(p,q)-Hahn Difference Operator

Duygu Malyalı

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Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazım Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Asst. Prof. Dr. Arran Fernandez Co-Supervisor	Prof. Dr. Mehmet Ali Özarslan Supervisor

Examining Committee

1. Prof. Dr. Hüseyin Aktuğlu
2. Prof. Dr. Mehmet Ali Özarslan
3. Assoc. Prof. Dr. Cem Kaanoğlu
4. Asst. Prof. Dr. Arran Fernandez
5. Asst. Prof. Dr. Müge Saadetoğlu

ABSTRACT

One of the main starting point for the theory of calculus is the differentiation operation, which is defined as follows. Firstly, divide the difference of two function values by the difference of the corresponding two arguments, and then take the limit as the two arguments converge to each other. The result of this limit is called the derivative of the original function.

Many variants of this basic operation have been proposed, giving rise to different theories and types of calculus. In this thesis, I will study some particular variants in which the limiting process is omitted but the two arguments in the quotient expression are linear functions of each other. The most basic one is the q-calculus (or quantum calculus), which is a particular case of both the (q,ω)-calculus (or Hahn calculus) and the (p,q)-calculus, which are the both special cases of the new type called (p,q)-Hahn calculus.

These approaches give more discrete theories than the original calculus, more applicable to quantum physics. But a lot of the structure remains the same: in all cases there are derivatives, integrals, product and chain rules, exponential and Appell functions. In this thesis, I will study important properties and special functions associated with each of these three known types of calculus, and finally, I introduce the new (p,q)-Hahn Calculus.

Keywords: q-calculus or quantum calculus, q,ω-calculus or Hahn Calculus, (p,q)-

calculus, (p,q)-Hahn Calculus, Exponential Functions, Appell Polynomials.

ÖZ

Analizin ana başlangıç noktalarından biri türev işlemdir ve şu şekilde tanımlanır. İlk olarak, 2 fonksiyonun farkını eş 2 argümanın farkına bölünür ve sonra bu iki argüman birbirine yaklaşana kadar limit alınır. Elde edilen limitin sonucuna da orjinal fonksiyonun türevi denir.

Farklı teorilere ve analiz tiplerine yol açan bu temel işlemin bir çok değişik biçimi sunulmuştur. Bu tezde, limit sürecinin atlandığı üç özel değişken üzerinde çalışacağım. Ancak bölüm kısmında yer alan 2 argüman birbirlerinin lineer fonksiyonlarıdır. En temel olanı ise q-calculusdur (veya quantum calculus)ve hem (q,ω)-calculus (veya Hahn calculus) hem de the (p,q)-calculus'un özel durumudur.

Bahsedilen 3 yaklaşım orijinal analizden ziyade ayrık matematiğe çekilebilir ve uygulanabilirlik açısından kuantum fiziğine daha yakındır. Fakat 3 durumda da, türevler, integraller, çarpım and zincir kuralları, üslü ve Appell fonksiyonları vardır. Bu tezde, analizin 3 tipi olan q-calculus,(q,ω)-calculus ve (p,q)-calculus'un önemli özellikleri ve özel fonskiyonlarla ilişkilerini çalışacağım ve son olarak, yeni elde ettiğim ve geliştirdiğim (p,q)-Hahn Calculus'un özelliklerini çalışacağım.

Anahtar kelimeler: q-calculus veya quantum calculus, q,ω-calculus veya Hahn Calculus, (p,q)-calculus, (p,q)-Hahn Calculus, Üstel Fonksiyonlar, Appell Polinomları.

DEDICATION

To My Family and Grandmother

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Chapter 1

INTRODUCTION

The minor subjects of this thesis are q-calculus (quantum calculus), q,ω-calculus (Hahn calculus) and (p,q)-calculus. This thesis is generally based on the paper [1–3].

From [4], in the 17. century, the theory of differential and integral calculus is studied for the first time by Newton and Leibniz. In their study, $f'(x)$ is defined by

$$f'(x) = \lim_{z \to 0} \frac{f(x+z) - f(x)}{z}.$$

Now, consider the following:

$$f'(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}.$$

This formula is equivalent to the following known derivative

$$\frac{f(x + (q-1)x) - f(x)}{(q-1)x}.$$

In the 20th century, F.H. Jackson studied this derivative and many of its results. Then Jackson defined the q-derivative as follows:

$$\mathcal{D}_q f(x) = \frac{f(qx) - f(x)}{qx - x}, \qquad q \in (0,1).$$

This q-derivative can be applied to functions that may not be differentiable. And it reduces to the ordinary derivative when $q \to 1$:

$$\lim_{q\to 1} \mathcal{D}_q f(x) = f'(x).$$

In the reference of [5], the Hahn difference operator $\mathcal{D}_{q,\omega}$ was defined in 1949 by Wolfgang Hahn. It is like differentiation with two extra parameters q, ω. It may be seen as the combination of the forward difference operator together with the q-difference operator. Combining the ideas of these two operators, namely

$$\Delta_\omega f(x) = \frac{f(x+\omega) - f(x)}{(x+\omega) - x},$$

and

$$\mathcal{D}_q f(x) = \frac{f(qx) - f(x)}{qx - x},$$

The Hahn difference operator is defined as

$$\mathcal{D}_{q,\omega} f(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x + \omega}.$$

This operator appears in many references such as [6] and [7]. Its right inverse is Jackson–Nörlund integration, which was introduced by Aldwoah [8–10]. This is a generalization of both the inverse of $\mathcal{D}_q$ and also the inverse of Δ_ω [9, 10].

The functions $E_{q,\omega}$ and $e_{q,\omega}$ are the Hahn equivalents of exponential functions [1]. I also studied new Hahn exponential-type functions from [11] which are called non-parametric.

The (p,q)-integers are defined in the reference [12], generalized q-calculus and used to represent certain quantum algebras in the reference [13].

From [14], recently (p,q)-calculus has also been applied in the theory of approximation. Let $p,q \in \mathbb{R}$ or $\mathbb{C}$. The (p,q)-derivative of $f(x)$ is defined by

$$\mathcal{D}_{(p,q)} f(x) = \frac{f(px) - f(qx)}{(p-q)x}.$$

Chapter 5 includes new operator, (p,q)-Hahn where we call it derivative and it is defined by

$$\mathcal{D}_{(p,q),\omega} f(x) = \frac{f(qx+\omega) - f(px)}{(q-p)x+\omega}.$$

In this thesis, there are six chapters. Introduction is the first Chapter which includes the aims, background information and literature reviews of my thesis. The second Chapter is about q-calculus definition and properties, including the big q-Appell and q-Appell polynomials. Chapter 3 contains the Hahn difference Operator, its theorems, definitions and properties, including the right inverse of the Hahn difference operator (Jackson–Nörlund Integration), Hahn exponential and trigonometric functions, and (q,ω)-Appell polynomials. Chapter 4 is about (p,q)-calculus; we studied (p,q)-derivative, (p,q)-integral, (p,q)-exponential functions, (p,q)-Appell polynomials and big (p,q)-Appell polynomials. In Chapter 5, we obtained the new (p,q)-Hahn difference operator and its properties. The Chapter 6 is the conclusion and it contains the main aim and results of my thesis.

Chapter 2

q-CALCULUS

This section is generally composed of references [2, 15]. This section is about the quantum calculus. This operator is called Jackson's q-difference operator or q-derivative and it is symbolized by $\mathcal{D}_q f(t)$ or $\frac{d_q f(t)}{d_q t}$ when applied to a function of f. The function f is defined on a q-geometric set of $\mathbb{A}$ which is subset of $\mathbb{R}$ or $\mathbb{C}$. The q-derivative is defined by [2]

$$\mathcal{D}_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}, \qquad t, qt \in \mathbb{A} \tag{2.1}$$

where $0 < q < 1$.

If we take limit when $q \to 1$

$$\lim_{q \to 1} \mathcal{D}_q f(t) = \lim \frac{f(qt) - f(t)}{(q-1)t} = \frac{df(t)}{dt} \tag{2.2}$$

we can obtain classical derivative of f.

In [15], the following obvious properties has shown

1) $\mathcal{D}_q$ is a linear operator, so

$$\mathcal{D}_q (f+g)(t) = \mathcal{D}_q f(t) + \mathcal{D}_q g(t). \tag{2.3}$$

2) If f is q-differentiable at t, then $f(qt) = f(t) - \mathcal{D}_q f(t)(q-1)t$.

3) If f is q-differentiable, then f is continuous.

Let f, g be functions which are q-differentiable at $t \in I$.

The following algebraic property of q-derivatives may be called the product rule:

$$\mathcal{D}_q(fg)(t) = g(t)\mathcal{D}_q[f(t)] + f(qt)\mathcal{D}_q[g(t)]. \tag{2.4}$$

By symmetry, if we interchange f with g, we can obtain that,

$$\mathcal{D}_q(fg)(t) = f(t)\mathcal{D}_q[g(t)] + g(qt)\mathcal{D}_q[f(t)]. \tag{2.5}$$

The quotient rule for q-derivatives is

$$\mathcal{D}_q\left(\frac{f}{g}\right)(t) = \frac{g(t)\mathcal{D}_q f(t) - f(t)\mathcal{D}_q g(t)}{g(t)g(qt)}, \tag{2.6}$$

where $g(t)g(qt) \neq 0$.

Example 2.0.1. Assume that $z \neq 0$ and take q-derivative of $f(z) = z^n$

$$\mathcal{D}_q z^n = \frac{(qz)^n - z^n}{qz - z} = \frac{(q^n - 1)}{(q-1)} z^{n-1}.$$

From this solution, $[n]_q$ is defined in [16] as follows:

$$[n]_q = 1 + q + q^2 + q^3 + \ldots + q^{n-1} = \frac{(q^n - 1)}{(q-1)} \tag{2.7}$$

and it is the q-analogue of n.

Finally, we can write q-derivative as follows:

$$\mathcal{D}_q z^n = [n]_q z^{n-1}. \tag{2.8}$$

In addition, $[n]_q!$ as follows:

$$[n]_q! = \begin{cases} [1]_q[2]_q...[n]_q, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

Remark 2.0.1. It may happen that $\mathcal{D}_q f(t)$, exists for a function g without being differentiable or even continuous at zero. For instance: $f : [0,1] \to R$ defined by

$$f(t) = \begin{cases} 1, & \text{if } t = \frac{1}{\sqrt{n}}, n \in \mathbb{N} \\ t, & \text{otherwise.} \end{cases}$$

Here $f(0) = 0$ but there is no $\lim_{t\to 0} f(t)$ so f is not continuous at $t = 0$.

As a result:

$$\mathcal{D}_q f(0) = \lim_{t\to\infty} \frac{f(tq) - f(0)}{tq} = \lim_{t\to\infty} \frac{tq}{tq} = 1.$$

Theorem 2.0.1 (Chain Rule for q-derivative [2]). *Firstly, there doesn't exist a general chain rule for q-derivatives. However if the differentiation of a function of the form $g(z(t))$ and $z(t)$ is equals to at^b, (a,b) are constants then chain rule exists for q-derivatives.*

Consider,

$$\begin{aligned} \mathcal{D}_q [g(z(t))] = \mathcal{D}_q \left[g(at^b) \right] &= \frac{g(aq^b t^b) - g(at^b)}{qt - t} \\ &= \frac{g(aq^b t^b) - g(at^b)}{aq^b t^b - at^b} . \frac{aq^b t^b - at^b}{qt - t} \\ &= \frac{g(q^b) - g(z)}{q^b - z} . \frac{z(qt) - z(t)}{qt - t}. \end{aligned}$$

Then we can obtain that

$$\mathcal{D}_q g\,(z(t)) = \left(\mathcal{D}_{q^b} g\right)(z(t)).\mathcal{D}_q z(t).$$

Firstly, let's give the definition of Taylor's Formula. In the classical calculus, an analytic function $f(t)$ has power series around $t = a$ as

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(t-a)^n}{n!}.$$

Theorem 2.0.2 (Generalized Taylor's Formula for Polynomials) [2]). *Let $a \in \mathbb{N}$, D_q be a linear map acting on the vector space of polynomials , and $(P_n(t))$ be a polynomial sequence such that*

1) $P_0(a) = 1$, $P_n(a) = 0$, $\quad \forall n > 0$;

2) $deg(P_n) = n$;

3) $D_q P_n(t) = P_{n-1}(t)$, $\quad \forall n > 0$, $D_q(1) = 0$

Then there is a generalized Taylor formula as follows:

$$f(t) = \sum_{n=0}^{N} (D_q^n f)(a) P_n(t). \tag{2.9}$$

Proof. Assume that J is the space of all polynomials of degree $\leq N$. From condition 2, the polynomials $P_0(t), P_1(t), P_2(t), ...P_N(t)$ are linearly independent and there is a rapid increase in degrees. However they create a basis for J; i.e., as a sample we can give any polynomial $f(t) \in J$ expressions as

$$f(t) = \sum_{n=0}^{N} c_k P_k(t), \tag{2.10}$$

where c_k is constant.

Interchanging with t and a then if we use condition 1 we can get $c_0 = f(a)$ as a result. After that, operator D_q continuously applied on both sides n times, where $1 \leq n \leq N$. Then, using condition 2 and 3

$$(D_q^n f)(t) = \sum_{k=0}^{N} c_k D_q^n P_k(t) = \sum_{k=0}^{N} c_k P_{k-n}(t).$$

Then putting $t = a$ and from condition 1, we have

$$c_n = (D_q^n f)(a), \quad 0 \leq n \leq N$$

Finally, we can obtain (2.10) from (2.9). □

2.1 q-Analogue of Power Function and q-Derivatives of Binomials

This section is composed of references [2]. Assume $\mathcal{D}_q$ is an operator acting linearly on the vector space of polynomials. Let a sequence of polynomials $P_0(t), P_1(t), P_2(t), \ldots$ satisfies the three conditions of theorem 2.0.2 . If $a = 0$, then we can choose,

$$P_n(t) = \frac{t^n}{[n]_q!}.$$

Using the result (2.8) for $n \geq 1$, we can write that

$$\mathcal{D}_q P_n(t) = \frac{\mathcal{D}_q t^n}{[n]_q!} = \frac{t^{n-1}}{[n-1]!} = P_{n-1}(t).$$

If $a \neq 0$ then $P_n(t)$ is not simply $\frac{(t-a)^n}{n!}$; for example,

$$\mathcal{D}_q \frac{(t-a)^2}{2!} \neq (t-a).$$

Now set

$$P_0(t) = 1.$$

In order that $\mathcal{D}_q P_1(t) = 1$ and $P_1(a) = 0$, we should have

$$P_1(t) = t - a.$$

In order that $\mathcal{D}_q P_2(t) = t - a$ and $P_2(a) = 0$, we should have

$$P_2(t) = \frac{t^2}{[2]} - at - \frac{a^2}{[2]} - a^2 = \frac{(t-a)(t-qa)}{[2]}.$$

Similarly for $n = 3$,

$$P_3(t) = \frac{(t-a)(t-qa)(t-q^2a)}{[2][3]}.$$

Finally, we can obtain that

$$P_n(t) = \frac{(t-a)}{[n]_q!}(t-qa)...(t-q^{n-1}a)$$

and when $a = 0$ we can obtain that

$$P_n(t) = \frac{t^n}{[n]_q!}.$$

.

Definition 2.1.1 (The q-Binomial formula, The q-polynomial coefficient, The q-factorial [17]). The q-factorial is given as

$$(a;q)_n = \begin{cases} 1, & n = 0; \\ \prod_{j=0}^{n-1}(1-q^j a), & n \geq 1; \\ \prod_{j=0}^{\infty}(1-q^j a), & n = \infty. \end{cases}$$

The q-binomial formula is defined by

$$(1-a)_q^n = (a;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)}(-1)^k a^k.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{((q;q)_{n-k}q;q)_k}, \qquad (k \leqslant n, \quad n,k \in \mathbb{N}). \tag{2.11}$$

Definition 2.1.2 ([2]). The q-analogue of $(t-a)^n$ is the polynomial

$$(t-a)_q^n = \begin{cases} 1, & \text{if } n = 0, \\ (t-a)(t-qa)...(t-q^{n-1}a), & \text{if } n \geq 1. \end{cases}$$

We note that,

$$(t-a)_q^{m+n} \neq (t-a)_q^m (t-a)_q^n.$$

Indeed,

$$(t-a)_q^{m+n} = (t-a)(t-qa)...(t-q^{m+n-1}a)$$
$$= \left((t-a)(t-qa)...(t-q^{n-1}(q^m a))\right).$$

The affirmative result is

$$(t-a)_q^{m+n} = (t-a)_q^m (t-q^m a)_q^{m+n}. \tag{2.12}$$

Putting $m=-n$, we get

$$(t-a)_q^{-n} = \frac{1}{(t-q^{-n}a)_q^n}.$$

So that $(t-a)_q^n$ is now defined for any integer n.

