_n(x)$	Euler Polynomials
$\mathfrak{E}^{(\alpha)}_{n,q}(x,y)$	q-Euler Polynomials
$\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)$	Bivariate q-Multiple Euler Polynomials
$\mathcal{G}_n(x)$	Genocchi Polynomials
$\mathfrak{G}_{n,q}(x)$	q-Genocchi Polynomials
$\mathcal{H}_n(x)$	Hermite Polynomials
$\mathfrak{H}_{n,q}(x)$	q-Hermite Polynomials
$\mathbb{H}_{n_1,n_2}(x)$	q-Multiple Hermite Polynomials
$\mathcal{H}^{\alpha_1,\alpha_2}_{n_1,n_2}(x)$	Multiple Hermite Polynomials
$\mathscr{H}^{\alpha_1,\alpha_2,\delta}_{q,n_1,n_2}(x)$	2-Iterated q-Multiple Generalized Hermite Polynomials
$\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)$	Bivariate q-Multiple Bernoulli-Euler Polynomials
$\mathbb{M}_{n_1,n_2}(x)$	q-Multiple Power Polynomials for $(x+1)^{n_1+n_2}$
$\mathbb{N}_{n_1,n_2}(x)$	q-Multiple Power Polynomials for $(x-1)^{n_1+n_2}$
$\mathcal{P}_{n_1,n_2}(x)$	Univariate Multiple Appell Polynomials
$\mathcal{R}_n(x)$	Appell Polynomials
$\mathcal{R}^{[2]}_n(x)$	2-Iterated Appell Polynomials
$\mathcal{S}_{n_1,n_2}(x,y)$	Bivariate Multiple Appell Polynomials
$\mathbb{P}_{n_1,n_2}(x)$	Univariate q-Multiple Appell Polynomials

$\mathbb{S}_{n_1,n_2}(x,y)$	Bivariate q-Multiple Appell Polynomials
$T_{n_1,n_2}(x)$	q-Appell Polynomials
${}_{\mathcal{B}}\mathcal{E}_n(x)$	Euler-Bernoulli Polynomials
${}_{\mathcal{E}}\mathcal{B}_n(x)$	Bernoulli-Euler Polynomials

Chapter 1

INTRODUCTION

The main purpose of this chapter is to familiarize the general reader with the expressions and notations that will appear very often in the following chapters. One of the basic but important parts of this chapter is dedicated to the implementation of notations and specific formulas related to Appell polynomials. The second section is devoted to 2-iterated Appell polynomials and the third section is devoted to multiple Appell polynomials, which are also related to Appel polynomials. Section 4 holds for q-Calculus-related notations and miscellaneous q-formulas. Finally, we finish the first chapter by giving the definition and properties of q-Appell.

1.1 Appell Polynomials

The Appell sequence class is a particular category of polynomial sequences that appears in numerous applications in pure and applied mathematics. Especially in recent years appell polynomials have special interest of many researcher since they have many application in diverse areas of physics and engineering [6, 7, 9, 14, 16]. In 1880, Appell [4] introduced a set of the nth degree polynomials $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ that satisfies the following differential relationship

$$\frac{d}{dx}\mathcal{R}_n(x) = n\mathcal{R}_{n-1}, \quad n \geq 1. \tag{1.1}$$

An equivalent definition of (1.1) is given in the following theorem.

Theorem 1.1: Let $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials. Then the followings are all equivalent [17]:

a. $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ is a sequence of Appell polynomials.

b. $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ can be defined by means of following generating function

$$A(t)e^{xt} = \sum_{n=0}^{\infty} \mathcal{R}_n(x)\frac{t^n}{n!} \tag{1.2}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t_n}{n!}, \quad (a_0 \neq 0). \tag{1.3}$$

c. $\mathcal{R}_n(x)$ can be written explicitly as

$$\mathcal{R}_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}x^k \tag{1.4}$$

where a_n $(n = 0,1,2,...)$ is in (1.3).

d. $\mathcal{R}_n(x)$ can be written in the following form

$$\mathcal{R}_n(x) = \left(\sum_{k=0}^{\infty} \frac{a_n}{k!}D^k\right) x^k \tag{1.5}$$

where $D = \frac{d}{dx}$.

e. $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ satisfies

$$\mathcal{R}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{R}_{n-k}y^k, \quad n = 0,1,2... \tag{1.6}$$

The Appell sequence class includes a lot of well known families, such as Bernoulli polynomials, Euler polynomials, Hermite polynomials and Laguerre polynomials, and so on. We will give some of these polynomials that we will use throughout the next chapters.

Example 1.1 (Bernoulli Polynomials): As a particular case, choosing $A(t)$ as $\frac{t}{e^t-1}$ in (1.2) gives us Bernoulli polynomials $\mathcal{B}_n(x)$ as

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1.7}$$

The explicit formula of $\mathcal{B}_n(x)$ is

$$\mathcal{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{n-k} x^k \tag{1.8}$$

where $\mathcal{B}_k$ are the Bernoulli numbers, given by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \mathcal{B}_k \frac{t^k}{k!}. \tag{1.9}$$

Example 1.2 (Euler Polynomials): Taking $A(t)$ as $\frac{2}{e^t+1}$ in (1.2), we can get the Euler polynomials $\mathcal{E}_n(x)$ as

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1.10}$$

The explicit formula of $\mathcal{E}_n(x)$ is

$$\mathcal{E}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{E}_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k} \tag{1.11}$$

where $\mathcal{E}_k$ are the Euler numbers, given by

$$\frac{2}{e^t + 1} = \sum_{k=0}^{\infty} \mathcal{E}_k \frac{t^k}{k!}. \tag{1.12}$$

Example 1.3 (Genocchi Polynomials): Taking $A(t)$ as $\frac{2t}{e^t+1}$ in (1.2), we can get the Genocchi polynomials $\mathcal{G}_n(x)$ as

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n(x) \frac{t^n}{n!}, \quad |t| < \pi \tag{1.13}$$

where the Genocchi numbers $\mathcal{G}_k$, given by

$$\frac{2t}{e^t + 1} = \sum_{k=0}^{\infty} \mathcal{G}_k \frac{t^k}{k!}. \tag{1.14}$$

Example 1.4 (Hermite Polynomials): Hermite polynomials $\mathcal{H}_n(x)$ have following generating function

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} \mathcal{H}_n(x) \frac{t^n}{n!}. \tag{1.15}$$

The explicit formula of $\mathcal{H}_n(x)$ is

$$\mathcal{H}_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)!}{m!(n-2m)!} (2x)^{n-2m}. \tag{1.16}$$

1.2 2-Iterated Appell Polynomials

In this section we will give basic definition and some properties for 2-iterated Appell polynomials from [13]. Let, $\mathcal{R}_n^{(1)}(x)$ and $\mathcal{R}_n^{(2)}(x)$ denote two distinguish Appell polynomials, which are defined by

$$A_1(t)e^{xt} = \sum_{n=0}^{\infty} \mathcal{R}_n^{(1)}(x) \frac{t^n}{n!} \tag{1.17}$$

and

$$A_2(t)e^{xt} = \sum_{n=0}^{\infty} \mathcal{R}_n^{(2)}(x) \frac{t^n}{n!} \tag{1.18}$$

respectively.

Definition 1.1 (2-iterated Appell Polynomials): 2-iterated Appell polynomials denoted by $\mathcal{R}_n^{[2]}(x)$ are defined via the following generating function:

$$A_1(t)A_2(t)e^{xt} = \sum_{n=0}^{\infty} \mathcal{R}_n^{[2]}(x) \frac{t^n}{n!}. \tag{1.19}$$

We list some examples to 2-iterated Appell polynomials below:

Example 1.5 (2-iterated Bernoulli Polynomials): The generating function for 2-iterated Bernoulli polynomials with notation $\mathcal{B}_n^{[2]}(x)$ is

$$\left(\frac{t}{e^t - 1}\right)^2 e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{[2]}(x) \frac{t^n}{n!} \tag{1.20}$$

where $\mathcal{B}_n^{[2]}(x)$ has series expression as

$$\mathcal{B}_n^{[2]}(x) = \sum_{r=0}^{n} \binom{n}{r} \mathcal{B}_r \mathcal{B}_{n-r}(x). \tag{1.21}$$

Example 1.6 (2-iterated Euler Polynomials): The generating function for 2-iterated

Euler polynomials with notation $\mathcal{E}_n^{[2]}(x)$ is

$$\left(\frac{2}{e^t+1}\right)^2 e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{[2]}(x)\frac{t^n}{n!} \tag{1.22}$$

where $\mathcal{E}_n^{[2]}(x)$ has series expression as

$$\mathcal{E}_n^{[2]}(x) = \sum_{r=0}^{n} \binom{n}{r} \mathcal{E}_r \mathcal{E}_{n-r}(x). \tag{1.23}$$

Example 1.7: The generating function for 2-iterated Bernoulli-Euler polynomials with notation ${}_{\mathcal{E}}\mathcal{B}_n(x)$ or 2-iterated Euler-Bernoulli polynomials with notation ${}_{\mathcal{B}}\mathcal{E}_n(x)$ are given by

$$\frac{2t}{e^{2t}-1}e^{xt} = \sum_{n=0}^{\infty} {}_{\mathcal{E}}\mathcal{B}_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{\mathcal{B}}\mathcal{E}_n(x)\frac{t^n}{n!} \tag{1.24}$$

where ${}_{\mathcal{E}}\mathcal{B}_n(x)$ and ${}_{\mathcal{B}}\mathcal{E}_n(x)$ have series expressions as

$${}_{\mathcal{B}}\mathcal{E}_n(x) = {}_{\mathcal{E}}\mathcal{B}_n(x) = \sum_{r=0}^{n} \binom{n}{r} \mathcal{E}_r \mathcal{B}_{n-r}(x) = \sum_{r=0}^{n} \binom{n}{r} \mathcal{B}_r \mathcal{E}_{n-r}(x). \tag{1.25}$$

1.3 Multiple Appell Polynomials

In 2011, Lee defined multiple Appell polynomials, prove an equivalence theorem for them and showed that the only orthogonal multiple Appell polynomials are the multiple Hermite polynomials. The multiple Appell polynomials are defined in the following definition.

Definition 1.2: If a double indexed polynomial sequence $\mathcal{P}_{n_1,n_2}(x)$ is defined by the generating relation

$$A(t_1,t_2)e^{x(t_1+t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathcal{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!} \tag{1.26}$$

then the multiple polynomial system $\{\mathcal{P}_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ is called multiple Appell polynomial where $A(t_1,t_2)$ has a series expansion

$$A(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}, \tag{1.27}$$

where $a_{0,0} \neq 0$

Theorem 1.2: [17] Let $\{\mathcal{P}_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ be a multiple polynomials system. Then the followings are all equivalent:

a. $\{\mathcal{P}_{n_1,n_2}(x)\}_{n=0}^{\infty}$ is a sequence of multiple Appell polynomials.

b. The polynomial sequence $P_{n_1,n_2}(x)$ can be represented explicitly as

$$\mathcal{P}_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}a_{n_1-k_1,n_2-k_2}x^{k_1+k_2} \tag{1.28}$$

where the double sequence $\{a_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ given in (1.27) and $a_{0,0} \neq 0$.

c. For every $n_1+n_2 \geq 1$, we have

$$\mathcal{P}'_{n_1,n_2}(x) = n_1\mathcal{P}_{n_1-1,n_2}(x) + n_2\mathcal{P}_{n_1,n_2-1}(x). \tag{1.29}$$

d. There exist a sequence $\{a_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ with $a_{0,0} \neq 0$ such that

$$\mathcal{P}_{n_1,n_2}(x) = \left\{\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\frac{(n_1+n_2-k_1-k_2)!}{(n_1+n_2)!}a_{k_1,k_2}D^{k_1+k_2}\right\}x^{n_1+n_2}. \tag{1.30}$$

e. $\mathcal{P}_{n_1,n_2}(x)$ satisfies the following relation

$$\mathcal{P}_{n_1,n_2}(x+y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\mathcal{P}_{n_1-k_1,n_2-k_2}(x)y^{k_1+k_2}. \tag{1.31}$$

for $n_1,n_2 = 0,1,2,\ldots$.

Example 1.8: Multiple Hermite polynomial $\{\mathcal{H}_{n_1,n_2}^{\alpha_1,\alpha_2}(x)\}_{n_1,n_2=0}^{\infty}$ has the generating function as follows

$$e^{\frac{\delta}{2}(t_1+t_2)^2+\delta(t_1+t_2)x+\alpha_1t_1+\alpha_2t_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathcal{H}_{n_1,n_2}^{\alpha_1,\alpha_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!} \tag{1.32}$$

where $\delta < 0$.

Remark 1.1: We can get generating function of the classical Hermite polynomials by taking $t_2 = 0$ in Equation (1.32).

1.4 q-Calculus

The study of q-calculus was introduced in the 1920s. Since the 1980s, the field of q-calculus has become a linkage between engineering science and mathematics. The q-standard notations and definitions are taken from [19] and [11]. Throughout this thesis, we will use basic definitions from q-calculus as follows.

