Numerical Solution of Volterra Integral Equations by Using Some Positive Operators

Sara Safarzadeh Falahhesar

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Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Nazım Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Mehmet Ali Özarslan
Co-Supervisor

Assoc. Prof. Dr. Suzan Cival Buranay
Supervisor

Examining Committee

1. Prof. Dr. Hüseyin Aktuğlu
2. Prof. Dr. Ayhan Aydın
3. Prof. Dr. Agamirza Bashirov
4. Prof. Dr. Ogün Doğru
5. Prof. Dr. Nazım Mahmodov
6. Prof. Dr. Mehmet Ali Özarslan
7. Assoc. Prof. Dr. Suzan Cival Buranay

ABSTRACT

The achievement of this research is bifurcated. Firstly, for the numerical solution of the first kind linear Fredholm and Volterra integral equations with smooth kernels a numerical method by using Modified Bernstein-Kantorovich operators is given. The unknown function in the first kind integral equation is approximated by using the Modified Bernstein-Kantorovich operators. Hence, by applying discretization the obtained linear equations are transformed into systems of algebraic linear equations. Due to the sensitivity of the solutions on the input data significant difficulties may be encountered, leading to instabilities in the results during actualization. Consequently, to improve on the stability of the solutions which implies the accuracy of the desired results, regularization features are built into the proposed numerical approach. More stable approximations to the solutions of the Fredholm and Volterra integral equations are obtained especially when high order approximations are used by the Modified Bernstein-Kantorovich operators. Test problems are constructed to show the computational efficiency, applicability and the accuracy of the method. Furthermore, the applicability of the proposed method on second kind Volterra integral equations with smooth kernels is also demonstrated with examples.

Secondly we give hybrid positive linear operators which are defined by using the Bernstein-Kantorovich and Modified Bernstein-Kantorovich operators on certain subintervals of $[0,1]$. Additionally, we consider second kind linear Volterra integral equations with weak singular kernels of the form $(x-t)^{-\nu}\widetilde{K}(x,t)$, $0<\nu<1$, where $\widetilde{K}$ is a smooth function. It is well known that the solution usually possess singularities at the initial point. Subsequently, we develop a combined method which uses the proposed hybrid operators and approximates the solution on the constructed subintervals. Two algorithms are developed through the given combined method and applied on some examples from the literature. Furthermore, numerical validation of the combined method is also given on first kind integral equations, by first utilizing regularization. Eventually, it is shown that the proposed combined method hence, the given computational algorithms are numerically stable and give acceptable accurate approximations to solutions with singularities.

Keywords: Volterra integral equations; Fredholm integral equations; Modified Bernstein-Kantorovich operators; Moore-Penrose inverse; Regularization; Weakly singular Volterra integral equations; Asymptotic rate of convergence; Error analysis; Numerically stable algorithm.

ÖZ

Bu araştırmanın başarısı ikiye ayrılır. İlk olarak, Modifiye Bernstein-Kantorovich operatörlerini kullanarak düz(smooth) çekirdekli birinci tür lineer Fredholm ve Volterra integral denklemlerinin çözümü için nümeriksel bir yöntem verilir. Birinci tür integral denkleminde bilinmeyen fonksiyon, Modifiye Bernstein-Kantorovich operatörlerini kullanalarak yaklaşık hesaplanır. Böylece, ayrıştrıma uygulanarak elde edilen lineer denklemler cebirsel lineer denklem sistemlerine dönüştürülür. Yöntemin nümeriksel olarak gerçekleşmesi aşamasında çözümlerin giriş verileri üzerindeki hassasiyeti elde edilen sonuçlarda kararsızlıklara yol açabilen önemli zorluklar oluşturabilir. Sonuç olarak, istenen yaklaşık çözümlerin kararlığını artırmak için ki bu nümeriksel çözümlerin doğruluğunu belirler, önerilen nümeriksel yöntemde düzenlileştirme özellikleri kullanılır. Fredholm ve Volterra integral denklemlerinin çözümlerinde, özellikle Modifiye Bernstein-Kantorovich operatörleri tarafından yüksek dereceli yaklaşımlar kullanıldığında daha kararlı yaklaşımlar elde edilir. Yöntemin hesaplama verimliliğini, uygulanabilirliğini ve doğruluğunu göstermek için test problemleri oluşturulur. Ayrıca, önerilen yöntemin düz (smooth) çekirdekli ikinci tür Volterra integral denklemleri üzerindeki uygulanabilirliği de örneklerle gösterilir.

İkinci olarak, $[0,1]$ arlığının belirli alt aralıklarında Bernstein-Kantorovich ve Modifiye Bernstein-Kantorovich operatörlerini kullanarak tanımlanmış hibrit lineer pozitif operatörler verilir. Ayrıca $\widetilde{K}$ düzgün bir fonksiyon olup $(x-t)^{-\nu}\widetilde{K}(x,t)$, $0<\nu<1$, şeklindeki zayıf singülerli çekirdeğe sahip ikinci tür lineer Volterra integral denklemleri dikkate alınır. Çözümün genellikle başlangıç noktasında singülerliğe sahip olduğu iyi bilinmektedir. Sonradan, önerilen hibrit operatörlerini kullanan ve çözümü hibrit operatörlerin tanımlandığı alt aralıklarda yaklaşık olarak hesaplayan bir birleşik yöntem oluşturulur. Verilen birleşik yöntem ile iki algoritma geliştirilir ve literatürden bazı örnekler üzerinde uygulanır. Üstelik, önce düzenlileştirme kullanarak birinci tür integral denklemler üzerinde de birleşik metodun nümeriksel doğrulaması yapılır. Sonuçta önerilen sayısal birleşik yöntemin, dolayısıyla verilen hesaplama algoritmalarının sayısal kararlı olduğu ve singülerliği olan çözümlere kabul edilebilir doğruluklu yaklaşımlar verdiği gösterilir.

Anahtar Kelimeler: Volterra integral denklemleri; Fredholm integral denklemleri; Modifiye Bernstein-Kantorovich operatörleri; Moore-Penrose ters; Düzenlileştirme; Zayıf singüler Volterra integral denklemleri; Asemptotik yakınsama hızı; Hata analizi; Sayısal olarak kararlı algoritma.

... Dedication

This thesis is dedicated to my family.

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$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
(a,b)	an open interval
$[a,b]$	a closed interval
$C(I)$	the set of all real-valued and continuous functions defined on I:=$[0,1]$
$VK1$	Volterra integral equation of the first kind
$VK2$	Volterra integral equation of the second kind,
$FK1$	Fredholm integral equation of the first kind
$FK2$	Fredholm integral equation of the second kind
$RFK1$	Regularized problem for the Fredholm integral equation of the first kind
$RVK1$	Regularized problem for the Volterra integral equation of the first kind
$VAK1$	Volterra Abel-type integral equation of the first kind
$VAK2$	Volterra Abel-type integral equation of the second kind
$RVAK1$	Regularized problem for the Volterra Abel-type integral equation of the first kind

Chapter 1

INTRODUCTION

1.1 Motivation and Review of Literature

The need of stable, reliable and time efficient methods for the numerical solution of Fredholm and Volterra integral equations of first kind with continuous and square integrable kernels is the first motivation of this research.

Fredholm and Volterra integral equations of the first kind play an important role in many problems from science and engineering. It is known that the Fredholm integral equations can be derived from boundary value problems with given boundary conditions. For example, Fredholm integral equations of the first kind arise in a mathematical model of the transport of fluorescein across the blood–retina barrier in the transient state and the subsequent diffusion of fluorescein in the vitreous body given in Larsen et al. [1]. Some other applications are in palaeoclimatology given in Anderssen and Saull [2], antenna design in Herrington [3], astrometry in Craig and Brown [4], image restoration in Andrews and Hunt [5]. The investigation of Volterra integral equations is very important in solving initial value problems of usual and fractional differential equations arising from the mathematical modelling of many scientific problems, including population dynamics, spread of epidemics, and semi-conductor devices, such as the biological fractional n-species delayed cooperation model of Lotka–Volterra type given in Tuladhar et al. [6]. Examples of Volterra integral equations of first kind can be extended to mathematical model of animal studies of the effect of the deposition of radioactive debris in the lung by

Hendry [7], the heat conduction problem in Bartoshevich [8], tautochrone problem of which Abel integral equation was derived by Abel [9], (see also Groetsch [10]), electroelastic of dynamics of a nonhomogeneous spherically isotropic piezoelectric hollow sphere problem in Ding et al. [11]. Additionally, the use of a dynamical model of Volterra integral equations in energy storage with renewable and diesel generation has been analysed in Sidorov et al. [12].

As a classical ill-posed problem, the numerical solution of Fredholm integral equations of the first kind has been investigated by many authors, such as an early study by Phillips [13] and a recent work by Neggal et al. [14]. The well-known early methods are the regularization methods given with a technique by Phillips in [13] and the Tikhonov regularization by Tikhonov in [15, 16]. In the Tikhonov method, a continuous functional is usually used and the minimizer for the corresponding functional is difficult to obtain. Consequently, several methods have been proposed to obtain an effective choice of the regularization parameter in Tikhonov method such as the discrepancy principle, the quasi-optimality criterion (see Groetsch [17], Bazan [18] and references therein). Further, in Caldwell [19], a direct quadrature method and a boundary-integral method were examined for solving Fredholm integral equations of the first kind. Additionally, a regularization technique which replaces ill-posed equations of the first kind by well-posed equations of the second kind was employed to produce meaningful results for comparison purposes. Later, the extrapolation technique by Brezinski et al. [20] and a modified Tikhonov regularization method to solve the Fredholm integral equation of the first kind under the assumption that measured data are contaminated with deterministic errors was given in Wen and Wei [21]. Recently, a variant of projected Tikhonov regularization method for solving Fredholm integral equations of the first kind was proposed in

Neggal et al. [14] in which for the subspace of projection, the Legendre polynomials were used.

Early studies for the solution of Volterra integral equations of the first kind involve the high order block by block methods in Hoog and Weiss [22, 23]. However, these methods suffer from the disadvantage of requiring additional evaluations of the kernels and the solution of systems of algebraic equations for each step. Later, Taylor [24] used inverted differentiation formulae, which the resulting methods were explicit corresponding to local differentiation formulae. As the author stated "the main disadvantage of this method is that weights must be calculated from the recurrence relation (2.9) and the differentiation formula must be chosen so that the Dahlquist root condition is satisfied". Integral equations of the first kind associated with strictly monotone Volterra integral operators were solved in Brunner [25] by projecting the exact solution of such an equation into the space $S_m^{(-1)}(Z_N)$ of piecewise polynomials of degree $m \geq 0$ possessing jump discontinuities on the set Z_N of knots. Besides, the asymptotic behavior of solutions to nonlinear Volterra integral equations was analysed in Hulbert and Reich [26]. The future-sequential regularization method and predictor-corrector regularization method for the approximation of Volterra integral problems of first kind with convolution kernel were given in Lamm [27] and Lamm [28], respectively. The numerical solution of Volterra integral equations of the first kind by sequential Tikhonov regularization coupled with several standard discretizations (collocation-based methods, rectangular quadrature, or midpoint quadrature) was given in Lamm and Eldén [29].

New approaches have been developed for the solution of integral equations that use the basis functions and transform the integral equation to the system of linear or nonlinear equations. One of these approaches is the use of wavelet basis. For the solution of the Abel integral equation, Legendre wavelets were used in Yousefi [30] and the wavelet basis were used in Maleknejad et al. [31] for the numerical solution of Volterra type integral equations of the first kind. Another approach is the use of polynomial approximations. In Mandal and Bhattacharya [32], Fredholm integral equations of the second kind and a simple hypersingular integral equation and a hypersingular integral equation of the second kind were numerically solved using Bernstein polynomials. At the same year, in Maleknejad et al. [33] numerical solution of linear and nonlinear Volterra integral equations, of the second kind by using Chebyshev polynomials was given. Afterwards, a new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation was given in Maleknejad et al. [34].

Recently, exhaustive studies on the use of CESTAC method for the solution of Volterra first type integral equations has been given in Noeiaghdam et al. [35] in which the control of accuracy on Taylor-collocation method to solve the weakly regular Volterra integral equations of the first kind has been studied. Furthermore, in Noeiaghdam et al. [36] that the numerical validation of the Adomian decomposition method for solving Volterra integral equation with discontinuous kernels was given.

In this research our second interest is the numerical solution of Volterra integral equations of second kind

$$f(x) + \int_0^x \varphi(x,t)\widehat{K}(x,t,f(t))\,dt = g(x)\,,\ x \in [0,l]\,, \tag{1.1}$$

with weak singularities. The function $\varphi(x,t)$ is the singular part of the kernel and is unbounded function in the domain of integration. Also, $g(x)$ and $\widehat{K}(x,t,f(t))$ denote the given smooth functions and $f(t)$ is the unknown function to be determined. In practical applications one very frequently encounters the linear counterpart of (1.1) in the form $\widehat{K}(x,t,f(t)) = \widetilde{K}(x,t)f(t)$. These equations arise in a number of important practical applications. Such as stated before in Bartoshevich [8] and Abel [9], heat transfer problem between solids and gasses under nonlinear boundary conditions in Mann and Wolf [37] also in Chambre [38] for nonlinear heat transfer problem and additionally in theory of superfluidity by Levinson [39]. Typical forms of $\varphi(x,t)$ such as

$$(x-t)^{-\nu},\ 0<\nu<1, \tag{1.2}$$

$$\left(x^2-t^2\right)^{-\nu},\ 0<\nu<1, \tag{1.3}$$

$$\log(x-t), \tag{1.4}$$

$$\frac{t^{\mu-1}}{x^{\mu}},\ \mu>1, \tag{1.5}$$

were considered in some works, (see Linz [40], Brunner [41] and Diago [42]) for the numerical solution of (1.1). Exhaustive studies analysing the existence and uniqueness of the solution of second kind Volterra integral equations with singular kernels under some smoothness conditions on the input functions and the differentiability properties of the solution exist in the literature, such as Miller and Feldstein [43], de Hoog and Weiss [44], Logan [45], Lubich [46], Brunner [41, 47], Brunner and van der Houwen [48] and additionally, Han [49].

Despite the practical importance of these equations only a few papers dealing with appropriate numerical methods were published till the last quarter of the twentieth century. Approaches, based in some way on the concept of product integration were given in the earliest research by Huber [50]. Later Wagner [51], and Noble [52] extended the results. Oules [53] treated the special case for which $\widehat{K}(x,t,f) = h(f)$.

A rigorous theoretical justification of the algorithms and an easy way of generalizing the results by using finite difference schemes based on product integration were investigated by Linz [40]. In the literature Volterra integral equations with kernels involving singular parts of (1.2) are also known as Volterra Abel-type integral equations. Subsequently, Atkinson [54] gave the numerical solution of an Abel integral equation by a product trapezoidal method.

Afterwards, with the use of advanced computers the numerical solution of weakly singular Volterra integral equations has gained more interest by many authors. In most practical examples, a smooth forcing function leads to a solution which has typically unbounded derivatives at the initial point. It is well-known that in this case high-order accuracy of product integration and collocation schemes is lost and convergence of order $1-v$ has been proved (see Brunner and Norset [55], Brunner and van der Houwen [48]). Again, if one is interested in finding an approximate solution which exhibits high-order accuracy, one may resort to approximation with polynomial splines on graded grids which reflect the singular behaviour of the exact solution near the initial point as given in Brunner [41], [47], Brunner and van der Houwen [48], and Brunner [56]. Also, as in Tang [57] by which the application of a class of spline collocation methods to first-order Volterra integro-differential equations (VIDEs) that contain a weakly singular kernel of the Abel-type (1.2) were given by using graded meshes.

On the other hand one may keep the uniform meshes but then use nonpolynomial spline approximating functions reflecting the singularity as given in Brunner [58], Riele [59]. It has been observed in numerical experiments given in Riele [59] that as x increases the errors appear to be of order $2-v$, who considered the case of particular practical importance $v = \frac{1}{2}$. Later on variable transformations followed by standard methods have also been considered by several authors. Galperin et al. [60], Baratella and Orsi [61], and Pedas and Vainikko [62] are examples of this kind of studies. Subsequently, Yousefi [63] gave a numerical solution of Abel integral equation by using Legendre wavelets. Most recently based on Picard iteration and a suitable quadrature formula, Micula [64] gave an iterative numerical method for the solution of linear fractional integral equations of the second kind,

$$f(x) + \frac{a(x)}{\Gamma(v)} \int_0^x b(t)(x-t)^{v-1} f(t)dt = g(x), \; 0 \leq x \leq l, \tag{1.6}$$

for $0 < v < 1$, where, $\Gamma(v) = \int_0^\infty e^{-x} x^{v-1} dx$ and a, b and $g \in [0,l] \to \mathbb{R}$ were assumed to be continuous functions.

1.2 The Achievements and the Organization of the Study

The main achievements of this research and the organization of the thesis are given as follows:

In Chapter 2, the Modified Bernstein-Kantorovich operators and asymptotic rate of convergence of these operators for $f \in C^2[0,1]$ is given.

In Chapter 3, using the Modified Bernstein-Kantorovich operators, a numerical approach is developed for the solution of Fredholm and Volterra integral equations of the first kind with continuous kernels. Furthermore, regularized integral equations are considered to obtain more smooth solutions especially when high-order approximations are used by Modified Bernstein-Kantorovich operators. The proposed approach is applied by building regularization features into the algorithm and perturbation error analysis are given.

In Chapter 4, test problems are conducted and theoretical results given in Chapter 3 are justified with obtained numerical result. The presented theoretical and numerical results in Chapters 2,3 and Chapter 4 respectively are published in Buranay et al. [65].

In Chapter 5, we consider Abel-type integral equations of the second kind and give the assumptions and smoothness results. Next, the hybrid operators are defined by using classical Bernstein-Kantorovich operators and Modified Bernstein-Kantorovich operators $K_{n,\alpha}$ where, $n \in \mathbb{N}$ and $\alpha > 0$ is constant. Further, for the numerical solution of the Abel-type integral equations of the second kind two algorithms are developed by giving a combined method for the values of $0 < \alpha < 1$ and $\alpha > 1$. Additionally, the numerical solution of first kind Volterra Abel-type integral equations are also investigated by first utilizing a regularization and then applying the given algorithms to the yielded second kind equations. Eventually, we give the convergence analysis of the constructed algorithms.

In Chapter 6, experimental investigations of the proposed combined method are provided by applying the constructed algorithms to the considered test problems of second kind linear Volterra Abel-type integral equations. Also first kind Volterra Abel-type integral equations are considered and the given algorithms are used after utilizing regularization techniques. Furthermore, it is numerically shown that the given method hence also the developed algorithms provide accurate and stable numerical approximations to the solution of the Volterra Abel-type integral equations. The obtained theoretical results in Chapter 5 and numerical results in Chapter 6 are presented at the Conference MICOPAM 2020-2021 in Buranay et al. [66, 67]. Also, this study is under review in an SCI Journal.

In Chapter 7, some concluding remarks are given.

Chapter 2

MODIFIED BERNSTEIN-KANTOROVICH OPERATORS

In this chapter the Modified Bernstein-Kantorovich operators and asymptotic rate of convergence of these operators for $f \in C^2[0,1]$ are given.

The Modified Bernstein-Kantorovich operators $K_{n,\alpha}(f;x)$ were used to approximate a function $f:[0,1] \to \mathbb{R}$ (Özarslan and Duman [68]) where,

$$K_{n,\alpha}(f;x) = \sum_{k=0}^{n} P_{n,k}(x) \int_{0}^{1} f\left(\frac{k+t^{\alpha}}{n+1}\right) dt, \tag{2.1}$$

and

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{2.2}$$

and $\alpha > 0$ is constant. For $\alpha = 1$, the equation (2.1) reduces to classical Bernstein-Kantorovich operator

$$K_n := K_{n,1}(f;x) = (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt. \tag{2.3}$$

Theorem 2.1: (Özarslan and Duman [68]) For each $\alpha > 0$ and every $f \in C[0,1]$ we have $K_{n,\alpha}(f) \rightrightarrows f$ on $[0,1]$, where the symbol $\rightrightarrows$ denotes the uniform convergence.

Lemma 2.1: (Buranay et al. [65]) For each fixed $n \in \mathbb{N}, \alpha > 0$ and $x \in [0,1]$ we have

$$\sup_{x\in[0,1]} |K_{n,\alpha}((t-x);x)| \leq \frac{\beta(\alpha)}{n+1}, \tag{2.4}$$

$$\sup_{x\in[0,1]}\left|K_{n,\alpha}\left((t-x)^2;x\right)\right|\leq\frac{1}{(n+1)^2}\left(\frac{n}{4}+\sigma(\alpha)\right), \tag{2.5}$$

where,

$$\beta(\alpha)=\begin{cases}\frac{1}{\alpha+1}\text{ , if } 0<\alpha<1\\ \frac{\alpha}{\alpha+1}\text{ , if } \alpha\geq 1\end{cases} \tag{2.6}$$

$$\sigma(\alpha)=\begin{cases}\frac{1}{2\alpha+1}\text{ , if } 0<\alpha\leq 1\\ \frac{2\alpha^2}{(\alpha+1)(2\alpha+1)}\text{ , if } \alpha\geq 1\end{cases} \tag{2.7}$$

Proof. From (2.1) it follows that

$$K_{n,\alpha}(t-x;x)=\frac{1}{n+1}\left(\frac{1}{\alpha+1}-x\right), \tag{2.8}$$

$$K_{n,\alpha}((t-x)^2;x)=\frac{1}{(n+1)^2}\left(nx(1-x)+\varphi(\alpha,x)\right), \tag{2.9}$$

where

$$\varphi(\alpha,x)=x^2-\frac{2x}{\alpha+1}+\frac{1}{2\alpha+1}. \tag{2.10}$$

For each fixed $n\in\mathbb{N},\alpha>0$ the inequality (2.4) is obtained from $\beta(\alpha)=\max_{x\in[0,1]}\left|\frac{1}{\alpha+1}-x\right|$. The function $\varphi(\alpha,x)>0$ for $\alpha>0$ on $x\in[0,1]$. Further, $\min_{x\in[0,1]}\varphi(\alpha,x)=\frac{\alpha^2}{(\alpha+1)^2(2\alpha+1)}$ occurring at $x=\frac{1}{\alpha+1}$ and $\max_{x\in[0,1]}\varphi(\alpha,x)=\sigma(\alpha)$ occurring at the end points of the interval $[0,1]$. Also using that $\max_{x\in[0,1]}|x(1-x)|=\frac{1}{4}$ yields (2.5). □

Next, we use the notations $\|q\|=\sup_{x\in[0,1]}|q|$ and $\|q\|_2=\left(\int_0^1|q(x)|^2dx\right)^{\frac{1}{2}}$ to present the maximum norm for $q\in C[0,1]$ and L^2-norm of the function $q\in L^2[0,1]$. Further we denote $\|Y\|_2=\sqrt{\left(\sum_{k=1}^{n}(Y(k))^2\right)}$ and $\|P\|_2=\sqrt{\rho(P^TP)}$ to present the discrete Euclidean norm of a vector $Y\in R^n$ and the spectral norm of a matrix $P\in R^{n\times n}$ respectively, where ρ is the spectral radius and P^T is the transpose of P.

Voronowskaja [69] gave the asymptotic rate of convergence of the Bernstein operators

$$B_n(f;x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n}\right), \tag{2.11}$$

using the linearity property of the Bernstein operators and Taylor formula at a point x as

$$\lim_{n\to\infty} n\left[(B_n(f;x)) - f(x))\right] = \frac{1}{2}x(1-x)f''(x). \tag{2.12}$$

Based on the analogous approach in Voronowskaja [69] we give the asymptotic rate of convergence of the Modified Bernstein-Kantorovich operators by the next theorem.

Theorem 2.2: (Buranay et al. [65]) If f is integrable in $[0,1]$, and admits a derivative of second order at some point $x \in [0,1]$ then

$$\lim_{n\to\infty} n\left[K_{n,\alpha}(f;x) - f(x)\right] = \left(\frac{1}{\alpha+1} - x\right) f'(x) + \frac{1}{2}x(1-x)f''(x). \tag{2.13}$$

Additionally, this limit is uniform if $f \in C^2[0,1]$, thus the rate of convergence of the operator $K_{n,\alpha}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n}\right)$ for $x \in [0,1]$.

Proof. Assume that f is integrable in $[0,1]$, and has second order derivative at a point $x \in [0,1]$ then from Taylor's formula at x we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2}f''(x) + (t-x)^2 E(t-x), \tag{2.14}$$

and $E(u) \to 0$ as $u \to 0$ and E is integrable function on $[-x, 1-x]$. Using the linearity property of the operators $K_{n,\alpha}$ and (2.8), (2.9) and (2.10) we have

$$\begin{aligned} K_{n,\alpha}(f;x) - f(x) &= \frac{1}{n+1}\left(\frac{1}{\alpha+1} - x\right) f'(x) \\ &+ \frac{1}{2(n+1)^2}\left(x^2 - \frac{2x}{\alpha+1} + \frac{1}{2\alpha+1} + nx(1-x)\right) f''(x) \\ &+ E(n,\alpha,x), \end{aligned} \tag{2.15}$$

where,

$$E(n,\alpha,x)=\sum_{k=0}^{n}P_{n,k}(x)\int_{0}^{1}\left(\frac{k+t^{\alpha}}{n+1}-x\right)^{2}E\left(\frac{k+t^{\alpha}}{n+1}-x\right)dt. \tag{2.16}$$

To show that the asymptotic rate of convergence is $O\left(\frac{1}{n}\right)$ it is sufficient to show that $\lim_{n\to\infty} nE(n,\alpha,x)=0$. Let $M_1=\sup_{u\in[-x,1-x]}|E(u)|$ and for arbitrary $\varepsilon>0$ there exist $\delta_1>0$ such that $|E(u)|<\varepsilon$ whenever $|u|<\delta_1$. For all $t\in[0,1]$ it follows that $\left|E\left(\frac{k+t^{\alpha}}{n+1}-x\right)\right|<\varepsilon+M_1\left(\frac{k+t^{\alpha}}{n+1}-x\right)^{2}/\delta_1^2$. Then let

$$\gamma_p(\alpha)=\prod_{k=1}^{p}(1+k\alpha),\ p=1,2,3,4. \tag{2.17}$$

Using Lemma 2.1 estimation (2.5) gives

$$\begin{aligned}|E(n,\alpha,x)| &\leq \varepsilon\left|K_{n,\alpha}\left((t-x)^2;x\right)\right|+\frac{M_1}{\delta_1^2}\left|K_{n,\alpha}\left((t-x)^4;x\right)\right|\\ &\leq \frac{\varepsilon}{(n+1)^2}\left(\frac{n}{4}+\sigma(\alpha)\right)+\frac{M_1\widetilde{M}(n,\alpha)}{\delta_1^2(n+1)^4\gamma_4(\alpha)},\end{aligned} \tag{2.18}$$

where, $\sigma(\alpha)$ is as given in (2.7) and $\widetilde{M}(n,\alpha)=\sup_{x\in[0,1]}|Q(n,\alpha,x)|$. In addition for a fixed α, $\widetilde{M}(n,\alpha)$ is second degree polynomial in n and $Q(n,\alpha,x)$ is

$$\begin{aligned}Q(\alpha,n,x) &= 1+6\alpha+11\alpha^2+6\alpha^3\\ &\quad+(1+4\alpha)(-4\gamma_2(\alpha)+(1+3\alpha)(11+\alpha(17+2\alpha))n)x\\ &\quad+\frac{\gamma_4(\alpha)}{\gamma_2(\alpha)}(6(1+\alpha)-(41+\alpha(87+22\alpha))n+3\gamma_2(\alpha)n^2)x^2\\ &\quad-2\frac{\gamma_4(\alpha)}{\gamma_1(\alpha)}(2+n(-25+3\alpha(-5+n)+3n))x^3\\ &\quad+\gamma_4(\alpha)(1+n(-20+3n))x^4.\end{aligned} \tag{2.19}$$

It is obvious from (2.18) and (2.19) that for n large enough we have $|nE(\alpha,n,x)|<\varepsilon$ and using (2.15) we obtain (2.13). If $f\in C^2[0,1]$ then this limit is uniform, thus the rate of convergence of the operator $K_{n,\alpha}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n}\right)$ for $x\in[0,1]$. □

Corollary 2.1: (Buranay et al. [65]) If $f\in\left(C^{\lambda}\cap L^2\right)([0,1])$ for $\lambda\geq 2$ then

$$|K_{n,\alpha}(f;x)-f(x)| \leq \frac{\|f'\|}{n+1}\left|\frac{1}{\alpha+1}-x\right| + \frac{1}{2}\frac{\|f''\|}{(n+1)^2}\left(nx(1-x)+\varphi(\alpha,x)\right), \tag{2.20}$$

$$\sup_{x\in[0,1]}|K_{n,\alpha}(f;x)-f(x)| \leq \frac{\|f'\|}{n+1}\beta(\alpha)+\frac{\|f''\|}{2(n+1)^2}\left(\frac{n}{4}+\sigma(\alpha)\right), \tag{2.21}$$

$$\begin{aligned}\|K_{n,\alpha}(f;x)-f\|_2 &\leq \left\|\frac{\|f'\|}{n+1}\left(\frac{1}{\alpha+1}-x\right)\right\|_2 + \left\|\frac{\|f''\|}{2(n+1)^2}\left(nx(1-x)+\varphi(\alpha,x)\right)\right\|_2, \\ &= \frac{\|f'\|}{n+1}\widetilde{\beta}(\alpha)+\frac{\|f''\|}{2(n+1)^2}\widetilde{\sigma}(n,\alpha),\end{aligned} \tag{2.22}$$

hold true where, $\varphi(\alpha,x)$ is the given function in (2.10)

$$\widetilde{\beta}(\alpha)=\frac{\sqrt{1-\alpha+\alpha^2}}{(1+\alpha)\sqrt{3}}, \tag{2.23}$$

$$\widetilde{\sigma}(n,\alpha)=\frac{\sqrt{\frac{\phi_1(\alpha)}{(\gamma_2(\alpha))^2}+\frac{\phi_2(n,\alpha)}{\gamma_2(\alpha)}+n^2}}{\sqrt{30}}, \tag{2.24}$$

$$\phi_1(\alpha)=6+2\alpha\left(3+4\alpha\left(1+\alpha(-1+3\alpha)\right)\right), \tag{2.25}$$

$$\phi_2(n,\alpha)=\left(3+\alpha(-1+6\alpha)\right)n, \tag{2.26}$$

and $\beta(\alpha), \sigma(\alpha)$ are as given in (2.6) and (2.7) respectively and $\gamma_2(\alpha)$ is the same as in (2.17).

Proof. The inequality (2.20) is the consequence of the Theorem 2.2. The proof of (2.21) is obtained by using (2.20), Lemma 2.1 and estimations (2.4), (2.5). For $\alpha>0$ and $n\in\mathbb{N}$ the proof of (2.22) follows from the integral values

$$\left(\int_0^1\left|\frac{1}{\alpha+1}-x\right|^2dx\right)^{\frac{1}{2}}=\widetilde{\beta}(\alpha),$$

$$\left(\int_0^1|nx(1-x)+\varphi(\alpha,x)|^2dx\right)^{\frac{1}{2}}=\widetilde{\sigma}(n,\alpha),$$

given in (2.23), (2.24) respectively. □

Chapter 3

NUMERICAL SOLUTION OF THE FREDHOLM AND VOLTERRA INTEGRAL EQUATIONS BY USING MODIFIED BERNSTEIN-KANTOROVICH OPERATORS

In this chapter, by using the Modified Bernstein–Kantorovich operators, a numerical approach is developed for the solution of Fredholm and Volterra integral equations of the first kind with continuous kernels. Furthermore, regularized integral equations are considered to obtain more smooth solutions especially when high-order approximations are used by Modified Bernstein-Kantorovich operators. The proposed approach is applied by building regularization features into the algorithm and perturbation error analysis are given. The result presented in this chapter are published in the research article in Buranay et al. [65].

3.1 Representation of the $K_{n,\alpha}$ Operators and Discretization of First Kind Integral Equations

We consider the Fredholm integral equation of the first kind (**FK1**)

$$Tf = \int_0^1 K(x,t)\, f(t)\, dt = g(x),\ 0 \le x \le 1, \tag{3.1}$$

and Volterra integral equations of the first kind (**VK1**)

$$\widehat{T}f = \int_0^x K(x,t)\, f(t)\, dt = g(x),\ 0 \le x \le 1, \tag{3.2}$$

Definition 3.1: (Groetsch [17], Hansen [70]) By means of the singular value expansion (SVE) any square integrable kernel $K(x,t)$ can be written in the form

$$K(x,t) = \sum_{i=0}^{\infty} \mu_i u_i(x) v_i(t). \tag{3.3}$$

The functions u_i, v_i are the singular functions of K and they are orthonormal with respect to the usual inner product $(.,.)$ and the number μ_i are the singular values of K. For degenerate kernels the infinite sum (3.3) is replaced with the finite sum upto the rank of the kernel. The system $\{u_i, v_i; \mu_i\}$ is called the singular system of K.

Let $\Psi : H_1 \to H_2$ be a compact linear operator on a real Hilbert space H_1, taking values in a real Hilbert space H_2. The next theorem is known as the Picard's theorem on the existence of the solutions of first kind equations.

Theorem 3.1: (Groetsch [17]) Let $\Psi : H_1 \to H_2$ be a compact linear operator with singular system $\{u_i, v_i; \mu_i\}$. In order that the equation $\Psi f = g$ have a solution it is necessary and sufficient that $g \in N(\Psi^*)^{\perp}$ (orthogonal complement of the nullspace of the adjoint of Ψ) and

$$\sum_{i=0}^{\infty} \mu_i^2 |(g, v_i)|^2 < \infty. \tag{3.4}$$

On the basis of Theorem 3.1 we consider the following hypothesis:

Hypothesis A:

1. The kernel $K(x,t)$ is continuous hence square integrable function on $[0,1] \times [0,1]$.
2. $g \in C[0,1]$ and for **FK1** $g \in N(T^*)^{\perp}$ and for **VK1** $g \in N\left(\widehat{T}^*\right)^{\perp}$, also the Picard's condition (3.4) is satisfied.

