Highly Accurate Implicit Schemes Using Hexagonal Grids for the Approximation of the Derivatives of the Solution of Two Dimensional Heat Equation

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ABSTRACT

In this thesis, the first type (Dirichlet) boundary value problem for the heat equation on a rectangle is considered. The research has two main successes.

Firstly, we give a two-stage implicit method of second order accuracy for the approximation of the first order derivatives of the solution with respect to the spatial variables. To approximate the solution at the first stage, the unconditionally stable two layer implicit method on hexagonal grids given by Buranay and Arshad in 2020 is used which converges with second order in space and time variable on the grids. At the second stage, for the approximation of first derivatives with respect to the spatial variables we propose special difference boundary value problems on hexagonal grids of which the boundary conditions are defined by using the obtained solution from the first stage. Further, uniform convergence of the solution of the constructed special difference boundary value problems to the corresponding exact derivatives on hexagonal grids with second order is shown.

Secondly, we give fourth order accurate implicit methods for the computation of the first order spatial derivatives and second order mixed derivatives involving the time derivative of the solution. These methods are constructed based on two stages: At the first stage of the methods, the solution is approximated by using the implicit scheme given by Buranay and Arshad in 2020 that gives fourth order of convergence in space and first order in time variables to the exact solution on the constructed hexagonal grids. For the approximation of the derivative of the solution to the heat equation with respect to the time variable an analogous scheme is devised. Subsequently, to approximate the first order spatial derivatives and the second order mixed derivatives of the solution difference boundary value problems on hexagonal grids are constructed at the second stages. Further, uniform convergence of these implicit schemes to the corresponding exact derivatives are shown.

Eventually, the developed second order and fourth order accurate two-stage implicit methods are used to solve some test problems and the numerical results illustrating the applicability and the accuracy of the methods are presented through tables and figures.

Keywords: Finite difference method; Hexagonal grid; Stability analysis; Two dimensional heat equation; Approximation of derivatives.

ÖZ

Bu tezde, dikdörtgen üzerindeki ısı denkleminin birinci türden (Dirichlet) sınır değer problemi alınmıştır. Araştırmanın iki ana başarısı vardır.

İlk olarak, ısı denkleminin çözümünün birinci mertebeden uzay değişkenlere göre türevlerinin ikinci dereceden doğruluklu yaklaşık çözümü için iki aşamalı kapalı bir yöntem veriyoruz. İlk aşamada çözümü yaklaşık olarak hesaplamak için Buranay ve Arshad tarafından 2020 de verilen uzay ve zaman değisterlerine göre ikinci mertebeden yakınsak altıgen ızgaralarda koşulsuz kararlı iki katmanlı kapalı metod kullanılmıştır. İkinci aşamada, birinci mertebeden uzay türevlerin yaklaşık çözümü için ilk aşamadan elde edilen çözümleri sınır koşullarınının belirlenmesi için kullanan altıgen ızgaralar üzerinde özel fark sınır değer problemleri önerilmiştir. Üstelik, oluşturulan özel fark sınır değer problemlerinin çözümünün karşılık gelen kesin türevlerine altıgen ızgaralar üzerinde ikinci mertebeden düzgün yakınsadığı gösterilir.

İkinci olarak, ısı denkleminin çözümünün birinci mertebeden uzay değişkenlere göre türevleri ve zaman değişkenini içeren ikinci mertebeden karma türevlerinin yaklaşık çözümü için dördüncü dereceden doğruluklu kapalı metodlar verilir. Bu metodlar iki aşamaya bağlı olarak oluşturulur. Yöntemlerin ilk aşamasında, çözüm, Buranay ve Arshad tarafından 2020'de verilen ve uzay değisterlerine göre dördüncü, zaman değisterlerine göre birinci mertebeden doğruluk ile altıgen ızgaralarda kesin çözüme yakınsama veren şemalar kullanılarak yaklaşık olarak hesaplanır. Isı denkleminin çözümünün zaman değişkenine göre türevinin yakınlaştırılması için benzer bir şema tasarlanmıştır. Daha sonra, çözümün birinci mertebeden uzay türevlerini ve ikinci mertebeden karma türevlerininin yaklaşımı için altıgen ızgaralardaki sınır değer

problemleri ikinci aşamada oluşturulur. Ayrıca, bu kapalı şemaların karşılık gelen kesin türevlerine düzgün yakınsaması gösterilir.

Sonunda, geliştirilen ikinci dereceden ve dördüncü dereceden doğruluklu iki aşamalı kapalı yöntemler bazı test problemlerini çözmek için kullanılır ve yöntemlerin uygulanabilirliğini ve doğruluğunu gösteren sayısal sonuçlar tablo ve şekiller aracılığı ile takdim edilir.

Anahtar Kelimeler: Sonlu fark yöntemi; Altıgen ızgara; Kararlılık analizi; İki boyutsal ısı denklemi; Türevlerin yaklaşımı.

Dedicated

To My Family

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BVP	Boundary Value Problem
CPU	Central Processing Unit
CT	Computational Time
FDM	Finite Difference Method
GH	Giga Hertz
SPD	Symmetric Positive Definite
TCT	Total Computational Time

Chapter 1

INTRODUCTION

1.1 Motivation

Numerical methods have gained considerable attention in many applications, since the exact solution of many problems arising in the models of chemistry, physics, biology, engineering, and many other fields of different sciences is an uphill task. Modeling of these problems leads us to consider a number of physical quantities, representing physical phenomena on a modeling domain. These physical quantities then occur in the model via functions or function derivatives of which for a considerable number of them the Newtonian concept of a derivative satisfies the complexity of the natural occurrences. However, "time's evolution and changes occurring in some systems do not happen in the same manner after a fixed or constant interval of time and do not follow the same routine as one would expect. For instance, a huge variation can occur in a fraction of a second, causing a major change that may affect the whole system's state forever" as stated in [1].

Consequently, the modeling of numerous phenomena in diverse scientific fields leads us to consider conventional or fractional boundary value problems of time dependent differential equations on a modeling domain such as the first and second type boundary value problems to heat equation or diffusion equation. For example, the Brownian motion problem in statistics is modeled by heat equation via the Fokker–Planck equation (Adriaan Fokker [2] and Max Planck [3]). It is also named as the Kolmogorov forward equation, who discovered the concept in 1931, see in [4] independently. The stock market fluctuations represent one of the several important real-world applications of the mathematical model of Brownian motion. It was first given in the PhD thesis titled as “The theory of speculation”, by Louis Bachelier (see Mandelbrot and Hudson [5]) in 1900.

Another representative sample of problems that mathematical modeling brings about the heat equation is the image processing problems appearing through many applied sciences from archaeology to zoology. Examples of archaeological investigations include a camcorder for 3D underwater reconstruction of archeological objects in the study of Meline et al. [6]. Furthermore, a recent investigation by Woźniak and Polap [7] gave soft trees with neural components as image processing technique for archeological excavations. In zoology, a study of image reconstruction problem by the application of magnetic resonance imaging was given by Ziegler et al. [8] and in medical sciences as medical image reconstruction was studied in Zeng [9]. Furthermore, tomography, and medical and industrial applications are archetypal examples where substantial mathematical manipulation is required. In some cases, the aim is humble denoising or de-blurring. Witkin [10] and Koenderink [11] gave the modeling of blurring of an image by the heat equation. Later, a problem of solving the reverse heat equation known as de-blurring is studied in Rudin et al. [12] and Guichard and Morel [13].

Additionally, in mathematical biology, Wolpert [14, 15] gave a phenomenological concept of pattern formation and differentiation known as positional information. The pre-programming of the cells for reacting to a chemical concentration and differentiate accordingly, into different kinds of cells such as cartilage cells was proposed. Afterwards, the animal coat patterns, pattern formation on growing domains as alligators, snakes and bacterial patterns were modeled by reaction diffusion equations in Murray [16]. Furthermore, therein, gliomas or glioblastomas, which are highly diffusive brain tumors, are analyzed and a mathematical model for the spatiotemporal dynamics of tumor growth was developed. Therefore, the basic model in dimensional form was given by the diffusion equation

$$\frac{\partial \overline{c}}{\partial \overline{t}} = \overline{\nabla} \mathbf{J} + \rho \overline{c}, \tag{1.1}$$

where $\overline{c}(\overline{x}, \overline{t})$ is the number of cells at a position $\overline{\mathbf{x}}$ and time $\overline{t}$, ρ represents the net rate of growth of cells including proliferation and death (or loss), and **J** diffusional flux of cells taken $\mathbf{J} = \overline{D}\overline{\nabla}\overline{c}$, where $\overline{D}(x)$ (distance^2/time) is the diffusion coefficient of cells in brain tissue and $\overline{\nabla}$ is the gradient operator.

In general, finding analytical solutions of these modeled problems is a difficult task or even not possible. Approximations are needed when a mathematical model is switched to a numerical model. Finite difference methods (FDM) are a class of numerical techniques for solving differential equations that each derivative appearing in the partial differential equation has to be replaced by a suitable divided difference of function values at the chosen grid points, see Grossman et al. [17]. In the last decade, the use of advanced computers has led to the widespread use of FDM in modern numerical analysis. Some recent studies are: for the solution of problems with both stiff and nonstiff components a second order diagonally-implicit-explicit multi-stage integration method given in Zang and Sendu [18]. An implicit method for numerical solution of singular and stiff initial value problem developed in Hasan et al. [19]. For the epidemic models latest studies include the Crank Nicolson difference scheme and iteration method used for finding the approximate solution of system of nonlinear observing epidemic model in Ashyralyev and Hincal [20]. In addition, the article by Ahmed et al. [21], in which a novel and time efficient positivity preserving numerical scheme was designed to find the solution of epidemic model involving a reaction-diffusion system in three dimension. Furthermore, we specify the fractional diffusion equation-based image denoising model constructed in Abirami et al. [22], by using Crank–Nicholson and Grünwald Letnikov difference schemes (CN–GL).

Apart from rectangular grids, hexagonal grids have been also used to develop finite difference methods for the approximate solution of modeled problems in many applied sciences for more than the half century. These studies include the hexagonal grid methods given in meteorological and oceanographic applications by Sadourney et al. [23]–Ničkovič et al. [33], of which favorable results were obtained compared with rectangular grids. Hexagonal grids were applied in reservoir simulation in Pruess and Bodvarsson [34] and it was shown that for seven-point floods, hexagonal grid method provides good numerical accuracy at substantially less computational work than rectangular grid method (five or nine point methods). Hexagonal grids were also used in the simulation of electrical wave phenomena propagated in two dimensional reserved-C type cardiac tissue in Lee et al. [35]. The exhibited linear and spiral waves were more efficient than similar computation carried out on rectangular finite volume schemes. Furthermore, hexagonal grids were applied to approximate the solution of the first type boundary value problem of the heat equation in Richtmyer and Morton [36], Buranay and Arshad [37], Arshad [38], convection-diffusion equation in Karaa [39], and Dirichlet type boundary value problem of the two dimensional Laplace equation in Dosiyev and Celiker [40]. In the most recent investigation by Buranay and Arshad [37] computation of the solution to the heat equation

$$\frac{\partial u}{\partial t} = \omega \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) + f(x_1, x_2, t), \tag{1.2}$$

on special polygons, where $\omega > 0$ and f is the heat source by using implicit schemes defined on hexagonal grids was given. Therein, under some smoothness assumptions of the solution, two implicit methods were developed both on two layers with 14-point that have convergence orders of $O\left(h^2+\tau^2\right)$ and $O\left(h^4+\tau\right)$ accordingly to the solution on the grids. It was assumed that the heat source and the initial and boundary functions are given such that the exact solution belongs to the Hölder space $C_{x,t}^{6+\alpha,3+\frac{\alpha}{2}}$, $0<\alpha<1$.

On the other hand, besides the solution of a modeled problem, the high accurate computation of the derivatives of the solution are fundamental to determine some important phenomena of the considered model problem. For example in the electrostatics the first derivatives of electrostatic potential function define electric field. As the calculation of ray tracing in electrostatic fields by the interpolation methods require the specification at each mesh point not only the potential function Φ but also the gradients $\left\{\frac{\partial\Phi}{\partial x_1},\frac{\partial\Phi}{\partial x_2}\right\}$ and the mixed derivative $\frac{\partial^2\Phi}{\partial x_1\partial x_2}$. Further, for the diffusion problem (1.1) the functions $\frac{\partial c}{\partial t}$ and $\mathbf{J}$ gives the rate of change of the cells and diffusional flux of cells, respectively.

1.2 Literature Review

In the literature, exhaustive studies exist for the approximation of the derivatives of the solution to Laplace's equation under some smoothness conditions of the boundary functions and compatibility conditions. For the 2D Laplace equation, research was conducted by Volkov [41] and Dosiyev and Sadeghi [42]. For the 3D Laplace equation on a rectangular parallelepiped, studies were given by Volkov [43] and Dosiyev and Sadeghi [44], and recently by Dosiyev and Abdussalam [45], and Dosiyev and Sarikaya [46].

For the heat equation, the derivative of the solution of one-dimensional heat equation with respect to the space variable was given in Buranay and Farinola [47]. Within this paper, two implicit schemes were developed that converge to the corresponding exact spatial derivative with $O\left(h^2+\tau\right)$ and $O\left(h^2+\tau^2\right)$ accordingly.

In regard to the equilateral triangulation with a regular hexagonal support, we remark the research by Barrera et al. [48] where a new class of quasi-interpolant was constructed which has remarkable properties such as high order of regularity and polynomial reproduction. Furthermore, on the Delaunay triangulation, we mention the study by Guessab [49] that approximations of differentiable convex functions on arbitrary convex polytopes were given. Further, optimal approximations were computed by using efficient algorithms accessed by the set of barycentric coordinates generated by the Delaunay triangulation.

1.3 The Achievements and Organization of the Study

The motivation of the contributions of this thesis is the need of highly accurate and time-efficient implicit methods for the computation of the derivatives of the solution of the heat Equation (1.2). Hence, in this study a second order accurate two-stage implicit method for the approximation of the first order spatial derivatives of the solution of the Dirichlet problem (1.2) on rectangle is developed. The smoothness condition $u \in C_{x,t}^{7+\alpha,\frac{7+\alpha}{2}}$, $0<\alpha<1$ in the Hölder space is assumed and uniform convergence on the grids to the respective spatial derivatives of $O\left(h^2+\tau^2\right)$ accuracy for $r=\frac{\omega\tau}{h^2}\leq\frac{3}{7}$ is proved. Subsequently, these achievements are given in Buranay et al. [50], [51]. Furthermore, fourth order accurate implicit methods are constructed for the approximation of the first order spatial derivatives and second order mixed derivatives of the solution involving the time derivative. It is assumed that $u \in C_{x,t}^{9+\alpha,\frac{9+\alpha}{2}}$, and uniform convergence on the grids to the respective spatial derivatives of $O\left(h^4+\tau\right)$ of accuracy for $r=\frac{\omega\tau}{h^2}\geq\frac{1}{16}$ is given. The obtained theoretical and numerical results are presented in Buranay et al. [52], [53].

The thesis is organized as follows: Chapter 2 has 3 sections. In Section 2.1, we consider the first type boundary value problem for the heat equation in (1.2) on a rectangle D. Hexagonal grid structure and basic notations are given. It is assumed that the heat source and the initial and boundary functions are given such that on $\overline{Q}_T = \overline{D} \times [0,T]$ the solution $u(x_1,x_2,t)$ belongs to the Hölder space $C_{x,t}^{7+\alpha,\frac{7+\alpha}{2}}\left(\overline{Q}_T\right)$, where $x = (x_1,x_2) \in \overline{D}, t \in [0,T]$, and $\overline{D}$ is the closure of D. Further, at the first stage, a two layer implicit method on hexagonal grids given in Buranay and Arshad [37] with $O\left(h^2+\tau^2\right)$ order of accuracy, where h and $\frac{\sqrt{3}}{2}h$ are the step sizes in space variables x_1 and x_2, respectively, and τ is the step size in time is used to approximate the solution $u(x_1,x_2,t)$. For the error function when $r \leq \frac{3}{7}$, we provide a pointwise prior estimation depending on $\rho(x_1,x_2,t)$, which is the distance from the current grid point to the surface of Q_T. In Section 2.2, and Section 2.3, the second stages of the two-stage implicit method for the approximation to the first order derivatives of the solution $u(x_1,x_2,t)$ with respect to the spatial variables x_1 and x_2 are proposed, respectively. It is proved that the constructed implicit schemes at the second stage are unconditionally stable (see Theorem 1 in Lax and Richtmyer [54] which gives the sufficient condition of stability). For $r = \frac{\omega\tau}{h^2} \leq \frac{3}{7}$, priory error estimations in maximum norm between the exact derivatives $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}$ and the obtained corresponding approximate solutions are provided giving $O\left(h^2+\tau^2\right)$ order of accuracy on the hexagonal grids.

In Chapter 3, a numerical example is constructed to support the theoretical results given in Chapter 2. We applied incomplete block preconditioning given in Buranay and Iyikal [55] (see also Concus et al. [56], Axelsson [57]) for the conjugate gradient method to solve the obtained algebraic systems of linear equations for various values of r.

In Chapter 4 we study hexagonal grid computation of the derivatives of the solution to the heat equation by using fourth order accurate two-stage implicit methods. We organize the chapter in sections as follows: In section 4.1 the first type boundary value problem (Dirichlet problem) for the heat Equation (1.2) on a rectangle D is considered. The smoothness of the solution u is taken from the Hölder space $C_{x,t}^{9+\alpha,\frac{9+\alpha}{2}}\left(\overline{Q}_T\right)$. At the first stage, an implicit scheme on hexagonal grids given in Buranay and Arshad [37] with $O\left(h^4+\tau\right)$ order of accuracy is used to approximate the solution $u\left(x_1,x_2,t\right)$. An analogous implicit method is also given to approximate the derivative of the solution with respect to time. In section 4.2 and section 4.3 at the second stages, computation of the first order spatial derivatives and second order mixed derivatives involving time derivatives of the solution $u\left(x_1,x_2,t\right)$ of (1.2) are developed. When $r=\frac{\omega\tau}{h^2}\geq\frac{1}{16}$ uniform convergence of the approximate derivative to the exact derivatives $\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial t}$, and $\frac{\partial^2 u}{\partial x_i \partial t}$, $i=1,2$ with order $O\left(h^4+\tau\right)$ of accuracy on the hexagonal grids are proved.

In Chapter 5, numerical examples are given and for the solution of the obtained algebraic linear systems preconditioned conjugate gradient method is used. The incomplete block matrix factorization of the M-matrices given in Buranay and Iyikal [55] (see also Concus et al. [56], Axelsson [57]) is applied for the preconditioning.

In Chapter 6 concluding results and remarks are given.

Chapter 2

HEXAGONAL GRID COMPUTATION OF THE DERIVATIVES OF THE SOLUTION TO THE HEAT EQUATION BY USING SECOND ORDER ACCURATE TWO-STAGE IMPLICIT METHODS

In this chapter, we consider the first type boundary value problem for the heat equation in (1.2) on a rectangle D. Hexagonal grid structure and basic notations are given. In the first stage of the two-stage method, a two layer implicit method on hexagonal grids given in Buranay and Arshad [37] with $O\left(h^2+\tau^2\right)$ order of accuracy is used to approximate the solution $u(x_1,x_2,t)$. For the error function, we provide a pointwise prior estimation depending on $\rho(x_1,x_2,t)$, which is the distance from the current grid point to the surface of Q_T. In the second stage of the two-stage implicit method, second stages for the approximation to the first order derivatives of the solution $u(x_1,x_2,t)$ with respect to the spatial variables x_1 and x_2 are proposed, respectively. It is proved that the constructed implicit schemes at the second stage are unconditionally stable. Priory error estimations in maximum norm between the exact derivatives $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}$ and the obtained corresponding approximate solutions are provided giving $O\left(h^2+\tau^2\right)$ order of accuracy on the hexagonal grids.

2.1 Dirichlet Problem of Heat Equation and Second Order Accurate Solution by Using Hexagonal Grids

Let $D=\{(x_1,x_2):0<x_1<a_1, 0<x_2<a_2\}$ be a rectangle, where we require a_2 to be multiple of $\sqrt{3}$. Next, let γ_j, $j=1,2,3,4$, be the sides of D that starting from the

side $x_1 = 0$ are labeled in anticlockwise direction. Furthermore, the boundary of D is shown by $S = \bigcup_{j=1}^{4} \gamma_j$. Next, we indicate the closure of D by $\overline{D} = D \cup S$. Let $x = (x_1, x_2)$ and $Q_T = D \times (0,T)$, with the lateral surface $S_T = \{(x,t) : x = (x_1,x_2) \in S, t \in [0,T]\}$ and $\overline{Q}_T$ is the closure of Q_T. Let s be a non-integer positive number, $C_{x,t}^{s,\frac{s}{2}}\left(\overline{Q}_T\right)$ be the Banach space of functions $u(x,t)$ that are continuous in $\overline{Q}_T$ together with all derivatives of the form

$$\frac{\partial^{\xi+s_1+s_2} u}{\partial t^{\xi} \partial x_1^{s_1} \partial x_2^{s_2}} \text{ for } 2\xi + s_1 + s_2 < s \tag{2.1}$$

with bounded norm

$$\|u\|_{C_{x,t}^{s,\frac{s}{2}}\left(\overline{Q}_T\right)} = \langle u \rangle_{Q_T}^{(s)} + \sum_{j=0}^{[s]} \langle u \rangle_{Q_T}^{(j)}, \tag{2.2}$$

where

$$\langle u \rangle_{Q_T}^{(j)} = \sum_{2\xi+s_1+s_2=j} \max_{\overline{Q}_T} \left| \frac{\partial^{\xi+s_1+s_2} u}{\partial t^{\xi} \partial x_1^{s_1} \partial x_2^{s_2}} \right|, j = 0,1,2,...,[s], \tag{2.3}$$

$$\langle u \rangle_{Q_T}^{(s)} = \langle u \rangle_{x}^{(s)} + \langle u \rangle_{t}^{\left(\frac{s}{2}\right)}, \tag{2.4}$$

$$\langle u \rangle_{x}^{(s)} = \sum_{2r+s_1+s_2=[s]} \left\langle \frac{\partial^{\xi+s_1+s_2} u}{\partial t^{\xi} \partial x_1^{s_1} \partial x_2^{s_2}} \right\rangle_{x}^{s-[s]}, \tag{2.5}$$

$$\langle u \rangle_{t}^{\left(\frac{s}{2}\right)} = \sum_{0<s-2\xi-s_1-s_2<2} \left\langle \frac{\partial^{\xi+s_1+s_2} u}{\partial t^{\xi} \partial x_1^{s_1} \partial x_2^{s_2}} \right\rangle_{t}^{\frac{s-2\xi-s_1-s_2}{2}}, \tag{2.6}$$

further, $\langle u \rangle_x^{\alpha}$, $\langle u \rangle_t^{\beta}$ for $\alpha, \beta \in (0,1)$ are defined as

$$\langle u \rangle_x^{\alpha} = \sup_{(x,t),\,(x',t) \in \overline{Q}_T} \frac{|u(x,t) - u(x',t)|}{|x - x'|^{\alpha}}, \tag{2.7}$$

$$\langle u \rangle_t^{\beta} = \sup_{(x,t),\,(x,t') \in \overline{Q}_T} \frac{|u(x,t) - u(x,t')|}{|t - t'|^{\beta}}. \tag{2.8}$$

Volkov [58] gave the differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on rectangle. On cylindrical domains with smooth boundary, the differentiability properties of solutions of the parabolic equations were given in Ladyženskaja et al. [59] and Friedman [60]. On regions with edges, Azzam and Kreyszig studied the smoothness of solutions of parabolic equations for the Dirichlet problem in [61] and for the mixed boundary value problem in [62].

Our interest is the following problem for the heat equation

$$\mathbf{BVP}(u) \qquad \frac{\partial u}{\partial t} = \omega\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + f(x_1,x_2,t) \text{ on } Q_T,$$

$$u(x_1,x_2,0) = \varphi(x_1,x_2) \text{ on } \overline{D},$$

$$u(x_1,x_2,t) = \phi(x_1,x_2,t) \text{ on } S_T, \tag{2.9}$$

where ω is positive constant. This problem is known as first type (Dirichlet) boundary value problem.

Let the heat source function $f(x_1,x_2,t)$ and the initial and boundary functions $\varphi(x_1,x_2)$ and $\phi(x_1,x_2,t)$, respectively, be given such that the BVP(u) has a unique solution u belonging to the Hölder class $C_{x,t}^{7+\alpha,\frac{7+\alpha}{2}}\left(\overline{Q}_T\right)$. Let $h>0$, with $h = a_1/N_1$, where N_1 is positive integer and assign D^h a hexagonal grid on D, with step size h, defined as the set of nodes

$$D^h = \left\{ x = (x_1,x_2) \in D : x_1 = \frac{i'-j'}{2}h,\ x_2 = \frac{\sqrt{3}(i'+j')}{2}h,\right.$$

$$\left. i' = 1,2,...;\ j' = 0 \pm 1 \pm 2,... \right\}. \tag{2.10}$$

Let γ_j^h, $j = 1,...,4$ be the set of nodes on the interior of γ_j and let $\widehat{\gamma}_j^h = \gamma_{j-1} \cap \gamma_j$ be the jth vertex of D, $S^h = \bigcup_{j=1}^{4} (\gamma_j^h \cup \widehat{\gamma}_j^h)$, $\overline{D}^h = D^h \cup S^h$. Further, let D^{*lh}, D^{*rh} denote the set of interior nodes whose distance from the boundary is $\frac{h}{2}$. The hexagons in this set will be referred as irregular hexagons with left ghost point as shown in Figure 2.1 or a right ghost point as presented in Figure 2.2, emerging through the left or right side of the rectangle, respectively. We also define the sets $D^{*h} = D^{*lh} \cup D^{*rh}$ and $D^{0h} = D^h \backslash D^{*h}$.

Next, let

$$\gamma_\tau = \left\{ t_k = k\tau,\ \tau = \frac{T}{M'},\ k = 1, ..., M' \right\}, \tag{2.11}$$

$$\bar{\gamma}_\tau = \left\{ t_k = k\tau,\ \tau = \frac{T}{M'},\ k = 0, ..., M' \right\}, \tag{2.12}$$

and the set of internal nodes and lateral surface nodes be defined by

$$D^h\gamma_\tau = D^h \times \gamma_\tau = \left\{ (x,t) : x = (x_1, x_2) \in D^h,\ t \in \gamma_\tau \right\}, \tag{2.13}$$

$$S_T^h = S^h \times \bar{\gamma}_\tau = \left\{ (x,t) : x = (x_1, x_2) \in S^h,\ t \in \bar{\gamma}_\tau \right\}, \tag{2.14}$$

accordingly. Let $D^{*lh}\gamma_\tau = D^{*lh} \times \gamma_\tau \subset D^h\gamma_\tau$ and $D^{*rh}\gamma_\tau = D^{*rh} \times \gamma_\tau \subset D^h\gamma_\tau$ and $D^{*h}\gamma_\tau = D^{*lh}\gamma_\tau \cup D^{*rh}\gamma_\tau$. In addition, $D^{0h}\gamma_\tau = D^h\gamma_\tau \backslash D^{*h}\gamma_\tau$ and $\overline{D^h\gamma_\tau}$ is the closure of $D^h\gamma_\tau$.

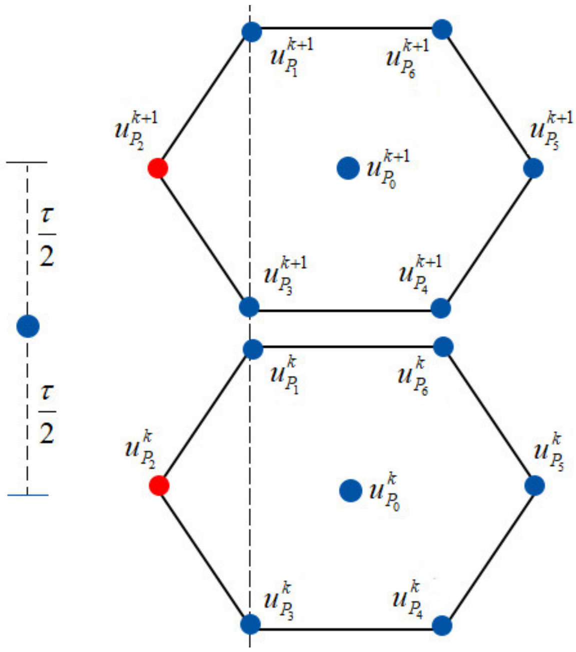


Figure 2.1: The illustration of an irregular hexagon with a left ghost point at time moments $t = k\tau$ and $(k+1)\tau$.

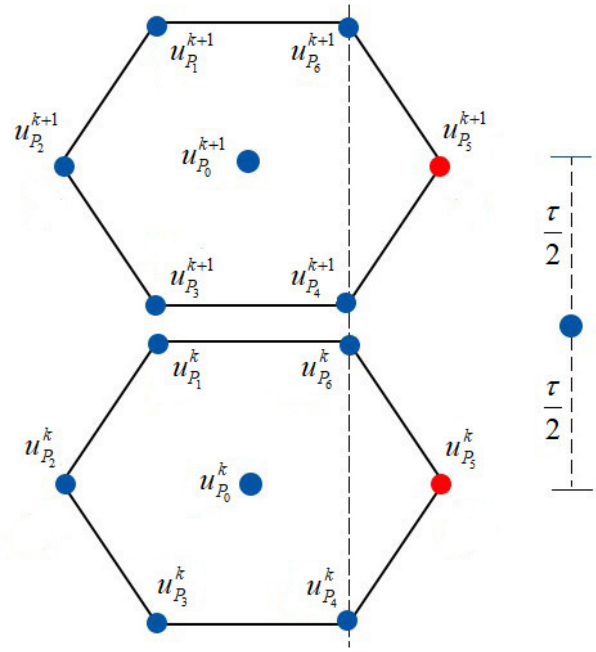


Figure 2.2: The illustration of an irregular hexagon with a right ghost point at time moments $t = k\tau$ and $(k+1)\tau$.

Let P_0 denote the center of the hexagon and $Patt\,(P_0)$ denote the pattern of the hexagon consisting the neighboring points $P_i, i = 1,...,6$. In addition, $u_{P_i}^{k+1}$ denotes the exact solution at the point P_i and $u_{P_A}^{k+1}$ denotes the value at the boundary point for the time moment $t+\tau$ as follows:

$$
\begin{aligned}
&u_{P_1}^{k+1} = u(x_1 - \frac{h}{2}, x_2 + \frac{\sqrt{3}}{2}h, t+\tau),\; u_{P_3}^{k+1} = u(x_1 - \frac{h}{2}, x_2 - \frac{\sqrt{3}}{2}h, t+\tau),\\
&u_{P_2}^{k+1} = u(x_1 - h, x_2, t+\tau),\; u_{P_5}^{k+1} = u(x_1 + h, x_2, t+\tau),\\
&u_{P_4}^{k+1} = u(x_1 + \frac{h}{2}, x_2 - \frac{\sqrt{3}}{2}h, t+\tau),\; u_{P_6}^{k+1} = u(x_1 + \frac{h}{2}, x_2 + \frac{\sqrt{3}}{2}h, t+\tau),\\
&u_{P_0}^{k+1} = u(x_1, x_2, t+\tau),\; u_{P_A}^{k+1} = u(\widehat{p}, x_2, t+\tau),\; (\widehat{p}, x_2, t+\tau) \in S_T^h,
\end{aligned}
$$

where the value of $\widehat{p} = 0$ if $P_0 \in D^{*lh}\gamma_\tau$ and $\widehat{p} = a_1$ if $P_0 \in D^{*rh}\gamma_\tau$. Analogously, the values $u_{P_i}^k$, $i = 0,...,6$ and $u_{P_A}^k$ present the exact solution at the same space coordinates of P_i, $i = 0,...,6$ and P_A, respectively, but at time level $t = k\tau$. Further, u_{h,τ,P_i}^{k+1}, $i = 0,...,6$, u_{h,τ,P_A}^{k+1}, and u_{h,τ,P_i}^{k}, $i = 0,...,6, u_{h,\tau,P_A}^{k}$ present the numerical solution at the same space coordinates of P_i, $i = 0,...,6$ and P_A for time moments $t+\tau$ and $t = k\tau$, respectively and $f_{P_0}^{k+\frac{1}{2}} = f(x_1, x_2, t+\frac{\tau}{2})$, and $f_{P_A}^{k+1} = f(\widehat{p}, x_2, t+\tau)$. The illustration of the exact solution at the irregular hexagons with a ghost point at time levels $t-\tau$, t and $t+\tau$ is given in Figure 2.3.

Buranay and Arshad [37] studied the numerical solution of the BVP(u) using hexagonal grids and gave the following difference problem (named as Difference Problem 1). We call this problem Stage 1$\left(H^{2nd}(u)\right)$ of the two-stage implicit method:

Stage 1 $\left(H^{2nd}(u)\right)$

$$\Theta_{h,\tau}u_{h,\tau}^{k+1} = \Lambda_{h,\tau}u_{h,\tau}^{k} + \psi \text{ on } D^{0h}\gamma_{\tau}, \tag{2.15}$$

$$\Theta_{h,\tau}^{*}u_{h,\tau}^{k+1} = \Lambda_{h,\tau}^{*}u_{h,\tau}^{k} + \Gamma_{h,\tau}^{*}\phi + \psi^{*} \text{ on } D^{*h}\gamma_{\tau}, \tag{2.16}$$

$$u_{h,\tau} = \varphi(x_1, x_2),\ t = 0 \text{ on } \overline{D}^{h}, \tag{2.17}$$

$$u_{h,\tau} = \phi(x_1, x_2, t) \text{ on } S_T^{h}, \tag{2.18}$$

for $k = 0, \ldots, M'-1$, where

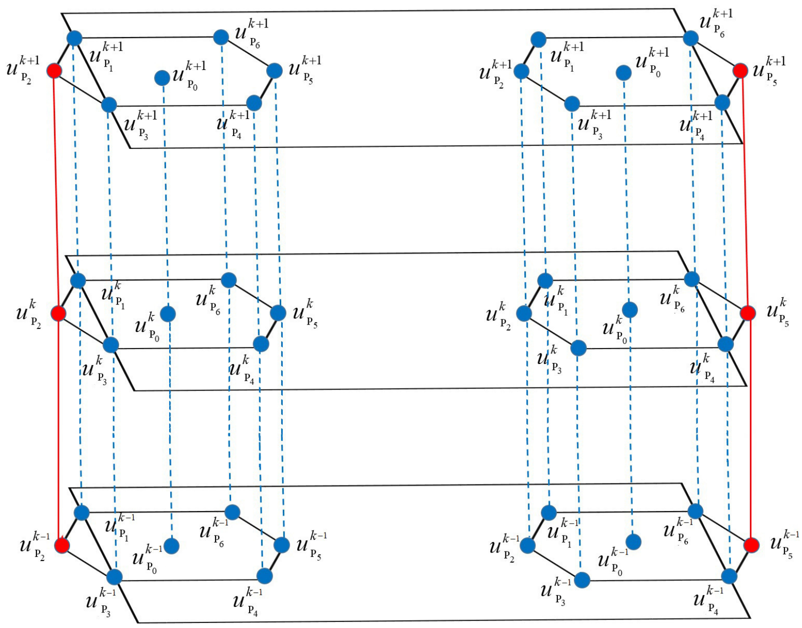


Figure 2.3: The illustration of the exact solution on the irregular hexagons with a ghost point at time levels $t-\tau$, t and $t+\tau$.

$$\psi = f_{P_0}^{k+\frac{1}{2}}, \tag{2.19}$$

$$\psi^* = f_{P_0}^{k+\frac{1}{2}} - \frac{1}{6} f_{P_A}^{k+\frac{1}{2}}, \tag{2.20}$$

$$\Theta_{h,\tau} u^{k+1} = \left(\frac{1}{\tau} + \frac{2\omega}{h^2}\right) u_{P_0}^{k+1} - \frac{\omega}{3h^2} \sum_{i=1}^{6} u_{P_i}^{k+1}, \tag{2.21}$$

$$\Lambda_{h,\tau} u^{k} = \left(\frac{1}{\tau} - \frac{2\omega}{h^2}\right) u_{P_0}^{k} + \frac{\omega}{3h^2} \sum_{i=1}^{6} u_{P_i}^{k}, \tag{2.22}$$

$$\begin{aligned}\Theta^*_{h,\tau} u^{k+1} = {} & \left(\frac{1}{\tau} + \frac{7\omega}{3h^2}\right) u_{P_0}^{k+1} - \frac{\omega}{3h^2} \left(u(p+\eta, x_2, t+\tau)\right. \\ & \left. + u(p, x_2 + \frac{\sqrt{3}}{2}h, t+\tau) + u(p, x_2 - \frac{\sqrt{3}}{2}h, t+\tau)\right),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\Lambda^*_{h,\tau} u^{k} = {} & \left(\frac{1}{\tau} - \frac{7\omega}{3h^2}\right) u_{P_0}^{k} + \frac{\omega}{3h^2} \left(u(p, x_2 + \frac{\sqrt{3}}{2}h, t)\right. \\ & \left. + u(p, x_2 - \frac{\sqrt{3}}{2}h, t) + u(p+\eta, x_2, t)\right),\end{aligned} \tag{2.24}$$

$$\begin{aligned}\Gamma^*_{h,\tau}\phi = {} & \frac{2\omega}{9h^2} \left(\phi(\widehat{p}, x_2 + \frac{\sqrt{3}}{2}h, t+\tau) + \phi(\widehat{p}, x_2 - \frac{\sqrt{3}}{2}h, t+\tau)\right. \\ & \left. + \phi(\widehat{p}, x_2 + \frac{\sqrt{3}}{2}h, t) + \phi(\widehat{p}, x_2 - \frac{\sqrt{3}}{2}h, t)\right) \\ & + \left(\frac{1}{6\tau} + \frac{8\omega}{9h^2}\right) \phi(\widehat{p}, x_2, t+\tau) + \left(-\frac{1}{6\tau} + \frac{8\omega}{9h^2}\right) \phi(\widehat{p}, x_2, t),\end{aligned} \tag{2.25}$$

and

$$\begin{cases} p = h, \widehat{p} = 0, \eta = \frac{h}{2} \text{ if } P_0 \in D^{*lh}\gamma_\tau, \\ p = a_1 - h, \widehat{p} = a_1, \eta = -\frac{h}{2} \text{ if } P_0 \in D^{*rh}\gamma_\tau. \end{cases} \tag{2.26}$$

We label the interior grid points using standard ordering as $L_j, j = 1, 2, ..., N$, and then obtain the algebraic linear system of equations in matrix form

$$A\widetilde{u}^{k+1} = B\widetilde{u}^{k} + \tau q_u^k, \tag{2.27}$$

as given in Buranay and Arshad [37] where $A, B \in R^{N\times N}$ are

$$A = \left(I + \frac{\omega\tau}{h^2}C\right),\ B = \left(I - \frac{\omega\tau}{h^2}C\right), \tag{2.28}$$

and

$$C = D_1 - \frac{1}{3} Inc \in R^{N \times N}, \tag{2.29}$$

and $\widetilde{u}^k, q_u^k \in R^N$. The matrix Inc is the neighboring topology and has the nonzero entries as unity for the points in the pattern of the hexagon center. In addition, I is the identity matrix, D_1 is a diagonal matrix with entries

$$d_{1,jj} = \begin{cases} 2 \text{ if } L_j \in D^{0h}\gamma_\tau \\ \frac{7}{3} \text{ if } L_j \in D^{*h}\gamma_\tau \end{cases}, \ j = 1,2,...,N. \tag{2.30}$$

Lemma 2.1: (Buranay and Arshad [37])

a) The matrix A in (2.27) is symmetric positive definite and an M-matrix

b) Also for $r = \frac{\omega\tau}{h^2} > 0$ the inequalities $\left\|A^{-1}\right\|_2 < 1$ and $\left\|A^{-1}B\right\|_2 < 1$ are valid.

Let

$$\varepsilon_{h,\tau}^u = u_{h,\tau} - u \text{ on } \overline{D^h\gamma_\tau}. \tag{2.31}$$

From (2.15)-(2.18) and (2.31), the error function $\varepsilon_{h,\tau}^u$ satisfies the following system as given in Buranay and Nouman [37]

$$\Theta_{h,\tau}\varepsilon_{h,\tau}^{u,k+1} = \Lambda_{h,\tau}\varepsilon_{h,\tau}^{u,k} + \Psi_1^{u,k} \text{ on } D^{0h}\gamma_\tau, \tag{2.32}$$

$$\Theta_{h,\tau}^*\varepsilon_{h,\tau}^{u,k+1} = \Lambda_{h,\tau}^*\varepsilon_{h,\tau}^{u,k} + \Psi_2^{u,k} \text{ on } D^{*h}\gamma_\tau, \tag{2.33}$$

$$\varepsilon_{h,\tau}^u = 0, \ t = 0 \text{ on } \overline{D}^h, \tag{2.34}$$

$$\varepsilon_{h,\tau}^u = 0 \text{ on } S_T^h, \tag{2.35}$$

where

$$\Psi_1^{u,k} = \Lambda_{h,\tau}u^k - \Theta_{h,\tau}u^{k+1} + \psi, \tag{2.36}$$

$$\Psi_2^{u,k} = \Lambda_{h,\tau}^*u^k - \Theta_{h,\tau}^*u^{k+1} + \Gamma_{h,\tau}^*\phi + \psi^*, \tag{2.37}$$

and ψ, ψ^*, and ϕ are the given functions in (2.15), (2.16), and (2.18), respectively.