Definition 1.3: For any non-negative n, the q-integer $[n]_q$ is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \quad q \neq 1, \quad n \in \mathbb{C}. \tag{1.33}$$

Definition 1.4: q-factorial $[n]_q!$ is given as

$$[n]_q! := \begin{cases} [n]_q[n-1]_q...[1]_q, & if n \in \mathbb{N} \\ 1 & if n = 0 \end{cases}$$

also,

$$[kn]_q!! = [kn]_q[kn-k]_q...[k]_q, \quad k \in \mathbb{Z}. \tag{1.34}$$

Definition 1.5: The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}(1-q^j a), \quad n \in \mathbb{N},$$
$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-q^j a), \quad |q| < 1, a \in \mathbb{C}. \tag{1.35}$$

Definition 1.6: The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad k = 0, 1, 2, ..., n, \quad (k \leqslant n, n \in \mathbb{N}). \tag{1.36}$$

Proposition 1.1: For the q-binomial coefficient, the following facts holds

i. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$,

ii. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$,

iii. $\begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} l \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ n-l \end{bmatrix}_q$.

Definition 1.7: The q-analogue of $(x+y)^n$ is specified as

$$(x+y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k = \prod_{k=0}^{n-1} (x+q^k y). \tag{1.37}$$

Definition 1.8: The q-binomial formula is known as

$$(1-a)_q^n = (a;q)_n = \prod_{j=0}^{n-1} (1-q^j a) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k. \tag{1.38}$$

Definition 1.9: The q-derivative D_q is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{(1-q)z}, \quad 0 < |q| < 1, \quad 0 \neq z \in \mathbb{C}. \tag{1.39}$$

Definition 1.10: There are two exponential functions used in q-calculus first one is given as

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)} \tag{1.40}$$

where $0 < |q| < 1$ and $|z| < |1-q|^{-1}$. Second one is given as

$$E_q(z) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1+(1-q)q^k z) \tag{1.41}$$

where $0 < |q| < 1$ and $z \in \mathbb{C}$. By simple calculation we can obtain that $e_q(z)E_q(-z) = 1$ and $E_q(z)e_q(-z) = 1$.

Remark 1.2: From the Definition 1.9 q-derivatives for exponential functions are

$$D_{q,z} e_q(z) = e_q(z) \tag{1.42}$$

and

$$D_{q,z} E_q(z) = E_q(qz). \tag{1.43}$$

Proposition 1.2: Consider any two arbitrary function $f(z)$ and $g(z)$. The followings hold:

a. If f is differentiable then $\lim_{q\to 1} D_q f(z) = \frac{df(z)}{dz}$ where $\frac{d}{dz}$ represents standard derivative known in calculus.

b. D_q is linear so for any two arbitrary constant numbers a and b we have

$$D_{q,z}(af(z)+bg(z)) = aD_{q,z}f(z)+bD_{q,z}g(z).$$

c. The q-derivative of the product is given by

$$D_{q,z}(fg)(z) = f(qz)D_{q,z}g(z)+g(z)D_{q,z}f(z).$$

d. q-quotient rule is given as

$$D_{q,z}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_{q,z}f(z)-f(z)D_{q,z}g(z)}{g(z)g(qz)}$$

where $g(z)g(qz)\neq 0$.

1.5 q-Appell Polynomials

This section provides a definition of q-Appell polynomials that have already been characterized by Al-Salam [2]. Their properties are also stated. We also mention some examples of q-Appell polynomials, which are q-Bernoulli polynomials and q-Euler polynomials, that we will use in the following chapters [19, 20].

Theorem 1.3: The q-Appell polynomials $\{T_n(x)\}_{n=0}^{\infty}$ are defined by the following generating function

$$A_q(t)e_q(tx) = \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{[n]_q!}, \quad 0<q<1 \tag{1.44}$$

where

$$A_q(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!} \quad a_0 \neq 0, \quad A_q(0)\neq 0. \tag{1.45}$$

Remark 1.3: We note that when $q \to 1$ in (1.44), q-Appell polynomials (1.44) becomes Appell polynomials (1.2). So we can think of a set of q-Appell polynomials as a generalization of Appell polynomials.

Definition 1.11 (Star Operator): For any two polynomial set $\mathscr{K}$ and $\mathscr{L}$ whose n^{th} components are, respectively

$$\mathscr{K}_n(x) = \sum_{k=0}^{n} a(n,k)x^k \tag{1.46}$$

and

$$\mathscr{L}_n(x) = \sum_{k=0}^{n} b(n,k)x^k \tag{1.47}$$

then $\mathscr{K} * \mathscr{L}$ is the polynomial set whose n^{th} component is

$$(\mathscr{K} * \mathscr{L})_n = \sum_{k=0}^{n} a(n,k)\mathscr{L}_k(x). \tag{1.48}$$

Theorem 1.4: Let $\mathscr{K}, \mathscr{L}, \mathscr{M} \in \mathbb{A}$ with determining functions $K(t), L(t)$ and $M(t)$ respectively, where $\mathbb{A}$ denoted the class of all q-Appell polynomials. Then

i. $\mathscr{K} + \mathscr{L} \in \mathbb{A}$ if $K(0) + L(0) \neq 0$,

ii. $K(t) + L(t)$ is the determining function of $\mathscr{K} + \mathscr{L}$,

iii. $\mathscr{K} + (\mathscr{L} + \mathscr{M}) = (\mathscr{K} + \mathscr{L}) + \mathscr{M}$.

Theorem 1.5: Let $\mathscr{K}, \mathscr{L}, \mathscr{M} \in \mathbb{A}$ with determining functions $K(t), L(t)$ and $M(t)$ respectively. Then

i. $\mathscr{K} * \mathscr{L} \in \mathbb{A}$,

ii. $\mathscr{K} * \mathscr{L} = \mathscr{L} * \mathscr{K}$,

iii. $K(t)L(t)$ is the determining of $\mathscr{K} * \mathscr{L}$,

iv. $\mathscr{K} * (\mathscr{L} * \mathscr{M}) = (\mathscr{K} * \mathscr{L}) * \mathscr{M}$.

Corollary 1.1: Let $\mathscr{K} \in \mathbb{A}$ and determining function of $\mathscr{K}$ is $A(t)$. Then there is a set $\mathscr{L} \in \mathbb{A}$ such that

$$\mathscr{K} * \mathscr{L} = \mathscr{L} * \mathscr{K} = I$$

where determining function of $\mathscr{L}$ is $A^{-1}(t)$ and $I = x^n$.

Corollary 1.2: For any $\mathscr{K} \in \mathbb{A}$ we can denote the $\mathscr{L} \in \mathbb{A}$ from Corollary 1.1 as $\mathscr{K}^{-1}$ since star operation of $\mathscr{K}$ and $\mathscr{L}$ gives identity element.

Corollary 1.3: From Theorem 1.5, Corollary 1.1 and Corollary 1.2 we can call that the system $(\mathbb{A}, *)$ is a commutative group.

Example 1.9 (q-Bernoulli Polynomials): As a particular case, choosing $A_q(t)$ as $\frac{t}{e_q(t)-1}$ in (1.44) gives us univariate q-Bernoulli polynomials $\mathfrak{B}_{n,q}(x)$ as

$$\frac{t}{e_q(t)-1} e_q(tx) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi \tag{1.49}$$

where q-Bernoulli numbers $\mathfrak{b}_{n,q}$ are given by

$$\frac{t}{e_q(t)-1} = \sum_{n=0}^{\infty} \mathfrak{b}_{n,q} \frac{t^n}{[n]_q!}. \tag{1.50}$$

Example 1.10 (Bivariate q-Bernoulli Polynomials): q-Bernoullie numbers $\mathfrak{B}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ are defined by the following generating functions [19]

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!}, \tag{1.51}$$

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx) E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} \tag{1.52}$$

where $q, \alpha \in \mathbb{C}$, $0 < |q| < 1$ and $|t| < 2\pi$.

Example 1.11 (q-Euler Polynomials): Choosing $A_q(t)$ as $\frac{2}{e_q(t)+1}$ in (1.44) gives as q-Euler polynomials $\mathfrak{E}_{n,q}(x)$ as

$$\frac{2}{e_q(t)+1}e_q(tx) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(x)\frac{t^n}{[n]_q!}, \quad |t| < \pi. \tag{1.53}$$

q-Euler numbers $\mathfrak{e}_{n,q}$ are given by

$$\frac{te_q(t)}{e_q(2t)+1} = \sum_{n=0}^{\infty} \mathfrak{e}_{n,q}\frac{t^n}{[n]_q!}. \tag{1.54}$$

The association to the Euler numbers is given by

$$\mathfrak{e}_{n,q} = 2^n \mathfrak{E}_{n,q}\left(\frac{1}{2}\right). \tag{1.55}$$

Example 1.12 (Bivariate q-Euler Polynomials): Generating functions of q-Euler numbers $\mathfrak{E}^{(\alpha)}_{n,q}$ and q-Euler polynomials $\mathfrak{E}^{(\alpha)}_{n,q}(x,y)$ are defined by [19]

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathfrak{E}^{(\alpha)}_{n,q}\frac{t^n}{[n]_q!}, \tag{1.56}$$

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}^{(\alpha)}_{n,q}(x,y)\frac{t^n}{[n]_q!} \tag{1.57}$$

where $q, \alpha \in \mathbb{C}$, $0 < |q| < 1$ and $|t| < \pi$.

Example 1.13 (Univariate q-Genocchi Polynomials): Univariate q-Genocchi polynomials $\mathfrak{G}_{n,q}(x)$ defined by the means of the generating function

$$A_q(t)e_q(tx) = \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(x)\frac{t^n}{[n]_q!} \tag{1.58}$$

where q-Genocchi numbers $\mathfrak{g}_{n,q}$ defined by the generating function

$$A_q(t) := \frac{2t}{e_q(t)+1} = \sum_{n=0}^{\infty} \mathfrak{g}_{n,q}\frac{t^n}{[n]_q!}. \tag{1.59}$$

Example 1.14 (Univariate q-Hermite Polynomials): Univariate q-Hermite polynomials $\mathfrak{H}_{n,q}(x)$ defined by the means of the generating function

$$A_q(t)e_q(tx) = \sum_{n=0}^{\infty} \mathfrak{H}_{n,q}(x)\frac{t^n}{[n]_q!} \tag{1.60}$$

where

$$A_q(t) := \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)} \frac{t^{2n}}{[2n]_q!!} \tag{1.61}$$

and $\mathfrak{H}_{n,q}(x)$ has following explicit form

$$\mathfrak{H}_{n,q}(x) = \sum_{n=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{k(k-1)[n]_q!}}{[2k]_q!![n-2k]_q!} x^{n-2k}. \tag{1.62}$$

Chapter 2

UNIVARIATE q-MULTIPLE APPELL POLYNOMIALS

In this chapter, we introduce univariate q-multiple Appell polynomials. We give the generating functions and features we have obtained for them.

Definition 2.1: A multiple polynomial system $\{\mathbb{P}_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ is named as univariate q-multiple Appell, if they defined via the generating function of the form

$$A_q(t_1,t_2)e_q(t_1x)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{2.1}$$

where A_q has the series expansion

$$A_q(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{2.2}$$

with $A_q(0,0) = a_{0,0} \neq 0$.

Remark 2.1: Taking $t_2 = 0$ in equation (2.1), univariate q-multiple Appell polynomials reduce to q-Appell polynomials.

Theorem 2.1 (Equivalence Theorem): The following statements are all equivalent to one another:

a. $\{\mathbb{P}_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ is a set of univariate q-multiple Appell polynomials.

b. The univariate q-multiple Appell polynomial $\{\mathbb{P}_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ possesses an explicit form given by

$$\mathbb{P}_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_2(k_2-1)}{2}} a_{n_1-k_1,n_2-k_2}x^{k_1+k_2}. \tag{2.3}$$

c. For every $n_1+n_2 \geq 1$, we have

$$D_{q,x}\mathbb{P}_{n_1,n_2}(x) = [n_2]_q\mathbb{P}_{n_1,n_2-1}(qx) + [n_1]_q\mathbb{P}_{n_1-1,n_2}(x). \tag{2.4}$$

d. There exists a sequence $\{a_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ with $a_{0,0} \neq 0$ such that

$$\mathbb{P}_{n_1,n_2}(x) = \left\{ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_2(k_2-1)}{2}} \frac{[n_1+n_2-k_1-k_2]!}{[n_1+n_2]!} a_{k_1,k_2} D_q^{k_1+k_2} \right\} x^{n_1+n_2}. \tag{2.5}$$

Proof. (a) $\Leftrightarrow$ (b) Since $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n b_k = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}b_k$, we have

$$\begin{aligned}
& A_q(t_1,t_2)e_q(t_1x)E_q(t_2x) \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2} \frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \sum_{k_1=0}^{\infty} \frac{x^{k_1}}{[k_1]_q!} t_1^{k_1} \sum_{k_2=0}^{\infty} q^{\frac{k_2(k_2-1)}{2}} \frac{x^{k_2}}{[k_2]_q!} t_2^{k_2} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} q^{\frac{k_2(k_2-1)}{2}} a_{n_1,n_2} x^{k_1+k_2} \frac{t_1^{n_1+k_1}t_2^{n_2+k_2}}{[n_1]_q![n_2]_q![k_1]_q![k_2]_q!} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \left(\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_2(k_2-1)}{2}} a_{n_1-k_1,n_2-k_2} x^{k_1+k_2} \right) \frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}.
\end{aligned} \tag{2.6}$$

by comparing coefficients of (2.1) and (2.6), we get the result.