Theorem 2.1.1 ([2]). *q-derivative of $(t-a)_q^n$ is given by*

$$\mathcal{D}_q(t-a)_q^n = [n]_q (t-a)_q^{n-1}, \qquad n \in \mathbb{Z}. \tag{2.13}$$

Proof. This theorem will be proven in 3 steps.

Fistly, Let's use mathematical induction.

Let take $n=1$ then $\mathcal{D}_q(t-a)_q^n = 1$ is true.

Then suppose that $n=k$ case is true so, $\mathcal{D}_q(t-a)_q^k = [k]_q(t-a)_q^{k-1}$. Now, we will prove that for $n = k+1$ is true:

$$\begin{aligned}
\mathcal{D}_q(t-a)_q^{k+1} &= \mathcal{D}_q\left[(t-a)(t-qa)...(t-q^{k-1}a)(t-q^k a)\right] \\
&= \mathcal{D}_q\left[(t-a)_q^k(t-q^k a)\right] \\
&= (t-a)_q^k + (qt-q^k a)\mathcal{D}_q(t-a)_q^k \\
&= (t-a)_q^k + (qt-q^k a)[k]_q(t-a)_q^{k-1} \\
&= (t-a)_q^k + q[k]_q(t-a)_q^{k-1}(t-q^{k-1}a) \\
&= (t-a)_q^k\,(1+q[k]) \\
&= [k+1]_q(t-a)_q^k.
\end{aligned}$$

Secondly, since $[0]_q = 0$ it is clear for $n = 0$.

Thirdly, assume that $n = -1, -2, \dots$. Then if $n = -n_1 < 0$, we get

$$(t-a)_q^k = \frac{1}{(t-q^{-n_1}a)_q^{n_1}}.$$

Taking q-derivative,

$$
\begin{aligned}
\mathcal{D}_q(t-a)_q^k &= \mathcal{D}_q \frac{1}{(t-q^{-n_1}a)_q^{n_1}} \\
&= -\frac{\mathcal{D}_q(t-q^{-n_1}a)_q^{n_1}}{(t-q^{-n_1}a)_q^{n_1}.(qt-q^{-n_1}a)_q^{n_1}} \\
&= -\frac{[n_1]_q(t-q^{-n_1}a)_q^{n_1-1}}{(t-q^{-n_1}a)_q^{n_1}.(qt-q^{-n_1}a)_q^{n_1}} \\
&= -\frac{[n_1]_q(t-q^{-n_1}a)_q^{n_1-1}}{(t-q^{-1}a).(t-q^{-n_1}a)_q^{n_1-1}(qt-q^{-n_1}a)_q^{n_1}} \\
&= -[n_1]_q \frac{1}{q^{n_1}(t-q^{-1}a)(t-q^{-2}a)\ldots(t-q^{-n_1-1}a)} \\
&= -[n_1]_q \frac{1}{(t-q^{-n_1-1}a)_q^{n_1+1}} \\
&= [n]_q(t-a)_q^{n-1}.
\end{aligned}
$$

□

2.2 q-Exponential and q-Trigonometric Functions

This section is generally composed of reference [2].

Definition 2.2.1. The two q-analogues of exponential functions are given by

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, \tag{2.14}$$

$$E_q(x) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_q!} = (-(1-q)x;q)_\infty, \tag{2.15}$$

where $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

Properties of q-exponentials [2]:

These q-analogues (2.14) and (2.15) satisfy:

$$e_q(x)E_q(-x) = 1. \tag{2.16}$$

$$e_q(x) = E_{\frac{1}{q}}(x). \tag{2.17}$$

$$e_{\frac{1}{q}}(x) = E_q(x). \tag{2.18}$$

They are q-equivalents of the original exponential function since

$$\lim_{q\to 1} e_q((1-q)z) = \lim_{q\to 1} E_q((1-q)z) = e^z.$$

If we take q-derivative of q-exponentials, we can obtain these

$$\begin{aligned}\mathcal{D}_q e_q(x) = \sum_{j=0}^{\infty} \frac{\mathcal{D}_q x^j}{[j]_q!} &= \sum_{j=1}^{\infty} \frac{[j]_q x^{j-1}}{[j]_q!} \\ &= \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!},\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_q E_q(x) &= \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}} \mathcal{D}_q x^j}{[j]_q!} \\ &= \sum_{j=1}^{\infty} \frac{q^{\frac{(j-1)(j-2)}{2}} q^{j-1} x^{j-1}}{[j-1]_q!} \\ &= \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}} q^j x^j}{[j]_q!}.\end{aligned}$$

In other words, we can say these

$$\mathcal{D}_q e_q(x) = e_q(x), \qquad \mathcal{D}_q E_q(x) = E_q(qx).$$

From reference of [2], in general, $e_q^x e_q^y \neq e_q^{x+y}$. Additive property of the q-exponentials has been supplied if $yx = qxy$. Since assume that

$$e_q^x e_q^y = \left(\sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}\right)\left(\sum_{k=0}^{\infty} \frac{y^k}{[k]_q!}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j y^k}{[j]_q![k]_q!}$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{[j+k]_q!}{[j]_q![k]_q!}\frac{x^j y^k}{[j+k]_q!}$$

If we change variable from j,k to j and $n = j+k$, then for n, j runs from 0 to n.We have

$$e_q^x e_q^y = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} x^j y^{n-j}\right)\frac{1}{[n]_q!} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{[n]_q!}$$

Thus, we have

$$e_q^x e_q^y = e_q^{x+y},$$

if $yx = qxy$.

Due to the commutation relation being not symmetric in x and y, we can obtain that

$$e_q^x e_q^y \neq e_q^y e_q^x.$$

Definition 2.2.2. [2] The q-trigonometric functions are given by

$$\sin_q z = \frac{e_q^{iz} - e_q^{-iz}}{2i}, \qquad Sin_q z = \frac{E_q^{iz} - E_q^{-iz}}{2i},$$
$$\cos_q z = \frac{e_q^{iz} + e_q^{-iz}}{2}, \qquad Cos_q z = \frac{E_q^{iz} + E_q^{-iz}}{2}.$$

From (2.18) we get

$$Sin_q z = \sin_{\frac{1}{q}} z, \qquad Cos_q z = \cos_{\frac{1}{q}} z.$$

Then we have

$$\cos_q z Cos_q z = \frac{e_q^{iz}E_q^{iz} + e_q^{-iz}E_q^{-iz} + 2}{4}, \qquad \sin_q z Sin_q z = -\frac{e_q^{iz}E_q^{iz} + e_q^{-iz}E_q^{-iz} - 2}{4}.$$

Hence, we get

$$\cos_q z Cos_q z + \sin_q z Sin_q z = 1.$$

Applying the q-derivative to q-trigonometric functions,

$$\mathcal{D}_q \sin_q z = \cos_q z, \qquad \mathcal{D}_q Sin_q z = Cos_q z,$$

$$\mathcal{D}_q \cos_q z = -\sin_q qz, \qquad \mathcal{D}_q Cos_q z = -Sin_q qz.$$

2.3 q-Integral

In [2], Jackson had identified the q-integral which is a right inverse of the q-derivative.

In [2], the q-integral over $[0,x]$ is defined as follows :

$$I_q = \int_0^x f(t) d_q t := x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k) \tag{2.19}$$

and then more generally over $[a,b]$ as follows:

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{2.20}$$

Note that

$$I(f) = \int_0^x f(t) dt = \lim_{q \uparrow 1} I_q(f)$$

Theorem 2.3.1 (Mean Value Theorem of q-Integral [18]). *Firstly, function of f is a continuous on $[0,a]$. Then $\forall\, q \in (0,1)$ and exists $\varepsilon \in [0,a]$ so*

$$I_q(f) = \int_0^b f(t) d_q t = b f(\varepsilon)$$

Proof. Since $f \in C[0,b]$ and assume that

$$m = \min\{f(x) : 0 \leq x \leq b\}, \qquad M = \max\{f(x) : 0 \leq x \leq b\}.$$

From $0 < q < 1$ we can write that

$$0 \leq bq^n \leq b$$

and

$$m \leq f(bq^n) \leq M.$$

Then we can write that

$$\begin{gathered}
mq^n \leq f(bq^n)q^n \leq Mq^n \\
\sum_{n=0}^{\infty} mq^n \leq \sum_{n=0}^{\infty} f(bq^n)q^n \leq \sum_{n=0}^{\infty} Mq^n \\
b(1-q)\sum_{n=0}^{\infty} mq^n \leq b(1-q)\sum_{n=0}^{\infty} f(bq^n)q^n \leq b(1-q)\sum_{n=0}^{\infty} Mq^n.
\end{gathered}$$

From the mean value theorem of q-integral

$$\begin{aligned}
bm &\leq \int_0^b f(t)d_qt \leq bM, \\
m &\leq \frac{1}{b}\int_0^b f(t)d_qt \leq M.
\end{aligned}$$

So, there exists $\varepsilon \in [0,a]$ such that

$$f(\varepsilon) = \frac{1}{b}\int_0^b f(t)d_qt.$$

□

Definition 2.3.1. Improper q-integrals on the interval $[0,+\infty)$ are explained as, if $q \in (0,1)$,

$$\int_0^\infty f(t)\,d_qt := \sum_{k=-\infty}^{\infty} \int_{q^{k+1}}^{q^k} f(t)\,d_qt, \tag{2.21}$$

and if $q > 1$,

$$\int_0^\infty f(t)\,d_qt := \sum_{k=-\infty}^{\infty} \int_{q^k}^{q^{k+1}} f(t)\,d_qt. \tag{2.22}$$

Theorem 2.3.2 (FT of q-calculus [2])**.** *If $g(x)$ is the q-derivative of $G(x)$ and is continuous at zero, then for any interval $[a,b]$ with*

$$\int_a^b g(x)\,d_qx = G(a) - G(b), \qquad 0 \leq a < b \leq \infty. \tag{2.23}$$

Proof. Assume that $G(x)$ is defined by the Jackson formula with the addition of a constant number,

$$G(x) = (1-q)x \sum_{k=0}^{\infty} q^k g(q^k x) + G(0).$$

From the definition, we have

$$\int_a^0 g(x)\,d_qx := (1-q)a \sum_{k=0}^{\infty} q^k g(q^k a).$$

Then we can obtain that

$$\int_0^a g(x)\,d_qx := G(a) - G(0).$$

For finite b the integral defined by,

$$\int_0^b g(x)\,d_qx := G(b) - G(0)$$

and

$$\int_a^b g(x)\,d_qx := \int_0^b g(x)\,d_qx - \int_0^a g(x)\,d_qx = G(b) - G(a).$$

Finally, if shifting a with q^{k+1} (or q^k) and b with q^k (or q^{k+1}), where $q \in (0,1)$ (or $q < 1$), and using definition (2.21), we can obtain that (2.23) is right for $b = \infty$ as well as if $\lim_{t\to\infty} G(x)$ exists. $\square$

Integration by parts in q-calculus is stated as

$$\int_\alpha^\beta f(x)\,d_qg(x) = f(\beta)g(\beta) - f(\alpha)g(\alpha) - \int_\alpha^\beta g(qx)\,d_qf(x). \tag{2.24}$$

It also satisfied if $\beta = \infty$. This can be applied to procure the q-Taylor formula with the Cauchy remainder term as follows.

Theorem 2.3.3. *[2] Suppose $\mathcal{D}_q^k f(t)$ is continuous at $t = 0$ an take $j \leq n+1$. Then, we can obtain q-Taylor's is stated as*

$$f(b) = \sum_{k=0}^{n} \left(\mathcal{D}_q^k f\right)(a)\frac{(b-a)_q^k}{[k]_q!} + \frac{1}{[n]_q!}\int_a^b \mathcal{D}_q^{n+1} f(t)(b-qt)_q^n\,d_qt. \tag{2.25}$$

Proof. Using Theorem 2.3.2, we have

$$f(b) - f(a) = \int_a^b \mathcal{D}_q f(t)\,d_qt := -\int_a^b \mathcal{D}_q f(t)\,d_q(b-t),$$

which proves (2.25) for $n=0$. Let (2.25) hold for $n-1$:

$$f(b)=\sum_{k=0}^{n-1}\left(\mathcal{D}_q^k f\right)(a)\frac{(b-a)_q^k}{[k]_q!}+\frac{1}{[n-1]_q!}\int_a^b \mathcal{D}_q^{n+1} f(t)(b-qt)_q^{n-1}\,d_qt.$$

From (2.7) and applying (2.24), we get

$$\begin{aligned}\int_a^b \mathcal{D}_q^{n+1} f(t)(b-qt)_q^{n-1}\,d_qt &= -\frac{1}{[n]_q}\int_a^b \mathcal{D}_q^n f(t)\,d_q(b-t)_q^n\\ &= \mathcal{D}_q^n f(a)\frac{(b-a)_q^n}{[n]_q}+\frac{1}{[n]_q}\int_a^b (b-qt)_q^n \mathcal{D}_q^{n+1} f(t) d_qt.\end{aligned}$$

So, the above proof can be completed by mathematical induction. □

2.4 q-Appell Polynomials

Appell polynomials are defined in 1880 by Paul Appell [19]. Al-Salam, in 1967, presented the q-Appell polynomials $A_{n,q}(x)_{n=0}^{\infty}$ and investigated their specialities in [20]. In [20], $A_{n,q}(x)$ is the q-Appell if it holds the below equation for $n=0,1,\ldots$:

$$\mathcal{D}_q(A_{n,q}(x))=[n]_q!A_{n-1,q}(x).$$

Definition 2.4.1 (q-Appell Polynomials [21])**.** The q-Appell polynomials are defined equivalently by

$$A_q(z;t):=A_q(t)e_q(tz)=\sum_{n=0}^{\infty} A_{n,q}(z)\frac{t^n}{[n]_q!}, \tag{2.26}$$

where the function $A_q(t)$ is

$$A_q(t):=\sum_{n=0}^{\infty}\frac{A_{n,q}}{[n]_q!}t^n, \qquad A(0)\neq 0, \qquad A_{n,q}:=A_{n,q}(0),$$

and

$$e_q(zt)=\sum_{k}^{\infty}\frac{z^k}{[k]!}.$$

In addition, particular cases of q-Appell polynomials are the q-Bernoulli polynomials, the q-Euler polynomials, and the q-Genocchi polynomials. These are the q-analogues of the original Bernoulli, Euler and Genocchi polynimials which are particular cases of Appell polynomials. The q-Bernoulli polynomials are given by

$$B_q(z,t) := \frac{t}{e_q(t)-1}e_q(zt) = \sum_{n=0}^{\infty} B_{n,q}(z)\frac{t^n}{n!} \qquad |t| < 2\pi,$$

and the q-Bernoulli numbers $b_{n,q}$ are given by

$$\frac{t}{e_q(t)-1} = \sum_{n=0}^{\infty} b_{n,q}\frac{t^n}{n!}.$$

The q-Euler polynomials are defined by

$$\mathbb{E}_q(z,t) := \frac{2}{e_q(t)+1}e_q(zt) = \sum_{n=0}^{\infty} \mathbb{E}_{n,q}(z)\frac{t^n}{n!} \qquad |t| < \pi,$$

and the q-Euler numbers $E_{n,q}$ are given by

$$\frac{te_q(t)}{e_q(2t)+1}e_q(zt) = \sum_{n=0}^{\infty} E_{n,q}(z)\frac{t^n}{n!}.$$

The q-Genocchi polynomials are defined by

$$\mathbb{G}_q(z,t) := \frac{2t}{e_q(t)+1}e_q(zt) = \sum_{n=0}^{\infty} \mathbb{G}_{n,q}(z)\frac{t^n}{n!}, \qquad |t| < \pi,$$

and the q-Genocchi numbers $g_{n,q}$ are given by

$$\frac{2t}{e_q(t)+1} = \sum_{n=0}^{\infty} g_{n,q}\frac{t^n}{n!}.$$

In the following, there are two major characterizations about q-Appell polynomials which were studied in [22] and [20].

Theorem 2.4.1. *[20, 22] Let $\{P_n(x)\}$ be a polynomial set. Then $P_n(x)$ is q-Appell iff there is a set of constants a_k with $a_0 \neq 0$ and from 2.11 we can obtain that*

$$P_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a_{n-k} x^k, \tag{2.27}$$

where for some fixed number a

$$a_k = (1-a)1-aq)..(1-aq^{k-1}), \qquad a_0 = 1.$$

Theorem 2.4.2. *[22] Let $\{P_n(x)\}$ be a polynomial set. Then $P_n(x)$ is q-Appell iff there is a formal power series*

$$A(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k, \qquad a_0 \neq 0, \tag{2.28}$$

such that

$$A(t)e_q(xt) = \sum_{k=0}^{\infty} \frac{P_n(x)t^n}{[n]_q!},$$

where

$$e_q(xt) = \sum_{k}^{\infty} \frac{x^k}{[k]_q!}.$$

In addition, we can say that a_k is same as these two theorems and the condition $a_0 \neq 0$.

