Existence Results for Boundary Value Problems of Fractional Type Differential Equations

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ABSTRACT

The theme of this thesis is based on the solutions of fractional differential equations. We investigate the existence and uniqueness results of the fractional differential equations with boundary value conditions. Mostly, in this thesis, one of the fractional differential equation which is the Caputo type fractional differential equation is used and also, for the boundary conditions, different types of boundary conditions are used such as nonlocal Katugampola fractional integral conditions and nonlinear boundary conditions. The existence and uniqueness results of solutions are discussed by using standard fixed point theorems such as Banach fixed point theorem, Leray-Schauder nonlinear alternative and Krasnoselskii's fixed point theorem. Furthermore, Perov's fixed point theorem is investigated for multivariable operators. Moreover, Ulam Hyers stable is studied. In addition, for the nonlinear boundary conditions of Caputo type fractional differential equation, parametrization technique is used. So, numerical analytic scheme is established for finding the successive approximations. Theories which are studied in this thesis are illustrated with examples.

Keywords: Fractional differential equations; Katugampola fractional integral; Caputo fractional derivative; Riemann-Liouville fractional integral; fixed point theorems; parametrization technique; successive approximations; multivariable operations.

ÖZ

Bu tezin konusu kesirli diferansiyel denklemlerin çözümüne dayanmaktadır. Tanımlanmış olan kesirli diferensiyel denklemlerin varlığı ve tek çözüm olma sonuçları araştırıldı. Bu tezde, çoğunlukla, kesirli diferansiyel denklemlerden biri olan Caputo tipi kesirli diferansiyel denklem kullanılmıştır. Ayrıca, sınır koşulları için, yerel olmayan Katugampola kesirli integral koşulları ve doğrusal olmayan sınır koşulları gibi farklı sınır koşulları uygulanmıştır. Çözümlerin varlığı ve tek olma sonuçları, Banach sabit nokta teoremi, Leray-Schauder'ın doğrusal olmayan alternatifi ve Krasnoselskii'nin sabit nokta teoremleri kullanılarak tartışılmıştır. Ayrıca, Perov'un sabit nokta teoremi çok değişkenli operatörler için incelenmiştir. Ek olarak, Caputo tipi kesirli diferensiyel denklemin doğrusal olmayan sınır koşulları için parametreleme tekniği kullanılmıştır. Böylece, ardışık yaklaşılanları bulmak için sayısal analitik şema kullanılmıştır. Ayrıca, bu tezde incelenen teoriler örneklerle gösterilmiştir.

Anahtar Kelimeler: Kesirli diferansiyel denklemler; Katugampola kesirli integral; Caputo kesirli türevi; Riemann-Liouville kesirli integral; sabit nokta teoremleri; parametrizasyon tekniği; ardışık yaklaşımlar; çok değişkenli işlemler.

To My Lovely Family

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Chapter 1

INTRODUCTION

In recent years, boundary value problems for nonlinear fractional differential equations have been studied by several researches. In fact, fractional differential equations have been played important role in physics, chemical technology, biology, economics, control theory, signal and image processing, see [9, 10, 11, 12, 22, 35, 55, 45] and the references cited therein.

Boundary value problems of fractional differential equations and inclusions involving different kind of boundary conditions such as nonlocal, integral and multipoint boundary conditions. The fractional integral boundary conditions were introduced lately in [11] and nonlocal conditions were presented by Bitsadze, see [22].

In chapter 3 and 4, we study the existence and uniqueness solutions of Caputo type fractional differential equation with Katugampola fractional integral boundary conditions. In chapter 3, we consider the Caputo type boundary value problem for $\alpha \in (2,3]$ and $\alpha \in (1,2]$ with Katugampola fractional integral boundary conditions. In chapter 4, we study the Caputo type boundary value problem for $\alpha \in (2,3]$ subjected to additional case of Katugampola fractional integral boundary conditions which considered in chapter 3. In both chapters, the first results are related with existence and uniqueness of the solutions and the results are based on Banach fixed point theorem. The second results of chapter 3 and chapter 4 are about the existence of the solutions and the results are proved by using Leray-Shauder and Krasnoselskii's fixed point theorem. Also, in both chapters, the obtained results are illustrated with several examples.

In chapter 5, we consider the Caputo type fractional differential equation of order $\alpha \in (0,1\,]$ with nonlinear boundary conditions. An appropriate parametrization technique is used to transform the nonlinear boundary conditions of the Caputo type fractional differential equation to the linear boundary conditions. Thus, the successive approximations are constructed for studying of Caputo type fractional differential equation with parameterized boundary conditions. Furthermore, uniform convergence of the successive approximations are discussed. Under the some assumptions, we state the relationship between the parameterized limit function and exact solution. Finally, in the last part of chapter 5, we give an example to illustrate the theory of this study.

In last chapter, we investigate the existence and uniqueness solutions of Caputo type fractional differential equations with parameterized boundary conditions. Also, Ulan-Hyers stability is discussed for the solution of Caputo type boundary value problem. The result is based on Perov-type fixed point theorem.

Chapter 2

PRELIMINARIES

In this section, we recall some basic definitions of fractional calculus and present some standard fixed point theorems which are needed for analyzing of the study.

Definition 2.0.1 *([35]) Let $\beta > 0$ and h be a continuous function from $(0,\infty) \to \mathbb{R}$. Then,*

$$J^{\beta}h(r) = \frac{1}{\Gamma(\beta)} \int_0^r (r-t)^{\beta-1} h(t)dt,$$

is called Riemann-Liouville fractional integral of order β. Here, the notation Γ is gamma function which is defined by

$$\Gamma(\beta) = \int_0^{\infty} e^{-s} s^{\beta-1} ds.$$

Definition 2.0.2 *([35]) Let $\beta > 0$ and h be a continuous function from $(0,\infty) \to \mathbb{R}$. Then,*

$$D_{0+}^{\beta}h(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1} h(s)ds, \; n-1 < \beta < n,$$

is called Riemann-Liouville fractional derivative of order β, where $n = [\beta]+1$, and $[\beta]$ denotes the integer part of real number β.

Definition 2.0.3 *Let $\beta > 0$ and h be a continuous function from $(0,\infty) \to \mathbb{R}$. Then,*

$$^{c}D^{\beta}h(t) = D_{0+}^{\beta}\left(h(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}h^{(k)}(0)\right),\ t > 0\ ,\ n-1 < \beta < n.$$

is called Caputo derivative of order β.

Lemma 2.0.4 *(Nonlinear alternative for single-valued maps) Given a Banach space R, let C be a closed, convex subset of R. Also, U be an open subset of C and $0 \in U$. Assume that $F : \bar{U} \to C$ is a continuous, compact (that is, $F(\bar{U})$ is a relatively compact subset of C) map. Then either*

a) *F has a fixed point in $\bar{U}$, or*

b) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Lemma 2.0.5 *([36]) (Krasnoselskii's fixed point theorem) Given a Banach space X, let N be a closed, bounded, convex and nonempty subset of a Banach space X. Let A,B be the operators such that (a) $Ax + By \in N$ whenever $x, y \in N$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in N$ such that $z = Az + Bz$.*

Remark 2.0.6 *Given a space $C^n[0,\infty)$, let $h \in C^n[0,\infty)$. Then,*

$$^{c}D^{\beta}h(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t \frac{h^{(n)}(s)}{(t-s)^{\beta+1-n}}ds = I^{n-\beta}h^{(n)}(t), t > 0,\ n-1 < \beta < n.$$

Definition 2.0.7 *([33]) Let* $q, \rho > 0$. *Then, for all* $t \in (0,\infty)$, *the following integral is called Katugampola integral of a function* $h(t)$ *with orders* q *and* ρ *as follows:*

$$^{\rho}I^{q}h(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1}h(s)}{(t^{\rho}-s^{\rho})^{1-q}} ds. \tag{2.0.1}$$

Remark 2.0.8 *The definition* (2.0.1) *is equivalent to the one for Riemann-Liouville fractional integral of order* $q > 0$ *when* $\rho = 1$, *while the famous Hadamard fractional integral is written as follows for* $\rho \to 0$:

$$\lim_{\rho \to 0} {}^{\rho}I^{q}h(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} \frac{h(s)}{s} ds.$$

Lemma 2.0.9 *([7]) Let* ρ, $q > 0$ *and* $\beta > 0$ *be the given constants. Then the following formula holds:*

$$^{\rho}I^{q}t^{\beta} = \frac{\Gamma\left(\frac{\beta+\rho}{\rho}\right)}{\Gamma\left(\frac{\beta+\rho q+\rho}{\rho}\right)} \frac{t^{\beta+\rho q}}{\rho^{q}}.$$

Lemma 2.0.10 *([35]) Let* $q > 0$ *and* $x \in C(0,T) \cap L(0,T)$. *Then,*

$$x(t) = k_1 t^{q-1} + k_2 t^{q-2} + ... + k_n t^{q-n},$$

is a unique solution of the fractional differential equation $D^q x(t) = 0$ *where* $k_i \in \mathbb{R}$, $i = 1,...,n$, *and* $n-1 < q < n$.

Lemma 2.0.11 *Let* $q > 0$ *and* $x \in C(0,T) \cap L(0,T)$. *Then,*

$$J^q D^q x(t) = x(t) + k_1 t^{q-1} + k_2 t^{q-2} + ... + k_n t^{q-n},$$

where $k_i \in \mathbb{R}$, $i = 1,...,n$, *and* $n-1 < q < n$.

Theorem 2.0.12 *Let $\{h_n\}_1^\infty$ be a bounded and equicontinuous sequence in $C(X)$. Then, the sequence $\{h_n\}$ has a uniformly convergent subsequence. In this statement,*

(a) *" $F \subset C(X)$ is bounded" means that there exists a positive constant $M < \infty$ such that $|h(x)| \leq M$ for each $x \in X$ and each $h \in F$ and*

(b) *" $F \subset C(X)$ is equicontinuous" means that: for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends on ε) such that for $x, y \in X$:*

$$d(x,y) < \delta \Longrightarrow |h(x) - h(y)| < \varepsilon \ \forall \ h \in F,$$

where d is the metric on X.

