The Interplay between Fractional Calculus and Complex Analysis

Chaima Bouzouina

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Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Nazım Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy in Mathematics.

Asst.Prof. Dr. Arran Fernandez
Supervisor

Examining Committee

1. Prof. Dr. Athanassios Fokas
2. Prof. Dr. Nazim Mahmudov
3. Prof. Dr. Necati Ozdemir
4. Assoc. Prof. Dr. Suzan Cival Buranay
5. Asst. Prof. Dr. Arran Fernandez

ABSTRACT

The usual definitions of fractional derivatives and integrals are very well-suited for a fractional generalisation of real analysis. But the basic building blocks of complex analysis are different: although fractional derivatives of complex-valued functions and to complex orders are well known, concepts such as the Cauchy–Riemann equations and d-bar derivatives have no analogues in the standard fractional calculus. In the current work, we propose a formulation of fractional calculus which is better suited to complex analysis and to all the tools and methods associated with this field. We consider some concrete examples and various fundamental properties of this fractional version of complex analysis.

Keywords: fractional derivatives, complex analysis, d-bar derivatives, Leibniz rule

ÖZ

Kesirli türevlerin ve integrallerin olağan tanımları, gerçek analizin kesirli bir genelleştirilmesi için çok uygundur. Ancak kompleks analizin temel yapı taşları farklıdır: kompleks değerli fonksiyonların kesirli türevleri ve kompleks emirler iyi bilinmesine rağmen, Cauchy-Riemann denklemleri ve d-bar türevleri gibi kavramların standart fraksiyonel kalkülüste analogları yoktur. Mevcut çalışmada, kompleks analize ve bu alanla ilişkili tüm araç ve yöntemlere daha uygun kesirli kalkülüsün formülasyonunu öneriyoruz. kompleks analizin bu kesirli versiyonunun bazı somut örneklerini ve çeşitli temel özelliklerini göz önünde bulunduruyoruz.

Anahtar Kelimeler: kesirli türevlerin, kompleks analiz, d-bar türevleri, Leibniz kuralı.

... Dedicated to my Mother and Father,

my three brothers: Tarik, Djalil and Mouhamed and only sister Nadjet Lenda

To my Bachelor degree supervisor Saadallah and to every single person that had a positive impact on me through out this journey I say

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Chapter 1

INTRODUCTION

The Fractional Calculus (FC) is the usual calculus' extension wherein the basic operations of differentiation and integration may be taken to arbitrary orders, not only repeated an integer number of times. The essential idea of this generalisation goes almost as far back as calculus itself. The first known reference is the correspondence of G. W. Leibniz and M. de l'Hopital in 1695 [22] where the question about the meaning of the semi-derivative has been raised. At that time, Leibniz had invented the notation of $d^n y/dx^n$ yet a simple modification of symbols inspired L'Hopital to ask him: *" What if n is $\frac{1}{2}$? "* Leibniz then replied prophetically: *"This is an apparent paradox from which, one day, useful consequences will be drawn"*.

This question attracted the attention of many mathematicians at the time, including Bernoulli when he had first mentioned the word derivatives of "general order" in 1695 [21]. Then came Euler in 1730 [9], to be the first to link between the interpolation and the question of n fractional.

In 1819, S.F. Lacroix referred to "a derivative of arbitrary order" in [20] which appears using Legendre's symbol for a generalised factorial starting by $y = x^m$, with m a natural number, he got $\dfrac{d^n y}{dx^n} = \dfrac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$, which answers the initial question for $n = 1/2$ as $\dfrac{d^{1/2} y}{dx^{1/2}} = \dfrac{2\sqrt{x}}{\sqrt{\pi}}$.

These advances in the field of FC encouraged J.B.J. Fourier [27] to give his definition of the fractional operators obtained by Fourier's integral representation of $f(x)$:

$$\frac{d^u}{dx^u}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)d\alpha \int_{-\infty}^{\infty} p^u cos[p(x-\alpha)+\frac{1}{2}u\pi]dp,$$

(where u is arbitrary).

The contribution of Abel in this field was huge [1,2] as he was the first to make use of fractional operations when he applied them in the solution of an integral equation that comes from the Tautochrone problem (i.e the problem evolves around determining the shape of the curve of a frictionless point mass so that the time of its slide down under the action of gravity, is independent of the starting point). He started with Abel's integral equation:

$$k = \int_0^x (x-t)^{-1/2} f(t)dt.$$

Then he operated with $d^{1/2}/dx^{1/2}$ on both sides of the equation to get:

$$\frac{d^{1/2}}{dx^{1/2}}k = \sqrt{\pi}f(x),$$

(with suitable conditions on f we have the fractional operators' property: $D^{1/2}D^{-1/2}f = D^0 f = f$) It's showing that the fractional derivative of a constant is not always zero which is a remarkable achievement of Abel that caused a mathematical controversy to be discussed.

Perhaps both Fourier's formula and Abel's solution have caught the interest of Liouville to then make the first significant research on the FC with three substantial memoirs and several more papers following. He would've used his definitions to solve problems from potential theory [27].

His theoretical work started with the famous result:

$$D^m e^{ax} = a^m e^{ax},$$

which was extended to derivatives of arbitrary order α as:

$$D^{\alpha}e^{ax} = a^{\alpha}e^{ax}.$$

He assumed that the fractional derivative of $f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$ may be given in a series form:

$$D^{\alpha}f(x) = \sum_{n=0}^{\infty} c_n a_n^{\alpha} e^{a_n x},$$

which is called Liouville's first formula for an arbitrary order derivative (i.e α can be rational, irrational or complex number) however this was applicable on the functions of the previously given form. So he formulated a second definition where he started off by the following integral:

$$I = \int_0^{\infty} u^{m-1}e^{-xu}du,$$

with $m > 0, x > 0$, the change of variable $xu = t$ and the definite integral of the gamma function yields

$$x^{-m} = \frac{1}{\Gamma(m)}I.$$

Then operating with D^{α} on both sides of the equation:

$$D^{\alpha}x^{-m} = \frac{(-1)^{\alpha}}{\Gamma(m)}\int_0^{\infty} u^{m+\alpha-1}e^{-xu}du,$$

from which we directly get Liouville's second definition of a fractional derivative:

$$D^{\alpha}x^{-m} = \frac{(-1)^{\alpha}\Gamma(m+\alpha)}{\Gamma(m)}x^{-m-\alpha},$$

where $m > 0$. We can say that Liouville was the founder of what will be later on referred to as Fractional Differential Equations (FDE). At this point of time there were different definitions of fractional operators with different uses. One of them was Liouville's second definition for the fractional derivative that would give in case we take $m = 0$ zero (i.e the fractional derivative of unity is zero).

On the contrary, by a generalisation of a case using Lacroix and Abel's version we obtain that the fractional derivative of order $\frac{1}{2}$ of unity, by letting $m = 0$ and $n = \frac{1}{2}$ is non-zero:

$$\frac{d^{1/2}}{dx^{1/2}}x^0 = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}x^{-1/2} = \frac{1}{\sqrt{\pi x}}. \tag{1.1}$$

From here onward, mathematicians were split between agreeing on this or that definition and justifying that choice.

Augustus De Morgan was wise enough to deal with this longstanding controversy when he said:"Both these systems may very possibly be parts of a more general system, but at present I incline to the conclusion that neither system has any claim to be considered as giving the form $D^n x^m$, though either may be a form" [27].

G.F.B. Riemann developed a generalisation of Taylor series [34] and derived:

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_c^x (x-t)^{\alpha-1} f(t)dt + \Psi(x),$$

where $\Psi(x)$ is a complementary function added by Riemann to avoid the problem of having different values for $c \neq c'$ for the case ${}_cD_x^{-\alpha}{}_{c'}D_x^{-\beta}$. Yet, the present-day definition has the complementary function removed from it (thanks to Cayley) as it caused confusion for the mathematicians about its meaning and necessity.

This one error of Riemann alongside with another made by Liouville when he didn't study the case of $x = 0$ which consequently led to a contradition. In addition to the previously mentioned dispute about which system is correct (Liouville or Lacroix), these reasons made the mathematicians lose trust in the theory of fractional operators.

Liouville and Hargreave worked on the generalisation of Leibniz's rule for nth derivative (n is not a natural number).

$$D^{\alpha}\left[f(x)g(x)\right] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D^{n} f(x) D^{\alpha-n} g(x). \tag{1.2}$$

Where D^n is the ordinary derivative and $D^{\alpha-n}$ a fractional derivative, and $\binom{\alpha}{n} = \dfrac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}$. This generalised Leibniz rule can be useful in many modern applications [Ross,Greer].

Sonin is considered as the first building block in what will be later known as Riemann-Liouville definition [36]. A.V. Letnikov made an extension of this work starting from the Cauchy's integral formula:

$$D^{n} f(z) = \frac{n!}{2\pi i} \int_{C} \frac{f(s)}{(s-z)^{n+1}} ds. \tag{1.3}$$

The generalisation is relatively intuitive however the integrand function contains a branch point instead of a pole which requires a branch cut in an appropriate contour.

The theory of generalised operators has reached a suitable level to be considered as a starting point for modern mathematicians after H. Laurent's publication in 1884. His contour was an open circuit in a Riemann surface in contrast to Letnikov's closed contour which resulted:

$${}_{c}D_{x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{c}^{x} (x-t)^{\alpha-1} f(t) dt, \tag{1.4}$$

where $\mathrm{Re}(\alpha) > 0$. When $x > c$, The previous definition coincides with Riemann's definition without the complementary function $\Psi(x)$.

The most used version is when $c = 0$. A sufficient condition for the convergence of the integral is

$$f\left(\frac{1}{x}\right) = O\left(x^{1-\varepsilon}\right), \tag{1.5}$$

with $\varepsilon > 0$. These functions are referred to as functions of Riemann class. When $c = -\infty$, the convergence condition becomes:

$$f(-x) = O\left(x^{-\alpha-\varepsilon}\right), \tag{1.6}$$

with $\varepsilon > 0$, $x \to \infty$. These functions are referred to as functions of Liouville class.

FC theory grew quickly, primarily as a mathematical foundation for a variety of applied disciplines such as Fractional Geometry, Fractional Differential Equations (FDE), and Fractional Dynamics. FC has so many applications currently that we may nearly conclude that no subject of modern engineering or science in general is unaffected by its tools and approaches. For example, applications can be found in rheology, bioengineering, electrical and mechanical engineering, viscoelasticity, acoustics, optics, robotics, chemical and statistical physics, control theory,... etc.

The fact that these new fractional-order models are often more accurate than integer-order models is the major reason for FC applications' success. The fascinating feature of the subject is that fractional operators are not local values; they consider the complete history of the process, allowing you to model non-local and dispersed effects that are common in natural phenomena.

FC can be considered as an excellent set of tools for describing hereditary properties and the memory of various materials and processes.

We refer to mathematical textbooks such as [18, 19, 27, 30, 35] for the general theory and others such as [16, 17, 25, 26, 39] for details of applications.

Conversely to the previously defined field, Complex analysis is a classical branch of mathematics widely known.

Complex numbers arose from the need to solve cubic equations rather than the quadratic ones, the commonly believed myth, as the later were already solved by Al-Khwarizmi (780-850) in his Algebra book.

If we consider the general cubic equation:

$$x^3 + ax^2 + bx + c = 0,$$

using the change of variable $s = x + \frac{a}{3}$ we can reduce the previous equation to:

$$s^3 + ps + q = 0.$$

If only positive values of s and positive coefficients are admitted then we have three forms of depressed cubic equations:

- $x^3 + px = q,$
- $x^3 = px + q,$
- $x^3 + q = px.$

Scipione del Ferro, a professor at the University of Bologna until 1526, was the first to find the solution for the first equation. The formulae of solutions for all three equations were then published in Cardano's Ars Magna (1545) [6]. In the solution for the second equation emmerged a difficluty namely a square root of a negative number:

$$x = u + v = \sqrt[3]{\frac{1}{2}q + w} + \sqrt[3]{\frac{1}{2}q - w}.$$

Where $w = \sqrt{\left(\frac{1}{2}q\right)^2 - \left(\frac{1}{3}p\right)^3}$. The "casus irreducibilis" is when the expression under the radical symbol in w is negative. In Ars Magna, Cardano ignores mentioning it which kept the suspicion by mathematicians regarding the negative roots which were described as 'false' at the time.

Rafael Bombelli completed Cardano's work in [5] where he discusses the "casus irreducibilis" fully. His great insight was treating $\sqrt{-1}$ as a number, and calling it "piu di meno" when he considers the second equation for $p = 15$ and $q = 4$ using Cardano's formula

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}.$$

By putting, $\sqrt[3]{2+\sqrt{-121}} = a+b\sqrt{-1}$ and $\sqrt[3]{2-\sqrt{-121}} = a-b\sqrt{-1}$, and after some algebraic manipulation he found $a = 2$, $b = 1$ substituting back into $x = a+b\sqrt{-1} + a-b\sqrt{-1} = 2a = 4$. Eventhough this trick worked only in a few cases, yet it was a major shift that helped to understand more about what's going to be referred to as imaginary numbers and how to manipulate them.

René Descartes (1596-1650) used the term "imaginary" in his book [8] saying: *"Neither the true nor the false [negative] roots are always real; but sometimes only imaginary."*

Later, John Wallis (1616-1703) notes that negative numbers can be represented geometrically, by a line with a zero in the middle, where positive numbers are points to the right of the zero mark , and negative numbers are to the left of zero. This helped in finding a geometric interpretation for complex roots of a quadratic equation with real coefficients.

To simplify the mathematical operations over these newly discovered numbers, Leonard Euler (1707-1783) standardised the notation of $i = \sqrt{-1}$ and the i being the first letter from the word "imaginary". He had major contribution to this field by using the imaginary numbers to solve quadratic and cubic equations, visualising complex numbers as points with Cartesian coordinates, and defining the complex exponential and proving $e^{\theta i} = \cos(\theta) + i\sin(\theta)$.

In 1831, the first rigorous definition of complex numbers was proposed by William Rowan Hamilton (1805-1865). He defined the ordered pairs and has seen the complex numbers as a couple of real numbers a and b i.e. $a + bi = (a, b)$. Then he defined the operations (addition and multiplication) of couples $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, bc + ad)$. This definition shows the consistency of the theory of complex numbers starting from the consistency of the real numbers.

The concept of analytic function first appeared in the memoir of Augustin-Louis Cauchy (1789-1857) published in 1825 where he initiated complex function theory. He dealt with complex integration describing a method to pass from the real to the imaginary field where the calculation of an improper integral becomes easier. He showed that the path doesn't change the value of the integral if there are no singularities.

By his publications, Cauchy had established the field of complex analysis, as one of the pillars pure mathematics stands on until today.

B. Riemann was the next major figure in complex analysis. He laid the groundwork for a universal theory of complex functions by establishing "Riemann surfaces" and defining holomorphic functions as complex single-valued functions on Riemann surfaces that fulfill the Cauchy-Riemann equations. Riemann introduces his mapping theorem towards the end of his dissertation: *"Two given simply connected plane surfaces can always be related to each other in such a way that each point of one surface corresponds to a point of the other, varying continuously with that point, with the corresponding smallest parts similar."* (Riemann 2004, 37)

Riemann may not have contributed as much to the theory as Cauchy did, but him recognising the necessity of the existence of a complex analysis really put the field on solid mathematical ground for the first time.

After this quick history on how both FC and Complex Analysis have started and developed over the years, It’s high time to speak about the link between both of them.

FC has gained success as it offered better description for the dynamics of complex systems from a variety of scientific and engineering disciplines since the order of operators is arbitrary: complex or real not just an integer. However, we find lots of examples where the functions themselves are complex-valued so how to deal with such a situation?

It seems that, despite the definitions and results laid out in standard sources such as [35, §22], much of the power of complex analysis has been neglected in the study of FC. This is unfortunate, because complex analysis is a very important and powerful branch of mathematics [3, 7]. The theory here has a very different structure from in real analysis, where each level of nth-order differentiability gives a different class C^n of functions; by contrast, complex differentiability is a very strong statement with implications including smoothness, existence of Taylor series, and unboundedness at infinity. Applications of complex analysis may be found both within mathematics and also in science and engineering.

The relationship between real and imaginary parts, both of a complex variable z and of a complex function $f(z)$ is one of the essential building blocks of complex analysis. The Cauchy–Riemann equations, on the differentiation of a complex function $f(z)$ with respect to the real and imaginary parts of z, form one of the first starting points of complex analysis. The Kramers-Kronig relations between the real and imaginary parts of an analytic function have many applications in physics [24, 29], while the d-bar formalism based on the Cauchy–Riemann equations has been very useful in solving assorted partial differential equations [3, 13, 14].

Unfortunately, none of these tools have discovered analogues in FC.

- There is no Cauchy–Riemann type relationship between the usual fractional derivatives of $f(z)$ w.r.t the real and imaginary parts of z.
- This means there are no Kramers–Kronig relations to be used in fractional control problems.
- A fractional version of the complex d-bar derivative has never been defined, so there is no d-bar formalism to be used in solving FDEs.

In the current work, we propose a definition of fractional complex derivatives which is specifically designed with these important ideas of complex analysis in mind. It's actually a fractionalisation of Wirtinger derivatives.

Similarly to the Cauchy–Riemann approach to complex analysis, we start with the usual definition of fractional differentiation in the real line. Instead of simply using the same formulae (2.3)–(2.4) in the complex case too, we define complex fractional derivatives with respect to $z = x + yi$ by directly combining the real fractional derivatives w.r.t x and y. In a similar way we can also define complex fractional derivatives w.r.t $\bar{z} = x - yi$ and thus obtain for the first time a d-bar fractional derivative.

The initial idea that inspired our work is a simple one, but the FC thus obtained turns out to have surprising depths and complexities. There are two different ways of defining the fractional complex partial derivatives, which are not equivalent either to each other or to the original Riemann-Liouville complex fractional derivative. However, both of them can be related back to the Riemann-Liouville derivative by a linear transformation with an initial-value error term, similar to the one linking between Riemann-Liouville and Caputo derivatives.

Chapter 2

PRELIMINARIES

Fractional calculus (FC) has expanded rapidly in the last few decades, partly due to the discovery of many real-world applications in assorted fields of science. The question of exactly how to define a derivative or integral to order α, when α is not a whole number, has not been uniquely answered. There are many different formulae and definitions: some of them dating back decades or centuries, others only a few years old [12, 35]. The different theories of FC thus obtained are not entirely separate, as many of them share similar properties and can be analysed using similar methods. This reflects the notion of a classification of the various operators of FC into broad classes [4], such as the class of fractional operators w.r.t functions [35, 37] or the class of fractional operators defined using analytic kernel functions [4, 12].

Let's first start with some special functions that we need all along this thesis:

2.1 Special functions

Gamma function

$\Gamma(z)$ is a continuous generalisation of $n!$ which is defined as the following

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2.1}$$

Two important formulae are:

Reflection formula: $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$.