Theorem 2.4.3. *[20, 23] Let $\{P_n(x)\}$ be a polynomial set. Then $P_n(x)$ is q-Appell iff there is a function $\beta(x;q) = \beta(x)$ of bounded variation on $(0,\infty)$ so that,*

1) $a_n = \int_0^\infty x^n d\beta(x)$ *exists* $\forall\, n = 0, 1, 2...$

2) $a_0 \neq 0$

3) $P_n(x) = \int_0^\infty (x+t)^n d\beta(x)$.

The determining function is then

$$A(t) = \int_0^\infty e(xt)d\beta(x).$$

Following [23], in the above theorem the set of $(x+t)_q^t$ is replaceable by any q-Appell set.

2.5 The Big q-Appell Polynomials

Definition 2.5.1. [24] The big q-Appell polynomials are defined by,

$$A_q(t)E_q\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,q}(x)\frac{t^n}{[n]_q!},$$

where

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{x^n}{[n]_q!}, \qquad (0<|q|<1; x \in \mathbb{C})$$

and

$$A_q(t) = \sum_{n=0}^{\infty} a_{n,q}\frac{t^n}{[n]_q!}.$$

In addition we can show that

$$\mathcal{D}_{q,x}\left(P_{n,q}\right)(x) = \frac{[n]_q}{q}P_{n-1,q}(qx).$$

where $[n]_q$ is known from (2.7).

Theorem 2.5.1. *[24] The folowing statements are all equivalent to each other:*

1) $\{P_{n,q}(x)\}_{n\in\mathbb{N}}$ *is a big q-Appell sequence*

2) The sequence $\{P_{n,q}(x)\}_{n\in\mathbb{N}}$ has an explicit form given by

$$P_{n,q}(x) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q a_{n-k,q} q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k.$$

3) The big q-Appell sequence $\{P_{n,q}(x)\}_{n\in\mathbb{N}}$ has a generating function

$$A_q(t)E_q\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,q}(x)\frac{t^n}{[n]_q!},$$

where

$$A_q(t) = \sum_{k=0}^{\infty} a_{k,q}\frac{t^k}{[k]_q!}.$$

Theorem 2.5.2 ([24]). *A recurrence relation satisfied by the big q-Appell polynomials is*

$$\left(\frac{x}{q} + \alpha_{0,q}\right) P_{n,q}(qx) + \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha_{n-k,q} P_{k,q}(qx) = P_{n+1,q}(x). \tag{2.29}$$

Chapter 3

HAHN CALCULUS

3.1 Definition of Hahn Difference Operator

This section is generally composed of reference [1]. In this section the Hahn difference operator and the related calculus is developed. In general the Hahn difference operator $\mathcal{D}_{q,\omega}$ composes both Δ_ω and $\mathcal{D}_q$. In other words, if you take limit $\omega \uparrow 0$ from Hahn difference operator you can obtain Jackson's q-difference operator and if we take limit $q \uparrow 1$ from Hahn difference operator we can obtain the Forward difference operator.

In [25, 26] $\mathcal{D}_{q,\omega}$ is given as

$$\mathcal{D}_{q,\omega} f(t) = \frac{f(qt+\omega) - f(t)}{(q-1)t+\omega}, \qquad t \neq \omega_0, \tag{3.1}$$

where $q \in [0,1]$ and $\omega > 0$ are constants and $\omega_0 := \frac{\omega}{1-q}$. We can say that $\mathcal{D}_{q,\omega} f(\omega_0) = f'(\omega_0)$, provided that derivative exists. So, (q,ω)-derivative of f is $\mathcal{D}_{q,\omega} f$. As a result, if $\mathcal{D}_{q,\omega} f(\omega_0)$ exists, we can say f is (q,ω)-differentiable. Assume that f is (q,ω)-differentiable on the interval I and $\mathcal{D}_{q,\omega} f = 0$ then f is a constant function. Then

$$f(t) = f(q^k t + \omega[k]_q), t \in I, \quad t \neq \omega_0 \qquad k \in \mathbb{N},$$

and therefore if we take limit, we can obtain that $f(t) = f(\omega)\ \forall\, t \in I, \quad t \neq \omega_0$.

The operator (2.1) is the first difference operator which Hahn's operator generalizes.

The second one is the $\Delta_\omega f(t)$, which is:

$$\Delta_\omega f(t) = \frac{f(t+\omega)-f(t)}{(t+\omega)-t}. \tag{3.2}$$

The Nörlund sum is the associated integral of the forward difference operator, and it is given as

$$\int_\infty^x f(t)\Delta_\omega t := -\omega \sum_{k=0}^{\infty} f(x+k\omega), \tag{3.3}$$

and

$$\int_a^b f(t)\Delta_\omega t := \omega \sum_{k=0}^{\infty} [f(a+k\omega)-f(b+k\omega)], \tag{3.4}$$

provided that the series is convergent.

Algebraic Properties of Hahn calculus [1]: Algebraic properties are like those of differentiation (linear, product, quotient) but here they are properties of q,ω-differentiation.

Let f,g be functions and they are q,ω-differentiable at $t \in I$, then we have linearity, product rule, quotient rule as follows:

$$\mathcal{D}_{q,\omega}(f+g)(t) = \mathcal{D}_{q,\omega}f(t) + \mathcal{D}_{q,\omega}g(t), \tag{3.5}$$

$$\mathcal{D}_{q,\omega}(fg)(t) = \mathcal{D}_{q,\omega}f(t)g(t) + f(qt+\omega)\mathcal{D}_{q,\omega}g(t), \tag{3.6}$$

$$\mathcal{D}_{q,\omega}\left(\frac{f}{g}\right)(t) = \frac{\mathcal{D}_{q,\omega}f(t)g(t) - f(t)\mathcal{D}_{q,\omega}g(t)}{g(t)g(qt+\omega)}, \tag{3.7}$$

where $g(t)g(qt+\omega)$ are not zero.

Example 3.1.1. The q-derivative of t^n is shown in example (2.0.1) and the forward difference operator is applied to t^n as follows:

$$\Delta_\omega t^n = \sum_{k=1}^{n} \binom{n}{k} \omega^{k-1} t^{n-k} = \sum_{k=0}^{n-1} \binom{n}{k} \omega^{n-k-1} t^k. \tag{3.8}$$

Here we consider (q,ω)-derivatives. If $f(t) = (\alpha t+\beta)^n$,

$$\mathcal{D}_{q,\omega}(\alpha t+\beta)^n = \alpha \sum_{k=0}^{n-1} (\alpha(qt+\omega)+\beta)^k (\alpha t+\beta)^{n-k-1}. \tag{3.9}$$

If $f(t) = (\alpha t+\beta)^{-n}$,

$$\mathcal{D}_{q,\omega}(\alpha t+\beta)^n = -\alpha \sum_{k=0}^{n-1} (\alpha(qt+\omega)+\beta)^{-n+k} (\alpha t+\beta)^{-k-1}, \tag{3.10}$$

where $\alpha, \beta \in R$ and $(\alpha(qt+\omega)+\beta)(\alpha t+\beta) \neq 0$.

If we apply $\alpha = 1$ and $\beta = 0$ then if we take q,ω-derivative, we can obtain that

$$\mathcal{D}_{q,\omega}(t)^n = \frac{(qt+\omega)^n - (t)^n}{(q-1)t+\omega} = \sum_{k=0}^{n-1} (qt+\omega) k t^{n-k-1}, \tag{3.11}$$

$$\mathcal{D}_{q,\omega}(t)^{-n} = \frac{(qt+\omega)^{-n} - (t)^{-n}}{(q-1)t+\omega} = \sum_{k=0}^{n-1} (qt+\omega) - n + k t^{-k-1}. \tag{3.12}$$

3.2 Theorems of Hahn Difference Operator

Theorem 3.2.1 (Leibniz Formula For Hahn difference operator [1])**.** *Let the functions f,g be given with existent q,ω-derivatives. Then the following equality is provided.*

$$\mathcal{D}^n_{q,\omega}(fg)(t) = \sum_{k=0}^{n} \binom{n}{k}_q (\mathcal{D}^{n-k}_{q,\omega} f)(q^k t + \omega[k]_q) \mathcal{D}^k_{q,\omega} g(t), \quad t \neq \omega_0. \tag{3.13}$$

Proof. Assume that $t \neq \omega_0$. We proceed by using induction. For the case $n = 1$, the result is already known as the original product rule (3.6).

Now, assume that (3.13) hold for $n = m$. We need to prove (3.13) for $n = m+1$ and get $r, k \in N$ and from $[k+1]_q - [k]_q = q^k$, we get

$$
\begin{aligned}
(\mathcal{D}_{q,\omega}^{z+1} f)(q^k t + \omega[k]_q) &= (\mathcal{D}_{q,\omega} D_{q,\omega}^{z} f)(q^k t + \omega[k]_q) \\
&= \frac{(D_{q,\omega}^{z} f)(q^{k+1} t + \omega[k+1]_q) - (D_{q,\omega}^{z} f)(q^k t + \omega[k]_q)}{(q^{k+1} + \omega[k+1]_q - (q^k t + \omega[k]_q)} \\
&= \frac{(D_{q,\omega}^{z} f)((qt+\omega)q^k + \omega[k]_q) - (D_{q,\omega}^{z} f)(q^k t + \omega[k]_q)}{(q^k(t(q-1)+\omega)} \\
&= q^{-k} \mathcal{D}_{q,\omega}(D_{q,\omega}^{z} f)(q^k t + \omega[k]_q).
\end{aligned}
$$

Now assume that $n = m+1$, and using the product rule from algebraic properties in 3.2, we obtain,

$$
\begin{aligned}
\mathcal{D}_{q,\omega}^{m+1}(fg)(t) &= \mathcal{D}_{q,\omega}(\mathcal{D}_{q,\omega}^{m}(fg)(t)) \\
&= \mathcal{D}_{q,\omega}\left[\sum_{k=0}^{m} \binom{m}{k}_q (\mathcal{D}_{q,\omega}^{m-k} f)(q^k t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k} g(t)\right] \\
&= \sum_{k=0}^{m} \binom{m}{k}_q \mathcal{D}_{q,\omega}((\mathcal{D}_{q,\omega}^{m-k} f)(q^k t + \omega[k]_q)) \mathcal{D}_{q,\omega}^{k} g(t) \\
&\qquad + \mathcal{D}_{q,\omega}\left[\sum_{k=0}^{m} \binom{m}{k}_q (\mathcal{D}_{q,\omega}^{m-k} f)(q^{k+1} t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k+1} g(t)\right] \\
&= \sum_{k=0}^{m} \binom{m}{k}_q q^k (\mathcal{D}_{q,\omega}^{m-k+1} f)(q^k t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k} g(t) \\
&\qquad + \mathcal{D}_{q,\omega}\left[\sum_{k=0}^{m} \binom{m}{k}_q (\mathcal{D}_{q,\omega}^{m-k} f)(q^{k+1} t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k+1} g(t)\right].
\end{aligned}
$$

From the known q-binomial coefficients property,

$$\begin{aligned}
\mathcal{D}_{q,\omega}^{m+1}(fg)(t) &= \sum_{k=1}^{m} \binom{m}{k}_q q^k (\mathcal{D}_{q,\omega}^{m-k+1} f)(q^k t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k} g(t) \\
&\qquad + \mathcal{D}_{q,\omega}^{m+1}(f(t)) g(t) + f(q^{m+1} t + \omega[m+1]_q) \mathcal{D}_{q,\omega}^{m+1} g(t) \\
&\qquad\quad + \sum_{k=1}^{m} \binom{m}{k-1}_q (\mathcal{D}_{q,\omega}^{m-k+1} f)(q^k t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k} g(t) \\
&= \sum_{k=1}^{m} \left(\binom{m}{k}_q q^k + \binom{m}{k-1}_q \right) (\mathcal{D}_{q,\omega}^{m-k+1} f)(q^k t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k} g(t) \\
&\qquad + \sum_{k=1}^{m} \binom{m}{k-1}_q (\mathcal{D}_{q,\omega}^{m-k+1} f)(q^k t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k} g(t) \\
&= \sum_{k=1}^{m+1} \binom{m+1}{k}_q (\mathcal{D}_{q,\omega}^{m-k+1} f)(q^k t + \omega[k]_q) \mathcal{D}_{q,\omega}^{k} g(t).
\end{aligned}$$

Finally, (3.13) is right for $n = m+1$ and all $n \in \mathbb{N}$. $\square$

Remark 3.2.1. If $t = \omega_0$ at (3.13), the original Leibniz rule is recovered. But if $\omega \uparrow 0$, we derive the q-Leibniz formula [5, 28, 29]

$$\mathcal{D}_q^n(fg)(t) = \sum_{k=0}^{n} \binom{n}{k}_q (\mathcal{D}_q^{n-k} f)(q^k t) \mathcal{D}_q^k g(t), \quad t \neq \omega_0. \tag{3.14}$$

Letting $q \downarrow 1$, then we obtain

$$\Delta_\omega^n(fg)(t) = \sum_{k=0}^{n} \binom{n}{k} (\Delta_\omega^{n-k} f)(t + k\omega) \Delta_\omega^k g(t) \tag{3.15}$$

which is the classical discrete Leibniz formula, [27].

Theorem 3.2.2. *[Chain Rule [1]] Assume that $g : I \to R$ is (q,ω)-differentiable and $f : R \to R$ is a usual C^1 function. There is some $c \in (qt+\omega, t)$ with*

$$\mathcal{D}_{q,\omega}(f \circ g)(t) = f'(g(c)) \mathcal{D}_{q,\omega} g(t). \tag{3.16}$$

Proof. Assume that $t \neq \omega_0$, we get

$$\mathcal{D}_{q,\omega}(f \circ g)(t) = \frac{f(g(qt+\omega)) - f(g(t))}{(q-1)t+\omega}, \qquad g(qt+\omega) \neq g(t).$$

If $g(qt+\omega) = g(t)$

$$\mathcal{D}_{q,\omega}(f \circ g)(t) = \mathcal{D}_{q,\omega}g(t) = 0.$$

Hence,

$$\mathcal{D}_{q,\omega}(f \circ g)(t) = \frac{f(g(qt+\omega)) - f(g(t))}{g(qt+\omega) - g(t)} \times \frac{g(qt+\omega) - g(t)}{(q-1)t+\omega} \tag{3.17}$$

and there exists τ between the point $g(t)$ and the point $g(qt+\omega)$ via

$$f'(\tau) = \frac{f(g(qt+\omega)) - f(g(t))}{g(qt+\omega) - g(t)}. \tag{3.18}$$

Finally, from (3.17) and (3.18) we can derive (3.16). In addition (3.16) is true in the classical sense for $t = \omega_0$. $\square$

3.3 Jackson–Nörlund Integration

This section is composed of reference [1]. In this section, the right inverse of the operator $\mathcal{D}_{q,\omega}$ is studied. It is called the q,ω-integral operator, and it is denoted by

$$\int_a^b f(t) d_{q,\omega} t.$$

Definition 3.3.1 (Definition of Jackson-Nörlund Integration)**.** Let $f : I \to \mathbb{R}$ be a function. The (q,ω)-integral of this function between two numbers c and b is given by (3.19) which uses (3.20):

$$\int_c^b f(t) d_{q,\omega} t := \int_{\omega_0}^b f(t) d_{q,\omega} t - \int_{\omega_0}^c f(t) d_{q,\omega} t, \qquad c, b \in I, \tag{3.19}$$

$$\int_{\omega_0}^{x} f(t)d_{q,\omega}t := (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(xq^k+\omega[k]_q), \quad x\in I. \tag{3.20}$$

ensured that the latter series converges. If f is (q,ω)-integrable over any interval $[c,b]$ in I, then f is (q,ω)-integrable on I. In addition the Jackson-Nörlund sum is the sum to the right hand side of (3.19).

In (3.20) and (3.19), while if $\omega \uparrow 0$ we can obtain (2.20) and (2.19), when $q \uparrow 1$ we can obtain (3.4).