Chapter 3

BOUNDARY VALUE PROBLEMS FOR NONLINEAR CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH KATUGAMPOLA FRACTIONAL INTEGRAL CONDITIONS

In chapter 3 , nonlinear differential equations are considered with Katugampola fractional integral boundary conditions. The aim of the study is to investigate the existence and uniqueness of solutions. In order to obtain the existence and uniqueness of solutions, some classical results of the fixed point theory are applied. Moreover, the obtained results are illustrated with the aid of examples. As a first problem, Caputo type fractional differential equation is considered for $2 < \alpha \leq 3$ as follows:

$$\begin{cases} {}^cD^{\alpha}x(t) = f(t,x(t)),\ t \in [0,T], \\ x(0) = 0,\ x(T) = \beta\ {}^{\rho}I^{q}\,x(\xi),\ 0 < \xi \leq T, \\ x'(T) = \gamma\,{}^{\rho}I^{q}\,x'(\eta),\ 0 < \eta \leq T, \end{cases} \tag{3.0.1}$$

and as a second problem, Caputo type fractional equation of order $1 < \alpha \leq 2$ is obtained as follows:

$$\begin{cases} {}^cD^{\alpha}x(t) = f(t,x(t)),\ t \in [0,T], \\ x(T) = \beta\ {}^{\rho}I^{q}\,x(\xi),\ 0 < \xi \leq T, \\ x'(T) = \gamma\,{}^{\rho}I^{q}\,x'(\eta),\ 0 < \eta \leq T. \end{cases} \tag{3.0.2}$$

In both problems, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, ${}^{\rho}I^{q}$ is the Katugampola fractional integral of order $q > 0$, $\rho > 0$. Also, the function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function in both (3.0.1) and (3.0.2) and $\beta, \gamma \in \mathbb{R}$.

The solutions of problem (3.0.1) and (3.0.2) are obtained for their associated linear problem. Consequently, the following lemmas are considered with the linear variant of the problems (3.0.1) and (3.0.2).

Lemma 3.0.1 *([56]) Let* $2 < \alpha \leq 3$ *and* $\beta, \gamma \in \mathbb{R}$ *. Then, for any* $y \in C([0,T], \mathbb{R})$ *,* x *is a solution of the following Caputo type fractional boundary value problem:*

$$\begin{cases} {}^{c}D^{\alpha}x(t) = y(t), \\ x(0) = 0,\ x(T) = \beta\ {}^{\rho}I^{q}\ x(\xi),\ 0 < \xi \leq T, \\ x'(T) = \gamma\ {}^{\rho}I^{q}\ x'(\eta),\ 0 < \eta \leq T, \end{cases} \tag{3.0.3}$$

if and only if

$$\begin{aligned} x(t) = J^{\alpha}y(t) &+ \frac{t}{\Delta}\left(2\omega_2(\gamma,\eta) + t\omega_1(\gamma,\eta)\right)\beta\ {}^{\rho}I^{q}\ J^{\alpha}y(\xi) \\ &+ \frac{t}{\Delta}\left(-\omega_3(\beta,\xi) + t\omega_2(\beta,\xi)\right)\gamma\ {}^{\rho}I^{q}\ J^{\alpha-1}y(\eta) \\ &- \frac{t}{\Delta}\left(2\omega_2(\gamma,\eta) + t\omega_1(\gamma,\eta)\right)J^{\alpha}y(T)) \\ &+ \frac{t}{\Delta}(\omega_3(\beta,\xi) - t\omega_2(\beta,\xi))J^{\alpha-1}y(T), \end{aligned} \tag{3.0.4}$$

where

$$\Delta = 2\omega_2(\beta,\xi)\,\omega_2(\gamma,\eta) + \omega_3(\beta,\xi)\,\omega_1(\gamma,\eta) \neq 0, \tag{3.0.5}$$

$$\omega_1(\beta,\xi) = \left(\beta\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)} - 1\right), \tag{3.0.6}$$

$$\omega_2(\beta,\xi) = \left(T - \beta\frac{\xi^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}\right), \tag{3.0.7}$$

$$\omega_3(\beta,\xi) = \left(T^2 - \beta\frac{\xi^{\rho q+2}}{\rho^q}\frac{\Gamma(\frac{2+\rho}{\rho})}{\Gamma(\frac{2+\rho q+\rho}{\rho})}\right). \tag{3.0.8}$$

Proof. By using Lemmas (2.0.10)-(2.0.11), the general solution of the fractional differential equation in (3.0.3) can be expressed as:

$$x(t) = c_0 + c_1 t + c_2 t^2 + J^\alpha y(t) \tag{3.0.9}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ are unknown arbitrary constants. Then, the Katugampola fractional integral operator is applied on (3.0.9) and simultaneously, Lemma (2.0.9) is used. Thus, the following equation is obtained:

$$\begin{aligned} {}^\rho I^q\, x(t) = {} & c_0\frac{t^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)} + c_1\frac{t^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})} \\ & + c_2\frac{t^{\rho q+2}}{\rho^q}\frac{\Gamma(\frac{2+\rho}{\rho})}{\Gamma(\frac{2+\rho q+\rho}{\rho})} + {}^\rho I^q\, J^\alpha y(t). \end{aligned} \tag{3.0.10}$$

After that, second boundary condition of (3.0.3) is applied on (3.0.10) and the equation (3.0.11) is found as follows:

$$\begin{aligned} J^\alpha y(T) + c_2 T^2 + c_1 T + c_0 = {} & \beta c_0\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)} + \beta c_1\frac{\xi^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})} \\ & + \beta c_2\frac{\xi^{\rho q+2}}{\rho^q}\frac{\Gamma(\frac{2+\rho}{\rho})}{\Gamma(\frac{2+\rho q+\rho}{\rho})} + \beta\, {}^\rho I^q\, J^\alpha y(\xi). \end{aligned} \tag{3.0.11}$$

Also, by using the fractional integral condition of (3.0.3), c_0 is equal to 0. Therefore, the equation (3.0.11) is rewritten as follows:

$$c_1\left(T-\beta\frac{\xi^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}\right)+c_2\left(T^2-\beta\frac{\xi^{\rho q+2}}{\rho^q}\frac{\Gamma(\frac{2+\rho}{\rho})}{\Gamma(\frac{2+\rho q+\rho}{\rho})}\right)$$
$$=\beta\,{}^\rho I^q\,J^\alpha y(\xi)-J^\alpha y(T). \tag{3.0.12}$$

Besides, the first derivative of the general solution of the fractional differential equation in (3.0.3) is written as follows:

$$x'(t)=c_1+2c_2t+J^{\alpha-1}y(t). \tag{3.0.13}$$

After applied the Katugampola fractional integral on (3.0.13) and used Lemma (2.0.9), the following equation is obtained:

$${}^\rho I^q x'(t)=c_1\frac{t^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}+2c_2\frac{t^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}$$
$$+\,{}^\rho I^q J^{\alpha-1}y(t). \tag{3.0.14}$$

By using the equations (3.0.13) and (3.0.14), following equation is obtained as follows:

$$J^{\alpha-1}y(T)+2c_2T+c_1$$
$$=\gamma c_1\frac{\eta^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}+\gamma 2c_2\frac{\eta^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}$$
$$+\gamma\,{}^\rho I^q\,J^{\alpha-1}y(\eta). \tag{3.0.15}$$

After collecting similar terms on one side, the following equation is obtained:

$$c_1\left(1-\gamma\frac{\eta^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right)+c_2\left(2T-2\gamma\frac{\eta^{\rho q+1}}{\rho^q}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}\right)$$
$$=\gamma\,{}^\rho I^q\,J^{\alpha-1}y(\eta)-J^{\alpha-1}y(T). \tag{3.0.16}$$

In order to find the values c_1 and c_2, the equations (3.0.12) and (3.0.16) are considered. The results are found as follows:

$$c_1 = \frac{1}{\Delta}\left(2\omega_2(\gamma,\eta)\left(\beta\, {}^{\rho}I^{q}\, J^{\alpha}y(\xi) - J^{\alpha}y(T)\right)\right.$$
$$\left. -\omega_3(\beta,\xi)\left(\gamma\, {}^{\rho}I^{q}\, J^{\alpha-1}y(\eta) - J^{\alpha-1}y(T)\right)\right)$$

and

$$c_2 = \frac{1}{\Delta}\left(\omega_2(\beta,\xi)\left(\gamma\, {}^{\rho}I^{q}\, J^{\alpha-1}y(\eta) - J^{\alpha-1}y(T)\right)\right.$$
$$\left. +\omega_1(\gamma,\eta)\left(\beta\, {}^{\rho}I^{q}\, J^{\alpha}y(\xi) - J^{\alpha}y(T)\right)\right)$$

As a final step, c_0, c_1 and c_2 are inserted in (3.0.9), then the formula (3.0.4) is obtained. Conversly, the general solution (3.0.4) satisfies the problem (3.0.3) by direct computation. The proof is completed. ■

The following lemma is involved with the solution of the problem (3.0.2). The proof is not provided as it based on the similar method with previous proof.

Lemma 3.0.2 *([56]) Let $1 < \alpha \leq 2$ and $\beta, \gamma \in \mathbb{R}$. Then, x is a solution of the following fractional boundary value problem for any $y \in C([0,T], \mathbb{R})$,*

$$\begin{cases} {}^{c}D^{\alpha}x(t) = y(t)\,, \\ x(T) = \beta\, {}^{\rho}I^{q}\, x(\xi),\ 0 < \xi \leq T, \\ x'(T) = \gamma\, {}^{\rho}I^{q}\, x'(\eta),\ 0 < \eta \leq T, \end{cases}$$

if and only if

$$x(t) = J^{\alpha}y(t) + \frac{1}{\omega_1(\beta,\xi)}\left(J^{\alpha}y(T) - \beta\, {}^{\rho}I^{q}\, J^{\alpha}y(\xi)\right)$$
$$+ \left(\frac{1}{\omega_1(\gamma,\eta)}\left(\frac{\omega_2(\beta,\xi)}{\omega_1(\beta,\xi)} + t\right)\left(J^{\alpha-1}y(T) - \gamma\, {}^{\rho}I^{q}\, J^{\alpha-1}y(\eta)\right)\right),$$

where

$$\omega_1(\beta,\xi) \neq 0 \text{ and } \omega_1(\gamma,\eta) \neq 0$$

which is given in (3.0.6).

3.1 Main Results

This section is dedicated to the main results concerning the existence and uniqueness results for the problems (3.0.1) – (3.0.2). The results are proved by using Banach fixed point theorem, Leray-Schauder nonlinear alternative and Krasnoselskii's fixed point theorem which are the standard fixed point theorems.