2.1.1 Pointwise Priory Estimation For the Error Function (2.32)-(2.35)

Consider the following systems

$$\Theta_{h,\tau}\widehat{q}_{h,\tau}^{k+1} = \Lambda_{h,\tau}\widehat{q}_{h,\tau}^{k} + \widehat{g}_1^k \text{ on } D^{0h}\gamma_\tau, \tag{2.38}$$

$$\Theta_{h,\tau}^{*}\widehat{q}_{h,\tau}^{k+1} = \Lambda_{h,\tau}^{*}\widehat{q}_{h,\tau}^{k} + \Gamma_{h,\tau}^{*}\widehat{q}_{\phi,h,\tau} + \widehat{g}_2^k \text{ on } D^{*h}\gamma_\tau, \tag{2.39}$$

$$\widehat{q}_{h,\tau} = \widehat{q}_{\varphi,h,\tau},\ t=0 \text{ on } \overline{D}^h, \tag{2.40}$$

$$\widehat{q}_{h,\tau} = \widehat{q}_{\phi,h,\tau} \text{ on } S_T^h, \tag{2.41}$$

$$\Theta_{h,\tau}\overline{q}_{h,\tau}^{k+1} = \Lambda_{h,\tau}\overline{q}_{h,\tau}^{k} + \overline{g}_1^k \text{ on } D^{0h}\gamma_\tau, \tag{2.42}$$

$$\Theta_{h,\tau}^{*}\overline{q}_{h,\tau}^{k+1} = \Lambda_{h,\tau}^{*}\overline{q}_{h,\tau}^{k} + \Gamma_{h,\tau}^{*}\overline{q}_{\phi,h,\tau} + \overline{g}_2^k \text{ on } D^{*h}\gamma_\tau, \tag{2.43}$$

$$\overline{q}_{h,\tau} = \overline{q}_{\varphi,h,\tau},\ t=0 \text{ on } \overline{D}^h, \tag{2.44}$$

$$\overline{q}_{h,\tau} = \overline{q}_{\phi,h,\tau} \text{ on } S_T^h, \tag{2.45}$$

for $k=0,...,M'-1$, where $\widehat{g}_1,\widehat{g}_2$ and $\overline{g}_1,\overline{g}_2$ are given functions. For every time level $k=0,...,M'-1$ the algebraic systems (2.38)-(2.41) and (2.42)-(2.45) can be written in matrix form

$$A\widehat{q}^{k+1} = B\widehat{q}^{k} + \tau\widehat{g}^{k}, \tag{2.46}$$

$$A\overline{q}^{k+1} = B\overline{q}^{k} + \tau\overline{g}^{k}, \tag{2.47}$$

accordingly, where A and B are the matrices given in (2.27) and $\widehat{q}^k,\overline{q}^k,\widehat{g}^k,\overline{g}^k \in R^N$. Furthermore, for $A=\left[a_{i,j}\right]$ and $B=\left[b_{i,j}\right]$, $i=1,2,...,N$ and $j=1,2,...,N$ of real matrices, we denote by $A>0$ $(A\geq 0)$ if $a_{i,j}>0$ $(a_{i,j}\geq 0)$ for all i,j. Also $A<B$ $(A\leq B)$ if $a_{i,j}<b_{i,j}$ $(a_{i,j}\leq b_{i,j})$. Analogous notation is also used for the vectors. Additionally, let w be a vector with coordinates w_j, $j=1,2,...,N$, the vector with coordinates $\left|w_j\right|$ is denoted by $|w|$.

Lemma 2.2: (Buranay et al. [51]) Let $\widehat{q}^{k+1}$ and $\overline{q}^{k+1}$ be the solutions of the difference equations (2.46) and (2.47) respectively. For $r=\frac{\omega\tau}{h^2}\leq\frac{3}{7}$, if

$$\overline{q}^0 \geq 0 \text{ and } \overline{g}^k \geq 0, \tag{2.48}$$

$$\left|\widehat{q}^0\right| \leq \overline{q}^0, \tag{2.49}$$

$$\left|\widehat{g}^k\right| \leq \overline{g}^k, \tag{2.50}$$

for $k = 0, ..., M' - 1$, then

$$\overline{q}^{k+1} \geq 0 \text{ and } \left|\widehat{q}^{k+1}\right| \leq \overline{q}^{k+1} \text{ for } k = 0, ..., M' - 1. \tag{2.51}$$

Proof. On the basis of Lemma 2.1, $A^{-1} \geq 0$ and if $r = \frac{\omega\tau}{h^2} \leq \frac{3}{7}$ then $B \geq 0$ and from (2.48) we have $\overline{g}^k \geq 0$, $k = 0, ..., M' - 1$ and $\overline{q}^0 \geq 0$. Then, assume that $\overline{q}^k \geq 0$ by using induction we have

$$\overline{q}^{k+1} = A^{-1}B\overline{q}^k + \tau A^{-1}\overline{g}^k \geq 0, \tag{2.52}$$

which gives $\overline{q}^{k+1} \geq 0$, $k = 0, ..., M' - 1$. In addition, $\left|\widehat{q}^0\right| \leq \overline{q}^0$ from (2.49). Next assume that $\left|\widehat{q}^k\right| \leq \overline{q}^k$, by using (2.50) and induction gives

$$\widehat{q}^{k+1} = A^{-1}B\widehat{q}^k + \tau A^{-1}\widehat{g}^k, \tag{2.53}$$

$$\begin{aligned} \left|\widehat{q}^{k+1}\right| &\leq A^{-1}B\left|\widehat{q}^k\right| + \tau A^{-1}\left|\widehat{g}^k\right| \\ &\leq A^{-1}B\overline{q}^k + \tau A^{-1}\overline{g}^k = \overline{q}^{k+1}. \end{aligned} \tag{2.54}$$

Thus, we obtain (2.51). □

Let

$$\begin{aligned} S_T\gamma_1 &= \gamma_1 \times (0,T] = \{(0,x_2,t) : (0,x_2) \in \gamma_1,\, t \in (0,T]\}, \\ S_T\gamma_2 &= \gamma_2 \times (0,T] = \{(x_1,0,t) : (x_1,0) \in \gamma_2,\, t \in (0,T]\}, \\ S_T\gamma_3 &= \gamma_3 \times (0,T] = \{(a_1,x_2,t) : (a_1,x_2) \in \gamma_3,\, t \in (0,T]\}, \\ S_T\gamma_4 &= \gamma_4 \times (0,T] = \{(x_1,a_2,t) : (x_1,a_2) \in \gamma_4,\, t \in (0,T]\}, \\ S_T\gamma_5 &= \left\{(x_1,x_2,0) : (x_1,x_2) \in \overline{D}\,,\, t = 0\right\}, \end{aligned} \tag{2.55}$$

and $S_T^h\gamma_i, i = 1,2,...,5$ define the corresponding sets of grid points. Furthermore, let

$F = \bigcup_{i=1}^{5} S_T\gamma_i$ denote the surface of Q_T.

Theorem 2.1: (Buranay et al. [51]) For the solution of the problem (2.32)-(2.35), the following inequality holds true

$$\left|\varepsilon_{h,\tau}^{u}\right| \leq d\Omega_1(h,\tau)\,\rho(x_1,x_2,t),\ \text{on}\ \overline{D^h\gamma_\tau}, \tag{2.56}$$

for $r = \frac{\omega\tau}{h^2} \leq \frac{3}{7}$ where

$$\Omega_1(h,\tau) = \frac{1}{24}\tau^2(1+6\omega)\beta^* + \frac{3\omega}{10}h^2\alpha^*, \tag{2.57}$$

$$\alpha^* = \max\left\{\max_{\overline{Q}_T}\left|\frac{\partial^4 u}{\partial x_1^4}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 u}{\partial x_2^4}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}\right|\right\}, \tag{2.58}$$

$$\beta^* = \max\left\{\max_{\overline{Q}_T}\left|\frac{\partial^3 u}{\partial t^3}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 u}{\partial x_2^2\partial t^2}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 u}{\partial x_1^2\partial t^2}\right|\right\}, \tag{2.59}$$

$$d = \max\left\{\frac{a_1}{2\omega}, \frac{a_2}{2\omega}, 1\right\}, \tag{2.60}$$

and u is the exact solution of BVP(u) and $\rho(x_1,x_2,t)$ is the distance from the current grid point in $\overline{D^h\gamma_\tau}$ to the surface F of Q_T.

Proof. We consider the system

$$\Theta_{h,\tau}\widehat{\varepsilon}_{h,\tau}^{u,k+1} = \Lambda_{h,\tau}\widehat{\varepsilon}_{h,\tau}^{u,k} + \Omega_1(h,\tau)\ \text{on}\ D^{0h}\gamma_\tau, \tag{2.61}$$

$$\Theta_{h,\tau}^{*}\widehat{\varepsilon}_{h,\tau}^{u,k+1} = \Lambda_{h,\tau}^{*}\widehat{\varepsilon}_{h,\tau}^{u,k} + \frac{5}{6}\Omega_1(h,\tau)\ \text{on}\ D^{*h}\gamma_\tau \tag{2.62}$$

$$\widehat{\varepsilon}_{h,\tau}^{u} = \widehat{\varepsilon}_{\varphi,h,\tau}^{u} = 0,\ t=0\ \text{on}\ \overline{D}^h, \tag{2.63}$$

$$\widehat{\varepsilon}_{h,\tau}^{u} = \widehat{\varepsilon}_{\phi,h,\tau}^{u} = 0\ \text{on}\ S_T^h, \tag{2.64}$$

and the majorant functions

$$\overline{\varepsilon}_1^u(x_1,x_2,t) = \frac{1}{2\omega}\Omega_1(h,\tau)\left(a_1x_1 - x_1^2\right) \geq 0\ \text{on}\ \overline{D^h\gamma_\tau}, \tag{2.65}$$

$$\overline{\varepsilon}_2^u(x_1,x_2,t) = \frac{1}{2\omega}\Omega_1(h,\tau)\left(a_2x_2 - x_2^2\right) \geq 0\ \text{on}\ \overline{D^h\gamma_\tau}, \tag{2.66}$$

$$\overline{\varepsilon}_3^u(x_1,x_2,t) = \Omega_1(h,\tau)\,t \geq 0\ \text{on}\ \overline{D^h\gamma_\tau}, \tag{2.67}$$

that each satisfies the next difference boundary value problem

$$\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{i,h,\tau} = \Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{i,h,\tau} + \Omega_1(h,\tau) \text{ on } D^{0h}\gamma_\tau, \tag{2.68}$$

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{u,k+1}_{i,h,\tau} = \Lambda^*_{h,\tau}\bar{\varepsilon}^{u,k}_{i,h,\tau} + \Gamma^*_{h,\tau}\bar{\varepsilon}^{u}_{i,\phi,h,\tau} + \frac{5}{6}\Omega_1(h,\tau) \text{ on } D^{*h}\gamma_\tau, \tag{2.69}$$

$$\bar{\varepsilon}^{u}_{i,h,\tau} = \bar{\varepsilon}^{u}_{i,\varphi,h,\tau} = \bar{\varepsilon}^{u}_{i}(x_1,x_2,0) \geq 0,\ t=0 \text{ on } \overline{D}^h, \tag{2.70}$$

$$\bar{\varepsilon}^{u}_{i,h,\tau} = \bar{\varepsilon}^{u}_{i,\phi,h,\tau} \geq 0 \text{ on } S^h_T, \tag{2.71}$$

The difference equations (2.68) and (2.69) are established by using the following results. First let us show that for regular grid points

$$\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{i,h,\tau} - \Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{i,h,\tau} = \Omega_1(h,\tau),\ i=1,2,3.$$

$$\begin{aligned}
\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{1,h,\tau} = {} & \frac{1}{2\omega}\Omega_1(h,\tau)\Bigg[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)\left(a_1x_1-x_1^2\right) \\
& + \left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(a_1\left(x_1+\frac{h}{2}\right)-\left(x_1+\frac{h}{2}\right)^2\right. \\
& + a_1\left(x_1-\frac{h}{2}\right)-\left(x_1-\frac{h}{2}\right)^2 + a_1(x_1-h) \\
& - (x_1-h)^2 + a_1\left(x_1-\frac{h}{2}\right)-\left(x_1-\frac{h}{2}\right)^2 + a_1\left(x_1+\frac{h}{2}\right) \\
& \left. + -(x_1+\frac{h}{2})^2 + a_1(x_1+h)-(x_1+h)^2\right)\Bigg] \\
= {} & \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{a_1x_1}{\tau}-\frac{x_1^2}{\tau}-\frac{h^2}{8\tau}+2\omega\right],
\end{aligned} \tag{2.72}$$

and

$$\begin{aligned}
\Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{1,h,\tau} = {} & \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{3}{4\tau}\left(a_1x_1-x_1^2\right)+\frac{1}{24\tau}\left(6a_1x_1-6x_1^2-3h^2\right)\right] \\
= {} & \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{a_1x_1}{\tau}-\frac{x_1^2}{\tau}-\frac{h^2}{8\tau}\right].
\end{aligned} \tag{2.73}$$

Using (2.72) and (2.73) gives

$$\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{1,h,\tau} - \Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{1,h,\tau} = \Omega_1(h,\tau).$$

$$
\begin{aligned}
\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{2,h,\tau} &= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)(a_2x_2-x_2^2)\right.\\
&+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(2a_2\left(x_2+\frac{\sqrt{3}h}{2}-\left(x_2+\frac{\sqrt{3}h}{2}\right)^2\right)\right.\\
&\left.\left.+\ 2\left(a_2\left(x_2-\frac{\sqrt{3}h}{2}\right)-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2\right)+2\left(a_2x_2-x_2^2\right)\right)\right]\\
&= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{a_2x_2}{\tau}-\frac{x_2^2}{\tau}-\frac{h^2}{8\tau}+2\omega\right], \qquad (2.74)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{2,h,\tau} &= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{3}{4\tau}(a_2x_2-x_2^2)+\frac{1}{24\tau}\left(6a_2x_2-6x_2^2-3h^2\right)\right]\\
&= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{a_2x_2}{\tau}-\frac{x_2^2}{\tau}-\frac{h^2}{8\tau}\right]. \qquad (2.75)
\end{aligned}
$$

Using (2.74) and (2.75) it follows that

$$
\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{2,h,\tau}-\Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{2,h,\tau}=\Omega_1(h,\tau).
$$

$$
\begin{aligned}
\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{3,h,\tau} &= \Omega_1(h,\tau)\left[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)(t+\tau)+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)(6(t+\tau))\right]\\
&= \Omega_1(h,\tau)\left[\frac{t}{\tau}+1\right], \qquad (2.76)
\end{aligned}
$$

$$
\Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{3,h,\tau} = \Omega_1(h,\tau)\left[\left(\frac{3}{4\tau}t+\frac{1}{24\tau}6t\right)\right] = \Omega_1(h,\tau)\left[\frac{t}{\tau}\right]. \qquad (2.77)
$$

From (2.76) and (2.77) we get

$$
\Theta_{h,\tau}\bar{\varepsilon}^{u,k+1}_{3,h,\tau}-\Lambda_{h,\tau}\bar{\varepsilon}^{u,k}_{3,h,\tau}=\Omega_1(h,\tau).
$$

Next let us show that for irregular grid points with a ghost point, the difference equation (2.69) is valid. We give the details only for irregular hexagons with a left ghost point as follows since for the case of a right ghost point it is analogous.

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{u,k+1}_{1,h,\tau} = \frac{1}{2\omega}\Omega_1(h,\tau)\left[\left(\frac{17}{24\tau}+\frac{14\omega}{3h^2}\right)(a_1x_1-x_1^2)+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\right.$$
$$\left.\times\left(a_1h-h^2+a_1h-h^2+\frac{3}{2}a_1h-\frac{9}{4}h^2\right)\right]$$
$$= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{17a_1x_1}{24\tau}-\frac{17x_1^2}{24\tau}+\frac{14\omega a_1x_1}{3h^2}-\frac{14\omega x_1^2}{3h^2}+\frac{7a_1h}{48\tau}\right.$$
$$\left.-\frac{17h^2}{96\tau}-\frac{7\omega a_1h}{3h^2}+\frac{17\omega}{6}\right]$$
$$= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{17a_1x_1}{24\tau}-\frac{17x_1^2}{24\tau}+\frac{7a_1h}{48\tau}-\frac{17h^2}{96\tau}\right]. \tag{2.78}$$

$$\Gamma^*_{h,\tau}\bar{\varepsilon}^{u}_{1,\phi,h,\tau} = 0, \tag{2.79}$$

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{u,k+1}_{1,h,\tau}-\Lambda^*_{h,\tau}\bar{\varepsilon}^{u,k}_{1,h,\tau} = \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{17a_1x_1}{24\tau}-\frac{17x_1^2}{24\tau}+\frac{14\omega a_1x_1}{3h^2}\right.$$
$$-\frac{14\omega x_1^2}{3h^2}+\frac{7a_1h}{48\tau}-\frac{17h^2}{96\tau}-\frac{7\omega a_1h}{3h^2}+\frac{17\omega}{6}-\frac{17a_1x_1}{24\tau}$$
$$\left.+\frac{17x_1^2}{24\tau}-\frac{7a_1h}{48\tau}+\frac{17h^2}{96\tau}\right]$$
$$= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{14\omega a_1x_1}{3h^2}-\frac{14\omega x_1^2}{3h^2}-\frac{7\omega a_1}{h}-\frac{17\omega}{6}\right] \tag{2.80}$$

from (2.78)-(2.80) and evaluating at $x_1=\frac{h}{2}$ gives

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{u,k+1}_{1,h,\tau}-\Lambda^*_{h,\tau}\bar{\varepsilon}^{u,k}_{1,h,\tau}-\Gamma^*_{h,\tau}\bar{\varepsilon}^{u}_{1,\phi,h,\tau} = \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{14\omega a_1}{6h}-\frac{14\omega h^2}{12h^2}\right.$$
$$\left.-\frac{7\omega a_1}{3h}+\frac{17\omega}{6}\right] = \frac{5}{6}\Omega_1(h,\tau).$$

Also,

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{u,k+1}_{2,h,\tau} = \frac{1}{2\omega}\Omega_1(h,\tau)\left[\left(\frac{17}{24\tau}+\frac{14\omega}{3h^2}\right)\left(a_2\left(x_2+\frac{\sqrt{3}h}{2}\right)-\left(x_2+\frac{\sqrt{3}h}{2}\right)^2\right.\right.$$
$$\left.\left.+\,a_2\left(x_2-\frac{\sqrt{3}h}{2}\right)-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2+a_2x_2-x_2^2\right)\right]$$
$$= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{20a_2x_2}{24\tau}-\frac{20x_2^2}{24\tau}+\frac{8\omega a_2x_2}{3h^2}-\frac{8\omega x_2^2}{3h^2}-\frac{h^2}{16\tau}+\omega\right] \tag{2.81}$$

$$\Lambda^*_{h,\tau}\bar{\varepsilon}^{u,k}_{2,h,\tau} = \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{17}{24\tau}(a_2x_2 - x_2^2) + \frac{1}{24\tau} - \left(3a_2x_2 - 3x_2^2 - \frac{3}{2}h^2\right)\right]$$
$$= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{20a_2x_2}{24\tau} - \frac{20x_2^2}{24\tau} - \frac{h^2}{16\tau}\right] \tag{2.82}$$

$$\Gamma^*_{h,\tau}\bar{\varepsilon}^{u}_{2,\phi,h,\tau} = \frac{1}{2\omega}\Omega_1(h,\tau)\Bigg[\left(-\frac{1}{36\tau} + \frac{4\omega}{9h^2}\right)\Bigg(a_2\left(x_2 + \frac{\sqrt{3}h}{2}\right)$$
$$- \left(x_2 + \frac{\sqrt{3}h}{2}\right)^2 + a_2\left(x_2 - \frac{\sqrt{3}h}{2}\right) - \left(x_2 - \frac{\sqrt{3}h}{2}\right)^2 + a_2x_2 - x_2^2\Bigg)$$
$$+ \left(\frac{1}{18\tau} + \frac{16\omega}{9h^2}\right)(a_2x_2 - x_2^2) + \frac{1}{36\tau}\Bigg(a_2\left(x_2 + \frac{\sqrt{3}h}{2}\right) - \left(x_2 + \frac{\sqrt{3}h}{2}\right)^2$$
$$+ a_2\left(x_2 - \frac{\sqrt{3}h}{2}\right) - \left(x_2 - \frac{\sqrt{3}h}{2}\right)^2 + a_2x_2 - x_2^2\Bigg)$$
$$- \frac{1}{18\tau}(a_2x_2 - x_2^2)\Bigg]$$
$$= \frac{1}{2\omega}\Omega_1(h,\tau)\Bigg[\left(-\frac{1}{36\tau} + \frac{4\omega}{9h^2}\right)\left(2a_2x_2 - 2x_2^2 - \frac{3}{2}h^2\right)$$
$$+ \left(\frac{1}{18\tau} + \frac{16\omega}{9h^2}\right)(a_2x_2 - x_2^2) + \frac{1}{36\tau}\left(2a_2x_2 - 2x_2^2 - \frac{3}{2}h^2\right)$$
$$- \frac{1}{18\tau}(a_2x_2 - x_2^2)\Bigg]$$
$$= \frac{1}{2\omega}\Omega_1(h,\tau)\left[\frac{8\omega a_2x_2}{3h^2} - \frac{8\omega x_2^2}{3h^2} - \frac{2}{3}\omega\right]. \tag{2.83}$$

From (2.81)-(2.83) we obtain

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{k+1}_{2,h,\tau} - \Lambda^*_{h,\tau}\bar{\varepsilon}^{k}_{2,h,\tau} - \Gamma^*_{h,\tau}\bar{\varepsilon}^{u}_{2,\phi,h,\tau} = \frac{1}{2\omega}\Omega_1(h,\tau)\Bigg[\frac{20a_2x_2}{24\tau}$$
$$- \frac{20x_2^2}{24\tau} + \frac{8\omega a_2x_2}{3h^2} - \frac{8\omega x_2^2}{3h^2} + -\frac{h^2}{16\tau} + \omega - \frac{20a_2x_2}{24\tau}$$
$$+ \frac{20x_2^2}{24\tau} + \frac{h^2}{16\tau} - \frac{8\omega a_2x_2}{3h^2} + \frac{8\omega x_2^2}{3h^2} + \frac{2}{3}\omega\Bigg]$$
$$= \frac{5}{6}\Omega_1(h,\tau).$$

Further,

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{k+1}_{3,h,\tau} = \Omega_1(h,\tau)(t+\tau)\left[\left(\frac{17}{24\tau} + \frac{14\omega}{3h^2}\right) + 3\left(\frac{1}{24\tau} - \frac{2\omega}{3h^2}\right)\right]$$
$$= \Omega_1(h,\tau)(t+\tau)\left[\frac{20}{24\tau} + \frac{8\omega}{3h^2}\right] \tag{2.84}$$

$$\Lambda^*_{h,\tau}\bar{\varepsilon}^k_{3,h,\tau} = \Omega_1(h,\tau)t\left[\frac{17}{24\tau}+\frac{3}{24\tau}\right] = \Omega_1(h,\tau)t\left[\frac{20}{24\tau}\right] \tag{2.85}$$

$$\begin{aligned}\Gamma^*_{h,\tau}\bar{\varepsilon}^u_{3,\phi,h,\tau} &= \Omega_1(h,\tau)(t+\tau)\left[2\left(-\frac{1}{36\tau}+\frac{4\omega}{9h^2}\right)+3\left(\frac{1}{18\tau}+\frac{16\omega}{9h^2}\right)\right] \\ &+\Omega_1(h,\tau)t\left[\frac{2}{36\tau}-\frac{1}{18\tau}\right] \\ &= \Omega_1(h,\tau)(t+\tau)\left[\frac{8\omega}{3h^2}\right]. \end{aligned} \tag{2.86}$$

From (2.84)-(2.86) results

$$\begin{aligned}\Theta^*_{h,\tau}\bar{\varepsilon}^{k+1}_{3,h,\tau} - \Lambda^*_{h,\tau}\bar{\varepsilon}^k_{3,h,\tau} - \Gamma^*_{h,\tau}\bar{\varepsilon}^u_{3,\phi,h,\tau} &= \Omega_1(h,\tau)(t+\tau)\left[\frac{20}{24\tau}+\frac{8\omega}{3h^2}\right] \\ &- \Omega_1(h,\tau)t\left[\frac{20}{24\tau}\right] - \Omega_1(h,\tau)(t+\tau)\left[\frac{8\omega}{3h^2}\right] \\ &= \frac{5}{6}\Omega_1(h,\tau)\end{aligned}$$

Consequently, for fixed $k \geq 0$ the difference problems (2.61)-(2.64) and (2.68)-(2.71) may be given in matrix form

$$A\widehat{\varepsilon}^{u,k+1} = B\widehat{\varepsilon}^{u,k} + \tau\widehat{e}^{u,k}, \tag{2.87}$$

$$A\bar{\varepsilon}^{u,k+1}_i = B\bar{\varepsilon}^{u,k}_i + \tau\bar{e}^{u,k}_i,\ i = 1,2,3, \tag{2.88}$$

respectively, and A and B are the matrices given in (2.27). Also, $\bar{e}^{u,k}_i, \bar{\varepsilon}^{u,k}_i, i = 1,2,3$ and $\widehat{\varepsilon}^{u,k}, \widehat{e}^{u,k} \in R^N$. From(2.57) and (2.61)-(2.71) results $\bar{\varepsilon}^{u,0}_i \geq 0$, $\left|\widehat{\varepsilon}^{u,0}\right| \leq \bar{\varepsilon}^{u,0}_i$, and $\bar{e}^{u,k}_i \geq 0$, and $\left|\widehat{e}^{u,k}\right| \leq \bar{e}^{u,k}_i$, $i = 1,2,3$, for $k = 0,...,M'-1$. On the basis of Lemma 2.2, we get $\left|\widehat{\varepsilon}^{u,k+1}\right| \leq \bar{\varepsilon}^{u,k+1}_i, k = 0,...,M'-1$ and using that $\Omega_1(h,\tau) \geq \left|\Psi^{u,k}_1\right|$ on $D^{0h}\gamma_\tau$, and $\frac{5}{6}\Omega_1(h,\tau) \geq \left|\Psi^{u,k}_2\right|$ on $D^{*h}\gamma_\tau$ gives

$$\left|\varepsilon^u_{h,\tau}\right| \leq \min_{i=1,2,3}\bar{\varepsilon}^u_i(x_1,x_2,t) \leq d\Omega_1(h,\tau)\rho(x_1,x_2,t) \text{ on } \overline{D^h\gamma_\tau} \tag{2.89}$$

□

2.2 Difference Problem Approximating $\frac{\partial u}{\partial x_1}$ on Hexagonal Grids with $O(h^2+\tau^2)$ Order of Accuracy

We use the notation $\partial_{x_1} f_{P_0}^{k+\frac{1}{2}} = \frac{\partial f}{\partial x_1}\Big|_{(x_1,x_2,t+\frac{\tau}{2})}$ and $\partial_{x_1} f_{P_A}^{k+\frac{1}{2}} = \frac{\partial f}{\partial x_1}\Big|_{(\widehat{p},x_2,t+\frac{\tau}{2})}$. Let the boundary value problem BVP(u) be given. We denote $p_i = \frac{\partial u}{\partial x_1}$ on $S_T\gamma_i, i=1,2,...,5$ and establish the following BVP for $v=\frac{\partial u}{\partial x_1}$.

Boundary Value Problem for $v=\frac{\partial u}{\partial x_1}$ $\left(\mathbf{BVP}\left(\frac{\partial u}{\partial x_1}\right)\right)$

$$Lv = \frac{\partial f(x_1,x_2,t)}{\partial x_1} \text{ on } Q_T,$$

$$v(x_1,x_2,t) = p_i \text{ on } S_T\gamma_i, i=1,2,...,5, \tag{2.90}$$

where, $f(x_1,x_2,t)$ is the given heat source function in (2.9) and

$$L \equiv \frac{\partial}{\partial t} - \omega\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right). \tag{2.91}$$

From $u \in C_{x,t}^{7+\alpha,\frac{7+\alpha}{2}}\left(\overline{Q}_T\right)$, we assume that the solution $v \in C_{x,t}^{6+\alpha,3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$.

We take

$$p_{1h}^{2^{nd}} = \begin{cases} \frac{1}{2h}\big(-3u(0,x_2,t)+4u_{h,\tau}(h,x_2,t) \\ \quad -u_{h,\tau}(2h,x_2,t)\big) \text{ if } P_0 \in D^{0h}\gamma_\tau \\ \frac{1}{3h}\left(-8u(0,x_2,t)+9u_{h,\tau}\left(\frac{h}{2},x_2,t\right)\right. \\ \quad \left.-u_{h,\tau}\left(\frac{3h}{2},x_2,t\right)\right) \text{ if } P_0 \in D^{*lh}\gamma_\tau \end{cases} \text{ on } S_T^h\gamma_1, \tag{2.92}$$

$$p_{3h}^{2^{nd}} = \begin{cases} \frac{1}{2h}\big(3u(a_1,x_2,t)-4u_{h,\tau}(a_1-h,x_2,t) \\ \quad +u_{h,\tau}(a_1-2h,x_2,t)\big) \text{ if } P_0 \in D^{0h}\gamma_\tau \\ \frac{1}{3h}\left(8u(a_1,x_2,t)-9u_{h,\tau}\left(a_1-\frac{h}{2},x_2,t\right)\right. \\ \quad \left.+u_{h,\tau}\left(a_1-\frac{3h}{2},x_2,t\right)\right) \text{ if } P_0 \in D^{*rh}\gamma_\tau \end{cases} \text{ on } S_T^h\gamma_3, \tag{2.93}$$

$$p_{ih} = \frac{\partial \phi(x_1,x_2,t)}{\partial x_1} \text{ on } S_T^h\gamma_i, i=2,4, \tag{2.94}$$

$$p_{5h} = \frac{\partial \varphi(x_1,x_2)}{\partial x_1} \text{ on } S_T^h\gamma_5. \tag{2.95}$$

Here, $\varphi(x_1,x_2)$, $\phi(x_1,x_2,t)$ are as given in BVP(u), presented in the Equation (2.9) and the solution of the difference problem in Stage 1 $\left(H^{2nd}(u)\right)$ is $u_{h,\tau}$. Further, we give the derivation of the forward and backward schemes in (2.92) and (2.93) for the irregular grid points that have a center $h/2$ units away from the boundary $x_1 = 0$ and $x_1 = a_1$. For the forward scheme of the irregular hexagons we define the grid points as follows:

$A : u(x_1,x_2,t)$

$B : u(x_1+\frac{h}{2},x_2,t)$

$C : u(x_1+\frac{3}{2}h,x_2,t)$

$$\begin{aligned} B : u\left(x_1+\frac{h}{2},x_2,t\right) &= u(x_1,x_2,t)+\frac{h}{2}\partial_{x_1}u(x_1,x_2,t) \\ &+\frac{1}{8}h^2\partial_{x_1}^2u(x_1,x_2,t) \\ &+\frac{1}{48}h^3\partial_{x_1}^3u(x_1+\alpha_1 h,x_2,t)\ . \end{aligned} \tag{2.96}$$

$$\begin{aligned} C : u\left(x_1+\frac{3h}{2},x_2,t\right) &= u(x_1,x_2,t)+\frac{3h}{2}\partial_{x_1}u(x_1,x_2,t) \\ &+\frac{9}{8}h^2\partial_{x_1}^2u(x_1,x_2,t) \\ &+\frac{9}{16}h^3\partial_{x_1}^3u(x_1+\alpha_2 h,x_2,t), \end{aligned} \tag{2.97}$$

where, $0<\alpha_1<\frac{1}{2}$ and $0<\alpha_2<\frac{3}{2}$. Multiplying the Equations (2.96) and (2.97) by 3 and $-\frac{1}{3}$ respectively we get

$$\begin{aligned} 3u\left(x_1+\frac{h}{2},x_2,t\right) &= 3u(x_1,x_2,t)+\frac{3h}{2}\partial_{x_1}u(x_1,x_2,t) \\ &+\frac{3}{8}h^2\partial_{x_1}^2u(x_1,x_2,t) \\ &+\frac{1}{16}h^3\partial_{x_1}^3u(x_1+\alpha_2 h,x_2,t) \end{aligned} \tag{2.98}$$

$$\begin{aligned} -\frac{1}{3}u\left(x_1+\frac{3h}{2},x_2,t\right) &= -\frac{1}{3}u(x_1,x_2,t)-\frac{h}{2}\partial_{x_1}u(x_1,x_2,t) \\ &-\frac{3}{8}h^2\partial_{x_1}^2u(x_1,x_2,t) \\ &-\frac{3}{16}h^3\partial_{x_1}^3u(x_1+\alpha_1 h,x_2,t) \end{aligned} \tag{2.99}$$

Adding (2.98) and (2.99) gives

$$-\frac{1}{3}u\left(x_1+\frac{3h}{2},x_2,t\right)+3u\left(x_1+\frac{h}{2},x_2,t\right)$$
$$=\frac{8}{3}u\left(x_1,x_2,t\right)+h\partial_{x_1}u\left(x_1,x_2,t\right)$$
$$-\frac{1}{8}h^3\partial_{x_1}^3u\left(\widetilde{x_1},x_2,t\right),\;\; x_1<\;\widetilde{x_1}<x_1+\frac{3h}{2} \tag{2.100}$$

$$\frac{1}{3h}\left(-8u\left(x_1,x_2,t\right)+9u\left(x_1+\frac{h}{2},x_2,t\right)-u\left(x_1+\frac{3h}{2},x_2,t\right)\right)$$
$$=\partial_{x_1}u\left(x_1,x_2,t\right)+O\left(h^2\right). \tag{2.101}$$

For the backward scheme for the irregular hexagons we take the grid point as follows:

$A:u(x_1,x_2,t)$

$B:u(x_1-\frac{h}{2},x_2,t)$

$C:u(x_1-\frac{3}{2}h,x_2,t)$

$$C:u\left(x_1-\frac{3h}{2},x_2,t\right)=u\left(x_1,x_2,t\right)-\frac{3h}{2}\partial_{x_1}u\left(x_1,x_2,t\right)$$
$$+\frac{9}{8}h^2\partial_{x_1}^2u\left(x_1,x_2,t\right)$$
$$-\frac{9}{16}h^3\partial_{x_1}^3u\left(x_1+\beta_1h,x_2,t\right), \tag{2.102}$$
$$B:u\left(x_1-\frac{h}{2},x_2,t\right)=u\left(x_1,x_2,t\right)-\frac{h}{2}\partial_{x_1}u\left(x_1,x_2,t\right)$$
$$+\frac{1}{8}h^2\partial_{x_1}^2u\left(x_1,x_2,t\right)$$
$$-\frac{1}{48}h^3\partial_{x_1}^3u\left(x_1+\beta_2h,x_2,t\right)\;, \tag{2.103}$$

where, $-\frac{3}{2}<\beta_1<0$ and $-\frac{1}{2}<\beta_2<0$. Multiplying the Equations (2.102) and (2.103) by $-\frac{1}{3}$ and 3 respectively we get

$$-\frac{1}{3}u\left(x_1-\frac{3h}{2},x_2,t\right)=-\frac{1}{3}u\left(x_1,x_2,t\right)+\frac{h}{2}\partial_{x_1}u\left(x_1,x_2,t\right)$$
$$-\frac{3}{8}h^2\partial_{x_1}^2u\left(x_1,x_2,t\right)$$
$$+\frac{3}{16}h^3\partial_{x_1}^3u\left(x_1+\beta_1h,x_2,t\right) \tag{2.104}$$

$$3u\left(x_1-\frac{h}{2},x_2,t\right)=3u\left(x_1,x_2,t\right)-\frac{3h}{2}\partial_{x_1}u\left(x_1,x_2,t\right)$$
$$+\frac{3}{8}h^2\partial_{x_1}^2u\left(x_1,x_2,t\right)$$
$$-\frac{1}{16}h^3\partial_{x_1}^3u\left(x_1+\beta_2h,x_2,t\right) \tag{2.105}$$

Adding (2.104) and (2.105) yields

$$-\frac{1}{3}u\left(x_1-\frac{3h}{2},x_2,t\right)+3u\left(x_1-\frac{h}{2},x_2,t\right)$$
$$=\frac{8}{3}u\left(x_1,x_2,t\right)-h\partial_{x_1}u\left(x_1,x_2,t\right)$$
$$+\frac{1}{8}h^3\partial_{x_1}^3u\left(\overline{x_1},x_2,t\right),\ \ x_1-\frac{3h}{2}<\ \overline{x_1}<x_1 \tag{2.106}$$

$$\frac{1}{3h}\left(8u\left(x_1,x_2,t\right)-9u\left(x_1-\frac{h}{2},x_2,t\right)+u\left(x_1-\frac{3h}{2},x_2,t\right)\right)$$
$$=\partial_{x_1}u\left(x_1,x_2,t\right)+O\left(h^2\right).$$

Lemma 2.3: (Buranay et al. [51]) The following inequality

$$\left|p_{ih}^{2^{nd}}\left(u_{h,\tau}\right)-p_{ih}^{2^{nd}}\left(u\right)\right|\leq 3d\Omega_1\left(h,\tau\right),\ \ i=1,3. \tag{2.107}$$

holds true for $r=\frac{\omega\tau}{h^2}\leq\frac{3}{7}$, where u is the solution of the boundary value problem BVP(u) and $u_{h,\tau}$ is the solution of Stage 1$\left(H^{2nd}\left(u\right)\right)$ and $\Omega_1\left(h,\tau\right)$ is as given in (2.57), d is as presented in (2.60).

Proof. From Theorem 2.1, and the equations (2.56), (2.92), and (2.93) when $P_0\in D^{0h}\gamma_\tau$, we have

$$\left|p_{ih}^{2^{nd}}\left(u_{h,\tau}\right)-p_{ih}^{2^{nd}}\left(u\right)\right|\leq\frac{1}{2h}\left(4hd\Omega_1\left(h,\tau\right)+2hd\Omega_1\left(h,\tau\right)\right)$$
$$\leq 3d\Omega_1\left(h,\tau\right),\ \ i=1,3 \text{ if } P_0\in D^{0h}\gamma_\tau, \tag{2.108}$$

where Ω_1 is as in (2.57) and d is the positive constant defined in (2.60). When $P_0\in D^{*h}\gamma_\tau$ yields

$$\left|p_{ih}^{2^{nd}}\left(u_{h,\tau}\right)-p_{ih}^{2^{nd}}\left(u\right)\right|\leq\frac{1}{3h}\left(9\frac{h}{2}d\Omega_{1}\left(h,\tau\right)+\frac{3h}{2}d\Omega_{1}\left(h,\tau\right)\right)$$

$$\leq 2d\Omega_{1}\left(h,\tau\right),\ \ i=1,3 \text{ if } P_{0}\in D^{*h}\gamma_{\tau}. \tag{2.109}$$

Thus, we obtain (2.107). □

Lemma 2.4: (Buranay et al. [51]) For $r=\frac{\omega\tau}{h^2}\leq\frac{3}{7}$ the following inequality

$$\max_{S_T^h\gamma_1\cup S_T^h\gamma_3}\left|p_{ih}^{2^{nd}}\left(u_{h,\tau}\right)-p_i\right|\leq M_1h^2+3d\Omega_1\left(h,\tau\right),\ \ i=1,3, \tag{2.110}$$

holds true where $u_{h,\tau}$ is the solution of the difference problem in Stage $1\left(H^{2nd}\right)$ and $M_1=\frac{1}{3}\max_{\overline{Q}_T}\left|\frac{\partial^3 u}{\partial x_1^3}\right|$ and Ω_1 and d are as given in (2.57) and (2.60), respectively.