(b) $\Leftrightarrow$ (c) From the generating function of $\{\mathbb{P}_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$, direct conclusion yield

$$
\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} D_q(\mathbb{P}_{n_1,n_2}(x))\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= D_{q,x}(A_q(t_1,t_2)e_q(t_1x)E_q(t_2x))\\
&= A_q(t_1,t_2)(t_2e_q(t_1qx)E_q(t_2qx)+t_1e_q(t_1x)E_q(t_2x))\\
&= t_2A_q(t_1,t_2)e_q(t_1qx)E_q(t_2qx)+t_1A_q(t_1,t_2)e_q(t_1x)E_q(t_2x)\\
&= t_2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(qx)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&+t_1\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left([n_2]_q\mathbb{P}_{n_1,n_2-1}(qx)+[n_1]_q\mathbb{P}_{n_1-1,n_2}(x)\right)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}
\end{aligned}
\tag{2.7}
$$

Thus the result follows from comparing the coefficients of $t_1^{n_1}t_2^{n_2}$.

(c) $\Leftrightarrow$ (a) Assume that (c) holds for $n_1+n_2\geq 1$ and

$$
\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}=A_q(x;t_1,t_2)e_q(t_1x)E_q(t_2x).
$$

Applying the q-derivative operator $D_{q,x}$ on both sides, we get

$$
\begin{aligned}
&D_{q,x}(A_q(x;t_1,t_2)e_q(t_1x)E_q(t_2x))\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} D_q(\mathbb{P}'_{n_1,n_2}(x))\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}([n_2]_q\mathbb{P}_{n_1,n_2-1}(qx)+[n_1]_q\mathbb{P}_{n_1-1,n_2}(x))\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(t_2\mathbb{P}_{n_1,n_2}(qx)+t_1\mathbb{P}_{n_1,n_2}(x))\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= t_2A_q(x;t_1,t_2)e_q(t_1qx)E_q(t_2qx)+t_1A_q(x;t_1,t_2)e_q(t_1x)E_q(t_2x)
\end{aligned}
$$

$D_{q,x}A_q(x;t_1,t_2)=0$ points that A is independent of x, therefore $A_q(x;t_1,t_2)=A_q(t_1,t_2)$.

(b) $\Rightarrow$ (d) The result of taking the q-derivative of $x^{n_1+n_2}$, k_1+k_2 times can be calculated as

$$D_{q,x}^{k_1+k_2}(x^{n_1+n_2}) = \frac{[n_1+n_2]_q!}{[n_1+n_2-k_1-k_2]_q!}x^{n_1+n_2-k_1-k_2} \tag{2.8}$$

and by using this solution we can prove (d) directly as follows

$$\begin{aligned}\mathbb{P}_{n_1,n_2}(x) &= \left\{\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{k_2(k_2-1)}{2}}\frac{[n_1+n_2-k_1-k_2]_q!}{[n_1+n_2]_q!}a_{k_1,k_2}D_{q,x}^{k_1+k_2}\right\}x^{n_1+n_2}\\ &= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{k_2(k_2-1)}{2}}a_{n_1-k_1,n_2-k_2}x^{k_1+k_2}.\end{aligned}$$

□

Theorem 2.2 (Recurrence Relations): The following recurrence relations for the univariate q-multiple Appell polynomials holds true:

$$\begin{aligned}\mathbb{P}_{n_1+1,n_2}(x) &= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\alpha_{n_1-k_1,n_2-k_2}\mathbb{P}_{k_1,k_2}(x)\\ &\quad+x\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q(q^{n_1-k_1}-1)\alpha_{n_1-k_1-1,n_2-k_2}\mathbb{P}_{k_1,k_2}(x)\\ &\quad+x\mathbb{P}_{n_1,n_2}(x),\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathbb{P}_{n_1,n_2+1}(x) &= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\beta_{n_1-k_1,n_2-k_2}\mathbb{P}_{k_1,k_2}(x)\\ &\quad+xq^{n_2}\mathbb{P}_{n_1,n_2}(x).\end{aligned} \tag{2.10}$$

where the double sequence α_{n_1,n_2} is given as

$$\frac{D_{q,t_1}A_q(t_1,t_2)}{A_q(t_1,t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\alpha_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{2.11}$$

and β_{n_1,n_2} is given as

$$\frac{D_{q,t_2}A_q(t_1,t_2)}{A_q(t_1,t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\beta_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{2.12}$$

Proof. Since $D_{q,t_1}A_q(t_1,t_2) = \frac{A_q(qt_1,t_2)-A_q(t_1,t_2)}{(q-1)t_1}$, we can get

$$A_q(qt_1,t_2) = (q-1)t_1 D_{q,t_1}A_q(t_1,t_2) + A_q(t_1,t_2). \tag{2.13}$$

On the other hand, it is clear that

$$\begin{aligned}
(q-1)t_1 \frac{D_{q,t_1}A_q(t_1,t_2)}{A_q(t_1,t_2)} &= (q-1)t_1 \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \alpha_{n_1,n_2} \frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= (q-1) \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \alpha_{n_1,n_2} \frac{t_1^{n_1+1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= (q-1) \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} [n_1]_q \alpha_{n_1-1,n_2} \frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= (q-1) \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{(1-q^{n_1})}{1-q} \alpha_{n_1-1,n_2} \frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} (q^{n_1}-1) \alpha_{n_1-1,n_2} \frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}.
\end{aligned} \tag{2.14}$$

Taking q-derivative with respect to t_1 of left hand side of (2.1) and using (2.13) and (2.14), we can easily write that

$$A_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$= D_{q,t_1}(A_q(t_1,t_2))e_q(t_1x)E_q(t_2x) + xA_q(qt_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$= D_{q,t_1}(A_q(t_1,t_2))e_q(t_1x)E_q(t_2x)$$

$$+ x((q-1)t_1 D_{q,t_1}A_q(t_1,t_2) + A_q(t_1,t_2))e_q(t_1x)E_q(t_2x))$$

$$= \frac{D_{q,t_1}(A_q(t_1,t_2))}{A_q(t_1,t_2)}e_q(t_1x)E_q(t_2x)$$

$$+ x(q-1)t_1\frac{D_{q,t_1}(A_q(t_1,t_2))}{A_q(t_1,t_2)}e_q(t_1x)E_q(t_2x) + xA_q(t_1,t_2))e_q(t_1x)E_q(t_2x)$$

$$= \sum_{n_1,n_2=0}^{\infty} \alpha_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\sum_{k_1,k_2=0}^{\infty}\mathbb{P}_{k_1,k_2}\frac{t_1^{k_1}t_2^{k_2}}{[k_1]_q![k_2]_q!}$$

$$+ x\sum_{n_1,n_2=0}^{\infty}(q^{n_1}-1)\alpha_{n_1-1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\sum_{k_1,k_2=0}^{\infty}\mathbb{P}_{k_1,k_2}\frac{t_1^{k_1}t_2^{k_2}}{[k_1]_q![k_2]_q!}$$

$$+ x\sum_{n_1,n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$= \sum_{n_1,n_2}^{\infty}\sum_{k_1,k_2=0}^{n_1,n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\alpha_{n_1-k_1,n_2-k_2}\mathbb{P}_{k_1,k_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$+ x\sum_{n_1,n_2}^{\infty}\sum_{k_1,k_2=0}^{n_1,n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q(q^{n_1-k_1}-1)\alpha_{n_1-k_1-1,n_2-k_2}\mathbb{P}_{k_1,k_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$+ x\sum_{n_1,n_2}^{\infty}\mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{2.15}$$

By taking q-derivative with respect to t_1 of the right hand side of (2.1) we can get

$$D_{q,t_1}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)$$

$$= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)[n_1]_q\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1-1]_q![n_2]_q!}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1+1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{2.16}$$

Comparing coeficients of (2.15) and (2.16), we can get (2.9).

The q-derivative with respect to t_2 of the left hand side of (2.1) is

$$
\begin{aligned}
&D_{q,t_2}(A_q(t_1,t_2)e_q(t_1x)E_q(t_2x))\\
&= D_{q,t_2}(A_q(t_1,t_2))e_q(t_1x)E_q(t_2x) + xA_q(t_1,qt_2)e_q(t_1x)E_q(qt_2x)\\
&= \frac{D_{q,t_2}A_q(t_1,t_2)}{A_q(t_1,t_2)}A_q(t_1,t_2)e_q(t_1x)E_q(t_2x)\\
&+ xA_q(t_1,qt_2)e_q(t_1x)E_q(qt_2x)\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\beta_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\sum_{k_1,k_2}^{\infty}\mathbb{P}_{k_1,k_2}\frac{t_1^{k_1}t_2^{k_2}}{[k_1]_q![k_2]_q!}\\
&+ x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)q^{n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1}^{n_1}\sum_{k_2}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\beta_{n_1-k_1,n_2-k_2}\mathbb{P}_{k_1,k_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&+ x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)q^{n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \qquad (2.17)
\end{aligned}
$$

The q-derivative of the right hand side of (2.1) with respect to t_2 is

$$
\begin{aligned}
&D_{q,t_2}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\mathbb{P}_{n_1,n_2}(x)[n_2]_q\frac{t_1^{n_1}t_2^{n_2-1}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\mathbb{P}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2-1}}{[n_1]_q![n_2-1]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{P}_{n_1,n_2+1}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \qquad (2.18)
\end{aligned}
$$

By comparing coeficients of (2.17) and (2.18), we can get (2.10).

$$
\mathbb{P}_{n_1,n_2+1}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}\begin{bmatrix}n_2\\k_2\end{bmatrix}\beta_{n_1-k_1,n_2-k_2}\mathbb{P}_{k_1,k_2}(x) + xq^{n_2}\mathbb{P}_{n_1,n_2}(x). \qquad (2.19)
$$

□

Corollary 2.1: By setting $q \to 1$ we can get the following recurrence relations for univariate multiple Appell polynomials $P_{n_1,n_2}(x)$ which is not mentioned in [17].

$$P_{n_1+1,n_2}(x) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} \binom{n_2}{k_2} \alpha_{n_1-k_1,n_2-k_2} P_{k_1,k_2}(x) + xP_{n_1,n_2}(x) \tag{2.20}$$

$$P_{n_1,n_2+1}(x) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} \binom{n_2}{k_2} \beta_{n_1-k_1,n_2-k_2} P_{k_1,k_2}(x) + xP_{n_1,n_2}(x) \tag{2.21}$$

where the double sequence α_{n_1,n_2} is given as

$$\frac{D_{t_1}A(t_1,t_2)}{A(t_1,t_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \alpha_{n_1,n_2} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!} \tag{2.22}$$

and β_{n_1,n_2} is given as

$$\frac{D_{t_2}A(t_1,t_2)}{A(t_1,t_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \beta_{n_1,n_2} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!}. \tag{2.23}$$

Chapter 3

BIVARIATE q-MULTIPLE APPELL POLYNOMIALS

In this chapter, we will use definition of univariate q-multiple Appell polynomials to define the bivariate q-multiple Appell polynomials and constitute their properties [34].

Definition 3.1: Bivariate q-multiple Appell polynomials which we denoted them by $\{\mathbb{S}_{n_1,n_2}(x,y)\}_{n_1,n_2=0}^{\infty}$ are defined by the generating relation

$$A_q(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{S}_{n_1,n_2}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{3.1}$$

where A_q has a series expansion

$$A_q(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}a_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{3.2}$$

with $A(0,0) = a_{0,0} \neq 0$.

Remark 3.1: Taking $t_2 = 0$ in equation (3.1), bivariate q-multiple Appell polynomials reduce to bivariate q-Appell polynomials. Example (1.10) and Example (1.12) are given as examples for bivariate q-Appell polynomials, in Chapter 1.