Without loss of generality the solution f of **FK1** and **VK1** denotes the pseudoinverse solution or the Moore-Penrose generalized inverse solution for **FK1** and **VK1**

$$f = T^{\dagger} g \text{ and } f = \widehat{T}^{\dagger} g, \tag{3.5}$$

respectively. Further, in order to determine the effect of $\alpha > 0$ in the numerical solution we represent the Modified Bernstein-Kantorovich operators (2.1) for $0 < \mu < 1$ in the form

$$\begin{aligned} K_{n,\alpha}(f;x) &= \sum_{k=0}^{n} P_{n,k}(x) \left(\int_0^{\mu} f\left(\frac{k+t^{\alpha}}{n+1}\right) dt + \int_{\mu}^{1} f\left(\frac{k+t^{\alpha}}{n+1}\right) dt \right) \\ &= \omega(n,\alpha) \sum_{k=0}^{n} P_{n,k}(x) \left(\frac{1}{\omega(n,\alpha)} \int_0^{\mu} f\left(\frac{k+t^{\alpha}}{n+1}\right) dt + \int_{\frac{k+\mu^{\alpha}}{n+1}}^{\frac{k+1}{n+1}} q(u)\, du \right), \end{aligned} \tag{3.6}$$

where

$$q(u) = \begin{cases} f(u)\left((n+1)u - k\right)^{\frac{1-\alpha}{\alpha}} \text{ if } \alpha \neq 1, \\ f(u) \text{ if } \alpha = 1, \end{cases} \tag{3.7}$$

$$\omega(n,\alpha) = \frac{(n+1)}{\alpha}. \tag{3.8}$$

For the numerical solution of **FK1** and **VK1** we approximate the function f by using the Modified Bernstein-Kantorovich operators in (3.6). We obtain the following equation for **FK1**

$$\omega(n,\alpha) \int_0^1 K(x,t) \sum_{k=0}^{n} P_{n,k}(t) \left(\frac{1}{\omega(n,\alpha)} \int_0^{\mu} f\left(\frac{k+t^{\alpha}}{n+1}\right) dt + \int_{\frac{k+\mu^{\alpha}}{n+1}}^{\frac{k+1}{n+1}} q(u)\, du \right) dt = g(x), \tag{3.9}$$

and for **VK1** we get

$$\omega(n,\alpha) \int_0^x K(x,t) \sum_{k=0}^{n} P_{n,k}(t) \left(\frac{1}{\omega(n,\alpha)} \int_0^{\mu} f\left(\frac{k+t^{\alpha}}{n+1}\right) dt + \int_{\frac{k+\mu^{\alpha}}{n+1}}^{\frac{k+1}{n+1}} q(u)\, du \right) dt = g(x). \tag{3.10}$$

Subsequently we take the grid points $x_j = \frac{j}{n} + \varepsilon,\ j = 0, 1, ..., n-1$ and $x_n = 1 - \varepsilon$, where $0 < \varepsilon < \frac{1}{2n}$. Then, the equations (3.9), (3.10) are transformed into algebraic systems of equations

$$AX = B, \text{ and } \widehat{A}X = B, \tag{3.11}$$

respectively, where the coefficient matrices A and $\widehat{A}$ have the entries

$$[A]_{j+1,k+1} = \omega(n,\alpha)\,[A_*]_{j+1,k+1} = \omega(n,\alpha)\int_0^1 K(x_j,t)\,P_{n,k}(t)dt, \tag{3.12}$$

$$\left[\widehat{A}\right]_{j+1,k+1} = \omega(n,\alpha)\left[\widehat{A}_*\right]_{j+1,k+1} = \omega(n,\alpha)\int_0^{x_j} K(x_j,t)\,P_{n,k}(t)dt, \tag{3.13}$$

$j = 0,1,...,n, k = 0,1,...,n$, and

$$X(k+1) = \frac{1}{\omega(n,\alpha)}\int_0^{\mu} f\left(\frac{k+t^{\alpha}}{n+1}\right)dt + \int_{\frac{k+\mu^{\alpha}}{n+1}}^{\frac{k+1}{n+1}} q(u)\,du,\ k = 0,1,...,n, \tag{3.14}$$

$$B(j+1) = g(x_j),\ j = 0,1,...,n. \tag{3.15}$$

$q(u)$ and $\omega(n,\alpha)$ are as given in (3.7) and (3.8) respectively. The coefficient matrices A and $\widehat{A}$ in (3.11) are ill-conditioned matrices and may be rank deficient or even singular matrices. Therefore, we consider the following minimum norm least squares problem for **FK1**

$$\min_{X\in S_1}\|X\|_2, \quad S_1 = \left\{X \in R^{n+1} \mid \|B - AX\|_2 = \min\right\}, \tag{3.16}$$

and for **VK1**

$$\min_{X\in S_2}\|X\|_2, \quad S_2 = \left\{X \in R^{n+1} \mid \left\|B - \widehat{A}X\right\|_2 = \min\right\}. \tag{3.17}$$

Lemma 3.1: (Buranay et al. [65]) The problems (3.16) and (3.17) have the unique minimum norm least squares solutions $X = A^{\dagger}B$ and $X = \widehat{A}^{\dagger}B$ respectively.

Proof. Proof is analogous to the proof of Theorem 1.2.10 in Björck [71]. □

By solving the algebraic systems (3.16) and (3.17) we get a numerical solution of the unknown X in (3.14) and denote this approximation by X_n. Further, let us use F_n to denote the obtained numerical approximation to f that is in the implicit form in X_n and obtained by using the proposed approach. Substituting F_n in (3.6) we get $K_{n,\alpha}(F_n;x)$ as

$$K_{n,\alpha}(F_n;x) = \omega(n,\alpha)\sum_{k=0}^{n} P_{n,k}(x)X_n(k+1), \tag{3.18}$$

Definition 3.2: (Björck [71]) The condition number of $U \in R^{m\times n}$ $(U \neq 0)$ is

$$\kappa(U) = \left\|U^{\dagger}\right\|_2 \|U\|_2 = \frac{\sigma_1}{\sigma_\tau},$$

where $\tau = rank(U) \leq \min(m,n)$, and $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_\tau > 0$ are the nonzero singular values of U.

Theorem 3.2: (Buranay et al. [65]) Consider **FK1** and **VK1** in (3.1), (3.2) respectively and assume that the conditions of the Hypothesis A are satisfied also the solution f belongs to $\left(C^{\lambda}\cap L^2\right)([0,1])$ for some $\lambda \geq 2$ then for **FK1**

$$\|K_{n,\alpha}(F_n) - f\|_2 \leq W_1(n,\alpha,f) + M_2W_2(n,\alpha,f)\frac{\kappa(A_*)}{\|A_*\|_2}, \tag{3.19}$$

and for **VK1**

$$\|K_{n,\alpha}(F_n) - f\|_2 \leq W_1(n,\alpha,f) + M_2W_2(n,\alpha,f)\frac{\kappa\left(\widehat{A}_*\right)}{\left\|\widehat{A}_*\right\|_2}, \tag{3.20}$$

hold true where,

$$W_1(n,\alpha,f) = \frac{\|f'\|}{n+1}\widetilde{\beta}(\alpha) + \frac{\|f''\|}{2(n+1)^2}\widetilde{\sigma}(n,\alpha), \tag{3.21}$$

$$W_2(n,\alpha,f) = \frac{\|f'\|}{n+1}\beta(\alpha) + \frac{\|f''\|}{2(n+1)^2}\left(\frac{n}{4} + \sigma(\alpha)\right). \tag{3.22}$$

and $\beta(\alpha)$, $\sigma(\alpha), \widetilde{\beta}(\alpha)$ and $\widetilde{\sigma}(n,\alpha)$ are given in (2.6), (2.7), (2.23) and (2.24) respectively. Furthermore, $M_2 = \|S\|_2$ where $S(j+1) = \sup_{t\in[0,1]}\left|K\left(x_j,t\right)\right|$, $x_j = \frac{j}{n} + \varepsilon$, $j = 0,1,...,n-1$ and $x_n = 1-\varepsilon$, and $0 < \varepsilon < \frac{1}{2n}$. Further, $K_{n,\alpha}(F_n;x)$ is the approximate solution obtained by the proposed method and A_* and $\widehat{A}_*$ are given in (3.12) and (3.13) respectively.

Proof. For **FK1** it follows that

$$\|K_{n,\alpha}(F_n) - f\|_2 \leq \|K_{n,\alpha}(f) - f\|_2 + \|K_{n,\alpha}(F_n) - K_{n,\alpha}(f)\|_2. \tag{3.23}$$

Based on Corollary 2.1 and the estimation (2.22) and taking (3.21) we obtain

$$\|K_{n,\alpha}(f) - f\|_2 \leq W_1(n,\alpha,f). \tag{3.24}$$

Next let $\overline{X}(k+1) = X_n(k+1) - X(k+1)$ for $k = 0,1,...,n$ from (3.6) and (3.18) and using that $\sum\limits_{k=0}^{n} P_{n,k}(x) = 1$ gives

$$\begin{aligned}\|K_{n,\alpha}(F_n) - K_{n,\alpha}(f)\|_2 &= \left(\int_0^1 \left|\omega(n,\alpha)\sum_{k=0}^{n} P_{n,k}(x)\overline{X}(k+1)\right|^2 dx\right)^{\frac{1}{2}} \\ &\leq \omega(n,\alpha)\left(\sum_{k=1}^{n+1}\left(\overline{X}(k)\right)^2\right)^{\frac{1}{2}}\left(\int_0^1\left|\sum_{k=0}^{n} P_{n,k}(x)\right|^2 dx\right)^{\frac{1}{2}}.\end{aligned} \tag{3.25}$$

It follows that

$$\|K_{n,\alpha}(F_n) - K_{n,\alpha}(f)\|_2 \leq \omega(n,\alpha)\left\|\overline{X}\right\|_2. \tag{3.26}$$

From Theorem 2.1, the operator $K_{n,\alpha}(f;x)$ uniformly converges to f for any $f \in C[0,1]$ and for any computationally acceptable small $\varepsilon > 0$,

$$|K_{n,\alpha}(f;x) - f(x)| < \varepsilon + \frac{2\|f\|}{\delta_1^2}K_{n,\alpha}((t-x)^2;x),$$

where, as usual, δ_1 comes from the uniform continuity of the function $f \in [0,1]$ and $K_{n,\alpha}((t-x)^2;x)$ is given in (2.9) (see Özarslan and Duman [68]). Therefore, for the numerical solution of **FK1** and **VK1** equations in (3.1), and (3.2) in accordance we assume

$$\int_0^1 K(x,t)K_{n,\alpha}(f;t)\,dt = g(x),\ 0 \leq x \leq 1, \tag{3.27}$$

$$\int_0^x K(x,t)K_{n,\alpha}(f;t)\,dt = g(x),\ 0 \leq x \leq 1, \tag{3.28}$$

respectively. If we substitute $F_n(x)$ instead of $f(x)$ in (3.27), (3.28) we get new function $\widehat{g}(x)$ on the right sides of these equations accordingly,

$$\int_0^1 K(x,t)\, K_{n,\alpha}(F_n;t)\, dt = \widehat{g}(x),\ 0 \le x \le 1, \tag{3.29}$$

$$\int_0^x K(x,t)\, K_{n,\alpha}(F_n;t)\, dt = \widehat{g}(x),\ 0 \le x \le 1. \tag{3.30}$$

Thus, for **FK1** using (3.27) and (3.29) and by taking the grid points $x_j = \frac{j}{n} + \varepsilon,\ j = 0,1,...,n-1$ and $x_n = 1-\varepsilon$, where $0 < \varepsilon < \frac{1}{2n}$ we obtain the algebraic system

$$A\overline{X} = \overline{B},\ \overline{B}(j+1) = \widehat{g}(x_j) - g(x_j),\ j = 0,1,...,n. \tag{3.31}$$

The minimum norm solution of the least squares problem for (3.31) is

$$\overline{X} = A^{\dagger}\overline{B}. \tag{3.32}$$

Thus

$$\omega(n,\alpha)\left\|\overline{X}\right\|_2 \le \omega(n,\alpha)\left\|A^{\dagger}\right\|_2 \left\|\overline{B}\right\|_2 = \left\|A_*^{\dagger}\right\|_2 \left\|\overline{B}\right\|_2, \tag{3.33}$$

and for **VK1**

$$\omega(n,\alpha)\left\|\overline{X}\right\|_2 \le \omega(n,\alpha)\left\|\widehat{A}^{\dagger}\right\|_2 \left\|\overline{B}\right\|_2 = \left\|\widehat{A}_*^{\dagger}\right\|_2 \left\|\overline{B}\right\|_2. \tag{3.34}$$

Next consider **FK1** and let $\widehat{g}(x) = \int_0^1 K(x,t)\, K_{n,\alpha}(f;t)\, dt$ and $g(x) = \int_0^1 K(x,t)\, f(t)\, dt$, then it follows that

$$\widehat{g}(x) - g(x) = \int_0^1 K(x,t)\left(K_{n,\alpha}(f;t) - f(t)\right) dt, \tag{3.35}$$

then using Corollary 2.1 and estimation (2.21) and (3.31) and (3.35) and taking $S(j+1) = \sup_{t\in[0,1]} \left|K(x_j,t)\right|$ for $j = 0,1,...,n$ and $M_2 = \|S\|_2$ and using (3.22) we get

$$\begin{aligned}
\left(\sum_{j=0}^{n} \left|\left(\widehat{g}(x_j) - g(x_j)\right)\right|^2\right)^{\frac{1}{2}} &= \left(\sum_{j=0}^{n}\left|\int_0^1 K(x_j,t)\left(K_{n,\alpha}(f;t) - f(t)\right) dt\right|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{j=0}^{n} (S(j+1))^2\right)^{\frac{1}{2}} \sup_{t\in[0,1]} \left|K_{n,\alpha}(f;t) - f(t)\right| \\
&\le M_2 W_2(n,\alpha,f).
\end{aligned} \tag{3.36}$$

Substituting (3.36) into (3.33) and the obtained result in (3.26) gives

$$\left\|K_{n,\alpha}(F_n)-K_{n,\alpha}(f)\right\|_2 \le M_2 W_2(n,\alpha,f)\left\|A_*^{\dagger}\right\|_2. \tag{3.37}$$

Further, using the estimations (3.24) and (3.37) in (3.23) and also from $\kappa(A_*)=\left\|A_*^{\dagger}\right\|_2\|A_*\|_2$ we get (3.19). Analogously, for **VK1** it follows that

$$\widehat{g}(x)-g(x)=\int_0^x K(x,t)\left(K_{n,\alpha}(f;t)-f(t)\right)dt. \tag{3.38}$$

Using Corollary 2.1 and estimation (2.21) and taking (3.22) we obtain

$$\begin{aligned}\left(\sum_{j=0}^{n}\left|\left(\widehat{g}(x_j)-g(x_j)\right)\right|^2\right)^{\frac{1}{2}} &= \left(\sum_{j=0}^{n}\left|\int_0^{x_j} K(x_j,t)\left(K_{n,\alpha}(f;t)-f(t)\right)dt\right|^2\right)^{\frac{1}{2}}\\ &\le \left(\sum_{j=0}^{n}(S(j+1))^2\right)^{\frac{1}{2}}\sup_{t\in[0,1]}\left|K_{n,\alpha}(f;t)-f(t)\right|\\ &\le M_2 W_2(n,\alpha,f).\end{aligned} \tag{3.39}$$

Next substituting (3.39) in (3.34) and the obtained result in (3.26) we get

$$\left\|K_{n,\alpha}(F_n)-K_{n,\alpha}(f)\right\|_2 \le M_2 W_2(n,\alpha,f)\left\|\widehat{A}_*^{\dagger}\right\|_2. \tag{3.40}$$

Therefore using the estimations (3.24) and (3.40) in (3.23) follows (3.20). □

Remark 3.1: If the matrix A in (3.12) and the matrix $\widehat{A}$ in (3.13) are invertible then $A^{\dagger}=A^{-1}$ and $\widehat{A}^{\dagger}=\widehat{A}^{-1}$and the inequalities (3.19) and (3.20) hold true.

3.2 Regularized Numerical Solution

The numerical solution of the general least squares problems (3.16) and (3.17) may be extremely difficult because the solution is very sensitive to the perturbations of the coefficient matrices A and $\widehat{A}$ and the right side vector B. This is reflected in the fact that $\kappa(A)$, and $\kappa(\widehat{A})$ are very large and increases as n increases which is the degree of the constructed polynomial by the Modified Bernstein- Kantorovich operator used for the approximation of the solution. High condition numbers of the matrices A and $\widehat{A}$ cause rounding errors that prevent the computation of an accurate numerical solution of the problems (3.16) and (3.17) respectively. Moreover, the obtained discrete problems are always perturbed by approximations such as the integrals given as the entries of A and $\widehat{A}$ are evaluated numerically. Therefore, even if we were able to solve the discrete algebraic problems (3.16) and (3.17) without rounding errors we would not obtain a "smooth" solution because of the oscillations in the singular vectors. By a smooth solution we mean "a solution which has some useful properties in common with the exact solution to the underlying and unknown unperturbed problem"as stated in Hansel [72]. Furthermore, the function g is typically a measured or observed quantity and hence in practice the true g is not available to us (see Tikhonov [15, 16] and Groetsch [17]).

Consider the first kind integral equation

$$\Lambda f = g. \tag{3.41}$$

In many practical problems one needs to solve an approximate equation

$$\widetilde{\Lambda} f = \widetilde{g}, \tag{3.42}$$

instead of the exact equation (3.41). The operator $\widetilde{\Lambda}$ is approximation to Λ and the function $\widetilde{g}$ is approximation to g satisfying

$$\left\| \widetilde{\Lambda} - \Lambda \right\| \leq \delta_1, \|\widetilde{g} - g\| \leq \delta_2,\ \delta = \max\{\delta_1, \delta_2\},$$

where $\delta > 0$ is called the noise level. Therefore we consider the following regularized problems for the Fredholm integral equation of the first kind (**RFK1**) (see Tikhonov [15] and [16] and Groetsch [17])

$$\int_0^1 K(x,t)\, f_\eta^\delta(t)\, dt + \eta(\delta)\, f_\eta^\delta(x) = g_\delta(x),\ 0 \leq x \leq 1, \tag{3.43}$$

and Volterra integral equations of the first kind (**RVK1**)

$$\int_0^x K(x,t)\, f_\eta^\delta(t)\, dt + \eta(\delta)\, f_\eta^\delta(x) = g_\delta(x),\ 0 \le x \le 1. \tag{3.44}$$

where, $\eta(\delta)$ is positive regularization parameter coordinated with δ. The regularized problems (3.43) and (3.44) can be presented in the operator equation as

$$\left(\Lambda + \eta(\delta)\widetilde{E}\right) f_\eta^\delta = g_\delta, \tag{3.45}$$

and $\widetilde{E}$ is the unity operator.

Hypothesis B: The following are assumed to be fulfilled:

a.

$$\left\|\left(\Lambda + \eta(\delta)\widetilde{E}\right)^{-1}\right\| \le \varkappa(\eta(\delta)), \tag{3.46}$$

where, $\varkappa(\eta)$ is a continuous function.

b. For $\varsigma \in (0,1)$ the noise level δ is selected such that for $\delta < \delta_0$ the inequality $\delta\varkappa(\eta(\delta)) \le \varsigma$ holds true.

c.

$$\lim_{\delta\to 0} \varkappa(\eta(\delta)) = \infty,\ \lim_{\delta\to 0} \eta(\delta) = 0,\ \lim_{\delta\to 0} \delta\varkappa(\eta(\delta)) = 0 \tag{3.47}$$

For the convergence of the regularized solutions of (3.43) and (3.44) see Tikhonov [15] and [16] and Groetsch [17]), Muftahov et al. [73] and the references therein. It is clear that (3.43) and (3.44) are second kind Fredholm and Volterra integral equations respectively. For the numerical solution of **RFK1** and **RVK1** by the proposed method $M(K_{n,\alpha})$ we take the grid points $x_j = \frac{j}{n} + \varepsilon,\ j = 0,1,...,n-1$ and $x_n = 1 - \varepsilon$, where $0 < \varepsilon < \frac{1}{2n}$ and is sufficiently small number also $\eta(\delta) > 0$ is called the regularization parameter. We assume the following algebraic equations for **RFK1**

$$\int_0^1 K(x_j,t) K_{n,\alpha}(f_\eta^\delta;t)dt + \omega(n,\alpha)\eta(\delta) X_\eta^\delta(j+1) = g_\delta(x_j), \tag{3.48}$$

and for **RVK1**

$$\int_0^{x_j} K\left(x_j,t\right) K_{n,\alpha}(f_\eta^\delta;t)dt + \omega(n,\alpha)\eta\left(\delta\right) X_\eta^\delta\left(j+1\right) = g_\delta\left(x_j\right), \tag{3.49}$$

for $j = 0,1,...,n$, $k = 0,1,...,n$. Then, the discrete regularized equations (3.48), (3.49) can be presented in matrix form

$$\widetilde{A}X_\eta^\delta = \widetilde{B},\ \widetilde{\widehat{A}}X_\eta^\delta = \widetilde{B}, \tag{3.50}$$

for the **RFK1** and for the **RVK1** respectively where,

$$X_\eta^\delta\left(k+1\right) = \frac{1}{\omega(n,\alpha)}\int_0^\mu f_\eta^\delta\left(\frac{k+t^\alpha}{n+1}\right)dt + \int_{\frac{k+\mu^\alpha}{n+1}}^{\frac{k+1}{n+1}} q_\eta^\delta\left(u\right)du,\ k = 0,1,...,n, \tag{3.51}$$

$$q_\eta^\delta\left(u\right) = \begin{cases} f_\eta^\delta\left(u\right)\left(\left(n+1\right)u-k\right)^{\frac{1-\alpha}{\alpha}} & \text{, if } \alpha \neq 1\ , \\ f_\eta^\delta\left(u\right) & \text{, if } \alpha = 1. \end{cases} \tag{3.52}$$

and the vector $\widetilde{B} \in R^{n+1}$

$$\widetilde{B}\left(j+1\right) = g_\delta\left(x_j\right),\ \ j = 0,1,...,n. \tag{3.53}$$

which can be written as $\widetilde{B} = B + \Delta B$ such that ΔB is the priori error level $\left\|\Delta B\right\|_2 \leq \delta$. Also $\widetilde{A} = A + \Delta A$ where A is the matrix in (3.12) and $\Delta A = \omega(n,\alpha)\eta\left(\delta\right)I + \Delta_1 A$, with the addition of diagonal matrix $\omega(n,\alpha)\eta\left(\delta\right)I$ and $\Delta_1 A$ which is the defect matrix of the numerical errors of the computation of the integrals in (3.48) with a predescribed error $\delta^* = \delta^*\left(\delta\right) \geq 0$, depending on δ. Analogously, $\widetilde{\widehat{A}} = \widehat{A} + \Delta\widehat{A}$ and $\widehat{A}$ is as in (3.13) and the matrix $\Delta\widehat{A} = \omega(n,\alpha)\eta\left(\delta\right)I + \Delta_1\widehat{A}$ has the defect matrix $\Delta_1\widehat{A}$ of the numerical errors of the computed integrals in (3.49) with a predescribed error $\delta^* = \delta^*\left(\delta\right) \geq 0$. Therefore, it is possible to choose $\eta\left(\delta\right), \delta^*$ such that $\left\|\Delta A\right\|_2 \leq h$ and $\left\|\Delta\widehat{A}\right\|_2 \leq h$. Clearly, the numbers h and δ are estimates of the errors of the approximate data $\left(\widetilde{A},\widetilde{B}\right), \left(\widetilde{\widehat{A}},\widetilde{B}\right)$ of the problem (3.11) for **FK1** and **VK1** respectively with the exact data $\left(A,B\right), \left(\widehat{A},B\right)$ accordingly. Thus, the given regularized systems (3.50) use h and

δ explicitly. In this connection, about the remarks on choosing the regularization parameter using the quasi-optimality and ratio criterion, see Bakushinskii [74] and for the data errors and an error estimation for ill-posed problems see Yagola et al. [75]. Next we consider the following general least squares problem for **RFK1**

$$\min_{X_\eta^\delta \in \widetilde{S}_1} \left\| X_\eta^\delta \right\|_2, \quad \widetilde{S}_1 = \left\{ X_\eta^\delta \in R^{n+1} \mid \left\| \widetilde{B} - \widetilde{A} X_\eta^\delta \right\|_2 = \min \right\}, \tag{3.54}$$

and for **RVK1**

$$\min_{X_\eta^\delta \in \widetilde{S}_2} \left\| X_\eta^\delta \right\|_2, \quad \widetilde{S}_2 = \left\{ X_\eta^\delta \in R^{n+1} \mid \left\| \widetilde{B} - \widetilde{\widehat{A}} X_\eta^\delta \right\|_2 = \min \right\}. \tag{3.55}$$

Theorem 3.3: (Theorem 1.4.2 in Björck [71]) If $rank(U+\Delta U) = rank(U)$ and $\widetilde{\eta} = \left\| U^\dagger \right\|_2 \left\| \Delta U \right\|_2 < 1$ then

$$\left\| (U+\Delta U)^\dagger \right\|_2 \leq \frac{1}{1-\widetilde{\eta}} \left\| U^\dagger \right\|_2.$$

Theorem 3.4: (Theorem 1.4.6 in Björck [71]) Assume that $rank(U+\Delta U) = rank(U)$ and let

$$\frac{\|\Delta U\|_2}{\|U\|_2} \leq \varepsilon_U, \quad \frac{\|\Delta B\|_2}{\|B\|_2} \leq \varepsilon_B. \tag{3.56}$$

Then if $\widetilde{\eta} = \kappa(U)\,\varepsilon_U < 1$ the perturbations ΔX and Δr in the least squares solution X and the residual $r = B - UX$ satisfy

$$\|\Delta X\|_2 \leq \frac{\kappa(U)}{1-\widetilde{\eta}} \left(\varepsilon_U \|X\|_2 + \varepsilon_B \frac{\|B\|_2}{\|U\|_2} + \varepsilon_U \kappa(U) \frac{\|r\|_2}{\|U\|_2} \right)$$
$$+ \varepsilon_U \kappa(U) \|X\|_2, \tag{3.57}$$

$$\|\Delta r\|_2 \leq \varepsilon_U \|X\|_2 \|U\|_2 + \varepsilon_B \|B\|_2 + \varepsilon_U \kappa(U) \|r\|_2. \tag{3.58}$$

Let $X_{\eta,n}^\delta$ denote the minimum norm solution obtained by solving the general least squares problems (3.54), (3.55). Further, $F_{\eta,n}^\delta$ denote the obtained approximation to function f_η^δ appearing implicitly in (3.51). Substituting $F_{\eta,n}^\delta$ in (3.6) we get

$K_{n,\alpha}\left(F^{\delta}_{\eta,n};x\right)$ as

$$K_{n,\alpha}\left(F^{\delta}_{\eta,n};x\right) = \omega(n,\alpha)\sum_{k=0}^{n} P_{n,k}(x)X^{\delta}_{\eta,n}(k+1). \tag{3.59}$$

We also present the residual error of the obtained algebraic linear system (3.11) for **FK1** by $r = B - AX$ ($r = B - \widehat{A}X$ for **VK1).** The regularized residual error of the system (3.50) for **RFK1** is $r^{\delta}_{\eta} = \widetilde{B} - \widetilde{A}X^{\delta}_{\eta}$ ($r^{\delta}_{\eta} = \widetilde{B} - \widetilde{\widehat{A}}X^{\delta}_{\eta}$ for **RVK1**). Furthermore, the corresponding numerical calculation of the regularized residual error is $r^{\delta}_{\eta,n} = \widetilde{B} - \widetilde{A}X^{\delta}_{\eta,n}$ ($r^{\delta}_{\eta,n} = \widetilde{B} - \widetilde{\widehat{A}}X^{\delta}_{\eta,n}$) accordingly. Next the following priory bound for the error of the approximation follows.

Theorem 3.5: (Buranay et al. [65]) Assume that the conditions of Hypothesis A are satisfied and the solution f^{δ}_{η} of (3.43) belongs to $\left(C^{\lambda}\cap L^{2}\right)([0,1])$ for some $\lambda \geq 2$. Consider the regularized linear system $\widetilde{A}X^{\delta}_{\eta} = \widetilde{B}$ given in (3.50) where $\widetilde{A} = A + \Delta A$ and A is the matrix in (3.12) and $\|\Delta A\|_2 \leq h$. Also $\widetilde{B} = B + \Delta B$ as in (3.53) and B is the vector in (3.15) and $\|\Delta B\|_2 \leq \delta$. Additionally $X^{\delta}_{\eta} = X + \Delta X$ and $r^{\delta}_{\eta} = r + \Delta r$ and let $S(j+1) = \sup_{t\in[0,1]}\left|K\left(x_j,t\right)\right|$ for $x_j = \frac{j}{n} + \varepsilon,\ j = 0,1,...,n-1$ and $x_n = 1 - \varepsilon$, where $0 < \varepsilon < \frac{1}{2n}$ and $M_2 = \|S\|_2$. Further,

$$\frac{\|\Delta A\|_2}{\|A\|_2} \leq \frac{h}{\|A\|_2} = \varepsilon_A, \quad \frac{\|\Delta B\|_2}{\|B\|_2} \leq \frac{\delta}{\|B\|_2} = \varepsilon_B. \tag{3.60}$$

If $rank(\widetilde{A}) = rank(A)$ and $\widetilde{\eta} = \kappa(A)\,\varepsilon_A < 1$ then

$$\left\|K_{n,\alpha}\left(F^{\delta}_{\eta,n}\right) - f^{\delta}_{\eta}\right\|_2 \leq W_1\left(n,\alpha,f^{\delta}_{\eta}\right) + \frac{M_2 W_2\left(n,\alpha,f^{\delta}_{\eta}\right) + \eta(\delta)\,W_3\left(n,f^{\delta}_{\eta}\right)}{1-\widetilde{\eta}}\frac{\kappa(A_*)}{\|A_*\|_2}, \tag{3.61}$$

$$\left\|X - X_{\eta,n}^{\delta}\right\|_2 \leq \frac{\kappa(A)}{(1-\widetilde{\eta})\|A\|_2}\left(h\|X\|_2 + \delta + \varepsilon_A \kappa(A)\|r\|_2\right) + \varepsilon_A \kappa(A)\|X\|_2$$
$$+ \frac{M_2 W_2\left(n,\alpha,f_{\eta}^{\delta}\right) + \eta(\delta) W_3\left(n,f_{\eta}^{\delta}\right)}{1-\widetilde{\eta}} \frac{\kappa(A)}{\|A\|_2}, \tag{3.62}$$

$$\left\|r - r_{\eta,n}^{\delta}\right\|_2 \leq h\|X\|_2 + \delta + \varepsilon_A \kappa(A)\|r\|_2$$
$$+ (1+\varepsilon_A)\frac{M_2 W_2\left(n,\alpha,f_{\eta}^{\delta}\right) + \eta(\delta) W_3\left(n,f_{\eta}^{\delta}\right)}{1-\widetilde{\eta}} \kappa(A), \tag{3.63}$$

hold true where, $\eta(\delta)$ is the regularization parameter and $W_1\left(n,\alpha,f_{\eta}^{\delta}\right)$, $W_2\left(n,\alpha,f_{\eta}^{\delta}\right)$ are as in (3.21) and (3.22) respectively. Also $W_3\left(n,f_{\eta}^{\delta}\right) = \frac{1}{\sqrt{n+1}}\left\|\frac{df_{\eta}^{\delta}}{dx}\right\|$ and A, A_* are as given in (3.12).