Lemma 3.3.1. *Let $f,g: I \to \mathbb{R}$ be function that are (q,ω)-integrable and let α,β,γ be three points in I. Then:*

i) $\int_{\alpha}^{\alpha} f(t)d_{q,\omega}t = 0, \qquad \int_{\beta}^{\alpha} f(t)d_{q,\omega}t = -\int_{\alpha}^{\beta} f(t)d_{q,\omega}t.$

ii) $\int_{\alpha}^{\beta} f(t)d_{q,\omega}t = \int_{\alpha}^{\gamma} f(t)d_{q,\omega}t + \int_{\gamma}^{\beta} f(t)d_{q,\omega}t.$

iii) $\int_{\alpha}^{\beta} (kf(t)+lg(t))d_{q,\omega}t = k\int_{\alpha}^{\beta} f(t)d_{q,\omega}t + l\int_{\alpha}^{\beta} g(t)d_{q,\omega}t.$

Theorem 3.3.1 (FT of Hahn calculus [1])**.** *For any function $f: I \to \mathbb{R}$ that is continuous at ω_0, if*

$$F(x) := \int_{\omega_0}^{x} f(t)d_{q,\omega}t, \qquad x\in I,$$

then we can say that F is also continuous at ω_0. Additionally, $\mathcal{D}_{q,\omega}F(x)$ exists at every point in I and

$$\mathcal{D}_{q,\omega}F(x) = f(x),$$

otherwise, $\forall\, a,b \in \mathbb{I}$,

$$\int_{a}^{b} \mathcal{D}_{q,\omega}F(x)_{q,\omega} = f(b)-f(a).$$

Proof. If $x = s$ and $s \in I$, from (3.20) we can say that

$$F(s) = (s(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(sq^k + \omega[k]_q),$$

is continuous at ω_0. Let us prove that $\mathcal{D}_{q,\omega}F(x)$ exists $\forall\, x \in I$. If $x = \omega_0$, then continuity of $f(x)$ is enough,

$$\begin{aligned}
\mathcal{D}_{q,\omega}F(\omega_0) &= \lim_{s\uparrow\omega_0} \frac{F(\omega_0)-F(s)}{\omega_0 - s} \\
&= \lim_{s\to\omega_0} \frac{-\int_{\omega_0}^{s} f(t)d_{q,\omega}t}{\omega_0 - s} \\
&= \lim_{s\uparrow\omega_0}(1-q)\sum_{k=0}^{\infty} q^k f(sq^k + \omega[k]_q \\
&= (1-q)\sum_{k=0}^{\infty} q^k f(\omega_0) \\
&= f(\omega_0).
\end{aligned}$$

Otherwise for $x \neq \omega_0$, let us prove that

$$\int_{\omega_0}^{x} \mathcal{D}_{q,\omega}f(t)d_{q,\omega}t = f(x) - f(\omega_0), \quad \textit{for} \text{ all } x \in \mathbb{I}.$$

From (3.20), the continuity of $f(x)$ for $x \in I$, using(2.1) and $[n+1]_q = \frac{1-q^{n+1}}{1-q}$,

$$\begin{aligned}
\int_{\omega_0}^{x} \mathcal{D}_{q,\omega}f(t)d_{q,\omega}t &= (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k(\mathcal{D}_{q,\omega})f(xq^k + \omega[k]_q) \\
&= (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k \frac{f(xq^{k+1}+\omega[k+1]_q) - f(xq^k+\omega[k]_q)}{xq^k(q-1)+\omega([k+1]_q-[k]_q)} \\
&= (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k \frac{f(xq^{k+1}+\omega[k+1]_q) - f(xq^k+\omega[k]_q)}{xq^k(q-1)+\omega q^k} \\
&= -\sum_{k=0}^{\infty}\Big(f(xq^{k+1}+\omega[k+1]_q) - f(xq^k+\omega[k]_q)\Big) \\
&= \lim_{n\to\infty}\Big(f(xq^k+\omega[k]_q) - f(xq^{k+1}+\omega[k+1]_q)\Big) \\
&= f(x) - f(\omega),
\end{aligned}$$

which completes the proof. $\square$

Lemma 3.3.2 (q,ω-integration by parts [1]). *Suppose $f,g : I \to \mathbb{R}$ are continuous at the point ω_0. From lemma (3.3.1) and the algebraic property of the product rule, we can obtain*

$$\int_c^b f(t)\cdot \mathcal{D}_{q,\omega} g(t) d_{q,\omega}t = f(t)g(t)|_c^b - \int_c^b \mathcal{D}_{q,\omega} f(t)\cdot g(qt+\omega) d_{q,\omega}t.$$

Theorem 3.3.2 (Mean Value Theorem [7]). *Let $g : \mathbb{I} \to X$ be continuous and then we can write that*

$$\int_t^{b(t)} g(\tau) d_{q,\omega}\tau = (b(t)-t)g(t)$$

3.4 Hahn Exponential Functions

Firstly, the q-analogue, q-shift factorial, q-binomial coefficients, q-factorial are known from other sections. Now several definitions and formulas from [11] will be gives which are used in this section .

Let us give the polynomial bases $\{(x)^n_{q,\omega}\}_{n\geq 0}$ and $\{[x]^n_{q,\omega}\}_{n\geq 0}$. They can be shown in the following way,

$$[x]^n_{q,\omega} = x(qx+\omega)(q^2x+[2]_q\omega)\ldots(q^{n-1}x+[n-1]_q\omega), \tag{3.21}$$

$$(x)^n_{q,\omega} = x\cdot(x-\omega)\cdot(x-[2]_q\omega)\cdot(x-[3]_q\omega)\ldots(x-[n-1]_q\omega). \tag{3.22}$$

These bases satisfy the following:

$$(x)^n_{q,0} = x^n, \qquad (x)^n_{1,0} = x^n,$$

$$[x]^n_{q,0} = q^{\frac{n(n-1)}{2}} x^n, \qquad [x]^n_{1,0} = x^n.$$

If we apply Hahn difference operator to polynomial bases, we can obtain:

$$\mathcal{D}_{q,\omega}(x)_{q,\omega}^{n}=[n]_q(x)_{q,\omega}^{n-1}, \tag{3.23}$$

$$\mathcal{D}_{q,\omega}[x]_{q,\omega}^{n}=[n]_q[qx+\omega]_{q,\omega}^{n-1} \tag{3.24}$$

where $q\in(0,1)$ and $\omega>0$ and $n\geq 1$.

Theorem 3.4.1. *[11] Assume that $e_q^{\omega}(x)$ is the (q,ω)-exponential function and $q\in(0,1)$ satisfy the following first order initial value problem*

$$\mathcal{D}_{q,\omega}f(x)=f(x), \tag{3.25}$$

$$f(0)=1. \tag{3.26}$$

what denoted by

$$e_q^{\omega}(x):=\sum_{n=0}^{\infty}\frac{1}{[n]_q!}(x)_{q,\omega}^{n}=\frac{(-\omega;q)_\infty}{((1-q)x-\omega;q)_\infty},\qquad |(1-q)x-\omega|<1. \tag{3.27}$$

Proof. We know that $\mathcal{D}_{q,\omega}(x)_{q,\omega}^{n}=[n]_q(x)_{q,\omega}^{n-1}$, so

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{[n]_q!}(x)_{q,\omega}^{n}$$

is a solution of the problem given by (3.25) and (3.26). To show that it is unique, let us write (3.25) in the form

$$f(x)=\frac{f(qx+\omega)-f(x)}{(q-1)x+\omega},$$

to obtain that

$$f(x)=f(qx+\omega)\cdot\frac{1}{1+(q-1)t+\omega}.$$

Applying this identity m times, we derive

$$f(x)=f(q^{m+1}x+[m+1]_q\omega)\prod_{k=0}^{m}\frac{1}{1+q^k((q-1)x+\omega)}.$$

Then allow $m\to\infty$, to obtain

$$f(x)=f(\omega_0)\prod_{k=0}^{m}\frac{1}{1+q^k((q-1)x+\omega)}. \tag{3.28}$$

Apply the initial condition (3.26), and from q-shift factorial definition, we get

$$f(\omega_0)=\prod_{k=0}^{m}(1+\omega q^k)=(-\omega;q)_\infty. \tag{3.29}$$

From (3.28) and (3.29), we have

$$f(x)=\prod_{k=0}^{m}\frac{(1+\omega q^k)}{1+q^k[(q-1)x+\omega)]}=\frac{(-\omega;q)_\infty}{((1-q)x-\omega;q)_\infty}.$$

As a result, the initial value problem (3.25) and (3.26) has one solution. □

Theorem 3.4.2. *[11] The (q,ω)-exponential function $E_q^\omega(x)$ and $q\in(0,1)$ satisfying the following first order initial value problem*

$$\mathcal{D}_{q,\omega}f(x)=f(qx+\omega), \tag{3.30}$$

$$f(0)=1, \tag{3.31}$$

is devoted by

$$E_q^\omega(x):=\sum_{n=0}^{\infty}\frac{[x]^n_{q,\omega}}{[n]_q!}=\frac{((q-1)x+\omega;q)_\infty}{(-\omega;q)_\infty},\qquad |\omega|<1. \tag{3.32}$$

Proof. The function

$$f(x) = \sum_{n=0}^{\infty} \frac{[x]_{q,\omega}^{n}}{[n]_q!}.$$

is a solution of the problem given by (3.30) and (3.31). From (3.30), we get

$$\mathcal{D}_{q,\omega} f(x) = \frac{f(qx+\omega) - f(x)}{(q-1)t+\omega} = f(qx+\omega),$$

so we can obtain that

$$f(x) = f(qx+\omega) \cdot (1 - [(q-1)t+\omega]).$$

Applying this identity m times, we derive

$$f(x) = f(q^{m+1}x + [m+1]_q \omega) \prod_{k=0}^{m} (1 - q^k[(q-1)x+\omega]).$$

Then allow $m \to \infty$, to obtain

$$f(x) = f(\omega_0) \prod_{k=0}^{\infty} (1 - q^k[(q-1)x+\omega]). \tag{3.33}$$

Apply the initial condition, and from the q-shift factorial definition, we get

$$f(\omega_0) = \frac{1}{\prod_{k=0}^{\infty}(1-\omega q^k)} = \frac{1}{(\omega;q)_\infty}. \tag{3.34}$$

From (3.33) and (3.34); we get

$$f(x) = \prod_{k=0}^{n} \frac{1 - q^k[(q-1)x+\omega)]}{(1-\omega q^k)} = \frac{((q-1)x+\omega;q)_\infty}{(\omega;q)_\infty}.$$

As a result, the first initial value problem (3.30) and (3.31) has one solution which (3.32) exists. □

Remark 3.4.1. [11] It is obvious that,

$$e_q^0(x) = e_q(x), \qquad e_1^0(x) = e^x, \qquad e_1^\omega(x) = (1+\omega)^{\frac{x}{\omega}},$$

$$E_q^0(x) = e_q(x), \qquad E_1^0(x) = e^x, \qquad E_1^\omega(x) = (1-\omega)^{\frac{-x}{\omega}}.$$

Then,

$$E_q^{-\omega}(-x)e_q^\omega(x) = 1.$$

3.5 Hahn Trigonometric Functions

The Hahn trigonometric functions are defined by

$$sin_q^\omega z = \frac{e_q^{\omega iz} - e_q^{-\omega iz}}{2i}, \qquad Sin_q^\omega z = \frac{E_q^{\omega iz} - E_q^{-\omega iz}}{2i},$$
$$\cos_q^\omega z = \frac{e_q^{\omega iz} + e_q^{-\omega iz}}{2}, \qquad Cos_q^\omega = \frac{E_q^{\omega iz} + E_q^{-\omega iz}}{2}.$$

and

$$Sin_q^\omega z = \sin_{\frac{1}{q}}^\omega z, \qquad Cos_q z = \cos_{\frac{1}{q}}^\omega z.$$

Then we have

$$\cos_q^\omega z Cos_q^\omega z = \frac{e_q^{\omega iz}E_q^{\omega iz} + e_q^{-\omega iz}E_q^{-\omega iz} + 2}{4},$$
$$\sin_q^\omega z Sin_q^\omega z = -\frac{e_q^{\omega iz}E_q^{\omega iz} + e_q^{-\omega iz}E_q^{-\omega iz} - 2}{4}.$$

Hence, we get

$$\cos_q^\omega z Cos_q^\omega z + \sin_q^\omega z Sin_q^\omega z = 1, \qquad where \quad \sin^2 z + \cos^2 z = 1.$$

Applying the q,ω-derivative to q-trigonometric functions, we get

$$\mathcal{D}_{q,\omega} \sin_q^\omega z = \cos_q^\omega z, \qquad \mathcal{D}_{q,\omega} Sin_q^\omega z = Cos_q^\omega z,$$

$$\mathcal{D}_{q,\omega} \cos_q^\omega z = -\sin_q^\omega qz, \qquad \mathcal{D}_q Cos_q^\omega z = -Sin_q^\omega qz.$$

3.6 $(\mathbf{q},\omega)$-Appell Polynomials

This section is composed of reference [11]. Sequences of (q,ω)-Appell polynomials are defined by the relation

$$\mathcal{D}_{q,\omega}P_n(x) = [n]_q P_{n-1}(x), n \geq 1. \tag{3.35}$$

For example, if we take q,ω-derivative of $\{(x)^n_{q,\omega}\}_{n\geq 0}$ then we get

$$\mathcal{D}_{q,\omega}(x)^n_{q,\omega} = [n]_q (x)^{n-1}_{q,\omega}, \tag{3.36}$$

So, we can say that $\{(x)^n_{q,\omega}\}_{n\geq 0}$ is (q,ω)-Appell polynomial sequence.

We show some characterizations of (q,ω)-Appell polynomials, equivalent to the (3.35), first of all the generating function defined by

$$A(t)e(xt) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!},$$

where $A(t) = \sum_{k=0}^{\infty} a_k t^k, a_0 \neq 0$.

Firstly from $(x)^n_{q,\omega}$ basis, we can defined that;

$$e_q^{\omega t}(x) := \sum_{n=0}^{\infty} \frac{(x)^n_{q,\omega} t^t}{[n]_q!} = \frac{(-\omega t;q)_\infty}{(t((1-q)x-\omega);q)_\infty}, |(1-q)x-\omega|<1. \tag{3.37}$$

Theorem 3.6.1. *[11] Let $\{P_n(.;\omega;q)_{n\geq 0}\}$ be a sequence of polynomials and then the following representations are equivalent.*

i) $P_n(.;\omega;q)_{n\geq 0}\}$ is (q,ω)-Appell.

ii) The polynomial sequence $\{P_n(.;\omega;q)_{n\geq 0}\}$ is defined by

$$P_n(x;\omega;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a_{n-k}(x)_{q,\omega}^k, \tag{3.38}$$

where $\{a_k\}_{k\geq 0}$ is a sequence of numbers independent of n with $a_0 \neq 0$.

iii) $\{P_n(.;\omega;q)\}_{n\geq 1}$ is defined by

$$A(t)e_q^{\omega t}(xt) = \sum_{n=0}^{\infty} P_n(x;\omega;q)\frac{t^n}{[n]_q!}, \tag{3.39}$$

where the function $A(t)$ is

$$A(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!}.$$

Proof. Show that (i)⇒(ii). Assume that (i) is true. Since the sequence $\{(x)_{q,\omega}^n\}_{n\geq 0}$ is (q,ω)-Appell, therefore:

$$P_n(x;\omega;q) = \sum_{k=0}^{n} \gamma_{n,k}(x)_{q,\omega}^k, \gamma_{n,n} \neq 0 \tag{3.40}$$

where $\gamma_{n,k}$ based on both n and k potentially. Using the operator $\mathcal{D}_{q,\omega}$ on this equation, we have

$$P_{n-1}(x;\omega;q) = \sum_{k=0}^{n-1} \gamma_{n,k+1}\frac{[k+1]_q}{[n]_q}(x)_{q,\omega}^k. \tag{3.41}$$

If changing n with $n+1$ in (3.41), we get

$$P_n(x;\omega;q) = \sum_{k=0}^{n} \gamma_{n+1,k+1}\frac{[k+1]_q}{[n+1]_q}(x)_{q,\omega}^k. \tag{3.42}$$

After that, comparing (3.42) with (3.40) gives,

$$\gamma_{n+1,k+1} = \frac{n+1}{k+1}\gamma_{n,k}. \tag{3.43}$$

Iterating (3.43) k times and taking $\gamma_{n-k,0} = a_{n-k}$, we have

$$\gamma_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q \gamma_{n-k,0} = \begin{bmatrix} n \\ k \end{bmatrix}_q a_{n-k}. \tag{3.44}$$

(ii)⇒(iii). Start from (3.38), multiply both sides by $\frac{t^n}{[n]_q!}$, some over $n = 0, 1, \ldots,$, use the Cauchy product of the series to obtain .