Duplication formula: $\Gamma(2z) = \dfrac{2^{2z-1}}{\sqrt{(\pi)}}\Gamma(z)\Gamma(z+\frac{1}{2})$.

Psi function

Using the derivative of the Gamma function we obtain a definition for a new special function:

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\log(\Gamma(z)).$$

Useful values: $\Psi(1) = -\gamma$, $\Psi(\frac{1}{2}) = -\gamma - \ln 4$, $\Psi(n) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}$, with γ as Euler's constant.

Fractional exponential function

It is denoted as $E_x(.,.)$ and defined as the following:

$$E_x(\alpha, a) = x^{\alpha}\sum_{n=0}^{\infty}\frac{(az)^n}{\Gamma(\alpha+n+1)}.$$

Hypergeometric function

It's a solution of a 2nd degree ODE with 3 singular points and it's defined as follows:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b+n)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)}\frac{x^n}{n!}.$$

In this research work, We will simply focus on the most used definition of fractional operators, Riemann—Liouville. This is the usual starting point for studying FC, the fundamental definition to which many others can be connected, and therefore it will be the commencement of our fractional d-bar derivatives' study.

2.2 Definition of Riemann-Liouville model

We take inspiration from the following theorem:

Theorem 2.1 (Cauchy formula for repeated integration): For $n \in \mathbb{N}$ we have

$$\int_c^x \int_c^{x_1} \int_c^{x_2} \dots \int_c^{x_{n-1}} f(x_n)dx_n \dots dx_1 = \frac{1}{(n-1)!}\int_c^x (x-t)^{n-1} f(t)dt. \qquad (2.2)$$

Proof. By induction, first for $n = 1$

$$\int_c^x f(x_1)dx_1 = \frac{1}{(1-1)!}\int_c^x (x-t)^{1-1} f(t)dt$$

is true.

Assume it's true for n and let's prove it for $n+1$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{n!}\int_c^x (x-t)^n f(t)dt\right) &= \frac{d}{dx}\left(\frac{1}{n!}\int_c^x \sum_{k=0}^{n}\binom{n}{k} x^k(-t)^{n-k} f(t)dt\right) \\
&= \sum_{k=0}^{n}\frac{1}{k!(n-k)!}\frac{d}{dx}\left[x^k \int_c^x (-t)^{n-k} f(t)dt\right] \\
&= \sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left[kx^{k-1}\int_c^x (-t)^{n-k} f(t)dt + x^k(-x)^{n-k} f(x)\right] \\
&= \sum_{k=1}^{n}\frac{1}{k!(n-k)!}x^{k-1}\int_c^x (-t)^{n-k} f(t)dt \\
&\qquad + \sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!(n-k)!}x^n f(x) \\
&= \sum_{k=0}^{n-1}\frac{1}{k!(n-1-k)!}\int_c^x x^k(-t)^{n-1-k} f(t)dt + (1-1)^n x^n f(x) \\
&= \frac{1}{(n-1)!}\int_c^x (x-t)^{n-1} f(t)dt
\end{aligned}$$

By integrating both sides:

$$\frac{1}{n!}\int_c^x (x-t)^n f(t)dt = \int_c^x \frac{1}{(n-1)!}\int_c^y (x-t)^{n-1} f(t)dtdy + C$$

at $x = c$ we get that $C = 0$. So the proof is complete. $\square$

Definition 2.1 (Riemann-Liouville fractional integral [27,30,35]): In the Riemann-Liouville model, the fractional integral to order α o $f(x)$, with constant of integration c, is defined as:

$$ {}_cI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_c^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi. \tag{2.3}$$

The requirements on f, α, x, and c may be set in various ways. We always have either $\alpha > 0$ (in the real case) or $\mathrm{Re}(\alpha) > 0$ (in the complex case). Usually the constant c is taken to be either 0 or $-\infty$, in which case the fractional integral is said to be respectively of Riemann type or Liouville type.

Definition 2.2 (Riemann-Liouville fractional derivative [27, 30, 35]): the fractional derivative to order α of $f(x)$, with constant of differentiation c, is defined as:

$$ {}_cD_x^{\alpha} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}_cI_x^{n-\alpha} f(x)\right), \quad n = \lfloor \mathrm{Re}(\alpha) \rfloor + 1. \tag{2.4}$$

Here we have either $\alpha > 0$ (in the real case) or $\mathrm{Re}(\alpha) \geq 0$ (in the complex case). Note that the number n is chosen so that $n \in \mathbb{N}$ and $n - \alpha$ is positive or has positive real part.

It's crucial to mention that the fractional integrals and derivatives defined by (2.3) and (2.4) can both be considered as two sides of the same coin. If we write ${}_cI_x^{\alpha} f(x) = {}_cD_x^{-\alpha} f(x)$, then we have a quantity ${}_cD_x^{\alpha} f(x)$ which is defined for all $\alpha \in \mathbb{C}$ and which we may call a **fractional differintegral**. It turns out [35] that this quantity is an entire function of α, and thus the fractional derivative formula (2.4) is an analytic continuation in α of the fractional integral formula (2.3). This demonstrates the naturality of the Riemann-Liouville definition of FC.

2.3 Properties of Riemann-Liouville operators

Properties of fractional operators are not always exactly what would be expected from extrapolating the known properties of classical calculus.

Semigroup Property

Riemann-Liouville fractional integrals have a semigroup property, namely the following identity:

Theorem 2.2: Let f be continuous on J for all x we have:

$$ {}_cI_x^{\alpha}\, {}_cI_x^{\beta} f(x) = {}_cI_x^{\alpha+\beta} f(x), \tag{2.5}$$

whenever $\mathrm{Re}(\beta) > 0$.

Proof. For $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\alpha) < 0$ then,

$$\begin{aligned}
{}_cI_x^{\alpha}\,{}_cI_x^{\beta} f(x) &= \frac{d^n}{dx^n}{}_cI_x^{\alpha+n}\,{}_cI_x^{\beta} f(x),\\
&= \frac{d^n}{dx^n}{}_cI_x^{\alpha+\beta+n} f(x),\\
&= {}_cI_x^{\alpha+\beta} f(x).
\end{aligned}$$

We need to prove it for integrals to complete the proof (i.e for $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\alpha) > 0$) :

$$\begin{aligned}
{}_cI_x^{\alpha}\,{}_cI_x^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)}\int_c^x (x-t)^{\alpha-1}\,{}_cI_x^{\beta} f(t)dt,\\
&= \frac{1}{\Gamma(\alpha)}\int_c^x (x-t)^{\alpha-1}\left[\frac{1}{\Gamma(\beta)}\int_c^t (t-u)^{\beta-1} f(u)du\right]dt,\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_c^x\int_c^t (x-t)^{\alpha-1}(t-u)^{\beta-1} f(u)dudt,\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_c^x f(u)\int_u^x (x-t)^{\alpha-1}(t-u)^{\beta-1} dtdu.
\end{aligned}$$

By this change of variable $v = \dfrac{t-u}{x-u}$ so $dv = \dfrac{dt}{x-u}$

$$\begin{aligned}
{}_cI_x^{\alpha}\,{}_cI_x^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_c^x f(u)\int_0^1 [(x-u)(1-v)]^{\alpha-1}[v(x-u)]^{\beta-1}(x-u)dvdu,\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_c^x f(u)(x-u)^{\alpha+\beta-1}\int_0^1 (1-v)^{\alpha-1}v^{\beta-1}dvdu,\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_c^x f(u)(x-u)^{\alpha+\beta-1}B(\alpha,\beta)du,\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_c^x f(u)(x-u)^{\alpha+\beta-1}du,\\
&={}_c I_x^{\alpha+\beta} f(x).
\end{aligned}$$

□

Composition rule

Riemann-Liouville fractional *derivatives* do not have semigroup property. Even for a standard n times repeated derivative, we have:

Theorem 2.3: Let n be a natural number and $D^n f$ be continuous. We then get:

$$ {}_cI_x^{\alpha}\left(D_x^n f(x)\right) = {}_cI_x^{\alpha-n} f(x) - \sum_{k=0}^{n-1} \frac{(x-c)^{k-n+\alpha}}{\Gamma(k-n+\alpha+1)} D_x^k f(c), \qquad \alpha \in \mathbb{C}. \tag{2.6} $$

Proof. Let $\mathrm{Re}(\alpha) \geq 0$ in all the following

1. **Fractional integral of an integer order derivative**

By induction, We start with $n = 1$:

$$ \begin{aligned} {}_cI_x^{\alpha} f'(x) &= \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f'(t)dt, \\ &= \frac{1}{\Gamma(\alpha)} \left[(x-t)^{\alpha-1} f(t)\right]_c^x - \frac{1}{\Gamma(\alpha)} \int_c^x (-\alpha+1)(x-t)^{\alpha-2} f(t)dt, \\ &= -\frac{1}{\Gamma(\alpha)} \left[(x-c)^{\alpha-1} f(c)\right] + \frac{\alpha-1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-2} f(t)dt, \\ &= \frac{1}{\Gamma(\alpha-1)} \int_c^x (x-t)^{\alpha-2} f(t)dt - \frac{(x-c)^{\alpha-1} f(c)}{\Gamma(\alpha)}, \\ &= {}_cI_x^{\alpha-1} f(x) - \frac{(x-c)^{\alpha-1} f(c)}{\Gamma(\alpha)}. \end{aligned} $$

So we have proven a commutation relation between ordinary differentiation and fractional integration:

$$ \frac{d}{dx} {}_cI_x^{\alpha} f(x) = {}_cI_x^{\alpha-1} f(x) = {}_cI_x^{\alpha} \frac{d}{dx} f(x) + \frac{(x-c)^{\alpha-1} f(c)}{\Gamma(\alpha)}. $$

We notice that when $f(c) = 0$ or when $c = -\infty$ f decays at infinity, we have a semi group property.

Now, assume it's true for n i.e.

$$ {}_cI_x^{\alpha}\left(D_x^n f(x)\right) = {}_cI_x^{\alpha-n} f(x) - \sum_{k=0}^{n-1} \frac{(x-c)^{k-n+\alpha}}{\Gamma(k-n+\alpha+1)} D_x^k f(c) $$

and we prove it for $n+1$.

$$
\begin{aligned}
{}_cI_x^\alpha\left(D_x^{n+1}f(x)\right) &= {}_cI_x^\alpha\left(D_x^n f'(x)\right)\\
&= {}_cI_x^{\alpha-n}f'(x) - \sum_{k=0}^{n-1}\frac{(x-c)^{k-n+\alpha}}{\Gamma(k-n+\alpha+1)}D_x^k f'(c)\\
&= {}_cI_x^{\alpha-n}f'(x) - \sum_{k=0}^{n-1}\frac{(x-c)^{k-n+\alpha}}{\Gamma(k-n+\alpha+1)}D_x^{k+1} f(c)\\
&= \left[{}_cI_x^{(\alpha-n)-1}f(x) - \frac{(x-c)^{(\alpha-n)-1}f(c)}{\Gamma(\alpha-n)}\right] - \sum_{k=1}^{n}\frac{(x-c)^{k-n+\alpha-1}}{\Gamma(k-n+\alpha)}f^{(k)}(c)\\
&= {}_cI_x^{\alpha-n-1}f(x) - \sum_{k=0}^{n}\frac{(x-c)^{k-n+\alpha-1}}{\Gamma(k-n+\alpha)}D_x^k f(c)
\end{aligned}
$$

2. **Fractional derivative of an integer order derivative**

Let $m = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$ so that

$$
{}_cD_x^\alpha f(x) = \frac{d^m}{dx^m}\,{}_cI_x^{m-\alpha}f(x)
$$

We use this definition together with the result we proved for "fractional integral of integer order derivative" to prove.

$$
\begin{aligned}
\frac{d^n}{dx^n}\,{}_cD_x^\alpha f(x) &= {}_cD_x^{\alpha+n}f(x)\\
&= \frac{d^{m+n}}{dx^{m+n}}\,{}_cI_x^{m-\alpha}f(x)\\
&= \frac{d^{m+n}}{dx^{m+n}}\left({}_cI_x^{m-\alpha+n}D_x^n f(x) + \sum_{k=0}^{n-1}\frac{(x-c)^{m-\alpha+k}D_x^k f(c)}{\Gamma(m-\alpha+k+1)}\right)\\
&= {}_cD_x^\alpha\left(D_x^n f(x)\right) + \sum_{k=0}^{n-1}\frac{D_x^k f(c)}{\Gamma(m-\alpha+k+1)}\frac{d^{m+n}}{dx^{m+n}}(x-c)^{m-\alpha+k}\\
&= {}_cD_x^\alpha\left(D_x^n f(x)\right) + \sum_{k=0}^{n-1}\frac{D_x^k f(c)}{\Gamma(m-\alpha+k+1)}\\
&\qquad\times\frac{\Gamma(m-\alpha+k+1)}{\Gamma(m-\alpha+k+1-(m+n))}(x-c)^{m-\alpha+k-m-n}\\
&= {}_cD_x^\alpha\left(D_x^n f(x)\right) + \sum_{k=0}^{n-1}\frac{(x-c)^{k-\alpha-n}D_x^k f(c)}{\Gamma(k-\alpha-n+1)}
\end{aligned}
$$

Proof is complete. □

Remark 2.1: This last relation is exactly the relationship between Riemann-Liouville and Caputo derivatives for $n = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$

$$ {}^{RL}_{c}D^{\alpha}_{x}f(x) = {}^{\mathscr{C}}_{c}D^{\alpha}_{x}f(x) + \sum_{k=0}^{n-1} \frac{(x-c)^{k-\alpha} D^{k}_{x}f(c)}{\Gamma(k-\alpha+1)} $$

Caputo derivative is defined as ${}^{\mathscr{C}}_{c}D^{\alpha}_{x}f(x) = \frac{d^n}{dx^n}{}^{RL}_{c}I^{n-\alpha}_{x}f(x)$ where $n = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$.

Many sources, including for the most part standard textbooks such as [27,30], use the convention that the variables x and α must be real. This assumption is often merely for simplicity, and because for many practical applications it is sufficient to have these variables in $\mathbb{R}$ or even just in a finite interval such as $[0,1]$. However, it is also possible to use the above formulae for complex α, and when x is complex there is an equivalent formula [28] using a contour integral in the complex plane instead of the line segment $[c,x]$. These ideas were discussed briefly in [30, §3.4] and [35, §22], and extended beyond the Riemann-Liouville model in [10], but rarely used in later research into the properties and applications of FC.

Leibniz rule

For example, we have the following fractional Leibniz-type formula [31,33]:

Theorem 2.4: For f is continuous and g is analytic on the interval $[c,x]$. For $\mathrm{Re}(\alpha) > 0$ we have

$$ {}_{c}D^{\alpha}_{x}\left(f(x)g(x)\right) = \sum_{j=0}^{\infty} \binom{\alpha}{j} {}_{c}D^{\alpha-j}_{x} f(x) {}_{c}D^{j}_{x} g(x), \qquad \alpha, x \in [c,x]. \tag{2.7} $$

Proof. g is analytic so

$$ g(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\, {}_{c}D^{j}_{x} g(x)(x-t)^j $$

with $t \in [c,x]$.

$$
\begin{aligned}
{}_cD_x^{\alpha}\left(f(x)g(x)\right) &= \frac{1}{\Gamma(-\alpha)}\int_c^x (x-t)^{-\alpha-1} f(t)\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\,{}_cD_x^j g(x)(x-t)^j dt \\
&= \frac{1}{\Gamma(-\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\,{}_cD_x^j g(x)\int_(x-t)^{j-\alpha-1} f(t)dt \\
&= \frac{1}{\Gamma(-\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\,{}_cD_x^j g(x)\left(\Gamma(j-\alpha)\,{}_cD_x^{\alpha-j} f(x)\right) \\
&= \sum_{j=0}^{\infty}\binom{\alpha}{j}\,{}_cD_x^{\alpha-j} f(x)\,{}_cD_x^j g(x)
\end{aligned}
$$

We used the following result $\frac{(-1)^j\Gamma(j-\alpha)}{\Gamma(-\alpha)j!} = \frac{\Gamma(1+\alpha)}{j!\Gamma(1-j+\alpha)}$ □

2.4 Example functions

Power function

For any $\alpha \in \mathbb{C}$ we have:

$$D^{\alpha}x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(-\alpha+1)}x^{\lambda-\alpha}$$

The special case when $\lambda = 0$

$$D^{\alpha}1 = \frac{1}{\Gamma(-\alpha+1)}x^{-\alpha}$$

It shows that a constant's Riemann-Liouville derivative is surprisingly not equal to zero.

Exponential function

For any $\alpha \in \mathbb{C}$ we have:

$$D^{\alpha}e^{ax} = E_x(-\alpha, a)$$

Where $E_x(.,.)$ is the fractional exponential function.

Logarithmic-type function

For any $\alpha \in \mathbb{C}$ and using:

$$D^{\alpha}\left[x^{\lambda}\ln x\right] = \frac{\Gamma(\lambda+1)}{\Gamma(-\alpha+1}x^{\lambda-\alpha}\left[\ln x + \Psi(\lambda+1) - \Psi(\lambda-\alpha+1)\right]$$

We obtain directly for $\lambda = 0$

$$D^{\alpha}\left[\ln x\right] = \frac{1}{\Gamma(-\alpha+1}x^{-\alpha}\left[\ln x + \gamma - \Psi(-\alpha+1)\right]$$

2.5 Complex partial and d-bar derivatives

We begin with a short review of the basics of complex analysis. It is necessary to review these in detail in order to see how we may extend them to a fractional scenario. Let us consider a function $f(z)$ of a complex variable $z = x + yi \in \mathbb{C}$. This can also be considered as a function $f(x,y)$ of two independent real variables $x, y \in \mathbb{R}$. As a bivariate real function, it can be differentiated in two possible ways:

$$\frac{\partial f}{\partial x}(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y).$$

Let us assume that f is real differentiable, i.e. that the above two partial derivatives are well-defined.

There is a bijective linear mapping between the pair of independent real variables x, y and the pair of complex variables $z, \bar{z}$, namely:

$$z = x + yi \qquad \text{and} \qquad \bar{z} = x - yi;$$

$$x = \tfrac{1}{2}(z + \bar{z}) \qquad \text{and} \qquad y = \tfrac{1}{2i}(z - \bar{z}).$$

This gives rise, via the bivariate chain rule, to another pair of partial derivatives which are usually denoted by $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ and defined by the Austrian mathematician Wilhelrn Wirtinger as follows:

Definition 2.3 (Wirtinger derivatives, [40]): The partial derivatives of a complex function f w.r.t z and $\bar{z}$, respectively, are defined as:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\cdot\frac{\partial f}{\partial x} + \frac{1}{2i}\cdot\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right);$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\cdot\frac{\partial f}{\partial x} - \frac{1}{2i}\cdot\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Note that these are *not* partial derivatives w.r.t two independent variables, although it is often useful to consider them as such for formal manipulations whose conclusions be justified rigorously. It is also easy to show that the sum, product, and quotient rules still hold.