Theorem 3.1: For all $x, y \in \mathbb{C}$, we have the q-analogue of bivariate q-multiple Appell polynomials as follows

$$\mathbb{S}_{n_1,n_2,q}(x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q a_{k_1,k_2}(x+y)_q^{n_1+n_2-k_1-k_2} \tag{3.3}$$

Proof. $$\sum_{n_1,n_2=0}^{\infty} \mathbb{S}_{n_1,n_2}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$= A_q(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x)$$
$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} a_{k_1,k_2}\frac{t_1^{k_1}t_2^{k_2}}{[k_1]_q![k_2]_q!}\sum_{n_1=0}^{\infty}(x+y)_q^{n_1}\frac{t_1^{n_1}}{[n_1]_q!}\sum_{n_2=0}^{\infty}(x+y)_q^{n_2}\frac{t_2^{n_2}}{[n_2]_q!}$$
$$= \sum_{n_1,n_2=0}^{\infty}\sum_{k_1,k_2=0}^{n_1,n_2}\begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q\begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q a_{k_1,k_2}(x+y)_q^{n_1+n_2-k_1-k_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

□

Theorem 3.2 (Equivalence Theorem): Let $\{\mathbb{S}_{n_1,n_2}(x,y)\}_{n_1,n_2=0}^{\infty}$ be a bivariate q-multiple Appell polynomial. Then the followings are all equivalent

a. $\{\mathbb{S}_{n_1,n_2}(x,y)\}_{n_1,n_2=0}^{\infty}$ is a set of bivariate q-multiple Appell polynomials.

b. There exists an Appell sequence $\{\mathbb{P}_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ such that

$$\mathbb{S}_{n_1,n_2}(x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q\begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_1(k_1-1)}{2}}\mathbb{P}_{n_1-k_1,n_2-k_2}(x)y^{k_1+k_2}. \tag{3.4}$$

c. For every $n_1+n_2 \geq 1$, we have

$$D_{q,x}\mathbb{S}_{n_1,n_2}(x,y) = [n_2]_q\mathbb{S}_{n_1,n_2-1}(qx,y)+[n_1]_q\mathbb{S}_{n_1-1,n_2}(x,y) \tag{3.5}$$

$$D_{q,y}\mathbb{S}_{n_1,n_2}(x,y) = [n_1]_q\mathbb{S}_{n_1-1,n_2}(x,qy)+[n_2]_q\mathbb{S}_{n_1,n_2-1}(x,y). \tag{3.6}$$

d. There exists a sequence $\{\mathbb{P}_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ with $a_{0,0} \neq 0$ such that

$$\mathbb{S}_{n_1,n_2}(x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q\begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_1(k_1-1)}{2}}\frac{[n_1+n_2-k_1-k_2]!}{[n_1+n_2]!}$$
$$\times \mathbb{P}_{k_1,k_2}(x)D_{q,y}^{k_1+k_2}y^{n_1+n_2}. \tag{3.7}$$

Proof. (a) ⇔ (b) Since we have $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_nb_k = \sum_{n=0}^{\infty}\sum_{k=0}^{n}a_{n-k}b_k$ we can get

$$
\begin{aligned}
&A(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x)\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\sum_{k_2=0}^{\infty}y^{k_2}\frac{t_2{}^{k_2}}{[k_2]_q!}\sum_{k_1=0}^{\infty}q^{\frac{k_1(k_1-1)}{2}}y^{k_1}\frac{t_1{}^{k_1}}{[k_1]_q!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}P_{n_1,n_2}(x)q^{\frac{k_1(k_1-1)}{2}}y^{k_1+k_2}\frac{t_1{}^{n_1+k_1}t_2{}^{n_2+k_2}}{[n_1]_q[n_2]_q[k_1]_q[k_2]_q}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left\{\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{k_1(k_1-1)}{2}}P_{n_1-k_1,n_2-k_2}(x)y^{k_1+k_2}\right\}\frac{t_1{}^{n_1}t_2{}^{n_2}}{[n_1]_q[n_2]_q} \qquad (3.8)
\end{aligned}
$$

we can obtain (b) by comparing coefficients of (3.1) and (3.8) .

(b) $\Leftrightarrow$ (c) From the generating function of $\{S_{n_1,n_2}(x,y)\}_{n_1,n_2=0}^{\infty}$ we can get

$$
\begin{aligned}
&D_{q,x}(A(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x))\\
&=A(t_1,t_2)e_q(t_2y)E_q(t_1y)((t_2e_q(t_1qx)E_q(t_2qx)+t_1e_q(t_1x)E_q(t_2x))\\
&=t_2A(t_1,t_2)e_q(t_1qx)e_q(t_2y)E_q(t_1y)E_q(t_2qx)\\
&+t_1A(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x)\\
&=t_2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}S_{n_1,n_2}(qx,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}+t_1\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}S_{n_1,n_2}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left([n_2]_qS_{n_1,n_2-1}(qx,y)+[n_1]_qS_{n_1-1,n_2}(x,y)\right)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \qquad (3.9)
\end{aligned}
$$

and

$$
\begin{aligned}
&D_{q,y}(A(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x))\\
&=A(t_1,t_2)e_q(t_1x)E_q(t_2x)((t_1e_q(t_2qy)E_q(t_1qy)+t_2e_q(t_2x)E_q(t_1x))\\
&=t_2A(t_1,t_2)e_q(t_1qx)e_q(t_2y)E_q(t_1y)E_q(t_2qx)\\
&+t_1A(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x)\\
&=t_1\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}S_{n_1,n_2}(x,qy)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}+t_2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}S_{n_1,n_2}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left([n_1]_qS_{n_1-1,n_2}(x,qy)+[n_2]_qS_{n_1,n_2-1}(x,y)\right)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \qquad (3.10)
\end{aligned}
$$

by comparing coefficients of (3.1) by (3.9) and (3.10), we can get (c).

(b) ⇔ (d) This is immediately apparent from

$$S_{n_1,n_2}(x,y) = \left\{ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_1(k_1-1)}{2}} \frac{[n_1+n_2-k_1-k_2]!}{[n_1+n_2]!} P_{k_1,k_2}(x) D_{q,y}^{k_1+k_2} \right\} y^{n_1+n_2}$$
$$= \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_1(k_1-1)}{2}} P_{n_1-k_1,n_2-k_2}(x) y^{k_1+k_2}$$

□

Theorem 3.3 (Recurrence Relations): Bivariate q-multiple Appell polynomials have recurrence relations as follows:

$$\begin{aligned} \mathbb{S}_{n_1+1,n_2}(x,y) = & \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q \alpha_{n_1-k_1,n_2-k_2} \mathbb{S}_{k_1,k_2}(x,y) \\ & + yq^{n_1} \mathbb{S}_{n_1,n_2}(x,y) \\ & + x \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q (q^{n_1-k_1}-1) \alpha_{n_1-k_1-1,n_2-k_2} \mathbb{S}_{k_1,k_2}(x,y) \\ & + x\mathbb{S}_{n_1,n_2}(x,y) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \mathbb{S}_{n_1,n_2+1}(x,y) = & \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q \beta_{n_1-k_1,n_2-k_2} \mathbb{S}_{k_1,k_2}(x,y) \\ & + xq^{n_2} \mathbb{S}_{n_1,n_2}(x,y) \\ & + \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q (q^{n_2-k_2}-1) \beta_{n_1-k_1,n_2-k_2-1} \mathbb{S}_{k_1,k_2}(x,y) \\ & + y\mathbb{S}_{n_1,n_2}(x,y) \end{aligned} \tag{3.12}$$

where generating function for α_{n_1,n_2} given as

$$\frac{D_{q,t_1} A_q(t_1,t_2)}{A_q(t_1,t_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \alpha_{n_1,n_2} \frac{t_1^{n_1} t_2^{n_2}}{[n_1]_q! [n_2]_q!} \tag{3.13}$$

and generating function of β_{n_1,n_2} given as

$$\frac{D_{q,t_2}A_q(t_1,t_2)}{A_q(t_1,t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\beta_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{3.14}$$

Proof. The proof is obtained as in the proof of Theorem 2.2. □

Corollary 3.1: As far as we know in the literature, there is no definition for bivariate usual Appell polynomials yet. By taking $q \to 1$, we can get recurrence relations for bivariate usual Appell polynomials $S_{n_1,n_2}(x)$ as follows

$$S_{n_1+1,n_2}(x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\alpha_{n_1-k_1,n_2-k_2}s_{k_1,k_2}(x,y)$$

$$+ yS_{n_1,n_2}(x,y) + xS_{n_1,n_2}(x,y) \tag{3.15}$$

$$S_{n_1,n_2+1}(x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\beta_{n_1-k_1,n_2-k_2}S_{k_1,k_2}(x,y)$$

$$+ xS_{n_1,n_2}(x,y) + yS_{n_1,n_2}(x,y). \tag{3.16}$$

where generating function for α_{n_1,n_2} and β_{n_1,n_2} given as

$$\frac{D_{t_1}A(t_1,t_2)}{A(t_1,t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\alpha_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}, \tag{3.17}$$

$$\frac{D_{t_2}A(t_1,t_2)}{A(t_1,t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\beta_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!} \tag{3.18}$$

respectively and $\{S_{n_1,n_2}(x,y)\}_{n_1,n_2}$ is defined via the generating relation

$$A(t_1,t_2)e^{(x+y)(t_1+t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}S_{n_1,n_2}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}. \tag{3.19}$$

Chapter 4

EXAMPLES OF q-MULTIPLE APPELL POLYNOMIALS

The main purpose of this chapter is to introduce some examples of univariate q-multiple Appell and bivariate q-multiple Appell polynomials. Let's consider two q-Appell polynomial

$$A_{1q}(t_1)e_q(t_1x) = \sum_{n_1=0}^{\infty} A_{n_1}^{(1)}(x)\frac{t_1^{n_1}}{[n_1]_q!} \tag{4.1}$$

and

$$A_{2q}(t_2)E_q(t_2x) = \sum_{n_2=0}^{\infty} A_{n_2}^{(2)}(x)\frac{t_2^{n_2}}{[n_2]_q!} \tag{4.2}$$

where

$$A_{1q}(t_1) = \sum_{k=0}^{\infty} \frac{a_{1k}}{[k]_q!}t_1^k \tag{4.3}$$

and

$$A_{2q}(t_2) = \sum_{k=0}^{\infty} \frac{a_{2k}}{[k]_q!}t_2^k \quad . \tag{4.4}$$

If we multiply (4.1) and (4.2), we obtain that

$$\begin{aligned} A_q(t_1,t_2)e_q(t_1x)E_q(t_2x) &= A_{1q}(t_1)e_q(t_1x)A_{2q}(t_2)E_q(t_2x) \\ &= \sum_{n_1=0}^{\infty} A_{n_1}^{(1)}(x)\frac{t_1^{n_1}}{[n_1]_q!}\sum_{n_2=0}^{\infty} A_{n_2}^{(2)}(x)\frac{t_2^{n_2}}{[n_2]_q!} \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A_{n_1}^{(1)}(x)A_{n_2}^{(2)}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathbb{A}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \end{aligned} \tag{4.5}$$

and this shows that

$$\mathbb{A}_{n_1,n_2}(x) = A_{n_1}^{(1)}(x)A_{n_2}^{(2)}(x). \tag{4.6}$$

Therefore, using this technique we can produce several q-multiple Appell polynomials from the known q-Appell polynomials. In this chapter, we will apply this technique to give some examples to the q-multiple Appell polynomials and we will call such an examples as trivial examples. Further we give a non-trivial example which is (4.66). Our examples include corollaries about the obtained results of the previous chapters. In some examples further results also given.

4.1 Trivial Examples

4.1.1 q-Multiple Power Polynomials

Definition 4.1: Taking $A_q(t_1,t_2) = E_q(t_1)e_q(t_2)$ in (2.1) gives us the following q-analogue of the generating function for $(x+1)^{n_1+n_2}$

$$E_q(t_1)e_q(t_2)e_q(t_1x)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{4.7}$$

where explicit form of $\mathbb{M}_{n_1,n_2}(x)$ is

$$\mathbb{M}_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_1(k_1-1)+k_2(k_2-1)}{2}} x^{k_1+k_2}. \tag{4.8}$$

By using q-binomial formula (1.38) and (4.8), we have

$$\mathbb{M}_{n_1,n_2}(x) = (x+1)(x+q)...(x+q^{n_1+n_2-1}), \quad \mathbb{M}_{0,0}(x) = 1. \tag{4.9}$$

Definition 4.2: q-Analogue of the generating function of $(x-1)^{n_1+n_2}$ can be defined by taking $A_q(t_1,t_2) = e_q(-t_1)E_q(-t_2)$ in (2.1) as

$$e_q(-t_1)E_q(-t_2)e_q(t_1x)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{4.10}$$

where explicit form of $\mathbb{N}_{n_1,n_2}(x)$ is

$$\mathbb{N}_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} (-1)^{k_1+k_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_1(k_1-1)+k_2(k_2-1)}{2}} x^{k_1+k_2}. \tag{4.11}$$

By using q-binomial formula (1.38) and (4.11), we have

$$\mathbb{N}_{n_1,n_2}(x) = (x-1)(x-q)...(x-q^{n_1+n_2-1}), \quad \mathbb{N}_{0,0}(x) = 1. \tag{4.12}$$

Theorem 4.1 (Recurrence Relations): q-Multiple Power polynomial $\mathbb{M}_{n_1,n_2}(x)$ satisfies the following recurrence relations

$$\mathbb{M}_{n_1+1,n_2}(x) = (q^{n_1} + x)\mathbb{M}_{n_1,n_2}(x), \tag{4.13}$$

$$\mathbb{M}_{n_1,n_2+1}(x) = (xq^{n_2} + 1)\mathbb{M}_{n_1,n_2}(x). \tag{4.14}$$

Proof. Lets take q-derivative both side of (4.7) with respect to t_1

$$\begin{aligned}
&D_{q,t_1}\left(E_q(t_1)e_q(t_2)e_q(t_1x)E_q(t_2x)\right)\\
&= E_q(qt_1)e_q(t_2)e_q(qt_1x)E_q(t_2x) + xE_q(t_1)e_q(t_2)e_q(t_1x)E_q(t_2x)\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)q^{n_1}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} + x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(q^{n_1} + x)\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&D_{q,t_1}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)\\
&= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)[n_1]_q\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1-1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1+1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}
\end{aligned} \tag{4.16}$$

by comparing coefficients of (4.16) and (4.18)

$$\mathbb{M}_{n_1+1,n_2}(x) = (q^{n_1} + x)\mathbb{M}_{n_1,n_2}(x). \tag{4.17}$$

Lets take q-derivative on both side of (4.7) with respect to t_2

$$\begin{aligned}
&D_{q,t_2}\left(E_q(t_1)e_q(t_2)e_q(t_1x)E_q(t_2x)\right)\\
&= xE_q(t_1)e_q(qt_2)e_q(t_1x)E_q(qt_2x) + E_q(t_1)e_q(t_2)e_q(t_1x)E_q(t_2x)\\
&= x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)q^{n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(xq^{n_2} + 1)\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}
\end{aligned} \tag{4.18}$$

$$D_{q,t_2}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\mathbb{M}_{n_1,n_2}(x)[n_2]_q\frac{t_1^{n_1}t_2^{n_2-1}}{[n_1]_q![n_2]_q!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\mathbb{M}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2-1}}{[n_1]_q![n_2-1]_q!}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{M}_{n_1,n_2+1}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{4.19}$$

By comparing coefficients (4.18) and (4.19), we get

$$\mathbb{M}_{n_1,n_2+1}(x)=(xq^{n_2}+1)\mathbb{M}_{n_1,n_2}(x) \tag{4.20}$$

which is the desired solution.