In order to prove the existence theorems for boundary value problems (3.0.1) – (3.0.2), the operators $S, \widehat{S} : C \to C$ are introduced as follows:

$$\begin{aligned}(Sx)(t) &= J^{\alpha} f(s,x(s))(t) - \frac{t}{\Delta}(2\omega_2(\gamma,\eta) \\ &\quad + t\omega_1(\gamma,\eta))J^{\alpha} f(s,x(s))(T) + \frac{t}{\Delta}(\omega_3(\beta,\xi) \\ &\quad - t\omega_2(\beta,\xi))J^{\alpha-1} f(s,x(s))(T) + \frac{t}{\Delta}(2\omega_2(\gamma,\eta) \\ &\quad + t\omega_1(\gamma,\eta))\beta\, {}^{\rho}I^{q}\, J^{\alpha} f(s,x(s))(\xi) + \frac{t}{\Delta}(-\omega_3(\beta,\xi) \\ &\quad + t\omega_2(\beta,\xi))\gamma\, {}^{\rho}I^{q}\, J^{\alpha-1} f(s,x(s))(\eta), \end{aligned} \tag{3.1.1}$$

$$\begin{aligned}(\widehat{S}x)(t) &= J^{\alpha} f(s,x(s))(t) + \frac{1}{\omega_1(\beta,\xi)}(J^{\alpha} f(s,x(s))(T) \\ &\quad - \beta\, {}^{\rho}I^{q}\, J^{\alpha} f(s,x(s))(\xi)) + \left(\frac{1}{\omega_1(\gamma,\eta)}\left(\frac{\omega_2(\beta,\xi)}{\omega_1(\beta,\xi)} + t\right)\right. \\ &\quad \left. \times J^{\alpha-1} f(s,x(s))(T) - \gamma\, {}^{\rho}I^{q}\, J^{\alpha-1} f(s,x(s))(\eta)\right).\end{aligned}$$

Also, the notations are defined as follows:

$$
\begin{aligned}
\Omega := {} & \frac{T\,|\beta|\,(2\,|\omega_2(\gamma,\eta)|+T\,|\omega_1(\gamma,\eta)|)}{|\Delta|} \\
& \times \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{1}{\Gamma(\alpha+1)} \\
& + \frac{T\,|\gamma|\,(|\omega_3(\beta,\xi)|+T\,|\omega_2(\beta,\xi)|)}{|\Delta|} \\
& \times \frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^q}\frac{1}{\Gamma(\alpha)},
\end{aligned}
\tag{3.1.2}
$$

$$
\begin{aligned}
\Omega_1 := {} & \frac{T^\alpha}{\Gamma(\alpha+1)} \\
& + \frac{T}{|\Delta|}\frac{T^\alpha}{\Gamma(\alpha+1)}(2\,|\omega_2(\gamma,\eta)|+T\,|\omega_1(\gamma,\eta)|) \\
& + \frac{T}{|\Delta|}\frac{T^{\alpha-1}}{\Gamma(\alpha)}(|\omega_3(\beta,\xi)|+T\,|\omega_2(\beta,\xi)|) \\
& + \Bigg(\frac{T\,|\beta|\,(2\,|\omega_2(\gamma,\eta)|+T\,|\omega_1(\gamma,\eta)|)}{|\Delta|} \\
& \times \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{1}{\Gamma(\alpha+1)}\Bigg) \\
& + \Bigg(\frac{T\,|\gamma|\,(|\omega_3(\beta,\xi)|+T\,|\omega_2(\beta,\xi)|)}{|\Delta|} \\
& \times \frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^q}\frac{1}{\Gamma(\alpha)}\Bigg),
\end{aligned}
\tag{3.1.3}
$$

and

$$
\begin{aligned}
\Theta := {} & \frac{|\beta|}{|\omega_1(\beta,\xi)|}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{1}{\Gamma(\alpha+1)} \\
& + \Bigg(\frac{|\gamma|}{|\omega_1(\gamma,\eta)|}\left(\frac{|\omega_2(\beta,\xi)|}{|\omega_1(\beta,\xi)|}+T\right) \\
& \times \frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^q}\frac{1}{\Gamma(\alpha)}\Bigg),
\end{aligned}
\tag{3.1.4}
$$

$$
\begin{aligned}
\Theta_1 :=& \frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \left(\frac{1}{|\omega_1(\beta,\xi)|\Gamma(\alpha+1)} \right. \\
&\times \left. \left(T^{\alpha} + \frac{|\beta|\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})} \frac{\xi^{\alpha+\rho q}}{\rho^{q}} \right) \right) \\
&+ \left(\frac{1}{|\omega_1(\gamma,\eta)|\Gamma(\alpha)} \left(\frac{|\omega_2(\beta,\xi)|}{|\omega_1(\beta,\xi)|} + T \right) \right. \\
&\times \left. \left(T^{\alpha-1} + \frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})} \frac{\eta^{\alpha-1+\rho q}}{\rho^{q}} \frac{|\gamma|}{\Gamma(\alpha)} \right) \right). \qquad (3.1.5)
\end{aligned}
$$

In the following subsections, existence and uniqueness results for problem (3.0.1) are stated and proved. Also, existence and uniqueness results for problem (3.0.2) are defined but the proofs are omitted because of the having similar results with problem (3.0.1).

3.1.1 Existence and Uniqueness Results

The first result is about the existence and uniqueness of the solution. The result is proved by Banach fixed point theorem.

Theorem 3.1.1 *([56]) Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A_1) *Let $L > 0$ and $x, y \in \mathbb{R}$. Then, $|f(t,x) - f(t,y)| \leq L\|x-y\|$, for all $t \in [0,T]$,*

(A_2) *$L\Omega_1 < 1$, then unique solution exists on $[0,T]$ for the boundary value problem (3.0.1).*

Proof. From the definition of S in $(3.1.1)$ and assumption (A_1), the following inequality is written for $x,\, y \in C$ and $\forall\, t \in [0,T]$ as follows:

$$
\begin{aligned}
& |(Sx)(t) - (Sy)(t)| \\
& \leq J^{\alpha} |f(s,x(s)) - f(s,y(s))| (T) \\
& + \frac{T|\beta|}{|\Delta|} (2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|)\ {}^{\rho}I^{q}\, J^{\alpha} |f(s,x(s)) - f(s,y(s))| (\xi) \\
& + \frac{T|\gamma|}{|\Delta|} (|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|)\ {}^{\rho}I^{q}\, J^{\alpha-1} |f(s,x(s)) - f(s,y(s))| (\eta) \\
& + \frac{T}{|\Delta|}(2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|) J^{\alpha} |f(s,x(s)) - f(s,y(s))| (T) \\
& + \frac{T}{|\Delta|}(|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|) J^{\alpha-1} |f(s,x(s)) - f(s,y(s))| (T) \\
& \leq L\|x-y\| J^{\alpha}(1)(T) + L\|x-y\| \left[\frac{T|\beta|}{|\Delta|} (2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|) \right. \\
& \times {}^{\rho}I^{q}\, J^{\alpha}(1)(\xi)] + L\|x-y\| \left[\frac{T|\gamma|}{|\Delta|} (|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|) \right. \\
& \times {}^{\rho}I^{q}\, J^{\alpha-1}(1)(\eta)\big] + L\|x-y\| \left[\frac{T}{|\Delta|}(2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|) \right. \\
& \times J^{\alpha}(1)(T)] + L\|x-y\| \frac{T}{|\Delta|}(|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|) J^{\alpha-1}(1)(T) \\
& \leq L \left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Delta|} \frac{T^{\alpha}}{\Gamma(\alpha+1)} (2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|) \right. \\
& \frac{T}{|\Delta|} \frac{T^{\alpha-1}}{\Gamma(\alpha)} (|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|) + \left(\frac{T|\beta|}{|\Delta|} T (2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|) \right. \\
& \left. \times \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})} \frac{\xi^{\alpha+\rho q}}{\rho^{q}} \frac{1}{\Gamma(\alpha+1)} \right) + \left(\frac{T|\gamma|}{|\Delta|} (|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|) \right. \\
& \left. \left. \times \frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})} \frac{\eta^{\alpha-1+\rho q}}{\rho^{q}} \frac{1}{\Gamma(\alpha)} \right) \right] \|x-y\| \\
& = L\Omega_1 \|x-y\|.
\end{aligned}
$$

Hence,

$$\|Sx - Sy\| \leq L\Omega_1 \|x - y\|.$$

As $L\Omega_1 < 1$, by the assumption (A_2) the operator $S : C \to C$ is a contraction map. Here C is a Banach space. As a result, by the Banach fixed point theorem the boundary value problem (3.0.1) has a unique solution on $[0,T]$. The proof is completed. ■

Theorem 3.1.2 *([56]) Given a continuous function f which is from $[0,T] \times \mathbb{R} \to \mathbb{R}$, let (A_1) holds. If*

$$L\Theta_1 < 1,$$

where Θ_1 is defined by (3.1.5), then the boundary value problem (3.0.2) has a unique solution on $[0,T]$.

3.1.2 Existence Results

The second result is related with existence of solutions. To prove this result, Leray-Schauder and Krasnoselskii's fixed point theorem are used.

Theorem 3.1.3 *([56]) Given a continuous function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A_3) *Let $\phi \in C([0,T],\mathbb{R})$ be a nonnegative function and $\Psi : [0,\infty) \to (0,\infty)$ be a non-decreasing function. Then,*

$$|f(t,u)| \leq \phi(t)\Psi(|u|) \text{ for any } (t,u) \in [0,T] \times \mathbb{R}$$

(A_4) *Let $N > 0$ be a positive constant such that*

$$\frac{N}{\Psi(N)\|\phi\|\Omega_1} > 1,$$

where Ω_1 *in* $(3.1.3)$, *then at least one solution exists for the problem* $(3.0.1)$ *on the interval* $[0,T]$.

Proof. A set

$$B_\ell = \{x \in C : \|x\| \le \ell\}$$

is defined as closed and bounded in $C([0,T],\mathbb{R})$. Existence of a solution of the problem $(3.0.1)$ is equivalent to the problem of finding fixed point of S from $B_\ell \to C([0,T],\mathbb{R})$. Then $\forall\, t \in [0,T]$, we have:

$$\begin{aligned}
\|(Sx)(t)\| \le\ & J^{\alpha}|f(s,x(s))|(T) \\
& + \bigg(\frac{T|\beta|(2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|)}{|\Delta|} \\
& \times {}^{\rho}I^{q}\, J^{\alpha}|f(s,x(s))|(\xi)\bigg) \\
& + \bigg(\frac{T|\gamma|(|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|)}{|\Delta|} \\
& \times {}^{\rho}I^{q}\, J^{\alpha-1}|f(s,x(s))|(\eta)\bigg) \\
& + \bigg(\frac{T(2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|)}{|\Delta|} \\
& \times J^{\alpha}|f(s,x(s))|(T)\bigg) \\
& + \bigg(\frac{T(|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|)}{|\Delta|} \\
& \times J^{\alpha-1}|f(s,x(s))|(T)\bigg) \\
& \le \Psi(\|x\|)\, J^{\alpha}\phi(s)(T) \\
& + \bigg(\Psi(\|x\|)\,\frac{|\beta|\,T(2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|)}{|\Delta|} \\
& \times {}^{\rho}I^{q}\, J^{\alpha}\phi(s)(\xi)\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg(\Psi(\|x\|)\frac{T|\gamma|(|\omega_3(\beta,\xi)|+T|\omega_2(\beta,\xi)|)}{|\Delta|}\\
&\times{}^{\rho}I^{q}J^{\alpha-1}\phi(s)(\eta)\Big)\\
&+\Bigg(\Psi(\|x\|)\frac{T(2|\omega_2(\gamma,\eta)|+T|\omega_1(\gamma,\eta)|)}{|\Delta|}\\
&\times J^{\alpha}\phi(s)(T))\\
&+\Bigg(\Psi(\|x\|)\frac{T(|\omega_3(\beta,\xi)|+T|\omega_2(\beta,\xi)|)}{|\Delta|}\\
&\times J^{\alpha-1}\phi(s)(T)\Big)\\
&\le\|\phi\|\Psi(\ell)\Bigg\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}\\
&+\frac{T}{|\Delta|}\frac{T^{\alpha}}{\Gamma(\alpha+1)}(2|\omega_2(\gamma,\eta)|+T|\omega_1(\gamma,\eta)|\\
&+\frac{T}{|\Delta|}\frac{T^{\alpha-1}}{\Gamma(\alpha)}(|\omega_3(\beta,\xi)|+T|\omega_2(\beta,\xi)|)\\
&+\Bigg(\frac{T|\beta|(2|\omega_2(\gamma,\eta)|+T|\omega_1(\gamma,\eta)|)}{|\Delta|}\\
&\times\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\xi^{\alpha+\rho q}}{\rho^{q}}\frac{1}{\Gamma(\alpha+1)}\Bigg)\\
&+\Bigg(\frac{T|\gamma|(|\omega_3(\beta,\xi)|+T|\omega_2(\beta,\xi)|)}{|\Delta|}\\
&\times\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^{q}}\frac{1}{\Gamma(\alpha)}\Bigg)\Bigg\}\\
&=\Psi(\ell)\|\phi\|\Omega_1<\ell.
\end{aligned}
$$

Secondly, we show that the map $S:C([0,T],)\to C([0,T],\mathbb{R})$ is completely continuous. Therefore, the map S sends bounded sets into relatively compact set of $C([0,T],\mathbb{R})$. Let τ_1,τ_2 be in the interval $[0,T]$. Also, let us consider $\tau_1<\tau_2$. Then,

$$
|(Sx)(\tau_2)-(Sx)(\tau_1)|\le|J^{\alpha}f(s,x(s))(\tau_2)-J^{\alpha}f(s,x(s))(\tau_1)|
$$

$$
\begin{aligned}
&+\left(\frac{|\tau_2-\tau_1|}{|\Delta|}(2|\omega_2(\gamma,\eta)|\ +(\tau_2+\tau_1)|\omega_1(\gamma,\eta)|)\right.\\
&\times|\beta|\ {}^{\rho}I^{q}\,J^{\alpha}|f(s,x(s))|(\xi)\Big)+\left(\frac{|\tau_2-\tau_1|}{|\Delta|}(|\omega_3(\beta,\xi)|\ +(\tau_2+\tau_1)|\omega_2(\beta,\xi)|)\right.\\
&\times|\gamma|\ {}^{\rho}I^{q}J^{\alpha-1}|f(s,x(s))|)(\eta)\Big)+\left(\frac{|\tau_2-\tau_1|}{|\Delta|}(2|\omega_2(\gamma,\eta)|\right.\\
&+(\tau_2+\tau_1)|\omega_1(\gamma,\eta)|)\ \times J^{\alpha}|f(s,x(s))|(T)\Big)+\left(\frac{|\tau_2-\tau_1|}{|\Delta|}(|\omega_3(\beta,\xi)|\right.\\
&+(\tau_2+\tau_1)|\omega_2(\beta,\xi)|)\ \times J^{\alpha-1}|f(s,x(s))|(T)\Big)\\
&\le\frac{\Psi(r)\,\|\phi\|}{\Gamma(\alpha)}\left[\int_0^{\tau_1}\left[(\tau_2-s)^{\alpha-1}\ -(\tau_1-s)^{\alpha-1}\right]ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\right]\\
&+\frac{\Psi(r)\,|\tau_2-\tau_1|}{|\Delta|}\{(2|\omega_2(\gamma,\eta)|)+(\tau_2+\tau_1)|\omega_1(\gamma,\eta)|\\
&\times|\beta|\ {}^{\rho}I^{q}\,J^{\alpha}\phi(s)(\xi)+(|\omega_3(\beta,\xi)|)\times|\gamma|\ {}^{\rho}I^{q}J^{\alpha-1}\phi(s)(\eta)\big)+(2|\omega_2(\gamma,\eta)|\\
&+(\tau_2+\tau_1)|\omega_1(\gamma,\eta)|)J^{\alpha}\phi(s))(T)\\
&+(|\omega_3(\beta,\xi)|\ +(\tau_2+\tau_1)|\omega_2(\beta,\xi)|)J^{\alpha-1}\phi(s)(T)\big)\}.
\end{aligned}
$$

From the previous inequality, it follows that the difference of $(Sx)(\tau_2)$ and $(Sx)(\tau_1)$ are not depent on x. Therefore, the right hand side of the inequality tends to zero. Thus, S is equicontinuous. Consequently, by the Arzela-Ascoli Theorem 2.0.12, the operator $S:C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ is completely continuous.

In the following step, we show that the map S has a fixed point. Let us define

$$A=\{x\in C([0,T],\mathbb{R}):\|x\|<N\}.$$

Then, the operator S is defined as continuous and completely continuous from $\overline{A}$ to $C([0,T],\mathbb{R})$. From the choice of A, for some $\eta\in(0,1)$, there is no x which belongs to ∂A such that $x=\eta Sx$. Here, for proving this result we use contradiction. Then, there exists $x \in \partial A$ such that $x = \eta Sx$ for some $\eta \in (0,1)$. Then,

$$\begin{aligned} \|x\| &= \|\mu Sx\| \leq \|Sx\| \\ &\leq \Psi(\|x\|)\,\|\phi\|\,\Omega_1 \end{aligned}$$

which means

$$\frac{\|x\|}{\Psi(\|x\|)\,\|\phi\|\,\Omega_1} \leq 1$$

This is contradict with

$$\frac{\|N\|}{\Psi(\|N\|)\,\|\phi\|\,\Omega_1} > 1.$$

Consequently, according to the Leray-Scahauder's nonlinear alternative, the operator S has a fixed point x in $\overline{A}$ which is the solution of the problem $(3.0.1)$. This completes the proof. ■

Theorem 3.1.4 *([56]) Given a continuous function f: $[0,T] \times \mathbb{R} \to \mathbb{R}$. Assume that (A_3) holds. If there exist a constant $N > 0$ such that*

$$\frac{N}{\Psi(M)\,\|\phi\|\,\Theta_1} > 1.$$

where Θ_1 is defined in $(3.1.5)$, then at least one solution exists on the inteval $[0,T]$ for the boundary value problem (3.0.2).

Theorem 3.1.5 *([56]) Given a continuous function $f: [0,T] \times \mathbb{R} \to \mathbb{R}$. Assume that (A_1) holds. Then, the function f satisfies the following assumptions:*

(A_5) *there exist a nonnegative function $\phi \in C([0,T], \mathbb{R})$ such that*

$$|f(t,u)| \leq \phi(t) \text{ for any } (t,u) \in [0,T] \times \mathbb{R}$$

(A_6) *$L\Omega < 1$ where Ω is defined in* (3.1.2). *Then the boundary value problem* (3.0.1) *has at least one solution on* $[0,T]$.

Proof. The new operators S_1 and S_2 are defined as

$$S_1x = J^{\alpha} f(s,x(s))(t) - \frac{t}{\Delta}(2\omega_2(\gamma,\eta)$$
$$+t\omega_1(\gamma,\eta))J^{\alpha} f(s,x(s))(T)$$
$$+\frac{t}{\Delta}(\omega_3(\beta,\xi) - t\omega_2(\beta,\xi))J^{\alpha-1} f(s,x(s))(T)$$

and

$$S_2x = \frac{t}{\Delta}(2\omega_2(\gamma,\eta) + t\omega_1(\gamma,\eta))\beta\, {}^{\rho}I^{q}\, J^{\alpha} f(s,x(s))(\xi)$$
$$+\frac{t}{\Delta}(-\omega_3(\beta,\xi) + t\omega_2(\beta,\xi)\gamma\, {}^{\rho}I^{q}\, J^{\alpha-1} f(s,x(s))(\eta)$$

First of all, we define $B_\ell = \{x \in C : \|x\| \leq \ell\}$ with

$$\ell \geq \|\phi\|\,\Omega_1,$$

and Ω_1 is defined in (3.1.3). Now, we show that $S_1x + S_2y \in B_\ell$. For any x, $y \in B_\ell$, we have :

$$\|S_1x + S_2y\| \leq J^{\alpha}|f(s,x(s))|(T)$$
$$+\left(\frac{T(2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|)}{|\Delta|}\; J^{\alpha}|f(s,x(s))|(T)\right)$$
$$+\left(\frac{T(|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|)}{|\Delta|}\; J^{\alpha-1}|f(s,x(s))|(T)\right)$$
$$+\left(\frac{T|\beta|(2|\omega_2(\gamma,\eta)| + T|\omega_1(\gamma,\eta)|)}{|\Delta|}\; {}^{\rho}I^{q}J^{\alpha}|f(s,y(s))|(\xi)\right)$$
$$+\left(\frac{T|\gamma|(|\omega_3(\beta,\xi)| + T|\omega_2(\beta,\xi)|)}{|\Delta|}\; {}^{\rho}I^{q}\, J^{\alpha-1}|f(s,y(s))|(\eta)\right)$$

$$
\begin{aligned}
&\leq J^{\alpha}\phi(s)(T) \\
&+\frac{T(2|\omega_2(\gamma,\eta)|+T|\omega_1(\gamma,\eta)|)}{|\Delta|}J^{\alpha}\phi(s)(T) \\
&+\frac{T(|\omega_3(\beta,\xi)|+T|\omega_2(\beta,\xi)|)}{|\Delta|}J^{\alpha-1}\phi(s)(T) \\
&+\Big(\frac{|\beta|T(2|\omega_2(\gamma,\eta)|+T|\omega_1(\gamma,\eta)|)}{|\Lambda|}\ {}^{\rho}I^{q}\,J^{\alpha}\phi(s)(\xi)\Big) \\
&+\Big(\frac{T|\gamma|(|\omega_3(\beta,\xi)|+T|\omega_2(\beta,\xi)|)}{|\Delta|}\ {}^{\rho}I^{q}\,J^{\alpha-1}\phi(s)(\eta)\Big) \\
&\leq \|\phi\|\Big\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T}{|\Delta|}\frac{T^{\alpha}}{\Gamma(\alpha+1)}(2|\omega_2(\gamma,\eta)| \\
&+T|\omega_1(\gamma,\eta)|)+\frac{T}{|\Delta|}\frac{T^{\alpha}}{\Gamma(\alpha+1)}(|\omega_3(\beta,\xi)| \\
&+T|\omega_2(\beta,\xi)|)+\Big(\frac{T|\beta|(2|\omega_2(\gamma,\eta)|+T|\omega_1(\gamma,\eta)|)}{|\Delta|} \\
&\times\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\xi^{\alpha+\rho q}}{\rho^{q}}\frac{1}{\Gamma(\alpha+1)}+\Big(\frac{T|\gamma|(|\omega_3(\beta,\xi)|+T|\omega_2(\beta,\xi)|)}{|\Delta|} \\
&\times\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^{q}}\frac{1}{\Gamma(\alpha)}\Big)\Big\} \\
&=\|\phi\|\,\Omega_1\leq\ell.
\end{aligned}
$$

So, it is obvious that $S_1x+S_2y\in B_{\ell}$. Also, the operator S_1 is compact and continuous and the proof was stated in second part of Theorem 3.1.1.