Proof. Since $u\in C_{x,t}^{7+\alpha,\frac{7+\alpha}{2}}\left(\overline{Q}_T\right)$, at the end points $\left(0,\eta\frac{\sqrt{3}}{2}h,k\tau\right)\in S_T^h\gamma_1$ and $\left(a_1,\eta\frac{\sqrt{3}}{2}h,k\tau\right)\in S_T^h\gamma_3$ of each line segment

$$\left[\left(x_1,\eta\frac{\sqrt{3}}{2}h,k\tau\right):0\leq x_1\leq a_1, 0\leq x_2=\eta\frac{\sqrt{3}}{2}h\leq a_2,\ 0\leq t=k\tau\leq T\right],$$

difference formulae (2.92) and (2.93) give the second order approximation of $\frac{\partial u}{\partial x_1}$, respectively. From the truncation error formula (see Burden and Faires [63]) it follows that

$$\max_{S_T^h\gamma_1\cup S_T^h\gamma_3}\left|p_{ih}^{2^{nd}}\left(u\right)-p_i\right|\leq\frac{h^2}{3}\max_{\overline{Q}_T}\left|\frac{\partial^3 u}{\partial x_1^3}\right|,\ i=1,3 \text{ if } P_0\in D^{0h}\gamma_\tau. \tag{2.111}$$

Analogously,

$$\max_{S_T^h\gamma_1\cup S_T^h\gamma_3}\left|p_{ih}^{2^{nd}}\left(u\right)-p_i\right|\leq\frac{h^2}{8}\max_{\overline{Q}_T}\left|\frac{\partial^3 u}{\partial x_1^3}\right|,\ i=1,3 \text{ if } P_0\in D^{*h}\gamma_\tau. \tag{2.112}$$

Using Lemma 2.3 and the estimations (2.111) and (2.112) follows (2.110). □

The numerical solution of BVP$\left(\frac{\partial u}{\partial x_1}\right)$ using hexagonal grids is developed as:

Stage 2$\left(H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)\right)$

$$\Theta_{h,\tau} v_{h,\tau}^{k+1} = \Lambda_{h,\tau} v_{h,\tau}^{k} + D_{x_1}\psi \text{ on } D^{0h}\gamma_\tau, \tag{2.113}$$

$$\Theta_{h,\tau}^{*} v_{h,\tau}^{k+1} = \Lambda_{h,\tau}^{*} v_{h,\tau}^{k} + \Gamma_{h,\tau}^{*} p_{1h}^{2^{nd}} + D_{x_1}\psi^{*} \text{ on } D^{*lh}\gamma_\tau, \tag{2.114}$$

$$\Theta_{h,\tau}^{*} v_{h,\tau}^{k+1} = \Lambda_{h,\tau}^{*} v_{h,\tau}^{k} + \Gamma_{h,\tau}^{*} p_{3h}^{2^{nd}} + D_{x_1}\psi^{*} \text{ on } D^{*rh}\gamma_\tau, \tag{2.115}$$

$$v_{h,\tau} = p_{ih}^{2^{nd}}\left(u_{h,\tau}\right) \text{ on } S_T^h\gamma_i, i = 1,3, \tag{2.116}$$

$$v_{h,\tau} = p_{ih} \text{ on } S_T^h\gamma_i, i = 2,4,5, \tag{2.117}$$

where $p_{1h}^{2^{nd}}, p_{3h}^{2^{nd}}$, and $p_{ih}, i = 2,4,5$ are defined by (2.92)-(2.95) and the operators $\Theta_{h,\tau}$, $\Lambda_{h,\tau}, \Theta_{h,\tau}^{*}, \Lambda_{h,\tau}^{*}$ and $\Gamma_{h,\tau}^{*}$ are the operators given in (2.21)-(2.25), respectively. Additionally,

$$D_{x_1}\psi = \partial_{x_1} f_{P_0}^{k+\frac{1}{2}}, \tag{2.118}$$

$$D_{x_1}\psi^{*} = \partial_{x_1} f_{P_0}^{k+\frac{1}{2}} - \frac{1}{6}\partial_{x_1} f_{P_A}^{k+\frac{1}{2}}, \tag{2.119}$$

Let

$$\varepsilon_{h,\tau}^{v} = v_{h,\tau} - v \text{ on } \overline{D^h\gamma_\tau}, \tag{2.120}$$

where $v = \frac{\partial u}{\partial x_1}$. From (2.113)-(2.117) and (2.120), we have

$$\Theta_{h,\tau}\varepsilon_{h,\tau}^{v,k+1} = \Lambda_{h,\tau}\varepsilon_{h,\tau}^{v,k} + \Psi_1^{v,k} \text{ on } D^{0h}\gamma_\tau, \tag{2.121}$$

$$\Theta_{h,\tau}^{*}\varepsilon_{h,\tau}^{v,k+1} = \Lambda_{h,\tau}^{*}\varepsilon_{h,\tau}^{v,k} + \Gamma_{h,\tau}^{*}\varepsilon_{h,\tau}^{*v} + \Psi_2^{v,k} \text{ on } D^{*h}\gamma_\tau, \tag{2.122}$$

$$\varepsilon_{h,\tau}^{v} = 0 \text{ on } S_T^h\gamma_i, i = 2,4,5, \tag{2.123}$$

$$\varepsilon_{h,\tau}^{v} = \varepsilon_{h,\tau}^{*v} = p_{ih}^{2^{nd}}\left(u_{h,\tau}\right) - p_i \text{ on } S_T^h\gamma_i,\ i = 1,3, \tag{2.124}$$

where

$$\Psi_1^{v,k} = \Lambda_{h,\tau} v^{k} - \Theta_{h,\tau} v^{k+1} + D_{x_1}\psi, \tag{2.125}$$

$$\Psi_2^{v,k} = \Lambda_{h,\tau}^{*} v^{k} - \Theta_{h,\tau}^{*} v^{k+1} + \Gamma_{h,\tau}^{*} p_i + D_{x_1}\psi^{*},\ i = 1,3. \tag{2.126}$$

Let

$$\theta_1 = \max\left\{ \max_{\overline{Q}_T}\left|\frac{\partial^4 v}{\partial x_1^4}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 v}{\partial x_2^4}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 v}{\partial x_1^2 \partial x_2^2}\right| \right\},$$

$$\sigma_1 = \max\left\{ \max_{\overline{Q}_T}\left|\frac{\partial^3 v}{\partial t^3}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 v}{\partial x_2^2 \partial t^2}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 v}{\partial x_1^2 \partial t^2}\right| \right\},$$

and

$$\theta = \max\left\{ \theta_1, \frac{40M_1}{3} + 12d\omega\alpha^* \right\}, \tag{2.127}$$

$$\sigma = \max\left\{ \sigma_1, 3d\beta^* \right\}, \tag{2.128}$$

where α^*, β^* are as given in (2.58), (2.59), respectively, and M_1 is as given in (2.110).

Theorem 2.2: (Buranay et al. [51]) In Stage $2(\frac{\partial u}{\partial x_1})$ the given implicit scheme is unconditionally stable.

Proof. Writing the algebraic linear system of equations (2.113)-(2.117) in matrix form

$$A\tilde{v}^{k+1} = B\tilde{v}^k + \tau q_v^k, \tag{2.129}$$

$k = 0, 1, ..., M' - 1$, where A and B are the matrices given in (2.27) and $\tilde{v}^k, q_v^k \in R^N$ and from assumption that v which is exact solution of the BVP$\left(\frac{\partial u}{\partial x_1}\right)$ belongs to $C_{x,t}^{6+\alpha, 3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$ and by using Lemma 2.1 and induction we get

$$\begin{aligned} \left\|\tilde{v}^{k+1}\right\|_2 &\leq \left\|A^{-1}B\right\|_2 \left\|\tilde{v}^k\right\|_2 + \tau \left\|A^{-1}\right\|_2 \left\|q_v^k\right\|_2 \\ &\leq \left\|\tilde{v}^0\right\|_2 + \tau \sum_{k'=0}^{k} \left\|q_v^{k'}\right\|_2 . \end{aligned} \tag{2.130}$$

Thus, Lax and Richtmyer sufficient condition for stability given in Theorem 1 of [54] is satisfied and the scheme is unconditionally stable. □

Theorem 2.3: (Buranay et al. [51]) The solution $v_{h,\tau}$ of the finite difference problem given in Stage $2\left(H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)\right)$ satisfies

$$\max_{D^h \gamma_\tau} \left|v_{h,\tau} - v\right| \leq \frac{\sigma}{12}(1 + 6\omega)(T + 1)\tau^2 + \frac{3\theta}{40}h^2\left(1 + a_1^2 + a_2^2\right), \tag{2.131}$$

for $r = \frac{\omega\tau}{h^2} \leq \frac{3}{7}$ where θ, σ are as given in (2.127), (2.128), respectively, and $v = \frac{\partial u}{\partial x_1}$ is the exact solution of BVP$\left(\frac{\partial u}{\partial x_1}\right)$.

Proof. Let

$$\Theta_{h,\tau}\widehat{\varepsilon}_{h,\tau}^{v,k+1} = \Lambda_{h,\tau}\widehat{\varepsilon}_{h,\tau}^{v,k} + \Omega_2(x_1) \text{ on } D^{0h}\gamma_\tau, \tag{2.132}$$

$$\Theta_{h,\tau}^{*}\widehat{\varepsilon}_{h,\tau}^{v,k+1} = \Lambda_{h,\tau}^{*}\widehat{\varepsilon}_{h,\tau}^{v,k} + \Gamma_{h,\tau}^{*}\widehat{\varepsilon}_{h,\tau}^{v*} + \Omega_2(x_1) - \frac{1}{6}\,\Omega_2(\widehat{p}) \text{ on } D^{*h}\gamma_\tau, \tag{2.133}$$

$$\widehat{\varepsilon}_{h,\tau}^{v} = 0 \text{ on } S_T^h\gamma_i, i = 2,4,5, \tag{2.134}$$

$$\widehat{\varepsilon}_{h,\tau}^{v} = \widehat{\varepsilon}_{h,\tau}^{v*} = p_{ih}^{2^{nd}}\left(u_{h,\tau}\right) - p_i \text{ on } S_T^h\gamma_i, i = 1,3, \tag{2.135}$$

where

$$\begin{aligned}\Omega_2(x_1) &= \frac{\sigma}{24a_1}(1+6\omega)\tau^2(2a_1 - x_1) + \frac{3\theta\omega}{10}h^2,\\ &\geq \frac{\sigma}{24}(1+6\omega)\tau^2 + \frac{3\theta\omega}{10}h^2 \geq \left|\Psi_1^{v,k}\right|,\end{aligned} \tag{2.136}$$

$$\begin{aligned}\Omega_2(x_1) - \frac{1}{6}\Omega_2(\widehat{p}) &= \begin{cases} (1+6\omega)\tau^2\left(\frac{5}{72} - \frac{h}{48a_1}\right) + \frac{\theta\omega}{4}h^2 \text{ if } P_0 \in D^{*lh}\gamma_\tau \\ (1+6\omega)\tau^2\left(\frac{5}{144} + \frac{h}{48a_1}\right) + \frac{\theta\omega}{4}h^2 \text{ if } P_0 \in D^{*rh}\gamma_\tau \end{cases}\\ &\geq \left|\Psi_2^{v,k}\right|,\end{aligned} \tag{2.137}$$

and $x_1 = \frac{h}{2}$ and $\widehat{p} = 0$ if $P_0 \in D^{*lh}\gamma_\tau$ and $x_1 = a_1 - \frac{h}{2}, \widehat{p} = a_1$ if $P_0 \in D^{*rh}\gamma_\tau$. We take the majorant function

$$\bar{\varepsilon}^{v}(x_1,x_2,t) = \bar{\varepsilon}_1^{v}(x_1,x_2,t) + \bar{\varepsilon}_2^{v}(x_1,x_2,t), \tag{2.138}$$

where

$$\bar{\varepsilon}_1^{v}(x_1,x_2,t) = \frac{\sigma\tau^2}{24a_1}(1+6\omega)(t+1)(2a_1 - x_1) \geq 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{2.139}$$

$$\bar{\varepsilon}_2^{v}(x_1,x_2,t) = \frac{3\theta}{40}h^2\left(1 + a_1^2 + a_2^2 - x_1^2 - x_2^2\right) \geq 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{2.140}$$

The function in (2.138) satisfies the difference problem

$$\Theta_{h,\tau}\bar{\varepsilon}_{h,\tau}^{v,k+1} = \Lambda_{h,\tau}\bar{\varepsilon}_{h,\tau}^{v,k} + \Omega_2(x_1) \text{ on } D^{0h}\gamma_\tau, \tag{2.141}$$

$$\Theta_{h,\tau}^{*}\bar{\varepsilon}_{h,\tau}^{v,k+1} = \Lambda_{h,\tau}^{*}\bar{\varepsilon}_{h,\tau}^{v,k} + \Gamma_{h,\tau}^{*}\bar{\varepsilon}_{h,\tau}^{v*} + \Omega_2(x_1) - \frac{1}{6}\Omega_2(\widehat{p}) \text{ on } D^{*h}\gamma_\tau, \tag{2.142}$$

$$\bar{\varepsilon}_{h,\tau}^{v} = \bar{\varepsilon}_{h,\tau}^{v*} = \bar{\varepsilon}_1^{v}(0,x_2,t) + \bar{\varepsilon}_2^{v}(0,x_2,t) \text{ on } S_T^h\gamma_1, \tag{2.143}$$

$$\bar{\varepsilon}_{h,\tau}^{v} = \bar{\varepsilon}_1^{v}(x_1,0,t) + \bar{\varepsilon}_2^{v}(x_1,0,t) \text{ on } S_T^h\gamma_2, \tag{2.144}$$

$$\bar{\varepsilon}_{h,\tau}^{v} = \bar{\varepsilon}_{h,\tau}^{v*} = \bar{\varepsilon}_1^{v}(a_1,x_2,t) + \bar{\varepsilon}_2^{v}(a_1,x_2,t) \text{ on } S_T^h\gamma_3, \tag{2.145}$$

$$\bar{\varepsilon}_{h,\tau}^{v} = \bar{\varepsilon}_1^{v}(x_1,a_2,t) + \bar{\varepsilon}_2^{v}(x_1,a_2,t) \text{ on } S_T^h\gamma_{4,} \tag{2.146}$$

$$\bar{\varepsilon}_{h,\tau}^{v} = \bar{\varepsilon}_1^{v}(x_1,x_2,0) + \bar{\varepsilon}_2^{v}(x_1,x_2,0) \text{ on } S_T^h\gamma_5. \tag{2.147}$$

In accordance, the following are used to establish the equations (2.141) and (2.142).

$$\begin{aligned}
\Theta_{h,\tau}\bar{\varepsilon}_{1,h,\tau}^{v,k+1} &= \frac{\sigma}{24a_1}(1+6\omega)(t+\tau+1)\tau^2\Bigg[\left(\frac{1}{\tau}+\frac{2\omega}{h^2}\right)(2a_1-x_1) \\
&\quad - \frac{\omega}{3h^2}\Bigg(2a_1-\left(x_1+\frac{h}{2}\right)+2a_1-\left(x_1-\frac{h}{2}\right)+2a_1-(x_1-h) \\
&\quad + 2a_1-\left(x_1-\frac{h}{2}\right)+2a_1-\left(x_1+\frac{h}{2}\right)+2a_1-(x_1+h)\Bigg)\Bigg] \\
&= \frac{\sigma}{24a_1}(1+6\omega)(t+\tau+1)\tau^2\left[\left(\frac{1}{\tau}+\frac{2\omega}{h^2}\right)(2a_1-x_1)\right],
\end{aligned} \tag{2.148}$$

$$\begin{aligned}
\Theta_{h,\tau}\bar{\varepsilon}_{2,h,\tau}^{v,k+1} &= \frac{3\theta h^2}{40}\Bigg[\left(\frac{1}{\tau}+\frac{2\omega}{h^2}\right)(a_1^2+a_2^2+1-x_1^2-x_2^2)-\frac{\omega}{3h^2}\Bigg(a_1^2+a_2^2+1 \\
&\quad -(x_1+\frac{h}{2})^2-(x_2-\frac{\sqrt{3}h}{2})^2+a_1^2+a_2^2+1-(x_1-\frac{h}{2})^2 \\
&\quad -(x_2+\frac{\sqrt{3}h}{2})^2+a_1^2+a_2^2+1-(x_1-h)^2-x_2^2+a_1^2+a_2^2+1 \\
&\quad -(x_1-\frac{h}{2})^2-(x_2-\frac{\sqrt{3}h}{2})^2+a_1^2+a_2^2+1-(x_1+\frac{h}{2})^2 \\
&\quad -(x_2-\frac{\sqrt{3}h}{2})^2+a_1^2+a_2^2+1-(x_1+h)^2-x_2^2\Bigg)\Bigg] \\
&= \frac{3\theta h^2}{40}\left[\frac{1}{\tau}(a_1^2+a_2^2+1-x_1^2-x_2^2)+2\omega\right].
\end{aligned} \tag{2.149}$$

Using (2.148) and (2.149) gives,

$$\Theta_{h,\tau}\bar{\varepsilon}_{h,\tau}^{v,k+1} = \Theta_{h,\tau}\bar{\varepsilon}_{1,h,\tau}^{v,k+1} + \Theta_{h,\tau}\bar{\varepsilon}_{2,h,\tau}^{v,k+1}$$
$$= \frac{\sigma}{24a_1}(1+6\omega)(t+\tau+1)\tau^2\left[\left(\frac{1}{\tau}-\frac{2\omega}{h^2}\right)(2a_1-x_1)\right]$$
$$+\frac{3\theta h^2}{40}\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)+2\omega\right], \tag{2.150}$$

$$\Lambda_{h,\tau}\bar{\varepsilon}_{1,h,\tau}^{v,k} = \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2\left[\left(\frac{1}{\tau}-\frac{2\omega}{h^2}\right)(2a_1-x_1)-\frac{\omega}{3h^2}(12a_1-6x_1)\right]$$
$$= \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2\left[\frac{1}{\tau}(2a_1-x_1)\right], \tag{2.151}$$
$$\Lambda_{h,\tau}\bar{\varepsilon}_{2,h,\tau}^{v,k} = \frac{3\theta h^2}{40}\left[\left(\frac{1}{\tau}+\frac{2\omega}{h^2}\right)\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)\right.$$
$$\left.+\frac{\omega}{3h^2}\left(6a_1^2+6a_2^2+6-6x_1^2-6x_2^2-6h^2\right)\right]$$
$$= \frac{3\theta h^2}{40}\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)-2\omega\right]. \tag{2.152}$$

Adding (2.151) and (2.152) yields

$$\Lambda_{h,\tau}\bar{\varepsilon}_{h,\tau}^{v,k} = \Lambda_{h,\tau}\bar{\varepsilon}_{1,h,\tau}^{v,k} + \Lambda_{h,\tau}\bar{\varepsilon}_{2,h,\tau}^{v,k}$$

$$= \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2\left[\frac{1}{\tau}(2a_1-x_1)\right]$$
$$+\frac{3\theta\omega h^2}{40}\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)-2\omega\right]. \tag{2.153}$$

Now using (2.150) and (2.153) it follows that

$$\Theta_{h,\tau}\bar{\varepsilon}_{h,\tau}^{v,k+1} - \Lambda_{h,\tau}\bar{\varepsilon}_{h,\tau}^{v,k} = \Omega_2(x_1)$$
$$= \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2(2a_1-x_1)+\frac{3\theta\omega^2h^2}{10}.$$

Subsequently we show that equation (2.142) hold true as follows:

$$\Theta_{h,\tau}^{*}\bar{\varepsilon}_{1,h,\tau}^{v,k+1} = \frac{\sigma}{24a_1}(1+6\omega)(t+\tau+1)\tau^2\left[\frac{1}{\tau}(2a_1-x_1)\right.$$
$$\left.+\frac{8\omega a_1}{3h^2}-\frac{4\omega x_1}{3h^2}+\frac{2\omega}{3h}\right] \tag{2.154}$$

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{v,k+1}_{2,h,\tau} = \frac{3\theta h^2}{40}\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)+\frac{4\omega a_1^2}{3h^2}+\frac{4\omega a_2^2}{3h^2}\right.$$
$$\left.+\frac{4\omega}{3h^2}-\frac{4\omega x_1^2}{3h^2}-\frac{4\omega x_2^2}{3h^2}+\frac{4\omega x_1}{3h}+\omega\right] \tag{2.155}$$

Adding (2.154) and (2.155) we get

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{v,k+1}_{h,\tau} = \Theta^*_{h,\tau}\bar{\varepsilon}^{v,k+1}_{1,h,\tau}+\Theta^*_{h,\tau}\bar{\varepsilon}^{v,k+1}_{2,h,\tau}$$
$$= \frac{\sigma}{24a_1}(1+6\omega)(t+\tau+1)\tau^2\left[\frac{1}{\tau}(2a_1-x_1)\right.$$
$$\left.+\frac{8\omega a_1}{3h^2}-\frac{4\omega x_1}{3h^2}+\frac{2\omega}{3h}\right]$$
$$+\frac{3\theta h^2}{40}\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)+\frac{4\omega a_1^2}{3h^2}+\frac{4\omega a_2^2}{3h^2}\right.$$
$$\left.+\frac{4\omega}{3h^2}-\frac{4\omega x_1^2}{3h^2}-\frac{4\omega x_2^2}{3h^2}+\frac{4\omega x_1}{3h}+\omega\right] \tag{2.156}$$

$$\Lambda^*_{h,\tau}\bar{\varepsilon}^{v,k}_{1,h,\tau} = \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2\left[\frac{1}{\tau}(2a_1-x_1)-\frac{8\omega a_1}{3h^2}\right.$$
$$\left.+\frac{4\omega x_1}{3h^2}-\frac{2\omega}{3h}\right] \tag{2.157}$$

$$\Lambda^*_{h,\tau}\bar{\varepsilon}^{v,k}_{2,h,\tau} = \frac{3\theta h^2}{40}\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)-\frac{4\omega a_1^2}{3h^2}-\frac{4\omega a_2^2}{3h^2}\right.$$
$$\left.-\frac{4\omega}{3h^2}+\frac{4\omega x_1^2}{3h^2}+\frac{4\omega x_2^2}{3h^2}-\frac{4\omega x_1}{3h}-\omega\right]. \tag{2.158}$$

Adding (2.157) and (2.158) gives

$$\Lambda^*_{h,\tau}\bar{\varepsilon}^{v,k}_{h,\tau} = \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2\left[\frac{1}{\tau}(2a_1-x_1)-\frac{8\omega a_1}{3h^2}\right.$$
$$\left.+\frac{4\omega x_1}{3h^2}-\frac{2\omega}{3h}\right]+\frac{3\theta h^2}{40}\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)\right.$$
$$\left.-\frac{4\omega a_1^2}{3h^2}-\frac{4\omega a_2^2}{3h^2}-\frac{4\omega}{3h^2}+\frac{4\omega x_1^2}{3h^2}+\frac{4\omega x_2^2}{3h^2}-\frac{4\omega x_1}{3h}-\omega\right], \tag{2.159}$$

$$\Gamma^*_{h,\tau}\bar{\varepsilon}^{v*}_{1,h,\tau} = \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2\left[\frac{16\omega a_1}{3h^2}\right]$$
$$+\frac{\sigma}{24a_1}(1+6\omega)\tau^3\left[\frac{8\omega a_1}{3h^2}+\frac{2a_1}{6\tau}\right], \tag{2.160}$$

$$\Gamma^*_{h,\tau}\bar{\varepsilon}^{\nu*}_{2,h,\tau} = \frac{3\theta h^2}{40}\left[\frac{8\omega}{3h^2}+\frac{8\omega a_1}{3h^2}+\frac{8\omega a_2}{3h^2}-\frac{8\omega x_2^2}{3h^2}-\frac{2\omega}{3}\right]. \qquad (2.161)$$

From (2.160) and (2.161) we get:

$$\begin{aligned}\Gamma^*_{h,\tau}\bar{\varepsilon}^{\nu*}_{h,\tau} &= \frac{\sigma}{24a_1}(1+6\omega)(t+1)\tau^2\left[\frac{16\omega a_1}{3h^2}\right]\\ &+\frac{\sigma}{24a_1}(1+6\omega)\tau^3\left[\frac{8\omega a_1}{3h^2}+\frac{2a_1}{6\tau}\right],\\ &+\frac{3\theta h^2}{40}\left[\frac{8\omega}{3h^2}+\frac{8\omega a_1}{3h^2}+\frac{8\omega a_2}{3h^2}-\frac{8\omega x_2^2}{3h^2}-\frac{2\omega}{3}\right], \qquad (2.162)\end{aligned}$$

By using (2.156), (2.159) and (2.162) we obtain

$$\Theta^*_{h,\tau}\bar{\varepsilon}^{\nu,k+1}_{h,\tau}-\Lambda^*_{h,\tau}\bar{\varepsilon}^{\nu,k}_{h,\tau}-\Gamma^*_{h,\tau}\bar{\varepsilon}^{\nu*}_{h,\tau} = \Omega_2(x_1)-\frac{1}{6}\Omega_2(\hat{p}), \qquad (2.163)$$

where the right side of (2.163) is as given in (2.137).

The algebraic system of equations (2.132)-(2.135) and (2.141)-(2.147) can be written in matrix form as

$$A\widehat{\varepsilon}^{\nu,k+1} = B\widehat{\varepsilon}^{\nu,k}+\tau\widehat{e}^{\nu,k}, \qquad (2.164)$$

$$A\bar{\varepsilon}^{\nu,k+1} = B\bar{\varepsilon}^{\nu,k}+\tau\bar{e}^{\nu,k}, \qquad (2.165)$$

respectively, for $k=0,...,M'-1$, where A,B are matrices as given in (2.27) and $\widehat{\varepsilon}^{\nu,k},\bar{\varepsilon}^{\nu,k},\widehat{e}^{\nu,k},\bar{e}^{\nu,k}\in R^N$. Using (2.136)-(2.147), we have $\bar{\varepsilon}^{\nu,0}\geq 0$, and $\bar{e}^{\nu,k}\geq 0$, and $\left|\widehat{e}^{\nu,k}\right|\leq\bar{e}^{\nu,k}$ for $k=0,...,M'-1$, and $\left|\widehat{\varepsilon}^{\nu,0}\right|\leq\bar{\varepsilon}^{\nu,0}$. Then, on the basis of Lemma 2.2, we get $\left|\widehat{\varepsilon}^{\nu,k+1}\right|\leq\bar{\varepsilon}^{\nu,k+1}$ for $k=0,...,M'-1$. From

$$\begin{aligned}\bar{\varepsilon}^{\nu}(x_1,x_2,t) &\leq \bar{\varepsilon}^{\nu}(0,0,T)\\ &= \frac{\sigma}{12}(1+6\omega)(T+1)\tau^2+\frac{3\theta}{40}h^2\left(1+a_1^2+a_2^2\right),\end{aligned}$$

and using (2.136) and (2.137) follows (2.131). □

2.3 Difference Problem Approximating $\frac{\partial u}{\partial x_2}$ on Hexagonal Grids with $O\left(h^2+\tau^2\right)$ Order of Accuracy

Additionally the notations $\partial_{x_2} f_{P_0}^{k+\frac{1}{2}} = \left.\frac{\partial f}{\partial x_2}\right|_{(x_1,x_2,t+\frac{\tau}{2})}$ and $\partial_{x_2} f_{P_A}^{k+\frac{1}{2}} = \left.\frac{\partial f}{\partial x_2}\right|_{(\widehat{p},x_2,t+\frac{\tau}{2})}$ and also $q_i = \frac{\partial u}{\partial x_2}$ on $S_T\gamma_i, i = 1,2,...,5$ are introduced. Let the problem BVP(u) be given, then we develop the next boundary value problem for $z = \frac{\partial u}{\partial x_2}$.

Boundary Value Problem for $z = \frac{\partial u}{\partial x_2}$ $\left(\mathbf{BVP}\left(\frac{\partial u}{\partial x_2}\right)\right)$

$$Lz = \frac{\partial f(x_1,x_2,t)}{\partial x_2} \text{ on } Q_T,$$

$$z(x_1,x_2,t) = q_i \text{ on } S_T\gamma_i, i = 1,2,...,5. \tag{2.166}$$

where the operator L is defined in (2.91) and $f(x_1,x_2,t)$ is the given function in (2.9). We assume that the solution $z \in C_{x,t}^{6+\alpha,3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$ and take

$$q_{2h}^{2nd} = \frac{1}{2\sqrt{3}h}\left(-3u(x_1,0,t)+4u_{h,\tau}\left(x_1,\sqrt{3}h,t\right)\right.$$
$$\left.-u_{h,\tau}\left(x_1,2\sqrt{3}h,t\right)\right) \text{ on } S_T^h\gamma_2, \tag{2.167}$$

$$q_{4h}^{2nd} = \frac{1}{2\sqrt{3}h}\left(3u(x_1,a_2,t)-4u_{h,\tau}\left(x_1,a_2-\sqrt{3}h,t\right)\right.$$
$$\left.+u_{h,\tau}\left(x_1,a_2-2\sqrt{3}h,t\right)\right) \text{ on } S_T^h\gamma_4, \tag{2.168}$$

$$q_{ih} = \frac{\partial \phi(x_1,x_2,t)}{\partial x_2} \text{ on } S_T^h\gamma_i, i = 1,3, \tag{2.169}$$

$$q_{5h} = \frac{\partial \varphi(x_1,x_2)}{\partial x_2} \text{ on } S_T^h\gamma_5, \tag{2.170}$$

where, the solution of the difference problem in Stage 1$\left(H^{2nd}(u)\right)$ is $u_{h,\tau}$ and $\varphi(x_1,x_2)$, $\phi(x_1,x_2,t)$ are as given in (2.9). We give the derivation of the formulae (2.167) and (2.168) as follows:

$A: u(x_1,x_2,t)$

$B: u(x_1,x_2+\sqrt{3}h,t)$

$C: u(x_1,x_2+2\sqrt{3}h,t)$

$$
\begin{aligned}
B: u\left(x_1, x_2+\sqrt{3}h, t\right) &= u(x_1,x_2,t)+\sqrt{3}h\partial_{x_2}u(x_1,x_2,t)\\
&+\frac{3}{2}h^2\partial_{x_2}^2u(x_1,x_2,t)\\
&+\frac{\sqrt{3}}{2}h^3\partial_{x_2}^3u(x_1,x_2+\xi_1h,t), \qquad (2.171)\\
C: u\left(x_1, x_2+2\sqrt{3}h, t\right) &= u(x_1,x_2,t)+2\sqrt{3}h\partial_{x_2}u(x_1,x_2,t)\\
&+6h^2\partial_{x_2}^2u(x_1,x_2,t)\\
&+4\sqrt{3}h^3\partial_{x_2}^3u(x_1,x_2+\xi_2h,t), \qquad (2.172)
\end{aligned}
$$

where, $0<\xi_1<\sqrt{3}$ and $0<\xi_2<2\sqrt{3}$. From (2.171) and (2.172) we get

$$
\begin{aligned}
&-4u\left(x_1,x_2+\sqrt{3}h,t\right)+u\left(x_1,x_2+2\sqrt{3}h,t\right)+3u(x_1,x_2,t)\\
&=-2\sqrt{3}h\partial_{x_2}u(x_1,x_2,t)+2\sqrt{3}h^3\partial_{x_2}^3u(x_1,\overline{x_2},t),\ x_2<\overline{x_2}<x_2+2\sqrt{3}h, \qquad (2.173)
\end{aligned}
$$

giving

$$
\begin{aligned}
&\frac{1}{2\sqrt{3}h}\left(-3u(x_1,x_2,t)+4u\left(x_1,x_2+\sqrt{3}h,t\right)-\left(x_1,x_2+2\sqrt{3}h,t\right)\right)\\
&=\partial_{x_2}u(x_1,x_2,t)+O\left(h^2\right). \qquad (2.174)
\end{aligned}
$$

Further, the validation of the backward difference scheme follows from

$A: u(x_1,x_2,t)$

$B: u(x_1,x_2-\sqrt{3}h,t)$

$C: (x_1,x_2-2\sqrt{3}h,t)$

$$
\begin{aligned}
B: u\left(x_1, x_2-\sqrt{3}h, t\right) &= u(x_1,x_2,t)-\sqrt{3}h\partial_{x_2}u(x_1,x_2,t)\\
&+\frac{3}{2}h^2\partial_{x_2}^2u(x_1,x_2,t)\\
&-\frac{3\sqrt{3}}{6}h^3\partial_{x_2}^3u\left(x_1,x_2+\widetilde{\xi}_1h,t\right), \qquad (2.175)\\
C: u\left(x_1, x_2-2\sqrt{3}h, t\right) &= u(x_1,x_2,t)-2\sqrt{3}h\partial_{x_2}u(x_1,x_2,t)\\
&+6h^2\partial_{x_2}^2u(x_1,x_2,t)\\
&-4\sqrt{3}h^3\partial_{x_2}^3u\left(x_1,x_2+\widetilde{\xi}_2h,t\right), \qquad (2.176)
\end{aligned}
$$

where, $-\sqrt{3} < \widetilde{\xi}_1 < 0$ and $-2\sqrt{3} < \widetilde{\xi}_2 < 0$. From (2.175) and (2.176) we get

$$-4u\left(x_1, x_2 - \sqrt{3}h, t\right) + u\left(x_1, x_2 - 2\sqrt{3}h, t\right) + 3u\left(x_1, x_2, t\right)$$
$$= 2\sqrt{3}hu_{x_2}\left(x_1, x_2, t\right) - 2\sqrt{3}h^3\partial^3_{x_2}u\left(x_1, \widetilde{x}_2, t\right),\ x_2 - 2\sqrt{3}h < \widetilde{x}_2 < x_2, \tag{2.177}$$

and

$$\frac{1}{2\sqrt{3}h}\left(3u\left(x_1, x_2, t\right) - 4u\left(x_1, x_2 - \sqrt{3}h, t\right) + u\left(x_1, x_2 - 2\sqrt{3}h, t\right)\right)$$
$$= \partial_{x_2}u\left(x_1, x_2, t\right) + O\left(h^2\right). \tag{2.178}$$

Lemma 2.5: (Buranay et al. [51]) The following inequality holds

$$\left|q^{2^{nd}}_{ih}\left(u_{h,\tau}\right) - q^{2^{nd}}_{ih}\left(u\right)\right| \leq 3d\Omega_1\left(h, \tau\right),\ \ i = 2, 4, \tag{2.179}$$

for $r = \frac{\omega\tau}{h^2} \leq \frac{3}{7}$, where u is the solution of the boundary value problem BVP(u) and $u_{h,\tau}$ is the solution of the difference problem (2.15)-(2.18) in Stage $1\left(H^{2nd}\left(u\right)\right)$ and $\Omega_1\left(h, \tau\right)$ is as in (2.57) and d is presented in (2.60).

Proof. Taking into consideration Theorem 2.1, and using (2.56), (2.167), and (2.168), we have

$$\left|q^{2^{nd}}_{ih}\left(u_{h,\tau}\right) - q^{2^{nd}}_{ih}\left(u\right)\right| \leq \frac{1}{2\sqrt{3}h}\left(4\sqrt{3}hd\Omega_1\left(h, \tau\right) + 2\sqrt{3}hd\Omega_1\left(h, \tau\right)\right)$$
$$\leq 3d\Omega_1\left(h, \tau\right),\ \ i = 2, 4, \tag{2.180}$$

thus, we obtain (2.179). □

Lemma 2.6: (Buranay et al. [51]) The following inequality is true

$$\max_{S^h_T\gamma_2 \cup S^h_T\gamma_4}\left|q^{2^{nd}}_{ih}\left(u_{h,\tau}\right) - q_i\right| \leq M_2h^2 + 3d\Omega_1\left(h, \tau\right),\ i = 2, 4, \tag{2.181}$$

for $r = \frac{\omega\tau}{h^2} \leq \frac{3}{7}$, where $M_2 = \max_{\overline{Q}_T}\left|\frac{\partial^3 u}{\partial x_2^3}\right|$ and $u_{h,\tau}$ is the solution of the difference problem in Stage $1\left(H^{2nd}(u)\right)$ and $\Omega_1\left(h, \tau\right)$ and d are as given in (2.57) and (2.60), respectively.

Proof. Since the exact solution $u \in C^{7+\alpha, \frac{7+\alpha}{2}}_{x,t}\left(\overline{Q}_T\right)$, at the end points $\left(\vartheta h, 0, k\tau\right) \in S^h_T\gamma_2$

and $(\vartheta h, a_2, k\tau) \in S_T^h \gamma_4$ of each line segment

$$[(\vartheta h, x_2, k\tau) : 0 \leq x_1 = \vartheta h \leq a_1, 0 \leq x_2 \leq a_2,\ 0 \leq t = k\tau \leq T],$$

difference formulas (2.167) and (2.168) give the second order approximation of $\frac{\partial u}{\partial x_2}$, respectively. From the truncation error formula (see [63]), it follows that

$$\max_{S_T^h \gamma_2 \cup S_T^h \gamma_4} \left| q_{ih}^{2^{nd}}(u) - q_i \right| \leq h^2 \max_{\overline{Q}_T} \left| \frac{\partial^3 u}{\partial x_2^3} \right|,\ i = 2, 4\ . \tag{2.182}$$

Taking $M_2 = \max_{\overline{Q}_T} \left| \frac{\partial^3 u}{\partial x_2^3} \right|$ and using Lemma 2.5 and the estimation (2.180) and (2.182) follows (2.181). □

Subsequently we establish the numerical solution of the BVP$\left(\frac{\partial u}{\partial x_2}\right)$ on hexagonal grids as the second stage by

Stage 2$\left(H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)\right)$

$$\Theta_{h,\tau} z_{h,\tau}^{k+1} = \Lambda_{h,\tau} z_{h,\tau}^{k} + D_{x_2}\psi \text{ on } D^{0h}\gamma_\tau, \tag{2.183}$$

$$\Theta_{h,\tau}^{*} z_{h,\tau}^{k+1} = \Lambda_{h,\tau}^{*} z_{h,\tau}^{k} + \Gamma_{h,\tau}^{*} q_{1h} + D_{x_2}\psi^{*} \text{ on } D^{*lh}\gamma_\tau, \tag{2.184}$$

$$\Theta_{h,\tau}^{*} z_{h,\tau}^{k+1} = \Lambda_{h,\tau}^{*} z_{h,\tau}^{k} + \Gamma_{h,\tau}^{*} q_{3h} + D_{x_2}\psi^{*} \text{ on } D^{*rh}\gamma_\tau, \tag{2.185}$$

$$z_{h,\tau} = q_{ih}^{2^{nd}}\left(u_{h,\tau}\right) \text{ on } S_T^h \gamma_i, i = 2, 4, \tag{2.186}$$

$$z_{h,\tau} = q_{ih} \text{ on } S_T^h \gamma_i, i = 1, 3, 5, \tag{2.187}$$

where $q_{2h}^{2^{nd}}, q_{4h}^{2^{nd}}$, and q_{ih}, $i = 1, 3, 5$ are defined by (2.167)-(2.170) and the operators $\Theta_{h,\tau}$, $\Lambda_{h,\tau}, \Theta_{h,\tau}^{*}, \Lambda_{h,\tau}^{*}$, and $\Gamma_{h,\tau}^{*}$ are the operators given in (2.21)-(2.25), respectively. In addition,

$$D_{x_2}\psi = \partial_{x_2} f_{P_0}^{k+\frac{1}{2}}, \tag{2.188}$$

$$D_{x_2}\psi^{*} = \partial_{x_2} f_{P_0}^{k+\frac{1}{2}} - \frac{1}{6}\partial_{x_2} f_{P_A}^{k+\frac{1}{2}}. \tag{2.189}$$

Let

$$\varepsilon_{h,\tau}^{z} = z_{h,\tau} - z \text{ on } \overline{D^h \gamma_\tau}. \tag{2.190}$$

From (2.183)-(2.187) and (2.190), we have

$$\Theta_{h,\tau}\varepsilon_{h,\tau}^{z,k+1} = \Lambda_{h,\tau}\varepsilon_{h,\tau}^{z,k} + \Psi_1^{z,k} \text{ on } D^{0h}\gamma_\tau, \tag{2.191}$$

$$\Theta_{h,\tau}^{*}\varepsilon_{h,\tau}^{z,k+1} = \Lambda_{h,\tau}^{*}\varepsilon_{h,\tau}^{z,k} + \Psi_2^{z,k} \text{ on } D^{*h}\gamma_\tau, \tag{2.192}$$

$$\varepsilon_{h,\tau}^{z} = 0 \text{ on } S_T^h\gamma_i, i = 1,3,5, \tag{2.193}$$

$$\varepsilon_{h,\tau}^{z} = q_{ih}^{2^{nd}}\left(u_{h,\tau}\right) - q_i \text{ on } S_T^h\gamma_i, i = 2,4, \tag{2.194}$$

where q_{ih} are defined by (2.167)-(2.170) and

$$\Psi_1^{z,k} = \Lambda_{h,\tau}z^k - \Theta_{h,\tau}z^{k+1} + D_{x_2}\psi, \tag{2.195}$$

$$\Psi_2^{z,k} = \Lambda_{h,\tau}^{*}z^k - \Theta_{h,\tau}^{*}z^{k+1} + \Gamma_{h,\tau}^{*}q_i + D_{x_2}\psi^{*}, i = 1,3. \tag{2.196}$$

Let

$$\kappa_1 = \max\left\{\max_{\overline{Q}_T}\left|\frac{\partial^4 z}{\partial x_1^4}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 z}{\partial x_2^4}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 z}{\partial x_1^2\partial x_2^2}\right|\right\}, \tag{2.197}$$

$$\delta_1 = \max\left\{\max_{\overline{Q}_T}\left|\frac{\partial^3 z}{\partial t^3}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 z}{\partial x_2^2\partial t^2}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^4 z}{\partial x_1^2\partial t^2}\right|\right\}, \tag{2.198}$$

and

$$\kappa = \max\left\{\kappa_1, \frac{40M_2}{3} + 12d\omega\alpha^{*}\right\}, \tag{2.199}$$

$$\delta = \max\left\{\delta_1, 3d\beta^{*}\right\}, \tag{2.200}$$

α^{*}, β^{*} are as given in (2.58), (2.59), respectively, and M_2 is the constant given in Lemma 2.6 and z is the solution of BVP$\left(\frac{\partial u}{\partial x_2}\right)$.

Theorem 2.4: (Buranay et al. [51]) In Stage 2$\left(H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)\right)$ the constructed implicit scheme is unconditionally stable.

Proof. The equations (2.183)-(2.187) can be given in matrix form:

$$A\tilde{z}^{k+1} = B\tilde{z}^{k} + \tau q_z^k, \tag{2.201}$$

for $k = 0, 1, ..., M' - 1$, where, A, B are as given in (2.27) and $\tilde{z}^k, q_z^k \in R^N$. Based on the assumption that z belongs to $C_{x,t}^{6+\alpha, 3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$ and using Lemma 2.1 and induction we get

$$\begin{aligned}\left\|\tilde{z}^{k+1}\right\|_2 &\leq \left\|A^{-1}B\right\|_2 \left\|\tilde{z}^{k}\right\|_2 + \tau \left\|A^{-1}\right\|_2 \left\|q_z^k\right\|_2 \\ &\leq \left\|\tilde{z}^{0}\right\|_2 + \tau \sum_{k'=0}^{k} \left\|q_z^{k'}\right\|_2 .\end{aligned} \tag{2.202}$$

Therefore, the scheme is unconditionally stable. □

Theorem 2.5: (Buranay et al. [51]) The solution $z_{h,\tau}$ of the finite difference problem given in Stage 2$\left(H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)\right)$ satisfies

$$\max_{D^h\gamma_\tau} \left|z_{h,\tau} - z\right| \leq \frac{\delta}{12}(1+6\omega)(T+1)\tau^2 + \frac{3\kappa}{40}\left(1 + a_1^2 + a_2^2\right)h^2, \tag{2.203}$$

for $r = \frac{\omega\tau}{h^2} \leq \frac{3}{7}$, where κ, δ are as given in (2.199), (2.200) respectively and $z = \frac{\partial u}{\partial x_2}$ is the exact solution of BVP$\left(\frac{\partial u}{\partial x_2}\right)$.