Remark 2.2: The Wirtinger derivatives can be considered in between the real derivative of a real function and the complex derivative of a complex function.
For $f(z) = f(x+yi) = u(x,y) + iv(x,y)$ we have the following:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\cdot\left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y} + i[\frac{\partial v(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y}]\right) \tag{2.8}$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\cdot\left(\frac{\partial u(x,y)}{\partial x} - \frac{\partial v(x,y)}{\partial y} + i[\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}]\right) \tag{2.9}$$

Furthermore, the existence of these derivatives relies only upon the real differentiability of f (w.r.t x and y) and does not imply complex differentiability of f with respect to z: it is possible to find a function f such that the 'partial derivative' $\frac{\partial f}{\partial z}$ exists but the usual complex derivative $\frac{\mathrm{d}f}{\mathrm{d}z}$ does not.

For a complex function f to be complex differentiable, the Cauchy—Riemann equations offer both necessary and sufficient conditions.
Namely, $f(z) = f(x+yi) = u(x,y) + iv(x,y)$ is complex differentiable if and only if it is both real differentiable and satisfies the following equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.10}$$

These equations are sometimes used as a definition of complex differentiability, in lieu of the more natural (equivalent) definition in terms of the existence of the limit

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}. \tag{2.11}$$

Another equivalent definition is in terms of the **d-bar derivative** defined in (2.9). Namely, the function $f(z) = f(x+yi)$ is complex differentiable if and only if it is both real differentiable and satisfies $\dfrac{\partial f}{\partial \bar{z}} = 0.$

Furthermore, if f is complex differentiable, then the 'partial' derivative $\frac{\partial f}{\partial z}$ defined by (2.8) is equal to the usual complex derivative $\frac{\mathrm{d}f}{\mathrm{d}z}$ defined by (2.11). Thus we see a fundamental connection between the concept of complex differentiability and the two quantities defined by (2.8)–(2.9) whose existence depends only on real differentiability.

How can this connection be extended to fractional derivatives? That is the fundamental issue this thesis seeks to address. With standard 1st-order differentiation, the real derivatives, complex derivatives, and complex partial derivatives all interconnect to one another nicely. These connections may be extended to nth-order differentiation, for $n \in \mathbb{N}$, by taking powers; for example, we have:

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$
$$\frac{\partial^3 f}{\partial z^3} = \frac{1}{8}\left(\frac{\partial^3 f}{\partial x^3} - 3i\frac{\partial^3 f}{\partial x^2 \partial y} - 3\frac{\partial^3 f}{\partial x \partial y^2} + i\frac{\partial^3 f}{\partial y^3}\right),$$

the second of which gives an expression for $\frac{\mathrm{d}^3 f}{\mathrm{d}z^3}$ when f is analytic, while the first shows that any analytic complex function is harmonic. With standard fractional-order differentiation, the connections are lost: the fractional αth derivatives with respect to x, y, and $z = x+yi$ are not interrelated, and the fractional d-bar derivative with respect to $\bar{z} = x - yi$ has never been defined at all.

Chapter 3

COMPLEX ANALYSIS AND ITS FRACTIONALISATION

In Chapter 2, we've provided some background on the differential operators of complex analysis in the non-fractional case, before defining our new operators and finding the correct function space for their domain. To illustrate this work, we consider how the operators apply to some important example functions. In Chapter 3, we'll conduct further analysis of the new operators: interpreting them as complex directional derivatives, proving their connection with the usual fractional derivatives, recovering a natural generalisation of the Leibniz rule, and investigating compositions of the new operators.

We emphasise in particular the very natural form of the Leibniz rule (Theorem 3.3 below) and the relationship between the fractional complex partial derivatives with respect to $z, \bar{z}$ and the usual fractional complex derivative with respect to z (Theorem 3.2 below). These results indicate that the definition we have proposed is the *natural* extension of Riemann-Liouville FC to the context of complex analysis.

Motivated by the linear relations (2.8)–(2.9), we shall now propose a new definition of FC in the complex plane which is, in our opinion, a better extension of the usual complex analysis than is provided by the Riemann-Liouville definition (2.3)–(2.4). As we shall see, in our new formulation fractional derivatives with respect to both z and $\bar{z}$ emerge naturally and with properties similar to those of the n-times repeated derivatives with respect to z and $\bar{z}$ which we considered above.

3.1 The definitions of fractional Wirtinger derivatives

Before stating our new definitions, we begin by showing how to derive them in a natural way motivated by the above discussion.

As we saw above, complex repeated derivatives with respect to both z and $\bar{z}$ can be defined by taking powers of $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and using the binomial theorem for powers of a sum of two terms:

Definition 3.1 (Integer order Wirtinger derivatives): For f a complex function with a complex variable $z = x + iy$, we have the following:

$$\frac{\partial^n f}{\partial z^n} = \frac{1}{2^n}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^n = \sum_{k=0}^{n}\binom{n}{k}(-i)^{n-k}\frac{\partial^n f}{\partial x^k \partial y^{n-k}}f = \sum_{k=0}^{n}\binom{n}{k}(-i)^k\frac{\partial^n f}{\partial x^{n-k}\partial y^k},$$
$$\frac{\partial^n f}{\partial \bar{z}^n} = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n f = \sum_{k=0}^{n}\binom{n}{k}i^{n-k}\frac{\partial^n f}{\partial x^k \partial y^{n-k}} = \sum_{k=0}^{n}\binom{n}{k}i^k\frac{\partial^n f}{\partial x^{n-k}\partial y^k}.$$

But the binomial theorem applies to fractional powers as well as integer powers. Therefore, the same idea can be used to define fractional iterations of the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ by taking fractional powers of $\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. We thus obtain the following definitions.

Definition 3.2 (Fractional order Wirtinger derivatives): The fractional complex partial derivative of a smooth (infinitely real differentiable) complex function $f(z,\bar{z}) = f(x+yi)$ with respect to z may be defined in either of two possible ways:

$${}^{1}_{c}\partial_z^{\alpha} f(z,\bar{z}) = 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}\,{}_{c}D_x^{\alpha-n}D_y^n f(x+yi), \tag{3.1}$$
$${}^{2}_{c}\partial_z^{\alpha} f(z,\bar{z}) = 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi(\alpha-n)/2}D_x^n\,{}_{c}D_y^{\alpha-n} f(x+yi), \tag{3.2}$$

provided these series converge. In each case c is a real constant, α is a complex parameter, and the real fractional derivative ${}_{c}D^{\alpha-n}$ is defined using Definition 2.1.

Definition 3.3: The fractional complex partial derivative of a smooth (infinitely real differentiable) complex function $f(z,\bar{z}) = f(x+yi)$ with respect to the complex conjugate $\bar{z}$ may be defined in either of two possible ways, corresponding respectively to (3.1) and (3.2) above:

$$ {}^{1}_{c}\partial^{\alpha}_{\bar{z}} f(z,\bar{z}) = 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} e^{i\pi n/2} {}_{c}D^{\alpha-n}_{x} D^{n}_{y} f(x+yi), \tag{3.3} $$

$$ {}^{2}_{c}\partial^{\alpha}_{\bar{z}} f(z,\bar{z}) = 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} e^{i\pi(\alpha-n)/2} D^{n}_{x} {}_{c}D^{\alpha-n}_{y} f(x+yi), \tag{3.4} $$

provided these series converge. In each case c is a real constant, α is a complex parameter, and the real fractional derivative ${}_{c}D^{\alpha-n}$ is defined using Definition 2.1.

The double formulae in each case are a new feature which arises in the fractional context. This corresponds to the fact that there are two possible binomial series for an expression $(a+b)^{\alpha}$ with $\alpha \notin \mathbb{N}$ – namely, one with integer powers of a and fractional powers of b, and the other vice versa. For numbers a and b, the two series would both converge to the same answer, according to which of a and b has larger magnitude. In our case, the a and b are not numbers but operations, and the two series are not necessarily equivalent; their convergence too is harder to analyse. As we shall see from some example cases below, the two ways of defining complex partial derivatives are not equivalent to each other in general, and neither of them is equivalent in general to the usual complex fractional derivative ${}_{c}D^{\alpha}_{z} f(z,\bar{z})$.

Remark 3.1: For the type-2 definitions, (3.2) and (3.4), we note that the expressions $i^{\alpha-n}$ and $(-i)^{\alpha-n}$ are not uniquely defined in general, due to the nature of complex powers. We have chosen the principal branches ($i = e^{i\pi/2}$ and $-i = e^{-i\pi/2}$) in order to preserve inversion properties such as $i^{\alpha-n} \cdot (-i)^{\alpha-n} = 1$.

Definition 3.4: Consider bivariate functions $\phi(x,y)$ defined for $x+yi$ in a two-dimensional disc $D(c,R)$ in the complex plane. We define the function space $C^{\infty}_{\mathfrak{B}}$ as follows:

$$C^{\infty}_{\mathfrak{B}}(D(c,R)) = \big\{\phi \in C^{\infty}(D(c,R)) \,:\, \exists B \in \mathbb{R}^{+} \text{ such that}$$
$$\forall n \in \mathbb{N}, |D^{n}_{x}\phi| \leq B \text{ and } \left|D^{n}_{y}\phi\right| \leq B\big\},$$

where the C^{∞} condition denotes infinite real differentiability w.r.t the real variables x and y.

Theorem 3.1: The fractional complex partial derivative operators ${}^{1}_{c}\partial^{\alpha}_{z}, {}^{1}_{c}\partial^{\alpha}_{\bar{z}}, {}^{2}_{c}\partial^{\alpha}_{z}, {}^{2}_{c}\partial^{\alpha}_{\bar{z}}$ defined above by (3.1)–(3.4) are well defined for any $f \in C^{\infty}_{\mathfrak{B}}(D(c,R))$ and for any $\alpha \in \mathbb{C}$ and finite $c \in \mathbb{R}$.

Proof. Suppose $f \in C^{\infty}_{\mathfrak{B}}$. Since $f(x+yi)$ is infinitely differentiable with respect to both x and y, clearly each term from each of the series in the definitions (3.1)–(3.4) is well-defined. It remains to show that the series converges in each case.

$$\begin{aligned}\left|{}^{1}_{c}\partial^{\alpha}_{z} f(z,\bar{z})\right| &= \left|2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}\,{}_{c}D^{\alpha-n}_{x}D^{n}_{y}f(x+iy)\right| \\ &\leq 2^{-\alpha}\sum_{n=0}^{\infty}\left|\binom{\alpha}{n}\right|\left\|{}_{c}D^{\alpha-n}_{x}D^{n}_{y}f(x+iy)\right\|_{L^{\infty}}\end{aligned}$$

It is sufficient to consider the tail of the series, which means we can assume $n \geq \lfloor \mathrm{Re}(\alpha)\rfloor + 1$ so that $D^{\alpha-n} = I^{n-\alpha}$ is a fractional integral. A standard bound for fractional integrals [35] gives us in this case:

$$\left\|{}_{c}D^{\alpha-n}_{x}D^{n}_{y}f(x+iy)\right\|_{L^{\infty}} \leq \frac{R^{n-\mathrm{Re}(\alpha)}}{(n-\mathrm{Re}(\alpha))|\Gamma(n-\alpha)|}\left\|D^{n}_{y}f\right\|_{L^{\infty}}$$

within the disc $D(c,R)$. From assumption, f satisfies the condition $|D^{n}_{y}f| \leq B$ for all $n \in \mathbb{N}$, where B is fixed independent of n.

So we are interested in the convergence of the series

$$\sum_{n=0}^{\infty}\left|\binom{\alpha}{n}\right|\frac{R^{n-\mathrm{Re}(\alpha)}}{(n-\mathrm{Re}(\alpha))|\Gamma(n-\alpha)|}||D_y^n f||_{L^\infty}\leq B\sum_{n=0}^{\infty}\left|\binom{\alpha}{n}\right|\frac{R^{n-\mathrm{Re}(\alpha)}}{(n-\mathrm{Re}(\alpha))|\Gamma(n-\alpha)|},$$

which converges absolutely by the ratio test.

In an identical way we can prove that ${}^1_c\partial^{\alpha}_{\bar{z}}f(z,\bar{z})$ is well defined. And for ${}^2_c\partial^{\alpha}_{z}f(z,\bar{z})$ and ${}^2_c\partial^{\alpha}_{\bar{z}}f(z,\bar{z})$, we need to take the condition $|D_x^n f|\leq B$ and follow the same steps as in the proof above with slight modifications. □

3.1.1 Application to some example functions

Having proposed the definitions for our fractional complex partial derivatives, we should investigate how they behave when applied to some elementary functions. In this section, we examine how the operators apply to exponential functions and power functions. This will demonstrate that our new derivatives are different from the standard fractional derivatives, and also that the two alternative definitions are different from each other.

Firstly, let us consider the case of $f(z,\bar{z})=e^{az}$ and $c=-\infty$:

$$
\begin{aligned}
{}^{1}_{-\infty}\partial_z^{\alpha} f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{-i\pi n/2}{}_{-\infty}D_x^{\alpha-n}D_y^n e^{a(x+yi)} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-i)^n a^{\alpha-n}e^{ax}(ai)^n e^{ayi} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} a^{\alpha}e^{a(x+iy)} \\
&= a^{\alpha}e^{az}; \\
{}^{2}_{-\infty}\partial_z^{\alpha} f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{-i\pi(\alpha-n)/2}D_x^n{-\infty}D_y^{\alpha-n}e^{a(x+yi)} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{-i\pi(\alpha-n)/2}D_x^n(e^{ax}){-\infty}D_y^{\alpha-n}(e^{ayi}) \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-i)^{\alpha-n}a^n e^{ax}(ai)^{\alpha-n}e^{ayi} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-i)^{\alpha-n}a^{\alpha}e^{a(x+iy)} \\
&= a^{\alpha}e^{az}.
\end{aligned}
$$

Similarly, under the assumption that $\mathrm{Re}(\alpha) > 0$, the fractional d-bar derivatives are:

$$
\begin{aligned}
{}^{1}_{-\infty}\partial_{\bar{z}}^{\alpha} f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{i\pi n/2}{}_{-\infty}D_x^{\alpha-n}D_y^n e^{a(x+yi)} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} i^{2n}(2a)^{\alpha}e^{a(x+yi)} \\
&= a^{\alpha}e^{a(x+yi)}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n = 0; \\
{}^{2}_{-\infty}\partial_{\bar{z}}^{\alpha} f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{i\pi(\alpha-n)/2}D^n_{x\,-\infty}D_y^{\alpha-n}e^{a(x+yi)} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} i^{2(\alpha-n)}(2a)^{\alpha}e^{a(x+yi)} \\
&= a^{\alpha}e^{a(x+yi)}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^{\alpha-n} = 0.
\end{aligned}
$$

(The assumption $\mathrm{Re}(\alpha) > 0$ is necessary for the series $\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n$ to converge. This does not violate the result of Theorem 3.1, which guarantees convergence of the series with no conditions on α, because in that theorem c was assumed to be finite, and here $c = -\infty$.)

In this case, the results for the fractional complex partial derivatives with respect to z agree precisely with the usual fractional derivatives using $c = -\infty$, and those with respect to $\bar{z}$ are zero just as expected for an analytic function. Our next example shows that this is not always true.

Let us consider $f(z,\bar{z}) = z^m$ with $m \in \mathbb{Z}_0^+$ and $c = 0$:

$$\begin{aligned}
{}_0^1\partial_z^\alpha f(z,\bar{z}) &= 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} e^{-i\pi n/2} {}_0D_x^{\alpha-n} D_y^n (x+yi)^m \\
&= 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} (-i)^n {}_0D_x^{\alpha-n} D_y^n \sum_{r=0}^{m} \binom{m}{r} x^r (yi)^{m-r} \\
&= 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} \sum_{r=0}^{m} \binom{m}{r} x^{r-\alpha+n} (yi)^{m-r-n}.
\end{aligned}$$

Now we take $k = m-r-n$, so that $m-\alpha-k = r-\alpha+n$ in the exponent, to get:

$$ {}_0^1\partial_z^\alpha f(z,\bar{z}) = \frac{m!}{(m-\alpha)!} z^{m-\alpha} - 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} \sum_{k=m-n+1}^{\infty} \frac{m!}{k!(m-\alpha-k)!} x^{m-\alpha-k} (yi)^k. $$

Similarly, for the type-2 derivative:

$$\begin{aligned}
{}_0^2\partial_z^\alpha f(z,\bar{z}) &= 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} e^{-i\pi(\alpha-n)/2} D_x^n {}_0D_y^{\alpha-n} (x+yi)^m \\
&= 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} (-i)^{\alpha-n} D_x^n {}_0D_y^{\alpha-n} \sum_{r=0}^{m} \binom{m}{r} x^{m-r} (yi)^r \\
&= 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} (-i)^{\alpha-n} \sum_{r=0}^{m} \binom{m}{r} D_x^n (x^{m-r}) {}_0D_y^{\alpha-n}((yi)^r) \\
&= 2^{-\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} \sum_{r=0}^{m} \frac{m!}{(m-\alpha)!} \frac{(m-\alpha)!}{(m-r-n)!(r-\alpha+n)!} x^{m-r-n} (yi)^{r-\alpha+n}.
\end{aligned}$$

Again taking $k = m-r-n$ so that $m-\alpha-k = r-\alpha+n$:

$$
\begin{aligned}
{}_0^2\partial_z^\alpha f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi(\alpha-n)/2}D_{x0}^{n}D_y^{\alpha-n}(x+yi)^m \\
&= 2^{-\alpha}\frac{m!}{(m-\alpha)!}\sum_{n=0}^{\infty}\binom{\alpha}{n}\sum_{k=-n}^{m-n}\frac{(m-\alpha)!}{k!(m-\alpha-k)!}x^k(yi)^{m-\alpha-k} \\
&= \frac{m!}{(m-\alpha)!}z^{m-\alpha}-2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\sum_{k=m-n+1}^{\infty}\frac{m!}{k!(m-\alpha-k)!}x^k(yi)^{m-\alpha-k}.
\end{aligned}
$$