Next step shows that the operator S_2 is contraction. We consider (A_1) to prove that S_2 is contraction:

$$
\begin{aligned}
&\|S_2x-S_2y\| \\
&\leq\Big(\frac{T|\beta|(2|\omega_2(\gamma,\eta)|+T|\omega_1(\gamma,\eta)|)}{|\Delta|} \\
&\times{}^{\rho}I^{q}\,J^{\alpha}|f(s,x(s))-f(s,y(s))|(\xi))
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\frac{T\,|\gamma|\,(|\omega_3\,(\beta,\xi)|+T\,|\omega_2\,(\beta,\xi)|)}{|\Delta|} \\
&\times {}^{\rho}I^{q}\,J^{\alpha-1}\,|f\,(s,x(s))-f\,(s,y(s))|\,(\eta)\Bigg) \\
&\le \Bigg(L\,\|x-y\|\,\frac{T\,|\beta|\,(2\,|\omega_2\,(\gamma,\eta)|+T\,|\omega_1\,(\gamma,\eta)|)}{|\Delta|} \\
&\times {}^{\rho}I^{q}\,J^{\alpha}(1)(\xi)\Bigg) \\
&+\Bigg(L\,\|x-y\|\,\frac{T\,|\gamma|\,(|\omega_3\,(\beta,\xi)|+T\,|\omega_2\,(\beta,\xi)|)}{|\Delta|} \\
&\times {}^{\rho}I^{q}\,J^{\alpha-1}(1)(\eta)\Bigg) \\
&\le L\Bigg\{\Bigg(\frac{T\,|\beta|\,(2\,|\omega_2\,(\gamma,\eta)|+T\,|\omega_1\,(\gamma,\eta)|)}{|\Delta|} \\
&\times\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\xi^{\alpha+\rho q}}{\rho^{q}}\frac{1}{\Gamma(\alpha+1)}\Bigg) \\
&+\Bigg(\frac{T\,|\gamma|\,(|\omega_3\,(\beta,\xi)|+T\,|\omega_2\,(\beta,\xi)|)}{|\Delta|} \\
&\times\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^{q}}\frac{1}{\Gamma(\alpha)}\Bigg)\Bigg\}\,\|x-y\| \\
&= L\Omega\,\|x-y\|\,.
\end{aligned}
$$

Therefore, we get:

$$\|S_2x-S_2y\|\le L\Omega\,\|x-y\|$$

As $L\Omega<1$ by (A_6). Hence the operator S_2 is contraction. Therefore, all the assumptions of Lemma (2.0.5) are satisfied. On the account of this, the problem (3.0.1) has at least one solution on $[0,T]$. ■

Theorem 3.1.6 *([56]) Given a continuous function $f\colon[0,T]\times\mathbb{R}\to\mathbb{R}$. Assume that the condition (A_5) holds. If*

$$L\Theta<1$$

where Θ is defined in (3.1.4). *Then the boundary value problem* (3.0.2) *has at least* *one solution on* $[0,T]$.

3.2 Applications

Example 1. Following boundary value problem is Caputo type fractional differential equation with Katugampola fractional integral boundary condition

$$\begin{cases} {}^{c}D^{5/2}x(t) = \frac{\sin^2(\pi t)}{2(e^t+9)}\left(\frac{|x(t)|}{|x(t)|+1}+1\right), \\ x(0)=0,\ x(1)=\frac{1}{2}\,{}^{2/3}I^{3}\,x(3/4), \\ x'(1)=\frac{1}{2}\,{}^{2/3}I^{3}\,x(2/3),\ t\in[0,1]. \end{cases} \tag{3.2.1}$$

Here , $\alpha = 5/2$, $T=1$, $\beta=1/2$, $\xi=3/4$, $\eta=2/3$, $\gamma=1/2$, $\rho=2/3$, $q=3$,

$$f(t,x) = \frac{\sin^2(\pi t)}{2(e^t+9)}\left(\frac{|x|}{|x|+1}+1\right).$$

Since $|f(t,x)-f(t,y)| \leq \frac{1}{10}\|x-y\|$. Therefore, (A_1) is satisfied. Using the given values, $\omega_1\left(\frac{1}{2},\frac{2}{3}\right) = -0.8750$, $\omega_1\left(\frac{1}{2},\frac{3}{4}\right) = -0.8418$, $\omega_2\left(\frac{1}{2},\frac{2}{3}\right) = 0.9873$, $\omega_2\left(\frac{1}{2},\frac{3}{4}\right) = 0.9819$, $\omega_3\left(\frac{1}{2},\frac{3}{4}\right) = 0.9970$. So, it is found that $\Delta = 1.0250$ and $\Omega_1 = 2.600$. Clearly, $L\Omega_1 \approx 0.26 < 1$. By using Theorem 3.1.1, the boundary value problem $(3.2.1)$ has a unique solution on $[0,1]$.

Example 2. Consider the following fractional boundary value problem

$$\begin{cases} {}^{c}D^{5/2}x(t) = \left(\frac{t^2+1}{10}\right)\left(\frac{x^2(t)}{|x(t)|+1}+\frac{\sqrt{|x(t)|}}{2(1+\sqrt{|x(t)|})}+\frac{1}{2}\right), \\ x(0)=0,\ x(1)=\frac{1}{2}\,{}^{2/3}I^{3}\,x(3/4), \\ x'(1)=\frac{1}{2}\,{}^{2/3}I^{3}\,x(2/3),\ t\in[0,1]. \end{cases} \tag{3.2.2}$$

Here , $\alpha = 5/2$, $T = 1$, $\beta = 1/2$, $\xi = 3/4$, $\eta = 2/3$, $\gamma = 1/2$, $\rho = 2/3$, $q = 3$,

$$|f(t,u)| = \left| \left(\frac{t^2+1}{10}\right) \left(\frac{u^2}{|u|+1} + \frac{\sqrt{|u|}}{2(1+\sqrt{|u|})} + \frac{1}{2}\right) \right|$$
$$\leq \frac{\left(t^2+1\right)\left(|u|+1\right)}{10}$$

So, $\phi(t) = \frac{t^2+1}{10}$ and $\Psi(|u|) = |u|+1$. Now, we need to show that

$$\frac{N}{\Psi(N)\,\|\phi\|\,\Omega_1} > 1$$

Therefore, we prove that

$$1 - \|\phi\|\,\Omega_1 \; > 0$$

Here $\|\phi\| = \frac{1}{5}$ and $\Omega_1 \; = 2.600$. So, $\|\phi\|\,\Omega_1 \; = 0.52 < 1$. Hence, by using Theorem 3.1.3, the boundary value problem (3.2.2) has at least one solution on $[0,1]$.

Example 3. Consider the following fractional boundary value problem

$$\begin{cases} {}^{c}D^{5/2}x(t) = \frac{\sin^2(\pi t)}{2(e^t+1)}\left(\frac{|x(t)|}{|x(t)|+1} + 1\right), \\ x(0) = 0,\; x(1) = \frac{1}{2}\,{}^{2/3}I^3\, x(3/4), \\ x'(1) = \frac{1}{2}\,{}^{2/3}I^3\, x(2/3),\; t \in [0,1]. \end{cases} \tag{3.2.3}$$

Here , $\alpha = 5/2$, $T = 1$, $\beta = 1/2$, $\xi = 3/4$, $\eta = 2/3$, $\gamma = 1/2$, $\rho = 2/3$, $q = 3$,

$$f(t,x) = \frac{\sin^2(\pi t)}{2(e^t+1)}\left(\frac{|x|}{|x|+1} + 1\right).$$

Since $|f(t,x) - f(t,y)| \leq \frac{1}{2}\|x-y\|$. It is clear that (A_1) is satisfied but when we consider (A_2) which is $L\Omega_1 = 1.3 \nless 1$. Therefore, (A_5) which is

$$|f(t,u)| \leq \frac{1}{2(e^t+9)} \leq \frac{1}{2} = \phi(t)$$

is satisfied. By using $(3.1.2)$, $\Omega = 0.0103$ is found. It is obvious that $L\Omega = 0.0052 < 1$. So, (A_6) is satisfied. Hence, by using Theorem 3.1.5, the boundary value problem $(3.2.3)$ has at least one solution on $[0,1]$.

Chapter 4

CAPUTO TYPE FRACTIONAL ORDER BOUNDARY VALUE PROBLEM WITH NONLOCAL INTEGRAL CONDITIONS

In this chapter, sufficient conditions of existence and uniqueness solutions are investigated for Caputo type fractional differential equations subjected to nonlocal Katugampola fractional integral boundary conditions as follows:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f(t,x(t)),\ t \in [0,T], \\ x(T) = \beta\ {}^{\rho}I^{q}x(\varepsilon),\ 0 < \varepsilon < T, \\ x'(T) = \gamma\ {}^{\rho}I^{q}x'(\eta),\ 0 < \eta < T, \\ x''(T) = \delta\ {}^{\rho}I^{q}x''(\zeta),\ 0 < \zeta < T, \end{cases} \tag{4.0.1}$$

where D^{α} is the Caputo fractional derivative of order $\alpha \in (2,3]$. ${}^{\rho}I^{q}$ is the Katugampola integral of orders $q > 0$, $\rho > 0$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Lemma 4.0.1 *([41]) Let $2 < \alpha \leq 3$ and $\beta, \gamma, \delta \in \mathbb{R}$. Then, for any $y \in C([0,T],\mathbb{R})$, there exists a solution x for the following fractional differential equation with Katugam-*

pola fractional integral conditions

$$\begin{cases} {}^cD^{\alpha}x(t) = y(t),\ t \in [0,T], \\ x(T) = \beta\ {}^{\rho}I^{q}x(\varepsilon),\ 0 < \varepsilon < T, \\ x'(T) = \gamma\ {}^{\rho}I^{q}x'(\eta),\ 0 < \eta < T, \\ x''(T) = \delta\ {}^{\rho}I^{q}x''(\zeta),\ 0 < \zeta < T, \end{cases} \tag{4.0.2}$$