Proof. For **RFK1** it follows that

$$\left\|K_{n,\alpha}\left(F_{\eta,n}^{\delta}\right) - f_{\eta}^{\delta}\right\|_2 \leq \left\|K_{n,\alpha}\left(f_{\eta}^{\delta}\right) - f_{\eta}^{\delta}\right\|_2 + \left\|K_{n,\alpha}\left(F_{\eta,n}^{\delta}\right) - K_{n,\alpha}\left(f_{\eta}^{\delta}\right)\right\|_2. \tag{3.64}$$

Based on Corollary 2.1 and the estimation (2.22) by replacing f with f_{η}^{δ} in estimation (2.22) and in (3.21) we obtain

$$\left\|K_{n,\alpha}\left(f_{\eta,n}^{\delta}\right) - f_{\eta}^{\delta}\right\|_2 \leq W_1\left(n,\alpha,f_{\eta}^{\delta}\right). \tag{3.65}$$

Let $\overline{X}_{\eta}^{\delta} = X_{\eta,n}^{\delta} - X_{\eta}^{\delta}$ then from (3.6) and (3.18) and using that $\sum_{k=0}^{n} P_{n,k}(x) = 1$, follows

$$\left\|K_{n,\alpha}\left(F_{\eta,n}^{\delta}\right) - K_{n,\alpha}\left(f_{\eta}^{\delta}\right)\right\|_2 = \left(\int_0^1 \left|\omega(n,\alpha)\sum_{k=0}^{n} P_{n,k}(x)\overline{X}_{\eta}^{\delta}(k+1)\right|^2 dx\right)^{\frac{1}{2}}$$
$$\leq \omega(n,\alpha)\left(\sum_{k=0}^{n}\left(\overline{X}_{\eta}^{\delta}(k+1)\right)^2\right)^{\frac{1}{2}}\left(\int_0^1\left|\sum_{k=0}^{n} P_{n,k}(x)\right|^2 dx\right)^{\frac{1}{2}}$$
$$= \omega(n,\alpha)\left\|\overline{X}_{\eta}^{\delta}\right\|_2, \tag{3.66}$$

where

$$\overline{X}_{\eta}^{\delta}(k+1) = \frac{1}{\omega(n,\alpha)} \int_{0}^{\mu} \left(F_{\eta,n}^{\delta}\left(\frac{k+t^{\alpha}}{n+1}\right) - f_{\eta}^{\delta}\left(\frac{k+t^{\alpha}}{n+1}\right)\right) dt$$

$$+ \int_{\frac{k+\mu^{\alpha}}{n+1}}^{\frac{k+1}{n+1}} \left(F_{\eta,n}^{\delta}(u) - f_{\eta}^{\delta}(u)\right) \left((n+1)u - k\right)^{\frac{1-\alpha}{\alpha}} du. \quad (3.67)$$

For the numerical solution of **RFK1** in (3.43) we use the grid points $x_j = \frac{j}{n} + \varepsilon,\ j = 0,1,...,n-1$ and $x_n = 1-\varepsilon$, where $0 < \varepsilon < \frac{1}{2n}$. We assume

$$\int_{0}^{1} K(x_j,t)\, K_{n,\alpha}\left(f_{\eta}^{\delta};t\right) dt + \omega(n,\alpha)\eta(\delta) X_{\eta}^{\delta}(j+1) = g_{\delta}(x_j), \quad (3.68)$$

where $\omega(n,\alpha) X_{\eta}^{\delta}(j+1)$ gives the average value of f_{η}^{δ} over the interval $\left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$. If we substitute $F_{\eta,n}^{\delta}$ instead of f_{η}^{δ} in (3.68) we get a new function $\widehat{g}_{\delta}$ on the right side of this equation

$$\int_{0}^{1} K(x_j,t)\, K_{n,\alpha}\left(F_{\eta,n}^{\delta};t\right) dt + \omega(n,\alpha)\eta(\delta) X_{\eta,n}^{\delta}(j+1) = \widehat{g}_{\delta}(x_j). \quad (3.69)$$

Thus, for **RFK1** from (3.68) and (3.69) we obtain

$$\widetilde{A}\overline{X}_{\eta}^{\delta} = \overline{\overline{B}}, \text{ and } \overline{\overline{B}}(j+1) = \widehat{g}_{\delta}(x_j) - g_{\delta}(x_j),\ j = 0,1,...,n, \quad (3.70)$$

where, $\overline{X}_{\eta}^{\delta}$ is as given in (3.67). The general least squares problem of (3.70) has the minimum norm solution

$$\overline{X}_{\eta}^{\delta} = \widetilde{A}^{\dagger}\overline{\overline{B}}. \quad (3.71)$$

Thus,

$$\left\|\overline{X}_{\eta}^{\delta}\right\|_2 \le \left\|\widetilde{A}^{\dagger}\right\|_2 \left\|\overline{\overline{B}}\right\|_2, \quad (3.72)$$

$$\omega(n,\alpha)\left\|\overline{X}_{\eta}^{\delta}\right\|_2 \le \left\|\widetilde{A}_{*}^{\dagger}\right\|_2 \left\|\overline{\overline{B}}\right\|_2. \quad (3.73)$$

Then let $\widehat{g}_{\delta}(x_j) = \int_{0}^{1} K(x_j,t)\, K_{n,\alpha}\left(f_{\eta}^{\delta};t\right) dt +\ \omega(n,\alpha)\eta(\delta) X_{\eta}^{\delta}(j+1)$ and $g_{\delta}(x_j) = \int_{0}^{1} K(x_j,t)\, f_{\eta}^{\delta}(t)\, dt +\ \eta(\delta) f_{\eta}^{\delta}(x_j)$ for $j = 0,1,...,n$ it follows that

$$\widehat{g}_{\delta}(x_j) - g_{\delta}(x_j) = \int_0^1 K(x_j,t)\left(K_{n,\alpha}\left(f_{\eta}^{\delta};t\right) - f_{\eta}^{\delta}(t)\right)dt$$

$$+\eta(\delta)\left(\omega(n,\alpha)X_{\eta}^{\delta}(j+1) - f_{\eta}^{\delta}(x_j)\right). \tag{3.74}$$

From the assumption that $f_{\eta}^{\delta} \in \left(C^{\lambda} \cap L^2\right)([0,1])$ for some $\lambda \geq 2$ it follows that $\sup\limits_{0\leq j\leq n}\left|\omega(n,\alpha)X_{\eta}^{\delta}(j+1) - f_{\eta}^{\delta}(x_j)\right| \leq \frac{1}{n+1}\left\|\frac{df_{\eta}^{\delta}}{dx}\right\|$. Let $W_3\left(n, f_{\eta}^{\delta}\right) = \frac{1}{\sqrt{n+1}}\left\|\frac{df_{\eta}^{\delta}}{dx}\right\|$, by taking $S(j+1) = \sup\limits_{t\in[0,1]}\left|K(x_j,t)\right|$ for $j = 0,1,...,n$ and $M_2 = \|S\|_2$ also on the basis of Corollary 2.1 and replacing f with f_{η}^{δ} in estimations (2.21) and (3.22) we obtain

$$\left(\sum_{j=0}^{n}\left|\left(\widehat{g}_{\delta}(x_j) - g_{\delta}(x_j)\right)\right|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=0}^{n}\left|\int_0^1 K(x_j,t)\left(K_{n,\alpha}\left(f_{\eta}^{\delta};t\right) - f_{\eta}^{\delta}(t)\right)dt\right|^2\right)^{\frac{1}{2}}$$

$$+\eta(\delta)\left(\sum_{j=0}^{n}\left|\omega(n,\alpha)X_{\eta}^{\delta}(j+1) - f_{\eta}^{\delta}(x_j)\right|^2\right)^{\frac{1}{2}} \tag{3.75}$$

$$\leq \left(\sum_{j=0}^{n}(S(j+1))^2\right)^{\frac{1}{2}} \sup_{t\in[0,1]}\left|K_{n,\alpha}\left(f_{\eta}^{\delta};t\right) - f_{\eta}^{\delta}(t)\right|$$

$$+\eta(\delta)W_3\left(n, f_{\eta}^{\delta}\right) \tag{3.76}$$

$$\leq M_2 W_2\left(n,\alpha, f_{\eta}^{\delta}\right) + \eta(\delta)W_3\left(n, f_{\eta}^{\delta}\right). \tag{3.77}$$

Substituting the estimation (3.77) into (3.73) and the result in (3.66) we get

$$\left\|K_{n,\alpha}\left(F_{\eta,n}^{\delta}\right) - K_{n,\alpha}\left(f_{\eta}^{\delta}\right)\right\|_2 \leq \left(M_2 W_2\left(n,\alpha, f_{\eta}^{\delta}\right) + \eta(\delta)W_3\left(n, f_{\eta}^{\delta}\right)\right)\left\|\widetilde{A}_*^{\dagger}\right\|_2. \tag{3.78}$$

Inserting (3.65) and (3.78) in (3.64) and on the basis of Theorem 3.3 and using that $\kappa(A_*) = \left\|A_*^{\dagger}\right\|_2 \|A_*\|_2$ we obtain (3.61). The inequality (3.62) is obtained by using

$$\left\|X - X_{\eta,n}^{\delta}\right\|_2 \leq \left\|X - X_{\eta}^{\delta}\right\|_2 + \left\|X_{\eta}^{\delta} - X_{\eta,n}^{\delta}\right\|_2, \tag{3.79}$$

and based on the Theorem 3.4 and the inequality (3.57) the first term on the right side of (3.79) is obtained as

$$\left\|X - X_\eta^\delta\right\|_2 \leq \frac{\kappa(A)}{(1-\widetilde{\eta})\|A\|_2}\left(h\|X\|_2 + \delta\right.$$
$$\left. + \varepsilon_A \kappa(A)\|r\|_2\right) + \varepsilon_A \kappa(A)\|X\|_2 . \tag{3.80}$$

Next on the basis of Theorem 3.3 and using (3.72), (3.77) and $\left\|A^\dagger\right\|_2 = \frac{\kappa(A)}{\|A\|_2}$ we get

$$\left\|X_\eta^\delta - X_{\eta,n}^\delta\right\|_2 \leq \frac{\left(M_2 W_2\left(n,\alpha,f_\eta^\delta\right) + \eta(\delta) W_3\left(n,f_\eta^\delta\right)\right)}{1-\widetilde{\eta}} \frac{\kappa(A)}{\|A\|_2}. \tag{3.81}$$

Inserting the estimations (3.80), (3.81) into (3.79) gives (3.62). To prove the inequality (3.63) we use

$$\left\|r - r_{\eta,n}^\delta\right\|_2 \leq \left\|r - r_\eta^\delta\right\|_2 + \left\|r_\eta^\delta - r_{\eta,n}^\delta\right\|_2, \tag{3.82}$$

and based on the Theorem 3.4 and the inequality (3.58) the first term on the right side of (3.82) is obtained as

$$\left\|r - r_\eta^\delta\right\|_2 \leq h\|X\|_2 + \delta + \varepsilon_A \kappa(A)\|r\|_2 . \tag{3.83}$$

The second error term on the right side of (3.82) satisfies

$$\left\|r_\eta^\delta - r_{\eta,n}^\delta\right\|_2 \leq \left\|\widetilde{A}\right\|_2 \left\|X_\eta^\delta - X_{\eta,n}^\delta\right\|_2, \tag{3.84}$$

using (3.81), (3.82) and (3.83), and that $\left\|\widetilde{A}\right\|_2 \leq \|A\|_2 + h$ and from (3.60) follows (3.63). □

Theorem 3.6: (Buranay et al. [65]) Assume that the conditions of Hypothesis A are satisfied and the solution f_η^δ of **RVK1** belongs to $\left(C^\lambda \cap L^2\right)([0,1])$ for some $\lambda \geq 2$. Consider the linear system $\widetilde{\widehat{A}} X_\eta^\delta = \widetilde{B}$ given in (3.50) where $\widetilde{\widehat{A}} = \widehat{A} + \Delta\widehat{A}$ and $\widehat{A}$ is the matrix in (3.13) and $\left\|\Delta\widehat{A}\right\|_2 \leq h$. Also $\widetilde{B} = B + \Delta B$ as in (3.53) and B is as in (3.15) and $\|\Delta B\|_2 \leq \delta$. Additionally $X_\eta^\delta = X + \Delta X$ and $r_\eta^\delta = r + \Delta r$ and let $S(j+1) = \sup\limits_{t\in[0,1]} \left|K\left(x_j,t\right)\right|$ for $x_j = \frac{j}{n} + \varepsilon$, $j = 0,1,...,n-1$ and $x_n = 1-\varepsilon$, where $0 < \varepsilon < \frac{1}{2n}$ and

$M_2 = \|S\|_2$. Further,

$$\frac{\left\|\Delta\widehat{A}\right\|_2}{\left\|\widehat{A}\right\|_2} \leq \frac{h}{\left\|\widehat{A}\right\|_2} = \varepsilon_{\widehat{A}}, \quad \frac{\|\Delta B\|_2}{\|B\|_2} \leq \frac{\delta}{\|B\|_2} = \varepsilon_B. \tag{3.85}$$

If $rank(\widetilde{\widehat{A}}) = rank(\widehat{A})$ and $\widetilde{\eta} = \kappa\left(\widehat{A}\right)\varepsilon_{\widehat{A}} < 1$ then

$$\left\|K_{n,\alpha}\left(F_{\eta,n}^{\delta}\right) - f_{\eta}^{\delta}\right\|_2 \leq W_1\left(n,\alpha,f_{\eta}^{\delta}\right) + \frac{M_2 W_2\left(n,\alpha,f_{\eta}^{\delta}\right) + \eta\left(\delta\right) W_3\left(n,f_{\eta}^{\delta}\right)}{1-\widetilde{\eta}} \frac{\kappa\left(\widehat{A}_*\right)}{\left\|\widehat{A}_*\right\|_2}, \tag{3.86}$$

$$\left\|X - X_{\eta,n}^{\delta}\right\|_2 \leq \frac{\kappa\left(\widehat{A}\right)}{(1-\widetilde{\eta})\left\|\widehat{A}\right\|_2}\left(h\|X\|_2 + \delta + \varepsilon_{\widehat{A}}\kappa\left(\widehat{A}\right)\|r\|_2\right) + \varepsilon_{\widehat{A}}\kappa\left(\widehat{A}\right)\|X\|_2 + \frac{M_2 W_2\left(n,\alpha,f_{\eta}^{\delta}\right) + \eta\left(\delta\right) W_3\left(n,f_{\eta}^{\delta}\right)}{1-\widetilde{\eta}} \frac{\kappa\left(\widehat{A}\right)}{\left\|\widehat{A}\right\|_2}, \tag{3.87}$$

$$\left\|r - r_{\eta,n}^{\delta}\right\|_2 \leq h\|X\|_2 + \delta + \varepsilon_{\widehat{A}}\kappa\left(\widehat{A}\right)\|r\|_2 + (1+\varepsilon_{\widehat{A}})\frac{M_2 W_2\left(n,\alpha,f_{\eta}^{\delta}\right) + \eta\left(\delta\right) W_3\left(n,f_{\eta}^{\delta}\right)}{1-\widetilde{\eta}}\kappa\left(\widehat{A}\right), \tag{3.88}$$

hold true where, $\eta\left(\delta\right)$ is the regularization parameter and $W_1\left(n,\alpha,f_{\eta}^{\delta}\right)$, $W_2\left(n,\alpha,f_{\eta}^{\delta}\right)$ are as in (3.21) and (3.22) respectively. Also $W_3\left(n,f_{\eta}^{\delta}\right) = \frac{1}{\sqrt{n+1}}\left\|\frac{df_{\eta}^{\delta}}{dx}\right\|$ and $\widehat{A}$ and $\widehat{A}_*$ are as given in (3.13).

Proof. Proof is analogous to the proof of Theorem 3.5. □

Chapter 4

EXPERIMENTAL INVESTIGATION OF THE GIVEN NUMERICAL APPROACH

For the theoretical results given in Chapter 3 we focus on the interval $[0,1]$, however for the numerical results we also consider examples on $[a,b]$ with the following extension of the Bernstein operators and Modified Bernstein-Kantorovich operators on the interval $[a,b]$

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} f\left(a + \frac{k}{n}(b-a)\right), \tag{4.1}$$

$$K_{n,\alpha}(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \frac{1}{b-a} \int_a^b f\left(a + \frac{k + \left(\frac{t-a}{b-a}\right)^{\alpha}}{(n+1)}(b-a)\right) dt, \tag{4.2}$$

respectively. All the computations in this section are performed using Mathematica in machine precision on a personal computer with properties AMD Ryzen 7 1800X Eight Core Processor 3.60GHz. We remark that the solution of the Volterra integral equations by using Bernstein polynomials was given in Maleknejad et al. [34]. All the considered test problems are also solved by using Bernstein operators (2.11) with the approach given in Maleknejad et al. [34], additionally regularization is applied. Further, the obtained algebraic system of equations by applying the methods $M(K_{n,\alpha})$ and $M(B_n)$ are solved using the pseudoinverse of the respective matrices. Let the following error grid functions be defined at the $N+1$ grid points $x_p = a + \frac{p(b-a)}{N}$, $p = 0,1,...,N$ over the interval $[a,b]$ as

$$\widetilde{E}_N\left[K_{n,\alpha}\left(F^{\delta}_{\eta,n};x_p\right)\right]=f(x_p)-K_{n,\alpha}\left(F^{\delta}_{\eta,n}(x_p)\right), \tag{4.3}$$

$$\widetilde{E}_N\left[B_n\left(F^{\delta}_{\eta,n};x_p\right)\right]=f(x_p)-B_n\left(F^{\delta}_{\eta,n}(x_p)\right). \tag{4.4}$$

Further, we use the following notations in tables and figures:

i) $M(K_{n,\alpha})$ presents the given approach by using the Modified Bernstein-Kantorovich operators $K_{n,\alpha}$.

ii) $M(B_n)$ presents the approach Maleknejad et al. [34] by using the Bernstein operators B_n.

iii) $Cond_{B_n}\left(\widetilde{A}\right)$ denotes the condition number of the perturbed matrix $\widetilde{A}$ obtained by the method $M(B_n)$ using LinearAlgebra‘Private‘MatrixConditionNumber command in Mathematica.

iv) $Cond_{K_{n,\alpha}}\left(\widetilde{A}\right)$ denotes the condition number of the perturbed matrix $\widetilde{A}$ obtained by the method $M(K_{n,\alpha})$ using LinearAlgebra‘Private‘MatrixConditionNumber command in Mathematica.

v) $RE_{\widetilde{E}_N}(K_{n,\alpha})$ denotes the root mean square error ($RMSE$) of the regularized solution

$$RE_{\widetilde{E}_N}(K_{n,\alpha})=\sqrt{\frac{1}{(N+1)}\sum_{p=0}^{N}\left(\widetilde{E}_N\left[K_{n,\alpha}\left(F^{\delta}_{\eta,n};x_p\right)\right]\right)^2},$$

obtained by $M(K_{n,\alpha})$.

vi) $RE_{\widetilde{E}_N}(B_n)]$ denotes $RMSE$ of the regularized solution

$$RE_{\widetilde{E}_N}(B_n)=\sqrt{\frac{1}{(N+1)}\sum_{p=0}^{N}\left(\widetilde{E}_N\left[B_n\left(F^{\delta}_{\eta,n};x_p\right)\right]\right)^2},$$

obtained by $M(B_n)$.

vii) $AE_{\widetilde{E}_N,x_p}(K_{n,\alpha})$ is the absolute error of the regularized solution

$\left|\widetilde{E}_N\left[K_{n,\alpha}\left(F^{\delta}_{\eta,n};x_p\right)\right]\right|$ at the point x_p.

viii) $AE_{\widetilde{E}_N,x_p}(B_n)$ is the absolute error of the regularized solution $\left|\widetilde{E}_N\left[B_n\left(F^{\delta}_{\eta,n};x_p\right)\right]\right|$ at the point x_p .

ix) $ME_{\widetilde{E}_N}(K_{n,\alpha})$ shows the maximum error (ME) of the regularized solution

$$\max_{0\leq p\leq N}\left|\widetilde{E}_N\left[K_{n,\alpha}\left(F^{\delta}_{\eta,n};x_p\right)\right]\right|.$$

x) $ME_{\widetilde{E}_N}(B_n)$ shows the maximum error ME of the regularized solution

$$\max_{0\leq p\leq N}\left|\widetilde{E}_N\left[B_n\left(F^{\delta}_{\eta,n};x_p\right)\right]\right|.$$

xi) (na) means that the specified method is not applied to the considered example.

xii) (ng) means that the absolute error is not given at the presented grid point by the specified method.

4.1 Application on Examples of Fredholm Integral Equations

We consider the following test problems of fist kind Fredholm integral equations which have been used as benchmark problems in the literature. For the implementation of the approach we have taken $\eta(\delta)=\delta$.

Example 1: $(Ex1)$ (Wen and Wei [21] and Baker et al. [76])

$$\int_0^1 e^{xt}f(t)\,dt=\frac{e^{x+1}-1}{x+1},\ 0\leq x\leq 1,$$

and the exact solution is $f(x)=e^x$.

Example 2: $(Ex2)$ (Wen and Wei [21])

$$\int_0^1 e^{-x+t}f(t)\,dt=\frac{3-3e^{-x}\cos(3)-e^{-x}x\sin(3)}{x^2+9},\ 0\leq x\leq 1,$$

where the exact solution is $f(x) = \sin(3x)$.

Example 3: $(Ex3)$ (Baker et al. [76])

$$\int_0^1 \sqrt{x^2+t^2} f(t)\,dt = \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - x^3,\ 0 \le x \le 1,$$

and the exact solution is $f(x) = x$.

Example 4: $(Ex4)$

$$\int_0^1 \frac{1}{\sqrt{1+t^{\frac{5}{2}}+x^2}} f(t)\,dt = \frac{4}{5}\sqrt{x^2+2} - \frac{4}{5}\sqrt{x^2+1},\ 0 \le x \le 1,$$

and the exact solution is $f(x) = x^{\frac{3}{2}}$.

Table 4.1 presents the *RMSE* with respect to n obtained by the proposed approach $M(K_{n,10})$ when $N = 51$ and $\varepsilon = 0.0001$, for the examples of **FK1** and when $\delta = 5 \times 10^{-12}$ for the Example 1, Example 2 and Example 4 and $\delta = 5 \times 10^{-9}$ for the Example 3. The absolute errors obtained by the method $M(K_{9,10})$ at the points $x_p = \frac{p}{8}$, $p = 0, 1, ..., 8$ for the examples **FK1** when $\varepsilon = 0.0001, n = 9$ and $\alpha = 10$ for the same values of δ as in Table 4.1 are demonstrated in Table 4.2. Further, Table 4.3 shows the same quantities as in Table 4.1 obtained by using the approach $M(B_n)$. Table 4.4, Table 4.5, Table 4.6 and Table 4.7 present the condition numbers of the perturbed matrices, *RMSE* with respect to the δ obtained by the proposed method $M(K_{8,1})$ and the method $M(B_8)$ when $\varepsilon = 0.0001$, and $N = 51$ for the Example 1, Example 2, Example 3 and Example 4 respectively. Table 4.8 presents the *RMSE* with respect to α obtained by the proposed approach $M(K_{9,\alpha})$, when $N = 51$ and $\varepsilon = 0.0001$, for the examples of **FK1.** In this Table the parameter δ is taken as $\delta = 5 \times 10^{-12}$ for the Example 1, Example 2 and Example 4 and $\delta = 5 \times 10^{-9}$ for the Example 3. This table shows that α also has an effect on the results and for the Example 1 and Example 4, when $\alpha = 10$ we obtain

Table 4.1: The *RMSE* for the examples of **FK1** with respect to n when $\varepsilon = 0.0001$ and $\alpha = 10, N = 51$ obtained by the method $M(K_{n,10})$.

n	$Ex1$ **FK1** $RE_{\widetilde{E}_{51}}(K_{n,10})$	$Ex2$ **FK1** $RE_{\widetilde{E}_{51}}(K_{n,10})$	$Ex3$ **FK1** $RE_{\widetilde{E}_{51}}(K_{n,10})$	$Ex4$ **FK1** $RE_{\widetilde{E}_{51}}(K_{n,10})$
2	0.00564194	0.01652300	1.065×10^{-8}	0.00585089
3	0.00036214	0.01575890	1.868×10^{-8}	0.00197745
4	0.00001859	0.00036009	5.625×10^{-8}	0.00095987
5	7.946×10^{-7}	0.00032459	1.552×10^{-7}	0.00073284
6	1.150×10^{-6}	0.00025850	1.195×10^{-6}	0.00070217
7	1.228×10^{-6}	0.00018178	0.00025021	0.00069018
8	2.988×10^{-6}	0.00012625	8.797×10^{-6}	0.00054624
9	1.126×10^{-6}	0.00009103	2.064×10^{-6}	0.00029078

lowest RMSE. Similarly, RMSE are minimum when $a = 0.1$ for the Example 2 and Example 3.

Figure 4.1 presents the *RMSE* with respect to α obtained by $M(K_{9,\alpha})$ for the examples of **FK1,** when $\varepsilon = 0.0001$, and $N = 51$. It can be viewed that the optimal value of α is $\alpha = 10$ for the Example 1, and Example 4 whereas, $\alpha = 0.1$ gives the lowest *RMSE* for the Example 2 and Example 3. Figure 4.2 illustrates the *RMSE* with respect to n obtained by the methods $M(K_{n,10})$ and $M(B_n)$ for the considered examples of **FK1** when $\varepsilon = 0.0001$ and $N = 51$. Also, for the data in Figure 4.1 and Figure 4.2 we take $\delta = 5 \times 10^{-12}$ for the Example 1, Example 2 and Example 4 and $\delta = 5 \times 10^{-9}$ for the Example 3. Figure 4.3 shows the *RMSE* with respect to δ obtained by the methods $M(K_{14,1})$ and $M(B_{14})$ for the examples of **FK1** when $\varepsilon = 0.0001$ and $N = 51$.

Table 4.9 shows the accuracy comparisons of the proposed approach with the known methods from the literature of which the errors in Baker et al. [76] are given in *ME* (maximum error) and other errors are given in *RMSE* for the Example 1, Example 2 and Example 3 of **FK1**. The data in the second row presents the results in Wen and

Table 4.2: The absolute errors at 9 points over [0,1] for the examples of **FK1** when $\varepsilon = 0.0001$, $n = 9$ and $\alpha = 10$ obtained by the method $M(K_{9,10})$.

x_p	$Ex1$ **FK1** $AE_{\widetilde{E}_8,x_p}(K_{9,10})$	$Ex2$ **FK1** $AE_{\widetilde{E}_8,x_p}(K_{9,10})$	$Ex3$ **FK1** $AE_{\widetilde{E}_8,x_p}(K_{9,10})$	$Ex4$ **FK1** $AE_{\widetilde{E}_8,x_p}(K_{9,10})$
0.000	1.654×10^{-6}	0.000192898	2.757×10^{-6}	0.00158317
0.125	3.161×10^{-7}	0.0000647276	1.081×10^{-6}	0.0000844532
0.250	2.285×10^{-7}	0.0000794108	1.402×10^{-6}	0.000210398
0.375	9.035×10^{-7}	6.929×10^{-6}	8.913×10^{-7}	0.000178284
0.500	4.513×10^{-7}	0.0000914048	1.856×10^{-6}	7.728×10^{-6}
0.625	7.493×10^{-7}	0.0000112388	3.405×10^{-6}	0.0000910857
0.750	1.033×10^{-6}	0.0001048730	1.154×10^{-6}	0.0000799071
0.875	4.732×10^{-7}	0.0001009200	6.462×10^{-6}	0.0000367732
1.000	3.398×10^{-6}	0.0003547500	0.0000147511	0.000193043

Table 4.3: The $RMSE$ for the examples of **FK1** with respect to n when $\varepsilon = 0.0001$ and $N = 51$ obtained by the method $M(B_n)$.

n	$Ex1$ **FK1** $RE_{\widetilde{E}_{51}}(B_n)$	$Ex2$ **FK1** $RE_{\widetilde{E}_{51}}(B_n)$	$Ex3$ **FK1** $RE_{\widetilde{E}_{51}}(B_n)$	$Ex4$ **FK1** $RE_{\widetilde{E}_{51}}(B_n)$
2	0.00564200	0.01652300	4.899×10^{-9}	0.00585089
3	0.00036214	0.01575890	1.162×10^{-8}	0.00197744
4	0.00001860	0.00036020	5.978×10^{-8}	0.00095794
5	7.899×10^{-7}	0.00032358	1.455×10^{-7}	0.00068048
6	1.470×10^{-6}	0.00024038	1.159×10^{-6}	0.00068950
7	1.163×10^{-6}	0.00016529	0.00011583	0.00069803
8	5.259×10^{-6}	0.00011862	8.864×10^{-6}	0.00054573
9	1.170×10^{-6}	0.00009103	2.064×10^{-6}	0.00029078

Table 4.4: Condition numbers and the *RMSE* for the Example 1 of **FK1** when $\varepsilon = 0.0001$ and $\alpha = 1, n = 8$.

δ	$Cond_{B_8}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}\left(B_8\right)$	$Cond_{K_{8,1}}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}\left(K_{8,1}\right)$
5×10^{-8}	5.136×10^{7}	0.00001356	3.750×10^{8}	7.004×10^{-6}
5×10^{-9}	4.082×10^{8}	6.675×10^{-6}	4.944×10^{9}	2.119×10^{-6}
5×10^{-10}	5.491×10^{9}	2.014×10^{-6}	4.424×10^{10}	1.276×10^{-6}
5×10^{-11}	4.859×10^{10}	1.264×10^{-6}	4.600×10^{11}	5.195×10^{-7}
5×10^{-12}	5.112×10^{11}	5.259×10^{-7}	4.419×10^{12}	1.972×10^{-6}
5×10^{-13}	4.900×10^{12}	2.005×10^{-6}	3.890×10^{13}	0.00002177
5×10^{-14}	4.274×10^{13}	0.00003837	4.276×10^{14}	0.00004387
5×10^{-15}	4.742×10^{14}	0.00015707	3.971×10^{15}	0.00067454
0	6.590×10^{16}	0.01001490	6.258×10^{16}	0.00218054

Table 4.5: Condition numbers and the *RMSE* for the Example 2 of **FK1** when $\varepsilon = 0.0001$ and $\alpha = 1, n = 8$.

δ	$Cond_{B_8}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}\left(B_8\right)$	$Cond_{K_{8,1}}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}\left(K_{8,1}\right)$
5×10^{-8}	3.207×10^{7}	0.00238070	4.013×10^{8}	0.00179104
5×10^{-9}	4.691×10^{8}	0.00189299	3.094×10^{9}	0.00058267
5×10^{-10}	3.412×10^{9}	0.00052648	2.712×10^{10}	0.00026338
5×10^{-11}	2.982×10^{10}	0.00025935	2.688×10^{11}	0.00012597
5×10^{-12}	2.988×10^{11}	0.00011862	2.741×10^{12}	0.00002467
5×10^{-13}	3.052×10^{12}	0.00002339	3.601×10^{13}	0.00003398
5×10^{-14}	4.182×10^{13}	0.00004678	2.068×10^{14}	0.00009016
5×10^{-15}	2.772×10^{14}	0.00004866	2.382×10^{15}	0.00041590
0	5.498×10^{16}	0.01668350	5.073×10^{16}	0.01305410

Table 4.6: Condition numbers and the *RMSE* for the Example 3 of **FK1** when $\varepsilon = 0.0001$ and $\alpha = 1$, $n = 8$.

δ	$Cond_{B_8}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}(B_8)$	$Cond_{K_{8,1}}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}(K_{8,1})$
5×10^{-8}	4.902×10^{7}	0.0000121677	3.919×10^{10}	0.00258407
5×10^{-9}	3.424×10^{9}	0.000028873	3.459×10^{9}	8.84178×10^{-6}
5×10^{-10}	3.898×10^{9}	0.0000109539	1.511×10^{11}	0.000186152
5×10^{-11}	1.006×10^{11}	0.000117269	2.740×10^{10}	0.0000283152
5×10^{-12}	2.715×10^{10}	0.0000278597	2.533×10^{10}	0.0000256703
5×10^{-13}	2.531×10^{10}	0.0000256237	2.514×10^{10}	0.0000254228
5×10^{-14}	2.513×10^{10}	0.0000254236	2.512×10^{10}	0.0000254052
5×10^{-15}	2.512×10^{10}	0.0000253998	2.512×10^{10}	0.0000253969
0	2.512×10^{10}	0.0000253957	2.512×10^{10}	0.0000254009

Table 4.7: Condition numbers and the *RMSE* for the Example 4 of **FK1** when $\varepsilon = 0.0001$ and $\alpha = 1$, $n = 8$.

δ	$Cond_{B_8}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}(B_8)$	$Cond_{K_{8,1}}\left(\widetilde{A}\right)$	$RE_{\widetilde{E}_{51}}(K_{8,1})$
5×10^{-8}	3.064×10^{7}	0.00051122	2.381×10^{8}	0.00048011
5×10^{-9}	2.645×10^{8}	0.00049303	2.367×10^{9}	0.00051823
5×10^{-10}	2.627×10^{9}	0.00051343	2.330×10^{10}	0.00054019
5×10^{-11}	2.593×10^{10}	0.00053620	2.336×10^{11}	0.00054578
5×10^{-12}	2.593×10^{11}	0.00054573	2.180×10^{12}	0.00051476
5×10^{-13}	2.398×10^{12}	0.00050975	2.530×10^{13}	0.00049179
5×10^{-14}	2.803×10^{13}	0.00050957	2.322×10^{14}	0.00069250
5×10^{-14}	2.573×10^{14}	0.00065452	2.488×10^{15}	0.00031208
0	5.559×10^{16}	0.04357480	7.397×10^{16}	0.03460650

Table 4.8: The *RMSE* for the examples of **FK1** with respect to α when $\varepsilon = 0.0001$, $N = 51$ and $\delta = 5 \times 10^{-12}$ for the Example 1, Example 2 and Example 4 and $\delta = 5 \times 10^{-9}$ for the Example 3.

α	$Ex1$ FK1 $RE_{\widetilde{E}_{51}}(K_{9,\alpha})$	$Ex2$ FK1 $RE_{\widetilde{E}_{51}}(K_{9,\alpha})$	$Ex3$ FK1 $RE_{\widetilde{E}_{51}}(K_{9,\alpha})$	$Ex4$ FK1 $RE_{\widetilde{E}_{51}}(K_{9,\alpha})$
0.0001	0.00944481	0.00235555	7.270×10^{-6}	0.00181866
0.001	0.00071458	0.00015984	7.373×10^{-6}	0.00182835
0.01	0.00033075	0.00002426	8.811×10^{-6}	0.00188793
0.1	0.00005143	0.00001386	1.681×10^{-6}	0.00276483
1	0.00001147	0.00002120	4.004×10^{-6}	0.04152750
10	1.126×10^{-6}	0.00009103	2.064×10^{-6}	0.00029078
100	1.699×10^{-6}	0.00023779	3.526×10^{-6}	0.00040206
1000	2.565×10^{-6}	0.00045454	2.775×10^{-6}	0.00037842
10,000	6.947×10^{-6}	0.00240884	0.00001448	0.00037095

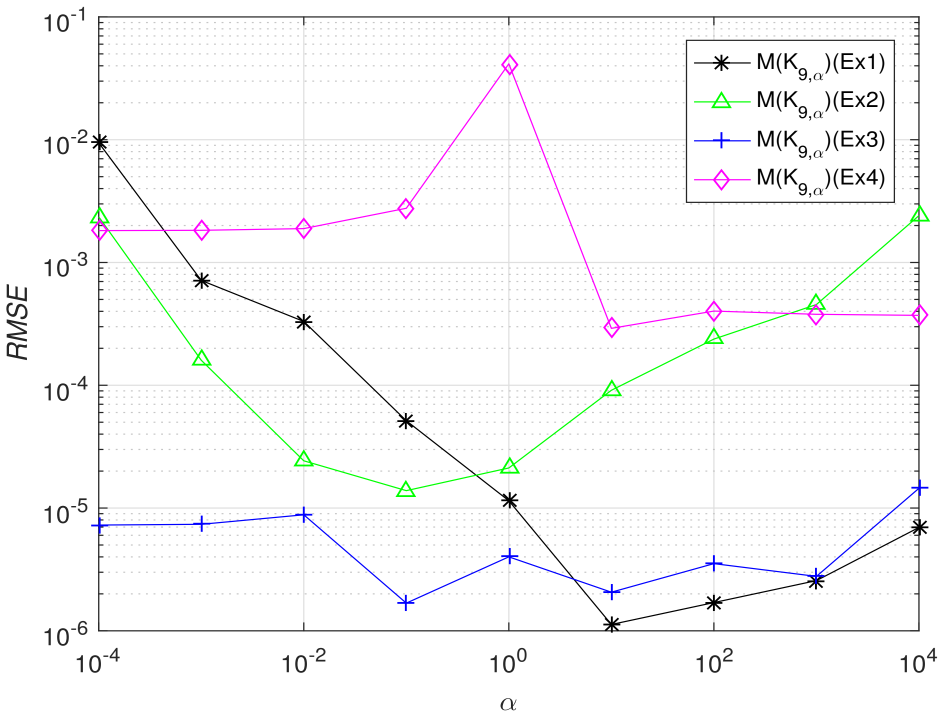

Figure 4.1: The *RMSE* with respect to α obtained by $M(K_{9,\alpha})$ for the examples of **FK1,** when $\varepsilon = 0.0001$, and $N = 51$.