□

Theorem 4.2 (Recurrence Relations): q-Multiple Power polynomial $\mathbb{N}_{n_1,n_2}(x)$ satisfies the following recurrence relations

$$\mathbb{N}_{n_1+1,n_2}(x)=(x-1)\mathbb{N}_{n_1,n_2}(x)+x(q^{n_1}-1)\mathbb{N}_{n_1-1,n_2}(x), \tag{4.21}$$

$$\mathbb{N}_{n_1,n_2+1}(x)=(x-1)q^{n_2}\mathbb{N}_{n_1,n_2}(x)-x(q^{n_2}-1)q^{n_2-1}\mathbb{N}_{n_1,n_2-1}(x). \tag{4.22}$$

Proof. By taking q-derivative of $e_q(-t_1)$ with respect to t_1 gives us

$$e_q(-qt_1)=(1-(q-1)t_1)e_q(-t_1)$$

Lets take q-derivative on both side with respect to t_1

$$D_{q,t_1}(e_q(-t_1)E_q(-t_2)e_q(t_1x)E_q(t_2x))$$

$$= xe_q(-qt_1)E_q(-t_2)e_q(t_1x)E_q(t_2x) - e_q(-t_1)E_q(-t_2)e_q(t_1x)E_q(t_2x)$$

$$= x(1-(q-1)t_1)e_q(t_1)E_q(-t_2)e_q(t_1x)E_q(t_2x)$$

$$- e_q(-t_1)E_q(-t_2)e_q(t_1x)E_q(t_2x)$$

$$= x\sum_{n_1,n_2=0}^{\infty} \mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$- x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(q^{n_1}-1)\mathbb{N}_{n_1-1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$- \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(x-1)\mathbb{N}_{n_1,n_2}(x) - x(q^{n_1}-1)\mathbb{N}_{n_1-1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{4.23}$$

$$\begin{aligned} &D_{q,t_1}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)\\ &= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{N}_{n_1,n_2}(x)[n_1]_q\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\ &= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1-1]_q![n_2]_q!}\\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{N}_{n_1+1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \end{aligned} \tag{4.24}$$

By comparing coefficients (4.23) and (4.24), we get

$$\mathbb{N}_{n_1+1,n_2}(x) = (x-1)\mathbb{N}_{n_1,n_2}(x) + x(q^{n_1}-1)\mathbb{N}_{n_1-1,n_2}(x).$$

Taking q-derivative of $E_q(-t_2)$ with respect to t_2 gives

$$E(-t_2) = (1+(q-1))t_2E(-qt_2).$$

Lets take q-derivative of the left hand side of (4.10) with respect to t_2, we get

$$D_{q,t_2}(e_q(-t_1)E_q(-t_2)e_q(t_1x)E_q(t_2x))$$

$$= -e_q(-t_1)E_q(-qt_2)e_q(t_1x)E_q(qt_2x) + xe_q(-t_1)E_q(-t_2)e_q(t_1x)E_q(qt_2x)$$

$$= -e_q(-t_1)E_q(-qt_2)e_q(t_1x)E_q(qt_2x)$$

$$+x(1+(q-1))t_2(e_q(-t_1)E_q(-qt_2)e_q(t_1x)E_q(qt_2x)$$

$$= -\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} q^{n_2}\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$+x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} q^{n_2}\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$+x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} q^{n_2-1}(q^{n_2}-1)\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} q^{n_2}(x-1)\mathbb{N}_{n_1,n_2}(x) + xq^{n_2-1}(q^{n_2}-1)\mathbb{N}_{n_1,n_2-1}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{4.25}$$

The q-derivative of the right-hand side of (4.10) will be

$$D_{q,t_2}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)$$
$$= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\mathbb{N}_{n_1,n_2}(x)[n_2]_q\frac{t_1^{n_1}t_2^{n_2-1}}{[n_1]_q![n_2]_q!}$$
$$= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\mathbb{N}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2-1}}{[n_1]_q![n_2-1]_q!}$$
$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{N}_{n_1,n_2+1}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{4.26}$$

By comparing coefficients (4.25) and (4.26), we get

$$\mathbb{N}_{n_1,n_2+1}(x) = q^{n_2}(x-1)\mathbb{N}_{n_1,n_2}(x) + xq^{n_2-1}(q^{n_2}-1)\mathbb{N}_{n_1,n_2-1}(x), \quad (n_1,n_2 = 0,1,2...)$$

Whence the result. □

Given two bivariate q-Appell polynomials in the form

$$A_q(t_1)e_q(t_1x)E_q(t_1y) = \sum_{n_1=0}^{\infty} A_{n_1}^{(1)}(x,y)\frac{t_1^{n_1}}{[n_1]_q!} \tag{4.27}$$

and

$$A_q(t_2)e_q(t_2x)E_q(t_2y) = \sum_{n_2=0}^{\infty} A_{n_2}^{(2)}(x,y)\frac{t_2^{n_2}}{[n_2]_q!}. \tag{4.28}$$

We can multiplying both sides of (4.27) and (4.28) to obtain

$$\begin{aligned}
&A_q(t_1,t_2)e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x) \\
&= A_q(t_1)e_q(t_1x)E_q(t_1y)A_q(t_2)e_q(t_2y)E_q(t_2x) \\
&= \sum_{n_1=0}^{\infty} A_{n_1}^{(1)}(x,y)\frac{t_1^{n_1}}{[n_1]_q!}\sum_{n_2=0}^{\infty} A_{n_2}^{(2)}(y,x)\frac{t_2^{n_2}}{[n_2]_q!} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A_{n_1}^{(1)}(x,y)A_{n_2}^{(2)}(y,x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathbb{A}_{n_1,n_2}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}.
\end{aligned} \tag{4.29}$$

Therefore,

$$\mathbb{A}_{n_1,n_2}(x,y) = A_{n_1}^{(1)}(x,y)A_{n_2}^{(2)}(y,x). \tag{4.30}$$

By applying the above technique we will give 3 trivial examples to bivariate q-multiple Appell polynomials.

4.1.2 Bivariate q-Multiple Bernoulli Polynomials

In this section, to define bivariate q-multiple Bernoulli polynomials and their features we will use definitions and properties of higher order q-Bernoulli polynomials from [19] and basics of q-calculus. Taking $A_q(t_1,t_2)$ in (3.1) as $\left(\frac{t_1}{e_q(t_1)-1}\right)^{\alpha}\left(\frac{t_2}{e_q(t_2)-1}\right)^{\alpha}$ gives us the following definition.

Definition 4.3: The bivariate q-multiple Bernoulli numbers $\mathbb{B}_{n_1,n_2,q}^{(\alpha)}$ is defined by means of the generating function:

$$\left(\frac{t_1}{e_q(t_1)-1}\right)^{\alpha}\left(\frac{t_2}{e_q(t_2)-1}\right)^{\alpha} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathbb{B}_{n_1,n_2,q}^{(\alpha)}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}, \quad |t| < 2\pi. \tag{4.31}$$

and bivariate q-multiple Bernoulli polynomials $\mathbb{B}_{n_1,n_2,q}^{(\alpha)}(x,y)$ in x,y of order α is defined by using the generating relation below

$$\left(\frac{t_1}{e_q(t_1)-1}\right)^{\alpha}\left(\frac{t_2}{e_q(t_2)-1}\right)^{\alpha} e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x)$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}, \quad |t|<2\pi. \tag{4.32}$$

Theorem 4.3: Let $\{\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)\}_{n_1,n_2=0}^{\infty}$ be a bivariate q-multiple Bernoulli polynomials. Then we have the following:

a. The bivariate q-multiple Bernoulli polynomials can be written as

$$\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)=\mathfrak{B}^{(\alpha)}_{n_1,q}(x,y)\mathfrak{B}^{(\alpha)}_{n_2,q}(y,x). \tag{4.33}$$

where

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)E_q(ty)=\sum_{n=0}^{\infty}\mathfrak{B}^{(\alpha)}_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{4.34}$$

b. For every $n_1+n_2\geq 1$, we have

$$D_{q,x}\left(\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)\right)=[n_1]_q\mathbb{B}^{(\alpha)}_{n_1-1,n_2,q}(x,y)+[n_2]_q\mathbb{B}^{(\alpha)}_{n_1,n_2-1,q}(qx,y), \tag{4.35}$$

$$D_{q,y}\left(\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)\right)=[n_2]_q\mathbb{B}^{(\alpha)}_{n_1,n_2-1,q}(x,y)+[n_1]_q\mathbb{B}^{(\alpha)}_{n_1-1,n_2,q}(x,qy). \tag{4.36}$$

Corollary 4.1: For all $x,y\in\mathbb{C}$, the bivariate q-multiple Bernoulli polynomials can be written explictly as

$$\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\mathbb{B}^{(\alpha)}_{k_1,k_2,q}(x+y)_q^{n_1+n_2-k_1-k_2}. \tag{4.37}$$

where $\mathbb{B}^{(\alpha)}_{k_1,k_2,q}$ is given in (4.31).

Corollary 4.2: The bivariate q-multiple Bernoulli polynomials $\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)$ can be represented in the following two ways:

$$\begin{aligned}
&\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,y)\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}\mathbb{B}^{(\alpha)}_{k_1,k_2,q}(x,0)y^{n_1+n_2-k_1-k_2}\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{B}^{(\alpha)}_{k_1,k_2,q}(0,y)x^{n_1+n_2-k_1-k_2}.
\end{aligned}\tag{4.38}$$

Corollary 4.3: In special, we get the following formulas by setting $x=0$ and $y=0$ in (4.38), respectively

$$\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(0,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}\mathbb{B}^{(\alpha)}_{k_1,k_2,q}y^{n_1+n_2-k_1-k_2},\tag{4.39}$$

$$\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,0)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{B}^{(\alpha)}_{k_1,k_2,q}x^{n_1+n_2-k_1-k_2}.\tag{4.40}$$

Corollary 4.4: By setting $x=1$ and $y=1$ in (4.38), respectively, we get

$$\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(1,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{B}^{(\alpha)}_{k_1,k_2,q}(0,y),\tag{4.41}$$

$$\mathbb{B}^{(\alpha)}_{n_1,n_2,q}(x,1)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}\mathbb{B}^{(\alpha)}_{k_1,k_2,q}(x,0).\tag{4.42}$$

4.1.3 Bivariate q-Multiple Euler Polynomials

This section holds for define bivariate q-multiple Euler polynomials and their features. We will use definitions and properties of higher order q-Euler polynomials from [19]. Taking $A_q(t_1,t_2)$ in (3.1) as $\left(\frac{2}{e_q(t_1)+1}\right)^\alpha\left(\frac{2}{e_q(t_2)+1}\right)^\alpha$ gives us the following definition.