Similarly, for the fractional d-bar derivatives:

$$
\begin{aligned}
{}_0^1\partial_{\bar{z}}^\alpha f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi n/2}{}_0D_x^{\alpha-n}D_y^{n}(x+yi)^m \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}i^{n+m-r}\sum_{r=0}^{m}\binom{m}{r}x^{r-\alpha+n}(y)^{m-r-n}\frac{(m-r)!r!}{(m-r-n)!(r-\alpha+n)!} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}i^{2n}\sum_{r=0}^{m}\frac{m!}{(m-r-n)!(r-\alpha+n)!}x^{r-\alpha+n}(yi)^{m-r-n} \\
&= \frac{m!}{(m-\alpha)!}z^{m-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n-\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n \\
&\qquad \times\sum_{k=m-n+1}^{\infty}\frac{m!}{k!(m-\alpha-k)!}x^{m-\alpha-k}(yi)^k \\
&= -2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n\sum_{k=m-n+1}^{\infty}\frac{m!}{k!(m-\alpha-k)!}x^{m-\alpha-k}(yi)^k.
\end{aligned}
$$

$$
\begin{aligned}
{}_0^2\partial_{\bar{z}}^\alpha f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi(\alpha-n)/2}D_{x0}^{n}D_y^{\alpha-n}(x+yi)^m \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}i^{\alpha-n+r}\sum_{r=0}^{m}\binom{m}{r}x^{m-r-n}(y)^{r-\alpha+n}\frac{(m-r)!r!}{(m-r-n)!(r-\alpha+n)!} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}i^{2(\alpha-n)}\sum_{r=0}^{m}\frac{m!}{(m-r-n)!(r-\alpha+n)!}x^{m-r-n}(yi)^{r-\alpha+n} \\
&= \frac{m!}{(m-\alpha)!}2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^{\alpha-n}\sum_{r=0}^{m}\frac{(m-\alpha)!}{(m-r-n)!(r-\alpha+n)!} \\
&\qquad \times x^{m-r-n}(yi)^{r-\alpha+n} \\
&= \frac{m!}{(m-\alpha)!}z^{m-\alpha}2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^{\alpha-n} \\
&\qquad -2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^{\alpha-n}\sum_{k=m-n+1}^{\infty}\frac{m!}{k!(m-\alpha-k)!}x^k(yi)^{m-\alpha-k} \\
&= -2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^{\alpha-n}\sum_{k=m-n+1}^{\infty}\frac{m!}{k!(m-\alpha-k)!}x^k(yi)^{m-\alpha-k};
\end{aligned}
$$

Example 3.1: To illustrate the above result for z^m, let us consider the simplest case

$m = 0$. We find the fractional d-bar derivatives of the constant function $f(z,\bar{z}) = z^0 = 1$ with constant of differintegration $c = 0$:

$$\begin{aligned}
{}^1_0\partial^{\alpha}_{\bar{z}}(1) &= -2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n\sum_{k=1-n}^{\infty}\frac{1}{k!(-\alpha-k)!}x^{-\alpha-k}(yi)^k \\
&= -2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n\sum_{k=0}^{\infty}\frac{1}{k!(-\alpha-k)!}x^{-\alpha-k}(yi)^k \\
&\qquad -2^{-\alpha}\sum_{k=n=0}\binom{\alpha}{n}(-1)^n\frac{1}{k!(-\alpha-k)!}x^{-\alpha-k}(yi)^k \\
&= -2^{-\alpha}\left[\alpha!x^{-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\alpha-n)!}\sum_{k=0}^{\infty}\frac{x^{-k}(yi)^k}{k!(-\alpha-k)!}-\frac{x^{-\alpha}}{(-\alpha)!}\right] \\
&= -2^{-\alpha}\left[\alpha!x^{-\alpha}\left[\frac{1}{\alpha!}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n\right]\left[\frac{1}{(-\alpha)!}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{yi}{x}\right)^k\right]-\frac{x^{-\alpha}}{(-\alpha)!}\right] \\
&= -2^{-\alpha}\left[\alpha!x^{-\alpha}(1-1)^{\alpha}\left(1+\frac{yi}{x}\right)^{-\alpha}\frac{1}{\alpha!(-\alpha)!}-\frac{x^{-\alpha}}{(-\alpha)!}\right] \\
&= \frac{(2x)^{-\alpha}}{\Gamma(1-\alpha)},
\end{aligned}$$

provided we assume $\mathrm{Re}(\alpha) > 0$ so that $(1-1)^{\alpha} = 0$. Similarly we have

$$\begin{aligned}
{}^2_0\partial^{\alpha}_{\bar{z}}(1) &= -2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^{\alpha-n}\sum_{k=1-n}^{\infty}\frac{1}{k!(-\alpha-k)!}x^{k}(yi)^{-\alpha-k} \\
&= -2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^{\alpha-n}\sum_{k=0}^{\infty}\frac{1}{k!(-\alpha-k)!}x^{k}(yi)^{-\alpha-k} \\
&\qquad +2^{-\alpha}\sum_{k=n=0}\binom{\alpha}{n}(-1)^{\alpha-n}\frac{1}{k!(-\alpha-k)!}x^{k}(yi)^{-\alpha-k} \\
&= -2^{-\alpha}\left[\alpha!e^{i\pi\alpha/2}y^{-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\alpha-n)!}\sum_{k=0}^{\infty}\frac{x^k(yi)^{-k}}{k!(-\alpha-k)!}-\frac{e^{i\pi\alpha/2}y^{-\alpha}}{(-\alpha)!}\right] \\
&= -2^{-\alpha}\alpha!e^{i\pi\alpha/2}y^{-\alpha}\left[\frac{1}{\alpha!}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n\right]\left[\frac{1}{(-\alpha)!}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{x}{yi}\right)^k\right] \\
&\qquad +2^{-\alpha}\frac{e^{i\pi\alpha/2}y^{-\alpha}}{(-\alpha)!} \\
&= -2^{-\alpha}\left[\alpha!i^{\alpha}y^{-\alpha}(1-1)^{\alpha}\left(1+\frac{x}{yi}\right)^{-\alpha}\frac{1}{\alpha!(-\alpha)!}-\frac{(-i)^{-\alpha}y^{-\alpha}}{(-\alpha)!}\right] \\
&= \frac{(-2iy)^{-\alpha}}{\Gamma(1-\alpha)},
\end{aligned}$$

again under the assumption that $\mathrm{Re}(\alpha) > 0$.

For complex partial derivatives with respect to z, we do not need to assume $\mathrm{Re}(\alpha) > 0$, because in this case the argument is like above but with $(1+1)^\alpha$ instead of $(1-1)^\alpha$:

$$\begin{aligned}
{}^{1}_{0}\partial_z^\alpha(1) &= \frac{1}{(-\alpha)!}z^{-\alpha} - 2^{-\alpha}\left[\alpha! x^{-\alpha}(1+1)^\alpha\left(1+\frac{yi}{x}\right)^{-\alpha}\frac{1}{\alpha!(-\alpha)!} - \frac{x^{-\alpha}}{(-\alpha)!}\right] \\
&= \frac{z^{-\alpha} - z^{-\alpha} + (2x)^{-\alpha}}{\Gamma(1-\alpha)} = \frac{(2x)^{-\alpha}}{\Gamma(1-\alpha)}; \\
{}^{2}_{0}\partial_z^\alpha(1) &= \frac{1}{(-\alpha)!}z^{-\alpha} - 2^{-\alpha}\left[\alpha!(-i)^\alpha y^{-\alpha}(1+1)^\alpha\left(1+\frac{x}{yi}\right)^{-\alpha}\frac{1}{\alpha!(-\alpha)!} - \frac{i^{-\alpha}y^{-\alpha}}{(-\alpha)!}\right] \\
&= \frac{z^{-\alpha} - z^{-\alpha} - (2iy)^{-\alpha}}{\Gamma(1-\alpha)} = \frac{(2iy)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}$$

All of these results can be confirmed by a simple manipulation directly from the definition of the fractional complex partial derivatives:

$$\begin{aligned}
{}_{c}\partial_{\bar{z}}^\alpha(1) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi n/2}\,{}_{c}D_x^{\alpha-n}D_y^n(1) \\
&= 2^{-\alpha}\binom{\alpha}{0}e^{i\pi 0/2}\,{}_{c}D_x^\alpha(1) = 2^{-\alpha}\frac{(x-c)^{-\alpha}}{\Gamma(1-\alpha)}, \\
{}^{2}_{c}\partial_{\bar{z}}^\alpha(1) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi(\alpha-n)/2}D_x^n\,{}_{c}D_y D_y^{\alpha-n}(1) \\
&= 2^{-\alpha}\binom{\alpha}{0}e^{i\pi\alpha/2}\,{}_{c}D_y^\alpha(1) = 2^{-\alpha}\frac{(ci-yi)^{-\alpha}}{\Gamma(1-\alpha)},
\end{aligned}$$

and similarly for the derivatives with respect to z as well as $\bar{z}$. Therefore, we have verified that our expressions for the fractional complex derivatives of z^m are correct in this case.

After this, we can easily get the fractional complex derivative of any constant function $f(z,\bar{z}) = K$, as a multiplication of K and the fractional complex derivative of 1.

3.2 Further properties of the fractional complex derivatives

3.2.1 Complex directional derivatives

In the usual complex analysis, it is well known that if a complex function F is analytic, then

$$F'(z) = \frac{\mathrm{d}F}{\mathrm{d}z} = \frac{\mathrm{d}F}{\mathrm{d}x} = -i\frac{\mathrm{d}F}{\mathrm{d}y}. \tag{3.5}$$

The proof of this identity is by considering the real derivatives with respect to x and y as *directional* derivatives in complex two-dimensional space. The complex derivative $\frac{\mathrm{d}F}{\mathrm{d}z}$ must be independent of the direction in which the limit in (2.11) is taken; by taking this limit in the horizontal and vertical directions, we obtain respectively $\frac{\mathrm{d}F}{\mathrm{d}x}$ and $-i\frac{\mathrm{d}F}{\mathrm{d}y}$. Is it possible to find a similar identity for *fractional* complex derivatives?

In the following two Lemmas, we prove identities analogous to (3.5) between the *fractional* derivatives with respect to z, x, and y, under the assumption that the function in question is analytic. In the fractional context, it is necessary to consider the constants of differentiation, and these will play a role in the relationships derived. Finally, we can use these two Lemmas to prove a theorem relating our new fractional complex partial derivatives to the usual fractional derivative with respect to z.

Lemma 3.1: If $F(z) = F(x+yi)$ is a complex analytic function, then the fractional differintegral with respect to the real part x can be written in terms of the fractional differintegral with respect to z:

$${}_{c}D_{x}^{\beta}\Big[F(x+yi)\Big] = {}_{c+yi}D_{z}^{\beta}F(z), \tag{3.6}$$

for any complex number β, where the right-hand side is defined by first taking the fractional integral ${}_{a}D_{z}^{\beta}F(z)$ with respect to z and then setting $a = c+yi$ afterwards.

Proof. Firstly, we rewrite the fractional x-integration operator in terms of z.

For $\mathrm{Re}(\beta) < 0$ (fractional *integration*) and for any L^1 function F, we have:

$$\begin{aligned}
{}_cD_x^\beta \left[F(x+yi)\right] &= \frac{1}{\Gamma(-\beta)} \int_c^x (x-\xi)^{-\beta-1} F(\xi+yi)\,\mathrm{d}\xi \\
&= \frac{1}{\Gamma(-\beta)} \int_{c+yi}^{x+yi} (x+yi-\zeta)^{-\beta-1} F(\zeta)\,\mathrm{d}\zeta \\
&= {}_{c+yi}D_z^\beta F(z),
\end{aligned}$$

where in the second step we substituted $\zeta = \xi + yi$, and where the variable z is taken to be $x+yi$ as usual.

For fractional differentiation, things become more complicated and we will need to use (3.5). For $\mathrm{Re}(\beta) \geq 0$ (fractional *differentiation*) and for a complex analytic function $F(z) = F(x+yi)$, we write $m = \lfloor \mathrm{Re}(\beta) \rfloor + 1$ and recall the definition:

$$ {}_{c+yi}D_z^\beta F(z) = {}_a D_z^\beta F(z)\Big|_{a=c+yi} = \left[\frac{\mathrm{d}^m}{\mathrm{d}z^m}\, {}_aD_z^{\beta-m} F(z)\right]_{a=c+yi}. $$

Since $F(z)$ is an analytic function, its Riemann-Liouville fractional integral ${}_aD_z^{\beta-m}F(z)$ is also analytic. So, by (3.5), applying $\frac{\mathrm{d}}{\mathrm{d}z}$ to this function is equivalent to applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to it, and we have:

$$ {}_{c+yi}D_z^\beta F(z) = \left[\frac{\mathrm{d}^m}{\mathrm{d}z^m}\, {}_aD_z^{\beta-m} F(z)\right]_{a=c+yi} = \left[\frac{\mathrm{d}^m}{\mathrm{d}x^m}\, {}_aD_z^{\beta-m} F(z)\right]_{a=c+yi}. $$

Now the quantity $c+yi$ is treated as a constant when differentiating with respect to x only. So setting $a = c+yi$ before or after the differentiation makes no difference here:

$$ {}_{c+yi}D_z^\beta F(z) = \left[\frac{\mathrm{d}^m}{\mathrm{d}x^m}\, {}_aD_z^{\beta-m} F(z)\right]_{a=c+yi} = \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left[{}_aD_z^{\beta-m} F(z)\right]_{a=c+yi}. $$

Since ${}_aD_z^{\beta-m}F(z)$ is a fractional *integral* ($\mathrm{Re}(\beta - m) < 0$), we can now use the result for fractional integrals:

$$\begin{aligned}
{}_{c+yi}D_z^\beta F(z) &= \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left[{}_aD_z^{\beta-m} F(z)\right]_{a=c+yi} = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\, {}_{c+yi}D_z^{\beta-m} F(z) \\
&= \frac{\mathrm{d}^m}{\mathrm{d}x^m}\, {}_cD_x^{\beta-m} F(x+yi) = {}_cD_x^\beta F(x+yi).
\end{aligned}$$

Thus, the result is proved for both fractional integrals and fractional derivatives. □

Lemma 3.2: If $F(z) = F(x+yi)$ is a complex analytic function, then the fractional differintegral with respect to the imaginary part y can be written in terms of the fractional differintegral with respect to z:

$$e^{-i\pi\beta/2} {}_cD_y^\beta \left[F(x+yi)\right] = {}_{x+ci} D_z^\beta F(z), \tag{3.7}$$

for any complex number β, where the right-hand side is defined by first taking the fractional integral ${}_aD_z^\beta F(z)$ with respect to z and then setting $a = x+ci$ afterwards.

Proof. We proceed by a similar method as in the previous proof. Firstly, for $\mathrm{Re}(\beta) < 0$ (fractional *integration*) and for any L^1 function F, we have:

$$\begin{aligned}
{}_cD_y^\beta \left[F(x+yi)\right] &= \frac{1}{\Gamma(-\beta)} \int_c^y (y-\eta)^{-\beta-1} F(x+\eta i)\,\mathrm{d}\xi \\
&= \frac{1}{\Gamma(-\beta)} \int_{x+ci}^{x+yi} (x+yi-\zeta)^{-\beta-1} i^{\beta+1} F(\zeta)\,\mathrm{d}\zeta \\
&= i^\beta {}_{x+ci}D_z^\beta F(z),
\end{aligned}$$

where in the second step we substituted $\zeta = x+\eta i$, and where the variable z is taken to be $x+yi$ as usual. This verifies the result for fractional integrals.

Now, for $\mathrm{Re}(\beta) \geq 0$ (fractional *differentiation*) and for a complex analytic function $F(z) = F(x+yi)$, we write $m = \lfloor \mathrm{Re}(\beta) \rfloor + 1$ and recall the definition:

$${}_{x+ci}D_z^\beta F(z) = {}_a D_z^\beta F(z)\bigg|_{a=x+ci} = \left[\frac{\mathrm{d}^m}{\mathrm{d}z^m} {}_aD_z^{\beta-m} F(z)\right]_{a=x+ci}.$$

Since $F(z)$ is an analytic function, its Riemann-Liouville fractional integral ${}_aD_z^{\beta-m} F(z)$ is also analytic. So, by (3.5), applying $\frac{\mathrm{d}}{\mathrm{d}z}$ to this function is equivalent to applying $-i\frac{\mathrm{d}}{\mathrm{d}y}$ to it, and we have:

$${}_{x+ci}D_z^\beta F(z) = \left[\frac{\mathrm{d}^m}{\mathrm{d}z^m} {}_aD_z^{\beta-m} F(z)\right]_{a=x+ci} = \left[(-i)^m \frac{\mathrm{d}^m}{\mathrm{d}y^m} {}_aD_z^{\beta-m} F(z)\right]_{a=x+ci}.$$

Now the quantity $x+ci$ is treated as a constant when differentiating with respect to y only. So setting $a = x+ci$ before or after the differentiation makes no difference here:

$$ {}_{x+ci}D_z^{\beta}F(z) = \left[(-i)^m \frac{\mathrm{d}^m}{\mathrm{d}y^m} {}_aD_z^{\beta-m}F(z)\right]_{a=x+ci} = (-i)^m \frac{\mathrm{d}^m}{\mathrm{d}y^m}\left[{}_aD_z^{\beta-m}F(z)\right]_{a=x+ci}. $$

Since ${}_aD_z^{\beta-m}F(z)$ is a fractional *integral* ($\mathrm{Re}(\beta-m)<0$), we can now use the result for fractional integrals:

$$ \begin{aligned} {}_{x+ci}D_z^{\beta}F(z) &= (-i)^m \frac{\mathrm{d}^m}{\mathrm{d}y^m}\left[{}_aD_z^{\beta-m}F(z)\right]_{a=x+ci} = (-i)^m \frac{\mathrm{d}^m}{\mathrm{d}y^m} {}_{x+ci}D_z^{\beta-m}F(z) \\ &= (-i)^m \frac{\mathrm{d}^m}{\mathrm{d}y^m}(-i)^{\beta-m} {}_cD_y^{\beta-m}F(x+yi) = (-i)^{\beta} {}_cD_y^{\beta}F(x+yi). \end{aligned} $$

Thus, the result is proved for both fractional integrals and fractional derivatives. □

Remark 3.2: In the statements of Lemma 3.1 and 3.2, it is necessary to state how we are defining fractional derivatives with non-constant "constants of integration", because there are two different possible definitions which may both seem intuitive to use. For fractional integrals ($\mathrm{Re}(\beta)<0$), there is only one sensible definition:

$$ {}_{c+yi}D_z^{\beta}F(z) = {}_aD_z^{\beta}F(z)\Big|_{a=c+yi} \quad , \quad {}_{x+ci}D_z^{\beta}F(z) = {}_aD_z^{\beta}F(z)\Big|_{a=x+ci}. \tag{3.8} $$

But for fractional derivatives ($\mathrm{Re}(\beta)\geq 0$), we might consider either of the following two possible definitions:

$$ {}_{c+yi}D_z^{\beta}F(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left({}_{c+yi}D_z^{\beta-m}F(z)\right) \quad , \quad {}_{x+ci}D_z^{\beta}F(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left({}_{c+yi}D_z^{\beta-m}F(z)\right); \tag{3.9} $$

$$ {}_{c+yi}D_z^{\beta}F(z) = {}_aD_z^{\beta}F(z)\Big|_{a=c+yi} \quad , \quad {}_{x+ci}D_z^{\beta}F(z) = {}_aD_z^{\beta}F(z)\Big|_{a=x+ci}. \tag{3.10} $$

However, the definitions (3.9) do not actually make sense. The expression inside the brackets is well-defined, using the definition (3.8) for fractional integrals, but it is not an analytic function of z, due to the extra x-dependence or y-dependence, and so the complex derivative $\frac{\mathrm{d}}{\mathrm{d}z}$ cannot be applied to it.