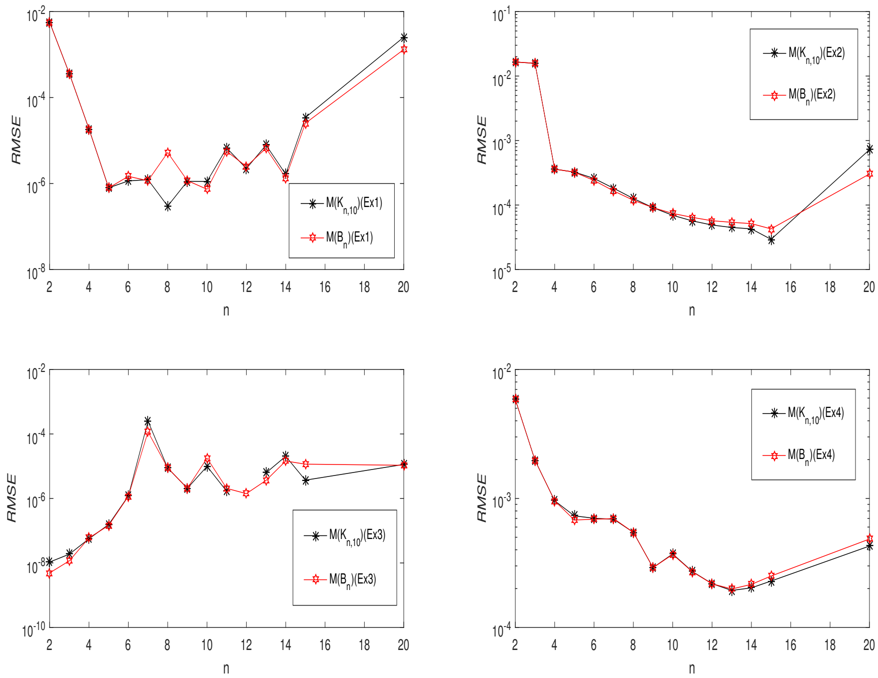

Figure 4.2: The *RMSE* with respect to n obtained by the methods $M(K_{n,10})$ and $M(B_n)$ for the examples of **FK1** when $\varepsilon = 0.0001$ and $N = 51$.

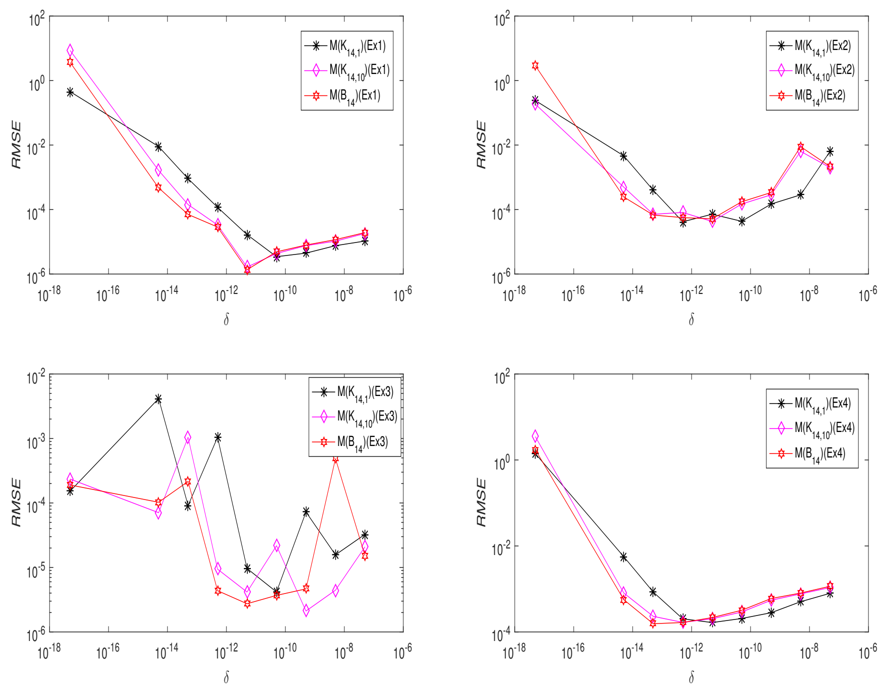

Figure 4.3: *RMSE* with respect to δ obtained by the methods $M(K_{14,1})$, $M(K_{14,10})$ and $M(B_{14})$ for the examples of **FK1** when $\varepsilon = 0.0001$ and $N = 51$.

Table 4.9: Accuracy comparison of the proposed approach with the methods from the literature for the Example 1, Example 2 and Example 3 of **FK1**.

Approach	Ex1 **FK1** Error	Ex2 **FK1** Error	Ex3 **FK1** Error
[21]	0.0084	0.0154	*na*
[76]	0.0001	*na*	0.0752
$M(K_{5,10})$	7.95×10^{-7}	0.00032	1.55×10^{-7}
$M(B_5)$	7.90×10^{-7}	0.00032	1.46×10^{-7}
$M(K_{12,10})$	2.23×10^{-6}	0.000049	1.71×10^{-6}
$M(B_{12})$	2.50×10^{-6}	0.000057	1.41×10^{-6}

Wei [21] for $n = 51$ and the error in the third row last column is from Table 1 ($s = 3$) given in Baker et al. [76]. The data in row 4 and row 5 are obtained by the methods $M(K_{5,10})$, and $M(B_5)$, respectively for $N = 51$, while the results in row 6 , row 7 are achieved by $M(K_{12,10}), M(B_{12})$ accordingly also for $N = 51$.

For the Example 4 the exact solution $f \in C^1[0,1]$. Hence, dealing with this test problem we provide comparisons between the methods $M(K_{n,\alpha})$, and $M(B_n)$ based on the regularization parameter $\eta(\delta)$ taken as δ and on the order n of the approximation in Figure 4.4 and Figure 4.5 respectively. Figure 4.4 shows the *RMSE* with respect to δ obtained by the methods $M(K_{8,0.1})$, $M(K_{8,1}), M(K_{8,10})$, and $M(B_8)$ for the Example 4 of **FK1** when $\varepsilon = 0.0001$ and $N = 51$. It can be viewed that for $\delta \leq 10^{-14}$ the given approach $M(K_{8,1})$, $M(K_{8,10})$ give more accurate results then $M(B_8)$. Figure 4.5 illustrates the *RMSE* with respect to n obtained by the methods $M(K_{n,0.0001}), M(K_{n,0.1}), M(K_{n,1}), M(K_{n,10})$, and $M(B_n)$ for the Example 4 of **FK1** when $\varepsilon = 0.0001$ and $N = 51, \delta = 5 \times 10^{-12}$. This figure show that $K_{n,1}$ and $K_{n,10}$ give more accurate results then B_n for large values of n that is for $n \geq 12$.

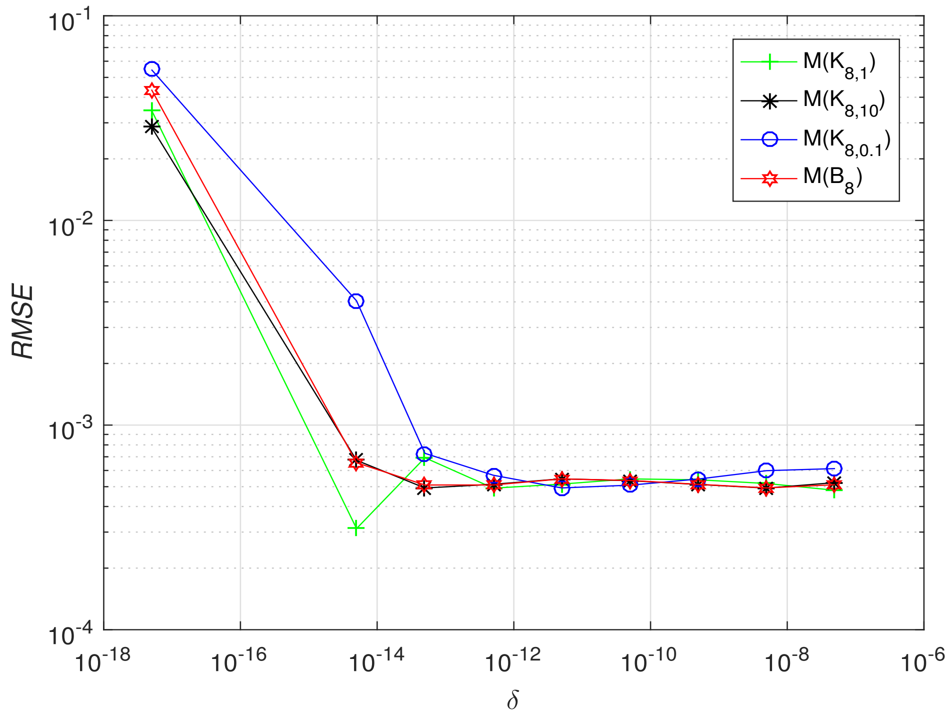

Figure 4.4: The *RMSE* with respect to δ obtained by the methods $M(K_{8,\alpha})$ for $\alpha = 0.0001, 0.1, 1, 10$ and $M(B_8)$ for the Example 4 of **FK1** when $\varepsilon = 0.0001$ and $N = 51$.

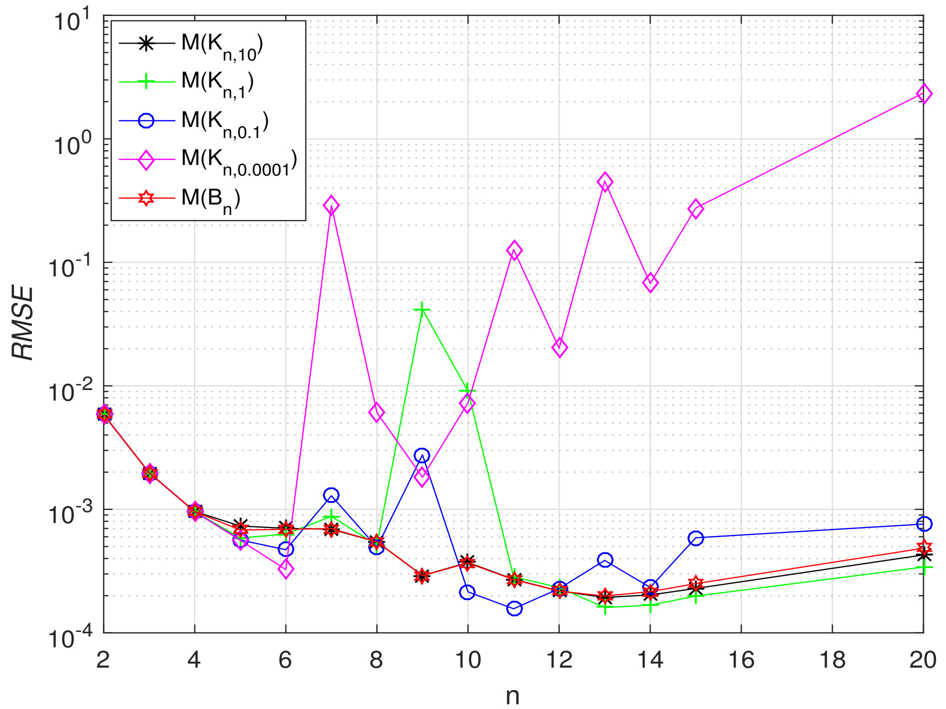

Figure 4.5: The *RMSE* with respect to n obtained by the methods $M(K_{n,\alpha})$ for $\alpha = 0.0001, 0.1, 1, 10$ and $M(B_n)$ for the Example 4 of **FK1** when $\varepsilon = 0.0001$ and $N = 51$

4.2 Applications on Volterra Integral Equations

In this section we also consider the second kind linear Volterra integral equation **(VK2)**.

$$f(x)+\phi\int_0^x K(x,t)f(t)dt=g(x)$$

where $\phi > 0$ is constant. For the numerical solution of **VK2** by the method $M\left(K_{n,\alpha}\right)$ and using the grid points $x_j=\frac{j}{n}+\varepsilon,\ j=0,1,...,n-1$ and $x_n=1-\varepsilon,\ 0<\varepsilon<\frac{1}{2n}$ results the following algebraic system of equations

$$\ddot{A}X=B, \tag{4.5}$$

where coefficient matrix $\ddot{A}$ has the entries

$$\left[\ddot{A}\right]_{j+1,k+1}=\omega(n,\alpha)\left(P_{n,k}(x_j)-\int\limits_0^{x_j}K\left(x_j,t\right)P_{n,k}(t)dt\right),\ j,k=0,1,...,n, \tag{4.6}$$

and the vectors X and B are as in (3.14) and (3.15) respectively. The following examples, are taken

Example 5: ($Ex5$) (Maleknejad et al. [33], Rashad [77])

$$f\left(x\right)-\int\limits_{-1}^{x}xtf\left(t\right)dt=e^{-x^2}-\frac{1}{2}\left(\frac{1}{e}-e^{-x^2}\right)x,\ -1\leq x\leq 1,$$

where the exact solution is $f\left(x\right)=e^{-x^2}$, $-1\leq x\leq 1$.

Example 6: ($Ex6$) (Maleknejad et al. [34], Polyamin [78])

$$f\left(x\right)-\int\limits_0^x e^x f\left(t\right)dt=\cos\left(x\right)-e^x\sin\left(x\right),\ 0\leq x\leq 1,$$

where, the exact solution is $f(x)=\cos(x)$, $0\leq x\leq 1$.

Example 7: ($Ex7$) (Taylor [24], Brunner [25])

$$\int_0^x (1+x-t) f(t)\, dt = x - 1 + e^{-x},$$

where, the exact solution is $f(x) = xe^{-x}$, and $x \in [0,3]$ in Taylor [24] and $x \in [0,10]$ in Brunner [25].

Example 8: $(Ex8)$ (Maleknejad et al. [34], Polyamin [78])

$$\int_0^x e^{x-t} f(t)\, dt = \sin(x), 0 \le x \le 1,$$

where the exact solution is $f(x) = \cos(x) - \sin(x),\ 0 \le x \le 1.$

Remark 4.1: The numerical solution of Example 5 of **VK2** by the method $M(K_{n,\alpha})$ is analogous by using the extension of the Modified Bernstein-Kantorovich operators (4.2) on the interval $[-1,1]$.

Table 4.10 presents the $RMSE$ with respect to n obtained by the proposed approach when $\alpha = 10, (M(K_{n,10}))$ and $\varepsilon = 0.001, N = 100$ for the Example 5, Example 6 of **VK2** and Example 7, Example 8 of **VK1.** Table 4.11 and Table 4.12 show the ME with respect to n obtained by the methods $M(K_{n,10})$ and $M(B_n)$ respectively when $\varepsilon = 0.001, N = 100$ for the considered examples of **VK2** and **VK1**.

From Table 4.10, Table 4.11 and Table 4.12 we conclude that the error is not improved for $n = 20$ for the Examples 6-8 due to the large condition numbers of the coefficient matrices. Table 4.13 demonstrates the $RMSE$ with respect to α obtained by the proposed approach when $n = 20$, and $\varepsilon = 0.001, N = 100$ for the considered examples of **VK2** and **VK1.** This Table shows that $M(K_{20,\alpha})$ gives stable solution with respect to α for the taken values of ε and δ. Further, in Tables 4.10-4.13 for the Example 7, when $x \in [0,3]$ it can be viewed that the results obtained by given approach are as good as the results obtained by using Bernstein operators.

Table 4.10: The *RMSE* for the Example 5, Example 6 of **VK2** and Example 7, Example 8 of **VK1** with respect to n when $\varepsilon = 0.001$, $N = 100$ obtained by $M(K_{n,10})$.

n	$Ex5$ VK2 $RE_{\widetilde{E}_{100}}(K_{n,10})$	$Ex6$ VK2 $RE_{\widetilde{E}_{100}}(K_{n,10})$	$Ex7$ VK1 $RE_{\widetilde{E}_{100}}(K_{n,10})$	$Ex8$ VK1 $RE_{\widetilde{E}_{100}}(K_{n,10})$
2	0.0499147	0.00324785	0.112084	0.00859961
3	0.0365226	0.00067710	0.0323648	0.00019578
4	0.00464224	8.702×10^{-6}	0.00714176	0.00003599
5	0.00317264	1.214×10^{-6}	0.00130879	5.551×10^{-7}
6	0.000392663	1.494×10^{-8}	0.000206789	8.283×10^{-8}
7	0.000263753	1.646×10^{-9}	0.0000287574	9.974×10^{-10}
8	0.0000299799	1.815×10^{-11}	3.571×10^{-6}	1.234×10^{-10}
9	0.0000202637	1.626×10^{-12}	4.002×10^{-7}	1.205×10^{-12}
10	2.0539×10^{-6}	2.734×10^{-14}	4.087×10^{-8}	6.730×10^{-14}
11	1.402×10^{-6}	1.140×10^{-14}	3.831×10^{-9}	1.464×10^{-13}
12	1.266×10^{-7}	1.684×10^{-14}	3.311×10^{-10}	1.895×10^{-13}
13	8.720×10^{-8}	6.657×10^{-15}	2.606×10^{-11}	1.676×10^{-13}
14	7.060×10^{-9}	5.783×10^{-15}	1.321×10^{-12}	3.036×10^{-13}
15	4.900×10^{-9}	2.555×10^{-14}	1.088×10^{-12}	3.113×10^{-13}
20	1.183×10^{-12}	3.155×10^{-12}	2.893×10^{-10}	1.767×10^{-10}

Table 4.11: The *ME* for the Example 5, Example 6 of **VK2** and Example 7, Example 8 of **VK1** with respect to n when $\varepsilon = 0.001$, $N = 100$ obtained by $M(K_{n,10})$.

n	$Ex5$ VK2 $ME_{\widetilde{E}_{100}}(K_{n,10})$	$Ex6$ VK2 $ME_{\widetilde{E}_{100}}(K_{n,10})$	$Ex7$ VK1 $ME_{\widetilde{E}_{100}}(K_{n,10})$	$Ex8$ VK1 $ME_{\widetilde{E}_{100}}(K_{n,10})$
2	0.077679	0.00774279	0.369287	0.0314081
3	0.0651687	0.00140263	0.13734	0.000817017
4	0.0106545	0.0000185564	0.0365845	0.000189993
5	0.00768217	2.947×10^{-6}	0.00771289	3.251×10^{-6}
6	0.0011101	3.299×10^{-8}	0.00135407	5.489×10^{-7}
7	0.00079366	4.587×10^{-9}	0.000204093	7.069×10^{-9}
8	0.0000986807	4.858×10^{-11}	0.0000269878	9.389×10^{-10}
9	0.0000699759	5.065×10^{-12}	3.180×10^{-6}	9.587×10^{-12}
10	7.625×10^{-6}	6.628×10^{-14}	3.382×10^{-7}	4.682×10^{-13}
11	5.455×10^{-6}	2.687×10^{-14}	3.276×10^{-8}	8.972×10^{-13}
12	5.121×10^{-7}	4.241×10^{-14}	2.909×10^{-9}	1.193×10^{-12}
13	3.735×10^{-7}	2.498×10^{-14}	2.341×10^{-10}	1.487×10^{-12}
14	3.143×10^{-8}	1.110×10^{-14}	1.101×10^{-11}	2.171×10^{-12}
15	2.277×10^{-8}	1.052×10^{-13}	8.665×10^{-12}	2.123×10^{-12}
20	6.127×10^{-12}	1.908×10^{-11}	2.800×10^{-9}	1.704×10^{-9}

Table 4.12: The ME for the Example 5, Example 6 of **VK2** and Example 7, Example 8 of **VK1** obtained by $M(B_n)$ for $N = 100$.

n	$Ex5$ VK2 $ME_{\widetilde{E}_{100}}(B_n)$	$Ex6$ VK2 $ME_{\widetilde{E}_{100}}(B_n)$	$Ex7$ VK1 $ME_{\widetilde{E}_{100}}(B_n)$	$Ex8$ VK1 $ME_{\widetilde{E}_{100}}(B_n)$
2	0.077679	0.00774279	0.369287	0.0314081
3	0.0651687	0.00140263	0.13734	0.000817017
4	0.0106545	0.0000185564	0.0365845	0.000189993
5	0.00768217	2.947×10^{-6}	0.00771289	3.251×10^{-6}
6	0.0011101	3.299×10^{-8}	0.00135407	5.488×10^{-7}
7	0.00079366	4.588×10^{-9}	0.000204093	7.071×10^{-9}
8	0.0000986807	4.859×10^{-11}	0.0000269878	9.395×10^{-10}
9	0.0000699759	5.022×10^{-12}	3.180×10^{-6}	9.609×10^{-12}
10	7.624×10^{-6}	5.440×10^{-14}	3.382×10^{-6}	8.811×10^{-13}
11	5.455×10^{-6}	2.065×10^{-14}	3.276×10^{-8}	2.442×10^{-14}
12	5.121×10^{-7}	2.554×10^{-14}	2.910×10^{-9}	3.484×10^{-13}
13	3.735×10^{-7}	2.809×10^{-14}	2.340×10^{-10}	8.926×10^{-13}
14	3.143×10^{-8}	4.508×10^{-14}	1.265×10^{-11}	4.927×10^{-13}
15	2.277×10^{-8}	1.196×10^{-13}	7.290×10^{-12}	3.042×10^{-13}
20	6.659×10^{-12}	1.908×10^{-11}	2.729×10^{-9}	1.659×10^{-9}

Table 4.13: The $RMSE$ for the Example 5, Example 6 of **VK2**, Example 7 and Example 8 of **VK1** with respect to α when $\varepsilon = 0.001$ and $N = 100$ obtained by $M(K_{20,\alpha})$.

α	$Ex5$ VK2 $RE_{\widetilde{E}_{100}}(K_{20,\alpha})$	$Ex6$ VK2 $RE_{\widetilde{E}_{100}}(K_{20,\alpha})$	$Ex7$ VK1 $RE_{\widetilde{E}_{100}}(K_{20,\alpha})$	$Ex8$ VK1 $RE_{\widetilde{E}_{100}}(K_{20,\alpha})$
0.0001	1.146×10^{-12}	3.174×10^{-12}	3.853×10^{-10}	3.340×10^{-9}
0.001	1.144×10^{-12}	3.188×10^{-12}	3.876×10^{-10}	3.341×10^{-9}
0.01	1.101×10^{-12}	3.189×10^{-12}	3.857×10^{-10}	3.346×10^{-9}
0.1	1.149×10^{-12}	3.116×10^{-12}	3.861×10^{-10}	3.343×10^{-9}
1	1.183×10^{-12}	1.183×10^{-12}	2.820×10^{-10}	3.342×10^{-9}
10	1.137×10^{-12}	3.181×10^{-12}	2.879×10^{-10}	3.341×10^{-9}
100	1.253×10^{-12}	3.244×10^{-12}	2.856×10^{-10}	3.316×10^{-9}
1000	1.136×10^{-12}	3.245×10^{-12}	2.856×10^{-10}	3.093×10^{-9}

The absolute errors for the Example 5 of **VK2** when $N = 8$ (9 points) obtained by the methods $M(K_{20,10})$ and $M(B_{20})$ are presented in Table 4.14. Additionally, in Table 4.15 absolute errors obtained by the methods $M(K_{10,10})$ and $M(B_{10})$ and by the approach given in Maleknejad et al. [33] (presented in the last column 3 of Table 1 in [33]) for the same example over the same grid points are compared when $n = 10$. It can be concluded from this table that the maximum error (ME) is 1.59792×10^{-6} by the methods $M(K_{10,10})$, $M(B_{10})$ and it is 1.593×10^{-6} by the method in Maleknejad et al. [33] and occurs at the same grid point $x_7 = 0.75$. Furthermore, Table 4.14 shows that the maximum error decreases down to 8.88623×10^{-13} by $M(K_{20,10})$ and to 9.83824×10^{-13} by $M(B_{20})$ over the same grid points. Table 4.16 shows the absolute errors (AE) at 7 points ($N = 6$) from the interval $x \in [0,3]$ for the Example 7 obtained by the methods $M(K_{15,10})$ and $M(B_{15})$ and by the method given in Taylor [24] (presented in the last column of Table 2 in [24]).

Table 4.17 gives AE at the points $x_p = p,\ p = 0,1,2,3,4,5$ from the interval $x \in [0,10]$ for the Example 7 obtained by the methods $M(K_{15,10})$ and $M(B_{15})$ and by the method given in Brunner [25] (given in the second column of Table 3.11 in [25]). We conclude from Table 4.16 and Table 4.17 that the presented AE by $M\left(K_{15,10}\right)$ are smaller than the given values from Taylor [24] and Brunner [25] respectively. However we should remark that precision of the computations were not mentioned in both of these references. Also $\delta = 5 \times 10^{-15}$ for the all considered examples of **VK2** and **VK1** by the methods $M(K_{n,\alpha})$ and $M(B_n)$.

Figure 4.6 illustrates the condition number of the matrix $\widetilde{\widehat{A}}$ in (3.50) when the method $M(K_{n,10})$ is applied for $n = 2,...,20$. The $RMSE$ and ME with respect to n obtained by the methods $M(K_{n,1})$, $M(K_{n,10})$ and $M(B_n)$ for the Example 5, Example 6 of **VK2** and

Table 4.14: The absolute errors at 9 points for the Example 5 of **VK2** obtained by the methods $M(K_{20,10})$ and $M(B_{20})$.

x_p	$AE_{\widetilde{E}_8,x_p}(K_{20,10})$	$AE_{\widetilde{E}_8,x_p}(B_{20})$
-1.0	6.667×10^{-13}	7.392×10^{-13}
-0.75	2.890×10^{-13}	3.290×10^{-13}
-0.50	1.831×10^{-13}	2.097×10^{-13}
-0.25	9.137×10^{-14}	1.081×10^{-13}
0	6.439×10^{-15}	1.332×10^{-15}
0.25	1.034×10^{-13}	1.106×10^{-13}
0.50	2.086×10^{-13}	2.295×10^{-13}
0.75	2.984×10^{-13}	3.302×10^{-13}
1.0	8.886×10^{-13}	9.838×10^{-13}

Table 4.15: Comparison of the absolute errors at 9 points for the Example 5 of **VK2** obtained by the methods $M(K_{10,10})$, $M(B_{10})$ and by the approach in Maleknejad et al. [33].

x_p	$AE_{\widetilde{E}_8,x_p}(K_{10,10})$	$AE_{\widetilde{E}_8,x_p}(B_{10})$	Maleknejad et al. [33]
-1.0	3.436×10^{-7}	3.436×10^{-7}	3.524×10^{-9}
-0.75	1.218×10^{-7}	1.218×10^{-7}	1.144×10^{-7}
-0.50	5.820×10^{-7}	5.820×10^{-7}	5.431×10^{-7}
-0.25	2.066×10^{-7}	2.066×10^{-7}	2.922×10^{-7}
0	2.805×10^{-11}	2.805×10^{-11}	0
0.25	2.536×10^{-7}	2.536×10^{-7}	3.396×10^{-7}
0.50	3.212×10^{-7}	3.212×10^{-7}	2.902×10^{-7}
0.75	1.598×10^{-6}	1.598×10^{-6}	1.593×10^{-6}
1.0	9.109×10^{-7}	9.109×10^{-7}	7.823×10^{-7}

Table 4.16: Comparison of the absolute errors at 7 points for the Example 7 of **VK1** obtained by the methods $M(K_{15,10})$, $M(B_{15})$ and by the approach in Taylor [24].

x_p	$AE_{\widetilde{E}_6,x_p}\left(K_{15,10}\right)$	$AE_{\widetilde{E}_6,x_p}\left(B_{15}\right)$	Taylor [24]
0	5.112×10^{-12}	5.108×10^{-12}	*ng*
0.5	7.105×10^{-15}	7.494×10^{-15}	2.7×10^{-7}
1.0	1.499×10^{-15}	2.609×10^{-15}	4.3×10^{-5}
1.5	1.332×10^{-15}	2.887×10^{-15}	*ng*
2.0	4.219×10^{-15}	4.996×10^{-15}	2.3×10^{-5}
2.5	1.219×10^{-14}	1.355×10^{-14}	*ng*
3.0	6.841×10^{-12}	7.290×10^{-12}	1.8×10^{-5}

Table 4.17: Comparison of the absolute errors when $N=10$ for the Example 7 of **VK1** obtained by the methods $M(K_{15,10})$, $M(B_{15})$ and by the approach from Brunner [25].

x_p	$AE_{\widetilde{E}_{10},x_p}\left(K_{15,10}\right)$	$AE_{\widetilde{E}_{10},x_p}\left(B_{15}\right)$	Brunner [25]
0	1.276×10^{-9}	1.276×10^{-9}	1.244×10^{-7}
1.0	2.069×10^{-8}	2.069×10^{-8}	3.128×10^{-8}
2.0	4.418×10^{-9}	4.418×10^{-9}	6.183×10^{-9}
3.0	3.748×10^{-10}	3.748×10^{-10}	*ng*
4.0	1.062×10^{-9}	1.062×10^{-9}	*ng*
5.0	1.870×10^{-10}	1.870×10^{-10}	4.87×10^{-10}

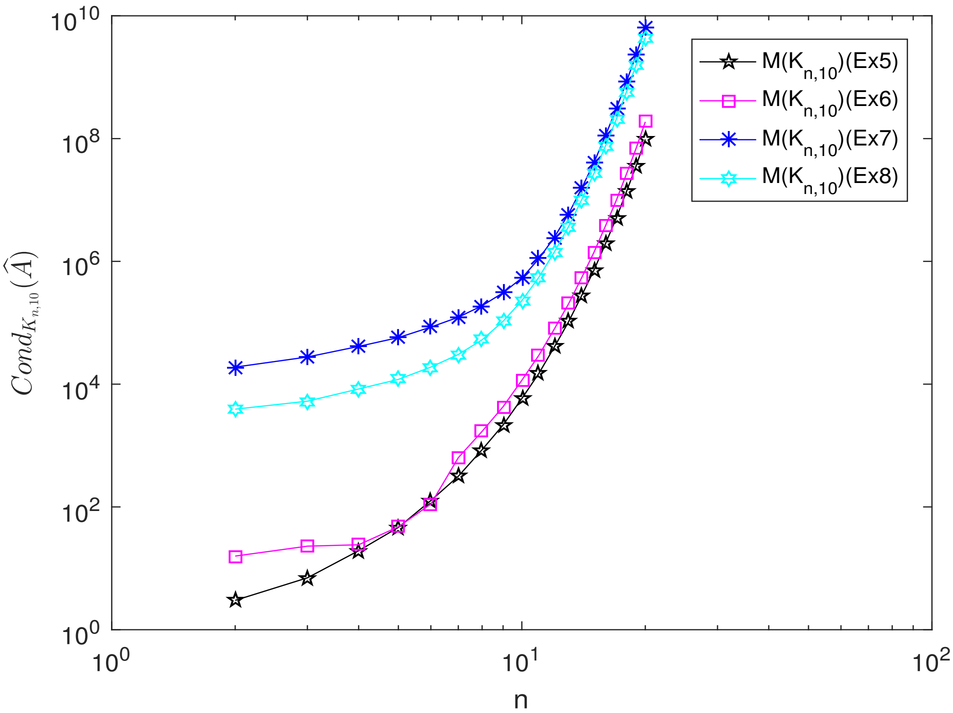

Figure 4.6: Condition number of the matrix $\widetilde{\widehat{A}}$ with respect to n obtained by the method $M(K_{n,10})$.

Example 7 and Example 8 of **VK1,** when $\varepsilon = 0.001$, and $N = 100$ are given in Figure 4.7 and Figure 4.8 respectively. Also, for the data in Figure 4.6, Figure 4.7 and Figure 4.8 the parameter δ is taken as $\delta = 5 \times 10^{-15}$ for the considered examples of **VK2** and **VK1**. It can be viewed from Figure 4.7 that for large n that is $n \geq 10$ the proposed method $M(K_{n,\alpha})$ for $\alpha = 1$ and $\alpha = 10$ gives more stable results than $M(B_n)$ for the Example 6 of **VK2**.

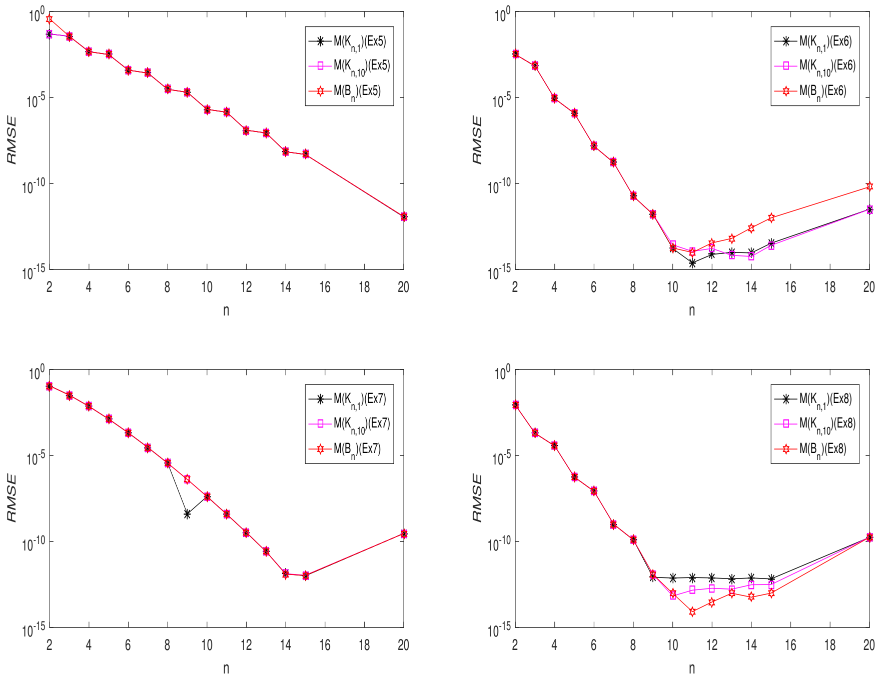

Figure 4.7: The $RMSE$ with respect to n obtained by the methods $M(K_{n,1}), M(K_{n,10})$ and $M(B_n)$ for the Example 5, Example 6 of **VK2** and Example 7 and Example 8 of **VK1** when $\varepsilon = 0.001$ and $N = 100$.