Definition 4.4: The bivariate q-multiple Euler numbers $\mathbb{E}^{(\alpha)}_{n_1,n_2,q}$ is defined by means of the generating function:

$$\left(\frac{2}{e_q(t_1)+1}\right)^\alpha\left(\frac{2}{e_q(t_2)+1}\right)^\alpha=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{E}^{(\alpha)}_{n_1,n_2,q}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},\quad |t|<2\pi,\tag{4.43}$$

and bivariate q-multiple Euler polynomials $\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)$ in x,y of order α is defined by using the generating function below

$$\left(\frac{2}{e_q(t_1)+1}\right)^{\alpha}\left(\frac{2}{e_q(t_2)+1}\right)^{\alpha} e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x)$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}, \quad |t|<2\pi. \tag{4.44}$$

Theorem 4.4: Let $\{\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)\}_{n_1,n_2=0}^{\infty}$ be a bivariate q-multiple Euler polynomials. Then the followings are all equivalent

a. The bivariate q-multiple Euler polynomials can be written as

$$\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)=\mathfrak{E}^{(\alpha)}_{n_1,q}(x,y)\mathfrak{E}^{(\alpha)}_{n_2,q}(y,x). \tag{4.45}$$

where

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha} e_q(tx)E_q(ty)=\sum_{n=0}^{\infty}\mathfrak{E}^{(\alpha)}_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{4.46}$$

b. For every $n_1+n_2\geq 1$, we have

$$D_{q,x}\left(\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)\right)=[n_1]_q\mathbb{E}^{(\alpha)}_{n_1-1,n_2,q}(x,y)+[n_2]_q\mathbb{E}^{(\alpha)}_{n_1,n_2-1,q}(qx,y), \tag{4.47}$$

$$D_{q,y}\left(\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)\right)=[n_2]_q\mathbb{E}^{(\alpha)}_{n_1,n_2-1,q}(x,y)+[n_1]_q\mathbb{E}^{(\alpha)}_{n_1-1,n_2,q}(x,qy). \tag{4.48}$$

Corollary 4.5: For all $x,y\in\mathbb{C}$, we have the bivariate q-multiple Euler polynomials can be written explicitly as

$$\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\mathbb{E}^{(\alpha)}_{k_1,k_2,q}(x+y)_q^{n_1+n_2-k_1-k_2} \tag{4.49}$$

where $\mathbb{E}^{(\alpha)}_{k_1,k_2,q}$ is given in (4.43).

Corollary 4.6: The bivariate q-multiple Euler polynomials $\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)$ can be represented in the following two ways:

$$\begin{aligned}
&\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,y)\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}\mathbb{E}^{(\alpha)}_{k_1,k_2,q}(x,0)y^{n_1+n_2-k_1-k_2}\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{E}^{(\alpha)}_{k_1,k_2,q}(0,y)x^{n_1+n_2-k_1-k_2}. \qquad (4.50)
\end{aligned}$$

Corollary 4.7: In special, we get the following formulas by setting $x=0$ and $y=0$ in (4.50), respectively

$$\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(0,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1))}{2}}\mathbb{E}^{(\alpha)}_{k_1,k_2,q}y^{n_1+n_2-k_1-k_2}, \qquad (4.51)$$

$$\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,0)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{E}^{(\alpha)}_{k_1,k_2,q}x^{n_1+n_2-k_1-k_2}. \qquad (4.52)$$

Corollary 4.8: By setting $x=1$ and $y=1$ in (4.50), respectively, we get

$$\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(1,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{E}^{(\alpha)}_{k_1,k_2,q}(0,y), \qquad (4.53)$$

$$\mathbb{E}^{(\alpha)}_{n_1,n_2,q}(x,1)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}\mathbb{E}^{(\alpha)}_{k_1,k_2,q}(x,0). \qquad (4.54)$$

4.1.4 Bivariate q-Multiple Bernoulli-Euler Polynomials

In this section, we will take $A_q(t_1,t_2)$ in (3.1) as $\left(\frac{t_1}{e_q(t_1)-1}\right)^{\alpha}\left(\frac{2}{e_q(t_2)+1}\right)^{\alpha}$ to get bivariate q-multiple Bernoulli-Euler polynomials which are another example for bivariate q-multiple Appell polynomials. We will use definitions and properties of higher order q-Bernoulli polynomials and q-Euler polynomials from [19]. With the definition below we will approach to main properties and corollaries of these polynomials.

Definition 4.5: The generating function of bivariate q-multiple Bernoulli-Euler numbers $\mathbb{K}^{(\alpha)}_{n_1,n_2,q}$ is given as

$$\left(\frac{t_1}{e_q(t_1)-1}\right)^{\alpha}\left(\frac{2}{e_q(t_2)+1}\right)^{\alpha}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{K}^{(\alpha)}_{n_1,n_2,q}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}, \quad |t|<2\pi, \qquad (4.55)$$

and bivariate q-multiple Bernoulli-Euler polynomials $\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)$ in x,y of order α is defined by using the generating relation below

$$\left(\frac{t_1}{e_q(t_1)-1}\right)^{\alpha}\left(\frac{2}{e_q(t_2)+1}\right)^{\alpha} e_q(t_1x)e_q(t_2y)E_q(t_1y)E_q(t_2x)$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}, \quad |t|<2\pi. \tag{4.56}$$

Theorem 4.5: Let $\{\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)\}_{n_1,n_2=0}^{\infty}$ be a bivariate q-multiple Bernoulli-Euler polynomials . Then we have the following:

a. The bivariate q-multiple Bernoulli-Euler polynomials can be written as

$$\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)=\mathfrak{B}^{(\alpha)}_{n_1,q}(x,y)\mathfrak{E}^{(\alpha)}_{n_2,q}(y,x). \tag{4.57}$$

where $\mathfrak{B}^{(\alpha)}_{n,q}(x,y)$ and $\mathfrak{E}^{(\alpha)}_{n,q}(x,y)$ given as in (4.34) and (4.46), respectively.

b. For every $n_1+n_2\geq 1$, we have

$$D_{q,x}\left(\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)\right)=[n_1]_q\mathbb{K}^{(\alpha)}_{n_1-1,n_2,q}(x,y)+[n_2]_q\mathbb{K}^{(\alpha)}_{n_1,n_2-1,q}(qx,y) \tag{4.58}$$

$$D_{q,y}\left(\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)\right)=[n_2]_q\mathbb{K}^{(\alpha)}_{n_1,n_2-1,q}(x,y)+[n_1]_q\mathbb{K}^{(\alpha)}_{n_1-1,n_2,q}(x,qy). \tag{4.59}$$

Corollary 4.9: For all $x,y\in\mathbb{C}$, the bivariate q-multiple Bernoulli-Euler polynomials can be written explicitly as

$$\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\mathbb{K}^{(\alpha)}_{k_1,k_2,q}(x+y)_q^{n_1+n_2-k_1-k_2}. \tag{4.60}$$

where $\mathbb{K}^{(\alpha)}_{k_1,k_2,q}$ is given as in (4.55).

Corollary 4.10: The bivariate q-multiple Bernoulli-Euler polynomials $\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)$ can be represented in the following two ways:

$$
\begin{aligned}
&\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,y)\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}\mathbb{K}^{(\alpha)}_{k_1,k_2,q}(x,0)y^{n_1+n_2-k_1-k_2}\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{K}^{(\alpha)}_{k_1,k_2,q}(0,y)x^{n_1+n_2-k_1-k_2}. \qquad (4.61)
\end{aligned}
$$

Corollary 4.11: In fact, by setting $x=0$ and $y=0$ in (4.61), we obtain the following formulas, respectively

$$
\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(0,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1))}{2}}\mathbb{K}^{(\alpha)}_{k_1,k_2,q}y^{n_1+n_2-k_1-k_2}, \qquad (4.62)
$$

$$
\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,0)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{K}^{(\alpha)}_{k_1,k_2,q}x^{n_1+n_2-k_1-k_2}. \qquad (4.63)
$$

Corollary 4.12: We can get following equalities by setting $x=1$ and $y=1$ in (4.61), respectively

$$
\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(1,y)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}\mathbb{K}^{(\alpha)}_{k_1,k_2,q}(0,y), \qquad (4.64)
$$

$$
\mathbb{K}^{(\alpha)}_{n_1,n_2,q}(x,1)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}\mathbb{K}^{(\alpha)}_{k_1,k_2,q}(x,0). \qquad (4.65)
$$

4.2 Non-trivial Examples

In this section we will provide non-trivial example for univariate q-multiple Appell polynomials.

4.2.1 q-Multiple Hermite Polynomials

In this section we define q-multiple Hermite polynomials as an non-trivial example for univariate q-multiple Appell polynomials. We further obtain their recurrence relation.

Definition 4.6: The q-multiple Hermite polynomials $\mathbb{H}_{n_1,n_2}(x)$ is defined by following generating relation

$$G_q(t_1,t_2)e_q(\delta t_1 x)E_q(\delta t_2 x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathbb{H}_{n_1,n_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \tag{4.66}$$

where $G_q(t_1,t_2)$ has a series expansion

$$G_q(t_1,t_2) = \sum_{m=0}^{\infty} q^{-m(m-1)}\delta^m \frac{(t_1+t_2)_q^{2m}}{[2m]_q!!}. \tag{4.67}$$

Theorem 4.6 (Recurrence Relation): q-multiple Hermite polynomials satisfies the following recurrence relation

$$\begin{aligned}\mathbb{H}_{n_1+1,n_2}(\delta x) &= \delta(qn_1+n_2)\mathbb{H}_{n_1,n_2}(\delta x)\\ &\quad + \delta^2 x(qn_1+n_2)(q^{n_1}-1)\mathbb{H}_{n_1-1,n_2}(\delta x)\\ &\quad + \delta x \mathbb{H}_{n_1,n_2}(\delta x).\end{aligned} \tag{4.68}$$

We will give the proof by applying Theorem 2.2 from univariate q-multiple Appell polynomials.

Proof. To prove (4.68), first we will try to calculate $\frac{D_{q,t_1}(G_q(t_1,t_2))}{G_q(t_1,t_2)}$. Lets calculate the q-derivative with respect to t_1 of $(t_1+t_2)_q^n$

$$D_{q,t_1}(t_1+t_2)_q^n = [n]_q(qt_1+t_2)_q^{n-1}. \tag{4.69}$$

From q-shifted factorial (1.35) we can calculate the following equality

$$\begin{aligned}(qt_1+t_2)^{2m+1} &= (qt_1+t_2)(qt_1+qt_2)(qt_1+q^2t_2)\dots(qt_1+q^{2m}t_2)\\ &= (qt_1+t_2)q^{2m}(t_1+t_2)_q^{2m}.\end{aligned} \tag{4.70}$$

By using (4.69) and (4.70) it is easy to show that q-derivative of $G_q(t_1,t_2)$ with respect to t_1 can be calculated following way

$$
\begin{aligned}
D_{q,t_1}(G_q(t_1,t_2)) &= \sum_{m=0}^{\infty} q^{-m(m-1)}\delta^m \frac{D_{q,t_1}(t_1+t_2)_q^{2m}}{[2m]_q!!} \\
&= \sum_{m=0}^{\infty} q^{-m(m-1)}\delta^m \frac{[2m]_q(qt_1+t_2)_q^{2m-1}}{[2m]_q!!} \\
&= \sum_{m=0}^{\infty} q^{-m(m-1)}\delta^m \frac{(qt_1+t_2)_q^{2m-1}}{[2m-2]_q!!} \\
&= \sum_{m=0}^{\infty} q^{-m(m-1)}q^{-2m}\delta^m \delta \frac{(qt_1+t_2)_q^{2m+1}}{[2m]_q!!} \\
&= \sum_{m=0}^{\infty} q^{-m(m-1)}q^{-2m}\delta^m \delta \frac{(qt_1+t_2)q^{2m}(t_1+t_2)_q^{2m}}{[2m]_q!!} \\
&= \delta(qt_1+t_2)\sum_{m=0}^{\infty} q^{-m(m-1)}\delta^m \frac{(t_1+t_2)_q^{2m}}{[2m]_q!!} \\
&= \delta(qt_1+t_2)G_q(t_1,t_2) \qquad (4.71)
\end{aligned}
$$

and

$$
\frac{D_{q,t_1}(G_q(t_1,t_2))}{G_q(t_1,t_2)} = \delta(qt_1+t_2). \qquad (4.72)
$$

Taking q-derivative of (4.66) on the right hand side with respect to t_1 we will get

$$
\begin{aligned}
&D_{q,t_1}(G_q(t_1,t_2)e_q(\delta xt_1)E_q(\delta xt_2)) \\
&= D_{q,t_1}(G_q(t_1,t_2))e_q(\delta xt_1)E_q(\delta xt_2) + \delta x G_q(qt_1,t_2)e_q(\delta xt_1)E_q(\delta xt_2) \\
&= \frac{D_{q,t_1}G_q(t_1,t_2)}{G_q(t_1,t_2)}G_q(t_1,t_2)e_q(\delta xt_1)E_q(\delta xt_2) \\
&+ \delta x(q-1)t_1\frac{D_{q,t_1}G_q(t_1,t_2)}{G_q(t_1,t_2)}G_q(t_1,t_2)e_q(\delta xt_1)E_q(\delta xt_2) \\
&+ \delta x G_q(t_1,t_2)e_q(\delta xt_1)E_q(\delta xt_2) \\
&= \delta(qn_1+n_2)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{H}_{n_1,n_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&+ \delta^2 x(qn_1+n_2)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(q^{n_1}-1)\mathbb{H}_{n_1-1,n_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&+ \delta x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{H}_{n_1,n_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!} \qquad (4.73)
\end{aligned}
$$

and q-derivative of left hand side of (4.66) is

$$
\begin{aligned}
&D_{q,t_1}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{H}_{n_1,n_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)\\
&=\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{H}_{n_1,n_2}(\delta x)[n_1]_q\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&=\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{H}_{n_1,n_2}(\delta x)\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1-1]_q![n_2]_q!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathbb{H}_{n_1+1,n_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}.
\end{aligned}
\tag{4.74}
$$

By comparing coefficients of (4.73) and (4.74) we obtain (4.68).