The definitions (3.9) and (3.10) are completely different from each other, since it makes a big difference whether we differentiate before or after setting the "constant" a to be a variable $c+yi$ or $x+ci$. In particular, we have just seen that (3.9) makes no sense but (3.10) is well-defined. Thus it is the definitions (3.10) that we use in the Lemmas above.

Theorem 3.2: For a complex analytic function $f(z,\bar{z})$, the fractional complex partial derivatives with respect to z can be written in the following forms for any $\alpha \in \mathbb{C}$:

$$ {}_c^1\partial_z^\alpha f(z,\bar{z}) = {}_{c+yi}D_z^\alpha f(z) - \sum_{k=0}^{\infty} \frac{2^{-\alpha}(x-c)^{k-\alpha}}{\Gamma(k-\alpha+1)} \left[\sum_{n=k+1}^{\infty} \binom{\alpha}{n} \right] f^{(k)}(c+yi) \tag{3.11} $$

$$ = {}_{c+yi}D_z^\alpha f(z) - \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(x-c)^{k-\alpha}}{2^\alpha (k+1)!\Gamma(-\alpha)} {}_2F_1(1,k+1-\alpha;k+2;-1) f^{(k)}(c+yi) \tag{3.12} $$

$$ {}_c^2\partial_z^\alpha f(z,\bar{z}) = {}_{x+ci}D_z^\alpha f(z) - \sum_{k=0}^{\infty} \frac{2^{-\alpha} e^{i\pi(k-\alpha)/2}(y-c)^{k-\alpha}}{\Gamma(k-\alpha+1)} \left[\sum_{n=k+1}^{\infty} \binom{\alpha}{n} \right] f^{(k)}(x+ci) \tag{3.13} $$

$$ = {}_{x+ci}D_z^\alpha f(z) - \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(yi-ci)^{k-\alpha}}{2^\alpha (k+1)!\Gamma(-\alpha)} {}_2F_1(1,k+1-\alpha;k+2;-1) f^{(k)}(x+ci), \tag{3.14} $$

where $f^{(k)}(a)$ denotes the kth derivative of $f(z,\bar{z})$ with respect to z evaluated at some point a, and ${}_2F_1$ is the hypergeometric function.

Proof. Since $f(z,\bar{z})$ is analytic, we have for every $n \in \mathbb{N}$ that

$$ D_z^n f(z,\bar{z}) = D_x^n f(x+yi) = (-i)^n D_y^n f(x+yi), $$

and that this also is an analytic function of z. Bearing these facts in mind, we proceed as follows:

Type-1 derivatives. We make use of Lemma 3.1 and the composition relation (2.6):

$$
\begin{aligned}
{}^{1}_{c}\partial_z^\alpha f(z,\bar z) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}{}_{c}D_x^{\alpha-n}D_y^n f(x+yi)\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}{}_{c}D_x^{\alpha-n}\Big(D_z^n f(z,\bar z)\Big)\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}{}_{c+yi}D_z^{\alpha-n}\Big(D_z^n f(z,\bar z)\Big)\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\Big[{}_{a}D_z^{\alpha-n}D_z^n f(z,\bar z)\Big]_{a=c+yi}\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\left[{}_{a}D_z^{\alpha} f(z,\bar z)-\sum_{k=0}^{n-1}\frac{(z-a)^{k-(\alpha-n)-n}}{\Gamma(k-(\alpha-n)-n+1)}D_z^k f(a)\right]_{a=c+yi}\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}{}_{c+yi}D_z^{\alpha} f(z,\bar z)-2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\sum_{k=0}^{n-1}\frac{(x-c)^{k-\alpha}}{\Gamma(k-\alpha+1)}D_z^k f(c+yi)\\
&= {}_{c+yi}D_z^{\alpha} f(z,\bar z)-\sum_{k=0}^{\infty}\frac{2^{-\alpha}(x-c)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left[\sum_{n=k+1}^{\infty}\binom{\alpha}{n}\right]f^{(k)}(c+yi).
\end{aligned}
$$

This proves the expression (3.11). To derive (3.12), it remains to prove the following identity for every $k\in\mathbb{Z}_0^+$:

$$
\frac{1}{\Gamma(k-\alpha+1)}\sum_{n=k+1}^{\infty}\binom{\alpha}{n}=\frac{(-1)^{k+1}}{(k+1)!\Gamma(-\alpha)}{}_2F_1(1,k+1-\alpha;k+2;-1). \qquad (3.15)
$$

We start from the hypergeometric function in (3.15), and expand it as a series:

$$
\begin{aligned}
{}_2F_1(1,k+1-\alpha;k+2;-1) &= \sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(k+1-\alpha+n)\Gamma(k+2)}{\Gamma(1)\Gamma(k+1-\alpha)\Gamma(k+2+n)}\cdot\frac{(-1)^n}{n!}\\
&= \sum_{n=0}^{\infty}(-1)^n\\
&\quad\times\frac{(k-\alpha+n)(k-\alpha+n-1)\dots(k-\alpha+2)(k-\alpha+1)}{(k+1+n)(k+n)\dots(k+3)(k+2)}\\
&= \sum_{n=0}^{\infty}\frac{(\alpha-k-1)(\alpha-k-2)\dots(\alpha-k-n+1)(\alpha-k-n)}{(k+2)(k+3)\dots(k+n)(k+n+1)}.
\end{aligned}
$$

After multiplying by another binomial-coefficient factor, this series becomes a truncated binomial series:

$$
\begin{aligned}
\binom{\alpha}{k+1}{}_2F_1(1,k+1-\alpha;k+2;-1)
&= \frac{\alpha(\alpha-1)\dots(\alpha-k+1)(\alpha-k)}{(k+1)!} \\
&\qquad \times \sum_{n=0}^{\infty} \frac{(\alpha-k-1)(\alpha-k-2)\dots(\alpha-k-n+1)(\alpha-k-n)}{(k+2)(k+3)\dots(k+n)(k+n+1)} \\
&= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-k-n+1)(\alpha-k-n)}{(k+n+1)!} \\
&= \sum_{n=0}^{\infty} \binom{\alpha}{k+n+1} = \sum_{n=k+1}^{\infty} \binom{\alpha}{n}.
\end{aligned}
$$

Now we have an identity between the hypergeometric function and the tail of the series.

To prove (3.15), it only remains to perform some manipulation of gamma functions:

$$
\begin{aligned}
\frac{1}{\Gamma(k-\alpha+1)}\binom{\alpha}{k+1} &= \frac{\Gamma(\alpha+1)}{\Gamma(k-\alpha+1)(k+1)!\Gamma(\alpha-k)} \\
&= \frac{\Gamma(\alpha+1)}{(k+1)!\frac{\pi}{\sin(\pi(\alpha-k))}} = \frac{\Gamma(\alpha+1)}{(k+1)!\frac{\pi}{(-1)^{k+1}\sin(\pi(\alpha+1))}} \\
&= \frac{(-1)^{k+1}\Gamma(\alpha+1)}{(k+1)!\Gamma(\alpha+1)\Gamma(-\alpha)} = \frac{(-1)^{k+1}}{(k+1)!\Gamma(-\alpha)}.
\end{aligned}
$$

Now we have verified (3.15) and hence (3.12).

Type-2 derivatives. We make use of Lemma 3.2 and the composition relation (2.6):

$$
\begin{aligned}
{}^{2}_{c}\partial_z^{\alpha} f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{-i\pi(\alpha-n)/2}\,{}_cD_y^{\alpha-n}D_x^n f(x+yi) \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{-i\pi(\alpha-n)/2}\,{}_cD_y^{\alpha-n}\Big(D_z^n f(z,\bar{z})\Big) \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\,{}_{x+ci}D_z^{\alpha-n}\Big(D_z^n f(z,\bar{z})\Big) \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\Big[{}_aD_z^{\alpha-n}D_z^n f(z,\bar{z})\Big]_{a=x+ci} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\left[{}_aD_z^{\alpha} f(z,\bar{z}) - \sum_{k=0}^{n-1}\frac{(z-a)^{k-(\alpha-n)-n}}{\Gamma(k-(\alpha-n)-n+1)}D_z^k f(a)\right]_{a=x+ci} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\,{}_{x+ci}D_z^{\alpha} f(z,\bar{z}) - 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\sum_{k=0}^{n-1}\frac{(yi-ci)^{k-\alpha}}{\Gamma(k-\alpha+1)}D_z^k f(c+yi) \\
&= {}_{x+ci}D_z^{\alpha} f(z,\bar{z}) - \sum_{k=0}^{\infty}\frac{2^{-\alpha}e^{i\pi(k-\alpha)/2}(y-c)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left[\sum_{n=k+1}^{\infty}\binom{\alpha}{n}\right] f^{(k)}(x+ci).
\end{aligned}
$$

This proves the expression (3.13), and again the alternative expression (3.14) follows by applying the identity (3.15). $\square$

The above Theorem establishes a connection between the fractional complex partial derivatives defined in this thesis and the usual fractional derivative with respect to z, for analytic function only. This is analogous to the fact that if $F(z)$ is an analytic function, then $\frac{\partial F}{\partial z} = \frac{\mathrm{d}F}{\mathrm{d}z}$ and $\frac{\partial F}{\partial \bar{z}} = 0$. In the fractional context, we do not have such simple identities as these, but instead identities which involve an extra added series of terms. This is similar to the relationship between Riemann-Liouville and Caputo fractional derivatives, which are equal up to an extra added series of initial value terms. It may be possible to view the fractional complex partial derivatives defined here as a third alternative way of defining fractional derivatives, alongside Riemann-Liouville and Caputo.

We can also derive similar results for the fractional derivatives w.r.t $\bar{z}$, although these results will later be rewritten in a different and more natural form.

Proposition 3.1: For $\mathrm{Re}(\alpha) > 0$, the fractional complex partial derivatives with respect to $\bar{z}$ can be written in the following forms:

$$
\begin{aligned}
{}^{1}_{c}\partial^{\alpha}_{\bar{z}} f(z,\bar{z}) &= \sum_{k=0}^{\infty} \frac{2^{-\alpha}(x-c)^{k-\alpha}}{\Gamma(k-\alpha+1)} \left[\sum_{n=k+1}^{\infty} (-1)^{n+1} \binom{\alpha}{n} \right] f^{(k)}(c+yi) && (3.16) \\
&= \sum_{k=0}^{\infty} \frac{-(x-c)^{k-\alpha}}{2^{\alpha}(k+1)!\Gamma(-\alpha)} {}_2F_1(1,k+1-\alpha;k+2;1) f^{(k)}(c+yi); && (3.17) \\
{}^{2}_{c}\partial^{\alpha}_{\bar{z}} f(z,\bar{z}) &= \sum_{k=0}^{\infty} \frac{2^{-\alpha} e^{i\pi(k+\alpha)/2}(y-c)^{k-\alpha}}{\Gamma(k-\alpha+1)} \left[\sum_{n=k+1}^{\infty} (-1)^{n+1} \binom{\alpha}{n} \right] f^{(k)}(x+ci) && (3.18) \\
&= \sum_{k=0}^{\infty} \frac{-e^{i\pi\alpha}(yi-ci)^{k-\alpha}}{2^{\alpha}(k+1)!\Gamma(-\alpha)} {}_2F_1(1,k+1-\alpha;k+2;1) f^{(k)}(x+ci), && (3.19)
\end{aligned}
$$

where $f^{(k)}(a)$ denotes the kth derivative of $f(z,\bar{z})$ with respect to z evaluated at some point a, and ${}_2F_1$ is the hypergeometric function.

Proof. The proof here is largely similar to the proof of the preceding Theorem.

Type-1 derivatives. The manipulation here is almost identical to the type-1 case in the previous proof, except with an extra factor of $(-1)^n$ in the summand:

$$\begin{aligned}
{}^{1}_{c}\partial^{\alpha}_{\bar{z}} f(z,\bar{z}) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{i\pi n/2}\,{}_{c}D^{\alpha-n}_{x} D^{n}_{y} f(x+yi) \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n e^{-i\pi n/2}\,{}_{c}D^{\alpha-n}_{x} D^{n}_{y} f(x+yi) \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n \left[{}_{a}D^{\alpha}_{z} f(z,\bar{z}) - \sum_{k=0}^{n-1}\frac{(z-a)^{k-(\alpha-n)-n}}{\Gamma(k-(\alpha-n)-n+1)} D^{k}_{z} f(a)\right]_{a=c+yi} \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n\,{}_{c+yi}D^{\alpha}_{z} f(z,\bar{z}) - 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n \\
&\qquad\qquad\qquad \times\sum_{k=0}^{n-1}\frac{(x-c)^{k-\alpha}}{\Gamma(k-\alpha+1)} D^{k}_{z} f(c+yi) \\
&= \sum_{k=0}^{\infty}\frac{2^{-\alpha}(x-c)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left[\sum_{n=k+1}^{\infty}(-1)^{n+1}\binom{\alpha}{n}\right] f^{(k)}(c+yi).
\end{aligned}$$

This proves (3.16). Now, looking at the proof of (3.15), we see that the argument -1 used for the hypergeometric function gives rise to a factor of $(-1)^n$ in the series, and the step at (**??**) gives rise to an extra outer factor of $(-1)^{k+1}$ in this case. Therefore we have the following result which can be proved in an identical way to (3.15):

$$\frac{1}{\Gamma(k-\alpha+1)}\sum_{n=k+1}^{\infty}\binom{\alpha}{n}(-1)^n = \frac{1}{(k+1)!\Gamma(-\alpha)}\,{}_{2}F_{1}(1,k+1-\alpha;k+2;1). \quad (3.20)$$

Substituting this new identity into (3.16) gives immediately (3.17).

Type-2 derivatives. The manipulation here is almost identical to the type-1 case in the previous proof, except with an extra factor of $(-1)^{\alpha-n}$ in the summand:

$$
\begin{aligned}
{}^2_c\partial^\alpha_{\bar z} f(z,\bar z) &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n} e^{i\pi(\alpha-n)/2} D^n_{xc} D^{\alpha-n}_y f(x+yi) \\
&= 2^{-\alpha} e^{i\pi\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n e^{-i\pi(\alpha-n)/2}\, {}_cD^{\alpha-n}_y D^n_x f(x+yi) \\
&= 2^{-\alpha} e^{i\pi\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n \\
&\qquad\qquad \left[{}_aD^\alpha_z f(z,\bar z) - \sum_{k=0}^{n-1}\frac{(z-a)^{k-(\alpha-n)-n}}{\Gamma(k-(\alpha-n)-n+1)} D^k_z f(a)\right]_{a=x+ci} \\
&= 2^{-\alpha} e^{i\pi\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n\, {}_{x+ci}D^\alpha_z f(z,\bar z) \\
&\qquad\qquad - 2^{-\alpha} e^{i\pi\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n \sum_{k=0}^{n-1}\frac{(yi-ci)^{k-\alpha}}{\Gamma(k-\alpha+1)} D^k_z f(c+yi) \\
&= 2^{-\alpha} e^{i\pi\alpha}\sum_{k=0}^{\infty}\frac{e^{i\pi(k-\alpha)/2}(y-c)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left[\sum_{n=k+1}^{\infty}(-1)^{n+1}\binom{\alpha}{n}\right] f^{(k)}(x+ci) \\
&= \sum_{k=0}^{\infty}\frac{2^{-\alpha} e^{i\pi(k+\alpha)/2}(y-c)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left[\sum_{n=k+1}^{\infty}(-1)^{n+1}\binom{\alpha}{n}\right] f^{(k)}(x+ci).
\end{aligned}
$$

This proves (3.18), and again by using (3.20) we derive (3.19). □

3.2.2 Leibniz rule

In calculus, an important early result is the product rule for differentiating a product of two functions. Its analogue for integrating a product of two functions is called integration by parts. The analogues of these two results for fractional derivatives and integrals have frequently been considered, some seminal results in this area being due to Hardy & Littlewood [15, 23] and Osler [31, 32]. Similar results have also been considered in some very general models of FC [12, 38].

Naturally it is also of interest for us to investigate a Leibniz rule analogue in the context of the new fractional complex partial derivatives. The result is given by the following theorem, which we note is in exactly the same form as the Leibniz rule (2.7) for the usual Riemann-Liouville fractional derivatives.

Theorem 3.3: If $f(z,\bar z)$ and $g(z,\bar z)$ are two functions in $C^\infty_{\mathfrak{B}}$, then so is their product

$f(z,\bar{z})g(z,\bar{z})$ and its fractional complex partial derivatives are given by:

$$
\begin{aligned}
{}^{1}_{c}\partial^{\alpha}_{z}\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{\gamma+n}\,{}^{1}_{c}\partial^{\alpha-\gamma-n}_{z}f(z,\bar{z})\,{}^{1}_{c}\partial^{\gamma+n}_{z}g(z,\bar{z}),\\
{}^{2}_{c}\partial^{\alpha}_{z}\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{\gamma+n}\,{}^{2}_{c}\partial^{\alpha-\gamma-n}_{z}f(z,\bar{z})\,{}^{2}_{c}\partial^{\gamma+n}_{z}g(z,\bar{z}),\\
{}^{1}_{c}\partial^{\alpha}_{\bar{z}}\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{\gamma+n}\,{}^{1}_{c}\partial^{\alpha-\gamma-n}_{\bar{z}}f(z,\bar{z})\,{}^{1}_{c}\partial^{\gamma+n}_{\bar{z}}g(z,\bar{z}),\\
{}^{2}_{c}\partial^{\alpha}_{\bar{z}}\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{\gamma+n}\,{}^{2}_{c}\partial^{\alpha-\gamma-n}_{\bar{z}}f(z,\bar{z})\,{}^{2}_{c}\partial^{\gamma+n}_{\bar{z}}g(z,\bar{z}),
\end{aligned}
$$

for any $\alpha,\gamma\in\mathbb{C}$.