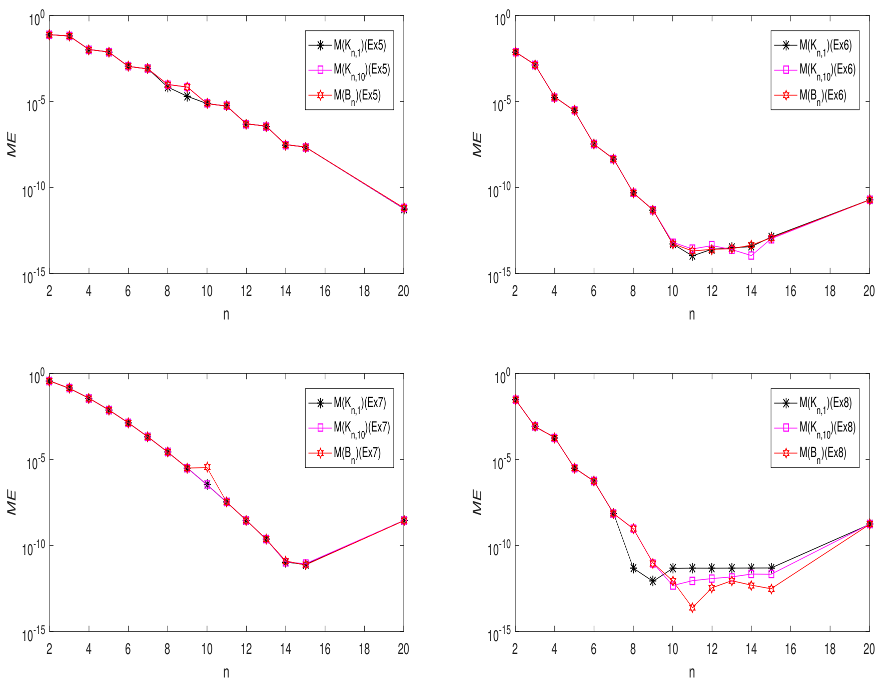

Figure 4.8: The ME with respect to n obtained by the methods $M(K_{n,1}), M(K_{n,10})$ and $M(B_n)$ for the Example 5, Example 6 of **VK2** and Example 7 and Example 8 of **VK1** when $\varepsilon = 0.001$ and $N = 100$.

Chapter 5

A COMBINED METHOD OF HYBRID OPERATORS FOR THE NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS WITH WEAK SINGULARITIES

In this chapter , we consider Abel-type integral equations of the second kind. The hybrid operators are defined on $[0,1]$ by using classical Bernstein-Kantorovich operators and Modified Bernstein-Kantorovich operators $K_{n,\alpha}$ where, $n \in \mathbb{N}$ and $\alpha > 0$ is constant. Further, for the numerical solution of the Abel-type integral equations of the second kind two algorithms are developed by giving a combined method for the values of $0 < \alpha < 1$ and $\alpha > 1$. Additionally, the first kind Volterra Abel-type integral equations are also numerically investigated by first utilizing a regularization and then applying the given algorithms to the yielded second kind equations. Eventually, we give the convergence analysis of the constructed algorithms.

5.1 Assumptions and Smoothness Results

We start with the existing useful definitions and results which are crucial in our investigation.

Definition 5.1: (Brunner et al. [79]) Let

$$D = \left\{(x,t) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, \ 0 \leq t < x\right\}. \tag{5.1}$$

The set $W^{k,v}(D)$ with $k \in \mathbb{N}$, $v \in \mathbb{R}$, $v < 1$ consists of all k times continuously differentiable functions $K : D \to \mathbb{R}$ satisfying

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^j K(x,t) \right| \le c \begin{cases} 1, & \text{if } v+i < 0, \\ 1 + |\log(x-t)|, & \text{if } v+i = 0, \\ (x-t)^{-v-i}, & \text{if } v+i > 0, \end{cases} \tag{5.2}$$

with a constant $c = c(K)$ for all $(x,t) \in D$ and all nonnegative integers i and j such that $i + j \le k$.

Definition 5.2: (Brunner et al. [79]) Let $I := [0,1]$ and $C^{k,v}(0,1]$, $k \in \mathbb{N}$, $v \in \mathbb{R}$, $v < 1$ be the set of all functions $f : I \to \mathbb{R}$, which are k times continuously differentiable in $(0,1]$ and such that the estimation

$$\left| f^{(i)}(x) \right| \le c \begin{cases} 1, & \text{if } i < 1-v, \\ 1 + |\log x|, & \text{if } i = 1-v, \\ x^{1-v-i}, & \text{if } i > 1-v, \end{cases} \tag{5.3}$$

holds with a constant $c = c(f)$ for all $x \in (0,1]$ and $i = 0, 1, ..., k$.

We consider the linear second kind Volterra Abel-type integral equations (**VAK2**) of the form

$$f(x) + \phi \int_0^x (x-t)^{-v} \widetilde{K}(x,t) f(t)\, dt = g(x),\ 0 \le x \le 1, \tag{5.4}$$

where, $0 < v < 1$, ϕ is constant while $K(x,t) = (x-t)^{-v}\, \widetilde{K}(x,t)$ is the kernel with the singular part $\varphi(x,t) = (x-t)^{-v}$. It follows from (5.2) that the kernel $K(x,t)$ of (5.4) possess a weak singularity as $t \to x$ and that if $\widetilde{K} \in C^k\left(\overline{D}\right)$ where $\overline{D} = \left\{(x,t) \in \mathbb{R}^2 :\ 0 \le t \le x \le 1\right\}$ then $K \in W^{k,v}(D)$. Further we denote the space $C(I) \cap C^k(0,1]$ by $C_I^k(0,1]$.

Theorem 5.1: (Brunner [41]) For the **VAK2** in (5.4) assume $g \in C^k(I)$ and $\widetilde{K} \in$

$C^k\left(\overline{D}\right)$, $k \in \mathbb{N}$. Then the integral equation (5.4) has unique solution $f \in C_I^k(0,1]$.

The smoothness result of the Theorem 5.1 is sharp in the following sense that even if $g \in C^\infty(I)$ the solution of **VAK2** have, in general singularities allowed by the space $C_I^k(0,1]$ (see also Kangro and Kangro [80]). Furthermore, it was also given in Brunner et al. [79] that when $g \in C^{k,v}(0,1]$, and $\widetilde{K} \in C^k\left(\overline{D}\right)$ then unique exact solution f of (5.4) belongs to $C^{k,v}(0,1]$. On the basis of Theorem 5.1 we assume that the following Hypothesis holds.

Hypothesis C: For the **VAK2** integral equations we assume $\widetilde{K}(x,t) \in C^2\left(\overline{D}\right)$ and $g \in C^2(I)$, and that neither function vanishes identically.

5.2 Hybrid Operators Defined by Bernstein Kantorovich and Modified-Bernstein Kantorovich Operators

The Modified Bernstein-Kantorovich operators $K_{n,\alpha}(f;x)$ where $\alpha > 0$ is constant were given to approximate a function $f : I \to \mathbb{R}$ (see Özarslan and Duman [68]). The extension of these operators on the interval $[a,b]$

$$K_{n,\alpha}(f;x) = \frac{1}{b-a}\sum_{k=0}^{n} P_{a,b}^{n,k}(x)\int_{a}^{b} f\left(a + \frac{k+\left(\frac{t-a}{b-a}\right)^{\alpha}}{(n+1)}(b-a)\right)dt, \tag{5.5}$$

where

$$P_{a,b}^{n,k}(x) = \binom{n}{k}\frac{(x-a)^{k}(b-x)^{n-k}}{(b-a)^{n}}, \tag{5.6}$$

was considered in Buranay et al. [65] for the solution of Fredholm and Volterra integral equations. For $\alpha = 1$, $K_{n,\alpha}(f;x)$ in (5.5) reduces to the classical Bernstein-Kantorovich operator

$$K_{n}(f;x) = \frac{1}{b-a}\sum_{k=0}^{n} P_{a,b}^{n,k}(x)\int_{a}^{b} f\left(a + \frac{k+\frac{t-a}{b-a}}{(n+1)}(b-a)\right)dt, \tag{5.7}$$

Lemma 5.1: (see Özarslan and Duman [68]) For any $\alpha > 1$ the order of approximation of $K_{n,\alpha}(f;x)$ to $f(x)$ is at least as good as that of the classical Bernstein-Kantorovich operators whenever $x \in \left[0, \frac{1}{3}\right)$. If $x \in \left(\frac{2}{3}, 1\right]$ then for any $0 < \alpha < 1$ the order of approximation of $K_{n,\alpha}(f;x)$ to $f(x)$ is at least as good as that of the classical Bernstein-Kantorovich operators.

Definition 5.3: Fix $\alpha > 0$ and let $0 < \widehat{\varepsilon} < \lambda$, where $\lambda = \frac{2\alpha+2}{6\alpha+3}$. For every $f \in C_{I}^{2}(0,1]$ we define the hybrid operators in three step as follows when $0 < \alpha < 1$

$$H_{m,\alpha}^{1-}(f;x) = \begin{cases} K_{n_0}(f;x) \text{ if } x \in [0,\widehat{\varepsilon}], \\ K_{n_1}(f;x) \text{ if } x \in [\widehat{\varepsilon},\lambda], \\ K_{n_2,\alpha}(f;x) \text{ if } x \in [\lambda,1], \end{cases} \tag{5.8}$$

and $\alpha > 1$

$$H_{m,\alpha}^{1+}(f;x) = \begin{cases} K_{n_0,\alpha}(f;x) \text{ if } x \in [0,\widehat{\varepsilon}], \\ K_{n_1,\alpha}(f;x) \text{ if } x \in [\widehat{\varepsilon},\lambda], \\ K_{n_2}(f;x) \text{ if } x \in [\lambda,1], \end{cases} \tag{5.9}$$

where, $m = n_0 + n_1 + n_2$ are positive integers.

Lemma 5.2: Let $e_i = t^i, i = 1,2$ for $t \in [a,b]$. Then for each fixed $n \in \mathbb{N}, \alpha > 0$,

$$K_{n,\alpha}(1;x) = 1, \tag{5.10}$$

$$K_{n,\alpha}(e_1;x) = \frac{nx+a}{n+1} + \frac{b-a}{(\alpha+1)(n+1)}, \tag{5.11}$$

$$\begin{aligned} K_{n,\alpha}(e_2;x) = {} & \frac{n(n-1)x^2}{(n+1)^2} + \left(n(3a+b) + \frac{2n(b-a)}{(\alpha+1)}\right)\frac{x}{(n+1)^2} \\ & + \frac{1}{(n+1)^2}\left(a^2 - abn + \frac{2a(b-a)}{(\alpha+1)}\right) \\ & + \frac{(b-a)^2}{(2\alpha+1)(n+1)^2}. \end{aligned} \tag{5.12}$$

Proof. The Bernstein operators for $f \in C[a,b]$ was given in Equation (4.1) as:

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} f(a + \frac{k}{n}(b-a))$$

and the proof of (5.10) is obtained by using

$$B_n(1;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} = 1$$

$$\begin{aligned} K_{n,\alpha}(1,x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \frac{1}{(b-a)} \int_a^b dt \\ &= B_n(1;x) \end{aligned}$$

Next the following summations are needed

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \frac{k}{n} = \frac{x-a}{b-a} \tag{5.13}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \left(\frac{k}{n}\right)^2 = \frac{1}{(b-a)^2} \left[x^2 \left(\frac{n-1}{n}\right) + x\left(\frac{b+a}{n} - 2a\right) - \frac{ab}{n} + a^2 \right]. \tag{5.14}$$

Consequently, using (5.5) it follows that

$$\begin{aligned} K_{n,\alpha}(t,x) &= \frac{1}{b-a} \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \\ &\times \int_a^b \left(a + \frac{k + \left(\frac{t-a}{b-a}\right)^\alpha}{n+1} (b-a) \right) dt \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \left(a + \frac{k(b-a)}{n+1} \right. \\ &+ \left. \frac{b-a}{(\alpha+1)(n+1)} \right) \\ &= a + \frac{n(x-a)}{(n+1)} + \frac{(b-a)}{(\alpha+1)(n+1)} \\ &= \frac{a+nx}{(n+1)} + \frac{(b-a)}{(\alpha+1)(n+1)} \end{aligned} \tag{5.15}$$

From (5.14) and (5.15) we obtain (5.11). Further,

$$\begin{aligned} K_{n,\alpha}(t^2,x) &= \frac{1}{b-a} \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \\ &\times \left(\int_a^b \left(a + \frac{k + (\frac{t-a}{b-a})^\alpha}{n+1} (b-a) \right)^2 dt \right) \\ &= \frac{1}{b-a} \sum_{k=0}^{n} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n} \\ &\times \left(\int_a^b \left(a^2 + 2a \frac{k + (\frac{t-a}{b-a})^\alpha}{n+1} (b-a) \right) dt \right. \\ &+ \left. \left(\frac{b-a}{n+1}\right)^2 \int_a^b \left(k + (\frac{t-a}{b-a})^\alpha \right)^2 dt \right) \end{aligned} \tag{5.16}$$

Transforming the integrating variable from t to $u = \frac{t-a}{b-a}$ Equation (5.16) yields

$$
\begin{aligned}
K_{n,\alpha}(t^2,x) &= \sum_{k=0}^{n}\binom{n}{k}\frac{(x-a)^k(b-x)^{n-k}}{(b-a)^n} \\
&\times \int_0^1 \left(a^2+2a\left(\frac{k+u^{\alpha}}{n+1}(b-a)\right)\right. \\
&\left.+\frac{(b-a)^2}{(n+1)^2}\left(k^2+2ku^{\alpha}+u^{2\alpha}\right)du\right. \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{(x-a)^k(b-x)^{n-k}}{(b-a)^n}\left(a^2+\frac{2ak}{n+1}(b-a)\right. \\
&\left.+\frac{2a(b-a)}{(\alpha+1)(n+1)}+\frac{(b-a)^2}{(n+1)^2}\left(k^2+\frac{2k}{(\alpha+1)}+\frac{1}{2\alpha+1}\right)\right). \qquad (5.17)
\end{aligned}
$$

Using (5.13) and (5.14) we get

$$
\begin{aligned}
K_{n,\alpha}(t^2,x) &= a^2+\frac{2an}{n+1}(x-a)+\frac{2a(b-a)}{(\alpha+1)(n+1)} \\
&+\frac{(b-a)^2}{(n+1)^2}\left(\frac{n^2}{(b-a)^2}\left(x^2\left(\frac{n-1}{n}\right)+x\left(\frac{b+a}{n}-2a\right)-\frac{ab}{n}+a^2\right)\right) \\
&+\frac{(b-a)^2}{(n+1)^2}\left(\frac{2n}{\alpha+1}\frac{x-a}{b-a}+\frac{1}{2\alpha+1}\right) \\
&= a^2+\frac{2an}{n+1}(x-a)+\frac{2a(b-a)}{(\alpha+1)(n+1)} \\
&+\frac{n^2}{(n+1)^2}\left(x^2\left(\frac{n-1}{n}\right)+x\left(\frac{b+a}{n}-2a\right)-\frac{ab}{n}+a^2\right) \\
&+\frac{(b-a)}{(n+1)^2}\left(\frac{2n\,(x-a)}{\alpha+1}\right)+\frac{(b-a)^2}{(2\alpha+1)\,(n+1)^2} \\
&= \frac{n(n-1)}{(n+1)^2}x^2+\frac{1}{(n+1)^2}\left(2an\,(n+1)+(b+a)\,n-2an^2+\frac{2(b-a)n}{\alpha+1}\right)x \\
&+\frac{1}{(n+1)^2}\left(a^2\,(n+1)^2-2a^2n\,(n+1)-abn+a^2n^2\right. \\
&\left.-(b-a)\left(\frac{2na}{\alpha+1}\right)+\frac{2a(b-a)\,(n+1)}{(\alpha+1)}\right)+\frac{(b-a)^2}{(2\alpha+1)\,(n+1)^2} \\
&= \frac{n(n-1)}{(n+1)^2}x^2+\frac{1}{(n+1)^2}\left((3a+b)\,n+\frac{2(b-a)n}{\alpha+1}\right)x \\
&+\frac{1}{(n+1)^2}\left(a^2-abn+\frac{2a\,(b-a)}{(\alpha+1)}\right) \\
&+\frac{(b-a)^2}{(n+1)^2}\frac{1}{2\alpha+1} \qquad (5.18)
\end{aligned}
$$

□

Lemma 5.3: For each fixed $n \in \mathbb{N}, \alpha > 0$ and $x \in [a,b]$ we have

$$\sup_{x\in[a,b]} |K_{n,\alpha}((t-x);x)| \leq \frac{\beta_{a,b}(\alpha)}{(n+1)}, \tag{5.19}$$

$$\sup_{x\in[a,b]} \left|K_{n,\alpha}\left((t-x)^2;x\right)\right| \leq \frac{1}{(n+1)^2}\theta_{a,b}(n,\alpha), \tag{5.20}$$

where,

$$\beta_{a,b}(\alpha) = \max\left\{\frac{b-a}{\alpha+1}, \frac{(b-a)\alpha}{\alpha+1}\right\}, \tag{5.21}$$

$$\theta_{a,b}(n,\alpha) = \frac{n}{4}(b-a)^2 + \sigma_{a,b}(\alpha), \tag{5.22}$$

$$\sigma_{a,b}(\alpha) = \max\left\{\left|\varphi^*_{a,b}(\alpha,a)\right|, \left|\varphi^*_{a,b}(\alpha,b)\right|, \left|\varphi^*_{a,b}\left(\alpha, \frac{b+a\alpha}{\alpha+1}\right)\right|\right\}, \tag{5.23}$$

and

$$\begin{aligned} \varphi^*_{a,b}(\alpha,x) &= x^2 + \frac{x}{\alpha+1}(-2(b+a\alpha)) \\ &\quad + a^2 + \frac{2a(b-a)}{\alpha+1} + \frac{(b-a)^2}{(2\alpha+1)}. \end{aligned} \tag{5.24}$$

Proof. On the basis of Lemma 5.2 it follows that

$$K_{n,\alpha}(t-x;x) = \frac{1}{n+1}\left(\frac{b+a\alpha}{\alpha+1} - x\right), \tag{5.25}$$

$$K_{n,\alpha}((t-x)^2;x) = \frac{1}{(n+1)^2}\left(n\psi^*_{a,b}(x) + \varphi^*_{a,b}(\alpha,x)\right), \tag{5.26}$$

where

$$\psi^*_{a,b}(x) = -x^2 + (a+b)x - ab, \tag{5.27}$$

and $\varphi^*_{a,b}(\alpha,x)$ is as given in (5.24). For each fixed $n \in \mathbb{N}, \alpha > 0$ the inequality (5.19) is obtained from the fact that $K_{n,\alpha}(t-x;x)$ takes its absolute maximum value at the end points $x = a$ or $x = b$. Hence, (5.19) follows. The function $\psi^*_{a,b}(x) \geq 0$ for $x \in [a,b]$ and takes its maximum value $\frac{(b-a)^2}{4}$ at $x = \frac{a+b}{2}$. Further $\varphi^*_{a,b}(\alpha,x)$ is quadratic function of x and $\sup_{x\in[a,b]}\left|\varphi^*_{a,b}(\alpha,x)\right| = \sigma_{a,b}(\alpha)$ where $\sigma_{a,b}(\alpha)$ is given in (5.23) thus (5.20) follows. □

Next, we use the notations $\|f\|_{[a,b]} = \sup\limits_{x\in[a,b]} |f|$ to present the maximum norm of $f \in C[a,b]$. Further we use $f' := \frac{df}{dx}, f'' := \frac{d^2f}{dx^2}$ and denote $\|Y\|_\infty = \max\limits_{1\le k\le N} |Y(k)|$ and $\|P\|_\infty = \max\limits_{1\le j\le N} \sum\limits_{k=1}^{N} \left|P_{j,k}\right|$ to present the discrete maximum norm of a vector Y in N dimensional real vector space R^N, and the maximum norm of $N\times N$ real matrix $P \in R^{N\times N}$ respectively.

Theorem 5.2: Let $0 < \widehat{\varepsilon} < \lambda$ where $\lambda = \frac{2\alpha+2}{6\alpha+3}$. Assume that the Hypothesis C holds, and f is the unique exact solution of (5.4) then there exist positive constants c_α^*, c_1^* both independent from $n_0 \in \mathbb{N}$ such that the following inequalities hold true

$$\sup_{x\in[0,\widehat{\varepsilon}]} \left|H_{m,\alpha}^{1+}(f;x) - f(x)\right| \le c_\alpha^* (n_0+1)^{\nu-1} = O\left(\frac{1}{n_0^{1-\nu}}\right). \tag{5.28}$$

$$\sup_{x\in[0,\widehat{\varepsilon}]} \left|H_{m,\alpha}^{1-}(f;x) - f(x)\right| \le c_1^* (n_0+1)^{\nu-1} = O\left(\frac{1}{n_0^{1-\nu}}\right). \tag{5.29}$$

Proof. Assume that the Hypothesis C holds. From Theorem 5.1 it follows that $f \in C_I^2(0,1]$. More precisely it has the form (see Brunner [41])

$$f(t) = g(t) + \sum_{s=1}^{\infty} \varphi_s(t)\, t^{s(1-\nu)},\ t \in I, \tag{5.30}$$

where $\varphi_s \in C^2(I)$, ($s \ge 1$ and note that $C^2(I) \subset C^{2,\nu}(0,1]$) and the series converges absolutely and uniformly on I. If ν is rational that is $\nu = \frac{\gamma}{q}$ with γ, q coprime then (5.30) can be written as

$$f(t) = h_0(t) + \sum_{s=1}^{q-1} h_s(t)\, t^{s(1-\nu)},\ t \in I, \tag{5.31}$$

with $h_s \in C^2(I)$ $(0 \le s \le q-1)$. For the sake of simplicity of notation we shall give the proofs for the case of rational ν; the generalization of the ideas involved in the subsequent arguments to irrational ν is straightforward. On the interval $[0,\widehat{\varepsilon}]$ the function f is not continuously differentiable unless $f(t) \equiv 0$. Further, from the assumptions in Hypothesis C, f is not identically zero. Moreover, since $h_s \in C^2(I)$

using Taylor expansion at x it follows that

$$h_s(t) = c_{1,s} + c_{2,s}(t-x) + R_s(t-x),\ t \in [0, \widehat{\varepsilon}], \tag{5.32}$$

where,

$$c_{1,s} := h_s(x),\ c_{2,s} := h_s^{'}(x) \tag{5.33}$$

$$R_s(t-x) = \frac{h_s^{''}(\xi_s)(t-x)^2}{2!},\ 0 < \xi_s < t. \tag{5.34}$$

From (5.31) and (5.32) we get

$$\begin{aligned} f(t) &= h_0(x) + h_0^{'}(x)(t-x) + R_0(t-x) \\ &+ \sum_{s=1}^{q-1} h_s(x) t^{s(1-v)} + \sum_{s=1}^{q-1} h_s^{'}(x) t^{s(1-v)}(t-x) \\ &+ \sum_{s=1}^{q-1} R_s(t-x) t^{s(1-v)}\ . \end{aligned} \tag{5.35}$$

For the proof of (5.28) on the base of (5.9) we use $H_{m,\alpha}^{1+}(f;x) = K_{n_0,\alpha}(f;x)$ for $x \in [0, \widehat{\varepsilon}]$ and from (5.35) it follows that

$$\begin{aligned} H_{m,\alpha}^{1+}(f;x) &= h_0(x) + h_0^{'}(x) K_{n_0,\alpha}((t-x);x) \\ &+ K_{n_0,\alpha}(R_0(t-x);x) + \sum_{s=1}^{q-1} h_s(x) K_{n_0,\alpha}\left(t^{s(1-v)};x\right) \\ &+ \sum_{s=1}^{q-1} h_s^{'}(x) K_{n_0,\alpha}\left(t^{s(1-v)}(t-x);x\right) \\ &+ \sum_{s=1}^{q-1} K_{n_0,\alpha}\left(R_s(t-x) t^{s(1-v)};x\right)\ . \end{aligned} \tag{5.36}$$

Next for $r \in Z^+ \cup \{0\}$

$$K_{n_0,\alpha}\left(t^{s(1-v)+r};x\right)=\frac{1}{\widehat{\varepsilon}}\sum_{k=0}^{n_0}P_{0,\widehat{\varepsilon}}^{n_0,k}(x)\int_0^{\widehat{\varepsilon}}\left(\frac{k+\left(\frac{t}{\widehat{\varepsilon}}\right)^{\alpha}}{n_0+1}\widehat{\varepsilon}\right)^{s(1-v)+r}dt$$

$$=\frac{1}{\widehat{\varepsilon}}P_{0,\widehat{\varepsilon}}^{n_0,0}(x)\int_0^{\widehat{\varepsilon}}\left(\frac{\left(\frac{t}{\widehat{\varepsilon}}\right)^{\alpha}}{n_0+1}\widehat{\varepsilon}\right)^{s(1-v)+r}dt$$

$$+\frac{\widehat{\varepsilon}^{s(1-v)+r-1}}{(n_0+1)^{s(1-v)+r}}\sum_{k=1}^{n_0}P_{0,\widehat{\varepsilon}}^{n_0,k}(x)\int_0^{\widehat{\varepsilon}}\left(k+\left(\frac{t}{\widehat{\varepsilon}}\right)^{\alpha}\right)^{s(1-v)+r}dt. \quad (5.37)$$

Then, for $k,s\in\mathbb{Z}^+$ and $r\in\mathbb{Z}^+\bigcup\{0\}$ using the integral

$$\frac{1}{\widehat{\varepsilon}}\int_0^{\widehat{\varepsilon}}\left(k+\left(\frac{t}{\widehat{\varepsilon}}\right)^{\alpha}\right)^{s(1-v)+r}dt=$$

$$k^{s(1-v)+r}\,{}_2F_1\left(\frac{1}{\alpha},s(1-v)+r,\frac{1}{\alpha}+1,\frac{-1}{k}\right), \quad (5.38)$$

where ${}_2F_1(a,b,c,z)$ is the Hypergeometric function given as

$${}_2F_1(a,b,c,z)=1+\frac{ab}{c}z$$

$$+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\ldots \quad (5.39)$$

and substituting (5.38) and (5.39) into (5.37) we get

$$K_{n_0,\alpha}\left(t^{s(1-v)+r};x\right)=\frac{\widehat{\varepsilon}^{s(1-v)+r}}{(n_0+1)^{s(1-v)+r}}\left(\frac{P_{0,\widehat{\varepsilon}}^{n_0,0}(x)}{\alpha(s(1-v)+r)+1}\right.$$

$$\left.+\sum_{k=1}^{n_0}P_{0,\widehat{\varepsilon}}^{n_0,k}k^{s(1-v)+r}\,{}_2F_1\left(\frac{1}{\alpha},(s(1-v)+r),\frac{1}{\alpha}+1,\frac{-1}{k}\right)\right)$$

$$=\frac{\widehat{\varepsilon}^{s(1-v)+r}}{(n_0+1)^{s(1-v)+r}}\left(\frac{P_{0,\widehat{\varepsilon}}^{n_0,0}(x)}{\alpha(s(1-v)+r)+1}\right. \quad (5.40)$$

$$\left.+\sum_{k=1}^{n_0}P_{0,\widehat{\varepsilon}}^{n_0,k}(x)k^{s(1-v)+r}\left(1+\frac{s(1-v)+r}{\alpha+1}\left(\frac{-1}{k}\right)+O\left(\frac{1}{k^2}\right)\right)\right).$$

From (5.40) it follows that we have

$$K_{n_0,\alpha}\left(t^{s(1-v)+r};x\right) = \frac{\widehat{\varepsilon}^{s(1-v)+r}}{(n_0+1)^{s(1-v)+r}} \frac{P_{0,\widehat{\varepsilon}}^{n_0,0}(x)}{\alpha(s(1-v)+r)+1}$$
$$+\frac{(n_0)^{s(1-v)+r}}{(n_0+1)^{s(1-v)+r}} B_{n_0}\left(t^{s(1-v)+r};x\right)$$
$$+\frac{\widehat{\varepsilon}^{s(1-v)+r}}{(n_0+1)^{s(1-v)+r}} \sum_{k=1}^{n_0} P_{0,\widehat{\varepsilon}}^{n_0,k}\left(-\frac{\varpi}{\alpha+1} k^{s(1-v)+r-1}\right.$$
$$\left.+O\left(k^{s(1-v)+r-2}\right)\right), \tag{5.41}$$

where $\varpi = s(1-v)+r$ and $B_{n_0}\left(t^{s(1-v)+r};x\right)$ is the Bernstein operator applied to $t^{s(1-v)+r}$ when $r \in \mathbb{Z}^+ \bigcup \{0\}$ and $s = 1,2,...,q-1$ that is

$$B_{n_0}\left(t^{s(1-v)+r};x\right) = \sum_{k=0}^{n_0} P_{0,\widehat{\varepsilon}}^{n_0,k}(x) \left(\frac{k}{n_0}\widehat{\varepsilon}\right)^{s(1-v)+r}, \tag{5.42}$$

and $P_{0,\widehat{\varepsilon}}^{n_0,k}$ is as defined in (5.6). Then from (5.36) and (5.41) we get

$$H_{m,\alpha}^{1+}(f;x) - f(x) = h_0'(x) K_{n_0,\alpha}((t-x);x) + K_{n_0,\alpha}(R_0(t-x);x)$$
$$+\sum_{s=1}^{q-1} h_s(x) \left(K_{n_0,\alpha}\left(t^{s(1-v)};x\right) - x^{s(1-v)}\right)$$
$$+\sum_{s=1}^{q-1} h_s'(x) K_{n_0,\alpha}\left(t^{s(1-v)}(t-x);x\right)$$
$$+\sum_{s=1}^{q-1} K_{n_0,\alpha}\left(R_s(t-x) t^{s(1-v)};x\right). \tag{5.43}$$

Therefore,

$$H_{m,\alpha}^{1+}(f;x) - f(x) = \frac{1}{n_0+1}\left(\frac{\lambda+\widehat{\varepsilon}\alpha}{\alpha+1} - x\right) h_0'(x) + \frac{1}{2(n_0+1)^2}\left(n_0 \psi_{0,\widehat{\varepsilon}}^*(x)\right.$$
$$\left.+\varphi_{0,\widehat{\varepsilon}}^*(\alpha,x)\right) h_0''(\xi_0) + \sum_{s=1}^{q-1} h_s(x) \left(\frac{1}{(n_0+1)^{s(1-v)}} \frac{P_{0,\widehat{\varepsilon}}^{n_0,0}(x)\widehat{\varepsilon}^{s(1-v)}}{\alpha s(1-v)+1}\right.$$
$$+\frac{(n_0)^{s(1-v)}}{(n_0+1)^{s(1-v)}} B_{n_0}\left(t^{s(1-v)};x\right) - x^{s(1-v)}$$
$$\left.+\frac{\widehat{\varepsilon}^{s(1-v)}}{(n_0+1)^{s(1-v)}} \sum_{k=1}^{n_0} P_{0,\widehat{\varepsilon}}^{n_0,k}(x)\left(-\frac{s(1-v)}{\alpha+1} k^{s(1-v)-1} + O\left(k^{s(1-v)-2}\right)\right)\right)$$

$$+\sum_{s=1}^{q-1} h_s^{'}(x)\left(K_{n_0,\alpha}\left(t^{s(1-v)+1};x\right)-xK_{n_0,\alpha}\left(t^{s(1-v)};x\right)\right)$$
$$+\frac{1}{2}\sum_{s=1}^{q-1} h_s^{''}(\xi_s)\left(K_{n_0,\alpha}\left(t^{s(1-v)+2};x\right)-2xK_{n_0,\alpha}\left(t^{s(1-v)+1};x\right)\right.$$
$$\left.+x^2K_{n_0,\alpha}\left(t^{s(1-v)};x\right)\right) \tag{5.44}$$

From (5.44) it follows that there exist a positive constant c_α^*, depending on $\widehat{\varepsilon},\alpha$ and the functions $h_1,h_1^{'},h_1^{''}$ and independent from n_0 such that the inequality (5.28) holds. The proof of (5.29) is analogous and follows by taking the value of $\alpha=1$, in the formulae (5.36)-(5.44). □

As a result of Theorem 5.2 the asymptotic rate of convergence of the hybrid operators defined in (5.8) and (5.9) on the interval $[0,\widehat{\varepsilon}]$ is $O\left(\frac{1}{n_0^{1-v}}\right)$. Based on the analogous approach in Voronowskaja [69], we give the asymptotic rate of convergence of these operators on the intervals $[\widehat{\varepsilon},\lambda]$ and $[\lambda,1]$ in the next two theorems.

Theorem 5.3: Let $\lambda=\frac{2\alpha+2}{6\alpha+3}$. If $\alpha>1$ and $f\in C_I^2(0,1]$, then the following holds true when $x\in[\widehat{\varepsilon},\lambda]$

$$\lim_{n_1\to\infty} n_1\left[H_{m,\alpha}^{1+}(f;x)-f(x)\right]=\left(\frac{\lambda+\widehat{\varepsilon}\alpha}{\alpha+1}-x\right)f'(x)+\frac{1}{2}\psi_{\widehat{\varepsilon},\lambda}^*(x)f''(x), \tag{5.45}$$

and when $x\in[\lambda,1]$

$$\lim_{n_2\to\infty} n_2\left[H_{m,\alpha}^{1+}(f;x)-f(x)\right]=\left(\frac{1+\lambda}{2}-x\right)f'(x)+\frac{1}{2}\psi_{\lambda,1}^*(x)f''(x). \tag{5.46}$$

The function $\psi_{a,b}^*$, is the same as given in (5.27). Additionally, since $f\in C^2[\widehat{\varepsilon},1]$ then the limit (5.45) is uniform on $[\widehat{\varepsilon},\lambda]$ and the rate of convergence of the operators $H_{m,\alpha}^{1+}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n_1}\right)$ for $x\in[\widehat{\varepsilon},\lambda]$. Also the limit (5.46) is uniform on $[\lambda,1]$ and the rate of convergence of the operators $H_{m,\alpha}^{1+}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n_2}\right)$ for $x\in[\lambda,1]$.