□

Remark 4.1: Taking $q \to 1$ in (4.66), we can get generating function of simple multiple Hermite polynomials $H_{n_1,n_2}(\delta x)$ as

$$
e^{\frac{\delta}{2}(t_1+t_2)^2+\delta(t_1+t_2)x}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}H_{n_1,n_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!} \tag{4.75}
$$

and simple multiple Hermite polynomials satisfies the following recurrence relation

$$
H_{n_1+1,n_2}(\delta x)=\delta(n_1+n_2)H_{n_1,n_2}(\delta x)+\delta x H_{n_1,n_2}(\delta x). \tag{4.76}
$$

Chapter 5

2-ITERATED q-MULTIPLE APPELL POLYNOMIALS

The polynomial set $\{\mathscr{U}_{n_1,n_2}(x)\}$ is denoted by a single $\mathscr{U}$ symbol and $\mathscr{U}_{n_1,n_2}(x)$ is referred to as the $(n_1,n_2)^{th}$ part of $\mathscr{U}$. Let's consider two polynomial set $\{\mathscr{U}_{n_1,n_2}(x)\}$ and $\{\mathscr{V}_{n_1,n_2}(x)\}$ from $\mathfrak{A}(q)$, such as

$$\mathscr{U}_{n_1,n_2}(x)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q u_{n_1,n_2}(k_1,k_2)x^{k_1+k_2} \tag{5.1}$$

and

$$\mathscr{V}_{n_1,n_2}(x)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q v_{n_1,n_2}(k_1,k_2)x^{k_1+k_2} \tag{5.2}$$

where $\mathfrak{A}(q)$ denotes the multiple polynomial family. We can define the q-star operation with a "$*_q$" symbol whose $(n_1,n_2)^{th}$ component is

$$\left(\mathscr{U}*_q\mathscr{V}\right)_{n_1,n_2}=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q u_{n_1,n_2}(k_1,k_2)\frac{\mathscr{V}_{k_1,k_2}(x)}{q^{\frac{k_2(k_2-1)}{2}}}. \tag{5.3}$$

If λ is a real or complex number, then $\lambda\mathscr{U}$ is the polynomial set whose $(n_1,n_2)^{th}$ component is $\lambda\mathscr{U}_{n_1,n_2}(x)$. Evidently, we have

$$\mathscr{U}+\mathscr{V}=\mathscr{V}+\mathscr{U} \text{ for all } \mathscr{U},\mathscr{V}$$

$$(\lambda\mathscr{U}*_q\mathscr{V})=(\mathscr{U}*_q\lambda\mathscr{V})=\lambda(\mathscr{U}*_q\mathscr{V}).$$

It's clear that "$*_q$" is not a commutative operation. Let us denote $\mathscr{A}(q)$ as the subclass of all univariate q-multiple Appell polynomials. Taking $A_q(t_1,t_2)$ as 1 in (2.1) gives us identity element of $\mathscr{A}(q)$ as q-multiple Appell set $I=\{x^{n_1+n_2}\}$.

In the below theorems we will choose $\mathscr{U},\mathscr{V},\mathscr{Y}$ from $\mathscr{A}(q)$. Therefore we can denote these polynomials explicitly as

$$A_q(t_1,t_2)e_q(t_1x)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathscr{U}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},$$

$$B_q(t_1,t_2)e_q(t_1x)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathscr{V}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},$$

$$C_q(t_1,t_2)e_q(t_1x)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mathscr{Y}_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},$$

where

$$A_q(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},$$

$$B_q(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} b_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},$$

$$C_q(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}.$$

Then the proof of the following theorem is straight forward.

Theorem 5.1: Let $\mathscr{U},\mathscr{V},\mathscr{Y} \in \mathscr{A}(q)$, then

i. $\mathscr{U}+\mathscr{V} \in \mathscr{A}(q)$ if $A_q(0,0)+B_q(0,0) \neq 0$,

ii. $\mathscr{U}+\mathscr{V}$ belongs to the determining function $A_q(t_1,t_2)+B_q(t_1,t_2)$,

iii. $\mathscr{U}+(\mathscr{V}+\mathscr{Y}) = (\mathscr{U}+\mathscr{V})+\mathscr{Y}$.

Theorem 5.2: If $\mathscr{U},\mathscr{V},\mathscr{Y} \in \mathscr{A}(q)$ then the following properties hold:

i. $\mathscr{U} *_q \mathscr{V} \in \mathscr{A}(q)$,

ii. $\mathscr{U} *_q \mathscr{V} = \mathscr{V} *_q \mathscr{U}$,

iii. $\mathscr{U} *_q \mathscr{V}$ belongs to the determining $A_q(t_1,t_2)B_q(t_1,t_2)$,

iv. $\mathscr{U} *_q (\mathscr{V} *_q \mathscr{Y}) = (\mathscr{U} *_q \mathscr{V}) *_q \mathscr{Y}$.

Proof. All properties can easily follow from the following. In fact using (5.3) and the fact that univariate q-multiple Appell polynomials have explicit form as

$$\mathscr{U}_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_2(k_2-1)}{2}} a_{n_1-k_1,n_2-k_2} x^{k_1+k_2} \tag{5.4}$$

we have

$$\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \left(\mathscr{U} *_q \mathscr{V}\right)_{n_1,n_2} \frac{t_1^{n_1} t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q q^{\frac{k_2(k_2-1)}{2}} a_{n_1-k_1,n_2-k_2} \frac{\mathscr{V}_{k_1,k_2}(x)}{q^{\frac{k_2(k_2-1)}{2}}} \frac{t_1^{n_1} t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q a_{n_1-k_1,n_2-k_2} \mathscr{V}_{k_1,k_2}(x) \frac{t_1^{n_1} t_2^{n_2}}{[n_1]_q![n_2]_q!} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2} \frac{t_1^{n_1} t_2^{n_2}}{[n_1]_q![n_2]_q!} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \mathscr{V}_{k_1,k_2}(x) \frac{t_1^{k_1} t_2^{k_2}}{[k_1]_q![k_2]_q!} \\
&= A_q(t_1,t_2) B_q(t_1,t_2) e_q(t_1 x) E_q(t_2 x).
\end{aligned}$$

□

The following corollary can be derived from this theorem.

Corollary 5.1: Let $\mathscr{U} \in \mathscr{A}(q)$. Then there is a set $\mathscr{V} \in \mathscr{A}(q)$ such that

$$\mathscr{U} *_q \mathscr{V} = \mathscr{V} *_q \mathscr{U} = I$$

and this shows that $A(t_1,t_2)^{-1}$ is the generating function of $\mathscr{V}$.

Corollary 5.2: For any $\mathscr{U} \in \mathscr{A}(q)$ we can denote the $\mathscr{V} \in \mathscr{A}(q)$ in Corollary 5.1 as $\mathscr{U}^{-1}$ since star operation of $\mathscr{U}$ and $\mathscr{V}$ gives identity element.

Corollary 5.3: We can call the system $(\mathscr{A}(q), *_q)$ a commutative group from Theorem 5.2, Corollary 5.1 and Corollary 5.2.

Corollary 5.4: As an application of Corollary 5.1 we can obtain

$$
\begin{aligned}
x^{n_1+n_2} &= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}(-1)^{k_1+k_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\mathbb{M}_{n_1-k_1,n_2-k_2}(x)\\
&= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\mathbb{N}_{k_1,k_2}(x).
\end{aligned}
\tag{5.5}
$$

where $\mathbb{M}_{n_1,n_2}(x)$ and $\mathbb{N}_{n_1,n_2}(x)$ are defined as in (4.7) and (4.10), respectively.

Definition 5.1 (2-Iterated q-Multiple Appell Polynomials): Let

$$
\mathscr{U}_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}\begin{bmatrix}n_2\\k_2\end{bmatrix}q^{\frac{k_2(k_2-1)}{2}}a_{n_1-k_1,n_2-k_2}x^{k_1+k_2} \tag{5.6}
$$

and

$$
\mathscr{V}_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}\begin{bmatrix}n_2\\k_2\end{bmatrix}q^{\frac{k_2(k_2-1)}{2}}b_{n_1-k_1,n_2-k_2}x^{k_1+k_2} \tag{5.7}
$$

be two q-multiple Appell polynomials. 2-iterated q-multiple Appell polynomials are defined by

$$
\mathscr{A}^{[2]}_{n_1,n_2}(x) = \left(\mathscr{U} *_q \mathscr{V}\right)_{n_1,n_2} = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q a_{n_1-k_1,n_2-k_2}\mathscr{V}_{k_1,k_2}(x). \tag{5.8}
$$

It can from the proof of the Theorem 5.2 that the generating function of the 2-iterated q-multiple Appell polynomials is written as

$$
A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathscr{A}^{[2]}_{q,n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}. \tag{5.9}
$$

Example 5.1 (2-Iterated q-Multiple Generalized Hermite Polynomials): Taking $A(t_1,t_2) = E_q(\alpha_1 t_1)e_q(\alpha_2 t_2)$ and $B_q(t_1,t_2)e_q(t_1x)E_q(t_2x) = G_q(t_1,t_2)e_q(\delta t_1 x)E_q(\delta t_2 x)$, we can define 2-iterated q-multiple generalized Hermite polynomials as follows

$$
E_q(\alpha_1 t_1)e_q(\alpha_2 t_2)G_q(t_1,t_2)e_q(\delta t_1 x)E_q(\delta t_2 x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathscr{H}^{\alpha_1,\alpha_2,\delta}_{q,n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q} \tag{5.10}
$$

where

$$
\mathscr{H}^{\alpha_1,\alpha_2,\delta}_{q,n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}\begin{bmatrix}n_2\\k_2\end{bmatrix}q^{\frac{n_1-k_1(n_1-k_1-1)}{2}}\alpha_1^{n_1-k_1}\alpha_2^{n_2-k_2}\mathbb{H}_{k_1,k_2}(\delta x). \tag{5.11}
$$

Proof. By direct calculation and the consideration of (4.66), we can clearly have that

$$\begin{aligned}
&E_q(\alpha_1 t_1)e_q(\alpha_2 t_2)G_q(t_1,t_2)e_q(\delta t_1 x)E_q(\delta t_2 x)\\
&=\sum_{n_1=0}^{\infty} q^{\frac{n_1(n_1-1)}{2}}\alpha_1^{n_1}\frac{t_1^{n_1}}{[n_1]_q!}\sum_{n_2=0}^{\infty}\alpha_2^{n_2}\frac{t_2^{n_2}}{[n_2]_q!}\sum_{k_1=0}^{\infty}\sum_{k_1=0}^{\infty}\mathbb{H}_{k_1,k_2}(\delta x)\frac{t_1^{k_1}t_2^{k_2}}{[k_1]_q![k_2]_q!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix} n_1\\ k_1\end{bmatrix}\begin{bmatrix} n_2\\ k_2\end{bmatrix} q^{\frac{n_1-k_1(n_1-k_1-1)}{2}}\alpha_1^{n_1-k_1}\alpha_2^{n_2-k_2}\mathbb{H}_{k_1,k_2}(\delta x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}
\end{aligned} \tag{5.12}$$

by comparing (5.10)and (5.12), we can obtain explicit form of $\mathscr{H}_{q,n_1,n_2}^{\alpha_1,\alpha_2,\delta}(x)$. □

Remark 5.1: By taking $\lim_{q\to 1}\mathscr{H}_{q,n_1,n_2}^{\alpha_1,\alpha_2,\delta}(x)$, we can obtain multiple Hermite polynomials $\mathcal{H}_{n_1,n_2}^{\alpha_1,\alpha_2}(x)$ as

$$e^{\frac{\delta}{2}(t_1+t_2)^2+\delta(t_1+t_2)x+\alpha_1 t_1+\alpha_2 t_2}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathcal{H}_{n_1,n_2}^{\alpha_1,\alpha_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!} \tag{5.13}$$

from [17].