if and only if

$$\begin{aligned} x(t) &= J^{\alpha}y(t) + \frac{1}{\varpi_1(\beta,\varepsilon)}\left(\beta^{\rho}I^{q}J^{\alpha}y(\varepsilon) - J^{\alpha}y(T)\right) \\ &- \frac{1}{\varpi_1(\gamma,\eta)}\left(\frac{\varpi_2(\beta,\varepsilon)}{\varpi_1(\beta,\varepsilon)} - t\right)\left(\gamma\ {}^{\rho}I^{q}J^{\alpha-1}y(\eta) - J^{\alpha-1}y(T)\right) \\ &+ \frac{1}{\varpi_1(\delta,\zeta)}\left(\frac{\varpi_3(\beta,\varepsilon)}{2\varpi_1(\beta,\varepsilon)} - \frac{\varpi_2(\beta,\varepsilon)\varpi_2(\gamma,\eta)}{\varpi_1(\beta,\varepsilon)\varpi_1(\gamma,\eta)}\right. \\ &\left.+\frac{\varpi_2(\gamma,\eta)t}{\varpi_1(\gamma,\eta)} - \frac{t^2}{2}\right)\left(J^{\alpha-2}y(T) - \delta\ {}^{\rho}I^{q}J^{\alpha-2}y(\zeta)\right), \end{aligned} \tag{4.0.3}$$

where

$$\varpi_1(\alpha,\xi) = \left(1 - \alpha\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right) \neq 0, \tag{4.0.4}$$

$$\varpi_2(\alpha,\xi) = \left(T - \alpha\frac{\xi^{\rho q+1}}{\rho^q}\frac{\Gamma\left(\frac{1+\rho}{\rho}\right)}{\Gamma\left(\frac{1+\rho q+\rho}{\rho}\right)}\right), \tag{4.0.5}$$

$$\varpi_3(\alpha,\xi) = \left(T^2 - \alpha\frac{\xi^{\rho q+2}}{\rho^q}\frac{\Gamma\left(\frac{2+\rho}{\rho}\right)}{\Gamma\left(\frac{2+\rho q+\rho}{\rho}\right)}\right). \tag{4.0.6}$$

Proof. Using Lemma (2.0.10)-(2.0.11), the general solution of the fractional differential equation in (4.0.2) can be written as

$$x(t) = c_0 + c_1 t + c_2 t^2 + J^{\alpha}y(t),\ c_0, c_1, c_2 \in \mathbb{R}. \tag{4.0.7}$$

From the first integral condition of the problem (4.0.2) is used and Katugampola integral is applied on $(4.0.7)$. Then, we obtain:

$$
\begin{aligned}
& c_0+c_1T+c_2T^2+J^{\alpha}y(T) \\
& =\beta c_0\frac{\varepsilon^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}+\beta c_1\frac{\varepsilon^{\rho q+1}}{\rho^q}\frac{\Gamma\left(\frac{1+\rho}{\rho}\right)}{\Gamma\left(\frac{1+\rho q+\rho}{\rho}\right)} \\
& +\beta c_2\frac{\varepsilon^{\rho q+2}}{\rho^q}\frac{\Gamma\left(\frac{2+\rho}{\rho}\right)}{\Gamma\left(\frac{2+\rho q+\rho}{\rho}\right)}+\beta\, {}^{\rho}I^qJ^{\alpha}y(\varepsilon).
\end{aligned}
$$

When the similar terms are collected in one part, we have the following equation:

$$
\begin{aligned}
& c_0\left(1-\beta\frac{\varepsilon^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right)+c_1\left(T-\beta c_1\frac{\varepsilon^{\rho q+1}}{\rho^q}\frac{\Gamma\left(\frac{1+\rho}{\rho}\right)}{\Gamma\left(\frac{1+\rho q+\rho}{\rho}\right)}\right) \\
& +c_2\left(T^2-\beta c_2\frac{\varepsilon^{\rho q+2}}{\rho^q}\frac{\Gamma\left(\frac{2+\rho}{\rho}\right)}{\Gamma\left(\frac{2+\rho q+\rho}{\rho}\right)}\right) \\
& =\beta\, {}^{\rho}I^qJ^{\alpha}y(\varepsilon)-J^{\alpha}y(T).
\end{aligned}
\tag{4.0.8}
$$

Rewriting the equation $(4.0.8)$ by using $(4.0.4)$, $(4.0.5)$ and $(4.0.6)$. We obtain:

$$
c_0\varpi_1(\beta,\varepsilon)+c_1\varpi_2(\beta,\varepsilon)+c_2\varpi_3(\beta,\varepsilon)=\beta\, {}^{\rho}I^qJ^{\alpha}y(\varepsilon)-J^{\alpha}y(T). \tag{4.0.9}
$$

Then, the derivative of $(4.0.7)$ is taken and second integral condition of $(4.0.2)$ is used. Therefore, we obtain:

$$
x'(T)=c_1+2c_2T+J^{\alpha-1}y(T). \tag{4.0.10}
$$

Now, the Katugampola integral is applied on $(4.0.10)$ and we have:

$$
\begin{aligned}
& c_1+2c_2T+J^{\alpha-1}y(T)=\gamma c_1\frac{\eta^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)} \\
& +2c_2\gamma\frac{\eta^{\rho q+1}}{\rho^q}\frac{\Gamma\left(\frac{1+\rho}{\rho}\right)}{\Gamma\left(\frac{1+\rho q+\rho}{\rho}\right)}+\gamma\, {}^{\rho}I^qJ^{\alpha-1}y(\eta).
\end{aligned}
\tag{4.0.11}
$$

The above equation $(4.0.11)$ implies that:

$$c_1\left(1-\gamma\frac{\eta^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right)+2c_2\left(T-\gamma\frac{\eta^{\rho q+1}}{\rho^q}\frac{\Gamma\left(\frac{1+\rho}{\rho}\right)}{\Gamma\left(\frac{1+\rho q+\rho}{\rho}\right)}\right)$$
$$=\gamma\,{}^{\rho}I^qJ^{\alpha-1}y(\eta)-J^{\alpha-1}y(T). \tag{4.0.12}$$

Also, by using (4.0.4) and (4.0.5), the equation (4.0.12) can be written as

$$c_1\varpi_1(\gamma,\eta)+2c_2\varpi_2(\gamma,\eta)=\gamma\,{}^{\rho}I^qJ^{\alpha-1}y(\eta)-J^{\alpha-1}y(T). \tag{4.0.13}$$

The last integral condition of (4.0.2) is used and Katugampola integral operator is applied on the second derivative of (4.0.10), then we have:

$$2c_2+J^{\alpha-2}y(T)=2\delta c_2\frac{\zeta^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}+\delta\,{}^{\rho}I^qJ^{\alpha-2}y(\zeta).$$

Hence, we obtain the following equation:

$$2c_2\left(1-\delta\frac{\zeta^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right)=\delta\,{}^{\rho}I^qJ^{\alpha-2}y(\zeta)-J^{\alpha-2}y(T). \tag{4.0.14}$$

By using (4.0.4), the equation (4.0.14) can be written as

$$2c_2\varpi_1(\delta,\zeta)=\delta\,{}^{\rho}I^qJ^{\alpha-2}y(\zeta)-J^{\alpha-2}y(T). \tag{4.0.15}$$

Furthermore, equation (4.0.15) implies that

$$c_2=\frac{1}{2\varpi_1(\delta,\zeta)}\left(\delta\,{}^{\rho}I^qJ^{\alpha-2}y(\zeta)-J^{\alpha-2}y(T)\right). \tag{4.0.16}$$

Substituting the values of (4.0.16) in (4.0.13), we get:

$$c_1=\frac{1}{\varpi_1(\gamma,\eta)}\left(\gamma\,{}^{\rho}I^qJ^{\alpha-1}y(\eta)-J^{\alpha-1}y(T)\right)$$
$$-\frac{\varpi_2(\gamma,\eta)}{\varpi_1(\gamma,\eta)\varpi_1(\delta,\zeta)}\left(\delta\,{}^{\rho}I^qJ^{\alpha-2}y(\zeta)-J^{\alpha-2}y(T)\right). \tag{4.0.17}$$

Now, substituting the values of (4.0.16) and (4.0.17) in (4.0.9), we obtain:

$$
\begin{aligned}
c_0 = & \frac{1}{\varpi_1(\beta,\varepsilon)}\left(\beta\, {}^{\rho}I^{q}J^{\alpha}y(\varepsilon) - J^{\alpha}y(T)\right) \\
& - \frac{\varpi_2(\beta,\varepsilon)}{\varpi_1(\beta,\varepsilon)\,\varpi_1(\gamma,\eta)}\left(\gamma\, {}^{\rho}I^{q}J^{\alpha-1}y(\eta) - J^{\alpha-1}y(T)\right) \\
& - \frac{\varpi_3(\beta,\varepsilon)}{2\varpi_1(\beta,\varepsilon)\,\varpi_1(\delta,\zeta)}\left(\delta\, {}^{\rho}I^{q}J^{\alpha-2}y(\zeta) - J^{\alpha-2}y(T)\right) \\
& + \frac{\varpi_2(\beta,\varepsilon)\,\varpi_2(\gamma,\eta)}{\varpi_1(\beta,\varepsilon)\,\varpi_1(\gamma,\eta)\varpi_1(\delta,\zeta)}\left(\delta\, {}^{\rho}I^{q}J^{\alpha-2}y(\zeta) - J^{\alpha-2}y(T)\right).
\end{aligned} \tag{4.0.18}
$$

Finally, substituting the values of (4.0.18), (4.0.17) and (4.0.16) in the equation (4.0.7), the general solution (4.0.3) of the problem (4.0.2) is obtained. It seems that converse is also true, when the direct computation is applied. ■

4.1 Main Results

In this section, several fixed point theorems are used to give sufficient conditions for existence (uniqueness) of solutions of (4.0.1) such as Banach contraction principle, the Krasnoselskii fixed point theorem, and the Leray Schauder nonlinear alternative.