Proof. Starting from the definitions (3.1)–(3.2), we apply first the standard Leibniz rule for nth-order derivatives and then the fractional Leibniz rule (2.7) with the appropriate choice of the arbitrary constant γ:

$$
\begin{aligned}
{}^{1}_{c}\partial^{\alpha}_{z}[f(z,\bar{z})g(z,\bar{z})] &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}{}_{c}D^{\alpha-n}_{x}D^{n}_{y}\big(f(z,\bar{z})g(z,\bar{z})\big) \\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}{}_{c}D^{\alpha-n}_{x}\left[\sum_{k=0}^{n}\binom{n}{k}D^{k}_{y}f(z,\bar{z})\cdot D^{n-k}_{y}g(z,\bar{z})\right] \\
&= 2^{-\alpha}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha}{k+l}e^{-i\pi(k+l)/2}\binom{k+l}{k}{}_{c}D^{\alpha-k-l}_{x}\left(D^{k}_{y}f(z,\bar{z})\cdot D^{n-k}_{y}g(z,\bar{z})\right) \\
&= 2^{-\alpha}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha}{k+l}e^{-i\pi(k+l)/2}\binom{k+l}{k}\sum_{m=-\infty}^{\infty}\binom{\alpha-k-l}{\gamma-l+m} \\
&\qquad\qquad \times_{c}D^{\alpha-\gamma-k-m}_{x}D^{k}_{y}f(z,\bar{z})\cdot_{c}D^{\gamma-l+m}_{x}D^{n-k}_{y}g(z,\bar{z}) \\
&= 2^{-\alpha}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\Gamma(\alpha+1)e^{-i\pi(k+l)/2}}{k!l!\Gamma(\alpha-\gamma-k-m+1)\Gamma(\gamma-l+m+1)} \\
&\qquad\qquad\qquad \times_{c}D^{\alpha-\gamma-k-m}_{x}D^{k}_{y}f(z,\bar{z})\cdot_{c}D^{\gamma-l+m}_{x}D^{n-k}_{y}g(z,\bar{z}) \\
&= 2^{-\alpha}\sum_{m=-\infty}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma-m+1)\Gamma(\gamma+m+1)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha-\gamma-m}{k} \\
&\qquad \times\binom{\gamma+m}{l}e^{-i\pi(k+l)/2}{}_{c}D^{\alpha-\gamma-k-m}_{x}D^{k}_{y}f(z,\bar{z})\cdot_{c}D^{\gamma-l+m}_{x}D^{n-k}_{y}g(z,\bar{z}) \\
&= \sum_{m=-\infty}^{\infty}\left[2^{-\alpha+\gamma+m}\sum_{k=0}^{\infty}\binom{\alpha-\gamma-m}{k}e^{-i\pi k/2}{}_{c}D^{\alpha-\gamma-k-m}_{x}D^{k}_{y}f(z,\bar{z})\right] \\
&\qquad \times\binom{\alpha}{\gamma+m}\left[2^{-\gamma-m}\sum_{l=0}^{\infty}\binom{\gamma+m}{l}e^{-i\pi l/2}{}_{c}D^{\gamma-l+m}_{x}D^{n-k}_{y}g(z,\bar{z})\right] \\
&= \sum_{m=-\infty}^{\infty}\binom{\alpha}{\gamma+m}{}^{1}_{c}\partial^{\alpha-\gamma-m}_{z}f(z,\bar{z}){}^{1}_{c}\partial^{\gamma+m}_{z}g(z,\bar{z});
\end{aligned}
$$

$$
\begin{aligned}
{}^{2}_{c}\partial_z^{\alpha}[f(z,\bar{z})g(z,\bar{z})] &= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi(\alpha-n)/2}D^{n}_{x\,c}D_y^{\alpha-n}f(z,\bar{z})g(z,\bar{z})\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi(\alpha-n)/2}{}_{c}D_y^{\alpha-n}\left[\sum_{k=0}^{n}\binom{n}{k}D_x^k f(z,\bar{z})\cdot D_x^{n-k}g(z,\bar{z})\right]\\
&= 2^{-\alpha}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha}{k+l}e^{-i\pi(\alpha-k-l)/2}\binom{k+l}{k}{}_{c}D_y^{\alpha-k-l}\left(D_x^k f(z,\bar{z})\cdot D_x^{n-k}g(z,\bar{z})\right)\\
&= 2^{-\alpha}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha}{k+l}e^{-i\pi(\alpha-k-l)/2}\binom{k+l}{k}\sum_{m=-\infty}^{\infty}\binom{\alpha-k-l}{\gamma-l+m}\\
&\qquad\qquad \times D^k_{x\,c}D_y^{\alpha-\gamma-k-m}f(z,\bar{z})\cdot D_x^{n-k}{}_{c}D_y^{\gamma-l+m}g(z,\bar{z})\\
&= 2^{-\alpha}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\Gamma(\alpha+1)e^{-i\pi(\alpha-k-l)/2}}{k!l!\Gamma(\alpha-\gamma-k-m+1)\Gamma(\gamma-l+m+1)}\\
&\qquad\qquad \times D^k_{x\,c}D_y^{\alpha-\gamma-k-m}f(z,\bar{z})\cdot D_x^{n-k}{}_{c}D_y^{\gamma-l+m}g(z,\bar{z})\\
&= 2^{-\alpha}\sum_{m=-\infty}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma-m+1)\Gamma(\gamma+m+1)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha-\gamma-m}{k}\binom{\gamma+m}{l}\\
&\qquad \times e^{-i\pi(\alpha-k-l)/2}D^k_{x\,c}D_y^{\alpha-\gamma-k-m}f(z,\bar{z})\cdot D_x^{n-k}{}_{c}D_y^{\gamma-l+m}g(z,\bar{z})\\
&= \sum_{m=-\infty}^{\infty}\left[2^{-\alpha+\gamma+m}\sum_{k=0}^{\infty}\binom{\alpha-\gamma-m}{k}e^{-i\pi(\alpha-\gamma-m-k)/2}D^k_{x\,c}D_y^{\alpha-\gamma-k-m}f(z,\bar{z})\right]\\
&\quad \times\binom{\alpha}{\gamma+m}\left[2^{-\gamma-m}\sum_{l=0}^{\infty}\binom{\gamma+m}{l}e^{-i\pi(\gamma+m-l)/2}D_x^{n-k}{}_{c}D_y^{\gamma-l+m}g(z,\bar{z})\right]\\
&= \sum_{m=-\infty}^{\infty}\binom{\alpha}{\gamma+m}{}^{2}_{c}\partial_z^{\alpha-\gamma-m}f(z,\bar{z}){}^{2}_{c}\partial_z^{\gamma+m}g(z,\bar{z}).
\end{aligned}
$$

Similarly, we can prove the results for the fractional complex derivatives with respect to $\bar{z}$. □

Corollary 3.1: Putting $\gamma = 0$ in the above Theorem, we obtain the simplest and most natural form of the fractional Leibniz rule. For $f, g \in C^{\infty}_{\mathfrak{B}}$, and for any $\alpha \in \mathbb{C}$:

$$
\begin{aligned}
{}_c^1\partial_z^\alpha\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{n}{}_c^1\partial_z^{\alpha-n}f(z,\bar{z})\partial_z^n g(z,\bar{z}),\\
{}_c^2\partial_z^\alpha\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{n}{}_c^2\partial_z^{\alpha-n}f(z,\bar{z})\partial_z^n g(z,\bar{z}),\\
{}_c^1\partial_{\bar{z}}^\alpha\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{n}{}_c^1\partial_{\bar{z}}^{\alpha-n}f(z,\bar{z})\partial_{\bar{z}}^n g(z,\bar{z}),\\
{}_c^2\partial_{\bar{z}}^\alpha\left(f(z,\bar{z})g(z,\bar{z})\right) &= \sum_{n=-\infty}^{\infty}\binom{\alpha}{n}{}_c^2\partial_{\bar{z}}^{\alpha-n}f(z,\bar{z})\partial_{\bar{z}}^n g(z,\bar{z}).
\end{aligned}
$$

Example 3.2: Let g be an analytic function. Then we know $\partial_{\bar{z}}^n g(z,\bar{z}) = 0$ for all $n \geq 1$, and using the Leibniz rule we can obtain simple expressions for the fractional d-bar derivatives of an analytic function. We have

$$
\begin{aligned}
{}_c^j\partial_{\bar{z}}^\alpha(f(z,\bar{z})g(z,\bar{z})) &= \sum_{n=0}^{\infty}\binom{\alpha}{n}{}_c^j\partial_{\bar{z}}^{\alpha-n}f(\bar{z}){}_c^j\partial_{\bar{z}}^n g(z)\\
&= \binom{\alpha}{0}{}_c^j\partial_{\bar{z}}^{\alpha-0}f(z,\bar{z}){}_c^j\partial_{\bar{z}}^0 g(z,\bar{z})\\
&={}_c^j\,\partial_{\bar{z}}^\alpha(f(z,\bar{z}))g(z,\bar{z}),
\end{aligned}
$$

for $j \in \{1,2\}$, and by taking $f(z,\bar{z}) = 1$ we obtain that, for any analytic function g:

$$
\begin{aligned}
{}_c^1\partial_{\bar{z}}^\alpha g(z,\bar{z}) &={}_c^1\,\partial_{\bar{z}}^\alpha(1)g(z,\bar{z}) = \frac{(2(x-c))^{-\alpha}}{\Gamma(1-\alpha)}g(z,\bar{z}),\\
{}_c^2\partial_{\bar{z}}^\alpha g(z,\bar{z}) &={}_c^2\,\partial_{\bar{z}}^\alpha(1)g(z,\bar{z}) = \frac{(2(ci-yi))^{-\alpha}}{\Gamma(1-\alpha)}g(z,\bar{z}),
\end{aligned}
$$

where we have used the results of Example 3.1 for the fractional d-bar derivatives of a constant function.

This provides a simpler expression for the fractional d-bar derivative of an analytic function than the one obtained in Proposition 3.1. However, both forms of the result are equivalent, as we can see from the following simplification of (3.17) and (3.19):

$$
\begin{aligned}
{}^1_c\partial^\alpha_{\bar z} f(z,\bar z) &= \sum_{k=0}^{\infty} \frac{-(x-c)^{k-\alpha}}{2^\alpha (k+1)!\Gamma(-\alpha)}\,{}_2F_1(1,k+1-\alpha;k+2;1) f^{(k)}(c+yi) \\
&= \sum_{k=0}^{\infty} \frac{-(x-c)^{k-\alpha}}{2^\alpha (k+1)!\Gamma(-\alpha)} \frac{k+1}{\alpha} f^{(k)}(c+yi) \\
&= \sum_{k=0}^{\infty} \frac{(x-c)^{k-\alpha}}{2^\alpha k!\Gamma(1-\alpha)} f^{(k)}(c+yi) \\
&= \frac{(2(x-c))^{-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{(x+yi-c-yi)^k}{k!} f^{(k)}(c+yi) \\
&= \frac{(2(x-c))^{-\alpha}}{\Gamma(1-\alpha)} f(z,\bar z); \\
{}^2_c\partial^\alpha_{\bar z} f(z,\bar z) &= \sum_{k=0}^{\infty} \frac{-e^{i\pi\alpha}(yi-ci)^{k-\alpha}}{2^\alpha (k+1)!\Gamma(-\alpha)}\,{}_2F_1(1,k+1-\alpha;k+2;1) f^{(k)}(x+ci) \\
&= \sum_{k=0}^{\infty} \frac{-e^{i\pi\alpha}(yi-ci)^{k-\alpha}}{2^\alpha (k+1)!\Gamma(-\alpha)} \frac{k+1}{\alpha} f^{(k)}(x+ci) \\
&= \sum_{k=0}^{\infty} \frac{e^{i\pi\alpha}(yi-ci)^{k-\alpha}}{2^\alpha k!\Gamma(1-\alpha)} f^{(k)}(x+ci) \\
&= \frac{(2(ci-yi))^{-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{(x+yi-x-ci)^k}{k!} f^{(k)}(x+ci) \\
&= \frac{(2(ci-yi))^{-\alpha}}{\Gamma(1-\alpha)} f(z,\bar z),
\end{aligned}
$$

Remark 3.3: The above work shows that the fractional d-bar derivative of an analytic function is not zero. The question thus arises: for which complex functions $f(z,\bar z)$ do we have ${}^j_c\partial^\alpha_{\bar z} f(z,\bar z) = 0$? This leads to a theory of fractionally polyanalytic functions which is beyond the scope of this thesis.

A related question is: which functions $f(z,\bar z)$ have other special properties with respect to the new operators, such as ${}^j_c\partial^\alpha_z f(z,\bar z) = f(z,\bar z)$ or ${}^j_c\partial^\alpha_{\bar z} f(z,\bar z) = f(z,\bar z)$? In the standard Riemann-Liouville or Caputo Fractional Calculus, answers to these questions would relate to the Mittag-Leffler function $E_\alpha(z^\alpha)$, a special function which serves as an eigenfunction for fractional derivative operators. But the Mittag-Leffler function no longer has such special properties with respect to the new complex operators: the function

$$E_\alpha(z^\alpha) = E_\alpha\big((x+yi)^\alpha\big)$$

does not behave as an eigenfunction under the operators ${}_cD_x^{\alpha-n}$ or ${}_cD_y^{\alpha-n}$. It seems that, in the setting of fractional complex partial derivatives, we may need to find new special functions to serve roles like that of the Mittag-Leffler function in the usual FC. This will form another direction for continuing research.

3.2.3 Composition properties

One of the most fundamental questions about fractional derivatives and integrals is how they work when composed together. Is the αth derivative or the βth derivative always going to be equal to the $(\alpha+\beta)$th derivative?

For the usual Riemann-Liouville FC, these questions were addressed a long time ago, with the results summarised in our identity (2.6) and the surrounding discussion. For the fractional complex partial derivatives, we shall consider this problem in a series of smaller stages as follows: .

- Firstly, in Lemma 3.3 and Lemma 3.4, we consider the composition of *single* complex partial derivatives ($\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$) with our fractional ones.
- Next, in Theorem 3.4 and Theorem 3.5, we consider the composition of *repeated* complex partial derivatives ($\frac{\partial^n}{\partial z^n}$ and $\frac{\partial^n}{\partial \bar{z}^n}$) with our fractional ones.
- Finally, in the full generality of Theorem 3.6 and Theorem 3.7, we consider the composition of our fractional complex partial derivatives with each other, to obtain the analogue of a semigroup property for these operators.

Lemma 3.3: Let f be a function in $C^\infty_{\mathfrak{B}}$; then we have the following:

$$\frac{\partial}{\partial z}\left({}_c^1\partial_z^{\alpha}[f(z,\bar{z})]\right) = {}_c^1\partial_z^{\alpha+1}[f(z,\bar{z})];$$
$$\frac{\partial}{\partial z}\left({}_c^2\partial_z^{\alpha}[f(z,\bar{z})]\right) = {}_c^2\partial_z^{\alpha+1}[f(z,\bar{z})];$$
$$\frac{\partial}{\partial \bar{z}}\left({}_c^1\partial_{\bar{z}}^{\alpha}[f(z,\bar{z})]\right) = {}_c^1\partial_{\bar{z}}^{\alpha+1}[f(z,\bar{z})];$$
$$\frac{\partial}{\partial \bar{z}}\left({}_c^2\partial_{\bar{z}}^{\alpha}[f(z,\bar{z})]\right) = {}_c^2\partial_{\bar{z}}^{\alpha+1}[f(z,\bar{z})].$$

Proof. This is a direct result of some manipulations of binomial-type series. We present the work only for the first identity, since the proofs for the other three are entirely similar.

$$\begin{aligned}
\frac{\partial}{\partial z}\left({}_c^1\partial_z^{\alpha}[f(z,\bar{z})]\right) &= \frac{\partial}{\partial z}\left(2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}{}_cD_x^{\alpha-n}D_y^n f(x+yi)\right)\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)_c D_x^{\alpha-n}D_y^n f(x+yi)\\
&= 2^{-\alpha-1}\sum_{n=0}^{\infty}\binom{\alpha}{n}\left[e^{-i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)+e^{-i\pi(n+1)/2}{}_cD_x^{\alpha-n}D_y^{n+1}f(x+yi)\right]\\
&= 2^{-\alpha-1}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)\\
&\qquad\qquad +2^{-\alpha-1}\sum_{k=1}^{\infty}\binom{\alpha}{k-1}e^{-i\pi k/2}{}_cD_x^{\alpha+1-k}D_y^k f(x+yi)\\
&= 2^{-\alpha-1}\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}+\binom{\alpha}{n-1}\right]e^{-i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)+\binom{\alpha}{0}{}_cD_x^{\alpha+1}f(x+yi)\\
&= 2^{-\alpha-1}\sum_{n=1}^{\infty}\binom{\alpha+1}{n}e^{-i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)+\binom{\alpha+1}{0}{}_cD_x^{\alpha+1}f(x+yi)\\
&= 2^{-(\alpha+1)}\sum_{n=0}^{\infty}\binom{\alpha+1}{n}e^{-i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi) = {}_c^1\partial_z^{\alpha+1}[f(z,\bar{z})].
\end{aligned}$$

□

Lemma 3.4: Let f be a function in $C_{\mathfrak{B}}^{\infty}$; then we have the following:

$$
\begin{aligned}
\frac{\partial}{\partial z}\left({}_c^1\partial_{\bar z}^{\alpha}[f(z,\bar z)]\right) &= {}_c^1\partial_{\bar z}^{\alpha+1}[f(z,\bar z)] + {}_c^1\partial_{\bar z}^{\alpha}[-iD_y f(z,\bar z)];\\
\frac{\partial}{\partial \bar z}\left({}_c^1\partial_{z}^{\alpha}[f(z,\bar z)]\right) &= {}_c^1\partial_{z}^{\alpha+1}[f(z,\bar z)] + {}_c^1\partial_{z}^{\alpha}[iD_y f(z,\bar z)];\\
\frac{\partial}{\partial z}\left({}_c^2\partial_{\bar z}^{\alpha} f(z,\bar z)\right) &= {}_c^2\partial_{\bar z}^{\alpha}[D_x f(z,\bar z)] - {}_c^2\partial_{\bar z}^{\alpha+1}[f(z,\bar z)];\\
\frac{\partial}{\partial \bar z}\left({}_c^2\partial_{z}^{\alpha} f(z,\bar z)\right) &= {}_c^2\partial_{z}^{\alpha}[D_x f(z,\bar z)] - {}_c^2\partial_{z}^{\alpha+1}[f(z,\bar z)].
\end{aligned}
$$

Proof. The proof of this result is similar to that for Lemma 3.3, but with an extra minus sign arising from the interchange of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ which ultimately gives the extra term on the right-hand side.

$$
\begin{aligned}
\frac{\partial}{\partial z}\left({}_c^1\partial_{\bar z}^{\alpha}[f(z,\bar z)]\right) &= \frac{\partial}{\partial z}\left(2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi n/2}{}_cD_x^{\alpha-n}D_y^n f(x+yi)\right)\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi n/2}\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right){}_cD_x^{\alpha-n}D_y^n f(x+yi)\\
&= 2^{-\alpha-1}\sum_{n=0}^{\infty}\binom{\alpha}{n}\left[e^{i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)+e^{i\pi(n-1)/2}{}_cD_x^{\alpha-n}D_y^{n+1}f(x+yi)\right]\\
&= 2^{-\alpha-1}\left[\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)\right.\\
&\qquad\left. -\sum_{k=1}^{\infty}\binom{\alpha}{k-1}e^{i\pi k/2}{}_cD_x^{\alpha+1-k}D_y^k f(x+yi)\right]\\
&= 2^{-\alpha-1}\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}-\binom{\alpha}{n-1}\right]e^{i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)+\binom{\alpha}{0}{}_cD_x^{\alpha+1}f(x+yi)\\
&= 2^{-\alpha-1}\sum_{n=1}^{\infty}\left[\binom{\alpha+1}{n}-2\binom{\alpha}{n-1}\right]e^{i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)\\
&\qquad+\binom{\alpha+1}{0}{}_cD_x^{\alpha+1}f(x+yi)\\
&= 2^{-(\alpha+1)}\sum_{n=0}^{\infty}\binom{\alpha+1}{n}e^{i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)\\
&\qquad-2^{-\alpha}\sum_{n=1}^{\infty}\binom{\alpha}{n-1}e^{i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)\\
&= {}_c^1\partial_{\bar z}^{\alpha+1}[f(z,\bar z)]-2^{-\alpha}\sum_{n=1}^{\infty}\binom{\alpha}{n-1}e^{i\pi n/2}{}_cD_x^{\alpha+1-n}D_y^n f(x+yi)\\
&= {}_c^1\partial_{\bar z}^{\alpha+1}[f(z,\bar z)] + {}_c^1\partial_{\bar z}^{\alpha}[-iD_y f(z,\bar z)].
\end{aligned}
$$

Similarly we obtain

$$\frac{\partial}{\partial \bar{z}}\left({}_c^1\partial_z^{\alpha}[f(z,\bar{z})]\right) = \frac{\partial}{\partial \bar{z}}\left(2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{-i\pi n/2}{}_cD_x^{\alpha-n}D_y^n f(x+yi)\right)$$

$$= {}_c^1\,\partial_z^{\alpha+1}[f(z,\bar{z})] + {}_c^1\,\partial_z^{\alpha}[iD_y f(z,\bar{z})],$$

where the change from $-i$ to i is a natural result of the switch between z and $\bar{z}$.