Proof. Let

$$\Theta_{h,\tau}\widehat{\varepsilon}_{h,\tau}^{z,k+1} = \Lambda_{h,\tau}\widehat{\varepsilon}_{h,\tau}^{z,k} + \Omega_3(x_2) \text{ on } D^{0h}\gamma_\tau, \tag{2.204}$$

$$\Theta_{h,\tau}^{*}\widehat{\varepsilon}_{h,\tau}^{z,k+1} = \Lambda_{h,\tau}^{*}\widehat{\varepsilon}_{h,\tau}^{z,k} + \frac{5}{6}\Omega_3(x_2) \text{ on } D^{*h}\gamma_\tau \tag{2.205}$$

$$\widehat{\varepsilon}_{h,\tau}^{z} = 0 \text{ on } S_T^h\gamma_i, i = 1,3,5, \tag{2.206}$$

$$\widehat{\varepsilon}_{h,\tau}^{z} = q_{ih}^{2^{nd}}\left(u_{h,\tau}\right) - q_i \text{ on } S_T^h\gamma_i, i = 2,4, \tag{2.207}$$

where $q_{2h}^{2^{nd}}, q_{4h}^{2^{nd}}, q_{ih}, i = 1,3,5$, are defined by (2.167)-(2.170) and

$$\begin{aligned}\Omega_3(x_2) &= \frac{\delta}{24a_2}(1+6\omega)\tau^2(2a_2 - x_2) + \frac{3\kappa\omega}{10}h^2 \\ &\geq \frac{\delta}{24}(1+6\omega)\tau^2 + \frac{3\kappa\omega}{10}h^2 \geq \left|\Psi_1^{z,k}\right|,\end{aligned} \tag{2.208}$$

$$\frac{5}{6}\Omega_3(x_2) = \frac{5\delta}{144a_2}(1+6\omega)\tau^2(2a_2 - x_2) + \frac{\kappa\omega}{4}h^2$$
$$\geq \frac{5\delta}{144}(1+6\omega)\tau^2 + \frac{\kappa\omega}{4}h^2 \geq \left|\Psi_2^{z,k}\right|. \tag{2.209}$$

We take the majorant function

$$\bar{\varepsilon}^z(x_1,x_2,t) = \bar{\varepsilon}_1^z(x_1,x_2,t) + \bar{\varepsilon}_2^z(x_1,x_2,t), \tag{2.210}$$

where

$$\bar{\varepsilon}_1^z(x_1,x_2,t) = \frac{\delta}{24a_2}\tau^2(1+6\omega)(t+1)(2a_2 - x_2) \geq 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{2.211}$$
$$\bar{\varepsilon}_2^z(x_1,x_2,t) = \frac{3\kappa}{40}h^2\left(1 + a_1^2 + a_2^2 - x_1^2 - x_2^2\right) \geq 0 \text{ on } \overline{D^h\gamma_\tau}. \tag{2.212}$$

The majorant function in (2.210) satisfies the difference problem

$$\Theta_{h,\tau}\bar{\varepsilon}_{h,\tau}^{z,k+1} = \Lambda_{h,\tau}\bar{\varepsilon}_{h,\tau}^{z,k} + \Omega_3(x_2) \text{ on } D^{0h}\gamma_\tau, \tag{2.213}$$
$$\Theta_{h,\tau}^*\bar{\varepsilon}_{h,\tau}^{z,k+1} = \Lambda_{h,\tau}^*\bar{\varepsilon}_{h,\tau}^{z,k} + \Gamma_{h,\tau}^*\bar{\varepsilon}_{h,\tau}^{z*} + \frac{5}{6}\Omega_3(x_2) \text{ on } D^{*h}\gamma_\tau, \tag{2.214}$$
$$\bar{\varepsilon}_{h,\tau}^z = \bar{\varepsilon}_{h,\tau}^{z*} = \bar{\varepsilon}_1^z(0,x_2,t) + \bar{\varepsilon}_2^z(0,x_2,t) \text{ on } S_T^h\gamma_1, \tag{2.215}$$
$$\bar{\varepsilon}_{h,\tau}^z = \bar{\varepsilon}_1^z(x_1,0,t) + \bar{\varepsilon}_2^z(x_1,0,t) \text{ on } S_T^h\gamma_2, \tag{2.216}$$
$$\bar{\varepsilon}_{h,\tau}^z = \bar{\varepsilon}_{h,\tau}^{z*} = \bar{\varepsilon}_1^z(a_1,x_2,t) + \bar{\varepsilon}_2^z(a_1,x_2,t) \text{ on } S_T^h\gamma_3, \tag{2.217}$$
$$\bar{\varepsilon}_{h,\tau}^z = \bar{\varepsilon}_1^z(x_1,a_2,t) + \bar{\varepsilon}_2^z(x_1,a_2,t) \text{ on } S_T^h\gamma_4, \tag{2.218}$$
$$\bar{\varepsilon}_{h,\tau}^z = \bar{\varepsilon}_1^z(x_1,x_2,0) + \bar{\varepsilon}_2^z(x_1,x_2,0) \text{ on } S_T^h\gamma_5. \tag{2.219}$$

We write the algebraic system of Equations (2.204)-(2.207) and (2.213)-(2.219) for fixed $k \geq 0$ in matrix form

$$A\widehat{\varepsilon}^{z,k+1} = B\widehat{\varepsilon}^{z,k} + \tau\widehat{e}^{z,k}, \tag{2.220}$$

$$A\bar{\varepsilon}^{z,k+1} = B\bar{\varepsilon}^{z,k} + \tau\bar{e}^{z,k}, \tag{2.221}$$

respectively, where A, B are as given in (2.27) and $\widehat{\varepsilon}^{z,k}, \bar{\varepsilon}^{z,k}, \widehat{e}^{z,k}, \bar{e}^{z,k} \in R^N$. Using (2.208)-(2.219), we get $\bar{e}^{z,k} \geq 0$ and $\left|\widehat{e}^{z,k}\right| \leq \bar{e}^{z,k}$ for $k = 0,1,\ldots,M'-1$ and $\bar{\varepsilon}^{z,0} \geq 0$, $\left|\widehat{\varepsilon}^{z,0}\right| \leq \bar{\varepsilon}^{z,0}$. Then, on the basis of Lemma 2.2 follows $\left|\widehat{\varepsilon}^{z,k+1}\right| \leq \bar{\varepsilon}^{z,k+1}$,

$k = 0, 1, ..., M' - 1$. From

$$\begin{aligned} \bar{\varepsilon}^z(x_1, x_2, t) &\leq \bar{\varepsilon}^z(0, 0, T) \\ &= \frac{\delta}{12}(1 + 6\omega)(T + 1)\tau^2 + \frac{3\kappa}{40}\left(1 + a_1^2 + a_2^2\right)h^2, \end{aligned} \tag{2.222}$$

and using (2.208), (2.209) follows (2.203). □

Chapter 3

EXPERIMENTAL INVESTIGATION OF THE SECOND ORDER ACCURATE IMPLICIT METHOD

To show the efficiency of the proposed two-stage implicit method we construct a test problem of which the exact solution is known. Further we take $D = \left\{(x_1,x_2) : 0 < x_1 < 1, 0 < x_2 < \frac{\sqrt{3}}{2}\right\}$, and $t \in [0,1]$. We used Mathematica in machine precision on a personal computer with the properties AMD Ryzen 7 1800X Eight Core Processor 3.60GHz. Moreover, the obtained linear algebraic systems of equations are solved by using incomplete block-matrix factorization of the block tridiagonal stiffness matrices which are symmetric $M-$matrices for the all considered pairs of (h,τ). Then these incomplete block-matrix factorizations are used as preconditioners for the conjugate gradient method as given in Buranay and Iyikal [55] (see also Concus et al. [56] and Axelsson [57]). Additionally, the notations given below are used in tables and figures:

$H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$ denotes the proposed two-stage implicit method on hexagonal grids for the approximation of the derivative $\frac{\partial u}{\partial x_1}$.

$H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$ denotes the proposed two-stage implicit method on hexagonal grids for the approximation of the derivative $\frac{\partial u}{\partial x_2}$.

$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$ presents the Central Processing Unit time in seconds ($CPUs$) per time level for the method $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$.

$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$ presents the Central Processing Unit time in seconds ($CPUs$) per time level for the method $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$.

$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$ shows the total Central Processing Unit time in seconds required for the solution at $t = 1$, by the method $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$.

$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$ shows the total Central Processing Unit time in seconds required for the solution at $t = 1$, by the method $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$.

For the approximation of the derivatives $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}$ we denote the given method by $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$, and $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$, respectively. Additionally, the corresponding solutions are denoted by $v_{2^{-\mu},2^{-\lambda}}$, and $z_{2^{-\mu},2^{-\lambda}}$, respectively, for $h = 2^{-\mu}$ and $\tau = 2^{-\lambda}$ where μ, λ are positive integers. On the grid points $\overline{D^h\gamma_\tau}$, which is the closure of $D^h\gamma_\tau$ we present the error function $\varepsilon_{h,\tau}$ obtained by $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$, and $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$ by $\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$ and by $\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, respectively. Furthermore, on the grid points the maximum errors $\max_{\overline{D^h\gamma_\tau}}\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right|$ and $\max_{\overline{D^h\gamma_\tau}}\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right|$ are presented by $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$ and $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$, accordingly. Further, we denote the order of convergence of the approximate solution $v_{2^{-\mu},2^{-\lambda}}$ to the exact solution $v = \frac{\partial u}{\partial x_1}$ obtained by using the two-stage implicit method $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$ by

$$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)} = \frac{\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)(2^{-\mu},2^{-\lambda})}\right\|_\infty}{\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)(2^{-(\mu+1)},2^{-(\lambda+1)})}\right\|_\infty}. \tag{3.1}$$

Analogously, the order of convergence of the approximate solution $z_{2^{-\mu},2^{-\lambda}}$ to the exact solution $z = \frac{\partial u}{\partial x_2}$ obtained by using the two-stage implicit method $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$ is given by

$$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)} = \frac{\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)(2^{-\mu},2^{-\lambda})}\right\|_\infty}{\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)(2^{-(\mu+1)},2^{-(\lambda+1)})}\right\|_\infty}. \tag{3.2}$$

Remark 3.1: We point out the numerical values in (3.1), (3.2) are $\approx 2^2$ showing the convergence of the approximate solution $v_{2^{-\mu},2^{-\lambda}}$ and $z_{2^{-\mu},2^{-\lambda}}$ converge to the respective exact solution $v=\frac{\partial u}{\partial x_1}$ and $z=\frac{\partial u}{\partial x_2}$ with second order both in the spatial variables x_1,x_2 and in time t.

Example 3.1: $\frac{\partial u}{\partial t}=0.5\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+f(x_1,x_2,t) \text{ on } Q_T,$

$$u(x_1,x_2,0)=0.0001\left(x_1^{\frac{57}{8}}(1-x_1)+\cos\left(x_2^{\frac{57}{8}}\right)\left(\frac{\sqrt{3}}{2}-x_2\right)\right) \text{ on } \overline{D},$$

$$u(x_1,x_2,t)=\widehat{u}(x_1,x_2,t) \text{ on } S_T,$$

where

$$\begin{aligned} f(x_1,x_2,t)=0.00035625\Bigg(&t^{\frac{41}{16}}-6.125x_1^{\frac{41}{8}}+8.125x_1^{\frac{49}{8}}\\ &+\left(\sqrt{3}\frac{3249}{912}-7.125x_2\right)x_2^{\frac{49}{4}}\cos\left(x_2^{\frac{57}{8}}\right)\\ &+\left(\sqrt{3}\frac{2793}{912}-8.125x_2\right)x_2^{\frac{41}{8}}\sin\left(x_2^{\frac{57}{8}}\right)\Bigg)\\ \widehat{u}(x_1,x_2,t)=0.0001\Bigg(&t^{\frac{57}{16}}+x_1^{\frac{57}{8}}(1-x_1)+\cos\left(x_2^{\frac{57}{8}}\right)\left(\frac{\sqrt{3}}{2}-x_2\right)\Bigg),\end{aligned}$$

are the heat source and exact solution. Table 3.1 demonstrates $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, maximum norm of the errors for $h=2^{-\mu},\mu=4,5,6,7$ when $\tau=2^{-\lambda},\lambda=13,14,15,16$, that is $r=\frac{0.5\tau}{h^2}\leq\frac{3}{7}$ and the order of convergence of $v_{h,\tau}$ to the exact derivatives $v=\frac{\partial u}{\partial x_1}$ with respect to h and τ obtained by using the constructed two-stage implicit method $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$. Table 3.2 shows $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, maximum norm of the errors for the same pairs of (h,τ) as in Table 3.1 and the order of convergence of $z_{h,\tau}$ to the exact derivative $z=\frac{\partial u}{\partial x_2}$ with respect to h and τ obtained by using the constructed two-stage implicit method $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$. Table 3.1 and Table 3.2 justify the theoretical results given such that the approximate solutions $v_{h,\tau}$ and $z_{h,\tau}$ of the proposed method converge to the corresponding exact derivatives $v=\frac{\partial u}{\partial x_1}$ and $z=\frac{\partial u}{\partial x_2}$ with second order both in the spatial variables x_1,x_2 and the time variable t for $r\leq\frac{3}{7}$, as given in Remark 3.1.

Table 3.3 presents the $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, maximum norm of the errors for $h = 2^{-\mu}, \mu = 4,5,6,7,8$ when $\tau = 2^{-\lambda}, \lambda = 8,9,10,11,12$, that is $r = \frac{0.5\tau}{h^2} > \frac{3}{7}$ and the order of convergence of $v_{h,\tau}$ to the exact derivative $v = \frac{\partial u}{\partial x_1}$ with respect to h and τ obtained by using the constructed two-stage implicit method $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$. Table 3.4 shows the $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, maximum norm of the errors for the same pairs of (h,τ) as in Table 3.3 and the order of convergence of $z_{h,\tau}$ to the exact derivative $z = \frac{\partial u}{\partial x_2}$ with respect to h and τ obtained by using the constructed two-stage implicit method $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$. Numerical results given in Table 3.3 and Table 3.4 demonstrate that when $r > \frac{3}{7}$, the approximate solutions $v_{h,\tau}$ and $z_{h,\tau}$ of the proposed method also converge with second order both in the spatial variables x_1, x_2 and the time variable t to their corresponding exact derivatives $v = \frac{\partial u}{\partial x_1}$ and $z = \frac{\partial u}{\partial x_2}$, as explained in remark 3.1.

Figure 3.1 illustrates the absolute error functions $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)(2^{-4},2^{-13})}\right|$, $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)(2^{-5},2^{-14})}\right|$, $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)(2^{-6},2^{-15})}\right|$, and $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)(2^{-7},2^{-16})}\right|$ at time moment $t = 0.2$ obtained by using $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$. Figure 3.2 demonstrates the absolute error functions $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)(2^{-4},2^{-13})}\right|$, $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)(2^{-5},2^{-14})}\right|$, $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)(2^{-6},2^{-15})}\right|$, and $\left|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)(2^{-7},2^{-16})}\right|$ at time moment $t = 0.2$ obtained by using $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$. The exact derivative $v = \frac{\partial u}{\partial x_1}$ and the grid function $v_{2^{-6},2^{-15}}$ for $h = 2^{-6}, \tau = 2^{-15}$ at time moment $t = 0.2$ obtained by using $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$ are presented in Figure 3.3. Further, Figure 3.4 shows the exact derivative $z = \frac{\partial u}{\partial x_2}$ and grid function $z_{2^{-6},2^{-15}}$ at time moment $t = 0.2$ obtained by using $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$.

Table 3.1: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$ when $r=\frac{0.5\tau}{h^2}\leq\frac{3}{7}$ for the Example 3.1 .

$r=\frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$
2^{-6}	$(2^{-4},2^{-13})$	0.03	197.34	9.34750×10^{-06}	3.1457
2^{-5}	$(2^{-5},2^{-14})$	0.09	1187.55	2.97147×10^{-06}	3.5508
2^{-4}	$(2^{-6},2^{-15})$	0.59	18501.80	8.36840×10^{-07}	3.7737
2^{-3}	$(2^{-7},2^{-16})$	3.69	144505.21	2.21757×10^{-07}	

Table 3.2: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$ when $r=\frac{0.5\tau}{h^2}\leq\frac{3}{7}$ for the Example 3.1 .

$r=\frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$
2^{-6}	$(2^{-4},2^{-13})$	0.02	181.88	3.72134×10^{-06}	1.7362
2^{-5}	$(2^{-5},2^{-14})$	0.13	1187.55	2.14336×10^{-06}	2.6720
2^{-4}	$(2^{-6},2^{-15})$	0.70	21557.80	8.02154×10^{-07}	3.2757
2^{-3}	$(2^{-7},2^{-16})$	4.09	169305.04	2.44880×10^{-07}	

Table 3.3: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$ when $r=\frac{0.5\tau}{h^2}>\frac{3}{7}$ for the Example 3.1 .

$r=\frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$
2^{-1}	$(2^{-4},2^{-8})$	0.02	4.75	9.34796×10^{-06}	3.1458
1	$(2^{-5},2^{-9})$	0.08	37.30	2.97159×10^{-06}	3.5508
2	$(2^{-6},2^{-10})$	0.42	347.70	8.36871×10^{-07}	3.7737
2^2	$(2^{-7},2^{-11})$	3.47	3988.83	2.21765×10^{-07}	3.8889
2^3	$(2^{-8},2^{-12})$	41.25	68313.10	5.70258×10^{-08}	

Table 3.4: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$ when $r = \frac{0.5\tau}{h^2} > \frac{3}{7}$ for the Example 3.1 .

$r = \frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$
2^{-1}	$\left(2^{-4},2^{-8}\right)$	0.03	7.52	3.72102×10^{-06}	1.7361
1	$\left(2^{-5},2^{-9}\right)$	0.13	64.38	2.14327×10^{-06}	2.6720
2	$\left(2^{-6},2^{-10}\right)$	0.59	533.53	8.02135×10^{-07}	3.2757
2^2	$\left(2^{-7},2^{-11}\right)$	3.83	5122.09	2.44877×10^{-07}	3.6202
2^3	$\left(2^{-8},2^{-12}\right)$	42.91	73957.51	$6.76426\times10^{-}08$	

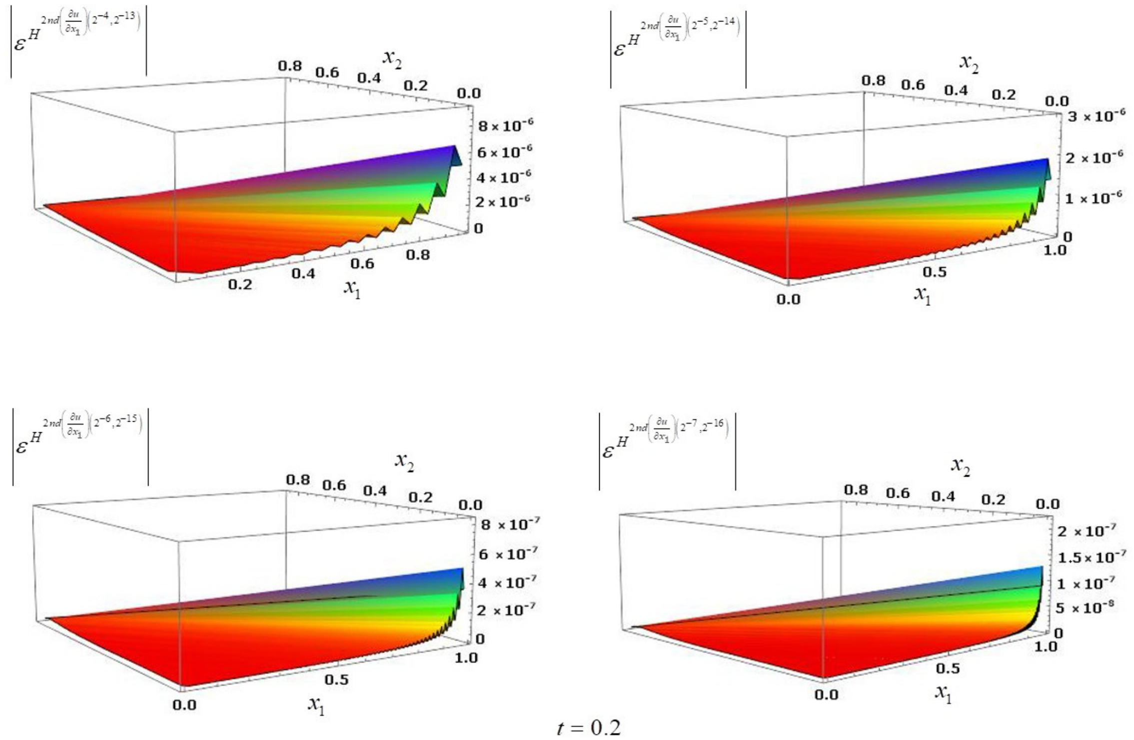
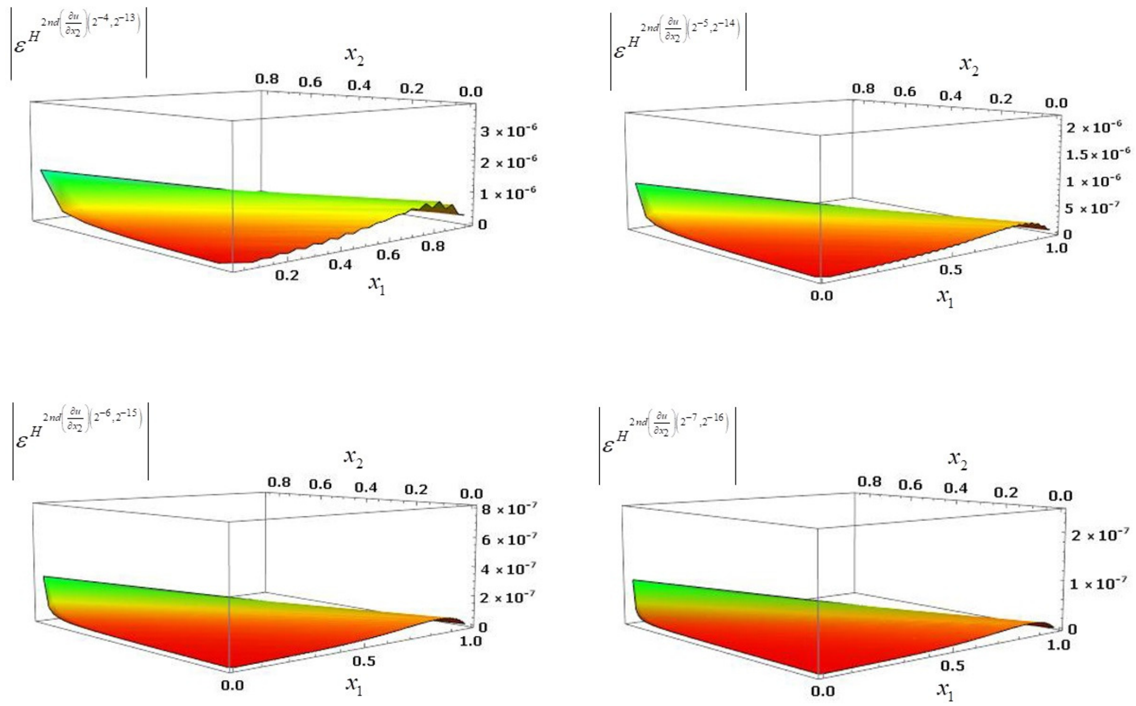


Figure 3.1: The grid function of absolute errors at time moment $t = 0.2$ achieved by $H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)$ for the Example 3.1.



$t = 0.2$

Figure 3.2: The grid function of absolute errors at time moment $t = 0.2$ achieved by $H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)$ for the Example 3.1.

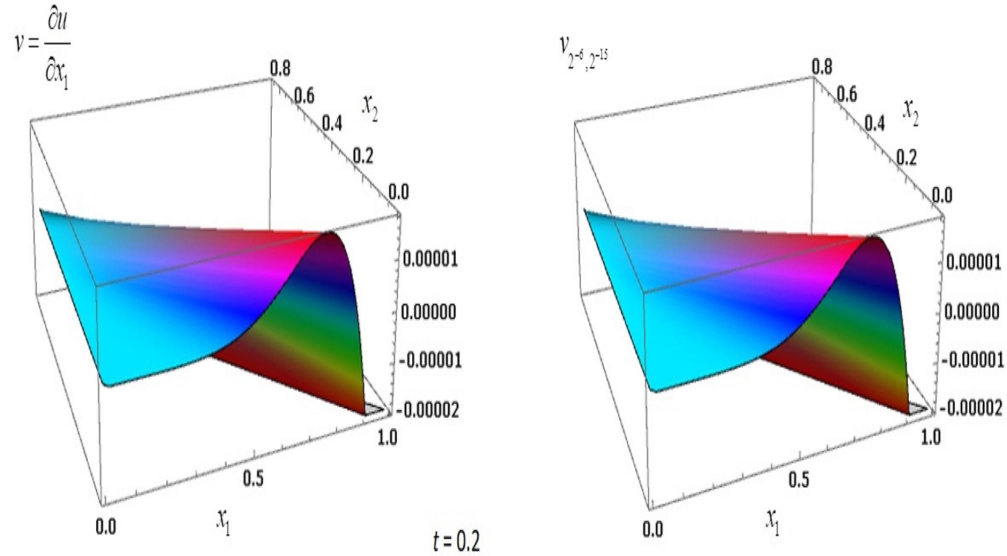


Figure 3.3: The exact solution $v = \frac{\partial u}{\partial x_1}$ and the approximate solution $v_{2^{-6},2^{-15}}$ at $t = 0.2$ for the Example 3.1.

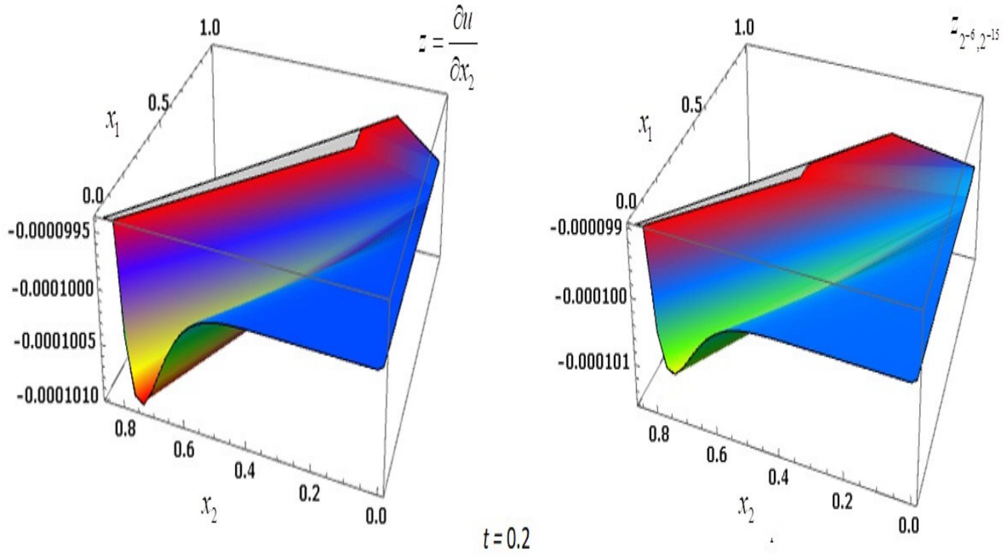


Figure 3.4: The exact solution $z=\frac{\partial u}{\partial x_2}$ and the approximate solution $z_{2^{-6},2^{-15}}$ at $t=0.2$ for the Example 3.1.

Table 3.5 shows the $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, maximum norm of the errors for $r\leq \frac{3}{7}$, and the order of convergence of $v_{h,\tau}$ to the exact derivative $v=\frac{\partial u}{\partial x_1}$ with respect to h and τ obtained when third order approximations for $v=\frac{\partial u}{\partial x_1}$

$$p_{1h}^{3^{rd}}=\begin{cases}\frac{1}{6h}\left(-11u(0,x_2,t)+18u_{h,\tau}(h,x_2,t)\right.\\ \left.-9u_{h,\tau}(2h,x_2,t)+2u_{h,\tau}(3h,x_2,t)\right) \text{ if } P_0\in D^{0h}\gamma_\tau\\ \frac{1}{60h}\left(-184u(0,x_2,t)+225u_{h,\tau}\left(\frac{h}{2},x_2,t\right)\right.\\ \left.-50u_{h,\tau}\left(\frac{3h}{2},x_2,t\right)+9u_{h,\tau}\left(\frac{5h}{2},x_2,t\right)\right) \text{ if } P_0\in D^{*lh}\gamma_\tau\end{cases} \text{ on } S_T^h\gamma_1, \quad (3.3)$$

$$p_{3h}^{3^{rd}}=\begin{cases}\frac{1}{6h}\left(11u(a_1,x_2,t)-18u_{h,\tau}(a_1-h,x_2,t)\right.\\ \left.+9u_{h,\tau}(a_1-2h,x_2,t)-2u_{h,\tau}(a_1-3h,x_2,t)\right) \text{ if } P_0\in D^{0h}\gamma_\tau\\ \frac{1}{60h}\left(184u(a_1,x_2,t)-225u_{h,\tau}\left(a_1-\frac{h}{2},x_2,t\right)\right.\\ \left.+50u_{h,\tau}\left(a_1-\frac{3h}{2},x_2,t\right)-9u_{h,\tau}\left(a_1-\frac{5h}{2},x_2,t\right)\right) \text{ if } P_0\in D^{*rh}\gamma_\tau\end{cases} \text{ on } S_T^h\gamma_3, \quad (3.4)$$

are used on $S_T^h\gamma_i, i=1,3$ for the Stage $2\left(H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)\right)$. Table 3.6 shows $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, maximum norm of the errors for $r\leq\frac{3}{7}$ and the order of convergence of $z_{h,\tau}$ to the exact derivative $z=\frac{\partial u}{\partial x_2}$ with respect to h and τ obtained when third order

Table 3.5: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$ when $r=\frac{0.5\tau}{h^2}\leq\frac{3}{7}$ and (3.3), (3.4) are used for the Example 3.1.

$r=\frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$
2^{-6}	$\left(2^{-4},2^{-13}\right)$	0.03	216.25	3.93819×10^{-06}	4.5863
2^{-5}	$\left(2^{-5},2^{-14}\right)$	0.09	1695.39	8.58690×10^{-07}	4.6031
2^{-4}	$\left(2^{-6},2^{-15}\right)$	0.63	18945.40	1.86547×10^{-07}	4.6131
2^{-3}	$\left(2^{-7},2^{-16}\right)$	3.67	218517.01	4.04385×10^{-08}	

approximations for $z=\frac{\partial u}{\partial x_2}$.

$$q_{2h}^{3rd}=\frac{1}{6\sqrt{3}h}\Big(-11u(x_1,0,t)+18u_{h,\tau}\left(x_1,\sqrt{3}h,t\right)$$
$$-9u_{h,\tau}\left(x_1,2\sqrt{3}h,t\right)+2u_{h,\tau}\left(x_1,3\sqrt{3}h,t\right)\Big)\text{ on }S_T^h\gamma_2, \tag{3.5}$$
$$q_{4h}^{3rd}=\frac{1}{6\sqrt{3}h}\Big(11u(x_1,a_2,t)-18u_{h,\tau}\left(x_1,a_2-\sqrt{3}h,t\right)$$
$$+9u_{h,\tau}\left(x_1,a_2-2\sqrt{3}h,t\right)-2u_{h,\tau}\left(x_1,a_2-3\sqrt{3}h,t\right)\Big)\text{ on }S_T^h\gamma_{4}, \tag{3.6}$$

are used on $S_T^h\gamma_i, i=2,4$ for the Stage $2\left(H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)\right)$. Table 3.7 presents $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, maximum norm of the errors for $r>\frac{3}{7}$, and the order of convergence of $v_{h,\tau}$ to the exact derivatives $v=\frac{\partial u}{\partial x_1}$ with respect to h and τ obtained by using the difference formulae (3.3), (3.4) on $S_T^h\gamma_i, i=1,3$ for the Stage $2\left(H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)\right)$. Table 3.8 gives $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, maximum norm of the errors for $r>\frac{3}{7}$, and the order of convergence of $z_{h,\tau}$ to the exact derivative $z=\frac{\partial u}{\partial x_2}$ with respect to h and τ obtained by using the difference formulae (3.5), (3.6) on $S_T^h\gamma_i, i=2,4$ for the Stage $2\left(H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)\right)$. Numerical results given in Table 3.5-Table 3.8 demonstrate that the approximate solution $v_{h,\tau}$ and $z_{h,\tau}$ of the proposed method converge to the corresponding exact derivatives $v=\frac{\partial u}{\partial x_1}$ and $z=\frac{\partial u}{\partial x_2}$ with second order both in the spatial variables x_1,x_2 and the time variable t with better error ratios.

Table 3.6: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$ when $r=\frac{0.5\tau}{h^2}\le\frac{3}{7}$ and (3.5) and (3.6) are used for the Example 3.1.

$r=\frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$
2^{-6}	$(2^{-4},2^{-13})$	0.03	251.27	3.37221×10^{-06}	2.5722
2^{-5}	$(2^{-5},2^{-14})$	0.13	2088.16	1.31103×10^{-06}	4.3277
2^{-4}	$(2^{-6},2^{-15})$	0.63	18945.40	3.02939×10^{-07}	4.4163
2^{-3}	$(2^{-7},2^{-16})$	3.85	234313.60	6.85956×10^{-08}	

Table 3.7: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$ when $r=\frac{0.5\tau}{h^2}>\frac{3}{7}$ and (3.3), (3.4) are used for the Example 3.1.

$r=\frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_1}\right)}$
2^{-1}	$(2^{-4},2^{-8})$	0.02	5.08	$3.93866\times^{-06}$	4.5862
1	$(2^{-5},2^{-9})$	0.08	38.19	$8.58815\times^{-07}$	4.6030
2	$(2^{-6},2^{-10})$	0.44	352.03	$1.86579\times^{-07}$	4.4176
2^2	$(2^{-7},2^{-11})$	3.52	3994.16	$4.22355\times^{-08}$	

Table 3.8: The $CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$, $\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$ and $\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$ when $r=\frac{0.5\tau}{h^2}>\frac{3}{7}$ and (3.5) and (3.6) are used for the Example 3.1.

$r=\frac{0.5\tau}{h^2}$	(h,τ)	$CT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$TCT^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$	$\left\|\varepsilon^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}\right\|_\infty$	$\Re^{H^{2nd}\left(\frac{\partial u}{\partial x_2}\right)}$
2^{-1}	$(2^{-4},2^{-8})$	0.02	5.89	$3.67669\times^{-06}$	2.8278
1	$(2^{-5},2^{-9})$	0.11	45.67	$1.30019\times^{-06}$	4.4268
2	$(2^{-6},2^{-10})$	0.50	414.27	$2.93712\times^{-07}$	4.5165
2^2	$(2^{-7},2^{-11})$	3.72	4475.91	$6.50300\times^{-08}$	

Chapter 4

HEXAGONAL GRID COMPUTATION OF THE DERIVATIVES OF THE SOLUTION TO THE HEAT EQUATION BY USING FOURTH ORDER ACCURATE TWO-STAGE IMPLICIT METHODS

In this chapter, we discuss hexagonal grid computation of the derivatives of the solution to the heat equation by using fourth order accurate two-stage implicit methods. We consider first type boundary value problem (Dirichlet problem) for the heat Equation (1.2) on a rectangle D. In the first stage of the two-stage an implicit scheme on hexagonal grids given in Buranay and Arshad [37] with $O\left(h^4+\tau\right)$ order of accuracy is used to approximate the solution $u(x_1,x_2,t)$. An analogous implicit method is also given to approximate the derivative of the solution with respect to time. In the second stage, computation of the first order spatial derivatives and second order mixed derivatives involving time derivatives of the solution $u(x_1,x_2,t)$ of (1.2) are developed. Uniform convergence of the approximate derivatives to the corresponding exact derivatives $\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial t}$, and $\frac{\partial^2 u}{\partial x_i \partial t}$, $i=1,2$ with order $O\left(h^4+\tau\right)$ of accuracy on the hexagonal grids are proved.

4.1 Hexagonal Grid Approximation of the Heat Equation and the Rate of Change by Using Fourth Order Accurate Difference Schemes

We assume that the initial and boundary functions $\varphi(x_1,x_2)$, $\phi(x_1,x_2,t)$, respectively, also the heat source function $f(x_1,x_2,t)$ possess the necessary smoothness and satisfy the conditions that the BVP(u) in (2.9) has unique solution $u \in C_{x,t}^{9+\alpha,\frac{9+\alpha}{2}}\left(\overline{Q}_T\right)$. We also

Table 4.1: Basic notations for the heat source function f and f_t.

f	f_t
$f_{P_0}^{k+1} = f(x_1, x_2, t+\tau)$	$f_{t,P_0}^{k+1} = \left.\frac{\partial f}{\partial t}\right\vert_{(x_1,x_2,t+\tau)}$
$f_{P_A}^{k+1} = f(\widehat{p}, x_2, t+\tau)$	$f_{t,P_A}^{k+1} = \left.\frac{\partial f}{\partial t}\right\vert_{(\widehat{p},x_2,t+\tau)}$
$f_{P_A}^{k} = f(\widehat{p}, x_2, t)$	$f_{t,P_A}^{k} = \left.\frac{\partial f}{\partial t}\right\vert_{(\widehat{p},x_2,t)}$
$\partial_{x_j} f_{P_A}^{k} = \left.\frac{\partial f}{\partial x_j}\right\vert_{(\widehat{p},x_2,t)}, \ j=1,2$	$\partial_{x_j} f_{t,P_A}^{k} = \left.\frac{\partial^2 f}{\partial x_j \partial t}\right\vert_{(\widehat{p},x_2,t)}, \ j=1,2$
$\partial_{x_j}^2 f_{P_0}^{k+1} = \left.\frac{\partial^2 f}{\partial x_j^2}\right\vert_{(x_1,x_2,t+\tau)}, \ j=1,2$	$\partial_{x_j}^2 f_{t,P_0}^{k+1} = \left.\frac{\partial^3 f}{\partial x_j^2 \partial t}\right\vert_{(x_1,x_2,t+\tau)}, \ j=1,2$
$\partial_{x_2}^2 \partial_{x_1} f_{P_0}^{k+1} = \left.\frac{\partial^3 f}{\partial x_2^2 \partial x_1}\right\vert_{(x_1,x_2,t+\tau)}$ $\partial_{x_1}^2 \partial_{x_2} f_{P_0}^{k+1} = \left.\frac{\partial^3 f}{\partial x_1^2 \partial x_2}\right\vert_{(x_1,x_2,t+\tau)}$	$\partial_{x_2}^2 \partial_{x_1} f_{t,P_0}^{k+1} = \left.\frac{\partial^4 f}{\partial x_2^2 \partial x_1 \partial t}\right\vert_{(x_1,x_2,t+\tau)}$ $\partial_{x_1}^2 \partial_{x_2} f_{t,P_0}^{k+1} = \left.\frac{\partial^4 f}{\partial x_1^2 \partial x_2 \partial t}\right\vert_{(x_1,x_2,t+\tau)}$

use the following notations in Table 4.1 to denote the values and partial derivatives of the heat source function f and $f_t = \frac{\partial f}{\partial t}$ with respect to the space variables.

4.1.1 Dirichlet Problem of Heat Equation and Difference Problem: Stage 1$\left(H^{4th}(u)\right)$

For computing numerically the solution of the BVP(u) we use the following difference problem given in Buranay and Arshad [37] and call this Stage 1$\left(H^{4th}(u)\right)$.

Stage 1$\left(H^{4th}(u)\right)$

$$\widetilde{\Theta}_{h,\tau} u_{h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau} u_{h,\tau}^{k} + \widetilde{\psi} \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^{*} u_{h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}^{*} u_{h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*}\phi + \widetilde{\psi}^{*} \text{ on } D^{*h}\gamma_\tau,$$

$$u_{h,\tau} = \varphi(x_1, x_2),\ t = 0 \text{ on } \overline{D}^h,$$

$$u_{h,\tau} = \phi(x_1, x_2, t) \ \text{ on } S_T^h, \tag{4.1}$$

$k = 0, ..., M'-1$, where φ, ϕ are the initial and boundary functions in (2.9), respectively, also

$$\widetilde{\psi} = f_{P_0}^{k+1} + \frac{1}{16}h^2\left(\partial_{x_1}^2 f_{P_0}^{k+1} + \partial_{x_2}^2 f_{P_0}^{k+1}\right), \tag{4.2}$$

$$\widetilde{\psi}^{*} = \frac{h^2}{96\tau\omega} f_{P_A}^{k+1} - \frac{h^2}{96\tau\omega} f_{P_A}^{k} - \frac{1}{6} f_{P_A}^{k+1} + f_{P_0}^{k+1} + \frac{1}{16}h^2\left(\partial_{x_1}^2 f_{P_0}^{k+1} + \partial_{x_2}^2 f_{P_0}^{k+1}\right), \tag{4.3}$$

$$\widetilde{\Theta}_{h,\tau}u^{k+1}=\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)u_{P_0}^{k+1}+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\sum_{i=1}^{6}u_{P_i}^{k+1}, \tag{4.4}$$

$$\widetilde{\Lambda}_{h,\tau}u^{k}=\frac{3}{4\tau}u_{P_0}^{k}+\frac{1}{24\tau}\sum_{i=1}^{6}u_{P_i}^{k}, \tag{4.5}$$

$$\begin{aligned}\widetilde{\Theta}^{*}_{h,\tau}u^{k+1}=&\left(\frac{17}{24\tau}+\frac{14\omega}{3h^2}\right)u_{P_0}^{k+1}+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\Bigg(u(p,x_2+\frac{\sqrt{3}}{2}h,t+\tau)\\&+u(p,x_2-\frac{\sqrt{3}}{2}h,t+\tau)+u(p+\eta,x_2,t+\tau)\Bigg),\end{aligned} \tag{4.6}$$

$$\begin{aligned}\widetilde{\Gamma}^{*}_{h,\tau}\phi=&\left(-\frac{1}{36\tau}+\frac{4\omega}{9h^2}\right)\left(\phi(\widehat{p},x_2+\frac{\sqrt{3}}{2}h,t+\tau)+\phi(\widehat{p},x_2-\frac{\sqrt{3}}{2}h,t+\tau)\right)\\&+\left(\frac{1}{18\tau}+\frac{16\omega}{9h^2}\right)\phi(\widehat{p},x_2,t+\tau)-\frac{1}{18\tau}\phi(\widehat{p},x_2,t)\\&+\frac{1}{36\tau}\left(\phi(\widehat{p},x_2+\frac{\sqrt{3}}{2}h,t)+\phi(\widehat{p},x_2-\frac{\sqrt{3}}{2}h,t)\right),\end{aligned} \tag{4.7}$$

$$\begin{aligned}\widetilde{\Lambda}^{*}_{h,\tau}u^{k}=&\frac{17}{24\tau}u_{P_0}^{k}+\frac{1}{24\tau}\Bigg(u(p,x_2+\frac{\sqrt{3}}{2}h,t)\\&+u(p,x_2-\frac{\sqrt{3}}{2}h,t)+u(p+\eta,x_2,t)\Bigg),\end{aligned} \tag{4.8}$$

and

$$\begin{cases} p=h,\widehat{p}=0,\eta=\frac{h}{2}\text{ if }P_0\in D^{*lh}\gamma_\tau,\\ p=a_1-h,\widehat{p}=a_1,\eta=-\frac{h}{2}\text{ if }P_0\in D^{*rh}\gamma_\tau.\end{cases} \tag{4.9}$$

4.1.2 Dirichlet Problem for the Rate of Change and Difference Problem: Stage 1$\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$

Further, for the computation of $\frac{\partial u}{\partial t}$, we construct the next boundary value problem denoted by $u_t=\frac{\partial u}{\partial t}$ which defines the rate of change function

BVP$\left(\frac{\partial u}{\partial t}\right)$

$$\frac{\partial u_t}{\partial t}=\omega\left(\frac{\partial^2 u_t}{\partial x_1^2}+\frac{\partial^2 u_t}{\partial x_2^2}\right)+f_t\ (x_1,x_2,t)\text{ on }Q_T,$$

$$u_t\,(x_1,x_2,0)=\widehat{\varphi}\ (x_1,x_2)\text{ on }\overline{D},$$

$$u_t\,(x_1,x_2,t)=\phi_t\ (x_1,x_2,t)\text{ on }S_T, \tag{4.10}$$

where
$$
\begin{aligned}
f_t &= \frac{\partial f(x_1,x_2,t)}{\partial t},\\
\widehat{\varphi} &= \omega\left(\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}\right)+f(x_1,x_2,0),\\
\phi_t &= \frac{\partial\phi(x_1,x_2,t)}{\partial t},
\end{aligned}
\tag{4.11}
$$

and φ,ϕ are the initial and boundary functions BVP(u) given in (2.9).