Proof. Assume that $f\in C_I^2(0,1]$, then $f\in C^2[\widehat{\varepsilon},1]$ and when $x\in[\widehat{\varepsilon},\lambda]$ we have

$H_{m,\alpha}^{1+}(f;x) = K_{n_1,\alpha}(f;x)$. From Taylor's formula at $x \in [\widehat{\varepsilon}, \lambda]$ we get

$$f(t) = f(x) + (t-x) f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 E(t-x), \tag{5.47}$$

with $E(u) \to 0$ as $u \to 0$ and it is integrable function on $[\widehat{\varepsilon}-x, \lambda - x]$. Using the linearity property of the operators $K_{n_1,\alpha}$ and (5.25), (5.26) we have

$$\begin{aligned} K_{n_1,\alpha}(f;x) - f(x) = {} & \frac{1}{n_1+1}\left(\frac{\lambda + \widehat{\varepsilon}\alpha}{\alpha+1} - x\right) f'(x) \\ & + \frac{1}{2(n_1+1)^2}\left(n_1 \psi^*_{\widehat{\varepsilon},\lambda}(x) + \varphi^*_{\widehat{\varepsilon},\lambda}(\alpha,x)\right) f''(x) \\ & + E_{\widehat{\varepsilon},\lambda}(n_1,\alpha,x), \end{aligned} \tag{5.48}$$

where, $\varphi^*_{\widehat{\varepsilon},\lambda}(\alpha,x)$, $\psi^*_{\widehat{\varepsilon},\lambda}(x)$ are the given functions in (5.24), (5.27) respectively and

$$E_{\widehat{\varepsilon},\lambda}(n_1,\alpha,x) = \frac{1}{\lambda - \widehat{\varepsilon}} \sum_{k=0}^{n_1} P_{\widehat{\varepsilon},\lambda}^{n_1,k}(x) \int_{\widehat{\varepsilon}}^{\lambda} (\tau - x)^2 E(\tau - x)\, dt. \tag{5.49}$$

Also $P_{\widehat{\varepsilon},\lambda}^{n_1,k}(x)$ is the polynomial in (5.6) and also $\tau = \widehat{\varepsilon} + \frac{k + \left(\frac{t-\widehat{\varepsilon}}{\lambda-\widehat{\varepsilon}}\right)^{\alpha}}{n_1+1}(\lambda - \widehat{\varepsilon})$. Let $c_1 = \sup_{u \in [\widehat{\varepsilon}-x, \lambda - x]} |E(u)|$. For arbitrary $\varepsilon^* > 0$ there exist $\delta^* > 0$ such that $|E(u)| < \varepsilon^*$ whenever $|u| < \delta^*$. For all $t \in [\widehat{\varepsilon}, \lambda]$ it follows that $|E(t-x)| < \varepsilon^* + c_1 (t-x)^2 / (\delta^*)^2$. Using Lemma 5.3 estimation (5.20) gives

$$\begin{aligned} \left|E_{\widehat{\varepsilon},\lambda}(n_1,\alpha,x)\right| & \leq \varepsilon^* \left|K_{n_1,\alpha}\left((t-x)^2;x\right)\right| + \frac{c_1}{(\delta^*)^2}\left|K_{n_1,\alpha}\left((t-x)^4;x\right)\right| \\ & \leq \frac{\varepsilon^*}{(n_1+1)^2}\left(\frac{n_1}{4}(\lambda - \widehat{\varepsilon})^2 + \sigma_{\widehat{\varepsilon},\lambda}(\alpha)\right) \\ & + \frac{c_1}{(\delta^*)^2 (n_1+1)^4} \widetilde{M}_{\widehat{\varepsilon},\lambda}(n_1,\alpha), \end{aligned} \tag{5.50}$$

where, $\sigma_{\widehat{\varepsilon},\lambda}(\alpha)$ is as given in (5.23) by taking $a = \widehat{\varepsilon}$ and $b = \lambda$ also $\widetilde{M}_{\widehat{\varepsilon},\lambda}$ is second degree polynomial in n_1 for a fixed $\alpha > 1$. To show that the asymptotic rate of convergence is $O\left(\frac{1}{n_1}\right)$ for $x \in [\widehat{\varepsilon}, \lambda]$ it is sufficient to show that $\lim_{n_1 \to \infty} n_1 E_{\widehat{\varepsilon},\lambda}(n_1,\alpha,x) = 0$. It is obvious from (5.50) that for n_1 large enough and for $x \in [\widehat{\varepsilon}, \lambda]$ we have $\left|n_1 E_{\widehat{\varepsilon},\lambda}(n_1,\alpha,x)\right| < \varepsilon^*$ and by multiplying both sides of (5.48) with n_1 and taking limit as $n_1 \to \infty$ follows (5.45). Since $f \in C_I^2(0,1]$ this limit is uniform on $[\widehat{\varepsilon},\lambda]$, thus the rate of convergence of the operator $H^{1+}_{m,\alpha}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n_1}\right)$ for $x\in[\widehat{\varepsilon},\lambda]$. Likewise using Definition 5.1, when $x\in[\lambda,1]$ $H^{1+}_{m,\alpha}(f;x)=K_{n_2}(f;x)$ From Taylor's formula at $x\in[\lambda,1]$ and from the linearity property of the operators K_{n_2} and (5.25), (5.26) we have

$$\begin{aligned} K_{n_2}(f;x)-f(x) &= \frac{1}{n_2+1}\left(\frac{1+\lambda}{2}-x\right)f'(x) \\ &+\frac{1}{2(n_2+1)^2}\left(n_2\psi^*_{\lambda,1}(x)+\varphi^*_{\lambda,1}(1,x)\right)f''(x) \\ &+E_{\lambda,1}(n_2,x), \end{aligned} \tag{5.51}$$

where, $\varphi^*_{\lambda,1}(1,x),\psi^*_{\lambda,1}(x)$ are the given functions in (5.24), (5.27) respectively. Here,

$$E_{\lambda,1}(n_2,x)=\frac{1}{1-\lambda}\sum_{k=0}^{n_2}P^{n_2,k}_{\lambda,1}(x)\int_{\lambda}^{1}(\tau^*-x)^2E(\tau^*-x)\,dt. \tag{5.52}$$

and $P^{n_2,k}_{\lambda,1}(x)$ is as given in (5.6) for $a=\lambda$ and $b=1$. Further, $\tau^*=\lambda+\frac{k+\frac{t-\lambda}{1-\lambda}}{n_2+1}(1-\lambda)$. Let $c_2=\sup\limits_{u\in[\lambda-x,1-x]}|E(u)|$. For arbitrary $\varepsilon^*>0$ there exist $\delta^*>0$ such that $|E(u)|<\varepsilon^*$ whenever $|u|<\delta^*$. For all $t\in[\lambda,1]$ it follows that $|E(t-x)|<\varepsilon^*+c_2(t-x)^2/(\delta^*)^2$. Using Lemma 5.3 estimation (5.20) gives

$$\begin{aligned} \left|E_{\lambda,1}(n_2,x)\right| &\le \varepsilon^*\left|K_{n_2}\left((t-x)^2;x\right)\right|+\frac{c_2}{(\delta^*)^2}\left|K_{n_2}\left((t-x)^4;x\right)\right| \\ &\le \frac{\varepsilon^*}{(n_2+1)^2}\left(\frac{n_2}{4}(1-\lambda)^2+\sigma_{\lambda,1}(1)\right) \\ &+\frac{c_2}{(\delta^*)^2(n_2+1)^4}\widetilde{M}_{\lambda,1}(n_2,1), \end{aligned} \tag{5.53}$$

where $\sigma_{\lambda,1}(1)$ is defined in (5.23) for $x\in[\lambda,1]$. Additionally, $\widetilde{M}_{\lambda,1}$ is second degree polynomial in n_2 for a fixed $\alpha>1$. It is obvious from (5.53) that for n_2 large enough we have $\left|n_2E_{\lambda,1}(n_2,x)\right|<\varepsilon^*$ and using (5.51) we obtain (5.2). From the assumption that $f\in C^2_I(0,1]$ this limit is uniform on $[\lambda,1]$, thus the rate of convergence of the operator $H^{1+}_{m,\alpha}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n_2}\right)$ for $x\in[\lambda,1]$. □

Theorem 5.4: Let $\lambda=\frac{2\alpha+2}{6\alpha+3}$. If $0<\alpha<1$ and $f\in C^2_I(0,1]$, then the following holds true for $x \in [\widehat{\varepsilon}, \lambda]$

$$\lim_{n_1 \to \infty} n_1 \left[H^{1-}_{m,\alpha}(f;x) - f(x) \right] = \left(\frac{\lambda + \widehat{\varepsilon}}{2} - x \right) f'(x) + \frac{1}{2} \psi^{*}_{\widehat{\varepsilon},\lambda}(x) f''(x), \tag{5.54}$$

and for $x \in [\lambda, 1]$

$$\lim_{n_2 \to \infty} n_2 \left[H^{1-}_{m,\alpha}(f;x) - f(x) \right] = \left(\frac{1 + \lambda\alpha}{\alpha + 1} - x \right) f'(x) + \frac{1}{2} \psi^{*}_{\lambda,1}(x) f''(x). \tag{5.55}$$

The function $\psi^{*}_{a,b}$, is as given in (5.27). Further, the limit (5.54) is uniform on $[\widehat{\varepsilon}, \lambda]$ and the rate of convergence of the operators $H^{1-}_{m,\alpha}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n_1}\right)$ for $x \in [\widehat{\varepsilon}, \lambda]$. Also the limit (5.55) is uniform on $[\lambda, 1]$ and the rate of convergence of the operators $H^{1-}_{m,\alpha}(f;x)$ to $f(x)$ is $O\left(\frac{1}{n_2}\right)$ for $x \in [\lambda, 1]$.

Proof. The proof can be given in a similar way to the proof of Theorem 5.3. □

Corollary 5.1: Let $0 < \widehat{\varepsilon} < \lambda$ and $\lambda = \frac{2\alpha+2}{6\alpha+3}$. If $\alpha > 1$ and $f \in C^2_I(0,1]$ then

$$\begin{aligned}\left\| H^{1+}_{m,\alpha}(f;x) - f(x) \right\|_{[\widehat{\varepsilon},\lambda]} \leq & \frac{\|f'\|_{[\widehat{\varepsilon},\lambda]}}{n_1 + 1} \beta_{\widehat{\varepsilon},\lambda}(\alpha) \\ & + \frac{\|f''\|_{[\widehat{\varepsilon},\lambda]}}{2(n_1+1)^2} \theta_{\widehat{\varepsilon},\lambda}(n_1, \alpha),\end{aligned} \tag{5.56}$$

$$\begin{aligned}\left\| H^{1+}_{m,\alpha}(f;x) - f(x) \right\|_{[\lambda,1]} \leq & \frac{\|f'\|_{[\lambda,1]}}{n_2 + 1} \beta_{\lambda,1}(1) \\ & + \frac{\|f''\|_{[\lambda,1]}}{2(n_2+1)^2} \theta_{\lambda,1}(n_2, 1),\end{aligned} \tag{5.57}$$

hold true where, the functions $\beta_{a,b}(\alpha), \theta_{a,b}(n,\alpha)$ are as given in (5.21) and (5.22) respectively.

Proof. From Theorem 5.3 using (5.48), we have

$$\sup_{x\in[\widehat{\varepsilon},\lambda]} \left| H_{m,\alpha}^{1+}(f;x) - f(x) \right|$$

$$= \sup_{x\in[\widehat{\varepsilon},\lambda]} \left| K_{n_1,\alpha}(f;x) - f(x) \right|$$
$$\leq \frac{1}{n_1+1} \sup_{x\in[\widehat{\varepsilon},\lambda]} \left| f'(x) \right| \sup_{x\in[\widehat{\varepsilon},\lambda]} \left| \left(\frac{\lambda+\widehat{\varepsilon}\alpha}{\alpha+1} - x \right) \right|$$
$$+ \frac{1}{2(n_1+1)^2} \sup_{x\in[\widehat{\varepsilon},\lambda]} \left| f''(x) \right| \sup_{x\in[\widehat{\varepsilon},\lambda]} \left| \left(n_1 \psi_{\widehat{\varepsilon},\lambda}^{*}(x) + \varphi_{\widehat{\varepsilon},\lambda}^{*}(\alpha,x) \right) \right| \tag{5.58}$$

where, $\varphi_{\widehat{\varepsilon},\lambda}^{*}(\alpha,x)$, $\psi_{\widehat{\varepsilon},\lambda}^{*}(x)$ are the functions given in (5.24), (5.27) respectively. Then using Lemma 5.3 and estimations (5.19), (5.20) we obtain (5.56). Likewise from Theorem 5.3 using (5.19), (5.20) and (5.51) the inequality (5.57) follows. □

Corollary 5.2: Let $0 < \widehat{\varepsilon} < \lambda$, where $\lambda = \frac{2\alpha+2}{6\alpha+3}$. If $0 < \alpha < 1$ and $f \in C_I^2(0,1]$ then

$$\left\| H_{m,\alpha}^{1-}(f;x) - f(x) \right\|_{[\widehat{\varepsilon},\lambda]} \leq \frac{\|f'\|_{[\widehat{\varepsilon},\lambda]}}{n_1+1} \beta_{\widehat{\varepsilon},\lambda}(1) + \frac{\|f''\|_{[\widehat{\varepsilon},\lambda]}}{2(n_1+1)^2} \theta_{\widehat{\varepsilon},\lambda}(n_1,1), \tag{5.59}$$

$$\left\| H_{m,\alpha}^{1-}(f;x) - f(x) \right\|_{[\lambda,1]} \leq \frac{\|f'\|_{[\lambda,1]}}{n_2+1} \beta_{\lambda,1}(\alpha) + \frac{\|f''\|_{[\lambda,1]}}{2(n_2+1)^2} \theta_{\lambda,1}(n_2,\alpha), \tag{5.60}$$

hold true where, $\beta_{a,b}(\alpha), \theta_{a,b}(n,\alpha)$ are as given in (5.21) and (5.22) respectively.

Proof. The proof is analogous as in the proof of Corollary 5.1. □

5.3 Combined Method of Hybrid Operators

Let $\lambda = \frac{2\alpha+2}{6\alpha+3}$. Further, we propose the following combined method in three step for the numerical solution of **VAK2.** We take the intervals $[l_0,l_1] = [0,\widehat{\varepsilon}]$, $[l_1,l_2] = [\widehat{\varepsilon},\lambda]$ and $[l_2,l_3] = [\lambda,1]$.

5.3.1 Combined Method for the Solution of VAK2 When $0 < \alpha < 1$

When $0 < \alpha < 1$, we use the hybrid operators $H_{m,\alpha}^{1-}$ given in (5.8) to approximate the solution of **VAK2** by the following algorithm of the proposed combined method.

Algorithm 5.1: Combined method for the solution of **VAK2** when $0 < \alpha < 1$.

Step 1: **(i)** For $p = 0,1$, take the grid points $x_j^{(p)} = l_p + \frac{j}{n_p}(l_{p+1} - l_p) + \varepsilon_p$, $j = 0,1,...,n_p - 1$ and $x_{n_p}^{(p)} = l_{p+1} - \varepsilon_p$, where $0 < \varepsilon_p < \frac{l_{p+1}-l_p}{2n_p}$.

(ii) Use the hybrid operator $H_{m,\alpha}^{1-}$ to approximate the unknown function f in (5.4) on the interval $[l_0, l_1]$

$$K_{n_0}(f;x) + \phi \int_{l_0}^{x} (x-t)^{-v} \widetilde{K}(x,t) K_{n_0}(f;t)\,dt = g(x), \tag{5.61}$$

and on $[l_1, l_2]$ as:

$$K_{n_1}(f;x) + \phi \int_{l_1}^{x} (x-t)^{-v} \widetilde{K}(x,t) K_{n_1}(f;t)\,dt$$
$$= g(x) - \phi \int_{l_0}^{l_1} (x-t)^{-v} \widetilde{K}(x,t) K_{n_0}\left(F_{n_0}^{1-};t\right) dt. \tag{5.62}$$

(iii) Evaluate the equations (5.61) and (5.62) at the grid points $x_j^{(p)}$, $p = 0,1$ respectively as given in Step 1 (i) to obtained the algebraic equations

$$K_{n_0}\left(f;x_j^{(0)}\right) + \phi \int_{l_0}^{x_j^{(0)}} (x_j^{(0)} - t)^{-v} \widetilde{K}\left(x_j^{(0)},t\right) K_{n_0}(f;t)\,dt = g\left(x_j^{(0)}\right), \tag{5.63}$$

for $j = 0,1,..,n_0$ and

$$K_{n_1}\left(f;x_j^{(1)}\right) + \phi \int_{l_1}^{x_j^{(1)}} (x_j^{(1)} - t)^{-v} \widetilde{K}\left(x_j^{(1)},t\right) K_{n_1}(f;t)\,dt$$
$$= g\left(x_j^{(1)}\right) - \phi \int_{l_0}^{l_1} (x_j^{(1)} - t)^{-v} \widetilde{K}\left(x_j^{(1)},t\right) K_{n_0}\left(F_{n_0}^{-};t\right) dt, \tag{5.64}$$

for $j = 0,1,..,n_1$. Also,

$$K_{n_p}\left(F_{n_p}^{1-};x\right) = \frac{(n_p+1)}{(l_{p+1}-l_p)} \sum_{k=0}^{n_p} \binom{n_p}{k} \frac{(x-l_p)^k \left(l_{p+1}-x\right)^{n_p-k}}{\left(l_{p+1}-l_p\right)^{n_p}} Y_{p,n_p}^{1-}(k+1), \tag{5.65}$$

and Y_{p,n_p}^{1-}, $p = 0,1$ are the unique numerical solution of the linear algebraic equations (5.63), (5.64) in matrix form

$$\frac{(n_p+1)}{(l_{p+1}-l_p)} A_p^{1-} Y_p^{1-} = B_p^{1-}, \tag{5.66}$$

provided A_p^{1-}, $p=0,1$ are nonsingular matrices. Further,

$$\left[A_p^{1-}\right]_{j+1,k+1} = \binom{n_p}{k}\left(\frac{\left(x_j^{(p)}-l_p\right)^k\left(l_{p+1}-x_j^{(p)}\right)^{n_p-k}}{\left(l_{p+1}-l_p\right)^{n_p}} + \phi\int_{l_p}^{x_j^{(p)}}(x_j^{(p)}-t)^{-\nu}\widetilde{K}\left(x_j^{(p)},t\right)\frac{(t-l_p)^k\left(l_{p+1}-t\right)^{n_p-k}}{\left(l_{p+1}-l_p\right)^{n_p}}dt\right), \tag{5.67}$$

$j=0,1,...,n_p$, $k=0,1,...,n_p$ and

$$Y_p^{1-}(k+1)=\int_{l_p+\frac{k}{n_1+1}(l_{p+1}-l_p)}^{l_p+\frac{k+1}{n_1+1}(l_{p+1}-l_p)} f(u)\,du,\ k=0,1,...,n_p, \tag{5.68}$$

$$B_0^{1-}(j+1)=g\left(x_j^{(0)}\right),\ j=0,1,...,n_0, \tag{5.69}$$

$$B_1^{1-}(j+1)=g\left(x_j^{(1)}\right)-Q_0^{1-}\left(x_j^{(1)},F_{n_0}^{1-}\right),\ j=0,1,...,n_1, \tag{5.70}$$

where,

$$Q_p^{1-}\left(x,F_{n_p}^{1-}\right)=\phi\int_{l_p}^{l_{p+1}}(x-t)^{-\nu}\widetilde{K}(x,t)\,K_{n_p}\left(F_{n_p}^{1-};t\right)dt,\ p=0,1. \tag{5.71}$$

Step 2: **(i)** Take $p=2$ and consider the grid points $x_j^{(2)}=l_2+\frac{j}{n_2}(l_3-l_2)+\varepsilon_2,\ j=0,1,...,n_2-1$ and $x_{n_2}^{(2)}=1-\varepsilon_2$, where $0<\varepsilon_2<\frac{l_3-l_2}{2n_2}$. Use the hybrid operator $H_{m,\alpha}^{1-}$ on the interval $[l_2,l_3]$ to approximate the unknown function f in (5.4) as:

$$K_{n_2,\alpha}(f;x)+\phi\int_{l_2}^{x}(x-t)^{-\nu}\widetilde{K}(x,t)\,K_{n_2,\alpha}(f;t)\,dt = g(x)-\phi\int_{l_0}^{l_1}(x-t)^{-\nu}\widetilde{K}(x,t)\,K_{n_0}\left(F_{n_0}^{1-};t\right)dt - \phi\int_{l_1}^{l_2}(x-t)^{-\nu}\widetilde{K}(x,t)\,K_{n_1}\left(F_{n_1}^{1-};t\right)dt, \tag{5.72}$$

where $K_{n_p}\left(F_{n_p}^{1-};x\right)$ for $p=0,1$ are obtained in Step 1 as given in (5.65).

(ii) Evaluate the equation (5.72) at the grid points $x_j^{(2)}$ for $j=1,2,..,n_2$ to obtain the following algebraic equations

$$K_{n_2,\alpha}\left(f;x_j^{(2)}\right)+\phi\int\limits_{l_2}^{x_j^{(2)}}(x_j^{(2)}-t)^{-\nu}\widetilde{K}\left(x_j^{(2)},t\right)K_{n_2,\alpha}(f;t)\,dt$$

$$=g\left(x_j^{(2)}\right)-\sum_{p=0}^{1}Q_p^{1-}\left(x_j^{(2)},F_{n_p}^{1-}\right), \tag{5.73}$$

where Q_p^{1-}, $p=0,1$ is defined in (5.71). The system (5.73) can be presented in the matrix form

$$\frac{(n_2+1)}{\alpha(l_3-l_2)}A_2^{1-}Y_2^{1-}=B_2^{1-}, \tag{5.74}$$

where,

$$\left[A_2^{1-}\right]_{j+1,k+1}$$

$$=\binom{n_2}{k}\left(\frac{\left(x_j^{(2)}-l_2\right)^k\left(l_3-x_j^{(2)}\right)^{n_2-k}}{(l_3-l_2)^{n_2}}\right.$$

$$\left.+\phi\int\limits_{l_2}^{x_j^{(2)}}\left(x_j^{(2)}-t\right)^{-\nu}\widetilde{K}\left(x_j^{(2)},t\right)\frac{(t-l_2)^k(l_3-t)^{n_2-k}}{(l_3-l_2)^{n_2}}dt\right), \tag{5.75}$$

$j=0,1,...,n_2$, $k=0,1,...,n_2$, and Y_2^{1-},B_2^{1-} are as

$$Y_2^{1-}(k+1)=\int\limits_{l_2+\frac{k}{n_2+1}(l_3-l_2)}^{l_2+\frac{k+1}{n_2+1}(l_3-l_2)}q_1(u)\,du,\ k=0,1,...,n_2, \tag{5.76}$$

$$q_1(u)=f(u)\left(\frac{(u-l_2)(n_2+1)}{l_3-l_2}-k\right)^{\frac{1-\alpha}{\alpha}}, \tag{5.77}$$

$$B_2^{1-}(j+1)=g\left(x_j^{(2)}\right)-\sum_{p=0}^{1}Q_p^{1-}\left(x_j^{(2)},F_{n_p}^{1-}\right),\ j=0,1,...,n_2. \tag{5.78}$$

(iii) Subsequently, if the matrix A_2^{1-} is nonsingular then the system (5.74) has the unique solution

$$Y_2^{1-}=\frac{\alpha(l_3-l_2)}{n_2+1}\left(A_2^{1-}\right)^{-1}B_2^{1-}. \tag{5.79}$$

Finally denote the numerical solution of Y_2^{1-} in (5.79) by Y_{2,n_2}^{1-} and let $F_{n_2,\alpha}^{1-}$ show the obtained numerical approximation to f over the interval $[\lambda,1]=[l_2,l_3]$ that is in the implicit form in Y_{2,n_2}^{1-}. Finally substitute $F_{n_2,\alpha}^{1-}$ in (5.5) to get $K_{n_2,\alpha}\left(F_{n_2,\alpha}^{1-};x\right)$ for $x \in [\lambda, 1]$ as

$$K_{n_2,\alpha}\left(F_{n_2,\alpha}^{1-};x\right) = \frac{n_2+1}{\alpha(1-\lambda)} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{(x-\lambda)^k (1-x)^{n_2-k}}{(1-\lambda)^{n_2}} Y_{2,n_2}^{1-}(k+1). \tag{5.80}$$

Let us denote the approximate solution obtained by the Algorithm 5.1 by

$$\widetilde{F}_{m,\alpha}^{1-} = \begin{cases} F_{n_0}^{1-} \text{ for } x \in [0, \widehat{\varepsilon}], \\ F_{n_1}^{1-} \text{ for } x \in [\widehat{\varepsilon}, \lambda], \\ F_{n_2,\alpha}^{1-} \text{ for } x \in [\lambda, 1]. \end{cases} \tag{5.81}$$

where, $m = n_0 + n_1 + n_2$. Substituting (5.81) into (5.8) we get

$$H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};x\right) = \begin{cases} K_{n_0}\left(F_{n_0}^{1-};x\right) \text{ if } x \in [0, \widehat{\varepsilon}], \\ K_{n_1}\left(F_{n_1}^{1-};x\right) \text{ if } x \in [\widehat{\varepsilon}, \lambda], \\ K_{n_2,\alpha}\left(F_{n_2,\alpha}^{1-};x\right) \text{ if } x \in [\lambda, 1]. \end{cases} \tag{5.82}$$

5.3.2 Combined Method for the Solution of VAK2 When $\alpha > 1$

When $\alpha > 1$ we use the hybrid operators $H_{m,\alpha}^{1+}$ defined in three step (5.9) to approximate the solution of (5.4) by using the following algorithm of the combined method.

Algorithm 5.2: Combined method for the solution of **VAK2** when $\alpha > 1$.

Step 1: **(i)** For $p = 0, 1$ take the grid points $x_j^{(p)} = l_p + \frac{j}{n_p}(l_{p+1} - l_p) + \varepsilon_p$, $j = 0, 1, ..., n_p - 1$ and $x_{n_p}^{(p)} = l_{p+1} - \varepsilon_p$, where $0 < \varepsilon_p < \frac{l_{p+1} - l_p}{2n_p}$.

(ii) Use the hybrid operator $H_{m,\alpha}^{1+}$ to approximate the unknown function f in (5.4) on the interval $[l_0, l_1]$ as:

$$K_{n_0,\alpha}(f;x) + \phi \int_{l_0}^{x} (x-t)^{-\nu} \widetilde{K}(x,t) K_{n_0,\alpha}(f;t)\, dt = g(x), \tag{5.83}$$

and on $[l_1, l_2]$

$$K_{n_1,\alpha}(f;x)+\phi\int_{l_1}^{x}(x-t)^{-\nu}\widetilde{K}(x,t)\,K_{n_1,\alpha}(f;t)\,dt$$

$$=g(x)-\phi\int_{l_0}^{l_1}(x-t)^{-\nu}\widetilde{K}(x,t)\,K_{n_0,\alpha}\left(F^{+}_{n_0,\alpha};t\right)dt. \tag{5.84}$$

(iii) Evaluate the equation (5.83) and (5.84) at the grid points $x_j^{(p)}, j=0,1,..,n_p,\ p=0,1$ respectively as given in Step 1 (i) to obtain the algebraic equations

$$K_{n_0,\alpha}\left(f;x_j^{(0)}\right)+\phi\int_{l_0}^{x_j^{(0)}}(x_j^{(0)}-t)^{-\nu}\widetilde{K}\left(x_j^{(0)},t\right)K_{n_0,\alpha}(f;t)\,dt=g\left(x_j^{(0)}\right), \tag{5.85}$$

for $j=0,1,...,n_0$ and

$$K_{n_1,\alpha}\left(f;x_j^{(1)}\right)+\phi\int_{l_1}^{x_j^{(1)}}(x_j^{(1)}-t)^{-\nu}\widetilde{K}\left(x_j^{(1)},t\right)K_{n_1,\alpha}(f;t)\,dt$$

$$=g\left(x_j^{(1)}\right)-\phi\int_{l_0}^{l_1}(x_j^{(1)}-t)^{-\nu}\widetilde{K}\left(x_j^{(1)},t\right)K_{n_0,\alpha}\left(F^{+}_{n_0,\alpha};t\right)dt, \tag{5.86}$$

for $j=0,1,...,n_1$. Also for $p=0,1$

$$K_{n_p,\alpha}\left(F^{1+}_{n_p,\alpha};x\right)=\frac{(n_p+1)}{\alpha(l_{p+1}-l_p)}\sum_{k=0}^{n_p}\binom{n_p}{k}\frac{(x-l_p)^k\left(l_{p+1}-x\right)^{n_p-k}}{\left(l_{p+1}-l_p\right)^{n_p}}Y^{1+}_{p,n_p}(k+1). \tag{5.87}$$

Further, Y^{1+}_{p,n_p} ,$p=0,1$ are the unique numerical solutions of the corresponding linear systems

$$\frac{(n_p+1)}{\alpha(l_{p+1}-l_p)}A_p^{1+}Y_p^{1+}=B_p^{1+}, \tag{5.88}$$

provided A_p^{1+} , $p=0,1$ are nonsingular matrices. Here,

$$\left[A_p^{1+}\right]_{j+1,k+1}$$

$$=\binom{n_p}{k}\left(\frac{\left(x_j^{(p)}-l_p\right)^k\left(l_{p+1}-x_j^{(p)}\right)^{n_p-k}}{\left(l_{p+1}-l_p\right)^{n_p}}\right.$$

$$\left.+\,\phi\int_{l_p}^{x_j^{(p)}}(x_j^{(p)}-t)^{-\nu}\widetilde{K}\left(x_j^{(p)},t\right)\frac{(t-l_p)^k\left(l_{p+1}-t\right)^{n_p-k}}{\left(l_{p+1}-l_p\right)^{n_p}}dt\right), \tag{5.89}$$

$j=0,1,...,n_p$, $k=0,1,...,n_p$ and

$$Y_0^{1+}(k+1)=\frac{\alpha(l_1-l_0)}{n_0+1}\int_{l_0}^{l_1} f\left(\frac{k+\left(\frac{t-l_0}{l_1-l_0}\right)^{\alpha}}{n_0+1}(l_1-l_0)\right)dt, \tag{5.90}$$

$$Y_1^{1+}(k+1)=\int_{l_1+\frac{k}{n_1+1}(l_2-l_1)}^{l_1+\frac{k+1}{n_1+1}(l_2-l_1)} q_2(u)\,du, \tag{5.91}$$

$$q_2(u)=f(u)\left(\frac{(u-l_1)(n_1+1)}{(l_2-l_1)}-k\right)^{\frac{1-\alpha}{\alpha}}, \tag{5.92}$$

$$B_0^{1+}(j+1)=g\left(x_j^{(0)}\right),\ j=0,1,...,n_0, \tag{5.93}$$

$$B_1^{1+}(j+1)=g\left(x_j^{(1)}\right)-Q_0^{1+}\left(x_j^{(1)},F_{n_0,\alpha}^{1+}\right),\ j=0,1,...,n_1. \tag{5.94}$$

Additionally,

$$Q_p^{1+}\left(x,F_{n_p,\alpha}^{1+}\right)=\phi\int_{l_p}^{l_{p+1}}(x-t)^{-v}\widetilde{K}(x,t)\,K_{n_p,\alpha}\left(F_{n_p,\alpha}^{1+};t\right)dt,\ p=0,1. \tag{5.95}$$

Step 2: **(i)** We take $p=2$ and the grid points $x_j^{(2)}=l_2+\frac{j}{n_2}(l_3-l_2)+\varepsilon_2$, $j=0,1,...,n_2-1$ and $x_{n_2}^{(2)}=1-\varepsilon_2$, where $0<\varepsilon_2<\frac{l_3-l_2}{2n_2}$. Use the hybrid operator $H_{m,\alpha}^{1+}$ on the interval $[l_2,l_3]$ to approximate the unknown function f in (5.4) as:

$$\begin{aligned}
&K_{n_2}(f;x)+\phi\int_{l_2}^{x}(x-t)^{-v}\widetilde{K}(x,t)\,K_{n_2}(f;t)\,dt\\
&=g(x)-\phi\int_{l_0}^{l_1}(x-t)^{-v}\widetilde{K}(x,t)\,K_{n_0,\alpha}\left(F_{n_0,\alpha}^{1+};t\right)dt\\
&-\phi\int_{l_1}^{l_2}(x-t)^{-v}\widetilde{K}(x,t)\,K_{n_1,\alpha}\left(F_{n_1,\alpha}^{1+};t\right)dt,
\end{aligned} \tag{5.96}$$

and, $K_{n_p,\alpha}\left(F_{n_p,\alpha}^{1+};x\right)$, $p=0,1$ are obtained in Step 1 as defined in (5.87).

(ii) Write the equation (5.96) at the grid points $x_j^{(2)}$, $j=1,2,...,n_2$ as given in Step 1 (i) and get the algebraic equations

$$K_{n_2}\left(f;x_j^{(2)}\right)+\phi\int_{l_2}^{x_j^{(2)}}(x_j^{(2)}-t)^{-v}\widetilde{K}\left(x_j^{(2)},t\right)K_{n_2}(f;t)\,dt$$
$$=g\left(x_j^{(2)}\right)-\sum_{p=0}^{1}Q_p^{1+}\left(x_j^{(2)},F_{n_p,\alpha}^{1+}\right), \tag{5.97}$$

where Q_p^{1+} is as presented in (5.95). Framing the system (5.97) in matrix form gives

$$\frac{(n_2+1)}{(l_3-l_2)}A_2^{1+}Y_2^{1+}=B_2^{1+}, \tag{5.98}$$

here,

$$\left[A_2^{1+}\right]_{j+1,k+1}$$
$$=\binom{n_2}{k}\left(\frac{\left(x_j^{(2)}-l_2\right)^k\left(l_3-x_j^{(2)}\right)^{n_2-k}}{(l_3-l_2)^{n_2}}\right.$$
$$\left.+\phi\int_{l_2}^{x_j^{(2)}}\left(x_j^{(2)}-t\right)^{-v}\widetilde{K}\left(x_j^{(2)},t\right)\frac{(t-l_2)^k(l_3-t)^{n_2-k}}{(l_3-l_2)^{n_2}}dt\right), \tag{5.99}$$

$j=0,1,...,n_2$, $k=0,1,...,n_2$, and Y_2^{1+},B_2^{1+} are as

$$Y_2^{1+}(k+1)=\int_{l_2+\frac{k}{n_2+1}(l_3-l_2)}^{l_2+\frac{k+1}{n_2+1}(l_3-l_2)}f(u)\,du,\ k=0,1,...,n_2, \tag{5.100}$$

$$B_2^{1+}(j+1)=g\left(x_j^{(2)}\right)-\sum_{p=0}^{1}Q_p^{1+}\left(x_j^{(2)},F_{n_p,\alpha}^{1+}\right),\ j=0,1,...,n_2. \tag{5.101}$$

(iii) Subsequently, if the matrix A_2^{1+} is nonsingular then the system (5.98) has the unique solution

$$Y_2^{1+}=\frac{(l_3-l_2)}{n_2+1}\left(A_2^{1+}\right)^{-1}B_2^{1+}. \tag{5.102}$$

Next denote the numerical solution of Y_2^{1+} in (5.102) by Y_{2,n_2}^{1+} and let $F_{n_2}^{1+}$ show the obtained numerical approximation to f over the interval $[l_2,l_3]$ that is in the implicit form in Y_{2,n_2}^{1+}. Finally substitute $F_{n_2}^{1+}$ in (5.7) to get $K_{n_2}\left(F_{n_2}^{1+};x\right)$ for $x\in[l_2,l_3]$ as

$$K_{n_2}\left(F_{n_2}^{1+};x\right)=\frac{n_2+1}{(l_3-l_2)}\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{(x-l_2)^k(l_3-x)^{n_2-k}}{(l_3-l_2)^{n_2}}Y_{2,n_2}^{1+}(k+1). \tag{5.103}$$

Let us denote the approximate solution obtained by the Algorithm 5.2 by

$$\widetilde{F}^{1+}_{m,\alpha} = \begin{cases} F^{1+}_{n_0,\alpha} \text{ for } x \in [0, \widehat{\varepsilon}], \\ F^{1+}_{n_1,\alpha} \text{ for } x \in [\widehat{\varepsilon}, \lambda], \\ F^{1+}_{n_2} \text{ for } x \in [\lambda, 1], \end{cases} \tag{5.104}$$

where, $m = n_0 + n_1 + n_2$. Substituting (5.104) into (5.8) we get

$$H^{1+}_{m,\alpha}\left(\widetilde{F}^{1+}_{m,\alpha}; x\right) = \begin{cases} K_{n_0,\alpha}\left(F^{1+}_{n_0,\alpha}; x\right) \text{ if } x \in [0, \widehat{\varepsilon}], \\ K_{n_1,\alpha}\left(F^{1+}_{n_1,\alpha}; x\right) \text{ if } x \in [\widehat{\varepsilon}, \lambda], \\ K_{n_2}\left(F^{1+}_{n_2}; x\right) \text{ if } x \in [\lambda, 1]. \end{cases} \tag{5.105}$$

5.3.3 Combined Method for the Regularized Solution of Linear Volterra Abel-type Integral Equations of First Kind

As is well known, Volterra integral equations of the first kind appear to be special cases of Fredholm integral equations of the first kind and are consequently classified among the lists of ill-conditioned problems solvable by the classical regularization means. The early theoretical development of a singular perturbation approach for regularizing first-kind Volterra problems is generally attributed to Sergeev [81] and Denisov [82] in the early 1970's, following the ideas of Lavrent'ev [83]. For this reason, the method is often referred to as Lavrent'ev's classical method, or the small parameter method. Subsequently, we consider the first kind Volterra Abel-type integral equations (**VAK1**)

$$\phi \int_0^x (x-t)^{-\nu} \widetilde{K}(x,t) f(t)\, dt = g(x),\ x \in I. \tag{5.106}$$

In this article the regularized processes construction is given by introducing the following equation

$$\phi \int_0^x (x-t)^{-\nu} \widetilde{K}(x,t) f^{\delta}_{\eta}(t)\, dt + \eta(\delta) f^{\delta}_{\eta}(x) = g_{\delta}(x),\ x \in I. \tag{5.107}$$

where, $\eta(\delta)$ is positive regularization parameter coordinated with δ. For the convergence of the regularized solution of (5.107) see Muftahov et al. [73] and the references therein. Obviously (5.107) is second kind Volterra integral equation with weak singularities. For the numerical solution of **RVAK1** by the proposed combined method, we apply the Algorithm 5.1 and Algorithm 5.2 for the equation (5.107) when $0 < \alpha < 1$ and $\alpha > 1$ respectively, by taking into consideration the coefficient $\eta(\delta)$ in the second term of the left side of equation (5.107).