Next,

$$\begin{aligned}
\frac{\partial}{\partial z}\left({}_c^2\partial_{\bar{z}}^{\alpha}f(z,\bar{z})\right) &= \frac{\partial}{\partial z}\left(2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi(\alpha-n)/2}D_{x\,c}^{n}D_y^{\alpha-n}f(x+yi)\right)\\
&= 2^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi(\alpha-n)/2}\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)D_{x\,c}^{n}D_y^{\alpha-n}f(x+yi)\\
&= 2^{-\alpha-1}\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi(\alpha-n)/2}D_x^{n+1}{}_cD_y^{\alpha-n}f(x+yi)\\
&\qquad + 2^{-\alpha-1}e^{i\pi(\alpha-n-1)/2}D_{x\,c}^{n}D_y^{\alpha+1-n}f(x+yi)\\
&= 2^{-\alpha-1}\left[\sum_{k=1}^{\infty}\binom{\alpha}{k-1}e^{i\pi(\alpha+1-k)/2}D_{x\,c}^{k}D_y^{\alpha+1-k}f(x+yi)\right.\\
&\qquad \left.+\sum_{n=0}^{\infty}\binom{\alpha}{n}e^{i\pi(\alpha-n-1)/2}D_{x\,c}^{n}D_y^{\alpha+1-n}f(x+yi)\right]\\
&= 2^{-\alpha-1}\sum_{n=1}^{\infty}\left[\binom{\alpha}{n-1}-\binom{\alpha}{n}\right]e^{i\pi(\alpha+1-n)/2}D_{x\,c}^{n}D_y^{\alpha+1-n}f(x+yi)\\
&\qquad + \binom{\alpha}{0}e^{i\pi(\alpha-1)/2}{}_cD_y^{\alpha+1}f(x+yi)\\
&= 2^{-\alpha-1}\sum_{n=1}^{\infty}\left[2\binom{\alpha}{n-1}-\binom{\alpha+1}{n}\right]e^{i\pi(\alpha+1-n)/2}D_{x\,c}^{n}D_y^{\alpha+1-n}f(x+yi)\\
&\qquad - \binom{\alpha+1}{0}e^{i\pi(\alpha+1)/2}{}_cD_x^{\alpha+1}f(x+yi)\\
&= 2^{-\alpha}\sum_{n=1}^{\infty}\binom{\alpha}{n-1}e^{i\pi(\alpha+1-n)/2}D_{x\,c}^{n}D_y^{\alpha+1-n}f(x+yi)\\
&\qquad - 2^{-(\alpha+1)}\sum_{n=0}^{\infty}\binom{\alpha+1}{n}e^{i\pi(\alpha+1-n)/2}D_{x\,c}^{n}D_y^{\alpha+1-n}f(x+yi)\\
&= 2^{-\alpha}\sum_{n=1}^{\infty}\binom{\alpha}{n-1}e^{i\pi(\alpha+1-n)/2}D_{x\,c}^{n}D_y^{\alpha+1-n}f(x+yi) - {}_c^2\,\partial_{\bar{z}}^{\alpha+1}[f(z,\bar{z})]\\
&= {}_c^2\,\partial_{\bar{z}}^{\alpha}[D_x f(z,\bar{z})] - {}_c^2\,\partial_{\bar{z}}^{\alpha+1}[f(z,\bar{z})].
\end{aligned}$$

Similarly we obtain

$$\frac{\partial}{\partial \bar{z}}\left({}_c^2\partial_z^\alpha f(z,\bar{z})\right) = {}_c^2\partial_z^\alpha[D_x f(z,\bar{z})] - {}_c^2\partial_z^{\alpha+1}[f(z,\bar{z})].$$

□

We can generalise the results of Lemmas 3.3 and 3.4 to the kth order of ordinary differentiation with respect to z and $\bar{z}$, as stated in the following theorem.

Theorem 3.4: Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ be arbitrary, and f be a function in $C^\infty_{\mathfrak{B}}$. We then have:

$$\frac{\partial^n}{\partial z^n}\left({}_c^1\partial_z^\alpha[f(z,\bar{z})]\right) = {}_c^1\partial_z^{\alpha+n} f(z,\bar{z});$$
$$\frac{\partial^n}{\partial z^n}\left({}_c^2\partial_z^\alpha[f(z,\bar{z})]\right) = {}_c^2\partial_z^{\alpha+n} f(z,\bar{z});$$
$$\frac{\partial^n}{\partial \bar{z}^n}\left({}_c^1\partial_{\bar{z}}^\alpha[f(z,\bar{z})]\right) = {}_c^1\partial_{\bar{z}}^{\alpha+n} f(z,\bar{z});$$
$$\frac{\partial^n}{\partial \bar{z}^n}\left({}_c^2\partial_{\bar{z}}^\alpha[f(z,\bar{z})]\right) = {}_c^2\partial_{\bar{z}}^{\alpha+n} f(z,\bar{z}).$$

Proof. This is a direct consequence of Lemma 3.3 by induction on n. □

Theorem 3.5: Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ be arbitrary, and f be a function in $C^\infty_{\mathfrak{B}}$. We then have:

$$\frac{\partial^n}{\partial z^n}\left({}_c^1\partial_{\bar{z}}^\alpha[f(z,\bar{z})]\right) = \sum_{k=0}^{n}\binom{n}{k}{}_c^1\partial_{\bar{z}}^{\alpha+k}\left[(-i)^{n-k}D_y^{n-k}f(z,\bar{z})\right]$$
$$\frac{\partial^n}{\partial \bar{z}^n}\left({}_c^1\partial_z^\alpha[f(z,\bar{z})]\right) = \sum_{k=0}^{n}\binom{n}{k}{}_c^1\partial_z^{\alpha+k}\left[i^{n-k}D_y^{n-k}f(z,\bar{z})\right]$$
$$\frac{\partial^n}{\partial z^n}\left({}_c^2\partial_{\bar{z}}^\alpha[f(z,\bar{z})]\right) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k{}_c^2\partial_{\bar{z}}^{\alpha+k}\left[D_x^{n-k}f(z,\bar{z})\right]$$
$$\frac{\partial^n}{\partial \bar{z}^n}\left({}_c^2\partial_z^\alpha[f(z,\bar{z})]\right) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k{}_c^2\partial_z^{\alpha+k}\left[D_x^{n-k}f(z,\bar{z})\right]$$

Proof. We will prove one of them and the others can be proved in an analogous way. We'll prove that

$$\frac{\partial^n}{\partial \bar{z}^n}\left({}_c^2\partial_z^\alpha[f(z,\bar{z})]\right) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\ {}_c^2\partial_z^{\alpha+k}\left[D_x^{n-k}f(z,\bar{z})\right],$$

using induction on n. For $n=1$, the result is that of Lemma 3.4. Suppose it's verified for n and prove it for $n+1$ as follows, using both the induction hypothesis and the result of Lemma 3.4:

$$
\begin{aligned}
\frac{\partial^{n+1}}{\partial\bar{z}^{n+1}}\left({}^2_c\partial^{\alpha}_z[f(z,\bar{z})]\right) &= \frac{\partial}{\partial\bar{z}}\left[\frac{\partial^n}{\partial\bar{z}^n}\left({}^2_c\partial^{\alpha}_z[f(z,\bar{z})]\right)\right] \\
&= \frac{\partial}{\partial\bar{z}}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^k\ {}^2_c\partial^{\alpha+k}_z\left[D_x^{n-k}f(z,\bar{z})\right]\right] \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\partial}{\partial\bar{z}}\left[{}^2_c\partial^{\alpha+k}_z\left[D_x^{n-k}f(z,\bar{z})\right]\right] \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\ \left({}^2_c\partial^{\alpha+k}_z\left[D_xD_x^{n-k}f(z,\bar{z})\right]-{}^2_c\,\partial^{\alpha+k+1}_z\left[D_x^{n-k}f(z,\bar{z})\right]\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\ {}^2_c\partial^{\alpha+k}_z\left[D_x^{n+1-k}f(z,\bar{z})\right] \\
&\qquad\qquad +\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}\ {}^2_c\partial^{\alpha+k+1}_z\left[D_x^{n+1-k-1}f(z,\bar{z})\right] \\
&= {}^2_c\,\partial^{\alpha}_z\left[D_x^{n+1}f(z,\bar{z})\right]+\sum_{k=1}^{n}\binom{n}{k}(-1)^k\ {}^2_c\partial^{\alpha+k}_z\left[D_x^{n+1-k}f(z,\bar{z})\right] \\
&+\sum_{k=1}^{n}\binom{n}{k-1}(-1)^k\ {}^2_c\partial^{\alpha+k}_z\left[D_x^{n+1-k}f(z,\bar{z})\right]+(-1)^{n+1}\ {}^2_c\partial^{\alpha+n+1}_z[f(z,\bar{z})] \\
&= {}^2_c\,\partial^{\alpha}_z\left[D_x^{n+1}f(z,\bar{z})\right]+\sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right](-1)^k\ {}^2_c\partial^{\alpha+k}_z\left[D_x^{n+1-k}f(z,\bar{z})\right] \\
&\qquad\qquad +(-1)^{n+1}\ {}^2_c\partial^{\alpha+n+1}_z[f(z,\bar{z})] \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k\ {}^2_c\partial^{\alpha+k}_z\left[D_x^{n+1-k}f(z,\bar{z})\right],
\end{aligned}
$$

which gives the required result for $n+1$. □

Theorem 3.6: Let α and β be two arbitrary complex numbers, and let f be a function in $C^{\infty}_{\mathfrak{B}}$. We then have:

$$
\begin{aligned}
{}_c^1\partial_z^\alpha({}_c^1\partial_z^\beta[f(z,\bar z)]) ={}& {}_c^1\partial_z^{\alpha+\beta}[f(z,\bar z)] - 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{q=0}^{\lfloor \mathrm{Re}(\beta)\rfloor} e^{-i\pi p/2}A_{p;q}(\alpha,\beta)\\
&\times \frac{(x-c)^{-\alpha+p-q-1}}{\Gamma(-\alpha+p-q)}\,{}_cD_x^{\beta-q-1}D_y^p f(c+yi);\\
{}_c^2\partial_z^\alpha({}_c^2\partial_z^\beta[f(z,\bar z)]) ={}& {}_c^2\partial_z^{\alpha+\beta}[f(z,\bar z)] - 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{q=0}^{\lfloor \mathrm{Re}(\beta)\rfloor} e^{-i\pi(\alpha+\beta-p)/2}A_{p;q}(\alpha,\beta)\\
&\times \frac{(y-c)^{-\alpha+p-q-1}}{\Gamma(-\alpha+p-q)}\,{}_cD_y^{\beta-q-1}D_x^p f(x+ci);\\
{}_c^1\partial_{\bar z}^\alpha({}_c^1\partial_{\bar z}^\beta[f(z,\bar z)]) ={}& {}_c^1\partial_{\bar z}^{\alpha+\beta}[f(z,\bar z)] - 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{q=0}^{\lfloor \mathrm{Re}(\beta)\rfloor} e^{i\pi p/2}A_{p;q}(\alpha,\beta)\\
&\times \frac{(x-c)^{-\alpha+p-q-1}}{\Gamma(-\alpha+p-q)}\,{}_cD_x^{\beta-q-1}D_y^p f(c+yi);\\
{}_c^2\partial_{\bar z}^\alpha({}_c^2\partial_{\bar z}^\beta[f(z,\bar z)]) ={}& {}_c^2\partial_{\bar z}^{\alpha+\beta}[f(z,\bar z)] - 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{q=0}^{\lfloor \mathrm{Re}(\beta)\rfloor} e^{i\pi(\alpha+\beta-p)/2}A_{p;q}(\alpha,\beta)\\
&\times \frac{(y-c)^{-\alpha+p-q-1}}{\Gamma(-\alpha+p-q)}\,{}_cD_y^{\beta-q-1}D_x^p f(x+ci),
\end{aligned}
$$

where the multiplier $A_{p;q}(\alpha,\beta)$ is defined for $\alpha,\beta\in\mathbb{C}$ and $p,q\in\mathbb{Z}_0^+$ by

$$
A_{p;q}(\alpha,\beta)=\sum_{n=0}^{\min(p,q)}\binom{\beta}{n}\binom{\alpha}{p-n}.
$$

Proof. We consider first the composition of fractional type-1 z-derivatives. Using the definition (3.1):

$$
\begin{aligned}
{}_c^1\partial_z^\alpha({}_c^1\partial_z^\beta[f(z,\bar z)]) &={}_c^1\partial_z^\alpha\Big(2^{-\beta}\sum_{n=0}^{\infty}\binom{\beta}{n}e^{-i\pi n/2}\,{}_cD_x^{\beta-n}D_y^n f(x+yi)\Big)\\
&=2^{-\beta}\sum_{n=0}^{\infty}\binom{\beta}{n}e^{-i\pi n/2}2^{-\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}e^{-i\pi k/2}\,{}_cD_x^{\alpha-k}D_y^k[{}_cD_x^{\beta-n}D_y^n f(x+yi)].
\end{aligned}
\tag{3.21}
$$

In the summand of (3.21), the (non-fractional) derivatives with respect to y can be composed directly with a semigroup property: $D_y^kD_y^n=D_y^{k+n}$. For the fractional differintegrals with respect to x, we recall the discuss around (2.6) of composition of Riemann-Liouville fractional differintegrals.

If $n \geq \lfloor \mathrm{Re}(\beta) \rfloor + 1$, then the inner differintegral ${}_cD_x^{\beta-n}$ is a fractional integral and we have a semigroup property: ${}_cD_x^{\alpha-k}{}_cD_x^{\beta-n} = {}_cD_x^{\alpha+\beta-k-n}$.

If $n \leq \lfloor \mathrm{Re}(\beta) \rfloor + 1$, then the inner differintegral ${}_cD_x^{\beta-n}$ is a fractional derivative, equal to

$$D_x^{\lfloor \mathrm{Re}(\beta-n) \rfloor+1}{}_cD_x^{\beta-\lfloor \mathrm{Re}(\beta) \rfloor-1}$$

by the definition (2.4) of the Riemann-Liouville derivative. So, in this case, the identity (2.6) gives us

$$\begin{aligned} {}_cD_x^{\alpha-k}{}_cD_x^{\beta-n}f(x+yi) &= {}_cD_x^{\alpha+\beta-k-n}f(x+yi) \\ &\quad - \sum_{m=0}^{\lfloor \mathrm{Re}(\beta-n) \rfloor} \frac{(x-c)^{m-\alpha+k-\lfloor \mathrm{Re}(\beta-n) \rfloor-1}}{\Gamma(m-\alpha+k-\lfloor \mathrm{Re}(\beta-n) \rfloor)}{}_cD_x^{\beta-\lfloor \mathrm{Re}(\beta) \rfloor+m-1}f(c+yi). \end{aligned}$$

From (3.21), we now have

$$\begin{aligned} {}_c^1\partial_z^{\alpha}({}_c^1\partial_z^{\beta}[f(z,\bar{z})]) &= 2^{-\alpha-\beta}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{\beta}{n}\binom{\alpha}{k}e^{-i\pi(n+k)/2}D_y^{k+n}{}_cD_x^{\alpha+\beta-(k+n)}f(x+yi) \\ &\quad - 2^{-\alpha-\beta}\sum_{n=0}^{\lfloor \mathrm{Re}(\beta) \rfloor}\sum_{k=0}^{\infty}\binom{\beta}{n}\binom{\alpha}{k}e^{-i\pi(n+k)/2}\sum_{m=0}^{\lfloor \mathrm{Re}(\beta-n) \rfloor} \\ &\quad \frac{(x-c)^{m-\alpha+k-\lfloor \mathrm{Re}(\beta-n) \rfloor-1}}{\Gamma(m-\alpha+k-\lfloor \mathrm{Re}(\beta-n) \rfloor)}{}_cD_x^{\beta-\lfloor \mathrm{Re}(\beta) \rfloor+m-1}D_y^{n+k}f(c+yi). \end{aligned}$$

In the first (double) sum, we write $p = k+n$ and use Vandermonde's identity

$$\sum_{k=0}^{p}\binom{\alpha}{k}\binom{\beta}{p-k} = \binom{\alpha+\beta}{p},$$

while in the second (triple) sum we replace m by $\lfloor \mathrm{Re}(\beta-n) \rfloor - m$, to get:

$$\begin{aligned} {}_c^1\partial_z^{\alpha}({}_c^1\partial_z^{\beta}[f(z,\bar{z})]) &= 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\binom{\alpha+\beta}{p}e^{-i\pi p/2}D_y^{p}{}_cD_x^{\alpha+\beta-p}f(x+yi) - 2^{-\alpha-\beta} \\ &\times \sum_{n=0}^{\lfloor \mathrm{Re}(\beta) \rfloor}\sum_{k=0}^{\infty}\binom{\beta}{n}\binom{\alpha}{k}e^{-i\pi(n+k)/2}\sum_{m=0}^{\lfloor \mathrm{Re}(\beta-n) \rfloor}\frac{(x-c)^{-\alpha+k-m-1}}{\Gamma(-\alpha+k-m)}{}_cD_x^{\beta-n-m-1}D_y^{n+k}f(c+yi). \end{aligned}$$