(iii) ⇒(i) Start from (3.39) and apply $\mathcal{D}_{q,\omega}$ on the both sides:

$$\sum_{n=1}^{\infty} \mathcal{D}_{q,\omega} P_n(x;\omega,q) \frac{t^n}{[n]_q!} = tA(t) = \sum_{n=0}^{\infty} P_n(x;\omega,q) \frac{t^n+1}{[n]_q!} = \sum_{n=1}^{\infty} [n]_q P_{n-1}(x;\omega,q) \frac{t^n}{[n]_q!}.$$

From (3.35), we obtain that,

$$\mathcal{D}_{q,\omega} P_n(x;\omega,q) = [n]_q P_{n-1}(x;\omega,q), \qquad n \geq 1.$$

□

Theorem 3.6.2. *[11] Suppose that $\{P_n(\cdot;\omega;q)\}_{n\geq 1}$ is a (q,ω)-Appell sequence of polynomials, and let the function A from (3.39) satisfy*

$$\frac{\mathcal{D}_q A(t)}{A(qt)} = \sum_{k=0}^{\infty} \alpha_k t^k, \tag{3.45}$$

and assume that the sequence $\{\beta_k\}_{k\geq 0}$ is given by

$$\sum_{k=0}^{\infty} \lim \beta_k t^k = (1+\omega t) \sum_{k=0}^{\infty} \lim \alpha_k t^k \Longleftrightarrow \beta_k = \alpha_k + \omega \alpha_{k-1}, \tag{3.46}$$

where $k \geq 0$ and $\alpha_{-1} = 0$. Then, the (q,ω)-Appell sequence has following properties.

i) A recurrence relation:

$$P_{n+1}(x;\omega,q) = (x + \beta_0 q^n - \omega[n]_q) P_n(x;\omega,q) + \sum_{k=0}^{\infty} \beta_k q^{n-k} \frac{[n]_q!}{[n-k]_q!} P_{n-k}(x;\omega,q)$$

ii) A difference equation:

$$\left[\sum_{k=2}^{n}\beta_{k-1}q^{n-k}\mathcal{D}_{q,\omega}^{k}+(x+\beta_0 q^{n-1}-\omega[n-1]_q\mathcal{D}_{q,\omega}-[n]_q\right]P_n(x;\omega,q)=0$$

Proof. (i) Taking q-derivative of (3.39), it becomes in (2.4) and (3.37) that

$$\mathcal{D}_q A(t)e_q^{\omega t}(x)(xqt)+\frac{x}{1+\omega t}A(t)e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}[n]_q P_n(x;\omega,q)\frac{t^{n-1}}{[n]_q!}. \tag{3.47}$$

Then, we get

$$(1+\omega t)\frac{\mathcal{D}_q A(t)}{A(qt)}\sum_{n=0}^{\infty}q^n P_n(x;\omega,q)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}xP_n(x;\omega,q)\frac{t^n}{[n]_q!} \tag{3.48}$$

$$=\sum_{n=0}^{\infty}[P_{n+1}(x;\omega,q)+\omega[n]_q P_n(x;\omega,q)]\frac{t^n}{[n]_q!}. \tag{3.49}$$

Using (3.45),(3.46) in (3.49) yields

$$\left[\sum_{k=0}^{\infty}\beta_k t^k\right]\left[\sum_{n=0}^{\infty}q^n P_n(x;\omega,q)\frac{t^n}{[n]_q!}\right]+\sum_{n=0}^{\infty}xP_n(x;\omega,q)\frac{t^n}{[n]_q!}$$

$$=\sum_{k=0}^{\infty}[P_{n+1}(x;\omega,q)+\omega[n]_q P_n(x;\omega,q)]\frac{t^n}{[n]_q!}. \tag{3.50}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in (3.50) then applying the Cauchy product, the result follows.

(ii) Putting $n-1$ instead of n in the recurrence relation, and multiplying it by $[n]_q$, leads to

$$[n]_q P_n(x;\omega,q) = (x+\beta_0 q^{n-1} - \omega[n-1]_q)[n]_q P_{n-1}(x;\omega,q)$$
$$+\sum_{k=2}^{n} \beta_{k-1} q^{n-k} \frac{[n]_q!}{[n-k]_q!} P_{n-k}(x;\omega,q).$$

By considering the equality $\mathcal{D}_{q,\omega}^{k} P_n(x;\omega,q) = \frac{[n]_q!}{[n-k]_q!} P_{n-k}(x;\omega,q)$, we achieve the stated result. $\square$

Chapter 4

(p, q)-CALCULUS

4.1 The (p, q)-derivative

Let p and q be two arbitrary numbers in $\mathbb{R}$ and $\mathbb{C}$. Then the (p,q)-derivative is defined by

$$\mathcal{D}_{(p,q)}f(t) = \frac{f(qt) - f(pt)}{(q-p)t}, \qquad t \neq 0, \tag{4.1}$$

and $\left(\mathcal{D}_{(p,q)}f\right)(0) = f'(0)$ provided that the function f is differentiable at 0. The (p,q)-derivative is investigated in [3] starting from the q-derivative which was given by (2.1). Let me introduce some notation from [30]. (p,q)-bracket or *twin basic number* and given by

$$[n]_{p,q} := p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = [n]_{q,p}, \qquad 0 < q < p. \tag{4.2}$$

By some simple algebra from (4.2), we obtain that

$$p\,[n]_{p,q} = p^n + p^{n-1}q + p^{n-2}q^2 + \cdots + p^2q^{n-1} + pq^{n-1},$$

$$q\,[n]_{p,q} = qp^{n-1} + p^{n-2}q^2 + p^{n-3}q^3 + \cdots + pq^{n-1} + q^n,$$

$$p\,[n]_{p,q} - q\,[n]_{p,q} = p^n - q^n,$$

Finally, we have

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{4.3}$$

From [31], if we take limit of (4.3) when $p \to 1$ we can obtain q-analogue of n. So

$$\lim_{p\to 1} [n]_{p,q} = [n]_q, \qquad q \neq 1.$$

The (p,q)-powers are defined by [3, 31]:

$$(t \ominus b)^n_{p,q} = (t-b)(pt-qb)...(pt^{n-1} - qb^{n-1}). \tag{4.4}$$

The (p,q)-factorial is defined by:

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \;\; n \geq 1, \qquad \text{and} \;\; [0]_{p,q}! = 1. \tag{4.5}$$

The (p,q)-binomial coefficient is,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}, \;\; 0 \leq k \leq n. \tag{4.6}$$

In [31], it is obvious that.

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}. \tag{4.7}$$

From [34], the (p,q)-analogues of Pascal's identity are defined by:

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} &= p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &= q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \end{aligned}$$

where $k = \{0, 1, 2, 3, ..., n\}$.

Proposition 4.1.1. *[3, 32] The product rules of (p,q)-derivative is as follows*

$$\mathcal{D}_{p,q}(f(t)g(t)) = f(pt)\mathcal{D}_{p,q}g(t) + g(qt)\mathcal{D}_{p,q}f(t) \tag{4.8}$$

$$= g(pt)\mathcal{D}_{p,q}f(t) + f(qt)\mathcal{D}_{p,q}g(t). \tag{4.9}$$

Proposition 4.1.2. *[32] The Quotient Rule for (p,q)-derivatives:*

$$\mathcal{D}_{p,q}\left(\frac{f(t)}{g(t)}\right) = \frac{g(qt)\mathcal{D}_{p,q}f(t) - f(qt)\mathcal{D}_{p,q}g(t)}{g(pt)g(qt)} = \frac{g(pt)\mathcal{D}_{p,q}f(t) - f(pt)\mathcal{D}_{p,q}g(t)}{g(pt)g(qt)},$$

provided $g(pt) \neq 0$ and $g(qt) \neq 0$.

Theorem 4.1.1 (Chain Rule for (p,q)-derivative [33]). *Firstly, there doesn't exist a general chain rule for (p,q)-derivatives. However if the differentiation of a function of the form $g(z(t))$ and $z(t)$ is equals to at^b, a,b are constants then chain rule exists for (p,q)-derivatives.*

Consider,

$$\begin{aligned}
\mathcal{D}_{p,q}[g(z(t))] = \mathcal{D}_{p,q}\left[g(at^b)\right] &= \frac{g(ap^bt^b) - g(at^bq^b)}{pt - qt} \\
&= \frac{g(ap^bt^b) - g(at^bq^b)}{ap^bt^b - at^bq^b} . \frac{ap^bt^b - at^bq^b}{pt - qt} \\
&= \frac{g(zp^b) - g(zq^b)}{zp^b - zq^b} . \frac{z(pt) - z(qt)}{pt - qt}.
\end{aligned}$$

Then we can obtain that

$$\mathcal{D}_q g(z(t)) = \left(\mathcal{D}_{p^b,q^b} g\right)(z(t)).\mathcal{D}_{p,q} z(t).$$

Theorem 4.1.2. *[32] Assume that f is an nth-order (p,q)-differentiable function.*

Then,

$$\left(\mathcal{D}^n_{p,q} f\right)(t) = \frac{q^{-\binom{n}{2}}}{t^n(p-q)^n} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{q^{\binom{k}{2}} f(tp^k q^{n-k})}{p^{k(2n-k-1)/2}} \tag{4.10}$$

Proof. From (4.1) and (4.10), for $n = 1$

$$\mathcal{D}_{(p,q)} f(t) = \frac{f(qt) - f(pt)}{(q-p)t} = \frac{f(qt)}{(q-p)t} + \frac{f(pt)}{(q-p)t} = [1]_{p,q}![qt, pt; f]$$

and for $n = 2$,

$$\begin{aligned}
\mathcal{D}^2_{(p,q)} f(t) &= \frac{\left(\mathcal{D}^n_{p,q} f\right) f(qt) - \left(\mathcal{D}^n_{p,q} f\right) f(pt)}{(q-p)t} \\
&= \frac{\frac{f(q^2t)-f(pqt)}{(q-p)t} - \frac{f(pqt)-f(p^2t)}{(q-p)t}}{(p-q)t} \\
&= (p+q)\left[\frac{f(q^2t)}{(q^2-p^2)(q-p)t^2q} - \frac{f(pqt)}{(q-p)^2t^2pq} + \frac{f(p^2t)}{(q^2-p^2)(q-p)t^2p}\right] \\
&= [2]_{p,q}![q^2t, pqt, p^2t; f],
\end{aligned}$$

and continuing to n, we obtain

$$\left(\mathcal{D}^n_{p,q} f\right)(t) = [n]_{p,q}![q^n t, q^{n-1}pt, ..., qp^{n-1}t, p^n t; f], \tag{4.11}$$

and as known from [32],

$$[t_0, t_1, ..., t_n; .] = \frac{[t_1, t_2, ..., t_n; .] - [t_0, t_1, ..., t_{n-1}; .]}{t_n - t_0}.$$

□

Theorem 4.1.3 ((p,q) Leibniz Rule [32])**.** *Let functions $f, g : \mathcal{D}_{p,q} \to \mathbb{C}$ be nth-order (p,q)-differentiable. After that,*

$$\mathcal{D}^n_{p,q}(fg)(t) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{p,q} \mathcal{D}^m_{p,q}(f)(tp^{n-m}) \mathcal{D}^{n-m}_{p,q}(g)(tq^m) \tag{4.12}$$

Proof. Assume that functions $f,g:\mathcal{D}_{p,q}\to\mathbb{C}$ are nth-order (p,q)-differentiable. Therefore, $(fg)(t)$ is nth-order (p,q)-differentiable and

$$\mathcal{D}^n_{p,q}(fg)(t)=[n]_{p,q}!\sum_{m=0}^{n}[q^nt,q^{n-1}pt,...,q^{n-m+1}p^{m-1}t,q^{n-m}p^kt;f]$$
$$\times[q^{n-m}p^mt,q^{n-m-1}p^{k+1}t,...,qp^{n-1}t,p^nt;g] \quad (4.13)$$

Proof of this theorem is clear with lemma and (4.11). □

Corollary 4.1.1. *[32] Assume that the function f be (p,q)-differentiable function of order n and $p,q\in\mathbb{C}$ such that $0<|q|<|p|\leqq 1$. Then*

$$f(x)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_{p,q}q^{m(m-n)}p^{\binom{m+1}{2}}(qt-pt)^m(\mathcal{D}^k_{p,q}f)\left(\frac{tp^{n-m}}{q^m}\right). \quad (4.14)$$

Proof. Since for $m\in 0,1,...,n$,

$$\begin{bmatrix}n\\k\end{bmatrix}_{\frac{1}{p},\frac{1}{q}}=\frac{[n]_{\frac{1}{p},\frac{1}{q}}!}{[n-m]_{\frac{1}{p},\frac{1}{q}}![m]_{\frac{1}{p},\frac{1}{q}}!}=\frac{(pq)^{-\binom{n}{2}}}{(pq)^{-\binom{n-m}{2}}(pq)^{-\binom{m}{2}}}\begin{bmatrix}n\\m\end{bmatrix}_{p,q}, \quad (4.15)$$

we can obtain that for $n=1$, we have

$$(\mathcal{D}_{\frac{1}{p},\frac{1}{q}}f)(t)=\frac{f(\frac{t}{q})-f(\frac{t}{p})}{(p-q)t}(pq)=pq(\mathcal{D}_{p,q}f)\left(\frac{t}{pq}\right)$$

and for $n=2$ we have

$$\begin{aligned}(\mathcal{D}^2_{\frac{1}{p},\frac{1}{q}}f)(t)&=\frac{pq(\mathcal{D}_{p,q}f)(\frac{t}{pq})-pq(\mathcal{D}_{p,q}f)(\frac{t}{pq})}{(\frac{1}{p}-\frac{1}{q})t}\\&=\frac{(pq)^2[(\mathcal{D}_{p,q}f)(\frac{t}{pq^2})-(\mathcal{D}_{p,q}f)(\frac{t}{pq})]}{(p-q)t}\\&=p^2q^2(\mathcal{D}^2_{p,q}f)\left(\frac{t}{p^2q^2}\right).\end{aligned}$$

Finally,

$$(\mathcal{D}^n_{\frac{1}{p},\frac{1}{q}}f)(t) = p^n q^n (\mathcal{D}^n_{p,q}f)(\frac{t}{p^n q^n})$$

for all n,

$$f(t) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{p,q} q^{m(m-n)} p^{\binom{m+1}{2}} (qt-pt)^m (\mathcal{D}^m_{p,q}f)\left(\frac{tp^{n-m}}{q^m}\right),$$

So, the proof is completed. □

4.2 The $(\mathbf{p},\mathbf{q})$-integral

In [3], the inverse of (p,q)-derivative is defined and they called it the (p,q)-integral.

Let g be any function and $b \in \mathbb{R}$. The (p,q)-integral of $g(t)$ on $[0,b]$ is descibed as follows [3, 30]

$$\int_0^b g(t)d_{p,q}t := b(q-p)\sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} g\left(\frac{p^k}{q^{k+1}}b\right), \qquad |\frac{p}{q}| < 1, \tag{4.16}$$

$$\int_0^b g(t)d_{p,q}t := t(p-q)\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} g\left(\frac{q^k}{p^{k+1}}b\right), \qquad |\frac{q}{p}| < 1. \tag{4.17}$$

For the (p,q)-integral on a semi-infinite interval $[0;+\infty]$ we define

$$\int_0^{\infty} g(t)d_{p,q}t := (p-q)\sum_{k=-\infty}^{\infty} \frac{q^k}{p^{k+1}} g\left(\frac{q^k}{p^{k+1}}\right), \qquad 0 < \frac{q}{p} < 1, \tag{4.18}$$

$$\int_0^{\infty} g(t)d_{p,q}t := (q-p)\sum_{k=-\infty}^{\infty} \frac{p^k}{q^{k+1}} g\left(\frac{p^k}{q^{k+1}}\right), \qquad \frac{q}{p} > 1. \tag{4.19}$$

From (4.17),

$$\begin{aligned}\int_0^\infty g(t)\mathcal{D}_{(p,q)}f(t)d_{p,q}t &= (p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}g(\frac{q^k}{p^{k+1}}x)\mathcal{D}_{(p,q)}f\left(\frac{q^k}{p^{k+1}}t\right)\\ &= (p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}g\left(\frac{q^k}{p^{k+1}}t\right)\frac{f(\frac{q^k}{p^k}t)-f(\frac{q^{k+1}}{p^{k+1}}t)}{(p-q)\frac{q^k}{p^{k+1}}t}\\ &= \sum_{k=0}^{\infty}g\left(\frac{q^k}{p^{k+1}}t\right)\left(f\left(\frac{q^k}{p^k}t\right)-f\left(\frac{q^{k+1}}{p^{k+1}}t\right)\right),\end{aligned}$$

in other words,

$$\int g(t)d_{p,q}f(t) = \sum_{k=0}^{\infty}g\left(\frac{q^k}{p^{k+1}}t\right)\left(f\left(\frac{q^k}{p^k}t\right)-f\left(\frac{q^{k+1}}{p^{k+1}}t\right)\right). \tag{4.20}$$

Theorem 4.2.1 (The FT of (p,q)-calculus [3]). *If $G(t)$ continuous at $t=0$ and its (p,q)-derivative is of g, after that*

$$\int_a^b g(t)d_{p,q}t = G(b)-G(a), \quad for \quad 0\le a\le b\le\infty. \tag{4.21}$$

Proof. The function $G(t)$ is given by

$$G(t) = t(p-q)\sum_{m=0}^{\infty}\frac{q^m}{p^{m+1}}g\left(\frac{q^m}{p^{m+1}}a\right)+G(0).$$

We can say that from (4.20),

$$\int_0^a g(t)d_{p,q}t = G(a)-G(0),$$

and

$$\int_0^b g(t)d_{p,q}t = G(b)-G(0),$$

and thence we get

$$\int_a^b g(t)d_{p,q}t = \int_0^b g(t)d_{p,q}t - \int_0^a g(t)d_{p,q}t = G(b)-G(a).$$

Exchanging a with $\frac{q^{m+1}}{p^{m+1}}$ and b with $\frac{q^m}{p^m}$ and considering (4.20), we can see that (4.21) will also be right for $b = \infty$. □

Remark 4.2.1. [3] If we take $p = 1$ in (4.19), we can obtain the well known Jackson Integral (2.19).