An operator $H : C \to C$ is defined on the problem (4.0.1) as

$$
\begin{aligned}
(Hx)(t) = & J^{\alpha}f(s,x(s))(t) + \frac{1}{\varpi_1(\beta,\varepsilon)}\left(\beta\, {}^{\rho}I^{q}J^{\alpha}f(s,x(s))(\varepsilon)\right. \\
& \left. - J^{\alpha}f(s,x(s))(T)\right) - \frac{1}{\varpi_1(\gamma,\eta)}\left(\frac{\varpi_2(\beta,\varepsilon)}{\varpi_1(\beta,\varepsilon)} - t\right) \\
& \times \left(\gamma\, {}^{\rho}I^{q}J^{\alpha-1}f(s,x(s))(\eta) - J^{\alpha-1}f(s,x(s))(T)\right) \\
& + \frac{1}{\varpi_1(\delta,\zeta)}\left(\frac{\varpi_3(\beta,\varepsilon)}{2\varpi_1(\beta,\varepsilon)} - \frac{\varpi_2(\beta,\varepsilon)\,\varpi_2(\gamma,\eta)}{\varpi_1(\beta,\varepsilon)\,\varpi_1(\gamma,\eta)}\right. \\
& \left. + \frac{\varpi_2(\gamma,\eta)t}{\varpi_1(\gamma,\eta)} - \frac{t^2}{2}\right)\left(J^{\alpha-2}f(s,x(s))(T)\right. \\
& \left. - \delta\, {}^{\rho}I^{q}J^{\alpha-2}f(s,x(s))(\zeta)\right).
\end{aligned} \tag{4.1.1}
$$

Also, the notations are defined as follows:

$$
\begin{aligned}
\Phi = & \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{|\varpi_1(\beta,\varepsilon)|\Gamma(\alpha+1)} \\
& \times \left(|\beta| \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})} \frac{\varepsilon^{\alpha+\rho q}}{\rho^q} + T^{\alpha} \right) \\
& + \frac{1}{|\varpi_1(\gamma,\eta)|\Gamma(\alpha)} \left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|} + T \right) \\
& \times \left(|\gamma| \frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})} \frac{\eta^{\alpha-1+\rho q}}{\rho^q} + T^{\alpha-1} \right) \\
& + \frac{1}{|\varpi_1(\delta,\zeta)|\Gamma(\alpha-1)} \left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|} \right. \\
& \left. + \frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|} + \frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|} + \frac{T^2}{2} \right) \\
& \times \left(|\delta| \frac{\Gamma(\frac{\alpha-2+\rho}{\rho})}{\Gamma(\frac{\alpha-2+\rho q+\rho}{\rho})} \frac{\zeta^{\alpha-2+\rho q}}{\rho^q} + T^{\alpha-2} \right)
\end{aligned} \tag{4.1.2}
$$

and

$$
\begin{aligned}
\Phi_1 = & \frac{|\beta|}{|\varpi_1(\beta,\varepsilon)|\Gamma(\alpha+1)} \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})} \frac{\varepsilon^{\alpha+\rho q}}{\rho^q} \\
& + \frac{|\gamma|}{|\varpi_1(\gamma,\eta|)\Gamma(\alpha)} \left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|} + T \right) \frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})} \frac{\eta^{\alpha-1+\rho q}}{\rho^q} \\
& + \frac{|\delta|}{|\varpi_1(\delta,\zeta)|\Gamma(\alpha-1)} \left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|} + \frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|} \right. \\
& \left. + \frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|} + \frac{T^2}{2} \right) \frac{\Gamma(\frac{\alpha-2+\rho}{\rho})}{\Gamma(\frac{\alpha-2+\rho q+\rho}{\rho})} \frac{\zeta^{\alpha-2+\rho q}}{\rho^q}.
\end{aligned} \tag{4.1.3}
$$

The following subsections are involved with existence and uniqueness results of the boundary value problem (4.0.1) by using Banach fixed point theorem, Leray Schauder nonlinear alternative, and Krasnoselskii's fixed point theorem.

4.1.1 Existence and Uniqueness Result:

Theorem 4.1.1 *([41]) Given a continuous function* $f\colon[0,T]\times\mathbb{R}\to\mathbb{R}$. *Assume that*

(S_1) $|f(t,x)-f(t,y)| \le L\|x-y\|$, *for all* $t\in[0,T]$, $L>0$, $x,y\in\mathbb{R}$ *;*

(S_2) $L\Phi < 1$ *where* Φ *is defined by* (4.1.2). *Then there exists a unique solution for boundary value problem* (4.0.1) *on the interval* $[0,T]$.

Proof. By using the operator H, which is defined by (4.1.1), we obtain:

$$
\begin{aligned}
|(Hx)(t)-(Hy)(t)| \leq\; & J^{\alpha}|f(s,x(s))-f(s,y(s))|(T) \\
& + \frac{|\beta|}{|\varpi_1(\beta,\varepsilon)|}\, {}^{\rho}I^{q}J^{\alpha}|f(s,x(s))-f(s,y(s))|(\varepsilon) \\
& + \frac{1}{|\varpi_1(\beta,\varepsilon)|} J^{\alpha}|f(s,x(s))-f(s,y(s))|(T) \\
& + \frac{|\gamma|}{|\varpi_1(\gamma,\eta)|}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right) \\
& \times {}^{\rho}I^{q}J^{\alpha-1}|f(s,x(s))-f(s,y(s))|(\eta) \\
& + \frac{1}{|\varpi_1(\gamma,\eta)|}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right) \\
& \times J^{\alpha-1}|f(s,x(s))-f(s,y(s))|(T) \\
& + \frac{1}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right. \\
& \left. + \frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right) \\
& \times J^{\alpha-2}|f(s,x(s))-f(s,y(s))|(T) \\
& + \frac{|\delta|}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right. \\
& \left. + \frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\ {}^{\rho}I^{q}J^{\alpha-2}\left|f(s,x(s))-f(s,y(s))\right|(\zeta)\\
&\le L\|x-y\|\left\{J^{\alpha}(1)(T)+\frac{1}{|\varpi_1(\beta,\varepsilon)|}\left(|\beta|\ {}^{\rho}I^{q}J^{\alpha}(1)(\varepsilon)\right.\right.\\
&\left.+J^{\alpha}(1)(T)\right)\\
&+\frac{1}{|\varpi_1(\gamma,\eta)|}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)\left(|\gamma|\ {}^{\rho}I^{q}J^{\alpha-1}(1)(\eta)\right.\\
&\left.+J^{\alpha-1}(1)(T)\right)\\
&+\frac{1}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right.\\
&\left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)\left(|\delta|\ {}^{\rho}I^{q}J^{\alpha-2}(1)(\zeta)\right.\\
&\left.\left.+J^{\alpha-2}(1)(T)\right)\right\}\\
&\le L\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{|\varpi_1(\beta,\varepsilon)|\Gamma(\alpha+1)}\right.\\
&\times\left(|\beta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\varepsilon^{\alpha+\rho q}}{\rho^{q}}+T^{\alpha}\right)\\
&+\frac{1}{|\varpi_1(\gamma,\eta)|\Gamma(\alpha)}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)\\
&\times\left(|\gamma|\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^{q}}+T^{\alpha-1}\right)\\
&+\frac{1}{|\varpi_1(\delta,\zeta)|\Gamma(\alpha-1)}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}\right.\\
&\left.+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)\\
&\left.\times\left(|\delta|\frac{\Gamma(\frac{\alpha-2+\rho}{\rho})}{\Gamma(\frac{\alpha-2+\rho q+\rho}{\rho})}\frac{\zeta^{\alpha-2+\rho q}}{\rho^{q}}+T^{\alpha-2}\right)\right\}\|x-y\|\\
&=L\Phi\|x-y\|,
\end{aligned}
$$

for any $x,y\in C$ and for each $t\in[0,T]$. This shows that $\|Hx-Hy\|\le L\Phi\|x-y\|$. As $L\Phi<1$, the operator $H:C\to C$ is a contraction mapping. That means, the boundary value problem (4.0.1) has a unique solution on $[0,T]$. ■

4.1.2 Existence Results:

Theorem 4.1.2 *([41]) Let* $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Assume that*

(S_3) *there exists a nonnegative function* $\Omega\in C([0,T],\mathbb{R})$ *and a nondecreasing function* $\Psi:[0,\infty)\to(0,\infty)$ *such that*

$$|f(t,u)|\leq\Omega(t)\Psi(|u|)\text{ for any }(t,u)\in[0,T]\times\mathbb{R}.$$

(S_4) *there exist a constant* $M>0$ *such that*

$$\frac{M}{\Psi(M)\,\|\Omega\|\,\Phi}>1,$$

where Φ *in* $(4.1.2)$.

Then problem $(4.0.1)$ *has at least one solution on* $[0,T]$.

Proof. Let $B_d=\{x\in C:\|x\|\leq d\}$ be a closed bounded subset in $C([0,T],\mathbb{R})$. Notice that the problem $(4.0.1)$ is equivalent to the problem of finding fixed point of H.

As a first step , we show that the operator H which is defined by $(4.1.1)$ maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. Then for $t\in[0,T]$ we have:

$$\begin{aligned}|H(x)(t)|\leq{}& J^{\alpha}|f(s,x(s))|(T)\\&+\frac{1}{|\varpi_1(\beta,\varepsilon)|}\left(|\beta|\ {}^{\rho}I^{q}J^{\alpha}|f(s,x(s))|(\varepsilon)\right.\\&\left.+J^{\alpha}|f(s,x(s))|(T)\right)\\&+\frac{1}{|\varpi_1(\gamma,\eta)|}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)\end{aligned}$$

$$\times\left(|\gamma| J^{\alpha-1}|f(s,x(s))|(\eta)+J^{\alpha-1}|f(s,x(s))|(T)\right)$$

$$+\frac{1}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right.$$

$$\left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)$$

$$\times\left(J^{\alpha-2}|f(s,x(s))|(T)+|\delta|\ {}^{\rho}I^{q}J^{\alpha-2}|f(s,x(s))|(\zeta)\right)$$

$$\leq\Psi(\|x\|)J^{\alpha}\Omega(s)(T)$$

$$+\frac{\Psi(\|x\|)}{|\varpi_1(\beta,\varepsilon)|}(|\beta|\ {}^{\rho}I^{q}J^{\alpha}\Omega(s)(\varepsilon)$$

$$+J^{\alpha}\Omega(s)(T))$$

$$+\frac{\Psi(\|x\|)}{|\varpi_1(\gamma,\eta)|}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)$$

$$\times\left(|\gamma|^{\rho}I^{q}J^{\alpha-1}\Omega(s)(\eta)+J^{\alpha-1}\Omega(s)(T)\right)$$

$$+\frac{\Psi(\|x\|)}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right.$$

$$\left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)$$

$$\times\left(J^{\alpha-2}\Omega(s)(T)+|\delta|\ {}^{\rho}I^{q}J^{\alpha-2}\Omega(s)(\zeta)\right)$$

$$\leq\|\Omega\|\Psi(d)\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{|\varpi_1(\beta,\varepsilon)|\Gamma(\alpha+1)}\right.$$

$$\times\left(|\beta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\varepsilon^{\alpha+\rho q}}{\rho^{q}}+T^{\alpha}\right)$$

$$+\frac{1}{|\varpi_1(\gamma,\eta)|\Gamma(\alpha)}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)$$

$$\times\left(|\gamma|\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^{q}}+T^{\alpha-1}\right)$$

$$+\frac{1}{|\varpi_1(\delta,\zeta)|\Gamma(\alpha-1)}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}\right.$$

$$\left.+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)$$

$$\times\left(|\delta|\frac{\Gamma(\frac{\alpha-2+\rho}{\rho})}{\Gamma(\frac{\alpha-2+\rho q+\rho}{\rho})}\frac{\zeta^{\alpha-2+\rho q}}{\rho^{q}}+T^{\alpha-2}\right)\Bigg\}$$

$$=\|\Omega\|\,\Psi(d)\Phi$$

which leads to $\|H(x)\|\leq\|\Omega\|\,\Psi(d)\Phi$. By (S_4) there exist $d>0$ such that $\Psi(d)\,\|\Omega\|\,\Phi<d$.