5.4 Convergence Analysis of Algorithm 5.1 and Algorithm 5.2

Theorem 5.5: Let the conditions of Hypothesis C be satisfied and let $0 < \widehat{\varepsilon} < \lambda$ where $\lambda = \frac{2\alpha+2}{6\alpha+3}$. If $0 < \alpha < 1$ and $H^{1-}_{m,\alpha}\left(\widetilde{F}^{1-}_{m,\alpha};x\right)$ is the approximate solution of **VAK2** in (5.4) obtained by the Algorithm 5.1 and f is the unique exact solution then

$$\begin{aligned}
&\sup_{x\in[0,1]}\left|H^{1-}_{m,\alpha}\left(\widetilde{F}^{1-}_{m,\alpha};x\right)-f(x)\right| \\
&\leq \sum_{p=0}^{2} L_p\left(n_p,\gamma_p^-,f\right)(1 \\
&+\left\|\left(A_p^{1-}\right)^{-1}\right\|_\infty\left(\phi M_{l_p,l_{p+1}}\frac{\left(l_{p+1}-l_p\right)^{1-v}}{1-v}+1\right)\Bigg), \qquad (5.108)
\end{aligned}$$

holds true where,

$$\gamma_p^- = \begin{cases} v, \text{ if } \ p=0, \\ 1, \text{ if } \ p=1, \\ \alpha, \text{ if } \ p=2, \end{cases}$$

$$\begin{aligned}
L_0\left(n_0,\gamma_0^-,f\right) &= c_1^*\frac{1}{\left(n_0+1\right)^{1-v}}, \\
L_1\left(n_1,\gamma_1^-,f\right) &= W_{l_1,l_2}\left(n_1,1,f\right), \\
L_2\left(n_{2,},\gamma_2^-,f\right) &= W_{l_2,l_3}\left(n_2,\alpha,f\right), \qquad (5.109)
\end{aligned}$$

and c_1^* is a positive constant independent from n_0. Also for $p=1,2$

$$\begin{aligned}
W_{l_p,l_{p+1}}\left(n_p,\alpha,f\right) = &\frac{\|f'\|_{[l_p,l_{p+1}]}}{n_p+1}\beta_{l_p,l_{p+1}}(\alpha) \\
&+\frac{\|f''\|_{[l_p,l_{p+1}]}}{2\left(n_p+1\right)^2}\theta_{l_p,l_{p+1}}\left(n_p,\alpha\right), \qquad (5.110)
\end{aligned}$$

$$M_{l_p,l_{p+1}} = \max_{x_j^{(p)} \in [l_p,l_{p+1}]} \left(\sup_{t \in [l_p,l_{p+1}]} \left| \widetilde{K}\left(x_j^{(p)},t\right) \right| \right), \tag{5.111}$$

and $x_j^{(p)}, j = 0,1,...,n_p$ for $p = 0,1,2$ are as defined in Algorithm 5.1. Also $0 < v < 1, \beta_{l_p,l_{p+1}}, \theta_{l_p,l_{p+1}}$ are as given in (5.21) and (5.22) respectively. Further, A_p^{1-}, $p = 0,1$ are the matrices given in (5.67) and A_2^{1-} is as given in (5.75).

Proof. Since the conditions of Hypothesis C are satisfied then on the basis of Theorem 5.1 **VAK2** has the unique exact solution $f \in C_I^2(0,1]$. We take the subintervals $[l_0,l_1] = [0,\widehat{\varepsilon}]$, $[l_1,l_2] = [\widehat{\varepsilon},\lambda]$ and $[l_2,l_3] = [\lambda,1]$ as defined in Algorithm 5.1. It follows that for $x \in [0,1]$

$$\sup_{x\in[0,1]} \left| H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};x\right) - f(x) \right| \leq \sup_{x\in[0,1]} \left| H_{m,\alpha}^{1-}(f;x) - f(x) \right|$$
$$+ \sup_{x\in[0,1]} \left| H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};x\right) - H_{m,\alpha}^{1-}(f;x) \right|. \tag{5.112}$$

From Theorem 5.2 using equation (5.29), and based on Corollary 5.1 and the estimations (5.59), (5.60) and from (5.109), (5.110) we obtain

$$\begin{aligned} \sup_{x\in[0,1]} \left| H_{m,\alpha}^{1-}(f;x) - f(x) \right| &\leq \sum_{p=0}^{2} \sup_{x\in[l_p,l_{p+1}]} \left| H_{m,\alpha}^{1-}(f;x) - f(x) \right| \\ &= c_1^* \frac{1}{(n_0+1)^{1-v}} \\ &+ W_{l_1,l_2}(n_1,1,f) + W_{l_2,l_3}(n_2,\alpha,f) \\ &= \sum_{p=0}^{2} L_p\left(n_p,\gamma_p^-,f\right). \end{aligned} \tag{5.113}$$

Next let

$$\overline{Y}_{p,n_p}^{1-}(k+1) = Y_{p,n_p}^{1-}(k+1) - Y_p^{1-}(k+1), \tag{5.114}$$

for $k = 0,1,...,n_p$ and $p = 0,1,2$. From (5.65), (5.82) and using the linearity property of the operator $H_{m,\alpha}^{1-}$ also from $\sum_{k=0}^{n_p} P_{l_p,l_{p+1}}^{n_p,k}(x) = 1$, it follows that

$$
\begin{aligned}
&\sup_{x\in[0,1]}\left|H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};x\right)-H_{m,\alpha}^{1-}(f;x)\right| = \sup_{x\in[0,1]}\left|H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-}-f;x\right)\right| \\
&\leq \sum_{p=0}^{1}\sup_{x\in\left[l_p,l_{p+1}\right]}\left|\frac{n_p+1}{\left(l_{p+1}-l_p\right)}\sum_{k=0}^{n_p}P_{l_p,l_{p+1}}^{n_p,k}(x)\,\overline{Y}_{p,n_p}^{1-}(k+1)\right| \\
&+\sup_{x\in[l_2,l_3]}\left|\frac{n_2+1}{\alpha\,(l_3-l_2)}\sum_{k=0}^{n_2}P_{l_2,l_3}^{n_2,k}(x)\,\overline{Y}_{2,n_2}^{1-}(k+1)\right| \\
&\leq \sum_{p=0}^{1}\frac{n_p+1}{\left(l_{p+1}-l_p\right)}\left(\max_{0\leq k\leq n_p}\left|\overline{Y}_{p,n_p}^{1-}(k+1)\right|\right) \\
&+\frac{n_2+1}{\alpha\,(l_3-l_2)}\left(\max_{0\leq k\leq n_2}\left|\overline{Y}_{2,n_2}^{1-}(k+1)\right|\right). \qquad (5.115)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sup_{x\in[0,1]}\left|H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};x\right)-H_{m,\alpha}^{1-}(f;x)\right| &\leq \sum_{p=0}^{1}\frac{n_p+1}{\left(l_{p+1}-l_p\right)}\left\|\overline{Y}_{p,n_p}^{1-}\right\|_{\infty} \\
&+\frac{n_2+1}{\alpha\,(l_3-l_2)}\left\|\overline{Y}_{2,n_2}^{1-}\right\|_{\infty}. \qquad (5.116)
\end{aligned}
$$

From Theorem 2.1, since the operator $H_{m,\alpha}^{1-}(f;x)$ uniformly converges to f on the constructed subintervals for any $f\in C[0,1]$ and for any $\varepsilon>0$, there exist m such that the inequality $\left|H_{m,\alpha}^{1-}(f;x)-f(x)\right|<\varepsilon$ holds. Therefore, for the numerical solution of **VAK2** when $0<\alpha<1$ and $x\in[0,1]$ we assume

$$
g(x)=H_{m,\alpha}^{1-}(f;x)+\phi\int_0^x (x-t)^{-\nu}\widetilde{K}(x,t)\,H_{m,\alpha}^{1-}(f;t)\,dt. \qquad (5.117)
$$

If we substitute $\widetilde{F}_{m,\alpha}^{1-}$ given in (5.81) instead of f in (5.117) we get new function $\widehat{g}(x)$ on the left sides of this equation.

$$
\begin{aligned}
\widehat{g}(x) &= H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};x\right) \\
&+\phi\int_0^x (x-t)^{-\nu}\widetilde{K}(x,t)\,H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};t\right)dt. \qquad (5.118)
\end{aligned}
$$

Then subtracting equation (5.117) from (5.118) gives

$$
\begin{aligned}
&\widehat{g}(x) - g(x) \\
&- \phi \int_{l_0}^{l_1} (x-t)^{-v} \widetilde{K}(x,t) H_{m,\alpha}^{1-}\left(F_{n_0}^{1-} - f;t\right) dt \\
&- \phi \int_{l_1}^{l_2} (x-t)^{-v} \widetilde{K}(x,t) H_{m,\alpha}^{1-}\left(F_{n_1}^{1-} - f;t\right) dt \\
&= H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-} - f;x\right) \\
&+ \phi \int_{l_2}^{x} (x-t)^{-v} \widetilde{K}(x,t) H_{m,\alpha}^{1-}\left(F_{n_2,\alpha}^{1-} - f;t\right) dt.
\end{aligned} \tag{5.119}
$$

Then taking the grid points $x_j^{(p)}, j = 0, 1, ..., n_p$ for $p = 0, 1, 2$ as defined in the Step 1 and Step 2 of Algorithm 5.1 we obtain the algebraic systems in matrix form for $p = 0, 1,$

$$
\frac{n_p + 1}{\left(l_{p+1} - l_p\right)} A_p^{1-} \overline{Y}_{p,n_p}^{1-} = \overline{B}_p^{1-}\ , \tag{5.120}
$$

$$
\overline{B}_0^{1-}(j+1) = \widehat{g}\left(x_j^{(0)}\right) - g\left(x_j^{(0)}\right), \tag{5.121}
$$

$$
\begin{aligned}
\overline{B}_1^{1-}(j+1) &= \widehat{g}\left(x_j^{(1)}\right) - g\left(x_j^{(1)}\right) \\
&\quad - Q_0^{1-}\left(x_j^{(1)}, F_{n_0}^{1-} - f\right),
\end{aligned} \tag{5.122}
$$

$j = 0, 1, ..., n_p$. Further for $p = 2$ we get

$$
\frac{n_2 + 1}{\alpha\left(l_3 - l_2\right)} A_2^{1-} \overline{Y}_{2,n_2}^{1-} = \overline{B}_2^{1-}\ , \tag{5.123}
$$

$$
\begin{aligned}
\overline{B}_2^{1-}(j+1) &= \widehat{g}\left(x_j^{(2)}\right) - g\left(x_j^{(2)}\right) \\
&\quad - \sum_{i=0}^{1} Q_i^{1-}\left(x_j^{(2)}, F_{n_i}^{1-} - f\right),
\end{aligned} \tag{5.124}
$$

$j = 0, 1, ..., n_2$. Thus if $A_p^{1-}, p = 0, 1, 2$ are invertible matrices then for $p = 0, 1$

$$
\frac{(n_p + 1)}{\left(l_{p+1} - l_p\right)} \left\| \overline{Y}_{p,n_p}^{1-} \right\|_\infty \leq \left\| \left(A_p^{1-}\right)^{-1} \right\|_\infty \left\| \overline{B}_p^{1-} \right\|_\infty, \tag{5.125}
$$

and for $p = 2$

$$
\frac{(n_2 + 1)}{\alpha\left(l_3 - l_2\right)} \left\| \overline{Y}_{2,n_2}^{1-} \right\|_\infty \leq \left\| \left(A_2^{1-}\right)^{-1} \right\|_\infty \left\| \overline{B}_2^{1-} \right\|_\infty. \tag{5.126}
$$

Next let $\widehat{g}(x) = H_{m,\alpha}^{1-}(f;x) + \phi\int_{0}^{x}(x-t)^{-v}\widetilde{K}(x,t)H_{m,\alpha}^{1-}(f;t)\,dt$ and $g(x) = f(x)+\phi\int_{0}^{x}(x-t)^{-v}\widetilde{K}(x,t)f(t)dt$ as given by **VAK2**. At the grid points $x_j^{(p)}$, $j=0,1,...,n_p$ we obtain for $p=0,1,2$

$$\sup_{x_j^{(p)}\in[l_p,l_{p+1}]}\left|\widehat{g}\left(x_j^{(p)}\right)-g\left(x_j^{(p)}\right)-\sum_{i=1}^{p}Q_{i-1}^{1-}\left(x_j^{(p)},F_{n_{i-1}}^{1-}-f\right)\right|$$

$$\leq \sup_{x_j^{(p)}\in[l_p,l_{p+1}]}\phi\left|\int_{l_p}^{x_j^{(p)}}(x_j^{(p)}-t)^{-v}\widetilde{K}\left(x_j^{(p)},t\right)\left(H_{m,\alpha}^{1-}(f;t)-f(t)\right)dt\right|$$

$$+\sup_{x_j^{(p)}\in[l_p,l_{p+1}]}\left|H_{m,\alpha}^{1-}\left(f;x_j^{(p)}\right)-f\left(x_j^{(p)}\right)\right|. \tag{5.127}$$

Using (5.111), (5.113) and that $\sup_{x_j^{(p)}\in[l_p,l_{p+1}]}\left|\int_{l_p}^{x_j^{(p)}}(x_j^{(p)}-t)^{-v}dt\right| = \frac{\left(l_{p+1}-l_p\right)^{1-v}}{1-v}$ when $0<v<1$, and taking $M_{l_p,l_{p+1}} = \max_{x_j^{(p)}\in[l_p,l_{p+1}]}\left(\sup_{t\in[l_p,l_{p+1}]}\left|\widetilde{K}\left(x_j^{(p)},t\right)\right|\right)$ as given in (5.111) gives

$$\sup_{x_j^{(p)}\in[l_p,l_{p+1}]}\left|\widehat{g}\left(x_j^{(p)}\right)-g(x_j^{(p)})-\sum_{i=1}^{p}Q_{i-1}^{1-}\left(x_j^{(p)},F_{n_{i-1}}^{1-}-f\right)\right|$$

$$\leq \phi M_{l_p,l_{p+1}}L_p\left(n_p,\gamma_p^-,f\right)\sup_{x_j^{(p)}\in[l_p,l_{p+1}]}\left|\int_{l_p}^{x_j^{(p)}}(x_j^{(p)}-t)^{-v}dt\right|$$

$$+L_p\left(n_p,\gamma_p^-,f\right)$$

$$=L_p\left(n_p,\gamma_p^-,f\right)\left(\phi M_{l_p,l_{p+1}}\frac{\left(l_{p+1}-l_p\right)^{1-v}}{1-v}+1\right). \tag{5.128}$$

Substituting (5.128) into (5.125) and the obtained result in (5.116) gives

$$\sup_{x\in[0,1]}\left|H_{m,\alpha}^{1-}\left(\widetilde{F}_{m,\alpha}^{1-};x\right)-H_{m,\alpha}^{1-}(f;x)\right|$$

$$\leq\sum_{p=0}^{2}L_p\left(n_p,\gamma_p^-,f\right)\left(\phi M_{l_p,l_{p+1}}\frac{\left(l_{p+1}-l_p\right)^{1-v}}{1-v}+1\right)\left\|\left(A_p^{1-}\right)^{-1}\right\|_\infty. \tag{5.129}$$

Substituting (5.113) and (5.129) into (5.112) follows (5.108). □

Theorem 5.6: Let the conditions of Hypothesis C be satisfied and let $0 < \widehat{\varepsilon} < \lambda$ where $\lambda = \frac{2\alpha+2}{6\alpha+3}$. If $\alpha > 1$ and $H^{1+}_{m,\alpha}\left(\widetilde{F}^{1+}_{m,\alpha};x\right)$ is the approximate solution of **VAK2** in (5.4) obtained by the Algorithm 5.2 and f is the unique exact solution then

$$\begin{aligned}
&\sup_{x\in[0,1]}\left|H^{1+}_{m,\alpha}\left(\widetilde{F}^{1+}_{m,\alpha};x\right) - f(x)\right| \\
&\leq \sum_{p=0}^{2}\widetilde{L}_p\left(n_p,\gamma_p^+,f\right)(1 \\
&+\left\|\left(A_p^{1+}\right)^{-1}\right\|_\infty\left(\phi M_{l_p,l_{p+1}}\frac{\left(l_{p+1}-l_p\right)^{1-v}}{1-v}+1\right)\Bigg), \qquad (5.130)
\end{aligned}$$

holds true where,

$$\gamma_p^+ = \begin{cases} v, \text{ if } p=0, \\ \alpha, \text{ if } p=1, \\ 1, \text{ if } p=2, \end{cases}$$

and

$$\begin{aligned}
\widetilde{L}_0\left(n_0,\gamma_0^+,f\right) &= c^*_\alpha\frac{1}{\left(n_0+1\right)^{1-v}}, \\
\widetilde{L}_1\left(n_1,\gamma_1^+,f\right) &= W_{l_1,l_2}\left(n_1,\alpha,f\right), \\
\widetilde{L}_2\left(n_2,\gamma_2^+,f\right) &= W_{l_2,l_3}\left(n_2,1,f\right), \qquad (5.131)
\end{aligned}$$

and c^*_α is a positive constant independent from n_0 and additionally, $W_{l_p,l_{p+1}}\left(n_p,\alpha,f\right)$, $p = 1,2$ are given in (5.110) and $M_{l_p,l_{p+1}}$for $p = 0,1,2$ is defined in (5.111). Also $0 < v < 1$ and A_p^{1+}, $p = 0,1$ are the matrices given in (5.89) and A_2^{1+} is as given in (5.99).

Proof. Based on the Algorithm 5.2 proof is analogous to the proof of Theorem 5.5 and follows by using the operators $H^{1+}_{m,\alpha}(f;x)$ given in (5.9), Theorem 5.2 equation (5.28) and Corollary 5.1 and the equations (5.56) and (5.57). □

Chapter 6

EXPERIMENTAL INVESTIGATION OF THE COMBINED METHOD

This chapter is devoted to experimental investigations of the proposed combined method by applying the constructed algorithms to the considered test problems of second kind linear Volterra Abel-type integral equations. Also first kind Volterra Abel-type integral equations are considered and the given algorithms are used after utilizing regularization techniques. Furthermore, it is numerically shown that the given method hence also the developed algorithms provide accurate and stable numerical approximations to the solution of the Volterra Abel-type integral equations.

All the computations in this chapter are performed using Mathematica in machine precision. We denote the numerical solution obtained by the Algorithm 5.1 by $\widetilde{F}^{1-}_{m,\alpha}$ and the solution obtained by Algorithm 5.2 by $\widetilde{F}^{1+}_{m,\alpha}$. Also $\varepsilon_p, p = 0,1,2$ in the Algorithm 5.1 and Algorithm 5.2 are taken as $\varepsilon_p = \frac{l_{p+1}-l_p}{n_p^2}$. Let the following error grid functions be defined at the $N+1$ grid points $x_s = \frac{s}{N}$, $s = 0,...,N$ over the interval $[0,1]$ as

$$E\left[H^{1-}_{m,\alpha}\left(\widetilde{F}^{1-}_{m,\alpha};x_s\right)\right] = f(x_s) - H^{1-}_{m,\alpha}\left(\widetilde{F}^{1-}_{m,\alpha}(x_s)\right), \tag{6.1}$$

$$E\left[H^{1+}_{m,\alpha}\left(\widetilde{F}^{1+}_{m,\alpha};x_s\right)\right] = f(x_s) - H^{1+}_{m,\alpha}\left(\widetilde{F}^{1+}_{m,\alpha}(x_s)\right), \tag{6.2}$$

For the implementation of the proposed combined method of hybrid operators we have considered two versions. In the first version we take two subintervals $[0,\lambda],[\lambda,1]$ of the interval $[0,1]$ and use hybrid linear and positive operators in two step when

$0 < \alpha < 1$

$$H^{1-}_{m,\alpha}(f;x) = \begin{cases} K_{n_0}(f;x) \text{ if } x \in [0,\lambda], \\ K_{n_1,\alpha}(f;x) \text{ if } x \in [\lambda,1], \end{cases} \tag{6.3}$$

and $\alpha > 1$

$$H^{1+}_{m,\alpha}(f;x) = \begin{cases} K_{n_0,\alpha}(f;x) \text{ if } x \in [0,\lambda], \\ K_{n_1}(f;x) \text{ if } x \in [\lambda,1], \end{cases} \tag{6.4}$$

where, $m = n_0 + n_1$ and n_0, n_1 are positive integers. We call this implementation as two step form and therefore, in the Algorithm 5.1 and Algorithm 5.2, step 1 is performed for $[l_0, l_1] = [0, \lambda]$ when $p = 0$ and step 2 is performed for $[l_1, l_2] = [\lambda, 1]$ when $p = 1$. In the second version we apply the combined method as same as given in the Algorithm 5.1 and Algorithm 5.2. We frame this implementation as three step form. Further, we use the following notations in tables and figures:

i) $M\left(H^{1-}_{m,\alpha}\right)_{2S}$, $M\left(H^{1-}_{m,\alpha}\right)_{3S}$ present the combined method in two and three steps respectively when $0 < \alpha < 1$.

ii) $M\left(H^{1+}_{m,\alpha}\right)_{2S}$, $M\left(H^{1+}_{m,\alpha}\right)_{3S}$ present the combined method in two and three steps accordingly when $\alpha > 1$.

iii) $D_{2S}(n_0,n_1), D_{3S}(n_0,n_1,n_2)$ show the degrees $n_p, p = 0,1$ for the two step and $n_p, p = 0,1,2$ for the three step applications of the given combined method, respectively.

iv) $Cond\left(A^{1-}_p\right)$, $Cond(A^{1+}_p)$, denote the condition number of the matrices A^{1-}_p, A^{+}_p obtained by the method $M\left(H^{1-}_{m,\alpha}\right)_{qS}$ and $M\left(H^{1+}_{m,\alpha}\right)_{qS}$, $q = 2,3$ for $p = 0,...,q-1$ respectively using LinearAlgebra‘Private ‘MatrixConditionNumber command in Mathematica.

v) $RE\left(H^{1-}_{m,\alpha}\right)_{qS}$, $q = 2,3$ denotes the root mean square error $RMSE$ of the solution

$$RE\left(H^{1-}_{m,\alpha}\right)_{qS}=\sqrt{\left(\frac{1}{N+1}\sum_{s=0}^{N}\left(E\left[H^{1-}_{m,\alpha}\left(\widetilde{F}^{1-}_{m,\alpha};x_s\right)\right]\right)^2\right)},$$

over $N+1$ points obtained by $M\left(H^{1-}_{m,\alpha}\right)_{qS}$, for $q=2,3$ steps.

vi) $RE\left(H^{1+}_{m,\alpha}\right)_{qS}$, $q=2,3$ denotes the root mean square error $RMSE$ of the solution

$$RE\left(H^{1+}_{m,\alpha}\right)_{qS}=\sqrt{\left(\frac{1}{N+1}\sum_{s=0}^{N}\left(E\left[H^{1+}_{m,\alpha}\left(\widetilde{F}^{1+}_{m,\alpha};x_s\right)\right]\right)^2\right)},$$

over $N+1$ points obtained by $M\left(H^{1+}_{m,\alpha}\right)_{qS}$, for $q=2,3$ steps.

viii) $AE_{x_s}\left(H^{1+}_{m,\alpha}\right)_{qS}$, $q=2,3$ shows the absolute error of the solution at the point x_s obtained by the method $M\left(H^{1+}_{m,\alpha}\right)_{qS}$, for $q=2,3$ steps.

6.1 Examples of Second Kind Volterra Abel-type Integral Equations

We consider the following test problems of second kind fractional Volterra integral equations.

Example 9: ($Ex9$),(Micula [64])

$$f(x)-\frac{0.01}{\Gamma(1/2)}\int_0^x\frac{x^{\frac{5}{2}}}{\sqrt{x-t}}f(t)\,dt=\sqrt{\pi}\,(1+x)^{-\frac{3}{2}}-0.02\frac{x^3}{1+x},\ x\in I,$$

and the exact solution is $f(x)=\sqrt{\pi}\,(1+x)^{-\frac{3}{2}}$, $x\in I$.

Example 10: ($Ex10$), (Micula [64])

$$f(x)-\frac{1}{27\Gamma(2/3)}\int_0^x\frac{t}{\sqrt[3]{x-t}}f(t)\,dt=\Gamma(2/3)x-\frac{1}{40}x^{\frac{8}{3}},\ x\in I,$$

where the exact solution is $f(x)=\Gamma(2/3)x$, $x\in I$.

Furthermore, we consider the next example which has a solution that has singular behaviour at the initial point $x=0$. For the considered examples in this section we take $\widehat{\varepsilon} = 0.05$.

Example 11: ($Ex11$), (Dixon [84], Abdalkhani [85] and Saeedi et al. [86])

$$f(x) + \int_0^x \frac{1}{\sqrt{x-t}} f(t)\,dt = \frac{1}{2}\pi x + \sqrt{x},\ x \in I,$$

where the exact solution is $f(x) = \sqrt{x}$, $x \in I$.

Table 6.1 presents $AE_{x_s}\left(H^{1-}_{33,0.01}\right)_{qS}$ and $AE_{x_s}\left(H^{1+}_{33,10}\right)_{qS}$ (the absolute errors (AE)) for the Example 9 at the points $x_s = \frac{s}{10}$, $s = 0, 1, ...10$ for $q = 2, 3$, steps with $\alpha = 0.01$, and $\alpha = 10$. Further, the results in this table are calculated for $D_{2S}(22, 11)$ and $D_{3S}(11, 11, 11)$. Figure 6.1 illustrates the data presented in Table 6.1. While, Figure 6.2 demonstrates $RE\left(H^{1-}_{m,0.01}\right)_{qS}$, and $RE\left(H^{1+}_{m,10}\right)_{qS}$, (root mean square error ($RMSE$)) for the Example 9 with respect to n_0 (α is fixed) for $q = 2, 3$, steps. Also, Figure 6.3 shows $RE\left(H^{1-}_{33,\alpha}\right)_{qS}$, and $RE\left(H^{1+}_{33,\alpha}\right)_{qS}$, with respect to α ($m = 33$ is fixed) by taking $D_{2S}(22, 11)$ for $q = 2$, and $D_{3S}(11, 11, 11)$ for $q = 3$. It can be viewed from Figure 6.3 that the $RMSE$ with respect to α obtained by $M\left(H^{1-}_{33,\alpha}\right)_{3S}$, $M\left(H^{1+}_{33,\alpha}\right)_{3S}$ behaves almost constant.

Table 6.2 illustrates a comparison of the maximum absolute errors (MAE) for the Example 10 obtained by $M\left(H^{1-}_{m,0.001}\right)_{qS}$, $M\left(H^{1+}_{m,10}\right)_{qS}$, for $q = 2, 3$ steps for $m = 12, 18, 24$ over the points $x_s = \frac{s}{N}$, $s = 0, 1, ..., N$ when $N = m$ and the iterative method given in Table 2 of Micula [64] with $n = 10$ iterations. The presented results for $m = 12, 18, 24$ by the given combined method are computed by taking $D_{2S}(8, 4), D_{2S}(13, 5)$ and $D_{2S}(21, 3)$ for the two step and $D_{3S}(4, 4, 4), D_{3S}(8, 5, 5)$ and $D_{3S}(18, 3, 3)$ for the three step applications. On the one hand, these results show that three stage application gives more accurate results then two stage case and both of the

Table 6.1: The AE for the Example 9 obtained by the proposed method $M\left(H^{1+}_{33,10}\right)_{2S}$, $M\left(H^{1+}_{33,10}\right)_{3S}$ and $M\left(H^{1-}_{33,0.01}\right)_{2S}$, $M\left(H^{1-}_{33,0.01}\right)_{3S}$.

x_s	$Ex9$ $M\left(H^{1+}_{33,10}\right)_{2S}$	$Ex9$ $M\left(H^{1+}_{33,10}\right)_{3S}$	$Ex9$ $M\left(H^{1-}_{33,0.01}\right)_{2S}$	$Ex9$ $M\left(H^{1-}_{33,0.01}\right)_{3S}$
0	3.610×10^{-9}	1.599×10^{-14}	2.389×10^{-9}	1.865×10^{-14}
0.1	2.810×10^{-9}	3.722×10^{-12}	1.928×10^{-10}	1.344×10^{-10}
0.2	1.692×10^{-8}	1.747×10^{-11}	3.604×10^{-11}	2.777×10^{-12}
0.3	3.643×10^{-10}	4.775×10^{-11}	4.370×10^{-11}	4.151×10^{-11}
0.4	2.134×10^{-11}	1.903×10^{-11}	7.795×10^{-11}	8.580×10^{-11}
0.5	1.183×10^{-10}	1.150×10^{-10}	1.536×10^{-10}	1.667×10^{-10}
0.6	1.927×10^{-10}	1.883×10^{-10}	2.851×10^{-10}	2.391×10^{-10}
0.7	2.767×10^{-10}	2.707×10^{-10}	2.275×10^{-10}	2.277×10^{-10}
0.8	3.911×10^{-10}	3.835×10^{-10}	2.965×10^{-10}	2.968×10^{-10}
0.9	4.526×10^{-10}	4.432×10^{-10}	4.236×10^{-10}	4.239×10^{-10}
1.0	5.166×10^{-10}	5.051×10^{-10}	4.838×10^{-10}	4.842×10^{-10}

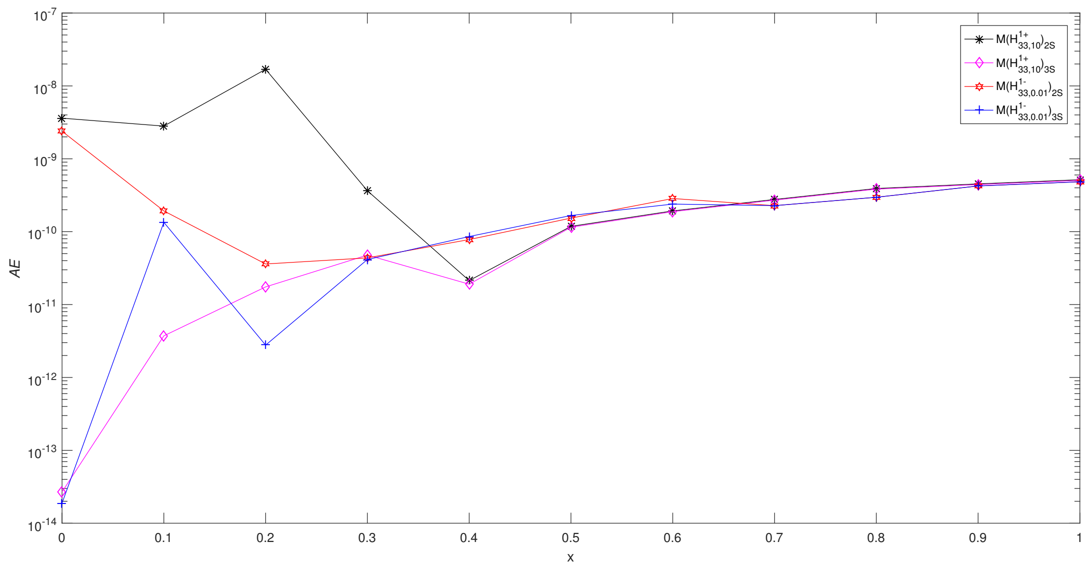

Figure 6.1: The AE for the Example 9 obtained by the combined method $M\left(H^{1+}_{33,10}\right)_{qS}$ and $M\left(H^{1-}_{33,0.01}\right)_{qS}$ for $q=2,3$ steps.