Theorem 5.3 (Recurrence Relation): 2-iterated q-multiple Appell polynomials satisfies the following recurrence relations

$$\begin{aligned}
&\mathscr{A}^{[2]}_{q,n_1+1,n_2}(x)\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\alpha_{n_1-k_1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\begin{bmatrix}k_1\\l_1\end{bmatrix}_q\begin{bmatrix}k_2\\l_2\end{bmatrix}_q(q^{n_1-k_1}-1)\alpha_{n_1-k_1-1,n_2-k_2}\\
&\times\beta_{k_1-l_1,k_2-l_2}\mathscr{A}^{[2]}_{q,l_1,l_2}(x)\\
&+\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\beta_{n_1-k_1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+x\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{l_1}^{k_1}\sum_{l_2}^{k_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\begin{bmatrix}k_1\\l_1\end{bmatrix}_q\begin{bmatrix}k_2\\l_2\end{bmatrix}_q(q^{n_1-k_1}-1)(q^{k_1-l_1}-1)\\
&\times\alpha_{n_1-k_1-1,n_2-k_2}\beta_{k_1-l_1-1,k_2-l_2}\mathscr{A}^{[2]}_{q,l_1,l_2}(x)\\
&+x\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q(q^{n_1-k_1}-1)\alpha_{n_1-k_1-1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+x\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q(q^{n_1-k_1}-1)\beta_{n_1-k_1-1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+x\mathscr{A}^{[2]}_{q,n_1,n_2}(x) \qquad (5.14)
\end{aligned}$$

$$
\begin{aligned}
&\mathscr{A}^{[2]}_{q,n_1,n_2+1}(x)\\
&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\delta_{n_1-k_1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\begin{bmatrix}k_1\\l_1\end{bmatrix}_q\begin{bmatrix}k_2\\l_2\end{bmatrix}_q(q^{n_2-k_2}-1)\delta_{n_1-k_1-1,n_2-k_2}\\
&\times\gamma_{k_1-l_1,k_2-l_2}\mathscr{A}^{[2]}_{q,l_1,l_2}(x)\\
&+\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\gamma_{n_1-k_1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+x\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{l_1}^{k_1}\sum_{l_2}^{k_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\begin{bmatrix}k_1\\l_1\end{bmatrix}_q\begin{bmatrix}k_2\\l_2\end{bmatrix}_q(q^{n_2-k_2}-1)(q^{k_2-l_2}-1)q^{l_2}\\
&\times\delta_{n_1-k_1,n_2-k_2-1}\gamma_{k_1-l_1,k_2-l_2-1}\mathscr{A}^{[2]}_{q,l_1,l_2}(x)\\
&+x\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q(q^{n_2-k_2}-1)q^{k_2}\delta_{n_1-k_1,n_2-k_2-1}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+x\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q q^{n_2-k_2}-1)q^{k_2}\gamma_{n_1-k_1,n_2-k_2-1}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\\
&+xq^{n_2}\mathscr{A}^{[2]}_{q,n_1,n_2}(x).
\end{aligned}
\tag{5.15}
$$

where the generating functions of α_{n_1,n_2}, β_{n_1,n_2}, δ_{n_1,n_2} and γ_{n_1,n_2} are given as

$$
\frac{D_{q,t_1}A_q(t_1,t_2)}{A_q(t_1,t_2)}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\alpha_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},\tag{5.16}
$$

$$
\frac{D_{q,t_1}B_q(t_1,t_2)}{B_q(t_1,t_2)}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\beta_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},\tag{5.17}
$$

$$
\frac{D_{q,t_2}A_q(t_1,t_2)}{A_q(t_1,t_2)}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\delta_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!},\tag{5.18}
$$

$$
\frac{D_{q,t_2}B_q(t_1,t_2)}{B_q(t_1,t_2)}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\gamma_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}.\tag{5.19}
$$

Proof. Taking q-derivative with respect to t_1 on the left side of (5.9), we get

$$D_{q,t_1}(A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x))$$

$$= D_{q,t_1}(A_q(t_1,t_2))B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$+ (A_q(qt_1,t_2)D_{q,t_1}(B_q(t_1,t_2))e_q(t_1x)E_q(t_2x) + xB_q(qt_1,t_2)e_q(qt_1x)E_q(t_2x)$$

$$= \frac{D_{q,t_1}(A_q(t_1,t_2))}{A_q(t_1,t_2)}A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$+ (q-1)t_1\frac{D_{q,t_1}(A_q(t_1,t_2))}{A_q(t_1,t_2)}\frac{D_{q,t_1}(B_q(t_1,t_2))}{B_q(t_1,t_2)}A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$+ \frac{D_{q,t_1}(B_q(t_1,t_2))}{B_q(t_1,t_2)}A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$+ x(q-1)^2t_1^2\frac{D_{q,t_1}(A_q(t_1,t_2))}{A_q(t_1,t_2)}\frac{D_{q,t_1}(B_q(t_1,t_2))}{B_q(t_1,t_2)}A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$+ x(q-1)t_1\frac{D_{q,t_1}(A_q(t_1,t_2))}{A_q(t_1,t_2)}A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$+ x(q-1)t_1\frac{D_{q,t_1}(B_q(t_1,t_2))}{B_q(t_1,t_2)}A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$+ A_q(t_1,t_2)B_q(t_1,t_2)e_q(t_1x)E_q(t_2x)$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\alpha_{n_1-k_1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q[n_2]_q}$$

$$+ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\begin{bmatrix}k_1\\l_1\end{bmatrix}_q\begin{bmatrix}k_2\\l_2\end{bmatrix}_q(q^{n_1-k_1}-1)\alpha_{n_1-k_1-1,n_2-k_2}$$

$$\times \beta_{k_1-l_1,k_2-l_2}\mathscr{A}^{[2]}_{q,l_1,l_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q[n_2]_q}$$

$$+ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\beta_{n_1-k_1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q[n_2]_q}$$

$$+ x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{l_1}^{k_1}\sum_{l_2}^{k_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q\begin{bmatrix}k_1\\l_1\end{bmatrix}_q\begin{bmatrix}k_2\\l_2\end{bmatrix}_q(q^{n_1-k_1}-1)\alpha_{n_1-k_1-1,n_2-k_2}$$

$$\times (q^{k_1-l_1}-1)\beta_{k_1-l_1-1,k_2-l_2}\mathscr{A}^{[2]}_{q,l_1,l_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q[n_2]_q}$$

$$+ x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q(q^{n_1-k_1}-1)\alpha_{n_1-k_1-1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q[n_2]_q}$$

$$+ x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_q\begin{bmatrix}n_2\\k_2\end{bmatrix}_q(q^{n_1-k_1}-1)\beta_{n_1-k_1-1,n_2-k_2}\mathscr{A}^{[2]}_{q,k_1,k_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q[n_2]_q}$$

$$+ x\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathscr{A}^{[2]}_{q,n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q[n_2]_q} \tag{5.20}$$

and the q-derivative of (5.9)'s right hand side with respect to t_1 is

$$
\begin{aligned}
&D_{q,t_1}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathscr{A}_{q,n_1,n_2}^{[2]}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\right)\\
&=\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathscr{A}_{q,n_1,n_2}^{[2]}(x)[n_1]_q\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1]_q![n_2]_q!}\\
&=\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\mathscr{A}_{q,n_1,n_2}^{[2]}(x)\frac{t_1^{n_1-1}t_2^{n_2}}{[n_1-1]_q![n_2]_q!}\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathscr{A}_{q,n_1+1,n_2}^{[2]}(x)\frac{t_1^{n_1}t_2^{n_2}}{[n_1]_q![n_2]_q!}
\end{aligned} \tag{5.21}
$$

Comparing coefficients of (5.20) and (5.21) gives us (5.14).

The proof of (5.15) is done in similar manner. □

REFERENCES

[1] Aceto, L., Malonek, H. R., & Tomaz, G. (2015). A unified matrix approach to the representation of Appell polynomials. *Integral Transforms and Special Functions, 26*(6), 426-441.

[2] Al-Salam, W. A. (1967). q-Appell polynomials. *Annali di Matematica Pura ed Applicata, 77(1)*, 31-45.

[3] Al-Salam, W. A., & Carlitz, L. (1965). Some Orthogonal q-Polynomials. *Mathematische Nachrichten, 30*(1-2), 47-61.

[4] Appell, P. (1880). Sur une classe de polynômes. In *Annales scientifiques de l'École normale supérieure* (Vol. 9, pp. 119-144).

[5] Aptekarev, A. I. (1998). Multiple orthogonal polynomials. *Journal of Computational and Applied Mathematics, 99*(1-2), 423-447.

[6] Bretti, G., & Ricci, P. E. (2004). Multidimensional extensions of the Bernoulli and Appell polynomials. *Taiwanese Journal of Mathematics, 8*(3), 415-428.

[7] Bretti, G., Cesarano, C., & Ricci, P. E. (2004). Laguerre-type exponentials and generalized Appell polynomials. *Computers & Mathematics with Applications, 48*(5-6), 833-839.

[8] Costabile, F. A. (2019). *Modern Umbral Calculus: An Elementary Introduction with Applications to Linear Interpolation and Operator Approximation Theory* (Vol. 72). Walter de Gruyter GmbH & Co KG.

[9] Costabile, F. A., & Longo, E. (2010). A determinantal approach to Appell polynomials. *Journal of computational and applied mathematics, 234*(5), 1528-1542.

[10] Euler, L. (1748). *Introductio in analysin infinitorum* (Vol. 2). MM Bousquet.

[11] Kac, V., & Cheung, P. (2001). *Quantum calculus*. Springer Science & Business Media.

[12] Keleshteri, M. E., & Mahmudov, N. I. (2015). A study on q-Appell polynomials from determinantal point of view. *Applied Mathematics and Computation, 260*, 351-369.

[13] Khan, S., & Raza, N. (2013). 2-Iterated Appell polynomials and related numbers. *Applied Mathematics and Computation, 219(17)*, 9469-9483.

[14] Khan, S., & Riyasat, M. (2015).A determinantal approach to Sheffer–Appell polynomials via monomiality principle. *Journal of Mathematical Analysis and Applications, 421*(1), 806-829.

[15] Khan, S., & Riyasat, M. (2016). Differential and integral equations for the 2-iterated Appell polynomials. *Journal of Computational and Applied*

Mathematics, 306, 116-132.

[16] Khan, S., Raza, N., & Ali, M. (2017). Finding mixed families of special polynomials associated with Appell sequences. *Journal of Mathematical Analysis and Applications, 447*(1), 398-418.

[17] Lee, D. (2011). On multiple Appell polynomials. *Proceedings of the American Mathematical Society, 139(6)*, 2133-2141.

[18] Lee, D. W. (2007). Properties of multiple Hermite and multiple Laguerre polynomials by the generating function. *Integral Transforms and Special Functions, 18*(12), 855-869.

[19] Mahmudov, N. I. (2013). *On a class of q-Bernoulli and q-Euler polynomials. Advances in Difference Equations, 2013(1)*, 1-11.

[20] Mahmudov, N. I. (2014). Difference equations of q-Appell polynomials. *Applied Mathematics and Computation, 245*, 539-543.

[21] Mahmudov, N. I., & Momenzadeh, M. (2014). On a Class of-Bernoulli,-Euler, and-Genocchi Polynomials. In *Abstract and Applied Analysis* (Vol. 2014). Hindawi.

[22] Özarslan, M. A. (2020). Approximation properties of Jain-Appell operators. *Applicable Analysis and Discrete Mathematics, 14*(3), 654-669.

[23] Özarslan, M. A. , & Gaboury, S. (2014). Srivastava–Pintér theorems for 2D-Appell polynomials and their applications. *Mathematical Methods in the Applied Sciences, 37*(15), 2198-2210.

[24] Özarslan, M. A., & Yılmaz, B. (2014). A set of finite order differential equations for the Appell polynomials. *Journal of Computational and Applied Mathematics, 259*, 108-116.

[25] Rainville, E. D. (1971). Special functions.

[26] Riyasat, M., Khan, S., & Nahid, T. (2017). q-difference equations for the composite 2D q-Appell polynomials and their applications. *Cogent Mathematics & Statistics, 4*(1), 1376972.

[27] Roman, S. (2005). *The umbral calculus* (pp. 437-472). Springer New York.

[28] Sheffer, I. M. (1945). Note on Appell polynomials. *Bulletin of the American Mathematical Society, 51*(10), 739-744.

[29] Srivastava, H. M. (1982). Some characterizations of appell and q-appell polynomials. *Annali di Matematica Pura ed Applicata, 130*(1), 321-329.

[30] Srivastava, H. M., Khan, S., & Riyasat, M. (2019). q-Difference equations for the 2-iterated q-Appell and mixed type q-Appell polynomials.*Arabian Journal of Mathematics, 8*(1), 63-77.

[31] Srivastava, H. M., Özarslan, M. A., &1Yaşar, B. Y. (2019). Difference equations for a class of twice-iterated Δ_h-Appell sequences of polynomials. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113*(3), 1851-1871.

[32] Srivastava, H. M., Özarslan, M. A., & Yılmaz, B. (2014). Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials. *Filomat, 28*(4), 695-708.

[33] Srivastava, H. M., Ricci, P. E., & Natalini, P. (2019). A family of complex Appell polynomial sets. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113*(3), 2359-2371.

[34] Srivastava, H. M., Yasmin, G., Muhyi, A., & Araci, S. (2019). *Certain results for the twice-iterated 2D q-Appell polynomials. Symmetry, 11(10)*, 1307.

[35] Srivastava, H. M., Yaşar, B. Y., & Özarslan, M. A. (2019). A class of big (p, q)-Appell polynomials and their associated difference equations. *Filomat, 33*(10), 3085-3121.

[36] Tempesta, P. (2008). On Appell sequences of polynomials of Bernoulli and Euler type. *Journal of mathematical analysis and applications, 341*(2), 1295-1310.

[37] Varma, S., Yaşar, B. Y., & Özarslan, M. A. (2019). Hahn–Appell polynomials and their d-orthogonality. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113*(3), 2127-2143.

[38] Yasmin, G., & Muhyi, A. (2019). Certain results of hybrid families of special polynomials associated with Appell sequences. *Filomat, 33*(12), 3833-3844.