Assuming $u_t\in C_{x,t}^{7+\alpha,\frac{7+\alpha}{2}}\left(\overline{Q}_T\right)$, fourth order accurate implicit schemes for the solution of the BVP$\left(\frac{\partial u}{\partial t}\right)$ is proposed with the following difference problem. This stage is called Stage $1\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$.

Stage 1$\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$

$$
\begin{aligned}
\widetilde{\Theta}_{h,\tau}u_{t,h,\tau}^{k+1} &= \widetilde{\Lambda}_{h,\tau}u_{t,h,\tau}^{k}+\widetilde{\psi}_t \text{ on } D^{0h}\gamma_\tau,\\
\widetilde{\Theta}^*_{h,\tau}u_{t,h,\tau}^{k+1} &= \widetilde{\Lambda}^*_{h,\tau}u_{t,h,\tau}^{k}+\widetilde{\Gamma}^*_{h,\tau}\phi_t+\widetilde{\psi}^*_t \text{ on } D^{*h}\gamma_\tau,\\
u_{t,h,\tau} &= \widehat{\varphi},\ t=0 \text{ on } \overline{D}^h,\\
u_{t,h,\tau} &= \phi_t(x_1,x_2,t) \text{ on } S_T^h,
\end{aligned}
\tag{4.12}
$$

$k=0,...,M'-1$, where the operators $\widetilde{\Theta}_{h,\tau}$, $\widetilde{\Lambda}_{h,\tau}$, $\widetilde{\Theta}^*_{h,\tau}$, $\widetilde{\Gamma}^*_{h,\tau}$ and $\widetilde{\Lambda}^*_{h,\tau}$ are presented in (4.4)–(4.8), respectively, and

$$
\widetilde{\psi}_t = f_{t,P_0}^{k+1}+\frac{1}{16}h^2\left(\partial_{x_1}^2 f_{t,P_0}^{k+1}+\partial_{x_2}^2 f_{t,P_0}^{k+1}\right), \tag{4.13}
$$

$$
\begin{aligned}
\widetilde{\psi}^*_t = {}& \frac{h^2}{96\tau\omega}f_{t,P_A}^{k+1}-\frac{h^2}{96\tau\omega}f_{t,P_A}^{k}-\frac{1}{6}f_{t,P_A}^{k+1}+f_{t,P_0}^{k+1}\\
&+\frac{1}{16}h^2\left(\partial_{x_1}^2 f_{t,P_0}^{k+1}+\partial_{x_2}^2 f_{t,P_0}^{k+1}\right).
\end{aligned}
\tag{4.14}
$$

4.1.3 $M-$Matrices and Convergence of Finite Difference Schemes in Stage 1$\left(H^{4th}(u)\right)$ and Stage 1$\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$

For a fixed time level $k\geq 0$ we present the equations (4.1) and (4.12) in matrix form with N unknown interior grid points L_j, $j=1,2,...,N$, labeled using standard ordering as

$$\widetilde{A}\widetilde{u}^{k+1} = \widetilde{B}\widetilde{u}^{k} + \tau\widetilde{q}_u^{k},$$

$$\widetilde{A}\widetilde{u}_t^{k+1} = \widetilde{B}\widetilde{u}_t^{k} + \tau\widetilde{q}_{u_t}^{k}, \tag{4.15}$$

respectively, where $\widetilde{A},\widetilde{B} \in R^{N\times N}$ and $\widetilde{u}^k, \widetilde{q}_u^k, \widetilde{u}_t^k, \widetilde{q}_{u_t}^k \in R^N$ and

$$\widetilde{A} = \left(\breve{E}_1 + \frac{1}{24}Inc + \frac{\omega\tau}{h^2}\widetilde{C}\right),\ \widetilde{B} = \left(\breve{E}_1 + \frac{1}{24}Inc\right), \tag{4.16}$$

$$\widetilde{C} = \breve{E}_2 - \frac{2}{3}Inc \in R^{N\times N}. \tag{4.17}$$

and Inc is the neighboring topology matrix, $\breve{E}_1, \breve{E}_2$ are diagonal matrices with entries

$$\left[\breve{E}_1\right]_{j,j} = \begin{cases} \frac{3}{4} \text{ if } L_j \in D^{0h}\gamma_\tau \\ \frac{17}{24} \text{ if } L_j \in D^{*h}\gamma_\tau \end{cases},\ j = 1,2,...,N, \tag{4.18}$$

$$\left[\breve{E}_2\right]_{j,j} = \begin{cases} 4 \text{ if } L_j \in D^{0h}\gamma_\tau \\ \frac{14}{3} \text{ if } L_j \in D^{*h}\gamma_\tau \end{cases},\ j = 1,2,...,N, \tag{4.19}$$

respectively (see Buranay and Arshad [37]).

Lemma 4.1: (Buranay and Arshad [37])

(a) The matrices $\widetilde{A}$ and $\widetilde{B}$ in (4.15) are symmetric positive definite (spd) matrices

(b) $\widehat{A} = I + \frac{\omega\tau}{h^2}\widetilde{B}^{-1}\widetilde{C}$ is spd matrix and $\left\|\widehat{A}^{-1}\right\|_2 < 1$.

Lemma 4.2: (Buranay et al. [53]) The matrix $\widetilde{A}$ in (4.15) is nonsingular $M-$matrix for $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$.

Proof. Taking into consideration Lemma 4.1, the matrix $\widetilde{A}$ is a spd matrix. Further, using the Equations (4.16)–(4.19), $\widetilde{A}$ is strictly diagonally dominant matrix with positive diagonal entries. Furthermore, off-diagonal entries are non-positive for $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$. Therefore, it is nonsingular $M-$matrix. □

Let

$$\xi^{u}_{h,\tau} = u_{h,\tau} - u \text{ on } \overline{D^h\gamma_\tau} \tag{4.20}$$

$$\xi^{u_t}_{h,\tau} = u_{t,h,\tau} - u_t \text{ on } \overline{D^h\gamma_\tau} \tag{4.21}$$

From (4.1) and (4.20) the error function (4.20) satisfies the following system as given in Buranay and Arshad: [37]

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}\xi^{u,k+1}_{h,\tau} &= \widetilde{\Lambda}_{h,\tau}\xi^{u,k}_{h,\tau} + \widetilde{\Psi}^{u,k}_{1} \text{ on } D^{0h}\gamma_\tau,\\
\widetilde{\Theta}^{*}_{h,\tau}\xi^{u,k+1}_{h,\tau} &= \widetilde{\Lambda}^{*}_{h,\tau}\xi^{u,k}_{h,\tau} + \widetilde{\Psi}^{u,k}_{2} \text{ on } D^{*h}\gamma_\tau,\\
\xi^{u}_{h,\tau} &= 0,\ t = 0 \text{ on } \overline{D}^{h},\\
\xi^{u}_{h,\tau} &= 0 \text{ on } S^{h}_{T},
\end{aligned} \tag{4.22}$$

where

$$\widetilde{\Psi}^{u,k}_{1} = \widetilde{\Lambda}_{h,\tau}u^{k} - \widetilde{\Theta}_{h,\tau}u^{k+1} + \widetilde{\psi}, \tag{4.23}$$

$$\widetilde{\Psi}^{u,k}_{2} = \widetilde{\Lambda}^{*}_{h,\tau}u^{k} - \widetilde{\Theta}^{*}_{h,\tau}u^{k+1} + \widetilde{\Gamma}^{*}_{h,\tau}\phi + \widetilde{\psi}^{*}, \tag{4.24}$$

and $\widetilde{\psi}, \widetilde{\psi}^{*}$ and ϕ are as presented in (4.1). Analogously, using (4.12) and (4.21) the error function (4.21) satisfies the following system:

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}\xi^{u_t,k+1}_{h,\tau} &= \widetilde{\Lambda}_{h,\tau}\xi^{u_t,k}_{h,\tau} + \widetilde{\Psi}^{u_t,k}_{1} \text{ on } D^{0h}\gamma_\tau,\\
\widetilde{\Theta}^{*}_{h,\tau}\xi^{u_t,k+1}_{h,\tau} &= \widetilde{\Lambda}^{*}_{h,\tau}\xi^{u_t,k}_{h,\tau} + \widetilde{\Psi}^{u_t,k}_{2} \text{ on } D^{*h}\gamma_\tau,\\
\xi^{u_t}_{h,\tau} &= 0,\ t = 0 \text{ on } \overline{D}^{h},\\
\xi^{u_t}_{h,\tau} &= 0 \text{ on } S^{h}_{T},
\end{aligned} \tag{4.25}$$

where

$$\widetilde{\Psi}^{u_t,k}_{1} = \widetilde{\Lambda}_{h,\tau}u^{k}_{t} - \widetilde{\Theta}_{h,\tau}u^{k+1}_{t} + \widetilde{\psi}_{t}, \tag{4.26}$$

$$\widetilde{\Psi}^{u_t,k}_{2} = \widetilde{\Lambda}^{*}_{h,\tau}u^{k}_{t} - \widetilde{\Theta}^{*}_{h,\tau}u^{k+1}_{t} + \widetilde{\Gamma}^{*}_{h,\tau}\phi_{t} + \widetilde{\psi}^{*}_{t}, \tag{4.27}$$

and $\phi_t, \widetilde{\psi}_t$, and $\widetilde{\psi}^{*}_{t}$ are the given functions in (4.11), (4.13) and (4.14) respectively.

Further, the following systems are considered:

$$\widetilde{\Theta}_{h,\tau}\widehat{w}_{h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}\widehat{w}_{h,\tau}^{k} + \widehat{\kappa}_1^k \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^{*}\widehat{w}_{h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}^{*}\widehat{w}_{h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*}\widehat{w}_{\phi,h,\tau} + \widehat{\kappa}_2^k \text{ on } D^{*h}\gamma_\tau,$$

$$\widehat{w}_{h,\tau} = \widehat{w}_{\varphi,h,\tau},\ t=0 \text{ on } \overline{D}^h,$$

$$\widehat{w}_{h,\tau} = \widehat{w}_{\phi,h,\tau} \text{ on } S_T^h, \tag{4.28}$$

$$\widetilde{\Theta}_{h,\tau}\overline{w}_{h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}\overline{w}_{h,\tau}^{k} + \overline{\kappa}_1^k \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^{*}\overline{w}_{h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}^{*}\overline{w}_{h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*}\overline{w}_{\phi,h,\tau} + \overline{\kappa}_2^k \text{ on } D^{*h}\gamma_\tau,$$

$$\overline{w}_{h,\tau} = \overline{w}_{\varphi,h,\tau},\ t=0 \text{ on } \overline{D}^h,$$

$$\overline{w}_{h,\tau} = \overline{w}_{\phi,h,\tau} \text{ on } S_T^h, \tag{4.29}$$

for $k=0,...,M'-1$, where $\widehat{\kappa}_1^k, \widehat{\kappa}_2^k$ and $\overline{\kappa}_1^k, \overline{\kappa}_2^k$ are given functions. The algebraic systems (4.28) and (4.29) at a fixed time level $k \geq 0$ may be given in matrix representation as

$$\widetilde{A}\widehat{w}^{k+1} = \widetilde{B}\widehat{w}^{k} + \tau\widehat{\kappa}^k, \tag{4.30}$$

$$\widetilde{A}\overline{w}^{k+1} = \widetilde{B}\overline{w}^{k} + \tau\overline{\kappa}^k, \tag{4.31}$$

accordingly. In these equations, $\widehat{w}^k, \overline{w}^k, \widehat{\kappa}^k, \overline{\kappa}^k \in R^N$ and the matrices $\widetilde{A}$ and $\widetilde{B}$ are given in (4.16).

Lemma 4.3: (Buranay et al. [53]) Let the solutions of (4.30) and (4.31) be presented by $\widehat{w}^{k+1}$ and $\overline{w}^{k+1}$, respectively, for $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$. If

$$\overline{w}^0 \geq 0 \text{ and } \overline{\kappa}^k \geq 0 \tag{4.32}$$

$$\left|\widehat{w}^0\right| \leq \overline{w}^0, \tag{4.33}$$

$$\left|\widehat{\kappa}^k\right| \leq \overline{\kappa}^k, \tag{4.34}$$

for $k=0,...,M'-1$ then

$$\left|\widehat{w}^{k+1}\right| \leq \overline{w}^{k+1},\ k=0,...,M'-1, \tag{4.35}$$

Proof. From Lemma 4.2, when $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$ the matrix $\widetilde{A}$ is nonsingular $M-$matrix therefore, $\widetilde{A}^{-1} \geq 0$. Furthermore, from (4.16) $\widetilde{B} \geq 0$ and using (4.32) it follows that $\overline{\kappa}^k \geq 0$, $k = 0, ..., M' - 1$ and $\overline{w}^0 \geq 0$. Further, assuming $\overline{w}^k \geq 0$ and from induction we achieve

$$\overline{w}^{k+1} = \widetilde{A}^{-1}\widetilde{B}\overline{w}^k + \tau A^{-1}\overline{\kappa}^k \geq 0, \tag{4.36}$$

which gives $\overline{w}^{k+1} \geq 0$ for $k = 0, ..., M' - 1$. Next, assume that $\left|\widehat{w}^k\right| \leq \overline{w}^k$ using (4.30)–(4.34), and by induction it follows that

$$\widehat{w}^{k+1} = \widetilde{A}^{-1}\widetilde{B}\widehat{w}^k + \tau\widetilde{A}^{-1}\widehat{\kappa}^k \tag{4.37}$$

$$\begin{aligned} \left|\widehat{w}^{k+1}\right| &\leq \widetilde{A}^{-1}\widetilde{B}\left|\widehat{w}^k\right| + \tau\widetilde{A}^{-1}\left|\widehat{\kappa}^k\right| \\ &\leq \widetilde{A}^{-1}\widetilde{B}\overline{w}^k + \tau\widetilde{A}^{-1}\overline{\kappa}^k = \overline{w}^{k+1}, \text{ for } k = 0, ..., M' - 1. \end{aligned} \tag{4.38}$$

□

Remark 4.1: Writing the implicit schemes on hexagonal grids for the problems (4.1) and (4.12) in the canonical form it follows that the maximum principle holds when $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$. Further, Lemma 4.3 is the consequence of comparison theorem (see Chapter 4, Section 4.2 Theorem 1 and Theorem 2 in Samarskii [64]) applied to the systems (4.28), (4.29).

Additionally, let

$$\begin{aligned} \mu_1(u) = \max\Bigg\{ &\max_{\overline{Q}_T}\left|\frac{\partial^5 u}{\partial x_1^4 \partial t}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^5 u}{\partial x_2^4 \partial t}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^5 u}{\partial x_1^2 \partial x_2^2 \partial t}\right|, \\ &\max_{\overline{Q}_T}\left|\frac{\partial^6 u}{\partial x_1^4 \partial x_2^2}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^6 u}{\partial x_1^2 \partial x_2^4}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^6 u}{\partial x_1^6}\right|, \max_{\overline{Q}_T}\left|\frac{\partial^6 u}{\partial x_2^6}\right|\Bigg\}, \end{aligned} \tag{4.39}$$

$$\mu_2(u) = \max_{\overline{Q}_T}\left|\frac{\partial^2 u}{\partial t^2}\right|. \tag{4.40}$$

Theorem 4.1: (Buranay et al. [53]) For the solution of the systems (4.22) and (4.25) when $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$, the following pointwise error estimations hold true:

$$\left|\xi_{h,\tau}^{u}(x_1,x_2,t)\right| \le d\widetilde{\Omega}_1(h,\tau)\,\rho(x_1,x_2,t) \text{ on } \overline{D^h\gamma_\tau}, \tag{4.41}$$

$$\left|\xi_{h,\tau}^{u_t}(x_1,x_2,t)\right| \le d\widetilde{\Omega}_{t,1}(h,\tau)\,\rho(x_1,x_2,t) \text{ on } \overline{D^h\gamma_\tau}, \tag{4.42}$$

respectively, where

$$\widetilde{\Omega}_1(h,\tau) = \frac{3}{5}\widetilde{\beta}\tau + \left(\frac{3}{160} + \frac{47}{2880}\omega\right)\widetilde{\alpha}h^4, \tag{4.43}$$

$$\widetilde{\Omega}_{t,1}(h,\tau) = \frac{3}{5}\widetilde{\beta}_t\tau + \left(\frac{3}{160} + \frac{47}{2880}\omega\right)\widetilde{\alpha}_t h^4, \tag{4.44}$$

and $\widetilde{\alpha} = \mu_1(u)$, $\widetilde{\alpha}_t = \mu_1(u_t)$ and $\widetilde{\beta} = \mu_2(u)$, $\widetilde{\beta}_t = \mu_2(u_t)$ and d is as given in (2.60) and u is the solution of BVP(u) and $\rho(x_1,x_2,t)$ is the function giving the distance from the considered hexagonal grid point $(x_1,x_2,t) \in \overline{D^h\gamma_\tau}$ to the surface of Q_T.

Proof. We give the proof of (4.41) by considering the auxiliary system

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}\widehat{\xi}_{h,\tau}^{u,k+1} &= \widetilde{\Lambda}_{h,\tau}\widehat{\xi}_{h,\tau}^{u,k} + \widetilde{\Omega}_1(h,\tau) \text{ on } D^{0h}\gamma_\tau,\\
\widetilde{\Theta}^{*}_{h,\tau}\widehat{\xi}_{h,\tau}^{u,k+1} &= \widetilde{\Lambda}^{*}_{h,\tau}\widehat{\xi}_{h,\tau}^{u,k} + \frac{5}{6}\widetilde{\Omega}_1(h,\tau) \text{ on } D^{*h}\gamma_\tau\\
\widehat{\xi}_{h,\tau}^{u} &= \widehat{\xi}_{\varphi,h,\tau}^{u} = 0,\ t=0 \text{ on } \overline{D}^h,\\
\widehat{\xi}_{h,\tau}^{u} &= \widehat{\xi}_{\phi,h,\tau}^{u} = 0 \text{ on } S_T^h,
\end{aligned} \tag{4.45}$$

and the majorant functions

$$\overline{\xi}_1^{u}(x_1,x_2,t) = \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left(a_1x_1 - x_1^2\right) \ge 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{4.46}$$

$$\overline{\xi}_2^{u}(x_1,x_2,t) = \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left(a_2x_2 - x_2^2\right) \ge 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{4.47}$$

$$\overline{\xi}_3^{u}(x_1,x_2,t) = \widetilde{\Omega}_1(h,\tau)\,t \ge 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{4.48}$$

which $\overline{\xi}_l^{u}(x_1,x_2,t)$, satisfy the following difference problem for $l = 1,2,3$, respectively.

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}_{l,h,\tau}^{u,k+1} &= \widetilde{\Lambda}_{h,\tau}\overline{\xi}_{l,h,\tau}^{u,k} + \widetilde{\Omega}_1(h,\tau) \text{ on } D^{0h}\gamma_\tau,\\
\widetilde{\Theta}^{*}_{h,\tau}\overline{\xi}_{l,h,\tau}^{u,k+1} &= \widetilde{\Lambda}^{*}_{h,\tau}\overline{\xi}_{l,h,\tau}^{u,k} + \widetilde{\Gamma}^{*}_{h,\tau}\overline{\xi}_{l,\phi,h,\tau}^{u*} + \frac{5}{6}\widetilde{\Omega}_1(h,\tau) \text{ on } D^{*h}\gamma_\tau,
\end{aligned}$$

$$\overline{\xi}^{u}_{l,h,\tau} = \overline{\xi}^{u}_{l,\varphi,h,\tau} = \overline{\xi}^{u}_{l}\left(x_1, x_2, 0\right) \geq 0,\ t = 0 \text{ on } \overline{D}^{h},$$

$$\overline{\xi}^{u}_{l,h,\tau} = \overline{\xi}^{u*}_{l,\phi,h,\tau} \geq 0 \text{ on } S^{h}_{T}. \tag{4.49}$$

For establishing (4.49) the following are used. First for regular interior grid points we have:

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{u,k+1}_{1,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1\left(h,\tau\right)\left[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)\left(a_1x_1 - x_1^2\right) + \left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(a_1\left(x_1+\frac{h}{2}\right)\right.\right.\\
&\quad -\left(x_1+\frac{h}{2}\right)^2 + a_1\left(x_1-\frac{h}{2}\right) - \left(x_1-\frac{h}{2}\right)^2 + a_1\left(x_1-h\right) - \left(x_1-h\right)^2\\
&\quad + a_1\left(x_1-\frac{h}{2}\right) - \left(x_1-\frac{h}{2}\right)^2 + a_1\left(x_1+\frac{h}{2}\right) - \left(x_1+\frac{h}{2}\right)^2\\
&\quad \left.\left. + a_1\left(x_1+h\right) - \left(x_1+h\right)^2\right)\right],\\
&= \frac{1}{2\omega}\widetilde{\Omega}_1\left(h,\tau\right)\left[\frac{a_1x_1}{\tau} - \frac{x_1^2}{\tau} - \frac{h^2}{8\tau} + 2\omega\right].
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{u,k}_{1,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1\left(h,\tau\right)\left[\frac{3}{4\tau}\left(a_1x_1 - x_1^2\right) + \frac{1}{24\tau}\left(6a_1x_1 - 6x_1^2 - 3h^2\right)\right]\\
&= \frac{1}{2\omega}\widetilde{\Omega}_1\left(h,\tau\right)\left[\frac{a_1x_1}{\tau} - \frac{x_1^2}{\tau} - \frac{h^2}{8\tau}\right].
\end{aligned} \tag{4.51}$$

Using equations (4.50) and (4.51) we can show that for $i = 1$

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{u,k+1}_{1,h,\tau} - \widetilde{\Lambda}_{h,\tau}\overline{\xi}^{u,k}_{1,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1\left(h,\tau\right)\left[\frac{a_1x_1}{\tau} - \frac{x_1^2}{\tau} - \frac{h^2}{8\tau} - \frac{a_1x_1}{\tau} + \frac{x_1^2}{\tau}\right.\\
&\quad \left. + \frac{h^2}{8\tau} + 2\omega\right]\\
&= \frac{1}{2\omega}\widetilde{\Omega}_1\left(h,\tau\right)\times 2\omega = \widetilde{\Omega}_1\left(h,\tau\right).
\end{aligned}$$

For $i = 2$, we obtain

$$
\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{u,k+1}_{2,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)\left(a_2x_2-x_2^2\right)\right.\\
&+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(2\left(a_2\left(x_2+\frac{\sqrt{3}h}{2}\right)-\left(x_2+\frac{\sqrt{3}h}{2}\right)^2\right)\right.\\
&\left.\left.+2\left(a_2\left(x_2-\frac{\sqrt{3}h}{2}\right)-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2\right)+2\left(a_2x_2-x_2^2\right)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{a_2x_2}{\tau}-\frac{x_2^2}{\tau}-\frac{h^2}{8\tau}+2\omega\right], \qquad (4.52)\\
\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{u,k}_{2,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{3}{4\tau}\left(a_2x_2-x_2^2\right)+\frac{1}{24\tau}\left(6a_2x_2-6x_2^2-3h^2\right)\right]\\
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{a_2x_2}{\tau}-\frac{x_2^2}{\tau}-\frac{h^2}{8\tau}\right]. \qquad (4.53)
\end{aligned}
$$

Using (4.52) and (4.53) gives

$$
\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{u,k+1}_{2,h,\tau}-\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{u,k}_{2,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{a_2x_2}{\tau}-\frac{x_2^2}{\tau}-\frac{h^2}{8\tau}-\frac{a_2x_2}{\tau}+\frac{x_2^2}{\tau}\right.\\
&\left.+\frac{h^2}{8\tau}+2\omega\right]\\
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\times 2\omega=\widetilde{\Omega}_1(h,\tau).
\end{aligned}
$$

Similarly, for $i=3$ we have

$$
\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{u,k+1}_{3,h,\tau} &= \widetilde{\Omega}_1(h,\tau)\left[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)(t+\tau)+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)(6(t+\tau))\right]\\
&= \widetilde{\Omega}_1(h,\tau)\left[\frac{t}{\tau}+1\right] \qquad (4.54)\\
\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{u,k}_{3,h,\tau} &= \widetilde{\Omega}_1(h,\tau)\left[\left(\frac{3}{4\tau}t+\frac{1}{24\tau}6t\right)\right]=\widetilde{\Omega}_1(h,\tau)\left[\frac{3t}{4\tau}+\frac{t}{4\tau}\right]\\
&= \widetilde{\Omega}_1(h,\tau)\left[\frac{t}{\tau}\right]. \qquad (4.55)
\end{aligned}
$$

Using (4.54) and (4.55) yields

$$
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{u,k+1}_{3,h,\tau}-\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{u,k}_{3,h,\tau}=\widetilde{\Omega}_1(h,\tau)\left[\frac{t}{\tau}+1\right]-\widetilde{\Omega}_1(h,\tau)\left[\frac{t}{\tau}\right]=\widetilde{\Omega}_1(h,\tau).
$$

Next for the irregular hexagons with a left ghost point for $i=1$, the following are achieved.

$$
\begin{aligned}
\widetilde{\Theta}^{*}_{h,\tau}\overline{\xi}^{u,k+1}_{1,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\Bigg[\left(\frac{17}{24\tau}+\frac{14\omega}{3h^2}\right)\left(a_1x_1-x_1^2\right)+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\Bigg(a_1h-h^2 \\
&\quad +a_1h-h^2+\frac{3}{2}a_1h-\frac{9}{4}h^2\Bigg)\Bigg] \\
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\Bigg[\frac{17a_1x_1}{24\tau}-\frac{17x_1^2}{24\tau}+\frac{14\omega a_1x_1}{3h^2}-\frac{14\omega x_1^2}{3h^2}+\frac{7a_1h}{48\tau} \\
&\quad -\frac{17h^2}{96\tau}-\frac{7\omega a_1h}{3h^2}+\frac{17\omega}{6}\Bigg].
\end{aligned} \tag{4.56}
$$

$$
\begin{aligned}
\widetilde{\Lambda}^{*}_{h,\tau}\overline{\xi}^{u,k}_{1,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\Bigg[\frac{17}{24\tau}\left(a_1x_1-x_1^2\right)+\frac{1}{24\tau}\Bigg(a_1h-h^2+a_1h-h^2 \\
&\quad +\frac{3}{2}a_1h-\frac{9}{4}h^2\Bigg)\Bigg] \\
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{17a_1x_1}{24\tau}-\frac{17x_1^2}{24\tau}+\frac{7a_1h}{48\tau}-\frac{17h^2}{96\tau}\right],
\end{aligned} \tag{4.57}
$$

$$
\widetilde{\Gamma}^{*}_{h,\tau}\overline{\xi}^{u*}_{1,\phi,h,\tau} = 0. \tag{4.58}
$$

Using equations (4.56), (4.57), and (4.58) with substituting $x_1=\frac{h}{2}$ for $i=1$ we have

$$
\widetilde{\Theta}^{*}_{h,\tau}\overline{\xi}^{u,k+1}_{1,h,\tau}-\widetilde{\Lambda}^{*}_{h,\tau}\overline{\xi}^{u,k}_{1,h,\tau}-\widetilde{\Gamma}^{*}_{h,\tau}\overline{\xi}^{u*}_{1,\phi,h,\tau}=\frac{5}{6}\widetilde{\Omega}_1(h,\tau).
$$

Consequently, for $i=2$, the following are valid:

$$
\begin{aligned}
\widetilde{\Theta}^{*}_{h,\tau}\overline{\xi}^{u,k+1}_{2,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\Bigg[\left(\frac{17}{24\tau}+\frac{14\omega}{3h^2}\right)\Bigg(a_2\left(x_2+\frac{\sqrt{3}h}{2}\right) \\
&\quad -\left(x_2+\frac{\sqrt{3}h}{2}\right)^2+a_2\left(x_2-\frac{\sqrt{3}h}{2}\right)-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2 \\
&\quad +a_2x_2-x_2^2\Bigg)\Bigg], \\
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{20a_2x_2}{24\tau}-\frac{20x_2^2}{24\tau}+\frac{8\omega a_2x_2}{3h^2}-\frac{8\omega x_2^2}{3h^2}-\frac{h^2}{16\tau}+\omega\right]
\end{aligned} \tag{4.59}
$$

$$
\widetilde{\Lambda}^{*}_{h,\tau}\overline{\xi}^{u,k}_{2,h,\tau} = \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{17}{24\tau}\left(a_2x_2-x_2^2\right)+\frac{1}{24\tau}-\left(3a_2x_2-3x_2^2-\frac{3}{2}h^2\right)\right]
$$

$$= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{20a_2x_2}{24\tau} - \frac{20x_2^2}{24\tau} - \frac{h^2}{16\tau}\right], \tag{4.60}$$

$$\begin{aligned}
\widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{u*}_{2,\phi,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\Bigg[\left(-\frac{1}{36\tau} + \frac{4\omega}{9h^2}\right)\Bigg(a_2\left(x_2 + \frac{\sqrt{3}h}{2}\right) \\
&- \left(x_2 + \frac{\sqrt{3}h}{2}\right)^2 + a_2\left(x_2 - \frac{\sqrt{3}h}{2}\right) - \left(x_2 - \frac{\sqrt{3}h}{2}\right)^2 \\
&+ a_2x_2 - x_2^2\Bigg) + \left(\frac{1}{18\tau} + \frac{16\omega}{9h^2}\right)\left(a_2x_2 - x_2^2\right) \\
&+ \frac{1}{36\tau}\Bigg(a_2\left(x_2 + \frac{\sqrt{3}h}{2}\right) - \left(x_2 + \frac{\sqrt{3}h}{2}\right)^2 \\
&+ a_2\left(x_2 - \frac{\sqrt{3}h}{2}\right) - \left(x_2 - \frac{\sqrt{3}h}{2}\right)^2 + a_2x_2 - x_2^2\Bigg) \\
&- \frac{1}{18\tau}\left(a_2x_2 - x_2^2\right)\Bigg] \\
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\frac{8\omega a_2x_2}{3h^2} - \frac{8\omega x_2^2}{3h^2} - \frac{2}{3}\omega\right].
\end{aligned} \tag{4.61}$$

Using equations (4.59), (4.60), and (4.61) we get

$$\begin{aligned}
\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{u,k+1}_{2,h,\tau} - \widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{u,k}_{2,h,\tau} - \widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{u*}_{2,\phi,h,\tau} &= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\Bigg[\frac{20a_2x_2}{24\tau} - \frac{20x_2^2}{24\tau} \\
&+ \frac{8\omega a_2x_2}{3h^2} - \frac{8\omega x_2^2}{3h^2} - \frac{h^2}{16\tau} + \omega - \frac{20a_2x_2}{24\tau} + \frac{20x_2^2}{24\tau} + \frac{h^2}{16\tau} \\
&- \frac{8\omega a_2x_2}{3h^2} + \frac{8\omega x_2^2}{3h^2} + \frac{2}{3}\omega\Bigg] \\
&= \frac{1}{2\omega}\widetilde{\Omega}_1(h,\tau)\left[\omega + \frac{2\omega}{3}\right] = \frac{5}{6}\widetilde{\Omega}_1(h,\tau),
\end{aligned}$$

Next, for $i = 3$, the following is obtained.

$$\begin{aligned}
\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{u,k+1}_{3,h,\tau} &= \widetilde{\Omega}_1(h,\tau)(t+\tau)\left[\left(\frac{17}{24\tau} + \frac{14\omega}{3h^2}\right) + 3\left(\frac{1}{24\tau} - \frac{2\omega}{3h^2}\right)\right] \\
&= \widetilde{\Omega}_1(h,\tau)(t+\tau)\left[\frac{20}{24\tau} + \frac{8\omega}{3h^2}\right],
\end{aligned} \tag{4.62}$$

$$\widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{u,k}_{3,h,\tau} = \widetilde{\Omega}_1(h,\tau)\,t\left[\frac{17}{24\tau} + \frac{3}{24\tau}\right]$$

$$= \widetilde{\Omega}_1(h,\tau)\, t\left[\frac{20}{24\tau}\right] \tag{4.63}$$

$$\widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{u*}_{3,\phi,h,\tau} = \widetilde{\Omega}_1(h,\tau)(t+\tau)\left[2\left(-\frac{1}{36\tau}+\frac{4\omega}{9h^2}\right)+3\left(\frac{1}{18\tau}+\frac{16\omega}{9h^2}\right)\right]$$

$$+\widetilde{\Omega}_1(h,\tau)\, t\left[\left(\frac{2}{36\tau}-\frac{1}{18\tau}\right)\right]$$

$$= \widetilde{\Omega}_1(h,\tau)(t+\tau)\left[\frac{8\omega}{3h^2}\right]. \tag{4.64}$$

Using (4.62), (4.63), and (4.64), it follows that

$$\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{u,k+1}_{3,h,\tau} - \widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{u,k}_{3,h,\tau} - \widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{u*}_{3,\phi,h,\tau} = \frac{5}{6}\widetilde{\Omega}_1(h,\tau).$$

In a similar way we can show that second equation of (4.49) holds true on $D^{*rh}\gamma_\tau$. Therefore, difference problems (4.45) and (4.49) in matrix form are

$$\widetilde{A}\widehat{\xi}^{u,k+1} = \widetilde{B}\widehat{\xi}^{u,k} + \tau\widehat{\eta}^{u,k}, \tag{4.65}$$

$$\widetilde{A}\overline{\xi}^{u,k+1}_i = \widetilde{B}\overline{\xi}^{u,k}_i + \tau\overline{\eta}^{u,k}_i,\ i = 1,2,3, \tag{4.66}$$

accordingly, and $\widetilde{A}$ and $\widetilde{B}$ are as given in (4.16) and $\overline{\eta}^{u,k}_i, \overline{\xi}^{u,k}_i, i = 1,2,3$ and $\widehat{\xi}^{u,k}, \widehat{\eta}^{u,k}, \in R^N$ satisfying $\overline{\xi}^{u,0}_i \geq 0$, $\left|\widehat{\xi}^{u,0}\right| \leq \overline{\xi}^{u,0}_i$, and $\overline{\eta}^{u,k}_i \geq 0$, and $\left|\widehat{\eta}^{u,k}\right| \leq \overline{\eta}^{u,k}_i$, $i = 1,2,3$, for $k = 0,...,M'-1$. Using that $\widetilde{\Omega}_1(h,\tau) \geq \left|\widetilde{\Psi}^{u,k}_1\right|$ on $D^{0h}\gamma_\tau$, and $\frac{5}{6}\widetilde{\Omega}_1(h,\tau) \geq \left|\widetilde{\Psi}^{u,k}_2\right|$ on $D^{*h}\gamma_\tau$ and on the basis of Lemma 4.3 we obtain

$$\left|\xi^u_{h,\tau}(x_1,x_2,t)\right| \leq \min_{i=1,2,3}\overline{\xi}^u_i(x_1,x_2,t) \leq d\widetilde{\Omega}_1(h,\tau)\rho(x_1,x_2,t) \text{ on } \overline{D^h\gamma_\tau}. \tag{4.67}$$

The proof of (4.42) is analogous and follows from Lemma 4.3 by taking the majorant functions

$$\overline{\xi}^{u_t}_1(x_1,x_2,t) = \frac{1}{2\omega}\widetilde{\Omega}_{t,1}(h,\tau)\left(a_1x_1 - x_1^2\right) \geq 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{4.68}$$

$$\overline{\xi}^{u_t}_2(x_1,x_2,t) = \frac{1}{2\omega}\widetilde{\Omega}_{t,1}(h,\tau)\left(a_2x_2 - x_2^2\right) \geq 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{4.69}$$

$$\overline{\xi}^{u_t}_3(x_1,x_2,t) = \widetilde{\Omega}_{t,1}(h,\tau)\, t \geq 0 \text{ on } \overline{D^h\gamma_\tau}, \tag{4.70}$$

where $\widetilde{\Omega}_{t,1}(h,\tau)$ is as given in (4.44). □

4.2 Second Stages of the Implicit Methods Approximating $\frac{\partial u}{\partial x_1}$ and $\frac{\partial^2 u}{\partial x_1 \partial t}$ with $O\left(h^4+\tau\right)$ Order of Convergence

4.2.1 Hexagonal Grid Approximation to $\frac{\partial u}{\partial x_1}$: Stage 2$\left(H^{4th}\left(\frac{\partial u}{\partial x_1}\right)\right)$

For obtaining fourth order accurate numerical approximation to $v=\frac{\partial u}{\partial x_1}$ first we apply the implicit method given in Stage 1$\left(H^{4th}(u)\right)$ and compute the approximate solution $u_{h,\tau}$. Next, we denote $p_i=\frac{\partial u}{\partial x_1}$ on $S_T\gamma_i, i=1,2,...,5$ and use the problem $\left(BVP\left(\frac{\partial u}{\partial x_1}\right)\right)$ given in Chapter 2.