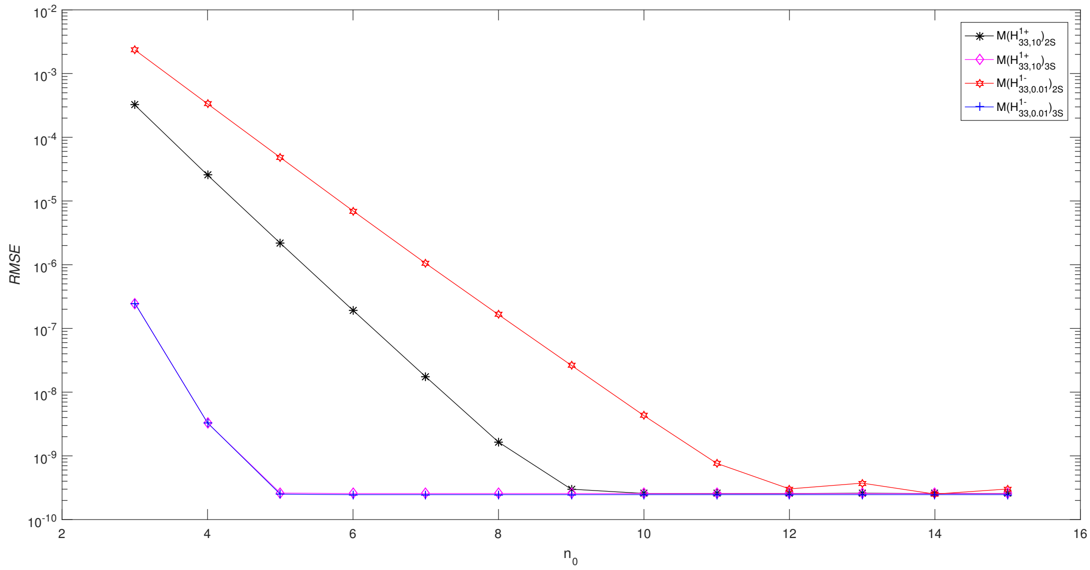

Figure 6.2: The *RMSE* with respect to n_0 obtained by the combined method $M\left(H^{1+}_{m,10}\right)_{qS}$ and $M\left(H^{1-}_{m,0.01}\right)_{qS}$ for $q = 2,3$ steps for the Example 9.

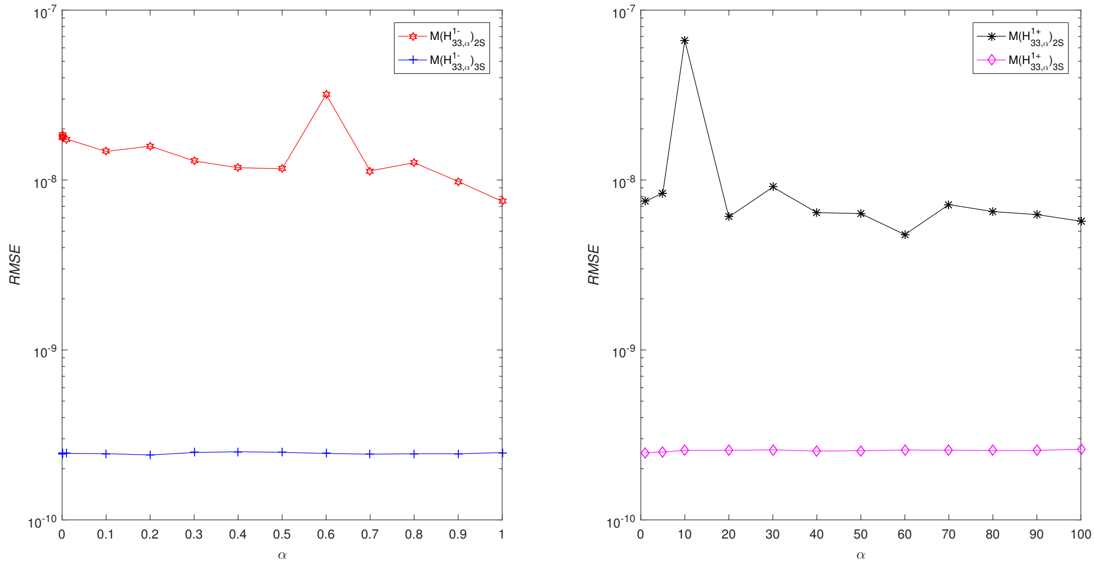

Figure 6.3: The *RMSE* with respect to α obtained by the given combined method $M\left(H^{1-}_{33,\alpha}\right)_{qS}, M\left(H^{1+}_{33,\alpha}\right)_{qS}$ for $q = 2,3$ steps for the Example 9.

Table 6.2: The comparison of the MAE for the Example 10 obtained by the methods $M\left(H^{1+}_{m,10}\right)_{qS}$ and $M\left(H^{1-}_{m,0.001}\right)_{qS}$ for $q=2,3$ and the iterative method given in Micula [64].

$Method$	$Ex10$ $MAE, m=12$	$Ex10$ $MAE, m=18$	$Ex10$ $MAE, m=24$
$M\left(H^{1-}_{m,0.001}\right)_{2S}$	1.656×10^{-9}	1.313×10^{-9}	7.113×10^{-10}
$M\left(H^{1-}_{m,0.001}\right)_{3S}$	1.656×10^{-9}	7.114×10^{-10}	1.569×10^{-11}
$M\left(H^{1+}_{m,10}\right)_{2S}$	4.220×10^{-8}	8.560×10^{-10}	6.769×10^{-10}
$M\left(H^{1+}_{m,10}\right)_{3S}$	8.504×10^{-10}	8.522×10^{-10}	1.761×10^{-11}
Micula [64], $n=10$	2.492919×10^{-5}	1.304662×10^{-5}	6.045722×10^{-6}

realizations of the proposed combined method give extremely accurate results when compared with the iterative method given in Table 2 of Micula [64].

Table 6.3 shows $AE_{x_s}\left(H^{1-}_{45,0.01}\right)_{qS}$ and $AE_{x_s}\left(H^{1+}_{45,10}\right)_{qS}$ over the points $x_s=\frac{s}{10}, s=0,1,..,10$ for the Example 11 for $q=2,3$ steps in accordance with $D_{2S}(35,10)$ and $D_{3S}(25,10,10)$. Furthermore, the singular behaviour of the exact solution at the initial point $x=0$ and the efficiency of the three step combined method is shown. Table 6.4 presents the comparison of the $RE\left(H^{1-}_{m,0.01}\right)_{3S}$, and $RE\left(H^{1+}_{m,10}\right)_{3S}$ for the Example 11 obtained by $M\left(H^{1-}_{m,0.01}\right)_{3S}$, $M\left(H^{1+}_{m,10}\right)_{3S}$ over the interval $[0.1,1]$ when $m=8,16,32,64,128$ with degrees $D_{3S}(\frac{m}{2},\frac{m}{4},\frac{m}{4})$ and the operational Haar method given in Table 1 of Saeedi et al. [86]. In this table for the method in Saeedi et al. [86] m denotes the resolution level of the wavelet.

Additionally, for the Example 11 in Table 6.5 we compare the maximum relative errors (MRE) over the points $x_s=\frac{s}{100}, s=1,..,100$ obtained by the proposed method $M\left(H^{1-}_{64,0.01}\right)_{qS}$, $M\left(H^{1+}_{64,10}\right)_{qS}$ for $q=2,3$ steps and the method given in Table 1 of Abdalkhani [85] when polynomials of second (P_2),fourth (P_4) and eight (P_8) degrees

Table 6.3: The AE for the Example 11 obtained by $M\left(H^{1-}_{45,0.01}\right)_{qS}$, $M\left(H^{1+}_{45,10}\right)_{qS}$ for $q = 2,3$ steps.

x_s	$Ex11$ $M\left(H^{1-}_{45,0.01}\right)_{2S}$	$Ex11$ $M\left(H^{1-}_{45,0.01}\right)_{3S}$	$Ex11$ $M\left(H^{1+}_{45,10}\right)_{2S}$	$Ex11$ $M\left(H^{1+}_{45,10}\right)_{3S}$
0	0.0192273	0.0061074	0.0159962	0.0061074
0.1	2.306×10^{-5}	1.172×10^{-5}	3.102×10^{-5}	2.623×10^{-6}
0.2	1.071×10^{-5}	2.328×10^{-6}	1.595×10^{-5}	9.739×10^{-7}
0.3	6.982×10^{-6}	2.726×10^{-6}	1.073×10^{-5}	5.014×10^{-7}
0.4	5.056×10^{-6}	2.319×10^{-6}	9.154×10^{-6}	4.526×10^{-7}
0.5	3.592×10^{-6}	2.204×10^{-6}	6.797×10^{-6}	3.779×10^{-7}
0.6	9.697×10^{-5}	3.565×10^{-7}	5.350×10^{-6}	3.215×10^{-7}
0.7	3.265×10^{-4}	4.236×10^{-7}	4.420×10^{-6}	2.805×10^{-7}
0.8	1.231×10^{-5}	5.997×10^{-7}	3.728×10^{-6}	2.671×10^{-7}
0.9	7.367×10^{-5}	6.046×10^{-7}	3.185×10^{-6}	2.370×10^{-7}
1.0	4.398×10^{-5}	5.117×10^{-7}	8.447×10^{-6}	1.328×10^{-7}

Table 6.4: The comparison of the $RMSE$ for the Example 11 obtained by the method $M\left(H^{1-}_{m,0.01}\right)_{3S}$, $M\left(H^{1+}_{m,10}\right)_{3S}$ and the approach in Saeedi et al. [86].

m	$Ex11$ $RE\left(H^{1-}_{m,0.01}\right)_{3S}$	$Ex11$ $RE\left(H^{1+}_{m,10}\right)_{3S}$	$Ex11$ Saeedi et al. [86]
8	1.065×10^{-3}	4.257×10^{-4}	1.5713×10^{-3}
16	3.880×10^{-4}	4.610×10^{-5}	4.5657×10^{-4}
32	1.820×10^{-5}	1.932×10^{-6}	1.3244×10^{-5}
64	1.114×10^{-6}	9.125×10^{-7}	3.8332×10^{-5}
128	7.153×10^{-7}	7.153×10^{-7}	1.1353×10^{-5}

Table 6.5: The comparison of the MRE for the Example 11 obtained by the methods $M\left(H^{1-}_{64,0.01}\right)_{qS}$, $M\left(H^{1-}_{64,0.01}\right)_{qS}$ for $q=2,3$ and the method given in Abdalkhani [85].

Method	$Ex11$ MRE
$M\left(H^{1-}_{64,0.01}\right)_{2S}$	0.0059026
$M\left(H^{1-}_{64,0.01}\right)_{3S}$	7.217×10^{-5}
$M\left(H^{1+}_{64,10}\right)_{2S}$	0.0028770
$M\left(H^{1+}_{64,10}\right)_{3S}$	7.217×10^{-5}
Abdalkhani [85], (P_2)	0.0756875
Abdalkhani [85], (P_4)	0.0408173
Abdalkhani [85], (P_8)	0.0307124

were employed. In this table data is computed by taking $D_{2S}(48,16), D_{3S}(32,16,16)$ for the given combined method when two and three step applications are applied respectively. Consequently, Table 6.3-Table 6.5 demonstrate the high accuracy and the stability of the proposed combined method. Finally for the Example 11 we present Figure 6.4 showing the maximum absolute errors (MAE) over the points $x_s=\frac{s}{10}, s=0,1,..,10$ and taking $D_{2S}(n_0,10), D_{3S}(n_0,10,10)$ that is with respect to n_0 obtained by the proposed method $M\left(H^{1-}_{m,0.01}\right)_{qS}$ and $M\left(H^{1+}_{m,10}\right)_{qS}$ for $q=2,3$ steps.

6.2 Examples of First Kind Volterra Abel-type Integral Equations

We consider the following test problems of first kind Volterra Abel-type integral equations. The regularization parameter $\eta(\delta)$ is taken as $\eta(\delta)=\delta^{\ 0.9}$ and $\delta=5\times10^{-15}$ and the results are calculated for $\widehat{\varepsilon}=0.0005$.

Example 12: $(Ex12)$

$$\int\limits_0^x \frac{t}{(x-t)^{\frac{1}{3}}} f(t)\,dt = \frac{x^{\frac{28}{15}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{11}{5}\right)}{\Gamma\left(\frac{43}{15}\right)},\ x\in I,$$

and the exact solution is $f(x)=\sqrt[5]{x}$, $x\in I$.

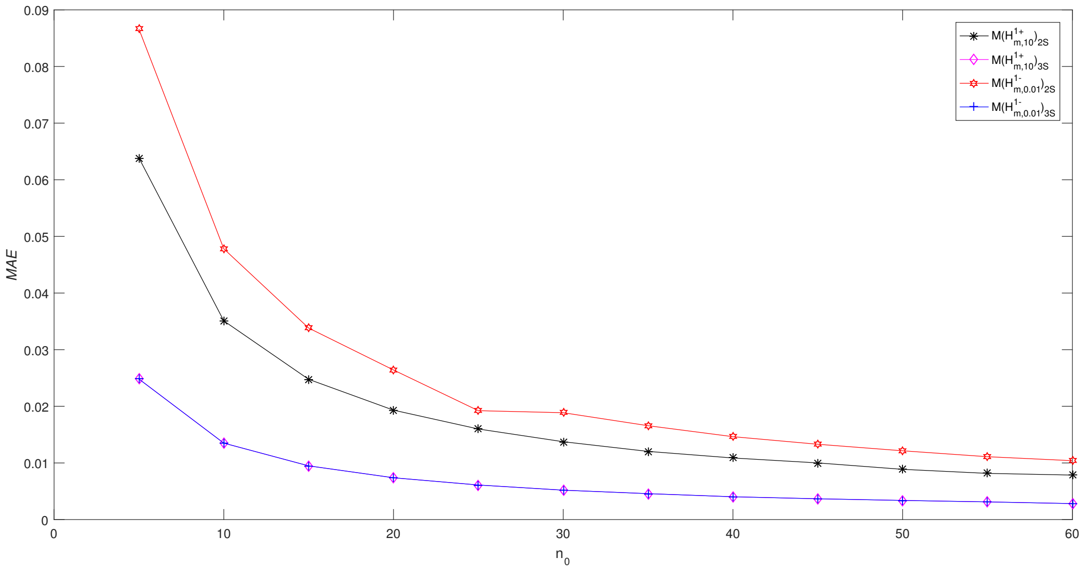

Figure 6.4: The MAE with respect to n_0 over the points $x_s = \frac{s}{10}, s = 0, 1, .., 10$ obtained by the proposed method $M\left(H^{1-}_{m,0.01}\right)_{qS}$ and $M\left(H^{1+}_{m,10}\right)_{qS}$ for $q = 2, 3$ steps for the Example 11.

Example 13: ($Ex13$), Yousefi [63],

$$\int_0^x \frac{1}{\sqrt{x-t}} f(t)\, dt = x, \; x \in I,$$

and the exact solution is $f(x) = \frac{2}{\pi}\sqrt{x}, \, x \in I$.

Example 14: ($Ex14$), Plato [87],

$$\frac{1}{\sqrt{\pi}} \int_0^x \frac{e^{-(x-t)}}{\sqrt{x-t}} f(t)\, dt = e^{-x}\left(x^5 + x^7 + x^9\right), \; x \in I,$$

and the exact solution is

$$f(x) = e^{-x}\left(\frac{5!}{\Gamma(5.5)}x^{4.5} + \frac{7!}{\Gamma(7.5)}x^{6.5} + \frac{9!}{\Gamma(9.5)}x^{8.5}\right), \; x \in I.$$

The solutions of the Example 12 and Example 13 have singular behaviour at $x = 0$. Table 6.6 presents $AE_{x_s}\left(H^{1-}_{m,0.2}\right)_{3S}$ and $AE_{x_s}\left(H^{1+}_{m,20}\right)_{3S}$ for the Example 12 at the points $x_s = \frac{s}{10}, \; s = 0, 1, ...10,$ for $m = 55, 64$ with degrees $D_{3S}(25, 20, 10)$ and

Table 6.6: The AE for the Example 12 obtained by the proposed method $M\left(H^{1+}_{m,20}\right)_{3S}$ and $M\left(H^{1-}_{m,0.2}\right)_{3S}$ for $m = 55, 64$.

x_s	$Ex12$ $M\left(H^{1+}_{55,20}\right)_{3S}$	$Ex12$ $M\left(H^{1+}_{64,20}\right)_{3S}$	$Ex12$ $M\left(H^{1-}_{55,0.2}\right)_{3S}$	$Ex12$ $M\left(H^{1-}_{64,0.2}\right)_{3S}$
0	0.0513282	0.0471597	0.0513282	0.0471597
0.1	1.099×10^{-6}	8.281×10^{-7}	1.004×10^{-5}	2.905×10^{-6}
0.2	1.920×10^{-7}	1.412×10^{-7}	1.112×10^{-6}	7.244×10^{-7}
0.3	1.739×10^{-5}	2.492×10^{-6}	3.021×10^{-7}	2.770×10^{-7}
0.4	1.134×10^{-5}	3.723×10^{-7}	3.139×10^{-8}	1.482×10^{-7}
0.5	1.376×10^{-6}	3.218×10^{-8}	2.299×10^{-5}	2.933×10^{-6}
0.6	5.220×10^{-7}	1.182×10^{-8}	1.153×10^{-4}	1.610×10^{-6}
0.7	2.893×10^{-7}	1.649×10^{-8}	9.272×10^{-6}	6.353×10^{-8}
0.8	2.835×10^{-8}	1.311×10^{-7}	3.304×10^{-6}	3.840×10^{-8}
0.9	1.962×10^{-7}	1.172×10^{-7}	2.148×10^{-6}	3.423×10^{-9}
1.0	1.906×10^{-6}	2.235×10^{-6}	8.048×10^{-5}	5.594×10^{-7}

$D_{3S}(32, 22, 10)$ accordingly.

Figure 6.5 and Figure 6.6 illustrate the maximum absolute errors (MAE) over the points $x_s = \frac{s}{10}, s = 0, 1, .., 10$ and $Cond\left(A^{1-}_1\right)$, $Cond\left(A^{1+}_1\right)$ accordingly with respect to n_0 by taking $D_{2S}(n_0, 10)$ and $D_{3S}(n_0, 22, 10)$ obtained by $M\left(H^{1-}_{m,0.2}\right)_{qS}$ and $M\left(H^{1+}_{m,20}\right)_{qS}$ for $q = 2, 3$ steps for the Example 12.

The $AE_{x_s}\left(H^{1-}_{m,0.05}\right)_{3S}$ and $AE_{x_s}\left(H^{1+}_{m,5}\right)_{3S}$ for the Example 13 at the points $x_s = \frac{s}{10}$, $s = 0, 1, ...10$, for $m = 55, 75$ are given in Table 6.7 with degrees $D_{3S}(30, 15, 10)$ and $D_{3S}(40, 20, 15)$ respectively. Further, Table 6.8 shows the $AE_{x_s}\left(H^{1-}_{50,0.7}\right)_{qS}$ and $AE_{x_s}\left(H^{1+}_{50,10}\right)_{qS}$ for the Example 14 at the same points for $q = 2, 3$ steps in accordance with $D_{2S}(40, 10)$ and $D_{3S}(25, 15, 10)$. Figure 6.7 illustrates the AE at the points $x_s = \frac{s}{10}, s = 0, 1, .., 10$ by taking $D_{2S}(35, 15)$ and $D_{3S}(20, 15, 15)$ for the Example 13 obtained by $M\left(H^{1-}_{50,0.9}\right)_{qS}$ and $M\left(H^{1+}_{50,5}\right)_{qS}$ for $q = 2, 3$ steps. From this

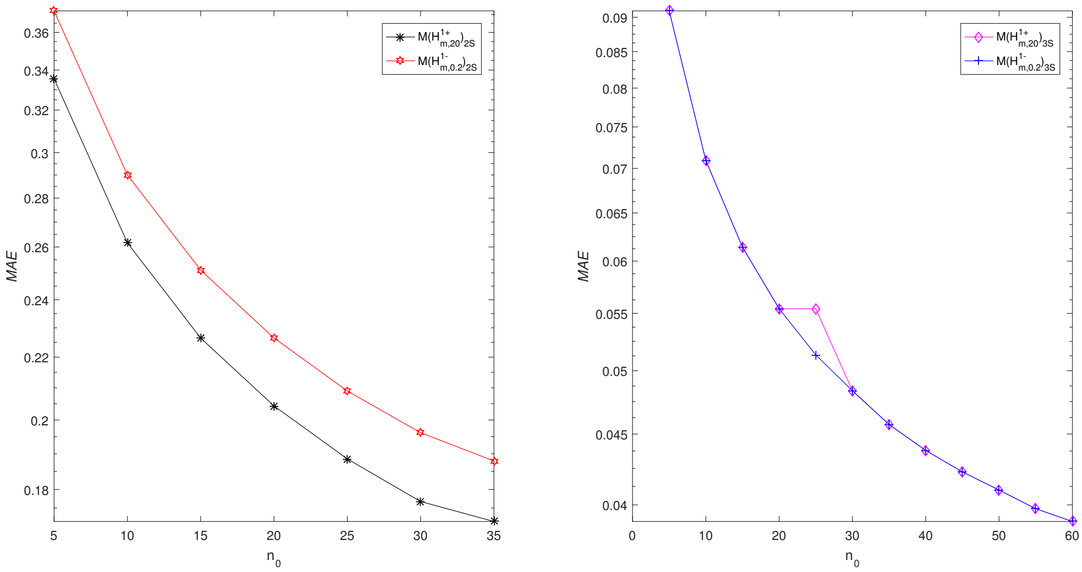

Figure 6.5: The MAE with respect to n_0 over the points $x_s = \frac{s}{10}, s = 0, 1, .., 10$ obtained by $M\left(H^{1-}_{m,0.2}\right)_{qS}$ and $M\left(H^{1+}_{m,20}\right)_{qS}$ for $q = 2, 3$ steps for Example 12.

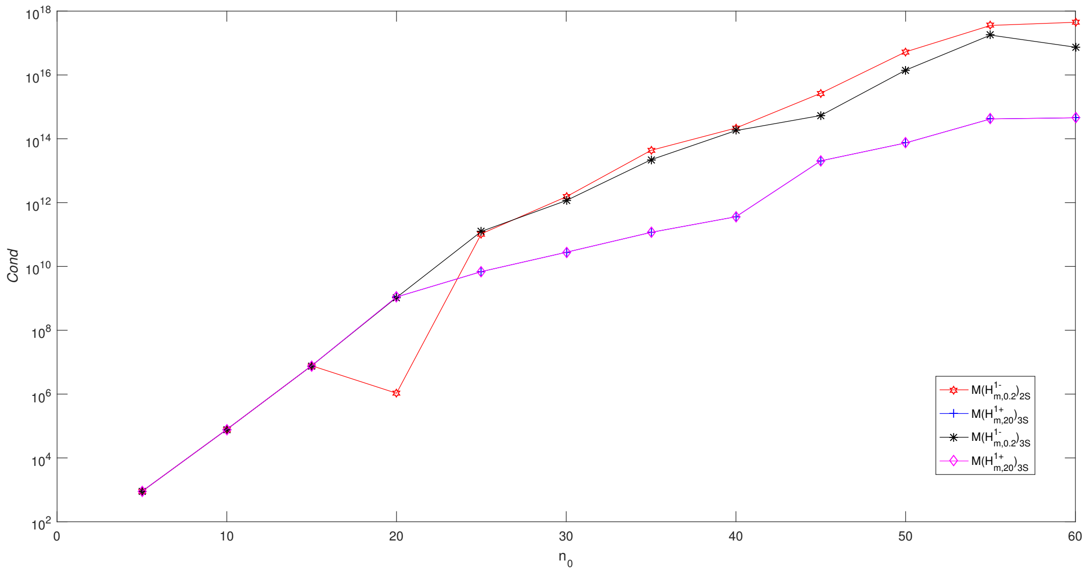

Figure 6.6: The $Cond\left(A^{1-}_1\right), Cond\left(A^{1+}_1\right)$ with respect to n_0 obtained by $M\left(H^{1-}_{m,0.2}\right)_{qS}$ and $M\left(H^{1+}_{m,20}\right)_{qS}$ for $q = 2, 3$ steps for the Example 12.

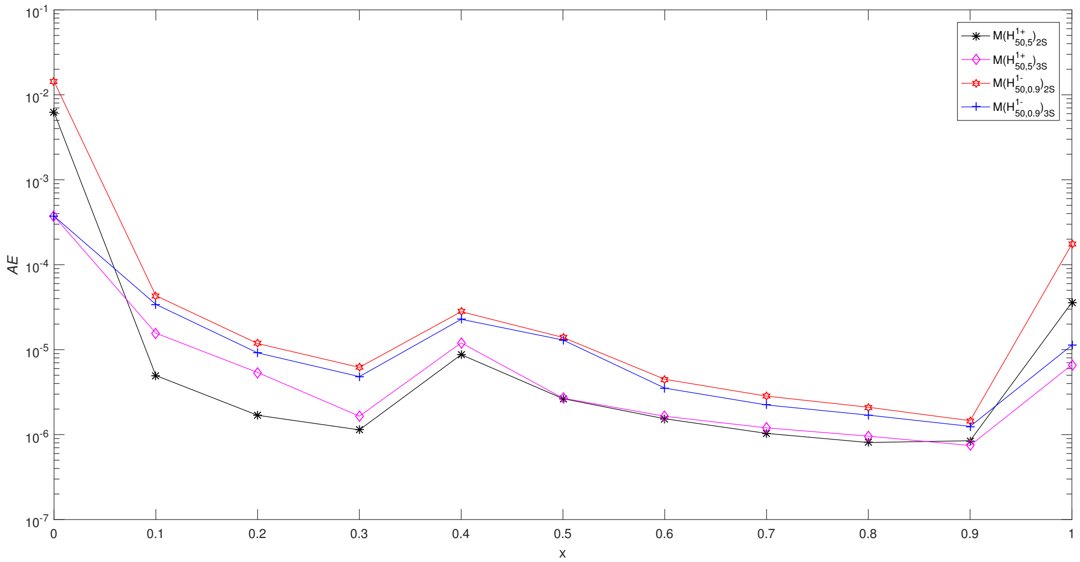

Figure 6.7: The AE for the Example 13 obtained by the combined method $M\left(H^{1+}_{50,5}\right)_{qS}$, and $M\left(H^{1-}_{50,0.9}\right)_{qS}$, for $q = 2,3$ steps.

figure it can be viewed that at the initial point $x = 0$ three step application of the given method gives better approximation. Furthermore, the MAE with respect to α obtained by the given combined method $M\left(H^{1-}_{50,\alpha}\right)_{qS}, M\left(H^{1+}_{50,\alpha}\right)_{qS}$ for $q = 2,3$ steps for the Example 14 are illustrated in Figure 6.8.

Table 6.7: The AE for the Example 13 obtained by the proposed method $M\left(H^{1+}_{55,5}\right)_{qS}$ and $M\left(H^{1-}_{55,0.05}\right)_{qS}$ for $q = 2,3$ steps.

x_s	$Ex13$ $M\left(H^{1+}_{55,5}\right)_{3S}$	$Ex13$ $M\left(H^{1+}_{75,5}\right)_{3S}$	$Ex13$ $M\left(H^{1-}_{55,0.05}\right)_{3S}$	$Ex13$ $M\left(H^{1-}_{75,0.05}\right)_{3S}$
0	2.768×10^{-4}	2.068×10^{-4}	2.768×10^{-4}	2.068×10^{-4}
0.1	1.564×10^{-5}	8.349×10^{-6}	3.924×10^{-5}	3.489×10^{-5}
0.2	5.391×10^{-6}	2.827×10^{-6}	2.020×10^{-5}	1.039×10^{-5}
0.3	1.650×10^{-6}	8.839×10^{-7}	1.102×10^{-5}	5.543×10^{-6}
0.4	1.204×10^{-5}	3.624×10^{-6}	7.379×10^{-6}	3.536×10^{-6}
0.5	2.685×10^{-6}	3.436×10^{-8}	5.745×10^{-6}	2.495×10^{-6}
0.6	1.644×10^{-6}	2.236×10^{-7}	9.393×10^{-5}	6.525×10^{-6}
0.7	1.202×10^{-6}	2.579×10^{-7}	1.347×10^{-5}	6.303×10^{-6}
0.8	9.596×10^{-7}	2.639×10^{-7}	5.185×10^{-6}	8.735×10^{-7}
0.9	7.452×10^{-7}	2.121×10^{-7}	3.692×10^{-6}	1.325×10^{-8}
1.0	6.641×10^{-6}	4.204×10^{-6}	7.559×10^{-6}	2.999×10^{-6}

Table 6.8: The AE for the Example 14 obtained by the proposed method $M\left(H^{1+}_{50,10}\right)_{qS}$ and $M\left(H^{1-}_{50,0.7}\right)_{qS}$ for $q = 2,3$.

x_s	$Ex14$ $M\left(H^{1+}_{50,10}\right)_{2S}$	$Ex14$ $M\left(H^{1+}_{50,10}\right)_{3S}$	$Ex14$ $M\left(H^{1-}_{50,0.7}\right)_{2S}$	$Ex14$ $M\left(H^{1-}_{50,0.7}\right)_{3S}$
0	7.278×10^{-10}	3.436×10^{-12}	4.543×10^{-10}	3.436×10^{-12}
0.1	2.995×10^{-11}	1.111×10^{-10}	2.845×10^{-11}	6.731×10^{-11}
0.2	5.420×10^{-10}	5.594×10^{-10}	5.416×10^{-10}	5.106×10^{-10}
0.3	2.721×10^{-9}	2.845×10^{-9}	2.712×10^{-9}	2.707×10^{-9}
0.4	6.362×10^{-9}	6.278×10^{-9}	9.023×10^{-9}	8.702×10^{-9}
0.5	1.621×10^{-8}	2.170×10^{-8}	2.205×10^{-8}	1.111×10^{-8}
0.6	5.626×10^{-8}	4.955×10^{-8}	4.637×10^{-8}	4.552×10^{-8}
0.7	9.989×10^{-8}	1.033×10^{-7}	1.011×10^{-7}	1.006×10^{-7}
0.8	2.067×10^{-7}	2.073×10^{-7}	2.000×10^{-7}	1.997×10^{-7}
0.9	3.656×10^{-7}	3.573×10^{-7}	3.651×10^{-7}	3.648×10^{-7}
1.0	6.115×10^{-7}	4.906×10^{-7}	6.301×10^{-7}	6.237×10^{-7}

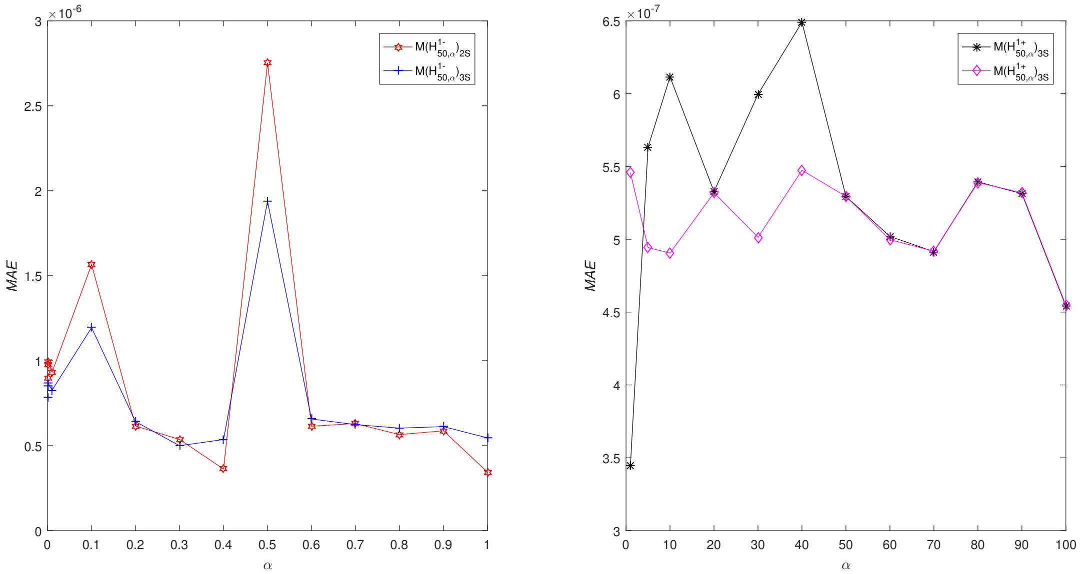

Figure 6.8: The MAE with respect to α obtained by the given combined method $M\left(H^{1-}_{50,\alpha}\right)_{qS}, M\left(H^{1+}_{50,\alpha}\right)_{qS}$ for $q = 2,3$ steps for the Example 14.

Chapter 7

CONCLUDING REMARKS

In this thesis we gave an approach that uses Modified Bernstein-Kantorovich operators to approximate the solution of the Fredholm and Volterra integral equations of first kind with smooth kernels. The method is developed first by representing the Modified Bernstein-Kantorovich operators such that the parameter α is also expressed explicitly in the operator. Further, the unknown function in the first kind integral equations is approximated by using the given form of the Modified Bernstein-Kantorovich operators so that the effect of α in the solution is analyzed. The obtained linear equations are transformed into system of algebraic linear equations. Furthermore, regularization technique is also applied to obtain more stable numerical solution when approximations are conducted using high order Modified Bernstein-Kantorovich operators. The proposed approach is simple and the obtained numerical results show that the accuracy is high even when low order approximations are used. Furthermore, hybrid positive linear operators which are defined by using the Bernstein-Kantorovich and Modified Bernstein-Kantorovich operators applied to certain subintervals of $[0,1]$ are given. A combined method using these operators is developed for solving the Volterra Abel-type integral equations of second kind. Additionally, the proposed combined method is also applied on the first kind Abel-type integral equations by first utilizing a regularization.

Remark 7.1: On the basis of experimental analysis we can conclude that the given numerical approach in Chapter 3 and the combined method in Chapter 5 consequently, the given algorithms are stable provided that there is no loss of significant numerical accuracy in the computations. The major reason of loosing accuracy in the numerical computations is due to the ill-conditioning of the coefficient matrices in the obtained algebraic linear systems. Therefore, preconditioning methods may be applied to precondition the obtained algebraic linear systems (see Buranay et al. [88] and Buranay and Iyikal [89], [90]). Furthermore, there are at least three ways to prevent loss of numerical accuracy in the computations, Kangro and Kangro [80]:

a. High precision arithmetics in computing can be used.

b. The exact formulas can be rewritten in a form that is not sensitive to round-off errors.

c. The system integrals may be computed numerically with sufficient accuracy.

Remark 7.2: The proposed combined method can be extended to solve the Volterra Abel-type integral equations on $[0,l]$, $l < \infty$.

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