Corollary 4.2.1. *[3] If $f(x)$ is a function whose ordinary derivative exists in a neighbourhood of $t = 0$ and it is continuous at the point $t = 0$, then we obtain that*

$$\int_a^b \mathcal{D}_{p,q} f(t) d_{p,q} t = f(b) - f(a). \tag{4.22}$$

Proof. By L'Hospital's rule,

$$\begin{aligned}\lim_{t\to 0} \mathcal{D}_{p,q} f(t) f(t) &= \lim_{t\to 0} \frac{f(pt) - f(qt)}{(p-q)t} \\ &= \lim_{t\to 0} \frac{pf'(pt) - qf'(qt)}{p-q} = f'(0).\end{aligned}$$

So defining $(\mathcal{D}_{p,q} f)(0) = f'(0)$ ensures that$\mathcal{D}_{p,q} f(t)$ is continuous at $t = 0$ and then (4.22) follows. □

From [3], an important property of the q-integral and (p,q)-integral, compared with ordinary integration, is that behaviour of a function at $t = 0$ can affect its integral on any interval. This can be seen from the definition and convergence conditions for the derinite (p,q)-integral.

For any functions $f(t)$ and $g(t)$ whose ordinary derivatives exist close to $t = 0$, the product rule (4.9) combined with the FTC gives

$$f(\beta)g(\beta) - f(\alpha)g(\alpha) = \int_{\alpha}^{\beta} f(pt)\cdot \mathcal{D}_{p,q}f(t)d_{p,q}t + \int_{\alpha}^{\beta} g(qt)\cdot \mathcal{D}_{p,q}f(t)d_{p,q}t,$$

or the following rule for (p,q)-integration by parts:

$$\int_{\alpha}^{\beta} f(pt)\mathcal{D}_{p,q}g(t)d_{p,q}t = f(\beta)g(\beta) - f(\alpha)g(\alpha) - \int_{\alpha}^{\beta} g(qt)\mathcal{D}_{p,q}f(t)d_{p,q}t.$$

Note that $b = \infty$ is possible.

4.3 (p, q)-Exponential Functions

The (p,q)-exponential functions from [31] are

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} z^n = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}}}{((p,q);(p,q))_n} z^n, \tag{4.23}$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} z^n = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{((p,q);(p,q))_n} z^n, \tag{4.24}$$

and we can say that,

$$e_{p,q}(x)E_{p,q}(-x) = 1.$$

4.4 (p, q)-Trigonometric Functions

The (p,q)-Trigonometric Functions can be defined by

$$sin_{p,q}z = \frac{e_{p,q}^{iz} - e_{p,q}^{-iz}}{2i}, \qquad Sin_{p,q}z = \frac{E_{p,q}^{iz} - E_{p,q}^{-iz}}{2i},$$

$$\cos_{p,q} z = \frac{e_{p,q}^{iz} + e_{p,q}^{-iz}}{2}, \qquad Cos_{p,q} = \frac{E_{p,q}^{iz} + E_{p,q}^{-iz}}{2}.$$

and

$$Sin_{p,q}z = \sin_{\frac{1}{p,q}} z, \qquad Cos_{p,q}z = \cos_{\frac{1}{p,q}} z.$$

Then we have

$$\cos_{p,q} z Cos_{p,q}z = \frac{e_{p,q}^{iz}E_{p,q}^{iz} + e_{p,q}^{-iz}E_{p,q}^{-iz} + 2}{4},$$

$$\sin_{p,q} z Sin_{p,q} z = -\frac{e^{iz}_{p,q}E^{iz}_{p,q} + e^{-iz}_{p,q}E^{-iz}_{p,q} - 2}{4}.$$

Hence, we get

$$\cos_{p,q} z Cos_{p,q} z + \sin_{p,q} z Sin_{p,q} z = 1, \qquad where \quad \sin^2 z + \cos^2 z = 1.$$

Applying the p,q-derivative to p,q-trigonometric functions,

$$\mathcal{D}_{p,q} \sin_{p,q} z = \cos_{p,q} z, \qquad \mathcal{D}_{p,q} Sin_{p,q} z = Cos_{p,q} z,$$

$$\mathcal{D}_{p,q} \cos_{p,q} z = -\sin_{p,q} qz, \qquad \mathcal{D}_{p,q} Cos_{p,q} z = -Sin_{p,q} qz.$$

4.5 The $(\mathbf{p},\mathbf{q})$-Appell Polynomials

This section is generally composed of reference [31].In this section we introduce sequences of (p,q)-Appell polynomials. We study some of their algebraic properties, recurrence relations, difference equations, etc. for such polynomials. In addition, if we replace p with 1, we can create the q-Appell polynomial sequences which were mentioned in chapter 2. If we replace p,q with 1, we gain some other Appell polynomial sequences which were studied in [35].

Definition 4.5.1. [31] Let a sequence of polynomials is $\{P_n(t)\}_{n\geq 0}^{\infty}$ and if

$$\mathcal{D}_{p,q} P_{n+1}(t) = [n+1]_{p,q} P_n(pt), \qquad n \geq 0. \tag{4.25}$$

it is called a (p,q)-Appell sequence.

For example from [3], the sequence $(x \ominus a)^n_{p,q}$ defined by (4.4). It can be checked that this satisfies the condition for being (p,q)-Appell.

Now, we prove the several characterization of (p,q)-Appell polynomial sequences.

Theorem 4.5.1. *[31] For any sequence of polynomials $\{P_n(t)\}_{n\geq 0}^{\infty}$ the following conditions are equivalent:*

i) The sequence $\{P_n(t)\}_{n\geq 0}^{\infty}$ is (p,q)-Appell.

ii) There is a sequence of numbers $(a_k)_{k\geq 0}$, with $a_0 \neq 0$, all independent of n, such that

$$P_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} a_k x^{n-k}, \qquad n \geq 0.$$

iii) The polynomial sequence is given by a generating function

$$A(t)e_{p,q}(xt) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{[n]_{p,q}!},$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!}.$$

iv) There is a sequence of numbers $(a_k)_{k\geq 0}$, with $a_0 \neq 0$, all independent of n, such that

$$P_n(x) = \left(\sum_{k=0}^{\infty} \frac{p^{\binom{n-k}{2}} a_k}{[k]_{p,q}!} D_{p,q}^k \right) x^n.$$

Proof. Firstly, we will show that $(i) \Longrightarrow (ii)$. Since $\{P_n(t)\}_{n=0}^{\infty}$ is a sequences of polynomials and

$$P_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n,k} p^{\binom{n-k}{2}} x^{n-k}, \qquad n = 1, 2, 3... \tag{4.26}$$

where $a_{n,k}$ may depend on n and $a_{n,k} \neq 0$. We will show that $a_{n,k}$ is independent of n.

Applying (4.1) to (4.26), we can obtain that,

$$P_{n-1}(x) = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n,k} p^{\binom{n-1-k}{2}} (px)^{n-1-k}, \qquad n = 1,2,3... \tag{4.27}$$

where $\mathcal{D}_{p,q} x^0 = 0$. Changing n with $n+1$ and Replacing x with xp^{-1} in (4.27) we get

$$\{P_n(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n+1,k} p^{\binom{n-k}{2}} x^{n-k}, \qquad n = 1,2,3... \tag{4.28}$$

Comparing (4.26) and (4.28), we get $a_{n+1,k} = a_{n,k}$ always, whence the $a_{n,k}$ do not depend on n.

Now, we will show that $(ii) \Longrightarrow (iii)$. From (ii), we get

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} a_k x^{n-k} \right) \frac{t^n}{[n]_{p,q}!} \\ &= \left(\sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!} \right) \left(\frac{p^{\binom{n}{2}}}{[n]_{p,q}!} (xt)^n \right) \\ &= A(t) e_{p,q}(xt). \end{aligned}$$

Finally, we will show that $(iii) \Longrightarrow (i)$. Let $P_n(t)\}_{n=0}^{\infty}$ be given by

$$A(t) e_{p,q}(xt) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{[n]_{p,q}!},$$

Using $\mathcal{D}_{p,q}$ on this, we obtain that

$$\begin{aligned} tA(t)e_{p,q}(pxt) &= \sum_{n=0}^{\infty} \mathcal{D}_{p,q}P_n(x)\frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} [n]_{p,q}P_{n-1}(px)\frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

Comparing t^n coefficients $\forall\, n$, we get (i). Then $(ii) \iff (iv)$ is clearly from $D^k_{p,q}t^n = 0$ for $k{>}n$. $\square$

Theorem 4.5.2 (Recurrence relations [31]). *Let $\{P_n(t)\}_{n\geq 0}$ be a (p,q)-Appell sequence of polynomials with generating function*

$$\mathbb{A}(x,t) = A(t)e_{p,q}(xt) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{[n]_{p,q}!}. \tag{4.29}$$

Then there is a recurrence relation, which is linear and homogeneous, given by

$$P_n\left(\frac{px}{q}\right) = \frac{1}{[n]_{p,q}}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \alpha_k P_{n-k}(x) + p^{n-14}q^{-1}xP_{n-1}(x), \tag{4.30}$$

where

$$t\frac{\mathcal{D}_{p,q}A(t)}{A(pt)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_{p,q}!}. \tag{4.31}$$

Proof. At the beginning, substituting x by xp in condition (3) and applying (4.1) (with respect to t) to both sides of condition (3) then multiplying by t, firstly we get

$$t\mathcal{D}^{\{t\}}_{p,q}\mathbb{A}(px,t) = t\sum_{n=0}^{\infty} [n]_{p,q}P_n(px)\frac{t^{n-1}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} [n]_{p,q}P_n(px)\frac{t^n}{[n]_{p,q}!}.$$

Secondly we have

$$
\begin{aligned}
t\mathcal{D}_{p,q}^{\{t\}}\mathbb{A}(px,t) &= t\left[\mathcal{D}_{p,q}^{\{t\}}\mathbb{A}A(t)e_{p,q}(pxt)\right] \\
&= t\left[A(pt)\mathcal{D}_{p,q}^{\{t\}}\mathbb{A}A(t)e_{p,q}(pxt)+e_{p,q}(pqxt)\mathcal{D}_{p,q}^{\{t\}}A(t)\right] \\
&= tpxA(pt)e_{p,q}(pqxt)+t\mathcal{D}_{p,q}A(t)e_{p,q}(pqxt) \\
&= A(pt)e_{p,q}(pqxt)\left(tpx+t\frac{\mathcal{D}_{p,q}^{\{t\}}A(t)}{A(pt)}\right) \\
&= \mathbb{A}(qx,pt)\left(tpx+t\frac{\mathcal{D}_{p,q}^{\{t\}}A(t)}{A(pt)}\right).
\end{aligned}
$$

These 2 equations imply that,

$$
\begin{aligned}
\sum_{n=0}^{\infty}[n]_{p,q}P_n(px)\frac{t^n}{[n]_{p,q}!} &= \mathbb{A}(qx,pt)\left(tpx+t\frac{\mathcal{D}_{p,q}^{\{t\}}A(t)}{A(pt)}\right) \\
&= \left(\sum_{n=0}^{\infty}p^nP_n(qx)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_{p,q}!}+tpx\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\alpha_k p^{n-k}P_{n-k}(qx)\right)\frac{t^n}{[n]_{p,q}!} \\
&+x\sum_{n=0}^{\infty}p^{n+1}P_n(qx)\frac{t^{n+1}}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\alpha_k p^{n-k}P_{n-k}(qx)\right)\frac{t^n}{[n]_{p,q}!} \\
&+x\sum_{n=0}^{\infty}[n]_{p,q}p^nP_{n-1}(qx)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}
$$

Comparing t^n coefficients, we gain

$$
[n]_{p,q}P_n(px) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\alpha_k p^{n-k}P_{n-k}(qx)+[n]_{p,q}p^nP_{n-1}(qx),
$$

After that, shifting x replaced by xp gives the result. $\square$

Theorem 4.5.3 (((p,q)-difference for Appell Polynomials [31]). *Let $\{P_n(t)\}_{n\geq 0}$ be a sequence of (p,q)-Appell polynomials with generating function which in (4.27). Consider (4.31), we get*

$$t\frac{\mathcal{D}_{p,q}^{\{t\}}A(t)}{A(pt)} = \sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_{p,q}!}, \tag{4.32}$$

which is valid near $t=0$. Then the following (p,q)-difference equation is satisfied:

$$\left(\sum_{k=0}^{n}\frac{\alpha_k}{[k]_{p,q}!}\mathcal{L}_p^{-k}\mathcal{D}_{p,q}^k + p^n q^{-1}x\mathcal{L}_p^{-1}\mathcal{D}_{p,q}\right)P_n(x) - [n]_{p,q}P_n\left(\frac{px}{q}\right) = 0,$$

where the operator $\mathcal{L}_p$ is defined by

$$\mathcal{L}_p^k f(x) = f(p^k x), \qquad k\in\mathbb{Z}.$$

Proof. The P_n satisfy the recursion formula, it is shown in (4.30). Since the sequence $\{P_n(t)\}_{n\geq 0}$ is (p,q)-Appell, we get

$$D_{p,q}^k P_n(x) = \frac{[n]_{p,q}!}{[n-k]_{p,q}!}P_{n-k}(p^k x), \qquad 0\leq k\leq n.$$

or in other words,

$$P_{n-k}(x) = \frac{[n-k]_{p,q}!}{[n]_{p,q}!}\mathcal{L}_p^{-k}D_{p,q}^k P_n(x), \qquad 0\leq k\leq n.$$

Then from (4.30),

$$\begin{aligned}
P_n\left(\frac{px}{q}\right) &= \frac{1}{[n]_{p,q}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \alpha_k \frac{[n-k]_{p,q}!}{[n]_{p,q}!} \mathcal{L}_p^{-k} D_{p,q}^k P_n(x) + p^n q^{-1} x P_{n-1}(x) \\
&= \frac{1}{[n]_{p,q}} \sum_{k=0}^{n} \frac{\alpha_k}{[k]_{p,q}!} \mathcal{L}_p^{-k} D_{p,q}^k P_n(x) + p^n q^{-1} x P_{n-1}(x) \\
&= \frac{1}{[n]_{p,q}} \left(\sum_{k=0}^{n} \frac{\alpha_k}{[k]_{p,q}!} \mathcal{L}_p^{-k} D_{p,q}^k + p^n q^{-1} x \mathcal{L}_p^{-k} \mathcal{D}_{p,q} \right) P_n(x)
\end{aligned}$$

□

4.6 The Big (p, q)-Appell Polynomials

This section is composed of reference [24]. In this section, we will present the big (p,q)- Appell polynomials, prove an equivalence theorem fulfilled by them, and obtain recurrence relations fulfilled by them.

Definition 4.6.1. From [24], the big (p,q)-Appell Polynomials are defined by the following relation, similar to (4.5) but with $E_{p,q}$ instead of $e_{p,q}$:

$$A_{p,q} E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!},$$

where

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}, \qquad 0 < \left|\frac{q}{p}\right| < 1; |x| < 1$$

and

$$A_{p,q} = \sum_{n=0}^{\infty} a_{n,p,q} \frac{t^n}{[n]_{p,q}!}.$$

The big (p,q)-Appell polynomials satisfy the following relation:

$$\left(\mathcal{D}_{p,q} P_{n,p,q}\right)(x) = \frac{[n]_{p,q}}{q} P_{n-1,p,q}(qx),$$

where $[n]_{p,q}$ is defined in (4.2).