Taking into consideration $u\in C_{x,t}^{9+\alpha,\frac{9+\alpha}{2}}\left(\overline{Q}_T\right)$, we require $v\in C_{x,t}^{8+\alpha,4+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$. Further, we take

$$p_{1h}^{4^{th}}=\begin{cases}\frac{1}{12h}\left(-25u(0,x_2,t)+48u_{h,\tau}(h,x_2,t)\right.\\ -36u_{h,\tau}(2h,x_2,t)+16u_{h,\tau}(3h,x_2,t)\\ \left.-3u_{h,\tau}(4h,x_2,t)\right)\text{ if }P_0\in D^{0h}\gamma_\tau,\\ \frac{1}{840h}\left(-2816u(0,x_2,t)+3675u_{h,\tau}\left(\frac{h}{2},x_2,t\right)\right.\\ -1225u_{h,\tau}\left(\frac{3h}{2},x_2,t\right)+441u_{h,\tau}\left(\frac{5h}{2},x_2,t\right)\\ \left.-75u_{h,\tau}\left(\frac{7h}{2},x_2,t\right)\right)\text{ if }P_0\in D^{*lh}\gamma_\tau,\end{cases}\quad\text{on } S_T^h\gamma_1, \tag{4.71}$$

$$p_{3h}^{4^{th}}=\begin{cases}\frac{1}{12h}\left(25u(a_1,x_2,t)-48u_{h,\tau}(a_1-h,x_2,t)\right.\\ +36u_{h,\tau}(a_1-2h,x_2,t)-16u_{h,\tau}(a_1-3h,x_2,t)\\ \left.+3u_{h,\tau}(a_1-4h,x_2,t)\right)\text{ if }P_0\in D^{0h}\gamma_\tau,,\\ \frac{1}{840h}\left(2816u(a_1,x_2,t)-3675u_{h,\tau}\left(a_1-\frac{h}{2},x_2,t\right)\right.\\ +1225u_{h,\tau}\left(a_1-\frac{3h}{2},x_2,t\right)-441u_{h,\tau}\left(a_1-\frac{5h}{2},x_2,t\right)\\ \left.+75u_{h,\tau}\left(a_1-\frac{7h}{2},x_2,t\right)\right)\text{ if }P_0\in D^{*rh}\gamma_\tau\end{cases}\quad\text{on } S_T^h\gamma_{3,} \tag{4.72}$$

$$p_{ih}=\frac{\partial\phi(x_1,x_2,t)}{\partial x_1}\text{ on } S_T^h\gamma_i, i=2,4, \tag{4.73}$$

$$p_{5h}=\frac{\partial\varphi(x_1,x_2)}{\partial x_1}\text{ on } S_T^h\gamma_5, \tag{4.74}$$

where $\varphi(x_1,x_2)$, $\phi(x_1,x_2,t)$ are as in (2.9), and $u_{h,\tau}$ is obtained by using Stage $1\left(H^{4th}(u)\right)$. The derivation of the forward difference formula (4.71) for the irregular grid points which have a center $\frac{h}{2}$ units away from the boundary $x_1=0$ is as follows:

$A: u(x_1,x_2,t)$

$B: u(x_1+\frac{h}{2},x_2,t)$

$C: u(x_1+\frac{3h}{2},x_2,t)$

$D: u(x_1+\frac{5h}{2},x_2,t)$

$E: u(x_1+\frac{7h}{2},x_2,t)$

$$\begin{aligned} B: u\left(x_1+\frac{h}{2},x_2,t\right) &= u(x_1,x_2,t)+\frac{h}{2}\partial_{x_1}u(x_1,x_2,t) \\ &+\frac{h^2}{8}\partial_{x_1}^2 u(x_1,x_2,t)+\frac{h^3}{48}\partial_{x_1}^3 u(x_1,x_2,t) \\ &+\frac{h^4}{384}\partial_{x_1}^4 u(x_1,x_2,t)+\frac{h^5}{3840}\partial_{x_1}^5 u(x_1+\upsilon_1 h,x_2,t), \end{aligned} \tag{4.75}$$

$$\begin{aligned} C: u\left(x_1+\frac{3h}{2},x_2,t\right) &= u(x_1,x_2,t)+\frac{3h}{2}\partial_{x_1}u(x_1,x_2,t) \\ &+\frac{9h^2}{8}\partial_{x_1}^2 u(x_1,x_2,t)+\frac{27h^3}{48}\partial_{x_1}^3 u(x_1,x_2,t) \\ &+\frac{81h^4}{384}\partial_{x_1}^4 u(x_1,x_2,t)+\frac{243h^5}{3840}\partial_{x_1}^5 u(x_1+\upsilon_2 h,x_2,t) \end{aligned} \tag{4.76}$$

$$\begin{aligned} D: u\left(x_1+\frac{5h}{2},x_2,t\right) &= u(x_1,x_2,t)+\frac{5h}{2}\partial_{x_1}u(x_1,x_2,t) \\ &+\frac{25h^2}{8}\partial_{x_1}^2 u(x_1,x_2,t)+\frac{125h^3}{48}\partial_{x_1}^3 u(x_1,x_2,t) \\ &+\frac{625h^4}{384}\partial_{x_1}^4 u(x_1,x_2,t)+\frac{3125h^5}{3840}\partial_{x_1}^5 u(x_1+\upsilon_3 h,x_2,t), \end{aligned} \tag{4.77}$$

$$\begin{aligned} E: u\left(x_1+\frac{7h}{2},x_2,t\right) &= u(x_1,x_2,t)+\frac{7h}{2}u_{x_1}(x_1,x_2,t) \\ &+\frac{49h^2}{8}\partial_{x_1}^2 u(x_1,x_2,t)+\frac{343h^3}{48}\partial_{x_1}^3 u(x_1,x_2,t) \\ &+\frac{2401h^4}{384}\partial_{x_1}^4 u(x_1,x_2,t)+\frac{16807h^5}{3840}\partial_{x_1}^5 u(x_1+\upsilon_4 h,x_2,t), \end{aligned} \tag{4.78}$$

where, $0<\upsilon_i<\frac{1}{2}+(i-1)$ for $i=1,\ldots,4$. By multiplying the equations (4.75), (4.76), (4.77) and (4.78) with $\frac{35}{8}, \frac{-35}{24}, \frac{21}{40}-\frac{5}{56}$ respectively and adding them we get the following:

$$\frac{35}{8}u\left(x_1+\frac{h}{2},x_2,t\right)-\frac{35}{24}u\left(x_1+\frac{3h}{2},x_2,t\right)+\frac{21}{40}u\left(x_1+\frac{5h}{2},x_2,t\right)$$
$$=-\frac{5}{56}u\left(x_1+\frac{7h}{2},x_2,t\right)-\frac{352}{105}u\left(x_1,x_2,t\right)$$
$$+h\partial_{x_1}u\left(x_1,x_2,t\right)-\frac{7h^5}{128}\partial_{x_1}^5u\left(x_1+\widetilde{\upsilon}h,x_2,t\right). \tag{4.79}$$

where, $0<\widetilde{\upsilon}<\frac{7}{2}$. Simplifying yields

$$\frac{1}{840}\left(3675u\left(x_1+\frac{h}{2},x_2,t\right)-1225u\left(x_1+\frac{3h}{2},x_2,t\right)\right.$$
$$\left.+441u\left(x_1+\frac{5h}{2},x_2,t\right)-75u\left(x_1+\frac{7h}{2},x_2,t\right)-2816u\left(x_1,x_2,t\right)\right)$$
$$=h\partial_{x_1}u\left(x_1,x_2,t\right)-\frac{7h^5}{128}\partial_{x_1}^5u\left(x_1+\widetilde{\upsilon}h,x_2,t\right) \tag{4.80}$$

hence

$$\frac{1}{840h}\left(-2816u\left(x_1,x_2,t\right)+3675u\left(x_1+\frac{h}{2},x_2,t\right)-1225u\left(x_1+\frac{5h}{2},x_2,t\right)\right.$$
$$\left.+441u\left(x_1+\frac{5h}{2},x_2,t\right)-75u\left(x_1+\frac{7h}{2},x_2,t\right)\right)$$
$$=\partial_{x_1}u\left(x_1,x_2,t\right)+O\left(h^4\right) \tag{4.81}$$

Lemma 4.4: (Buranay et al. [53]) Let u be the solution of BVP(u) in (2.9) and $u_{h,\tau}$ be the solution of (4.1) in Stage $1\left(H^{4th}\left(u\right)\right)$. Then, it holds that

$$\left|p_{ih}^{4^{th}}\left(u_{h,\tau}\right)-p_{ih}^{4^{th}}\left(u\right)\right|\leq 15d\widetilde{\Omega}_1\left(h,\tau\right),\ \ i=1,3, \tag{4.82}$$

where $\widetilde{\Omega}_1\left(h,\tau\right)$ in (4.43) and d in (2.60) was defined.

Proof. Using (4.71), (4.72) and from Theorem 4.1, and using (4.41) when $P_0\in D^{0h}\gamma_\tau$ gives

$$\left|p_{ih}^{4^{th}}\left(u_{h,\tau}\right)-p_{ih}^{4^{th}}\left(u\right)\right|\leq\frac{1}{12h}\Big(48hd\widetilde{\Omega}_1\left(h,\tau\right)+36\left(2h\right)d\widetilde{\Omega}_1\left(h,\tau\right)$$
$$+\,16\left(3h\right)d\widetilde{\Omega}_1\left(h,\tau\right)+3\left(4h\right)d\widetilde{\Omega}_1\left(h,\tau\right)\Big)$$
$$\leq 15d\widetilde{\Omega}_1\left(h,\tau\right),\ \ i=1,3,\ \text{if } P_0\in D^{0h}\gamma_\tau, \tag{4.83}$$

where $\widetilde{\Omega}_1\left(h,\tau\right)$ in (4.43) and d in (2.60) was defined. In the case $P_0\in D^{*h}\gamma_\tau$ it follows that

$$\left|p_{ih}^{4^{th}}\left(u_{h,\tau}\right)-p_{ih}^{4^{th}}\left(u\right)\right|\leq\frac{1}{840h}\left(3675\frac{h}{2}d\widetilde{\Omega}_1\left(h,\tau\right)+1225\frac{3h}{2}d\widetilde{\Omega}_1\left(h,\tau\right)\right.$$
$$\left.+\,441\frac{5h}{2}d\widetilde{\Omega}_1\left(h,\tau\right)+75\frac{7h}{2}d\widetilde{\Omega}_1\left(h,\tau\right)\right)$$
$$\leq 6d\widetilde{\Omega}_1\left(h,\tau\right),\ \ i=1,3 \text{ if } P_0\in D^{*h}\gamma_\tau. \tag{4.84}$$

Therefore, (4.82) follows. □

Lemma 4.5: (Buranay et al. [53]) Let $u_{h,\tau}$ be the solution of the problem (4.1) in Stage $1\left(H^{4th}\left(u\right)\right)$. Then, it holds that

$$\max_{S_T^h\gamma_1\cup S_T^h\gamma_3}\left|p_{ih}^{4^{th}}\left(u_{h,\tau}\right)-p_i\right|\leq\widetilde{M}_1h^4+15d\widetilde{\Omega}_1\left(h,\tau\right),\ \ i=1,3, \tag{4.85}$$

where $\widetilde{M}_1=\frac{1}{5}\max\limits_{\overline{Q}_T}\left|\frac{\partial^5u}{\partial x_1^5}\right|$ and $\widetilde{\Omega}_1\left(h,\tau\right)$ in (4.43) and d in (2.60) was defined.

Proof. On the basis of the assumption $u\in C_{x,t}^{9+\alpha,\frac{9+\alpha}{2}}\left(\overline{Q}_T\right)$, it follows that at the points $(0,x_2,k\tau)\in S_T^h\gamma_1$ and $(a_1,x_2,k\tau)\in S_T^h\gamma_3$ of each line segment

$$\left[\left(x_1,\eta\frac{\sqrt{3}}{2}h,k\tau\right):0\leq x_1\leq a_1,0\leq x_2=\eta\frac{\sqrt{3}}{2}h\leq a_2,\ 0\leq t=k\tau\leq T\right],$$

we obtain fourth order approximation of $\frac{\partial u}{\partial x_1}$ by the formulae (4.71) and (4.72). From the truncation error formula (see Burden and Faires [63]) results

$$\max_{S_T^h\gamma_1\cup S_T^h\gamma_3}\left|p_{ih}^{4^{th}}\left(u\right)-p_i\right|\leq\frac{h^4}{5}\max_{\overline{Q}_T}\left|\frac{\partial^5u}{\partial x_1^5}\right|,\ i=1,3 \text{ if } P_0\in D^{0h}\gamma_\tau. \tag{4.86}$$

Analogously,

$$\max_{S_T^h\gamma_1\cup S_T^h\gamma_3}\left|p_{ih}^{4^{th}}\left(u\right)-p_i\right|\leq\frac{7h^4}{128}\max_{\overline{Q}_T}\left|\frac{\partial^5u}{\partial x_1^5}\right|,\ i=1,3 \text{ if } P_0\in D^{*h}\gamma_\tau, \tag{4.87}$$

Using Lemma 4.4 and the estimations (4.86) and (4.87) follows (4.85). □

Subsequently, for a fourth order numerical solution of BVP$\left(\frac{\partial u}{\partial x_1}\right)$ we propose the following problem and call this Stage 2$\left(H^{4th}\left(\frac{\partial u}{\partial x_1}\right)\right)$.

Stage 2$\left(H^{4th}\left(\frac{\partial u}{\partial x_1}\right)\right)$

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}v_{h,\tau}^{k+1} &= \widetilde{\Lambda}_{h,\tau}v_{h,\tau}^{k} + \widetilde{D}_{x_1}\widetilde{\psi} \text{ on } D^{0h}\gamma_\tau,\\
\widetilde{\Theta}_{h,\tau}^{*}v_{h,\tau}^{k+1} &= \widetilde{\Lambda}_{h,\tau}^{*}v_{h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*}p_{1h}^{4^{th}}\left(u_{h,\tau}\right) + \widetilde{D}_{x_1}\widetilde{\psi}^{*} \text{ on } D^{*lh}\gamma_\tau\\
\widetilde{\Theta}_{h,\tau}^{*}v_{h,\tau}^{k+1} &= \widetilde{\Lambda}_{h,\tau}^{*}v_{h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*}p_{3h}^{4^{th}}\left(u_{h,\tau}\right) + \widetilde{D}_{x_1}\widetilde{\psi}^{*} \text{ on } D^{*rh}\gamma_\tau\\
v_{h,\tau} &= p_{ih}^{4^{th}}\left(u_{h,\tau}\right) \text{ on } S_T^h\gamma_i, i = 1,3,\\
v_{h,\tau} &= p_{ih} \text{ on } S_T^h\gamma_i,\ i = 2,4,5 \qquad (4.88)
\end{aligned}$$

where $p_{1h}^{4^{th}}, p_{3h}^{4^{th}},\ p_{ih}\ , i = 2,4,5$ are defined by (4.71)–(4.74) and the operators $\widetilde{\Theta}_{h,\tau}$, $\widetilde{\Lambda}_{h,\tau}, \widetilde{\Theta}_{h,\tau}^{*}, \widetilde{\Gamma}_{h,\tau}^{*}$ and $\widetilde{\Lambda}_{h,\tau}^{*}$ are given in (4.4)-(4.8), respectively. Furthermore,

$$\widetilde{D}_{x_1}\widetilde{\psi} = \partial_{x_1}f_{P_0}^{k+1} + \frac{1}{16}h^2\left(\partial_{x_1}^3 f_{P_0}^{k+1} + \partial_{x_2}^2\partial_{x_1}f_{P_0}^{k+1}\right), \qquad (4.89)$$

$$\begin{aligned}
\widetilde{D}_{x_1}\widetilde{\psi}^{*} = \frac{h^2}{96\tau\omega}\partial_{x_1}f_{P_A}^{k+1} - \frac{h^2}{96\tau\omega}\partial_{x_1}f_{P_A}^{k} - \frac{1}{6}\partial_{x_1}f_{P_A}^{k+1} + \partial_{x_1}f_{P_0}^{k+1}\\
+ \frac{1}{16}h^2\left(\partial_{x_1}^3 f_{P_0}^{k+1} + \partial_{x_2}^2\partial_{x_1}f_{P_0}^{k+1}\right) \qquad (4.90)
\end{aligned}$$

Let

$$\xi_{h,\tau}^{v} = v_{h,\tau} - v \text{ on } \overline{D^h\gamma_\tau}, \qquad (4.91)$$

where $v = \frac{\partial u}{\partial x_1}$. From (4.88) and (4.91) we have

$$\begin{aligned}
\widetilde{\Theta}_{h,\tau}\xi_{h,\tau}^{v,k+1} &= \widetilde{\Lambda}_{h,\tau}\xi_{h,\tau}^{v,k} + \widetilde{\Psi}_1^{v,k} \text{ on } D^{0h}\gamma_\tau,\\
\widetilde{\Theta}_{h,\tau}^{*}\xi_{h,\tau}^{v,k+1} &= \widetilde{\Lambda}_{h,\tau}^{*}\xi_{h,\tau}^{v,k} + \widetilde{\Gamma}_{h,\tau}^{*}\xi_{h,\tau}^{*v} + \widetilde{\Psi}_2^{v,k} \text{ on } D^{*h}\gamma_\tau\\
\xi_{h,\tau}^{v} &= 0 \text{ on } S_T^h\gamma_i,\ i = 2,4,5,\\
\xi_{h,\tau}^{v} &= \xi_{h,\tau}^{*v} = p_{ih}^{4^{th}}\left(u_{h,\tau}\right) - p_i \text{ on } S_T^h\gamma_i,\ i = 1,3. \qquad (4.92)
\end{aligned}$$

where

$$\widetilde{\Psi}_1^{v,k} = \widetilde{\Lambda}_{h,\tau} v^k - \widetilde{\Theta}_{h,\tau} v^{k+1} + \widetilde{D}_{x_1} \widetilde{\psi}, \quad (4.93)$$

$$\widetilde{\Psi}_2^{v,k} = \widetilde{\Lambda}_{h,\tau}^* v^k - \widetilde{\Theta}_{h,\tau}^* v^{k+1} + \widetilde{\Gamma}_{h,\tau}^* p_i + \widetilde{D}_{x_1} \widetilde{\psi}^*, \ i = 1,3. \quad (4.94)$$

Next, let $\widetilde{\theta}_1 = \mu_1(v)$, $\widetilde{\sigma}_1 = \mu_2(v)$, where μ_1, μ_2 are given in (4.39), (4.40), respectively, and

$$\widetilde{\theta} = \max \left\{ \widetilde{\theta}_1, \frac{\widetilde{M}_1}{\rho} + 15\frac{d}{\rho}\left(\frac{3}{160} + \frac{47\omega}{2880}\right)\widetilde{\alpha} \right\}, \quad (4.95)$$

$$\widetilde{\sigma} = \max \left\{ \widetilde{\sigma}_1, 15d\widetilde{\beta} \right\}, \quad (4.96)$$

where $\widetilde{\alpha} = \mu_1(u), \widetilde{\beta} = \mu_2(u)$ and d in (2.60), also $\widetilde{M}_1$ is as given in Lemma 4.5 and $\rho = \frac{3}{640\omega} + \frac{47}{11520}$.

Theorem 4.2: (Buranay et al. [53]) The solution $v_{h,\tau}$ of the finite difference problem given in Stage 2$\left(H^{4th}\left(\frac{\partial u}{\partial x_1}\right)\right)$ satisfies

$$\max_{D^h\gamma_\tau} \left| v_{h,\tau} - v \right| \leq \frac{6}{5}\widetilde{\sigma}(T+1)\tau + \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\left(1 + a_1^2 + a_2^2\right)\widetilde{\theta} h^4, \quad (4.97)$$

for $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$ where $\widetilde{\theta}, \widetilde{\sigma}$ are as given in (4.95), (4.96), respectively, and $v = \frac{\partial u}{\partial x_1}$ is the exact solution of BVP$\left(\frac{\partial u}{\partial x_1}\right)$.

Proof. Consider the next system

$$\widetilde{\Theta}_{h,\tau} \widehat{\xi}_{h,\tau}^{v,k+1} = \widetilde{\Lambda}_{h,\tau} \widehat{\xi}_{h,\tau}^{v,k} + \widetilde{\Omega}_2(x_1) \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^* \widehat{\xi}_{h,\tau}^{v,k+1} = \widetilde{\Lambda}_{h,\tau}^* \widehat{\xi}_{h,\tau}^{v,k} + \widetilde{\Gamma}_{h,\tau}^* \widehat{\xi}_{h,\tau}^{v*} + \widetilde{\Omega}_2(x_1) - \frac{1}{6}\widetilde{\Omega}_2(\widehat{p}) \text{ on } D^{*h}\gamma_\tau,$$

$$\widehat{\xi}_{h,\tau}^{v} = 0 \text{ on } S_T^h\gamma_i, \ i = 2,4,5,$$

$$\widehat{\xi}_{h,\tau}^{v} = \widehat{\xi}_{h,\tau}^{v*} = p_{ih}^{4^{th}}\left(u_{h,\tau}\right) - p_i \text{ on } S_T^h\gamma_i, \ i = 1,3, \quad (4.98)$$

where

$$\widetilde{\Omega}_2(x_1) = \frac{3}{5a_1}\widetilde{\sigma}\tau(2a_1 - x_1) + \left(\frac{3}{160} + \frac{47}{2880}\omega\right)\widetilde{\theta}h^4,$$

$$\geq \frac{3}{5}\widetilde{\sigma}\tau + \left(\frac{3}{160} + \frac{47}{2880}\omega\right)\widetilde{\theta}h^4 \geq \left|\widetilde{\Psi}_1^{v,k}\right|, \tag{4.99}$$

$$\widetilde{\Omega}_2(x_1) - \frac{1}{6}\widetilde{\Omega}_2(\widehat{p}) = \begin{cases} \widetilde{\sigma}\tau\left(1 - \frac{3h}{10a_1}\right) + \left(\frac{1}{64} + \frac{47}{3456}\omega\right)\widetilde{\theta}h^4 \text{ if } P_0 \in D^{*lh}\gamma_\tau, \\ \widetilde{\sigma}\tau\left(\frac{1}{2} + \frac{3h}{10a_1}\right) + \left(\frac{1}{64} + \frac{47}{3456}\omega\right)\widetilde{\theta}h^4 \text{ if } P_0 \in D^{*rh}\gamma_\tau, \end{cases}$$

$$\geq \left|\widetilde{\Psi}_2^{v,k}\right|. \tag{4.100}$$

Further, $x_1 = \frac{h}{2}$ and $\widehat{p} = 0$ if $P_0 \in D^{*lh}\gamma_\tau$ and $x_1 = a_1 - \frac{h}{2}, \widehat{p} = a_1$ if $P_0 \in D^{*rh}\gamma_\tau$. We take the majorant function

$$\overline{\xi}^v(x_1, x_2, t) = \overline{\xi}_1^v(x_1, x_2, t) + \overline{\xi}_2^v(x_1, x_2, t), \tag{4.101}$$

where

$$\overline{\xi}_1^v(x_1, x_2, t) = \frac{3}{5a_1}\widetilde{\sigma}\tau(t+1)(2a_1 - x_1) \text{ on } \overline{D^h\gamma_\tau},$$

$$\overline{\xi}_2^v(x_1, x_2, t) = \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\widetilde{\theta}h^4\left(1 + a_1^2 + a_2^2 - x_1^2 - x_2^2\right) \text{ on } \overline{D^h\gamma_\tau}.$$

The function in (4.101) satisfies the difference problem

$$\widetilde{\Theta}_{h,\tau}\overline{\xi}_{h,\tau}^{v,k+1} = \widetilde{\Lambda}_{h,\tau}\overline{\xi}_{h,\tau}^{v,k} + \widetilde{\Omega}_2(x_1) \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^*\overline{\xi}_{h,\tau}^{v,k+1} = \widetilde{\Lambda}_{h,\tau}^*\overline{\xi}_{h,\tau}^{v,k} + \widetilde{\Gamma}_{h,\tau}^*\overline{\xi}_{h,\tau}^{v*} + \widetilde{\Omega}_2(x_1) - \frac{1}{6}\widetilde{\Omega}_2(\widehat{p}) \text{ on } D^{*h}\gamma_\tau,$$

$$\overline{\xi}_{h,\tau}^v = \overline{\xi}_{h,\tau}^{v*} = \overline{\xi}_1^v(0, x_2, t) + \overline{\xi}_2^v(0, x_2, t) \text{ on } S_T^h\gamma_1,$$

$$\overline{\xi}_{h,\tau}^v = \overline{\xi}_1^v(x_1, 0, t) + \overline{\xi}_2^v(x_1, 0, t) \text{ on } S_T^h\gamma_2,$$

$$\overline{\xi}_{h,\tau}^v = \overline{\xi}_{h,\tau}^{v*} = \overline{\xi}_1^v(a_1, x_2, t) + \overline{\xi}_2^v(a_1, x_2, t) \text{ on } S_T^h\gamma_3,$$

$$\overline{\xi}_{h,\tau}^v = \overline{\xi}_1^v(x_1, a_2, t) + \overline{\xi}_2^v(x_1, a_2, t) \text{ on } S_T^h\gamma_4,$$

$$\overline{\xi}_{h,\tau}^v = \overline{\xi}_1^v(x_1, x_2, 0) + \overline{\xi}_2^v(x_1, x_2, 0) \text{ on } S_T^h\gamma_5. \tag{4.102}$$

The equation (4.102) is established using the following. Let us first give the validation $\widetilde{\Theta}_{h,\tau}\overline{\xi}_{h,\tau}^{v,k+1} - \widetilde{\Lambda}_{h,\tau}\overline{\xi}_{h,\tau}^{v,k} = \widetilde{\Omega}_2(x_1)$ on $D^{0h}\gamma_\tau$.

$$
\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{v,k+1}_{1,h,\tau} &= \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)(2a_1-x_1)+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(2a_1-(x_1+h)\right.\right.\\
&\quad +2a_1-\left(x_1+\frac{h}{2}\right)+2a_1-\left(x_1-\frac{h}{2}\right)+2a_1-(x_1-h)+2a_1-\left(x_1-\frac{h}{2}\right)\\
&\quad \left.\left.+2a_1-\left(x_1+\frac{h}{2}\right)\right)\right]\\
&= \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{6a_1}{4\tau}-\frac{3x_1}{4\tau}+\frac{8\omega a_1}{h^2}-\frac{4\omega x_1}{h^2}+\frac{2a_1}{4\tau}-\frac{x_1}{4\tau}-\frac{24\omega a_1}{3h^2}\right.\\
&\quad \left.+\frac{12\omega x_1}{3h^2}\right],\\
&= \frac{3}{5a_1}\tilde{\sigma}(t+\tau+1)\left[2a_1-x_1\right]. \qquad (4.103)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{v,k+1}_{2,h,\tau} &= \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\left(\frac{3}{4\tau}+\frac{4\omega}{h^2}\right)(a_1^2+a_2^2+1-x_1^2-x_2^2)\right.\\
&\quad +\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(6+6a_1^2+6a_2^2-(x_1+h)^2-x_2^2-\left(x_1+\frac{h}{2}\right)^2-\left(x_2+\frac{\sqrt{3}h}{2}\right)^2\right.\\
&\quad \left.\left.-\left(x_1-\frac{h}{2}\right)^2-\left(x_2+\frac{\sqrt{3}h}{2}\right)^2-(x_1-h)^2-x_2^2-\left(x_1-\frac{h}{2}\right)^2-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2\right)\right],
\end{aligned}
$$

$$
= \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2-\frac{h^2}{8}\right)+4\omega\right]. \qquad (4.104)
$$

Adding (4.103) and (4.104) we get

$$
\begin{aligned}
\widetilde{\Theta}_{h,\tau}\overline{\xi}^{v,k+1}_{h,\tau} &= \frac{3}{5a_1}\tilde{\sigma}(t+\tau+1)\left[2a_1-x_1\right]\\
&\quad +\left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{1}{\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2-\frac{h^2}{8}\right)\right.\\
&\quad \left.+4\omega\right] \qquad (4.105)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{v,k}_{1,h,\tau} &= \frac{3\tilde{\sigma}}{5a_1}\tau(t+1)\left[\frac{3}{4\tau}(2a_1-x_1)+\frac{1}{24\tau}(12a_1-6x_1)\right]\\
&= \frac{3\tilde{\sigma}}{5a_1}(t+1)\left[2a_1-x_1\right] \qquad (4.106)
\end{aligned}
$$

$$\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{v,k}_{2,h,\tau} = \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{3}{4\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)\right.$$

$$\left.+\frac{1}{24\tau}\left(6+6a_1^2+6a_2^2-6x_1^2-6x_2^2-6h^2\right)\right]$$

$$=\left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{1}{\tau}+\frac{a_1^2}{\tau}+\frac{a_2^2}{\tau}-\frac{x_1^2}{\tau}-\frac{x_2^2}{\tau}-\frac{h^2}{8\tau}\right]. \qquad (4.107)$$

From (4.106) and (4.107) we get

$$\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{v,k}_{h,\tau} = \frac{3\tilde{\sigma}}{5a_1}(t+1)\left[2a_1-x_1\right]+\left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{1}{\tau}+\frac{a_1^2}{\tau}+\frac{a_2^2}{\tau}\right.$$

$$\left.-\frac{x_1^2}{\tau}-\frac{x_2^2}{\tau}-\frac{h^2}{8\tau}\right]. \qquad (4.108)$$

Now using (4.105) and (4.108) we get

$$\widetilde{\Theta}_{h,\tau}\overline{\xi}^{v,k+1}_{h,\tau}-\widetilde{\Lambda}_{h,\tau}\overline{\xi}^{v,k}_{h,\tau} = \frac{3\tilde{\sigma}}{5a_1}\tau(2a_1-x_1)+\left(\frac{3}{120\omega}+\frac{47}{2880}\omega\right)\tilde{\theta}h^4$$

$$=\widetilde{\Omega}_2(x_1).$$

Second, we show that

$$\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{v,k+1}_{h,\tau}-\widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{v,k}_{h,\tau}-\widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{v*}_{h,\tau}=\widetilde{\Omega}_2(x_1)-\tfrac{1}{6}\widetilde{\Omega}_2(\widehat{p}) \text{ on } D^{*h}\gamma_\tau,$$

$$\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{v,k+1}_{1,h,\tau} = \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\left(\frac{17}{24\tau}+\frac{14\omega}{3h^2}\right)(2a_1-x_1)\right.$$

$$\left.+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(2a_1-h+2a_1-h+2a_1-\frac{3h}{2}\right)\right]$$

$$=\frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{5a_1}{3\tau}-\frac{h}{2\tau}-\frac{16\omega a_1}{3h^2}\right], \qquad (4.109)$$

$$\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{v,k+1}_{2,h,\tau} = \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\Theta}h^4\left[\left(\frac{17}{24\tau}+\frac{14\omega}{3h^2}\right)\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right)\right.$$

$$+\left(\frac{1}{24\tau}-\frac{2\omega}{3h^2}\right)\left(1+a_1^2+a_2^2-h^2-\left(x_2+\frac{\sqrt{3}h}{2}\right)^2\right.$$

$$\left.\left.+1+a_1^2+a_2^2-h^2-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2+1+a_1^2+a_2^2-\frac{9h^2}{4}-x_2^2\right)\right]$$

$$= \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{5}{6\tau} + \frac{5a_1^2}{6\tau} + \frac{5a_2^2}{6\tau} - \frac{17x_1^2}{24\tau} - \frac{5x_2^2}{6\tau} + \frac{8\omega}{3h^2}\right.$$
$$\left. + \frac{8\omega a_1^2}{3h^2} + \frac{8\omega a_2^2}{3h^2} - \frac{14\omega x_1^2}{3h^2} - \frac{8\omega x_2^2}{3h^2} + \frac{23\omega}{6} - \frac{23h^2}{96\tau}\right]. \tag{4.110}$$

Adding (4.109) and (4.110) yields

$$\begin{aligned}
\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{v,k+1}_{h,\tau} &= \widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{v,k+1}_{1,h,\tau} + \widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{v,k+1}_{2,h,\tau} \\
&= \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{5a_1}{3\tau} - \frac{h}{2\tau} - \frac{16\omega a_1}{3h^2}\right] \\
&+ \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{5}{6\tau} + \frac{5a_1^2}{6\tau} + \frac{5a_2^2}{6\tau} - \frac{17x_1^2}{24\tau}\right. \\
&- \frac{5x_2^2}{6\tau} + \frac{8\omega}{3h^2} + \frac{8\omega a_1^2}{3h^2} + \frac{8\omega a_2^2}{3h^2} - \frac{14\omega x_1^2}{3h^2} - \frac{8\omega x_2^2}{3h^2} \\
&\left. + \frac{23\omega}{6} - \frac{23h^2}{96\tau}\right].
\end{aligned} \tag{4.111}$$

Also,

$$\begin{aligned}
\widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{v*}_{1,\phi,h,\tau} &= \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\left(-\frac{1}{36\tau} + \frac{4\omega}{9h^2}\right)(2a_1+2a_1)\right. \\
&\left. + \left(\frac{1}{18\tau} + \frac{16\omega}{9h^2}\right)(2a_1)\right] \\
&+ \frac{3}{5a_1}\tilde{\sigma}(t+\tau+1)\left[\frac{1}{36\tau}(2a_1+2a_1) - \frac{1}{18\tau}(2a_1)\right] \\
&= \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{16\omega a_1}{3h^2}\right],
\end{aligned} \tag{4.112}$$

$$\widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{v*}_{2,h,\tau} = \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\Bigg[\left(-\frac{1}{36\tau}+\frac{4\omega}{9h^2}\right)\Bigg(a_1^2+a_2^2+1 - \left(x_2+\frac{\sqrt{3}h}{2}\right)^2 + a_1^2+a_2^2+1-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2\Bigg) + \left(\frac{1}{18\tau}+\frac{16\omega}{9h^2}\right)\left(1+a_1^2+a_2^2-x_2^2\right) + \frac{1}{36\tau}\Bigg(a_1^2+a_2^2+1-\left(x_2+\frac{\sqrt{3}h}{2}\right)^2+a_1^2+a_2^2+1-\left(x_2-\frac{\sqrt{3}h}{2}\right)^2\Bigg) - \frac{1}{18\tau}\left(a_1^2+a_2^2+1-x_2^2\right)\Bigg]$$

$$= \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{8\omega}{3h^2}+\frac{8\omega a_1^2}{3h^2}+\frac{8\omega a_2^2}{3h^2}-\frac{8\omega x_2^2}{3h^2}-\frac{2\omega}{3}\right].r \quad (4.113)$$

Adding (4.112) and (4.113) it follows that

$$\widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{v*}_{\phi,h,\tau} = \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{16\omega a_1}{3h^2}\right]+\left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{8\omega}{3h^2}+\frac{8\omega a_1^2}{3h^2}+\frac{8\omega a_2^2}{3h^2}-\frac{8\omega x_2^2}{3h^2}-\frac{2\omega}{3}\right]. \quad (4.114)$$

$$\widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{v,k}_{1,h,\tau} = \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{17}{24\tau}(2a_1-x_1)+\frac{1}{24\tau}\left(2a_1 - h+2a_1-h+2a_1-\frac{3h}{2}\right)\right]$$

$$= \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{5a_1}{3\tau}-\frac{h}{2\tau}\right], \quad (4.115)$$

$$\widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{v,k}_{2,h,\tau} = \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{17}{24\tau}\left(a_1^2+a_2^2+1-x_1^2-x_2^2\right) + \frac{1}{24\tau}\left(3+3a_1^2+3a_2^2-3x_2^2\right)\right]$$

$$= \left(\frac{3}{640\omega}+\frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{5}{6\tau}+\frac{5a_1^2}{6\tau}+\frac{5a_2^2}{6\tau} - \frac{17x_1^2}{24\tau}-\frac{23h^2}{96\tau}-\frac{5x_2^2}{6\tau}\right], \quad (4.116)$$

Adding (4.115) and (4.116) we get

$$\widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{v,k}_{2,h,\tau} = \frac{3}{5a_1}\tilde{\sigma}\tau(t+\tau+1)\left[\frac{5a_1}{3\tau} - \frac{h}{2\tau}\right] + \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\tilde{\theta}h^4\left[\frac{5}{6\tau} + \frac{5a_1^2}{6\tau} + \frac{5a_2^2}{6\tau} - \frac{17x_1^2}{24\tau} - \frac{23h^2}{96\tau} - \frac{5x_2^2}{6\tau}\right]. \quad (4.117)$$

using (4.111), (4.114), and (4.117) gives

$$\widetilde{\Theta}^*_{h,\tau}\overline{\xi}^{v,k+1}_{h,\tau} - \widetilde{\Lambda}^*_{h,\tau}\overline{\xi}^{v,k}_{h,\tau} - \widetilde{\Gamma}^*_{h,\tau}\overline{\xi}^{v*}_{h,\tau} = \widetilde{\Omega}_2(x_1) - \frac{1}{6}\widetilde{\Omega}_2(\widehat{p}) \quad (4.118)$$

where the right side of equation (4.118) is as given in (4.100).

Next, for $k = 0, ..., M' - 1$, we put the equations (4.98) and (4.102) in matrix form as

$$\widetilde{A}\widehat{\xi}^{v,k+1} = \widetilde{B}\widehat{\xi}^{v,k} + \tau\widehat{\eta}^{v,k}, \quad (4.119)$$

$$\widetilde{A}\overline{\xi}^{v,k+1} = \widetilde{B}\overline{\xi}^{v,k} + \tau\overline{\eta}^{v,k}, \quad (4.120)$$

where $\widetilde{A}, \widetilde{B}$ are as given in (4.16) and $\widehat{\xi}^{v,k}, \overline{\xi}^{v,k}, \widehat{\eta}^{v,k}, \overline{\eta}^{v,k} \in R^N$. Using (4.99)–(4.102) we have $\overline{\xi}^{v,0} \geq 0$, and $\overline{\eta}^{v,k} \geq 0$, and $\left|\widehat{\eta}^{v,k}\right| \leq \overline{\eta}^{v,k}$ for $k = 0, ..., M' - 1$, and $\left|\widehat{\xi}^{v,0}\right| \leq \overline{\xi}^{v,0}$. Then Lemma 4.3 implies that $\left|\widehat{\xi}^{v,k+1}\right| \leq \overline{\xi}^{v,k+1}$. Furthermore,

$$\overline{\xi}^{v}(x_1, x_2, t) \leq \overline{\xi}^{v}(0, 0, T) = \frac{6}{5}\widetilde{\sigma}(T+1)\tau + \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\left(1 + a_1^2 + a_2^2\right)\widetilde{\theta}h^4,$$

yielding (4.97). □

4.2.2 Boundary Value Problem for $\frac{\partial^2 u}{\partial x_1 \partial t}$ and Hexagonal Grid Approximation: Stage 2 $\left(H^{4th}\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)\right)$

First, we construct BVP($\frac{\partial u}{\partial t}$) and obtain the approximate solution $u_{t,h,\tau}$ by using the implicit method given in Stage 1 $\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$. Next, we denote $p_{t,i} = \frac{\partial^2 u}{\partial x_1 \partial t}$ on $S_T\gamma_i, i = 1, 2, ..., 5$ and propose the below problem for $v_t = \frac{\partial^2 u}{\partial x_1 \partial t}$.

Boundary Value Problem $\left(\textbf{BVP}\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)\right)$

$$Lv_t = \frac{\partial^2 f(x_1,x_2,t)}{\partial x_1 \partial t} \text{ on } Q_T,$$

$$v_t(x_1,x_2,t) = p_{t,i} \text{ on } S_T\gamma_i, i = 1,2,...,5. \qquad (4.121)$$

From $u \in C_{x,t}^{9+\alpha,\frac{9+\alpha}{2}}\left(\overline{Q}_T\right)$, we assume that the solution $v_t \in C_{x,t}^{6+\alpha,3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$. We take

$$p_{t,1h}^{4^{th}} = \begin{cases} \frac{1}{12h}\left(-25u_t(0,x_2,t) + 48u_{t,h,\tau}(h,x_2,t)\right. \\ -36u_{t,h,\tau}(2h,x_2,t) + 16u_{t,h,\tau}(3h,x_2,t) \\ \left. -3u_{t,h,\tau}(4h,x_2,t)\right) \text{ if } P_0 \in D^{0h}\gamma_\tau, \\ \frac{1}{840h}\left(-2816u_t(0,x_2,t) + 3675u_{t,h,\tau}\left(\frac{h}{2},x_2,t\right)\right. \\ -1225u_{t,h,\tau}\left(\frac{3h}{2},x_2,t\right) + 441u_{t,h,\tau}\left(\frac{5h}{2},x_2,t\right) \\ \left. -75u_{t,h,\tau}\left(\frac{7h}{2},x_2,t\right)\right) \text{ if } P_0 \in D^{*lh}\gamma_\tau, \end{cases} \text{ on } S_T^h\gamma_1, \qquad (4.122)$$

$$p_{t,3h}^{4^{th}} = \begin{cases} \frac{1}{12h}\left(25u_t(a_1,x_2,t) - 48u_{t,h,\tau}(a_1-h,x_2,t)\right. \\ +36u_{t,h,\tau}(a_1-2h,x_2,t) - 16u_{t,h,\tau}(a_1-3h,x_2,t) \\ \left. +3u_{t,h,\tau}(a_1-4h,x_2,t)\right) \text{ if } P_0 \in D^{0h}\gamma_\tau, \\ \frac{1}{840h}\left(2816u_t(a_1,x_2,t) - 3675u_{t,h,\tau}\left(a_1-\frac{h}{2},x_2,t\right)\right. \\ +1225u_{t,h,\tau}\left(a_1-\frac{3h}{2},x_2,t\right) - 441u_{t,h,\tau}\left(a_1-\frac{5h}{2},x_2,t\right) \\ \left. +75u_{t,h,\tau}\left(a_1-\frac{7h}{2},x_2,t\right)\right) \text{ if } P_0 \in D^{*rh}\gamma_\tau, \end{cases} \text{ on } S_T^h\gamma_{3,} \qquad (4.123)$$

$$p_{t,ih} = \frac{\partial \phi_t(x_1,x_2,t)}{\partial x_1} \text{ on } S_T^h\gamma_i, i = 2,4, \qquad (4.124)$$

$$p_{t,5h} = \frac{\partial \widehat{\varphi}(x_1,x_2)}{\partial x_1} \text{ on } S_T^h\gamma_5, \qquad (4.125)$$

where $\widehat{\varphi}(x_1,x_2)$ and $\phi_t(x_1,x_2,t)$ are as given in (4.11) and $u_{t,h,\tau}$ is the approximate solution achieved by using Stage 1$\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$.