Now the first (single) sum is simply the series for ${}_c^1\partial_z^{\alpha+\beta}[f(z,\bar{z})]$, while the second (triple) sum can be rewritten by setting $p = k+n$ and $q = n+m$ so that

$$\sum_{n=0}^{\lfloor \mathrm{Re}(\beta) \rfloor}\sum_{k=0}^{\infty}\sum_{m=0}^{\lfloor \mathrm{Re}(\beta-n) \rfloor} = \sum_{p=0}^{\infty}\sum_{q=0}^{\lfloor \mathrm{Re}(\beta) \rfloor}\sum_{n=0}^{\min(p,q)}.$$

So we have

$$
\begin{aligned}
&{}^1_c\partial_z^{\alpha}({}^1_c\partial_z^{\beta}[f(z,\bar{z})]) = {}^1_c\,\partial_z^{\alpha+\beta}[f(z,\bar{z})] \\
&-2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{q=0}^{\lfloor\mathrm{Re}(\beta)\rfloor}\sum_{n=0}^{\min(p,q)}\binom{\beta}{n}\binom{\alpha}{p-n}e^{-i\pi p/2}\frac{(x-c)^{-\alpha+p-q-1}}{\Gamma(-\alpha+p-q)}\,{}_cD_x^{\beta-q-1}D_y^{p}f(c+yi),
\end{aligned}
$$

which is the required result for fractional type-1 z-derivatives. The results in the other three cases can be proved in an analogous way. □

Corollary 3.2: The fractional complex partial differintegrals have a semigroup property for any fractional derivative or integral of a fractional integral. In other words, for any $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}(\beta)<0$, and for any $f\in C^{\infty}_{\mathfrak{B}}$, we have:

$$
\begin{aligned}
{}^1_c\partial_z^{\alpha}({}^1_c\partial_z^{\beta}[f(z,\bar{z})]) &= {}^1_c\,\partial_z^{\alpha+\beta}[f(z,\bar{z})];\\
{}^2_c\partial_z^{\alpha}({}^2_c\partial_z^{\beta}[f(z,\bar{z})]) &= {}^2_c\,\partial_z^{\alpha+\beta}[f(z,\bar{z})];\\
{}^1_c\partial_{\bar{z}}^{\alpha}({}^1_c\partial_{\bar{z}}^{\beta}[f(z,\bar{z})]) &= {}^1_c\,\partial_{\bar{z}}^{\alpha+\beta}[f(z,\bar{z})];\\
{}^2_c\partial_{\bar{z}}^{\alpha}({}^2_c\partial_{\bar{z}}^{\beta}[f(z,\bar{z})]) &= {}^2_c\,\partial_{\bar{z}}^{\alpha+\beta}[f(z,\bar{z})].
\end{aligned}
$$

Proof. This is a special case of the previous theorem, where the sum from 0 to $\lfloor\mathrm{Re}(\beta)\rfloor$ in every case is vacuous. □

Theorem 3.7: Let α and β be two arbitrary complex numbers, and let f be a function in $C^{\infty}_{\mathfrak{B}}$. We then have:

$$
\begin{aligned}
{}^1_c\partial_z^{\beta}\left[{}^1_c\partial_{\bar{z}}^{\alpha}f(z,\bar{z})\right] &= \sum_{n=0}^{\infty}\binom{\beta}{n}{}^1_c\partial_{\bar{z}}^{\alpha+\beta-n}\left[e^{-i\pi n/2}D_y^nf(x+iy)\right]-S^1_{z,\bar{z}};\\
{}^1_c\partial_{\bar{z}}^{\beta}\left[{}^1_c\partial_z^{\alpha}f(z,\bar{z})\right] &= \sum_{n=0}^{\infty}\binom{\beta}{n}{}^1_c\partial_z^{\alpha+\beta-n}\left[e^{i\pi n/2}D_y^nf(x+iy)\right]-S^1_{\bar{z},z};\\
{}^2_c\partial_z^{\beta}\left[{}^2_c\partial_{\bar{z}}^{\alpha}f(z,\bar{z})\right] &= \sum_{n=0}^{\infty}\binom{\beta}{n}{}^2_c\partial_{\bar{z}}^{\alpha+\beta-n}\left[e^{-i\pi(\beta-n)}D_x^nf(x+iy)\right]-S^2_{z,\bar{z}};\\
{}^2_c\partial_{\bar{z}}^{\beta}\left[{}^2_c\partial_z^{\alpha}f(z,\bar{z})\right] &= \sum_{n=0}^{\infty}\binom{\beta}{n}{}^2_c\partial_z^{\alpha+\beta-n}\left[e^{i\pi(\beta-n)}D_x^nf(x+iy)\right]-S^2_{\bar{z},z},
\end{aligned}
$$

where

$$S^1_{z,\bar{z}} = 2^{-\alpha-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\lfloor \mathrm{Re}(\alpha) \rfloor} \left[\sum_{k=0}^{\min(p,q)} \binom{\beta}{p-k} (-1)^k \binom{\alpha}{k} \right] \times \frac{(x-c)^{-\beta+p-q-1}}{\Gamma(-\beta+p-q)} (-i)^p {}_cD_x^{\alpha-q-1} D_y^p f(c+yi),$$

$$S^1_{\bar{z},z} = 2^{-\alpha-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\lfloor \mathrm{Re}(\alpha) \rfloor} \left[\sum_{k=0}^{\min(p,q)} \binom{\beta}{p-k} (-1)^k \binom{\alpha}{k} \right] \times \frac{(x-c)^{-\beta+p-q-1}}{\Gamma(-\beta+p-q)} i^p {}_cD_x^{\alpha-q-1} D_y^p f(c+yi),$$

$$S^2_{z,\bar{z}} = 2^{-\alpha-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\lfloor \mathrm{Re}(\alpha) \rfloor} \left[\sum_{k=0}^{\min(p,q)} \binom{\beta}{p-k} (-1)^k \binom{\alpha}{k} \right] \times \frac{(y-c)^{-\beta+p-q-1}}{\Gamma(-\beta+p-q)} e^{i\pi(\alpha-\beta+p)/2} D_x^p {}_cD_y^{\alpha-q-1} f(x+ci),$$

$$S^2_{\bar{z},z} = 2^{-\alpha-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\lfloor \mathrm{Re}(\alpha) \rfloor} \left[\sum_{k=0}^{\min(p,q)} \binom{\beta}{p-k} (-1)^k \binom{\alpha}{k} \right] \times \frac{(y-c)^{-\beta+p-q-1}}{\Gamma(-\beta+p-q)} e^{-i\pi(\alpha-\beta+p)/2} D_x^p {}_cD_y^{\alpha-q-1} f(x+ci).$$

Proof. For the first case with the type-1 derivatives, we have

$$\begin{aligned}
{}^1_c\partial_z^\beta \left[{}^1_c\partial_{\bar{z}}^\alpha f(z,\bar{z}) \right] &= 2^{-\beta} \sum_{n=0}^{\infty} \binom{\beta}{n} e^{-i\pi n/2} {}_cD_x^{\beta-n} D_y^n \left[{}^1_c\partial_{\bar{z}}^\alpha f(z,\bar{z}) \right] \\
&= 2^{-(\alpha+\beta)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{\beta}{n} \binom{\alpha}{k} e^{-i\pi(n-k)/2} {}_cD_x^{\beta-n} {}_cD_x^{\alpha-k} D_y^{n+k} f(z,\bar{z}) \\
&= 2^{-(\alpha+\beta)} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{[Re(\alpha)]} + \sum_{k=[\mathrm{Re}(\alpha)]+1}^{\infty} \right] \binom{\beta}{n} \binom{\alpha}{k} e^{-i\pi(n-k)/2} \\
&\qquad \times {}_cD_x^{\beta-n} {}_cD_x^{\alpha-k} D_y^{n+k} f(z,\bar{z}).
\end{aligned}$$

Using Theorem 3.9 and comparing the coefficients of ${}_cD_x^{\alpha+n-p} D_y^p f(z,\bar{z})$, we have

$$\sum_{k+m=p} \binom{n}{k} \binom{\alpha}{m} i^{m-k} = \sum_{k+m=p} 2^k \binom{n}{k} \binom{\alpha+n-k}{m} i^{m-k},$$

or in other words

$$\sum_{k+m=p} \binom{n}{k} \binom{\alpha}{m} (-1)^k = \sum_{k+m=p} \binom{n}{k} \binom{\alpha+n-k}{m} (-2)^k,$$

from which we can deduce the following for general α, β:

$$\sum_{k+m=p} \binom{\beta}{k} \binom{\alpha}{m} (-1)^k = \sum_{k+m=p} \binom{\beta}{k} \binom{\alpha+\beta-k}{m} (-2)^k.$$

Going back to the main argument, we have

$$
\begin{aligned}
{}_c^1\partial_z^{\beta}\left[{}_c^1\partial_{\bar{z}}^{\alpha} f(z,\bar{z})\right] &= 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{n+k=p}\left[\binom{\beta}{n}\binom{\alpha}{k}(-1)^n\right]e^{i\pi(n+k)/2} \\
&\qquad\qquad \times {}_c D_x^{\alpha+\beta-(n+k)} D_y^{n+k} f(x+iy) - S^1_{z,\bar{z}} \\
&= 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{n+k=p}\left[\binom{\beta}{n}\binom{\alpha+\beta-n}{k}(-2)^n\right]e^{i\pi(n+k)/2} \\
&\qquad\qquad \times {}_c D_x^{\alpha+\beta-(n+k)} D_y^{n+k} f(x+iy) - S^1_{z,\bar{z}} \\
&= \sum_{p=0}^{\infty}\sum_{n+k=p} 2^{-\alpha-\beta+n}\binom{\beta}{n}\binom{\alpha+\beta-n}{k}(-1)^n e^{i\pi(n+k)/2} \\
&\qquad\qquad \times {}_c D_x^{\alpha+\beta-(n+k)} D_y^{n+k} f(x+iy) - S^1_{z,\bar{z}} \\
&= \sum_{n=0}^{\infty}\binom{\beta}{n} 2^{-\alpha-\beta+n}\sum_{k=0}^{\infty}\binom{\alpha+\beta-n}{k} e^{i\pi k/2} \\
&\qquad\qquad \times {}_c D_x^{(\alpha+\beta-n)-k} D_y^{k}\left[e^{-i\pi n/2} D_y^n[f(x+iy)]\right] - S^1_{z,\bar{z}} \\
&= \sum_{n=0}^{\infty}\binom{\beta}{n}\, {}_c^1\partial_{\bar{z}}^{\alpha+\beta-n}\left[e^{-i\pi n/2} D_y^n f(x+iy)\right] - S^1_{z,\bar{z}},
\end{aligned}
$$

where

$$
\begin{aligned}
S^1_{z,\bar{z}} &= 2^{-\alpha-\beta}\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor \mathrm{Re}(\alpha)\rfloor}\binom{\beta}{n}\binom{\alpha}{k} e^{-i\pi(n-k)/2}\sum_{m=0}^{\lfloor \mathrm{Re}(\alpha-k)\rfloor}\frac{(x-c)^{-\beta+n-m-1}}{\Gamma(-\beta+n-m)} \\
&\qquad \times {}_c D_x^{\alpha-k-m-1} D_y^{n+k} f(c+yi) \\
&= 2^{-\alpha-\beta}\sum_{p=0}^{\infty}\sum_{q=0}^{\lfloor \mathrm{Re}(\alpha)\rfloor}\sum_{k=0}^{\min(p,q)}\binom{\beta}{p-k}\binom{\alpha}{k} e^{-i\pi(p-2k)/2}\frac{(x-c)^{-\beta+p-q-1}}{\Gamma(-\beta+p-q)} \\
&\qquad \times {}_c D_x^{\alpha-q-1} D_y^{p} f(c+yi),
\end{aligned}
$$

by a similar argument as in the proof of Theorem 3.6, this time setting $p = k+n$ and $q = k+m$. This completes the proof of the first of our four results.

Exactly the same approach with i and $-i$ swapped yields the second of the four results. For the type-2 derivatives, we have

$$
\begin{aligned}
{}^2_c\partial^\beta_z\left[{}^2_c\partial^\alpha_{\bar z} f(z,\bar z)\right] &= 2^{-\beta}\sum_{n=0}^\infty \binom{\beta}{n} e^{-i\pi(\beta-n)/2}\,{}_cD_y^{\beta-n} D_x^n \left[{}^1_c\partial^\alpha_{\bar z} f(z,\bar z)\right] \\
&= 2^{-(\alpha+\beta)}\sum_{n=0}^\infty\sum_{k=0}^\infty \binom{\beta}{n}\binom{\alpha}{k} e^{i\pi(\alpha-\beta+n-k)/2}\,{}_cD_y^{\beta-n}\,{}_cD_y^{\alpha-k}D_x^{n+k} f(z,\bar z) \\
&= 2^{-(\alpha+\beta)}\sum_{n=0}^\infty \left[\sum_{k=0}^{[\mathrm{Re}(\alpha)]} + \sum_{k=[\mathrm{Re}(\alpha)]+1}^{\infty}\right] \binom{\beta}{n}\binom{\alpha}{k} e^{i\pi(\alpha-\beta+n-k)/2} \\
&\qquad \times {}_cD_y^{\beta-n}\,{}_cD_y^{\alpha-k}D_x^{n+k} f(z,\bar z)
\end{aligned}
$$

Using the previous result

$$
\sum_{k+m=p}\binom{\beta}{k}\binom{\alpha}{m}(-1)^k = \sum_{k+m=p}\binom{\beta}{k}\binom{\alpha+\beta-k}{m}(-2)^k,
$$

we have

$$
\begin{aligned}
{}^2_c\partial^\beta_z\left[{}^2_c\partial^\alpha_{\bar z} f(z,\bar z)\right] &= 2^{-\alpha-\beta}\sum_{p=0}^\infty\sum_{n+k=p}\left[\binom{\beta}{n}\binom{\alpha}{k} e^{-i\pi(\beta-n)}\right] e^{i\pi(\alpha+\beta-n-k)/2} \\
&\qquad \times {}_cD_y^{\alpha+\beta-(n+k)} D_x^{n+k} f(x+iy) - S^2_{z,\bar z} \\
&= 2^{-\alpha-\beta}\sum_{p=0}^\infty\sum_{n+k=p}\left[\binom{\beta}{n}\binom{\alpha+\beta-n}{k}2^n\right] e^{-i\pi(\beta-n)} e^{i\pi(\alpha+\beta-n-k)/2} \\
&\qquad \times {}_cD_y^{\alpha+\beta-(n+k)} D_x^{n+k} f(x+iy) - S^2_{z,\bar z} \\
&= \sum_{n=0}^\infty \binom{\beta}{n} 2^{-\alpha-\beta+n}\sum_{k=0}^\infty \binom{\alpha+\beta-n}{k} e^{i\pi(\alpha+\beta-n-k)/2} \\
&\qquad \times {}_cD_y^{(\alpha+\beta-n)-k} D_x^k\left[e^{-i\pi(\beta-n)} D_x^n[f(x+yi)]\right] - S^2_{z,\bar z} \\
&= \sum_{n=0}^\infty \binom{\beta}{n}\, {}^2_c\partial^{\alpha+\beta-n}_{\bar z}\left[e^{-i\pi(\beta-n)} D_x^n f(x+iy)\right] - S^2_{z,\bar z},
\end{aligned}
$$

where

$$
\begin{aligned}
S^2_{z,\bar z} &= 2^{-\alpha-\beta}\sum_{n=0}^\infty\sum_{k=0}^{\lfloor\mathrm{Re}(\alpha)\rfloor}\binom{\beta}{n}\binom{\alpha}{k} e^{i\pi(\alpha-\beta+n-k)/2}\sum_{m=0}^{\lfloor\mathrm{Re}(\alpha-k)\rfloor}\frac{(y-c)^{-\beta+n-m-1}}{\Gamma(-\beta+n-m)} \\
&\qquad \times {}_cD_y^{\alpha-k-m-1} D_x^{n+k} f(x+ci) \\
&= 2^{-\alpha-\beta}\sum_{p=0}^\infty\sum_{q=0}^{\lfloor\mathrm{Re}(\alpha)\rfloor}\sum_{k=0}^{\min(p,q)}\binom{\beta}{p-k}\binom{\alpha}{k} e^{i\pi(\alpha-\beta+p-2k)/2}\frac{(y-c)^{-\beta+p-q-1}}{\Gamma(-\beta+p-q)} \\
&\qquad \times {}_cD_x^p D_y^{\alpha-q-1} f(x+ci),
\end{aligned}
$$

Exactly the same approach with i and $-i$ swapped yields the last of the four results. $\square$

Chapter 4

CONCLUSION

In this thesis, we have discovered a new way of uniting the fields of Fractional Calculus (FC) and complex analysis. Usually FC has been done either with real variables, or with complex variables in a way which does not preserve some powerful properties of complex analysis such as the Cauchy–Riemann equations. By starting from the basic operations of complex analysis – partial differentiation with respect to x, y, z, and $\bar{z}$ – we have constructed fractional versions of all these operations: most importantly, a fractional d-bar (Cauchy–Riemann) operator, which will be vital in the future in creating a fractional version of the theory and applications of d-bar analysis. We used the usual Riemann-Liouville FC for real variables, but the complex fractional derivatives we discovered are not equivalent to the Riemann-Liouville ones.

We made our theory solid by finding a function space to act as the domain for the fractional complex partial derivatives, and by studying how these operators apply to some important elementary functions. We also discovered a connection between the fractional complex partial derivatives and the usual fractional derivatives with respect to z, which we believe is analogous to the connection between Riemann-Liouville and Caputo fractional derivatives.

Finally, we proved some results which help to demonstrate the naturality of our definitions. The Leibniz rule for fractional complex partial derivatives of a product function has exactly the same form as the Leibniz rule for standard fractional derivatives of a product function. Furthermore, the results on compositions of our operators are analogous to the composition results for standard fractional operators: we have a semigroup property for integrals, and a quasi-semigroup property with an extra error term for derivatives.

This research opens the door for many new research directions, uniting the power of complex analysis with the scope of FC. We expect that the theory developed here will be useful in the future for solving fractional partial differential equations and for better understanding the dynamics of complex systems. Complex d-bar derivatives and their associated equations have applications in optics and medical imaging, and it will be important to examine fractional versions of those equations. Furthermore, there are many more aspects to be investigated from the pure mathematical angle. We have found a Leibniz rule already; perhaps it will also be possible to find a chain rule or fractional Taylor series for our new operators. It may also be possible to discover fractional analogues for some of the powerful theorems of complex analysis, such as Cauchy's theorem or the residue theorem. When two fields of mathematics are combined in a new way, the possibilities are almost endless.

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