In addition, replacing p with 1 in definition of the big (p,q)-Appell polynomials, we can obtain the big q-Appell polynomials defined by

$$A_q E_q\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,q}(x)\frac{t^n}{[n]_q!},$$

where

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{x^n}{[n]_q!}, \qquad 0<|q|<1; x \in \mathbb{C}.$$

Theorem 4.6.1. *[24] The following statements are all equivalent to one another:*

1) $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ *is a big* (p,q)*-Appell sequence.*

2) The sequence $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ *has an explicit form given by*

$$P_{n,p,q}(x) = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q} q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k.$$

3) The sequence $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ *has a generating function,*

$$A_{p,q}E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!},$$

where

$$A_{p,q}(t) = \sum_{k=0}^{\infty} a_{k,p,q}\frac{t^k}{[k]_{p,q}!}.$$

Proof. Fistly, we will prove that $(1) \Longrightarrow (2)$. Assuming, condition (1) is correct, we get

$$P_{n,p,q}(x) = \sum_{k=0}^{n} a_{n,k,p,q}[x]_{p,q}^k, \tag{4.33}$$

where,

$$[x]^k_{p,q} = x\left(\frac{q}{p}x\right)\left(\frac{q^2}{p^2}x\right)\cdots\left(\frac{q^{k-1}}{p^{k-1}}x\right) = \frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}x^k.$$

and the (p,q)-derivative of $[x]^k_{p,q}$ is given by,

$$\mathcal{D}_{p,q}[x]^k_{p,q} = [k]_{p,q}\frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}x^{k-1}.$$

Applying the (p,q)-derivative on both sides of (4.33) and using (4.6), we have

$$\frac{[n]_{p,q}}{q}P_{n-1,p,q}(qx) = \sum_{k=1}^{n} a_{n,k,p,q}\frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}x^{k-1}[k]_{p,q},$$

i.e.

$$P_{n-1,p,q}(qx) = \frac{q}{[n]_{p,q}}\sum_{k=1}^{n} a_{n,k,p,q}\frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}x^{k-1}[k]_{p,q}.$$

Shifting k with $k+1$ in the above equation, we get

$$P_{n-1,p,q}(qx) = \frac{q}{[n]_{p,q}}\sum_{k=0}^{n-1} a_{n,k+1,p,q}\frac{q^{\binom{k+1}{2}}}{p^{\binom{k+1}{2}}}x^{k}[k+1]_{p,q}.$$

Now replacing n by $n+1$ and qx by x, we get

$$P_{n,p,q}(x) = \frac{q}{[n+1]_{p,q}}\sum_{k=0}^{n} a_{n+1,k+1,p,q}\frac{q^{\binom{k+1}{2}}}{p^{\binom{k+1}{2}}}\left(\frac{x}{p}\right)^k[k+1]_{p,q}.$$

Then comparing (4.33) and the above equation, we obtain

$$a_{n,k,p,q} = \frac{p^{k-1}}{q}\frac{[n]_{p,q}}{[k]_{p,q}}a_{n-1,k-1,p,q}.$$

If we iterate the above equation k times, we have

$$a_{n,k,p,q} = \frac{p^{\binom{k}{2}}}{q^k}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,0,p,q}.$$

Setting $a_{n-k,0,p,q} = a_{n-k,p,q}$ and adding $a_{n,k,p,q}$ to (4.33), we create that

$$P_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q} q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k.$$

Secondly, we will prove $(2) \Longrightarrow (3)$. Assume that

$$P_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q} q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k.$$

Summing both sides of the above equation from $n = 0$ to $n = \infty$ and multiplying by $\frac{t^n}{[n]_{p,q}!}$, we can obtain

$$\sum_{n=0}^{\infty} P_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q} q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k \frac{t^n}{[n]_{p,q}!}.$$

Now, using the Cauchy product in the above equation, we get

$$\begin{aligned} \sum_{n=0}^{\infty} P_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q} a_{n,p,q} q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k \frac{t^{n+k}}{[n+k]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_{n,p,q}}{[n]_{p,q}![k]_{p,q}!} q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k t^{n+k} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{[n]_{p,q}!} \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{\left(\frac{xt}{q}\right)^k}{[k]_{p,q}!}, \end{aligned}$$

which, in statement (3) and the series expression for the big (p,q)-exponential function, yields

$$\sum_{n=0}^{\infty} P_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = A_{p,q}(t) E_{p,q}\left(\frac{xt}{q}\right).$$

Thirdly, I will prove (3) $\Longrightarrow$ (1) . Let, $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ has a generating function and it is

$$A_{p,q}(t)E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}.$$

Then take the (p,q)-derivative by x on both sides of the above equation, we can obtain that

$$A_{p,q}(t)\mathcal{D}_{p,q,x}\left(E_{p,q}\right)\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} \mathcal{D}_{p,q,x}\left(P_{n,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!},$$

and

$$\mathcal{D}_{p,q,x}\left(E_{p,q}\right) = \frac{t}{q}E_{p,q}(xt).$$

So,

$$A_{p,q}(t)\frac{t}{q}E_{p,q}(xt) = \sum_{n=0}^{\infty} \mathcal{D}_{p,q,x}\left(P_{n,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!}.$$

Replacing the left side by the corresponding series, we have

$$\frac{1}{q}\sum_{n=0}^{\infty}\left(P_{n,p,q}(qx)\right)\frac{t^{n+1}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{D}_{p,q,x}\left(P_{n,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!},$$

in other words,

$$\frac{1}{q}\sum_{n=0}^{\infty}[n]_{p,q}\left(P_{n-1,p,q}(qx)\right)\frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{D}_{p,q,x}\left(P_{n,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!}.$$

Then, comparing $\frac{t^n}{[n]_{p,q}!}$ coefficients, we can obtain that

$$\mathcal{D}_{p,q,x}\left(P_{n,p,q}(x)\right) = \frac{[n]_{p,q}}{q}P_{n-1,p,q}(qx).$$

Finally, it is shown that $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ be a big (p,q)-Appell sequence. $\square$

Theorem 4.6.2. *[24] The big (p,q)-Appell polynomials satisfy*

$$\frac{xp^n}{q}P_{n,p,q}\left(\frac{q}{p}x\right)+P_{n,p,q}(qx)\alpha_{0,p,q}+\sum_{k=0}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\alpha_{n-k,p,q}P_{k,p,q}(qx)=P_{n+1,p,q}(x). \tag{4.34}$$

Proof. Differentiating both sides of the generating function of the big (p,q)-Appell polynomials with respect to t, we get

$$A_{p,q}(pt)\mathcal{D}_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right)+A_{p,q}(t)E_{p,q}(xt)\frac{\mathcal{D}_{p,q,t}\left(A_{p,q}(t)\right)}{A_{p,q}(t)}=\sum_{n=0}^{\infty}P_{n+1,p,q}(x)\frac{t^n}{[n]_{p,q}!}, \tag{4.35}$$

and

$$\begin{aligned}\mathcal{D}_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right)&=\mathcal{D}_{p,q,t}\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{(\frac{xt}{q})^n}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}q^{\binom{n}{2}}q^{-n}x^n\frac{\mathcal{D}_{p,q,t}(t^n)}{[n]_{p,q}!}.\end{aligned}$$

Then inserting $\mathcal{D}_{p,q,t}(t^n)=t^{n-1}[n]_{p,q}$ and after some series manipulations

$$\mathcal{D}_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right)=\frac{x}{q}E_{p,q}(xt).$$

Now, inserting the above equation in (4.35), we can obtain

$$\frac{x}{q}A_{p,q}(pt)E_{p,q}(xt)+A_{p,q}(t)E_{p,q}(xt)\frac{\mathcal{D}_{p,q,t}\left(A_{p,q}(t)\right)}{A_{p,q}(t)}=\sum_{n=0}^{\infty}P_{n+1,p,q}(x)\frac{t^n}{[n]_{p,q}!}. \tag{4.36}$$

Now, if we define

$$\frac{\mathcal{D}_{p,q,t}\left(A_{p,q}(t)\right)}{A_{p,q}(t)}=\sum_{n=0}^{\infty}\alpha_{n,p,q}\frac{t^n}{[n]_{p,q}!},$$

then from (4.36) we can obtain that

$$\frac{x}{q}\sum_{n=0}^{\infty}P_{n,p,q}\left(\frac{q}{p}x\right)\frac{(pt)^n}{[n]_{p,q}!}+\sum_{k=0}^{\infty}P_{k,p,q}(qx)\frac{t^k}{[k]_{p,q}!}\sum_{n=0}^{\infty}\alpha_{n,p,q}\frac{t^n}{[n]_{p,q}!}=\sum_{n=0}^{\infty}P_{n+1,p,q}(x)\frac{t^n}{[n]_{p,q}!}.$$

Now, applying the Cauchy product,

$$\frac{x}{q}\sum_{n=0}^{\infty}P_{n,p,q}\left(\frac{q}{p}x\right)\frac{(pt)^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}P_{k,p,q}(qx)\alpha_{n-k,p,q}\frac{t^n}{[n]_{p,q}!}=\sum_{n=0}^{\infty}P_{n+1,p,q}(x)\frac{t^n}{[n]_{p,q}!}.$$

Equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ in the above equation, we get

$$\frac{xp^n}{q}P_{n,p,q}\left(\frac{q}{p}x\right)+\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}P_{k,p,q}(qx)\alpha_{n-k,p,q}=P_{n+1,p,q}(x),$$

$$\frac{xp^n}{q}P_{n,p,q}\left(\frac{q}{p}x\right)+P_{n,p,q}(qx)\alpha_{0,p,q}+\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}P_{k,p,q}(qx)\alpha_{n-k,p,q}=P_{n+1,p,q}(x)$$

. $\square$

Chapter 5

$(\mathbf{p},\mathbf{q})$-HAHN CALCULUS

This Chapter is very important because it constitutes the purpose of this thesis. In this chapter, the (p,q)-Hahn difference operator is introduced and its properties are investigated. First, we give some definitions and formulas.

5.1 Definition of $(\mathbf{p},\mathbf{q})$-Hahn Difference Operator

In general the (p,q)-Hahn difference operator $\mathcal{D}_{(p,q),\omega}$ unifies Hahn operator and (p,q)-derivative. The definition is

$$\mathcal{D}_{(p,q),\omega}f(t) = \frac{f(qt+\omega)-f(pt)}{(q-p)t+\omega}, \tag{5.1}$$

where $p,q \in [0,1]$ and ω>0.

The particular cases os the operator (5.1) can be found by taking the limits as.

$$\lim_{\substack{q\to 1\\ p\to 1\\ \omega\to 0}} \mathcal{D}_{(p,q),\omega}f(t) = \lim_{\substack{q\to 1\\ p\to 1\\ \omega\to 0}} \frac{f(qt+\omega)-f(pt)}{(q-p)t+\omega} = f'(t), \tag{5.2}$$

$$\lim_{\substack{q\to 1\\ p\to 1}} \mathcal{D}_{(p,q),\omega}f(t) = \lim_{\substack{q\to 1\\ p\to 1}} \frac{f(qt+\omega)-f(pt)}{(q-p)t+\omega} = \frac{f(t+\omega)-f(t)}{(t+\omega)-t} = \Delta_\omega f(t), \tag{5.3}$$

$$\lim_{\omega\to 0} \mathcal{D}_{(p,q),\omega}f(t) = \lim_{\omega\to 0} \frac{f(qt+\omega)-f(pt)}{(q-p)t+\omega} = \frac{f(qt)-f(pt)}{(q-p)t} = \mathcal{D}_{p,q}f(t), \tag{5.4}$$

$$\lim_{p\to 1} \mathcal{D}_{(p,q),\omega}f(t) = \lim_{p\to 1} \frac{f(qt+\omega)-f(pt)}{(q-p)t+\omega} = \frac{f(qt+\omega)-f(t)}{(q-1)t+\omega} = \mathcal{D}_{q,\omega}f(t). \tag{5.5}$$

5.2 Differentiation

Theorem 5.2.1. *Let f be a function. f is $(p,q),\omega$-differentiable if $\mathcal{D}_{p,q,\omega}f(\omega_0)$ exists where $\omega_0 = \frac{\omega}{1-\frac{p}{q}}$.*

Proof. Assume that $\omega_0 = \frac{\omega}{1-\frac{p}{q}}$. Taking limit of $\mathcal{D}_{p,q,\omega}f(\omega_0)$ gives

$$\lim_{\substack{q\to 1\\ p\to 1\\ \omega\to 0}} \mathcal{D}_{(p,q),\omega}f(\omega_0) = f'(\omega_0).$$

So, since $\mathcal{D}_{p,q,\omega}f(\omega_0)$ exists. then f is $(p,q),\omega$-differentiable. □

Theorem 5.2.2. *Assume that $\mathcal{D}_{(p,q),\omega}f(t) = 0$, $\forall\, t \in R$, then f is constant.*

Proof. From (3.1) we can write that,

$$\mathcal{D}_{\frac{q}{p},\omega}f(pt) = \frac{f(\frac{q}{p}pt+\omega) - f(pt)}{(\frac{q}{p}-1)pt+\omega}.$$

It is obvious that

$$\mathcal{D}_{(p,q),\omega}f(t) = \mathcal{D}_{\frac{q}{p},\omega}f(pt).$$

On the other hand,

$$\mathcal{D}_{(p,q),\omega}f\left(\frac{t}{p}\right) = \frac{f(\frac{q}{p}t+\omega) - f(pt)}{(\frac{q}{p}-1)t+\omega}.$$

It's easy to see that

$$\mathcal{D}_{(p,q),\omega}f\left(\frac{t}{p}\right) = \mathcal{D}_{\frac{q}{p},\omega}f(t).$$

So, from this notation $\mathcal{D}_{(p,q),\omega}f(\frac{t}{p}) = 0\ \forall\, t \in R$ means that $f(\frac{q}{p}t+\omega) = f(t)$.

Let $h_{\frac{q}{p}}(t) = \frac{q}{p}t+\omega$, then $f(t) = f(h_{\frac{q}{p}}(t))\ \forall\, t \in R$. So

$$f(t) = f(h_{\frac{q}{p}}(t)) = f(h^2_{\frac{q}{p}}(t)) = f(h^3_{\frac{q}{p}}(t)) = \dots$$

If $f(t) = f(h_{\frac{q}{p}^n})$ for all n, then in the limit $n \to \infty$ we have $\frac{q}{p}^n \to \omega_0$ and therefore $f(t) = f(\omega_0)$. So, f is constant. $\square$

5.3 Algebraic Properties of $(\mathbf{p},\mathbf{q})$-Hahn Difference Operator

Assume that f, g be p, q, ω-differentiable on the interval I, then

Linearity:

$$\mathcal{D}_{(p,q),\omega}(f+g)(t) = \mathcal{D}_{(p,q),\omega}f(t) + \mathcal{D}_{(p,q),\omega}g(t).$$

Product Rule:

$$\begin{aligned}
\mathcal{D}_{(p,q),\omega}(fg)(t) &= \frac{f(qt+\omega)g(qt+\omega) - f(pt)g(pt)}{(q-p)t+\omega} \\
&= \frac{f(qt+\omega)g(qt+\omega) - f(qt+\omega)g(pt) + f(qt+\omega)g(pt) - f(pt)g(pt)}{(q-p)t+\omega} \\
&= \frac{f(qt+\omega)\left[g(qt+\omega) - g(pt)\right]}{(q-p)t+\omega} + \frac{g(pt)\left[f(qt+\omega) - f(pt)\right]}{(q-p)t+\omega} \\
&= f(qt+\omega)\mathcal{D}_{(p,q),\omega}g(t) + g(pt)\mathcal{D}_{(p,q),\omega}f(t)
\end{aligned}$$

and if we add $\pm g(qt+\omega)f(pt)$

$$\begin{aligned}
\mathcal{D}_{(p,q),\omega}(fg)(t) &= \frac{f(qt+\omega)g(qt+\omega) - f(pt)g(pt)}{(q-p)t+\omega} \\
&= \frac{f(qt+\omega)g(qt+\omega) - g(qt+\omega)f(pt) + g(qt+\omega)f(pt) - f(pt)g(pt)}{(q-p)t+\omega} \\
&= g(qt+\omega)\mathcal{D}_{(p,q),\omega}f(t) + f(pt)\mathcal{D}_{(p,q),\omega}g(t)
\end{aligned}$$

Quotient Rule:

$$\begin{aligned}
\mathcal{D}_{(p,q),\omega}\left(\frac{f}{g}\right)(t) &= \frac{\frac{f(qt+\omega)}{g(qt+\omega)} - \frac{f(pt)}{g(pt)}}{(q-p)t+\omega} \\
&= \frac{\frac{f(qt+\omega)g(pt)-g(pt)f(pt)+g(pt)f(pt)-f(pt)g(qt+\omega)}{g(qt+\omega)g(pt)}}{(q-p)t+\omega} \\
&= \frac{g(pt)\frac{f(qt+\omega)-f(pt)}{(q-p)t+\omega} - f(pt)\frac{g(qt+\omega)-g(pt)}{(q-p)t+\omega}}{g(qt+\omega)g(pt)} \\
&= \frac{g(pt)\mathcal{D}_{(p,q),\omega}f(t) - f(pt)\mathcal{D}_{(p,q),\omega}g(t)}{g(qt+\omega)g(pt)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_{(p,q),\omega}\left(\frac{f}{g}\right)(t) &= \frac{\frac{f(qt+\omega)}{g(qt+\omega)} - \frac{f(pt)}{g(pt)}}{(q-p)t+\omega} \\
&= \frac{\frac{f(qt+\omega)g(pt)-g(qt+\omega)f(qt+\omega)+g(qt+\omega)f(qt+\omega)-f(pt)g(qt+\omega)}{g(qt+\omega)g(pt)}}{(q-p)t+\omega} \\
&= \frac{g(qt+\omega)\frac{f(qt+\omega)-f(pt)}{(q-p)t+\omega} - f(qt+\omega)\frac{g(qt+\omega)-g(pt)}{(q-p)t+\omega}}{g(qt+\omega)g(pt)} \\
&= \frac{g(qt+\omega)\mathcal{D}_{(p,q),\omega}f(t) - f(qt+\omega)\mathcal{D}_{(p,q),\omega}g(t)}{g(qt+\omega)g(pt)}.
\end{aligned}$$

Chapter 6

CONCLUSIONS

In conclusion, I will summarize the general results of q-calculus, Hahn calculus, (p,q)-calculus and $(p,q),\omega$-calculus with table 1. Then we will give the relationship between q-Appell, Hahn Appell polynomials, (p,q)-Appell polynomials, the big q-Appell polynomials, the big (p,q)-Appell with table 2. In table 3, we will compare q-calculus, Hahn calculus, (p,q)-calculus and $(p,q),\omega$-calculus with classical derivative. Finally, in table 4 we will show same properties of q-calculus, Hahn calculus, (p,q)-calculus and $(p,q),\omega$-calculus.