For a fourth order accurate hexagonal grid approximation of BVP$\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)$, we propose **Stage 2**$\left(H^{4th}\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)\right)$:

$$\widetilde{\Theta}_{h,\tau} v_{t,h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau} v_{t,h,\tau}^{k} + \widetilde{D}_{x_1} \widetilde{\psi}_t \text{ on } D^{0h}\gamma_\tau,$$
$$\widetilde{\Theta}_{h,\tau}^{*} v_{t,h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}^{*} v_{t,h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*} p_{t,1h}^{4^{th}}\left(u_{t,h,\tau}\right) + \widetilde{D}_{x_1} \widetilde{\psi}_t^{*} \text{ on } D^{*lh}\gamma_\tau$$
$$\widetilde{\Theta}_{h,\tau}^{*} v_{t,h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}^{*} v_{t,h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*} p_{t,3h}^{4^{th}}\left(u_{t,h,\tau}\right) + \widetilde{D}_{x_1} \widetilde{\psi}_t^{*} \text{ on } D^{*rh}\gamma_\tau$$
$$v_{t,h,\tau} = p_{t,ih}^{4^{th}}\left(u_{t,h,\tau}\right) \text{ on } S_T^h\gamma_i, i = 1,3,$$
$$v_{t,h,\tau} = p_{t,ih} \text{ on } S_T^h\gamma_i,\ i = 2,4,5 \tag{4.126}$$

where $p_{t,1h}^{4^{th}}, p_{t,3h}^{4^{th}}, p_{t,ih}$, $i = 2,4,5$ are defined by (4.122)–(4.125) and the operators $\widetilde{\Theta}_{h,\tau}$, $\widetilde{\Lambda}_{h,\tau}, \widetilde{\Theta}_{h,\tau}^{*}, \widetilde{\Lambda}_{h,\tau}^{*}$ and $\widetilde{\Gamma}_{h,\tau}^{*}$ are the operator given in (4.4)–(4.8), respectively. Furthermore, $v_{t,h,\tau}$ is the numerical solution of (4.126) and

$$\widetilde{D}_{x_1} \widetilde{\psi}_t = \partial_{x_1} f_{t,P_0}^{k+1} + \frac{1}{16} h^2 \left(\partial_{x_1}^3 f_{t,P_0}^{k+1} + \partial_{x_2}^2 \partial_{x_1} f_{t,P_0}^{k+1}\right), \tag{4.127}$$
$$\widetilde{D}_{x_1} \widetilde{\psi}_t^{*} = \frac{h^2}{96\tau\omega} \partial_{x_1} f_{t,P_A}^{k+1} - \frac{h^2}{96\tau\omega} \partial_{x_1} f_{t,P_A}^{k} - \frac{1}{6} \partial_{x_1} f_{t,P_A}^{k+1} + \partial_{x_1} f_{t,P_0}^{k+1} + \frac{1}{16} h^2 \left(\partial_{x_1}^3 f_{t,P_0}^{k+1} + \partial_{x_2}^2 \partial_{x_1} f_{t,P_0}^{k+1}\right). \tag{4.128}$$

Let

$$\xi_{h,\tau}^{v_t} = v_{t,h,\tau} - v_t \text{ on } \overline{D^h\gamma_\tau}, \tag{4.129}$$

where $v_t = \frac{\partial^2 u}{\partial x_1 \partial t}$. From (4.126) and (4.129), we have

$$\widetilde{\Theta}_{h,\tau} \xi_{h,\tau}^{v_t,k+1} = \widetilde{\Lambda}_{h,\tau} \xi_{h,\tau}^{v_t,k} + \widetilde{\Psi}_1^{v_t,k} \text{ on } D^{0h}\gamma_\tau,$$
$$\widetilde{\Theta}_{h,\tau}^{*} \xi_{h,\tau}^{v_t,k+1} = \widetilde{\Lambda}_{h,\tau}^{*} \xi_{h,\tau}^{v_t,k} + \widetilde{\Gamma}_{h,\tau}^{*} \xi_{h,\tau}^{*v_t} + \widetilde{\Psi}_2^{v_t,k} \text{ on } D^{*h}\gamma_\tau,$$
$$\xi_{h,\tau}^{v_t} = 0 \text{ on } S_T^h\gamma_i,\ i = 2,4,5,$$
$$\xi_{h,\tau}^{v_t} = \xi_{h,\tau}^{*v_t} = p_{t,ih}^{4^{th}}\left(u_{t,h,\tau}\right) - p_{t,i} \text{ on } S_T^h\gamma_i,\ i = 1,3, \tag{4.130}$$

where

$$\widetilde{\Psi}_1^{v_t,k} = \widetilde{\Lambda}_{h,\tau} v_t^k - \widetilde{\Theta}_{h,\tau} v_t^{k+1} + \widetilde{D}_{x_1} \widetilde{\psi}_t, \tag{4.131}$$

$$\widetilde{\Psi}_2^{v_t,k} = \widetilde{\Lambda}_{h,\tau}^* v_t^k - \widetilde{\Theta}_{h,\tau}^* v_t^{k+1} + \widetilde{\Gamma}_{h,\tau}^* p_{t,i} + \widetilde{D}_{x_1} \widetilde{\psi}_t^*, \ i = 1,3. \tag{4.132}$$

Let $\widetilde{\theta}_{t,1} = \mu_1(v_t)$, $\widetilde{\sigma}_{t,1} = \mu_2(v_t)$ where μ_1, μ_2 are given in (4.39), (4.40), respectively, and let

$$\widetilde{\theta}_t = \max\left\{\widetilde{\theta}_{t,1}, \frac{\widetilde{M}_{t,1}}{\rho} + 15\frac{d}{\rho}\left(\frac{3}{160} + \frac{47\omega}{2880}\right)\widetilde{\alpha}_t\right\}, \tag{4.133}$$

$$\widetilde{\sigma}_t = \max\left\{\widetilde{\sigma}_{t,1}, 15d\widetilde{\beta}_t\right\}, \tag{4.134}$$

where $\widetilde{\alpha}_t = \mu_1(u_t)$, $\widetilde{\beta}_t = \mu_2(u_t)$ and d is as given in (2.60). Furthermore, $\widetilde{M}_{t,1} = \frac{1}{5}\max\limits_{\overline{Q}_T}\left|\frac{\partial^5 u_t}{\partial x_1^5}\right|$ and $\rho = \frac{3}{640\omega} + \frac{47}{11520}$.

Theorem 4.3: (Buranay et al. [53]) The solution $v_{t,h,\tau}$ achieved by using Stage $2\left(H^{4th}\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)\right)$ satisfies

$$\max_{D^h \gamma_\tau}\left|v_{t,h,\tau} - v_t\right| \leq \frac{6}{5}\widetilde{\sigma}_t (T+1)\tau + \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\widetilde{\theta}_t \left(1 + a_1^2 + a_2^2\right) h^4, \tag{4.135}$$

for $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$ where $\widetilde{\theta}_t, \widetilde{\sigma}_t$ are presented in (4.133), (4.134), respectively, and $v_t = \frac{\partial^2 u}{\partial x_1 \partial t}$ is the exact solution of BVP$\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)$.

Proof. The proof basically is analogous with the proof of Theorem 4.2 and follows from the assumption $v_t \in C_{x,t}^{6+\alpha, 3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$. □

4.3 Second Stages of the Implicit Methods Approximating $\frac{\partial u}{\partial x_2}$ and $\frac{\partial^2 u}{\partial x_2 \partial t}$ with $O(h^4 + \tau)$ Order of Convergence

4.3.1 Boundary Value Problem for $\frac{\partial u}{\partial x_2}$ and Hexagonal Grid Approximation: Stage $2\left(H^{4th}\left(\frac{\partial u}{\partial x_2}\right)\right)$

Let the BVP(u) be given. First, we apply Stage $1\left(H^{4th}(u)\right)$ and obtain the approximate solution $u_{h,\tau}$ on the hexagonal grids. Then, by denoting $q_i = \frac{\partial u}{\partial x_2}$ on $S_T\gamma_i, i = 1,2,...,5$ we use the boundary value problem $BVP\left(\frac{\partial u}{\partial x_2}\right)$ for $z = \frac{\partial u}{\partial x_2}$, given in Chapter 2. We take

$$q_{2h}^{4^{th}} = \frac{1}{12\sqrt{3}h}\Big(-25u(x_1,0,t) + 48u_{h,\tau}\left(x_1,\sqrt{3}h,t\right) - 36u_{h,\tau}\left(x_1,2\sqrt{3}h,t\right)$$
$$+ 16u_{h,\tau}\left(x_1,3\sqrt{3}h,t\right) - 3u_{h,\tau}\left(x_1,4\sqrt{3}h,t\right)\Big) \text{ on } S_T^h\gamma_2, \tag{4.136}$$

$$q_{4h}^{4^{th}} = \frac{1}{12\sqrt{3}h}\Big(25u(x_1,a_2,t) - 48u_{h,\tau}\left(x_1,a_2-\sqrt{3}h,t\right) + 36u_{h,\tau}\left(x_1,a_2-2\sqrt{3}h,t\right)$$
$$- 16u_{h,\tau}\left(x_1,a_2-3\sqrt{3}h,t\right) + 3u_{h,\tau}\left(x_1,a_2-4\sqrt{3}h,t\right)\Big) \text{ on } S_T^h\gamma_4, \tag{4.137}$$

$$q_{ih} = \frac{\partial\phi(x_1,x_2,t)}{\partial x_2} \text{ on } S_T^h\gamma_i, i = 1,3, \tag{4.138}$$
$$q_{5h} = \frac{\partial\varphi(x_1,x_2)}{\partial x_2} \text{ on } S_T^h\gamma_5, \tag{4.139}$$

and $\varphi(x_1,x_2)$, $\phi(x_1,x_2,t)$ given in (2.9) are the initial and boundary functions, respectively, $u_{h,\tau}$ is the solution taken by using Stage $1\left(H^{4th}(u)\right)$. Further we give the derivation of the forward difference formula (4.136) as follows: Let

$A : u(x_1,x_2,t)$

$B : u(x_1,x_2+\sqrt{3}h,t)$

$C : u(x_1,x_2+2\sqrt{3}h,t)$

$D : u(x_1,x_2+3\sqrt{3}h,t)$

$E : u(x_1,x_2+4\sqrt{3}h,t)$

$$B : u\left(x_1,x_2+\sqrt{3}h,t\right) = u(x_1,x_2,t) + \sqrt{3}h\partial_{x_2}u(x_1,x_2,t)$$
$$+ \frac{3}{2}h^2\partial_{x_2}^2u(x_1,x_2,t) + \frac{\sqrt{3}}{2}h^3\partial_{x_2}^3u(x_1,x_2,t)$$
$$+ \frac{9}{24}h^4\partial_{x_2}^4u(x_1,x_2,t) + \frac{3\sqrt{3}}{40}h^5\partial_{x_2}^5u(x_1,x_2+\omega_1h,t), \tag{4.140}$$

$$C : u\left(x_1,x_2+2\sqrt{3}h,t\right) = u(x_1,x_2,t) + 2\sqrt{3}h\partial_{x_2}u(x_1,x_2,t)$$
$$+ 6h^2\partial_{x_2}^2u(x_1,x_2,t) + 4\sqrt{3}h^3\partial_{x_2}^3u(x_1,x_2,t)$$
$$+ \frac{18}{3}h^4\partial_{x_2}^4u(x_1,x_2,t) + \frac{12\sqrt{3}}{5}h^5\partial_{x_2}^5u(x_1,x_2+\omega_2h,t), \tag{4.141}$$

$$D: u\left(x_1, x_2 + 3\sqrt{3}h, t\right) = u(x_1, x_2, t) + 3\sqrt{3}h\partial_{x_2}u(x_1, x_2, t)$$
$$+ \frac{27}{2}h^2\partial_{x_2}^2 u(x_1, x_2, t) + \frac{27}{2}\sqrt{3}h^3\partial_{x_2}^3 u(x_1, x_2, t)$$
$$+ \frac{243}{8}h\partial_{x_2}^4 u(x_1, x_2, t) + \frac{729\sqrt{3}}{40}h^5\partial_{x_2}^5 u(x_1, x_2 + \omega_3 h, t), \tag{4.142}$$

$$E: u\left(x_1, x_2 + 4\sqrt{3}h, t\right) = u(x_1, x_2, t) + 4\sqrt{3}h\partial_{x_2}u(x_1, x_2, t)$$
$$+ 24h^2\partial_{x_2}^2 u(x_1, x_2, t) + 32\sqrt{3}h^3\partial_{x_2}^3 u(x_1, x_2, t)$$
$$+ 96h^4\partial_{x_2}^4 u(x_1, x_2, t) + \frac{384\sqrt{3}}{5}h^5\partial_{x_2}^5 u(x_1, x_2 + \omega_4 h, t), \tag{4.143}$$

where, $0 < \omega_i < \sqrt{3}i,\ i = 1,..,4$. Multiplying the equations (4.140)–(4.143) with $\frac{4\sqrt{3}}{3}, -\sqrt{3}, \frac{4\sqrt{3}}{9}, \frac{-\sqrt{3}}{12}$ respectively and adding the resulting equations we get the following:

$$\frac{4\sqrt{3}}{3}u\left(x_1, x_2 + \sqrt{3}h, t\right) - \sqrt{3}u\left(x_1, x_2 + 2\sqrt{3}h, t\right)$$
$$+ \frac{4\sqrt{3}}{9}u\left(x_1, x_2 + 3\sqrt{3}h, t\right) - \frac{\sqrt{3}}{12}u\left(x_1, x_2 + 4\sqrt{3}h, t\right)$$
$$= \frac{25\sqrt{3}}{36}u(x_1, x_2, t) + h\partial_{x_2}u(x_1, x_2, t) - \frac{9}{5}h^5\partial_{x_2}^5 u(x_1, \widetilde{x}_2, t), \tag{4.144}$$

where $x_2 \leq \widetilde{x}_2 < x_2 + 4\sqrt{3}h$. Simplifying equation (4.144) yields

$$\frac{1}{12\sqrt{3}h}\left(48u\left(x_1, x_2 + \sqrt{3}h, t\right) - 36u\left(x_1, x_2 + 2\sqrt{3}h, t\right)\right.$$
$$\left. + 16u\left(x_1, x_2 + 3\sqrt{3}h, t\right) - 3u\left(x_1, x_2 + 4\sqrt{3}h, t\right) - 25u(x_1, x_2, t)\right)$$
$$= \partial_{x_2}u(x_1, x_2, t) - \frac{9}{5}h^5\partial_{x_2}^5 u(x_1, \widetilde{x}_2, t). \tag{4.145}$$

Therefore,

$$\frac{1}{12\sqrt{3}h}\left(-25u(x_1, x_2, t) + 48u\left(x_1, x_2 + \sqrt{3}h, t\right) - 36u\left(x_1, x_2 + 2\sqrt{3}h, t\right)\right.$$
$$\left. + 16u\left(x_1, x_2 + 3\sqrt{3}h, t\right) - 3u\left(x_1, x_2 + 4\sqrt{3}h, t\right)\right)$$
$$= \partial_{x_2}u(x_1, x_2, t) + O\left(h^4\right). \tag{4.146}$$

In a similar way we one can obtain fourth order accurate backward difference formula for approximating $\partial_{x_2}u$ as:

$$
\frac{1}{12\sqrt{3}h}\left(25u\left(x_1,x_2,t\right)-48u\left(x_1,x_2+\sqrt{3}h,t\right)+36u\left(x_1,x_2+2\sqrt{3}h,t\right)\right.
$$
$$
\left.-16u\left(x_1,x_2+3\sqrt{3}h,t\right)+3u\left(x_1,x_2+4\sqrt{3}h,t\right)\right)
$$
$$
=\partial_{x_2}u\left(x_1,x_2,t\right)+O\left(h^4\right). \tag{4.147}
$$

Lemma 4.6: (Buranay et al. [53]) Let u be the solution of BVP(u) in (2.9) and $u_{h,\tau}$ be the approximation achieved by using Stage 1$\left(H^{4th}\left(u\right)\right)$. Then, the following inequality holds true

$$
\left|q_{ih}^{4^{th}}\left(u_{h,\tau}\right)-q_{ih}^{4^{th}}\left(u\right)\right|\leq 15d\widetilde{\Omega}_1\left(h,\tau\right),\ \ i=2,4, \tag{4.148}
$$

for $r\geq\frac{1}{16}$ where, $\widetilde{\Omega}_1\left(h,\tau\right)$ is given in (4.43) and d is defined in (2.60).

Proof. From Theorem 4.1, and using (4.136), (4.137), we have

$$
\begin{aligned}
\left|q_{ih}^{4^{th}}\left(u_{h,\tau}\right)-q_{ih}^{4^{th}}\left(u\right)\right| &\leq \frac{1}{12\sqrt{3}h}\Big(48\sqrt{3}hd\widetilde{\Omega}_1\left(h,\tau\right)+36(2\sqrt{3}hd\widetilde{\Omega}_1\left(h,\tau\right))\\
&\quad+16(3\sqrt{3}hd\widetilde{\Omega}_1\left(h,\tau\right))+3(4\sqrt{3}hd\widetilde{\Omega}_1\left(h,\tau\right))\Big)\\
&\leq 15d\widetilde{\Omega}_1\left(h,\tau\right),\ \ i=2,4.
\end{aligned} \tag{4.149}
$$

Thus, we obtain (4.148). □

Lemma 4.7: (Buranay et al. [53]) Let $\widetilde{M}_2=\frac{9}{5}\max_{\overline{Q}_T}\left|\frac{\partial^5u}{\partial x_2^5}\right|$ and $u_{h,\tau}$ be the approximation taken by using Stage 1$\left(H^{4th}\left(u\right)\right)$. Then, the following inequality is true:

$$
\max_{S_T^h\gamma_2\cup S_T^h\gamma_4}\left|q_{ih}^{4^{th}}\left(u_{h,\tau}\right)-q_i\right|\leq\widetilde{M}_2h^4+15d\widetilde{\Omega}_1\left(h,\tau\right),\ i=2,4, \tag{4.150}
$$

where $\widetilde{\Omega}_1\left(h,\tau\right)$ is given in (4.43) and d is defined in (2.60).

Proof. From $u\in C_{x,t}^{9+\alpha,\frac{9+\alpha}{2}}\left(\overline{Q}_T\right)$, at the points $(x_1,0,k\tau)\in S_T^h\gamma_2$ and $(x_2,a_2,k\tau)\in S_T^h\gamma_4$ of each line segment

$$
\left[(\sigma h,x_{2,}k\tau):0\leq x_1=\sigma h\leq a_1,\ 0\leq x_2\leq a_2,\ 0\leq t=k\tau\leq T\right],
$$

we get fourth order approximation of $\frac{\partial u}{\partial x_2}$ by the difference Formulas (4.136) and

(4.137). Then, the truncation error in (4.145) yields

$$\max_{S_T^h\gamma_2 \cup S_T^h\gamma_4} \left| q_{ih}^{4^{th}}(u) - q_i \right| \leq \frac{9}{5} h^4 \max_{\overline{Q}_T} \left| \frac{\partial^5 u}{\partial x_2^5} \right|, \; i = 2,4 \; . \tag{4.151}$$

Taking $\widetilde{M}_2 = \frac{9}{5} \max_{\overline{Q}_T} \left| \frac{\partial^5 u}{\partial x_2^5} \right|$ and using Lemma 4.6 and the estimation (4.148) and (4.151) follows (4.150). □

Second stage of the fourth order accurate implicit method for the numerical solution to BVP$\left(\frac{\partial u}{\partial x_2}\right)$ is given as follows:

Stage 2$\left(H^{4th}\left(\frac{\partial u}{\partial x_2}\right)\right)$

$$\widetilde{\Theta}_{h,\tau} z_{h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau} z_{h,\tau}^{k} + \widetilde{D}_{x_2} \widetilde{\psi} \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}^*_{h,\tau} z_{h,\tau}^{k+1} = \widetilde{\Lambda}^*_{h,\tau} z_{h,\tau}^{k} + \widetilde{\Gamma}^*_{h,\tau} q_{1h} + \widetilde{D}_{x_2} \widetilde{\psi}^* \text{ on } D^{*lh}\gamma_\tau,$$

$$\widetilde{\Theta}^*_{h,\tau} z_{h,\tau}^{k+1} = \widetilde{\Lambda}^*_{h,\tau} z_{h,\tau}^{k} + \widetilde{\Gamma}^*_{h,\tau} q_{3h} + \widetilde{D}_{x_2} \widetilde{\psi}^* \text{ on } D^{*rh}\gamma_\tau,$$

$$z_{h,\tau} = q_{ih} \text{ on } S_T^h\gamma_i, i = 1,3,5,$$

$$z_{h,\tau} = q_{ih}^{4^{th}} \text{ on } S_T^h\gamma_i, i = 2,4, \tag{4.152}$$

where $q_{ih}^{4^{th}}, i = 2,4$ and q_{ih} , $i = 1,3,5$ are defined by (4.136)–(4.139) and the operators $\widetilde{\Theta}_{h,\tau}$, $\widetilde{\Lambda}_{h,\tau}, \widetilde{\Theta}^*_{h,\tau}$, $\widetilde{\Gamma}^*_{h,\tau}$ and $\widetilde{\Lambda}^*_{h,\tau}$ are the operators given in (4.4)–(4.8) respectively. Furthermore, $z_{h,\tau}$ is the numerical solution and

$$\widetilde{D}_{x_2} \widetilde{\psi} = \partial_{x_2} f_{P_0}^{k+1} + \frac{1}{16} h^2 \left(\partial_{x_1}^2 \partial_{x_2} f_{P_0}^{k+1} + \partial_{x_2}^3 f_{P_0}^{k+1} \right), \tag{4.153}$$

$$\widetilde{D}_{x_2} \widetilde{\psi}^* = \frac{h^2}{96\tau\omega} \partial_{x_2} f_{P_A}^{k+1} - \frac{h^2}{96\tau\omega} \partial_{x_2} f_{P_A}^{k} - \frac{1}{6} \partial_{x_2} f_{P_A}^{k+1} + \partial_{x_2} f_{P_0}^{k+1} + \frac{1}{16} h^2 \left(\partial_{x_1}^2 \partial_{x_2} f_{P_0}^{k+1} + \partial_{x_2}^3 f_{P_0}^{k+1} \right) \tag{4.154}$$

Let

$$\xi_{h,\tau}^z = z_{h,\tau} - z \text{ on } \overline{D^h\gamma_\tau}. \tag{4.155}$$

From (4.152) and (4.155), we have

$$\widetilde{\Theta}_{h,\tau}\xi_{h,\tau}^{z,k+1} = \widetilde{\Lambda}_{h,\tau}\xi_{h,\tau}^{z,k} + \widetilde{\Psi}_1^{z,k} \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^{*}\xi_{h,\tau}^{z,k+1} = \widetilde{\Lambda}_{h,\tau}^{*}\xi_{h,\tau}^{z,k} + \widetilde{\Psi}_2^{z,k} \text{ on } D^{*h}\gamma_\tau,$$

$$\xi_{h,\tau}^{z} = 0 \text{ on } S_T^h\gamma_i, i = 1,3,5,$$

$$\xi_{h,\tau}^{z} = q_{ih}^{4th}\left(u_{h,\tau}\right) - q_i \text{ on } S_T^h\gamma_i, i = 2,4, \tag{4.156}$$

where q_{2h}^{4th}, q_{4h}^{4th} are defined by (4.136), (4.137) accordingly, and

$$\widetilde{\Psi}_1^{z,k} = \widetilde{\Lambda}_{h,\tau}z^k - \widetilde{\Theta}_{h,\tau}z^{k+1} + \widetilde{D}_{x_2}\widetilde{\psi}, \tag{4.157}$$

$$\widetilde{\Psi}_2^{z,k} = \widetilde{\Lambda}_{h,\tau}^{*}z^k - \widetilde{\Theta}_{h,\tau}^{*}z^{k+1} + \widetilde{\Gamma}_{h,\tau}^{*}q_i + \widetilde{D}_{x_2}\widetilde{\psi}^{*}, i = 1,3. \tag{4.158}$$

Further, let $\widetilde{\lambda}_1 = \mu_1(z)$, $\widetilde{\delta}_1 = \mu_2(z)$ where μ_1, μ_2 are given in (4.39), (4.40), respectively, and

$$\widetilde{\lambda} = \max\left\{\widetilde{\lambda}_1, \frac{\widetilde{M}_2}{\rho} + 15\frac{d}{\rho}\left(\frac{3}{160} + \frac{47\omega}{2880}\right)\widetilde{\alpha}\right\} \tag{4.159}$$

$$\widetilde{\delta} = \max\left\{\widetilde{\delta}_1, 15d\widetilde{\beta}\right\} \tag{4.160}$$

where $\widetilde{\alpha} = \mu_1(u)$, $\widetilde{\beta} = \mu_2(u)$ and d is presented in (2.60) and $\widetilde{M}_2$ is as given in Lemma 4.7 and z is the solution of BVP$\left(\frac{\partial u}{\partial x_2}\right)$.

Theorem 4.4: (Buranay et al. [53]) The solution $z_{h,\tau}$ achieved from Stage $2\left(H^{4th}\left(\frac{\partial u}{\partial x_2}\right)\right)$ satisfies

$$\max_{\overline{D}^h\gamma_\tau}\left|z_{h,\tau} - z\right| \le \frac{6}{5}\widetilde{\delta}(T+1)\tau + \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\widetilde{\lambda}\left(1 + a_1^2 + a_2^2\right)h^4, \tag{4.161}$$

for $r = \frac{\omega\tau}{h^2} \ge \frac{1}{16}$, where $\widetilde{\lambda}, \widetilde{\delta}$ are as given in (4.159), (4.160), respectively, and $z = \frac{\partial u}{\partial x_2}$ is the exact solution of BVP$\left(\frac{\partial u}{\partial x_2}\right)$.

Proof. We take the system

$$\widetilde{\Theta}_{h,\tau}\widehat{\xi}_{h,\tau}^{z,k+1} = \widetilde{\Lambda}_{h,\tau}\widehat{\xi}_{h,\tau}^{z,k} + \widetilde{\Omega}_3(x_2) \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^{*}\widehat{\xi}_{h,\tau}^{z,k+1} = \widetilde{\Lambda}_{h,\tau}^{*}\widehat{\xi}_{h,\tau}^{z,k} + \frac{5}{6}\widetilde{\Omega}_3(x_2) \text{ on } D^{*h}\gamma_\tau,$$

$$\widehat{\xi}_{h,\tau}^{z} = 0 \text{ on } S_T^h\gamma_i, i = 1,3,5,$$

$$\widehat{\xi}_{h,\tau}^{z} = q_{ih}^{4^{th}}\left(u_{h,\tau}\right) - q_i \text{ on } S_T^h\gamma_i, i = 2,4. \tag{4.162}$$

$q_{2h}^{4^{th}}$, $q_{4h}^{4^{th}}$ are defined by (4.136), (4.137) accordingly and

$$\widetilde{\Omega}_3(x_2) = \frac{3}{5a_2}\widetilde{\delta}\tau(2a_2 - x_2) + \left(\frac{3}{160} + \frac{47}{2880}\omega\right)\widetilde{\lambda}h^4,$$
$$\geq \frac{3}{5}\widetilde{\delta}\tau + \left(\frac{3}{160} + \frac{47}{2880}\omega\right)\widetilde{\lambda}h^4 \geq \left|\Psi_1^{z,k}\right| \tag{4.163}$$

$$\frac{5}{6}\widetilde{\Omega}_3(x_2) = \frac{1}{2a_2}\widetilde{\delta}\tau(2a_2 - x_2) + \left(\frac{1}{64} + \frac{47}{3456}\omega\right)\widetilde{\lambda}h^4,$$
$$\geq \frac{1}{2}\widetilde{\delta}\tau + \left(\frac{1}{64} + \frac{47}{3456}\omega\right)\widetilde{\lambda}h^4 \geq \left|\Psi_2^{z,k}\right|. \tag{4.164}$$

Furthermore, construct the following majorant function:

$$\overline{\xi}^{z}(x_1,x_2,t) = \overline{\xi}_1^{z}(x_1,x_2,t) + \overline{\xi}_2^{z}(x_1,x_2,t), \tag{4.165}$$

where

$$\overline{\xi}_1^{z}(x_1,x_2,t) = \frac{3}{5a_2}\widetilde{\delta}\tau(t+1)(2a_2 - x_2) \text{ on } \overline{D^h\gamma_\tau},$$
$$\overline{\xi}_2^{z}(x_1,x_2,t) = \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\widetilde{\lambda}h^4\left(1 + a_1^2 + a_2^2 - x_1^2 - x_2^2\right) \text{ on } \overline{D^h\gamma_\tau},$$

which satisfies the difference problem

$$\widetilde{\Theta}_{h,\tau}\overline{\xi}_{h,\tau}^{z,k+1} = \widetilde{\Lambda}_{h,\tau}\overline{\xi}_{h,\tau}^{z,k} + \widetilde{\Omega}_3(x_2) \text{ on } D^{0h}\gamma_\tau,$$

$$\widetilde{\Theta}_{h,\tau}^{*}\overline{\xi}_{h,\tau}^{z,k+1} = \widetilde{\Lambda}_{h,\tau}^{*}\overline{\xi}_{h,\tau}^{z,k} + \widetilde{\Gamma}_{h,\tau}^{*}\overline{\xi}_{h,\tau}^{z*} + \frac{5}{6}\widetilde{\Omega}_3(x_2) \text{ on } D^{*h}\gamma_\tau,$$

$$\overline{\xi}_{h,\tau}^{z} = \overline{\xi}_{h,\tau}^{z*} = \overline{\xi}_1^{z}(0,x_2,t) + \overline{\xi}_2^{z}(0,x_2,t) \text{ on } S_T^h\gamma_1,$$

$$\overline{\xi}_{h,\tau}^{z} = \overline{\xi}_1^{z}(x_1,0,t) + \overline{\xi}_2^{z}(x_1,0,t) \text{ on } S_T^h\gamma_2,$$

$$\overline{\xi}^{z}_{h,\tau} = \overline{\xi}^{z*}_{h,\tau} = \overline{\xi}^{z}_{1}(a_1,x_2,t) + \overline{\xi}^{z}_{2}(a_1,x_2,t) \text{ on } S^h_T\gamma_3,$$

$$\overline{\xi}^{z}_{h,\tau} = \overline{\xi}^{z}_{1}(x_1,a_2,t) + \overline{\xi}^{z}_{2}(x_1,a_2,t) \text{ on } S^h_T\gamma_4,$$

$$\overline{\xi}^{z}_{h,\tau} = \overline{\xi}^{z}_{1}(x_1,x_2,0) + \overline{\xi}^{z}_{2}(x_1,x_2,0) \text{ on } S^h_T\gamma_5. \tag{4.166}$$

By writing (4.162) and (4.166) in matrix form as

$$\widetilde{A}\widehat{\xi}^{z,k+1} = \widetilde{B}\widehat{\xi}^{z,k} + \tau\widehat{\eta}^{z,k}, \tag{4.167}$$

$$\widetilde{A}\overline{\xi}^{z,k+1} = \widetilde{B}\overline{\xi}^{z,k} + \tau\overline{\eta}^{z,k}, \tag{4.168}$$

respectively, where $\widetilde{A},\widetilde{B}$ are as given in (4.16) and $\widehat{\xi}^{z,k},\overline{\xi}^{z,k},\widehat{\eta}^{z,k},\overline{\eta}^{z,k} \in R^N$ and using (4.163)–(4.166) we get $\overline{\eta}^{z,k} \geq 0$ and $\left|\widehat{\eta}^{z,k}\right| \leq \overline{\eta}^{z,k}$ for $k = 0,1,...,M'-1$ and $\overline{\xi}^{z,0} \geq 0$, $\left|\widehat{\xi}^{z,0}\right| \leq \overline{\xi}^{z,0}$. Then, on the basis of Lemma 4.3 follows $\left|\widehat{\xi}^{z,k+1}\right| \leq \overline{\xi}^{z,k+1}$, $k = 0,1,...,M'-1$. From

$$\overline{\xi}^{z}(x_1,x_2,t) \leq \overline{\xi}^{z}(0,0,T)$$
$$= \frac{6}{5}\widetilde{\delta}(T+1)\tau + \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\widetilde{\lambda}\left(1+a_1^2+a_2^2\right)h^4,$$

follows (4.161). □

4.3.2 Boundary Value Problem for $\frac{\partial^2 u}{\partial x_2 \partial t}$ and Hexagonal Grid Approximation: Stage 2$\left(H^{4th}\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)\right)$

Let the BVP(u) be given. Then, as the first step we apply the Stage 1$\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$ and obtain the approximate solution $u_{t,h,\tau}$ on the hexagonal grids. Subsequently, denote $q_{t,i} = \frac{\partial^2 u}{\partial x_2 \partial t}$ on $S_T\gamma_i, i = 1,2,...,5$ and develop the next problem for $z_t = \frac{\partial^2 u}{\partial x_2 \partial t}$.

Boundary Value Problem for $\frac{\partial^2 u}{\partial x_2 \partial t}$ $\left(\textbf{BVP}\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)\right)$

$$Lz_t = \frac{\partial^2 f(x_1,x_2,t)}{\partial x_2 \partial t} \text{ on } Q_T,$$

$$z_t(x_1,x_2,t) = q_{t,i} \text{ on } S_T\gamma_i, i = 1,2,...,5, \tag{4.169}$$

We assume $z_t \in C_{x,t}^{6+\alpha,3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$. We take

$$q_{t,2h}^{4^{th}} = \frac{1}{12\sqrt{3}h}\left(-25u_t(x_1,0,t) + 48u_{t,h,\tau}\left(x_1,\sqrt{3}h,t\right) - 36u_{t,h,\tau}\left(x_1,2\sqrt{3}h,t\right)\right.$$
$$\left. + 16u_{t,h,\tau}\left(x_1,3\sqrt{3}h,t\right) - 3u_{t,h,\tau}\left(x_1,4\sqrt{3}h,t\right)\right) \text{ on } S_T^h\gamma_2, \tag{4.170}$$

$$q_{t,4h}^{4^{th}} = \frac{1}{12\sqrt{3}h}\left(25u_t(x_1,a_2,t) - 48u_{t,h,\tau}\left(x_1,a_2-\sqrt{3}h,t\right) + 36u_{t,h,\tau}\left(x_1,a_2-2\sqrt{3}h,t\right)\right.$$
$$\left. - 16u_{t,h,\tau}\left(x_1,a_2-3\sqrt{3}h,t\right) + 3u_{t,h,\tau}\left(x_1,a_2-4\sqrt{3}h,t\right)\right) \text{ on } S_T^h\gamma_4, \tag{4.171}$$

$$q_{t,ih} = \frac{\partial\phi_t(x_1,x_2,t)}{\partial x_2} \text{ on } S_T^h\gamma_i, i=1,3, \tag{4.172}$$

$$q_{t,5h} = \frac{\partial\widehat{\varphi}(x_1,x_2)}{\partial x_2} \text{ on } S_T^h\gamma_5, \tag{4.173}$$

where $\widehat{\varphi}(x_1,x_2)$ and $\phi_t(x_1,x_2,t)$ are as given in (4.11) and $u_{t,h,\tau}$ is the approximate solution taken by Stage $1\left(H^{4th}\left(\frac{\partial u}{\partial t}\right)\right)$.

For a stable fourth order accurate numerical solution of BVP$\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)$ we propose the next problem:

Stage 2$\left(H^{4th}\left(\frac{\partial^2 u}{\partial x_2\partial t}\right)\right)$

$$\widetilde{\Theta}_{h,\tau}z_{t,h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}z_{t,h,\tau}^{k} + \widetilde{D}_{x_2}\widetilde{\psi}_t \text{ on } D^{0h}\gamma_\tau,$$
$$\widetilde{\Theta}_{h,\tau}^{*}z_{t,h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}^{*}z_{t,h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*}q_{t,1h} + \widetilde{D}_{x_2}\widetilde{\psi}_t^{*} \text{ on } D^{*lh}\gamma_\tau,$$
$$\widetilde{\Theta}_{h,\tau}^{*}z_{t,h,\tau}^{k+1} = \widetilde{\Lambda}_{h,\tau}^{*}z_{t,h,\tau}^{k} + \widetilde{\Gamma}_{h,\tau}^{*}q_{t,3h} + \widetilde{D}_{x_2}\widetilde{\psi}_t^{*} \text{ on } D^{*rh}\gamma_\tau$$
$$z_{t,h,\tau} = q_{t,ih} \text{ on } S_T^h\gamma_i, i=1,3,5,$$
$$z_{t,h,\tau} = q_{t,ih}^{4^{th}} \text{ on } S_T^h\gamma_i, i=2,4 \tag{4.174}$$

where $q_{ih}^{4^{th}}, i=2,4$ and q_{ih} , $i=1,3,5$ are defined by (4.136)–(4.139) and the operators $\widetilde{\Theta}_{h,\tau}$, $\widetilde{\Lambda}_{h,\tau}, \widetilde{\Theta}_{h,\tau}^{*}$, $\widetilde{\Gamma}_{h,\tau}^{*}$ and $\widetilde{\Lambda}_{h,\tau}^{*}$ are the operators given in (4.4)–(4.8) respectively. Additionally,

$$\widetilde{D}_{x_2}\widetilde{\psi}_t = \partial_{x_2} f_{t,P_0}^{k+1} + \frac{1}{16}h^2\left(\partial_{x_1}^2 \partial_{x_2} f_{t,P_0}^{k+1} + \partial_{x_2}^3 f_{t,P_0}^{k+1}\right), \tag{4.175}$$

$$\widetilde{D}_{x_2}\widetilde{\psi}_t^* = \frac{h^2}{96\tau\omega}\partial_{x_2} f_{t,P_A}^{k+1} - \frac{h^2}{96\tau\omega}\partial_{x_2} f_{t,P_A}^{k} - \frac{1}{6}\partial_{x_2} f_{t,P_A}^{k+1} + \partial_{x_2} f_{t,P_0}^{k+1}$$
$$+ \frac{1}{16}h^2\left(\partial_{x_1}^2 \partial_{x_2} f_{t,P_0}^{k+1} + \partial_{x_2}^3 f_{t,P_0}^{k+1}\right). \tag{4.176}$$

Let

$$\xi_{h,\tau}^{z_t} = z_{t,h,\tau} - z_t \text{ on } \overline{D^h\gamma_\tau}, \tag{4.177}$$

from (4.174) and (4.177) we have

$$\widetilde{\Theta}_{h,\tau}\xi_{h,\tau}^{z_t,k+1} = \widetilde{\Lambda}_{h,\tau}\xi_{h,\tau}^{z_t,k} + \widetilde{\Psi}_1^{z_t,k} \text{ on } D^{0h}\gamma_\tau,$$
$$\widetilde{\Theta}_{h,\tau}^*\xi_{h,\tau}^{z_t,k+1} = \widetilde{\Lambda}_{h,\tau}^*\xi_{h,\tau}^{z_t,k} + \widetilde{\Psi}_2^{z_t,k} \text{ on } D^{*h}\gamma_\tau$$
$$\xi_{h,\tau}^{z_t} = 0 \text{ on } S_T^h\gamma_i, i = 1,3,5$$
$$\xi_{h,\tau}^{z_t} = q_{t,ih}^{4^{th}}\left(u_{h,\tau}\right) - q_{t,i} \text{ on } S_T^h\gamma_i, i = 2,4. \tag{4.178}$$

where $q_{t,2h}^{4^{th}}$, $q_{t,4h}^{4^{th}}$, $q_{t,ih}$, $i = 1,3,5$ are defined by (4.170)-(4.173) accordingly and

$$\widetilde{\Psi}_1^{z_t,k} = \widetilde{\Lambda}_{h,\tau}z_t^k - \widetilde{\Theta}_{h,\tau}z_t^{k+1} + \widetilde{D}_{x_2}\widetilde{\psi}_t, \tag{4.179}$$

$$\widetilde{\Psi}_2^{z_t,k} = \widetilde{\Lambda}_{h,\tau}^*z_t^k - \widetilde{\Theta}_{h,\tau}^*z_t^{k+1} + \widetilde{\Gamma}_{h,\tau}^*q_{t,i} + \widetilde{D}_{x_2}\widetilde{\psi}_t^*, i = 1,3. \tag{4.180}$$

Let $\widetilde{\lambda}_{t,1} = \mu_1(z_t)$, $\widetilde{\delta}_{t,1} = \mu_2(z_t)$, where μ_1, μ_2 are given in (4.39), (4.40), respectively, and

$$\widetilde{\lambda}_t = \max\left\{\widetilde{\lambda}_{t,1}, \frac{\widetilde{M}_{t,2}}{\rho} + 15\frac{d}{\rho}\left(\frac{3}{160} + \frac{47\omega}{2880}\right)\widetilde{\alpha}_t\right\}, \tag{4.181}$$

$$\widetilde{\delta}_t = \max\left\{\widetilde{\delta}_{t,1}, 15d\widetilde{\beta}_t\right\}, \tag{4.182}$$

where $\widetilde{\alpha}_t = \mu_1(u_t)$, $\widetilde{\beta}_t = \mu_2(u_t)$ and d is presented in (2.60) also $\widetilde{M}_{t,2} = \frac{9}{5}\max_{\overline{Q}_T}\left|\frac{\partial^5 u_t}{\partial x_2^5}\right|$ and $\rho = \frac{3}{640\omega} + \frac{47}{11520}$ and z_t is the solution of BVP$\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)$.

Theorem 4.5: (Buranay et al. [53]) The solution $z_{t,h,\tau}$ achieved by Stage $2\left(H^{4th}\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)\right)$ satisfies

$$\max_{\overline{D^h \gamma_\tau}} \left| z_{t,h,\tau} - z_t \right| \leq \frac{6}{5}\widetilde{\delta}_t \left(T+1\right)\tau + \left(\frac{3}{640\omega} + \frac{47}{11520}\right)\widetilde{\lambda}_t \left(1 + a_1^2 + a_2^2\right) h^4, \quad (4.183)$$

for $r = \frac{\omega\tau}{h^2} \geq \frac{1}{16}$, where $\widetilde{\lambda}_t, \widetilde{\delta}_t$ are positive constants given in (4.181), (4.182), respectively, and $z_t = \frac{\partial^2 u}{\partial x_2 \partial t}$ is the exact solution of BVP$\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)$.

Proof. The proof is analogous to the proof of Theorem 4.4, and follows from the assumption $z_t \in C_{x,t}^{6+\alpha, 3+\frac{\alpha}{2}}\left(\overline{Q}_T\right)$. □

Chapter 5

EXPERIMENTAL INVESTIGATIONS OF THE FOURTH ORDER ACCURATE TWO-STAGE IMPLICIT METHODS

The proposed fourth order two-stage implicit methods are applied on two test problems such that for the first example the exact solution is known. However, for the second example the exact solution is not given. We take $D=\left\{(x_1,x_2):0<x_1<1,0<x_2<\frac{\sqrt{3}}{2}\right\}$, and $t\in[0,1]$. Further, Mathematica is used for the realization of the algorithms in machine precision. Also we used preconditioned conjugate gradient method with the preconditioning approach given in Buranay and Iyikal [55] (see also Concus et al. [56] and Axelsson [57]). We define the following:

$H^{4th}\left(\frac{\partial u}{\partial x_i}\right), i=1,2$ is the given fourth order method for the computation $\frac{\partial u}{\partial x_i}$, $i=1,2$, respectively.

$H^{4th}\left(\frac{\partial^2 u}{\partial x_i \partial t}\right), i=1,2$ is the given fourth order method for the computation $\frac{\partial^2 u}{\partial x_i \partial t}$, $i=1,2$, seriatim.

$CT^{H^{4th}}_{\frac{\partial u}{\partial x_i}}, i=1,2$ presents the $CPUs$ for one time level spend by the method $H^{4th}\left(\frac{\partial u}{\partial x_i}\right)$, $i=1,2$, accordingly.

$CT^{H^{4th}}_{\frac{\partial^2 u}{\partial x_i \partial t}}, i=1,2$ shows the $CPUs$ for one time level spend by the method $H^{4th}\left(\frac{\partial^2 u}{\partial x_i \partial t}\right)$, $i=1,2$, respectively.

Furthermore, $v_{2^{-\mu},2^{-\lambda}}$, $z_{2^{-\mu},2^{-\lambda}}$, $u_{t,2^{-\mu},2^{-\lambda}}$, and $v_{t,2^{-\mu},2^{-\lambda}}$, $z_{t,2^{-\mu},2^{-\lambda}}$ are the computed grid functions obtained by the methods $H^{4th}\left(\frac{\partial u}{\partial x_i}\right)$, $i=1,2$, $H^{4th}\left(\frac{\partial u}{\partial t}\right)$ and $H^{4th}\left(\frac{\partial^2 u}{\partial x_i \partial t}\right)$, $i=1,2$, accordingly for $h=2^{-\mu}$ and $\tau=2^{-\lambda}$ where μ,λ are positive integers. The error function $\varepsilon_{h,\tau}$ on the set $\overline{D^h\gamma_\tau}$ obtained by $H^{4th}\left(\frac{\partial u}{\partial x_i}\right)$, $i=1,2$ for $h=2^{-\mu},\tau=2^{-\lambda}$ is presented by $\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i}}\left(2^{-\mu},2^{-\lambda}\right), i=1,2$ while the error function resulting by the methods $H^{4th}\left(\frac{\partial^2 u}{\partial x_i \partial t}\right)$, $i=1,2$ are shown with $\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i \partial t}}\left(2^{-\mu},2^{-\lambda}\right), i=1,2$, respectively. Furthermore,

$$\max_{\overline{D^h\gamma_\tau}}\left|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i}}\left(2^{-\mu},2^{-\lambda}\right)\right| = \left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i}}\right\|_\infty, i=1,2, \tag{5.1}$$

$$\max_{\overline{D^h\gamma_\tau}}\left|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i \partial t}}\left(2^{-\mu},2^{-\lambda}\right)\right| = \left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i \partial t}}\right\|_\infty, i=1,2. \tag{5.2}$$

Further, we denote the order of convergence of the approximate solution $v_{2^{-\mu},2^{-\lambda}}$ and $z_{2^{-\mu},2^{-\lambda}}$ to the functions $v=\frac{\partial u}{\partial x_1}$ and $z=\frac{\partial u}{\partial x_2}$ obtained by using the fourth order implicit method $H^{4th}\left(\frac{\partial u}{\partial x_i}\right), i=1,2$ by

$$\Re^{H^{4th}}_{\frac{\partial u}{\partial x_i}} = \frac{\left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i}}(2^{-\mu},2^{-\lambda})\right\|_\infty}{\left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i}}(2^{-(\mu+1)},2^{-(\lambda+4)})\right\|_\infty} i=1,2. \tag{5.3}$$

Furthermore, the order of convergence of the approximate solutions $v_{t,2^{-\mu},2^{-\lambda}}$ and $z_{t,2^{-\mu},2^{-\lambda}}$ to their corresponding exact solutions $v_t=\frac{\partial^2 u}{\partial x_1 \partial t}$ and $z_t=\frac{\partial^2 u}{\partial x_2 \partial t}$ obtained by $H^{4th}\left(\frac{\partial^2 u}{\partial x_i \partial t}\right), i=1,2$ are given by

$$\Re^{H^{4th}}_{\frac{\partial^2 u}{\partial x_i \partial t}} = \frac{\left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i \partial t}}(2^{-\mu},2^{-\lambda})\right\|_\infty}{\left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i \partial t}}(2^{-(\mu+1)},2^{-(\lambda+4)})\right\|_\infty}, \ i=1,2. \tag{5.4}$$

Remark 5.1: We remark that the computed values of (5.3) and (5.4) are $\approx 2^4$ showing the fourth order convergence of the given methods in x_1, x_2 and linear convergence in t.

Example 5.1:
$$\frac{\partial u}{\partial t}=0.25\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+f(x_1,x_2,t) \text{ on } Q_T,$$
$$u(x_1,x_2,0)=0.005x_1^{9+\alpha}+0.03x_2^{9+\alpha}+1+x_1x_2 \text{ on } \overline{D},$$
$$u(x_1,x_2,t)=\widehat{u}(x_1,x_2,t) \text{ on } S_T,$$

where
$$f(x_1,x_2,t)=-\left(\frac{9+\alpha}{2}\right)t^{\frac{7+\alpha}{2}}\sin\left(t^{\frac{7+\alpha}{2}}\right)$$
$$-x_1x_2e^{-t}-0.25(9+\alpha)(8+\alpha)\left[0.005x_1^{7+\alpha}+0.03x_2^{7+\alpha}\right]$$
$$\widehat{u}(x_1,x_2,t)=0.005x_1^{9+\alpha}+0.03x_2^{9+\alpha}+\cos(t^{\frac{9+\alpha}{2}})+x_1x_2e^{-t}.$$

present the heat source and the exact solution respectively and we take $\alpha=0.5$. For the Example 5.1, Table 5.1 demonstrates $CT_{\frac{\partial u}{\partial x_i}}^{H^{4th}}$, $\left\|\varepsilon_{\frac{\partial u}{\partial x_i}}^{H^{4th}}\right\|_\infty$ and $\Re_{\frac{\partial u}{\partial x_i}}^{H^{4th}}$ $i=1,2$ achieved by $H^{4th}\left(\frac{\partial u}{\partial x_i}\right)$, $i=1,2$ respectively while Table 5.2 shows $CT_{\frac{\partial^2 u}{\partial x_i \partial t}}^{H^{4th}}$, $\left\|\varepsilon_{\frac{\partial^2 u}{\partial x_i \partial t}}^{H^{4th}}\right\|_\infty$ and $\Re_{\frac{\partial^2 u}{\partial x_i \partial t}}^{H^{4th}}$ $i=1,2$ taken by the method $H^{4th}\left(\frac{\partial^2 u}{\partial x_i \partial t}\right)$, $i=1,2$ accordingly. Tables 5.1 and 5.2 justify the theoretical results given such that the approximate solutions $v_{h,\tau}, z_{h,\tau}, v_{t,h,\tau}$ and $z_{t,h,\tau}$ converge to the corresponding exact functions $v=\frac{\partial u}{\partial x_1}$ and $z=\frac{\partial u}{\partial x_2}, v_t=\frac{\partial^2 u}{\partial x_1 \partial t}$ and $z_t=\frac{\partial^2 u}{\partial x_2 \partial t}$ with fourth order in spatial variables and first order in time for $r \geq \frac{1}{16}$, as explained in Remark 5.1 and presented in the fourth and last columns of Tables 5.1 and 5.2. Moreover, the last two rows in Tables 5.1 and 5.2 demonstrate that the order of convergence is also $O\left(h^4+\tau\right)$ when $r<\frac{1}{16}$.

Figures 5.1 and 5.2 illustrate the grid functions $\left|\varepsilon_{\frac{\partial u}{\partial x_i}}^{H^{4th}}(2^{-4},2^{-3})\right|$, $\left|\varepsilon_{\frac{\partial u}{\partial x_i}}^{H^{4th}}(2^{-5},2^{-7})\right|$, $\left|\varepsilon_{\frac{\partial u}{\partial x_i}}^{H^{4th}}(2^{-6},2^{-11})\right|$ and $\left|\varepsilon_{\frac{\partial u}{\partial x_i}}^{H^{4th}}(2^{-7},2^{-15})\right|$, $i=1,2$, respectively, when $t=0.8$ obtained by the corresponding method $H^{4th}\left(\frac{\partial u}{\partial x_i}\right)$, $i=1,2$ for the Example 5.1. Figures 5.3 and 5.4 demonstrate the grid functions $\left|\varepsilon_{\frac{\partial^2 u}{\partial x_i \partial t}}^{H^{4th}}(2^{-4},2^{-3})\right|$, $\left|\varepsilon_{\frac{\partial^2 u}{\partial x_i \partial t}}^{H^{4th}}(2^{-5},2^{-7})\right|$,

Table 5.1: $CT^{H^{4th}}_{\frac{\partial u}{\partial x_i}}$, $\left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_i}}\right\|_\infty$ for $i = 1,2$ and the convergence orders of $v_{h,\tau}$ and $z_{h,\tau}$ to their exact respective derivatives for the Example 5.1.

(h,τ)	$CT^{H^{4th}}_{\frac{\partial u}{\partial x_1}}$	$\left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_1}}\right\|_\infty$	$\Re^{H^{4th}}_{\frac{\partial u}{\partial x_1}}$	$CT^{H^{4th}}_{\frac{\partial u}{\partial x_2}}$	$\left\|\varepsilon^{H^{4th}}_{\frac{\partial u}{\partial x_2}}\right\|_\infty$	$\Re^{H^{4th}}_{\frac{\partial u}{\partial x_2}}$
$(2^{-4},2^{-3})$	0.33	4.5384×10^{-3}	14.634	0.31	5.3873×10^{-3}	14.595
$(2^{-5},2^{-7})$	20.55	3.1012×10^{-4}	15.901	19.03	3.6911×10^{-4}	15.895
$(2^{-6},2^{-11})$	1309.02	1.9503×10^{-5}	15.991	1220.01	2.3222×10^{-5}	15.992
$(2^{-7},2^{-15})$	82622.60	1.2196×10^{-6}		78092.10	1.4521×10^{-6}	
$(2^{-4},2^{-11})$	79.27	1.8788×10^{-5}	15.980	73.06	2.0209×10^{-5}	16.006
$(2^{-5},2^{-15})$	5209.05	1.1757×10^{-6}		4880.77	1.2626×10^{-6}	

Table 5.2: $CT^{H^{4th}}_{\frac{\partial^2 u}{\partial x_i\partial t}}$, $\left\|\varepsilon^{H^{4th}}_{\frac{\partial^2 u}{\partial x_i\partial t}}\right\|_\infty$, for $i = 1,2$ and the convergence orders of $v_{t,h,\tau}$ and $z_{t,h,\tau}$ to their exact respective derivatives for the Example 5.1.

(h,τ)	$CT^{H^{4th}}_{\frac{\partial^2 u}{\partial x_1\partial t}}$	$\left\|\varepsilon^{H^{4th}}_{\frac{\partial^2 u}{\partial x_1\partial t}}\right\|_\infty$	$\Re^{H^{4th}}_{\frac{\partial^2 u}{\partial x_1\partial t}}$	$CT^{H^{4th}}_{\frac{\partial^2 u}{\partial x_2\partial t}}$	$\left\|\varepsilon^{H^{4th}}_{\frac{\partial^2 u}{\partial x_2\partial t}}\right\|_\infty$	$\Re^{H^{4th}}_{\frac{\partial^2 u}{\partial x_2\partial t}}$
$(2^{-4},2^{-3})$	0.41	4.42644×10^{-6}	15.451	0.39	4.2937×10^{-6}	15.401
$(2^{-5},2^{-7})$	24.78	2.8648×10^{-7}	15.925	22.593	2.7879×10^{-7}	15.892
$(2^{-6},2^{-11})$	1595.03	1.7989×10^{-8}	15.997	1436.69	1.7543×10^{-8}	15.993
$(2^{-7},2^{-15})$	100555.00	1.1245×10^{-9}		92543.1	1.0969×10^{-10}	
$(2^{-4},2^{-11})$	96.94	1.8392×10^{-8}	15.997	88.61	1.7381×10^{-8}	15.920
$(2^{-5},2^{-16})$	6414.28	1.1497×10^{-9}		5733.49	1.0918×10^{-9}	

$\left|\varepsilon^{H^{4th}}_{\frac{\partial^2 u}{\partial x_i\partial t}}(2^{-6},2^{-11})\right|$ and $\left|\varepsilon^{H^{4th}}_{\frac{\partial^2 u}{\partial x_i\partial t}}(2^{-7},2^{-15})\right|$ for $i = 1,2$ respectively, for $t = 0.8$ achieved by applying the corresponding method $H^{4th}\left(\frac{\partial^2 u}{\partial x_i\partial t}\right)$, $i = 1,2$ for the Example 5.1.

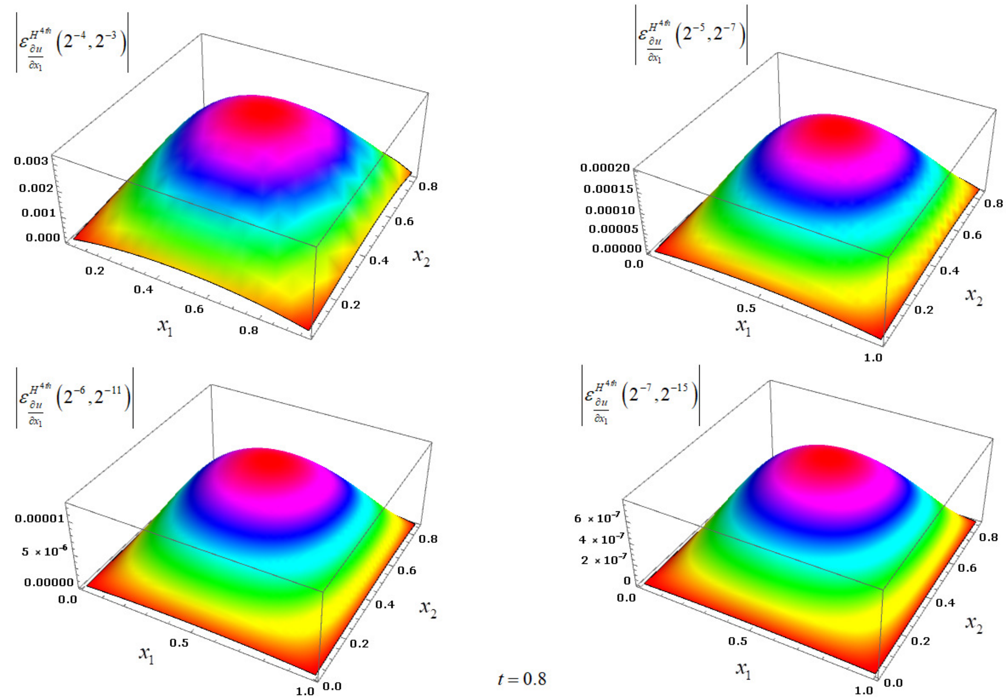


Figure 5.1: The grid function of absolute errors when $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial u}{\partial x_1}\right)$ for the Example 5.1.

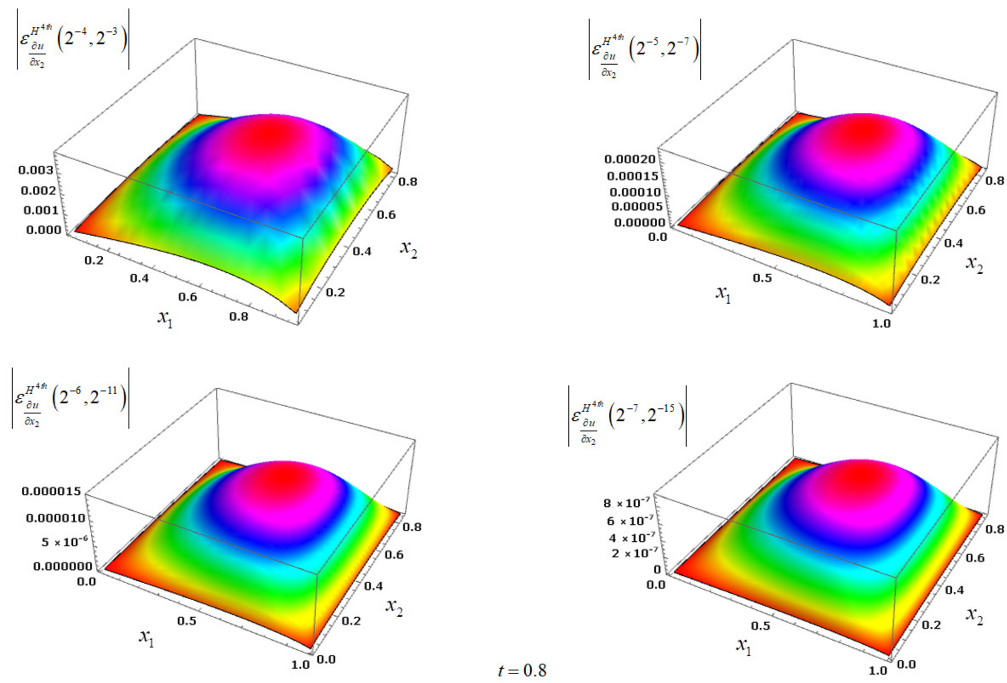


Figure 5.2: The grid function of absolute errors when $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial u}{\partial x_2}\right)$ for the Example 5.1.

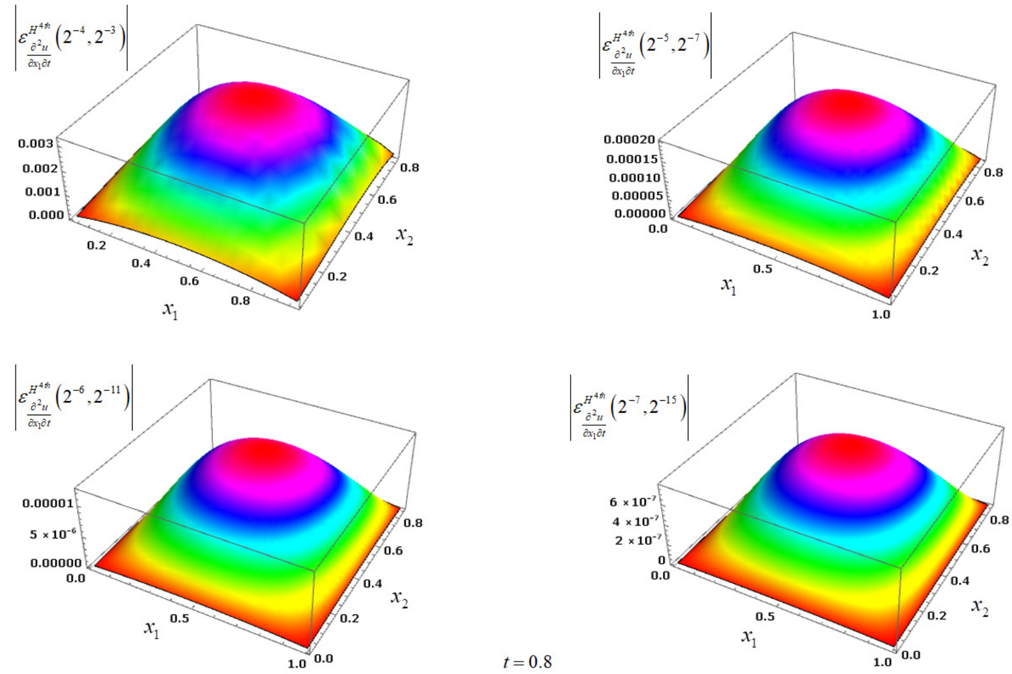


Figure 5.3: The grid function of absolute errors when $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)$ for the Example 5.1.

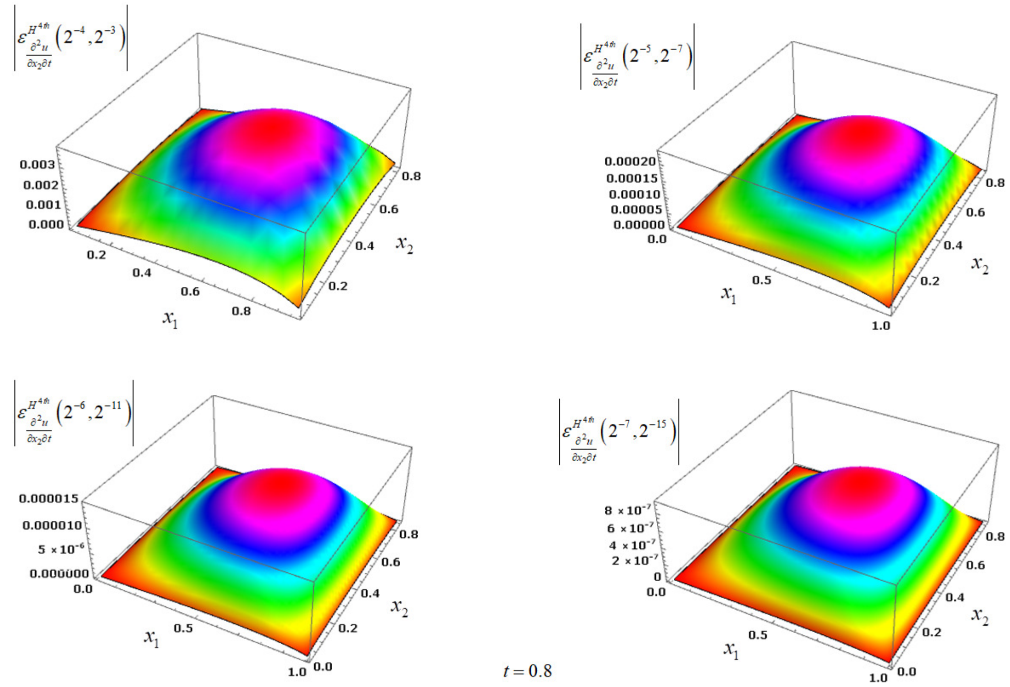


Figure 5.4: The grid function of absolute errors when $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)$ for the Example 5.1.

Example 5.2:
$$\frac{\partial u}{\partial t} = 0.25\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + f(x_1,x_2,t) \text{ on } Q_T,$$
$$u(x_1,x_2,0) = 0.01x_1x_2(1-x_1)\left(\frac{\sqrt{3}}{2} - x_2\right) \text{ on } \overline{D},$$
$$u(x_1,x_2,t) = 0 \text{ on } S_T.$$

The heat source function is
$$f(x_1,x_2,t) = -0.01x_1x_2(1-x_1)\left(\frac{\sqrt{3}}{2} - x_2\right)\sin t$$
$$+0.005\left(x_1(1-x_1) + x_2\left(\frac{\sqrt{3}}{2} - x_2\right)\right)\cos t.$$

The problem in Example 5.2 is a benchmark problem such that the solution is not provided. An analogous problem with zero heat source was also considered in Henner et al. [65]. By applying the proposed methods $H^{4th}\left(\frac{\partial u}{\partial x_i}\right), i = 1,2,$ we obtain the approximate solutions $v_{2^{-\mu},2^{-\lambda}}$ and $z_{2^{-\mu},2^{-\lambda}}$ accordingly at every time level for the considered values $\mu = 5,6,7$ and $\lambda = 7,11,15$. Tables 5.3 and 5.4 present $v_{2^{-\mu},2^{-\lambda}}(x_1,x_2,t)$ and $z_{2^{-\mu},2^{-\lambda}}(x_1,x_2,t)$, respectively, at the grid points $\left(0.125, \frac{\sqrt{3}}{8}, 1\right)$, $\left(0.25, \frac{\sqrt{3}}{8}, 1\right)$, $\left(0.375, \frac{\sqrt{3}}{8}, 1\right)$, $\left(0.5, \frac{\sqrt{3}}{8}, 1\right)$, $\left(0.625, \frac{\sqrt{3}}{8}, 1\right)$, $\left(0.75, \frac{\sqrt{3}}{8}, 1\right)$ and $\left(0.875, \frac{\sqrt{3}}{8}, 1\right)$ and the corresponding order of convergence $\Re^{H^{4th}}_{\frac{\partial u}{\partial x_i}}(P)$ for $i = 1,2$ at the grid point $P(x_1,x_2,t)$ given as
$$\Re^{H^{4th}}_{\frac{\partial u}{\partial x_1}}(P) = \left|\frac{v_{2^{-5},2^{-7}}(P) - v_{2^{-6},2^{-11}}(P)}{v_{2^{-6},2^{-11}}(P) - v_{2^{-7},2^{-15}}(P)}\right|, \quad (5.5)$$
$$\Re^{H^{4th}}_{\frac{\partial u}{\partial x_2}}(P) = \left|\frac{z_{2^{-5},2^{-7}}(P) - z_{2^{-6},2^{-11}}(P)}{z_{2^{-6},2^{-11}}(P) - z_{2^{-7},2^{-15}}(P)}\right|. \quad (5.6)$$

By the same way Tables 5.5 and 5.6 show $v_{t,2^{-\mu},2^{-\lambda}}(x_1,x_2,t)$ and $z_{t,2^{-\mu},2^{-\lambda}}(x_1,x_2,t)$, respectively, at the the considered grids and the corresponding convergence orders $\Re^{H^{4th}}_{\frac{\partial^2 u}{\partial x_i \partial t}}(P)$ for $i = 1,2$ at the point $P(x_1,x_2,t)$ defined as

Table 5.3: The numerical solution $v_{h,\tau}$ at seven points when $t = 1$, and the convergence orders obtained by $H^{4th}(\frac{\partial u}{\partial x_1})$ for the Example 5.2.

P	$v_{2^{-5},2^{-7}}(P)$	$v_{2^{-6},2^{-11}}(P)$	$v_{2^{-7},2^{-15}}(P)$	$\Re_{\frac{\partial u}{\partial x_1}}^{H^{4th}}(P)$
$\left(0.125, \frac{\sqrt{3}}{8}, 1\right)$	0.000569713036	0.000569841548	0.000569849555	16.052
$\left(0.25, \frac{\sqrt{3}}{8}, 1\right)$	0.000379748416	0.000379890609	0.000379899468	16.049
$\left(0.375, \frac{\sqrt{3}}{8}, 1\right)$	0.000189857076	0.000189944236	0.000189949667	16.048
$\left(0.5, \frac{\sqrt{3}}{8}, 1\right)$	5.22×10^{-16}	-3.27×10^{-17}	1.87×10^{-18}	16.046
$\left(0.625, \frac{\sqrt{3}}{8}, 1\right)$	−0.000189857076	−0.000189944236	−0.000189949667	16.048
$\left(0.75, \frac{\sqrt{3}}{8}, 1\right)$	−0.000379748416	−0.000379890609	−0.000379899468	16.049
$\left(0.875, \frac{\sqrt{3}}{8}, 1\right)$	−0.000569713036	−0.000569841548	−0.00056984955	16.052

Table 5.4: The numerical solution $z_{h,\tau}$ at seven points when $t = 1$, and the convergence orders obtained by $H^{4th}(\frac{\partial u}{\partial x_2})$ for the Example 5.2.

P	$z_{2^{-5},2^{-7}}(P)$	$z_{2^{-6},2^{-11}}(P)$	$z_{2^{-7},2^{-15}}(P)$	$\Re_{\frac{\partial u}{\partial x_2}}^{H^{4th}}(P)$
$\left(0.125, \frac{\sqrt{3}}{8}, 1\right)$	0.000255810101	0.000255886243	0.000255890985	16.052
$\left(0.25, \frac{\sqrt{3}}{8}, 1\right)$	0.000438524584	0.000438661691	0.000438670233	16.052
$\left(0.375, \frac{\sqrt{3}}{8}, 1\right)$	0.000548151240	0.000548326834	0.000548337774	16.052
$\left(0.5, \frac{\sqrt{3}}{8}, 1\right)$	0.000584693185	0.000584881865	0.000584893620	16.052
$\left(0.625, \frac{\sqrt{3}}{8}, 1\right)$	0.000548151240	0.000548326834	0.000548337774	16.052
$\left(0.75, \frac{\sqrt{3}}{8}, 1\right)$	0.000438524584	0.000438661691	0.000438670233	16.052
$\left(0.875, \frac{\sqrt{3}}{8}, 1\right)$	0.000255810101	0.000255886242	0.000255890985	16.052

$$\Re_{\frac{\partial^2 u}{\partial x_1 \partial t}}^{H^{4th}}(P) = \left| \frac{v_{t,2^{-5},2^{-7}}(P) - v_{t,2^{-6},2^{-11}}(P)}{v_{t,2^{-6},2^{-11}}(P) - v_{t,2^{-7},2^{-15}}(P)} \right|, \tag{5.7}$$

$$\Re_{\frac{\partial^2 u}{\partial x_2 \partial t}}^{H^{4th}}(P) = \left| \frac{z_{t,2^{-5},2^{-7}}(P) - z_{t,2^{-6},2^{-11}}(P)}{z_{t,2^{-6},2^{-11}}(P) - z_{t,2^{-7},2^{-15}}(P)} \right|. \tag{5.8}$$

The computed solutions $v_{2^{-7},2^{-15}}$ and $z_{2^{-7},2^{-15}}$ achieved by using the corresponding two-stage method $H^{4th}\left(\frac{\partial u}{\partial x_i}\right), i = 1,2$ are demonstrated in Figures 5.5 and 5.6 for the time levels $t = 0.2$ and $t = 0.8$. Figures 5.7 and 5.8 illustrate the approximate solutions $v_{t,2^{-7},2^{-15}}$ and $z_{t,2^{-7},2^{-15}}$ taken by using the respective two-stage method $H^{4th}\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)$,

Table 5.5: The numerical solution $v_{t,h,\tau}$ at seven points when $t = 1$, and the convergence orders obtained by $H^{4th}(\frac{\partial^2 u}{\partial x_1 \partial t})$ for the Example 5.2.

P	$v_{t,2^{-5},2^{-7}}(P)$	$v_{t,2^{-6},2^{-11}}(P)$	$v_{t,2^{-7},2^{-15}}(P)$	$\Re H^{4th}_{\frac{\partial^2 u}{\partial x_1 \partial t}}(P)$
$\left(0.125, \frac{\sqrt{3}}{8}, 1\right)$	−0.000887304144	−0.000887477357	−0.000887488206	15.966
$\left(0.25, \frac{\sqrt{3}}{8}, 1\right)$	−0.000591460365	−0.000591646827	−0.000591658507	15.964
$\left(0.375, \frac{\sqrt{3}}{8}, 1\right)$	−0.000295709687	−0.000295822129	−0.000295829173	15.963
$\left(0.5, \frac{\sqrt{3}}{8}, 1\right)$	7.22×10^{-18}	3.33×10^{-19}	-9.86×10^{-20}	15.957
$\left(0.625, \frac{\sqrt{3}}{8}, 1\right)$	0.0002957096868	0.000295822129	0.000295829173	15.963
$\left(0.75, \frac{\sqrt{3}}{8}, 1\right)$	0.0005914603655	0.000591646827	0.000591658507	15.964
$\left(0.875, \frac{\sqrt{3}}{8}, 1\right)$	0.0008873041426	0.000887477357	0.000887488206	15.966

Table 5.6: The numerical solution $z_{t,h,\tau}$ at seven points when $t = 1$, and the convergence orders obtained by $H^{4th}(\frac{\partial^2 u}{\partial x_2 \partial t})$ for the Example 5.2.

P	$z_{t,2^{-5},2^{-7}}(P)$	$z_{t,2^{-6},2^{-11}}(P)$	$z_{t,2^{-7},2^{-15}}(P)$	$\Re H^{4th}_{\frac{\partial^2 u}{\partial x_2 \partial t}}(P)$
$\left(0.125, \frac{\sqrt{3}}{8}, 1\right)$	−0.000398417531	−0.000398520228	−0.000398526661	15.966
$\left(0.25, \frac{\sqrt{3}}{8}, 1\right)$	−0.000682992442	−0.000683176968	−0.000683188526	15.966
$\left(0.375, \frac{\sqrt{3}}{8}, 1\right)$	−0.000853734894	−0.000853970855	−0.000853985635	15.965
$\left(0.5, \frac{\sqrt{3}}{8}, 1\right)$	−0.000910648720	−0.0009109021308	−0.000910918003	15.966
$\left(0.625, \frac{\sqrt{3}}{8}, 1\right)$	−0.000853734894	−0.0008539708553	−0.000853985635	15.966
$\left(0.75, \frac{\sqrt{3}}{8}, 1\right)$	−0.000682992442	−0.000683176968	−0.000683188526	15.965
$\left(0.875, \frac{\sqrt{3}}{8}, 1\right)$	−0.000398417531	−0.000398520228	−0.000398526661	15.966

$i = 1, 2$ for time levels $t = 0.2$ and $t = 0.8$.

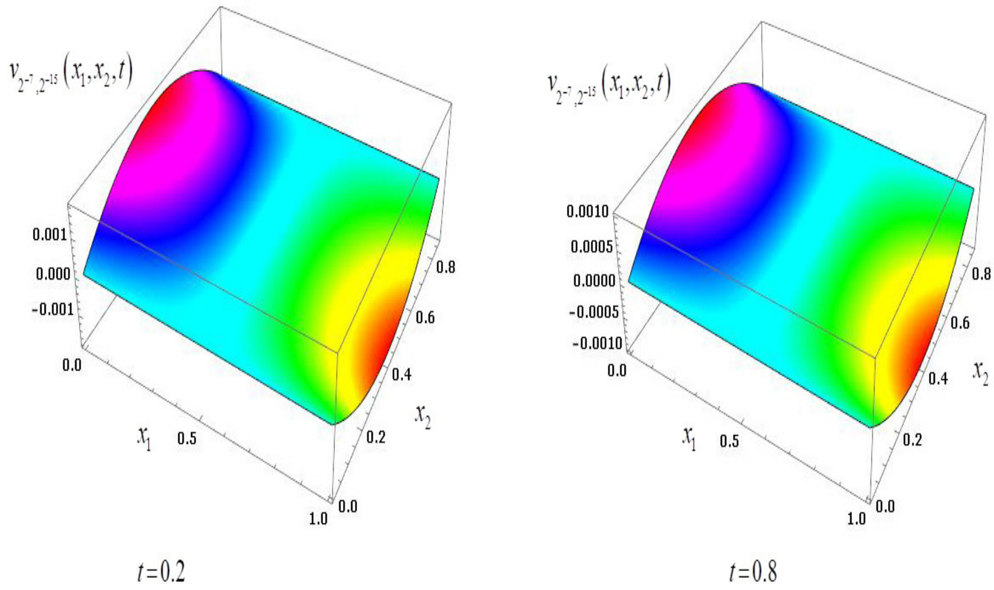


Figure 5.5: The approximate solution $v_{2^{-7},2^{-15}}$ at time levels $t = 0.2$ and $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial u}{\partial x_1}\right)$ for the Example 5.2.

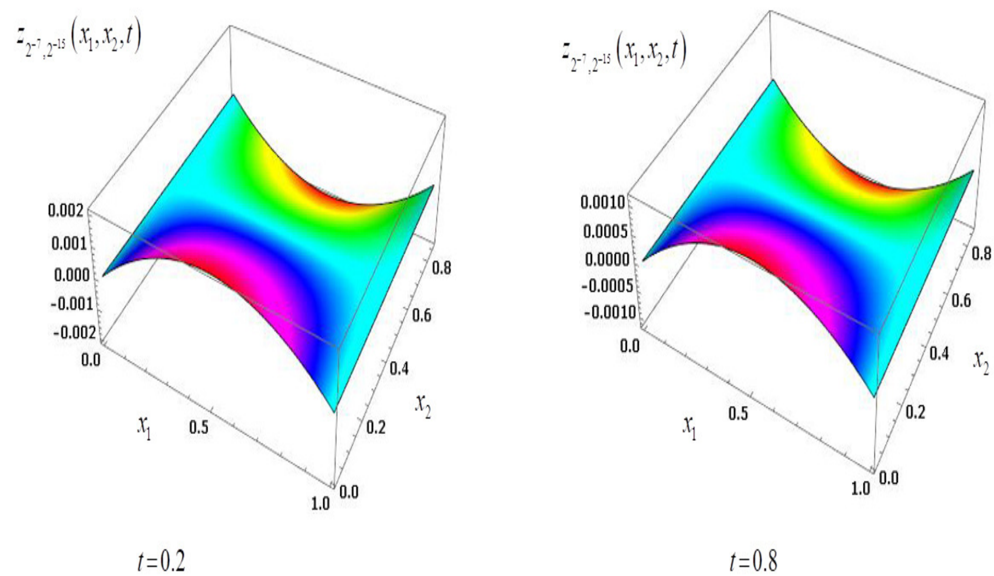


Figure 5.6: The approximate solution $z_{2^{-7},2^{-15}}$ at time levels $t = 0.2$ and $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial u}{\partial x_2}\right)$ for the Example 5.2.

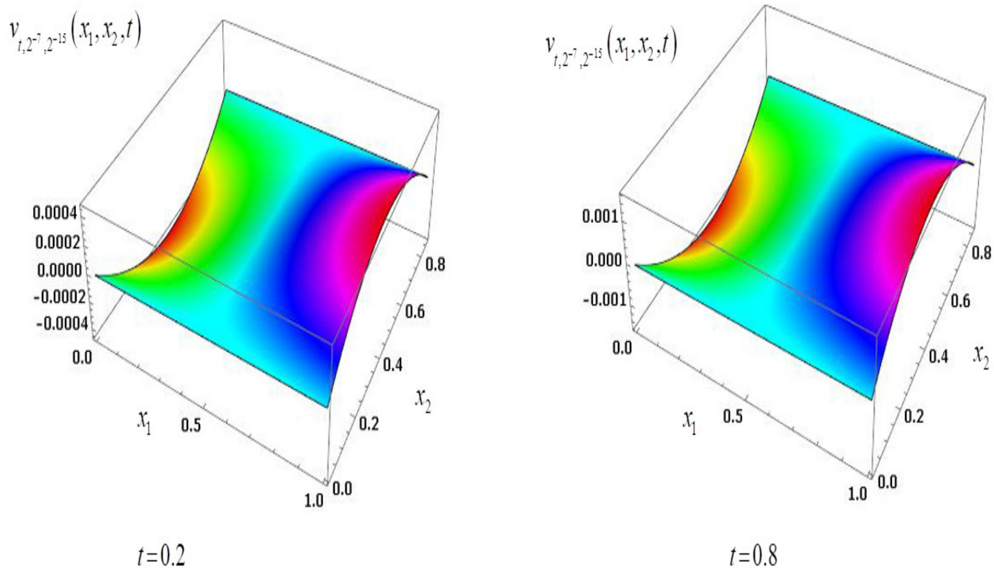


Figure 5.7: The approximate solution $v_{t,2^{-7},2^{-15}}$ at time levels $t = 0.2$ and $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial^2 u}{\partial x_1 \partial t}\right)$ for the Example 5.2.

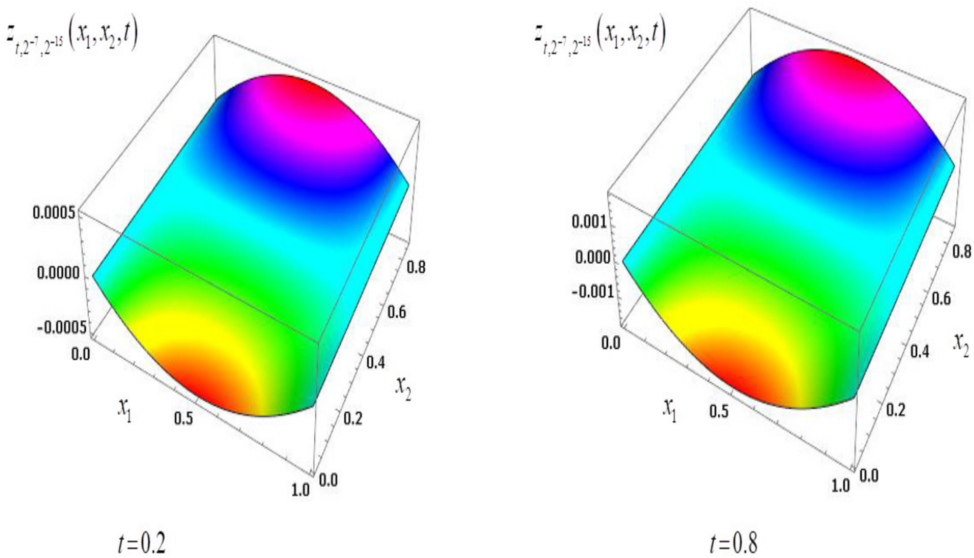


Figure 5.8: The approximate solution $z_{t,2^{-7},2^{-15}}$ at time levels $t = 0.2$ and $t = 0.8$ obtained by the method $H^{4th}\left(\frac{\partial^2 u}{\partial x_2 \partial t}\right)$ for the Example 5.2.

Chapter 6

CONCLUSION AND FINAL REMARKS

In this thesis we developed numerical methods using implicit schemes defined on hexagonal grids for computing the derivatives of the solution to Dirichlet problem of the heat equation on a rectangle. We gave highly accurate two-stage implicit methods on hexagonal grids for the approximation of the first order derivatives of the solution with respect to the spatial variables and second order mixed derivatives involving the time derivative. At the first stage, for the error function, we obtained a pointwise prior estimation depending on $\rho(x_1, x_2, t)$, which is the distance from the current grid point to the surface of Q_T. At the second stage, we constructed special difference problems for the approximation of the first order spatial derivatives with the two-stage implicit methods of second order and fourth order accuracy. In the case, when second order accurate implicit method is used uniform convergence of $O\left(h^2+\tau^2\right)$ order of accuracy to the corresponding exact derivatives $\frac{\partial u}{\partial x_i}, i=1,2$ when $r=\frac{\omega\tau}{h^2}\leq\frac{3}{7}$ is proved. When fourth order accurate implicit methods are used uniform convergence of $O\left(h^4+\tau\right)$ of the constructed difference schemes on the hexagonal grids to the respective exact derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial t}, i=1,2$ for $r=\frac{\omega\tau}{h^2}\geq\frac{1}{16}$ is shown.

Furthermore, the given two-stage implicit methods are applied on some test problems and the given theoretical order of convergence of the implicit methods are validated with the obtained numerical order of convergence and demonstrated by using tables and figures.

Remark 6.1: The approximation of the first order partial derivatives of solution of first type boundary value problem of heat equation in three space dimension is a challenging problem. The methodology given in this research may be used to construct highly accurate implicit splitting schemes (fractional step methods) and alternating direction methods (ADI) (see Peaceman and Rachford [66], Douglas [67], Bagrinovskii and Godunov [68], and Marchuk [69]).

Remark 6.2: Additionally, the numerical computation of the spatial derivatives of the solution of the time-fractional structure of the heat equation is a second interesting problem. The given approach may be extend on rectangular or triangular grids to give approximate solution of the spatial derivatives. For example the time-space fractional convection-diffusion equation, see Gu et al. [70], in which for the solution a fast iterative method with a second order implicit difference scheme was studied.

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