Table 1: The general results of q-calculus, Hahn calculus, (p,q)-calculus and (p,q),ω -calculus.

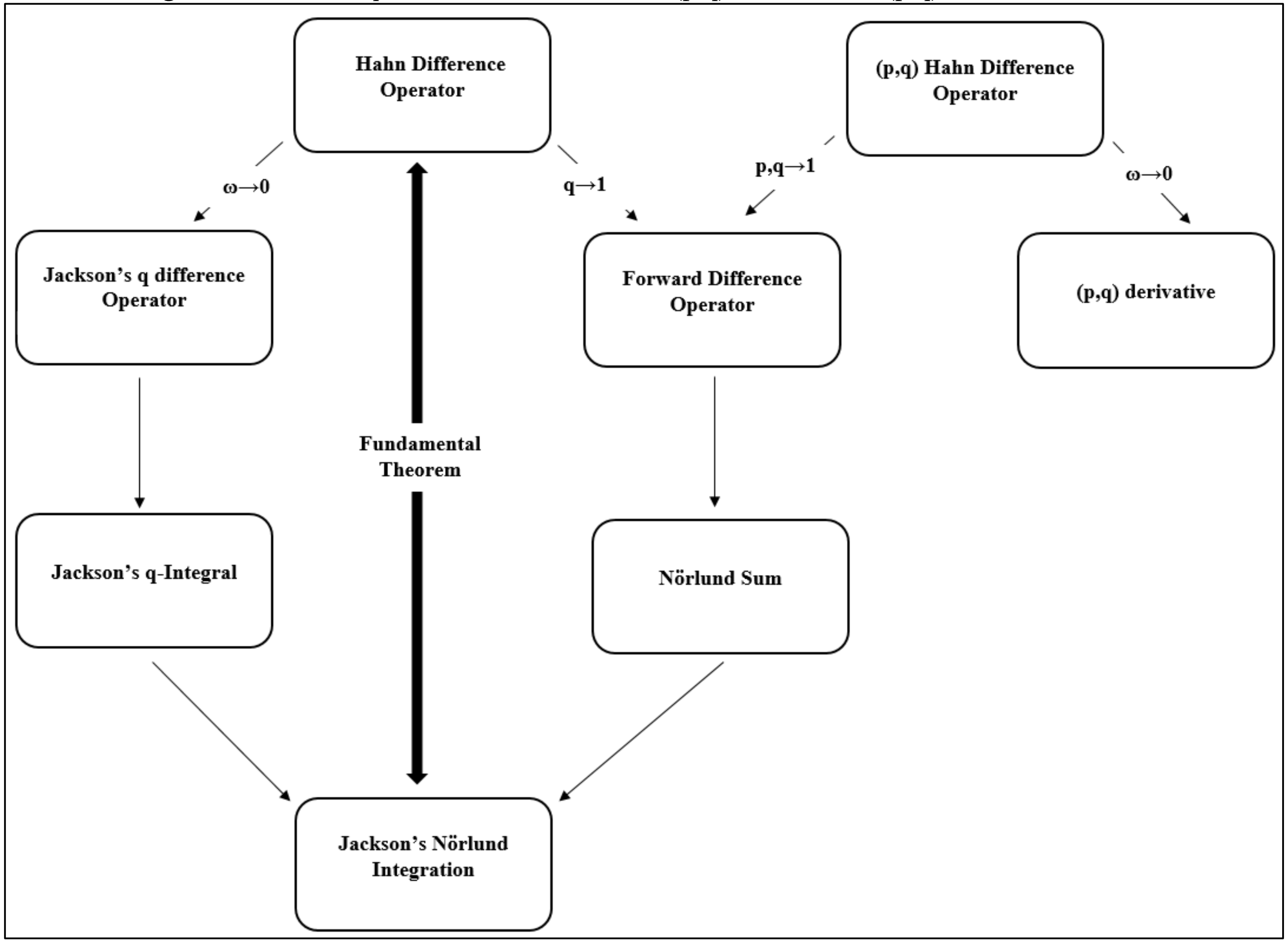

Table 2: The general results of q-Appell, Hahn Appell, (p,q)-Appell, the big q-Appell, the big (p,q)-Appell polynimials.

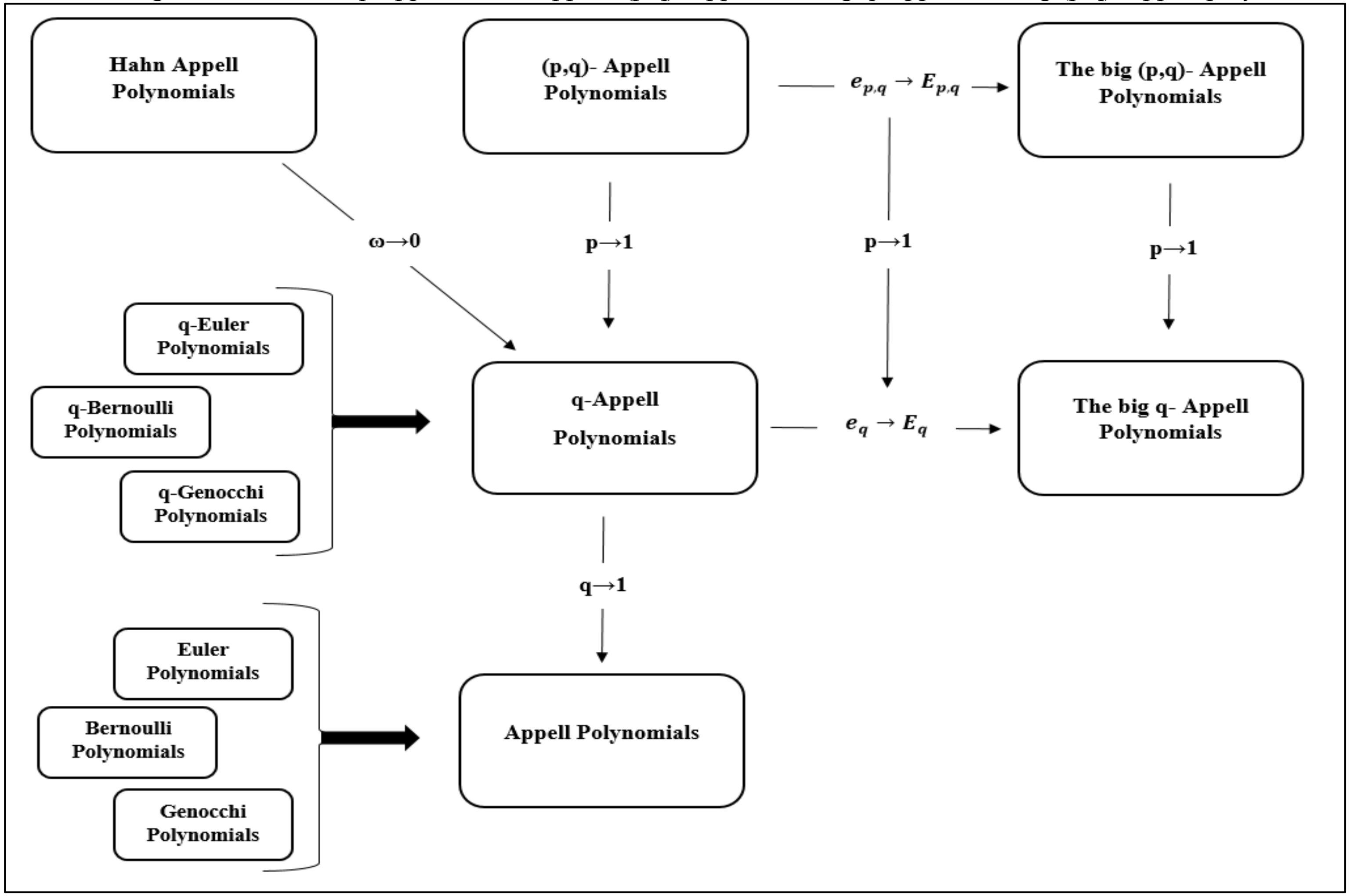

Table 3: The general results of q-calculus, Hahn calculus, (p,q)-calculus and (p,q),ω-calculus with classical derivative.

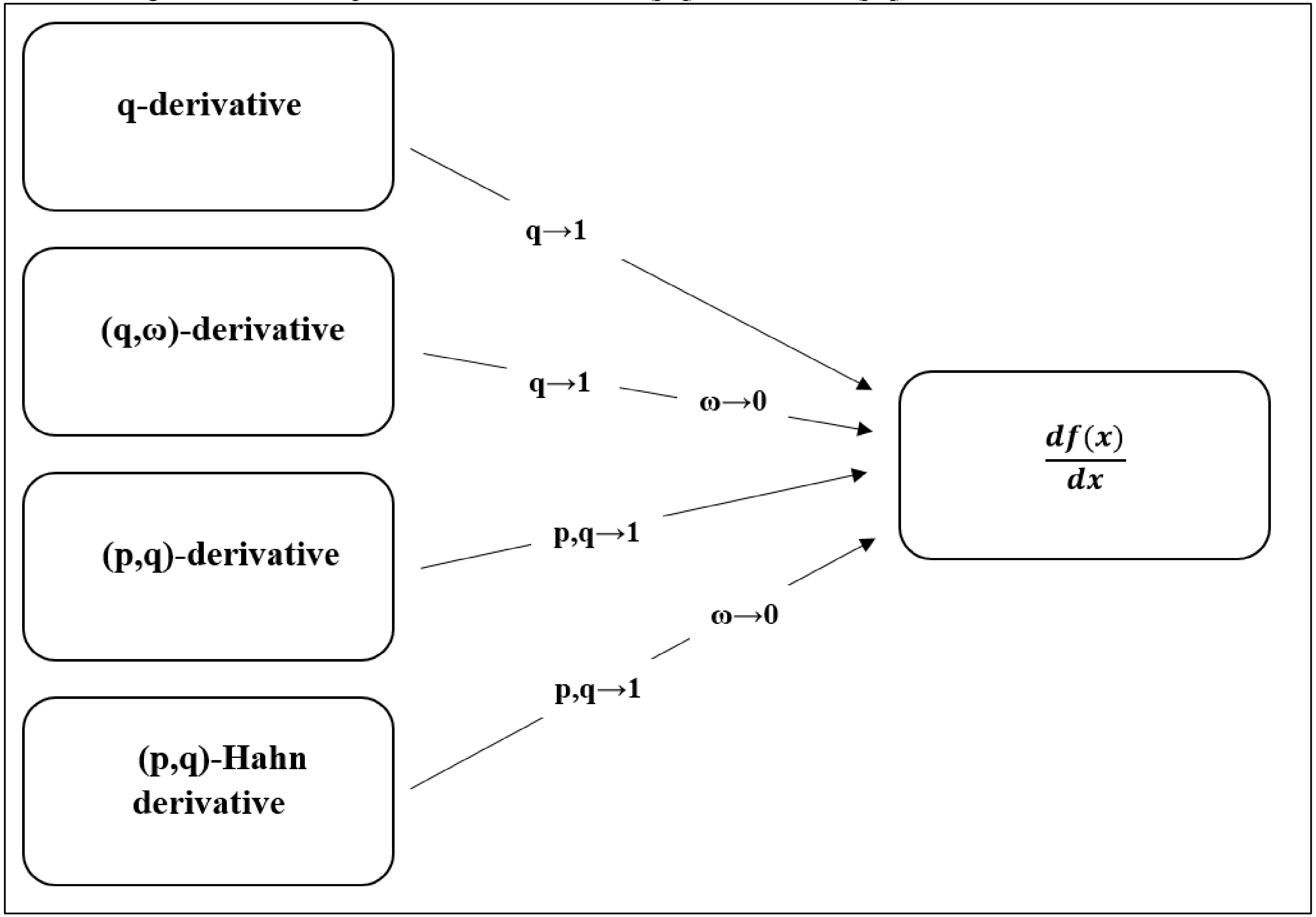

Table 4: Properties of q-calculus, Hahn calculus, (p,q)-calculus and (p,q),ω-calculus.

	$q-Calculus$	$Hahn\ Calculus$	$(p,q)-Calculus$	$(p,q)-Hahn\ Calculus$
Linearity	$D_q\,(f+g)(t)=D_qf(t)+D_qg(t)$	$D_{q,\omega}\,(f+g)(t)=D_{q,\omega}f(t)+D_{q,\omega}g(t)$	$D_{p,q}\,(f+g)(t)=D_{p,q}f(t)+D_{p,q}g(t)$	$D_{p,q,\omega}\,(f+g)(t)=D_{p,q,\omega}f(t)+D_{p,q,\omega}g(t)$
Product Rule	$D_q\,(fg)(t)=f(t)D_qg(t)+g(qt)D_qf(t)$	$D_{q,\omega}\,(fg)(t)=g(t)D_{q,\omega}f(t)+f(qt+\omega)D_{q,\omega}g(t)$	$D_{p,q}\,(fg)(t)=g(qt)D_{p,q}f(t)+f(pt)D_{p,q}g(t)$	$D_{(p,q),\omega}\,(fg)(t)=g(pt)D_{(p,q),\omega}f(t)+f(qt+\omega)D_{(p,q),\omega}g(t)$
Quetient Rule	$D_q\left(\frac{f}{g}\right)(t)=\frac{g(t)D_qf(t)-f(t)D_qg(t)}{g(qt)g(t)}$	$D_{q,\omega}\left(\frac{f}{g}\right)(t)=\frac{g(t)D_{q,\omega}f(t)-f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$	$D_{p,q}\left(\frac{f}{g}\right)(t)=\frac{g(pt)D_{p,q}f(t)-f(pt)D_{p,q}g(t)}{g(qt)g(pt)}$	$D_{(p,q),\omega}\left(\frac{f}{g}\right)(t)=\frac{g(pt)D_{(p,q),\omega}f(t)-f(pt)D_{(p,q),\omega}g(t)}{g(qt+\omega)g(pt)}$
Binomial Coefficient	$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[n-k]_q!\,[k]_q!}$ $\boldsymbol{q}$**-binomial coefficients**	$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[n-k]_q!\,[k]_q!}$ $\boldsymbol{q}$**-binomial coefficients**	$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}$ $\boldsymbol{p,q}$**-binomial coefficients**	
Exponential, Trigonometric Function	$e_q(x),\ E_q(x)$ $Sin_q,\ sin_q,\ \cos_q,\ \mathrm{Cos}_q$	$e_q^{\omega}(x),\ E_q^{\omega}(x)$ $\mathrm{Sin}_q^{\omega},\ \sin_q^{\omega},\cos_q{}^{\omega},\ \mathrm{Cos}_q{}^{\omega}$	$e_{p,q}(x)\ ,\ E_{p,q}(x)$ $Sin_{p,q},\ sin_{p,q},\ \cos_{p,q},\ \mathrm{Cos}_{p,q}$	
Integral	$\int_0^X f(t)d_qt=x(1-q)\sum_{k=0}^{\infty}q^kf(x\,q^k)$	$\int_{\omega_0}^X f(t)d_{q,\omega}t=(x(1-q)-\omega)\sum_{k=0}^{\infty}q^kf(x\,q^k+\omega[k]_q)$	$\int_0^X f(t)d_{p,q}t=\left(x(p-q)\right)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f(\frac{q^k}{p^{k+1}})$	
Fundamental Theorem	Fundamental theorem for $q-Calculus$	Fundamental theorem for $Hahn\ Calculus$	Fundamental theorem for $(p,q)-Calculus$	
Leibniz Formula	Leibniz Formula for $q-Calculus$	Leibniz Formula for $Hahn\ Calculus$	Leibniz Formula for $(p,q)-Calculus$	
Chain Rule	Chain rule for $q-Calculus$	Chain rule for $Hahn\ Calculus$	Chain rule for $(p,q)-Calculus$	
Appell Polynomial	$q-Appell\ polynomial$ $The\ big\ q-Appell\ polynomial$	$q,\omega-Appell\ polynomial$	$(p,q)-Appell\ polynomial$ $The\ big\ (p,q)-Appell\ polynomial$	
Analogue of n	$[n]_q=\frac{q^n-1}{q-1}$	$[n]_q=\frac{q^n-1}{q-1}$	$[n]_{p,q}=\frac{p^n-q^n}{p-q}$	

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