Next, we show that the map $H:C([0,T],\mathbb{R})\rightarrow C([0,T],\mathbb{R})$ is completely continuous. Therefore, to prove that the map H is completely continuous, we show that H is a map from bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let choose $t_{1,}t_2$ from the interval $[0,T]$ and also $t_1<t_2$. Then we have:

$$\begin{aligned}
&|(Hx)(t_2)-(Hx)(t_1)|\\
&\leq|J^{\alpha}f(s,x(s))(t_2)-J^{\alpha}f(s,x(s))(t_1)|\\
&+\frac{|t_2-t_1|}{|\varpi_1(\gamma,\eta)|}\left(|\gamma|^{q}J^{\alpha-1}|f(s,x(s)|(\eta)+J^{\alpha-1}|f(s,x(s))|(T)\right)\\
&+\frac{1}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_2(\gamma,\eta)|}{|\varpi_1(\gamma,\eta)|}|t_2-t_1|+\frac{|t_2^2-t_1^2|}{2}\right)\\
&\times\left(J^{\alpha-2}|f(s,x(s))|(T)+|\delta|\ {}^{\rho}I^{q}J^{\alpha-2}|f(s,x(s))|(\zeta)\right)\\
&\leq\frac{\Psi(\|x\|)\Omega(s)}{\Gamma(\alpha)}\left[\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right]\\
&+|t_2-t_1|\frac{\Psi(\|x\|)}{|\varpi_1(\gamma,\eta)|}\left(|\gamma|J^{\alpha-1}\Omega(s)(\eta)+J^{\alpha-1}\Omega(s)(T)\right.\\
&\left.+\left(\frac{|\varpi_2(\gamma,\eta)|}{|\varpi_1(\delta,\zeta)|}+\frac{|t_2+t_1|}{2}\frac{|\varpi_1(\gamma,\eta)|}{|\varpi_1(\delta,\zeta)|}\right)\left(J^{\alpha-2}\Omega(s)(T)+|\delta|\ {}^{\rho}I^{q}J^{\alpha-2}\Omega(s)(\zeta)\right)\right)\\
&\leq\frac{\Psi(d)\,\|\Omega\|}{\Gamma(\alpha)}\left[\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right]
\end{aligned}$$

$$+\left|t_2-t_1\right|\frac{\Psi(d)\|\Omega\|}{|\varpi_1(\gamma,\eta)|}\left\{\frac{1}{\Gamma(\alpha)}\left(T^{\alpha-1}+|\gamma|\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^q}\right)\right.$$

$$\left.+\frac{1}{\Gamma(\alpha-1)}\left(\frac{|\varpi_2(\gamma,\eta)|}{|\varpi_1(\delta,\zeta)|}+\frac{|t_2+t_1|}{2}\frac{|\varpi_1(\gamma,\eta)|}{|\varpi_1(\delta,\zeta)|}\right)\left(T^{\alpha-2}+|\delta|\frac{\Gamma(\frac{\alpha-2+\rho}{\rho})}{\Gamma(\frac{\alpha-2+\rho q+\rho}{\rho})}\frac{\zeta^{\alpha-2+\rho q}}{\rho^q}\right)\right\}.$$

It is clear that the right hand side of the previous inequality is independent from x. Therefore, as $t_2-t_1\to 0$, right hand side of the inequality tends to zero . That means, H is equicontinuous and by the Arzelà-Ascoli Theorem (2.0.12), the operator $H:C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ is completely continuous.

In view of (S_4), there exist a positive M such that $\|x\|\neq M$. Let us set

$$U=\{x\in C([0,T],\mathbb{R}):\|x\|<M\}.$$

Then the operator $H:\overline{U}\to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\mu Hx$ for some $\mu\in(0,1)$. It can be proved by using contraction. Assume that there exist $x\in\partial U$ such that $x=\mu Hx$ for some $\mu\in(0,1)$. Then,

$$\|x\|=\|\mu Hx\|\leq\|Hx\|\leq\Psi(\|x\|)\|\Omega\|\Phi$$

$$\frac{\|x\|}{\Psi(\|x\|)\|\Omega\|\Phi}\leq 1$$

This is contradict with

$$\frac{\|x\|}{\Psi(\|x\|)\|\Omega\|\Phi}>1.$$

Consequently, by the nonlinear alternative of Leray-Schauder type, we conclude that H has a fixed point $x\in\overline{U}$, which is a solution of the problem $(4.0.1)$. This completes the proof. ■

Theorem 4.1.3 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function and condition (S_1) holds. In addition, the function f satisfies the assumptions:*

(S_5) *there exist a nonnegative function $\Omega\in C([0,T],\mathbb{R})$ such that*

$$|f(t,u)|\le\Omega(t)$$

for any $(t,u)\in[0,T]\times\mathbb{R}$.

(S_6) $L\Phi_1<1$, *where Φ_1 is defined by* $(4.1.3)$.

Then the boundary value problem $(4.0.1)$ has at least one solution on $[0,T]$.

Proof. We first define the new operators H_1 and H_2 as

$$\begin{aligned}(H_1x)(t)&=J^{\alpha}f(s,x(s))(t)-\frac{1}{\varpi_1(\beta,\varepsilon)}J^{\alpha}f(s,x(s))(T)\\&+\frac{1}{\varpi_1(\gamma,\eta)}\left(\frac{\varpi_2(\beta,\varepsilon)}{\varpi_1(\beta,\varepsilon)}-t\right)J^{\alpha-1}f(s,x(s))(T)\\&+\frac{1}{\varpi_1(\delta,\zeta)}\left(\frac{\varpi_3(\beta,\varepsilon)}{2\varpi_1(\beta,\varepsilon)}-\frac{\varpi_2(\beta,\varepsilon)\varpi_2(\gamma,\eta)}{\varpi_1(\beta,\varepsilon)\varpi_1(\gamma,\eta)}\right.\\&\left.+\frac{\varpi_2(\gamma,\eta)t}{\varpi_1(\gamma,\eta)}-\frac{t^2}{2}\right)J^{\alpha-2}f(s,x(s))(T)\end{aligned}\tag{4.1.4}$$

and

$$\begin{aligned}(H_2x)(t)&=\frac{\beta}{\varpi_1(\beta,\varepsilon)}\;{}^{\rho}I^{q}J^{\alpha}f(s,x(s))(\varepsilon)\\&-\frac{\gamma}{\varpi_1(\gamma,\eta)}\left(\frac{\varpi_2(\beta,\varepsilon)}{\varpi_1(\beta,\varepsilon)}-t\right)\;{}^{\rho}I^{q}J^{\alpha-1}f(s,x(s))(\eta)\\&-\frac{\delta}{\varpi_1(\delta,\zeta)}\left(\frac{\varpi_3(\beta,\varepsilon)}{2\varpi_1(\beta,\varepsilon)}-\frac{\varpi_2(\beta,\varepsilon)\varpi_2(\gamma,\eta)}{\varpi_1(\beta,\varepsilon)\varpi_1(\gamma,\eta)}\right.\\&\left.+\frac{\varpi_2(\gamma,\eta)t}{\varpi_1(\gamma,\eta)}-\frac{t^2}{2}\right)\;{}^{\rho}I^{q}J^{\alpha-2}f(s,x(s))(\zeta).\end{aligned}\tag{4.1.5}$$

Then, we consider a closed, bounded, convex and nonempty subset of Banach space X as

$$B_d=\{x\in C:\|x\|<d\}\ \text{with}\ \|\Omega\|\,\Phi\le d,$$

where Φ is defined by $(4.1.2)$. Now, we show that $H_1x+H_2y \in B_d$ for any $x,y \in B_d$, where H_1 and H_2 are denoted by (4.1.4), (4.1.5) respectively.

$$
\begin{aligned}
\|H_1x+H_2y\| \leq\; & J^{\alpha}\left|f(s,x(s))\right|(T) \\
& + \frac{1}{|\varpi_1(\beta,\varepsilon)|}\left(J^{\alpha}\left|f(s,x(s))\right|(T)+|\beta|\ {}^{\rho}I^{q}J^{\alpha}\left|f(s,x(s))\right|(\varepsilon)\right) \\
& + \frac{1}{|\varpi_1(\gamma,\eta)|}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right) \\
& \times \left(J^{\alpha-1}\left|f(s,x(s))\right|(T)+|\gamma|\ {}^{\rho}I^{q}J^{\alpha-1}f(s,x(s))(\eta)\right) \\
& + \frac{1}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)|\,|\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)|\,|\varpi_1(\gamma,\eta)|}\right. \\
& \left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right) \\
& \times \left(J^{\alpha-2}\left|f(s,x(s))\right|(T)+|\delta|\ {}^{\rho}I^{q}J^{\alpha-2}\left|f(s,x(s))\right|(\zeta)\right) \\
& \leq J^{\alpha}\Omega(s)(T) \\
& + \frac{1}{|\varpi_1(\beta,\varepsilon)|}\left(J^{\alpha}\Omega(s)(T)+|\beta|\ {}^{\rho}I^{q}J^{\alpha}\Omega(s)(\varepsilon)\right) \\
& + \frac{1}{|\varpi_1(\gamma,\eta)|}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right) \\
& \times \left(J^{\alpha-1}\Omega(s)(T)+|\gamma|\ {}^{\rho}I^{q}J^{\alpha-1}\Omega(s)(\eta)\right) \\
& + \frac{1}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)|\,|\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)|\,|\varpi_1(\gamma,\eta)|}\right. \\
& \left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right) \\
& \times \left(J^{\alpha-2}\Omega(s)(T)+|\delta|\ {}^{\rho}I^{q}J^{\alpha-2}\Omega(s)(\zeta)\right) \\
& \leq \|\Omega\|\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{|\varpi_1(\beta,\varepsilon)|\Gamma(\alpha+1)}\right.
\end{aligned}
$$

$$\times\left(T^{\alpha}+|\beta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\varepsilon^{\alpha+\rho q}}{\rho^{q}}+\right)$$

$$+\frac{1}{|\varpi_1(\gamma,\eta)|\Gamma(\alpha)}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)$$

$$\times\left(T^{\alpha-1}+|\gamma|\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^{q}}\right)$$

$$+\frac{1}{|\varpi_1(\delta,\zeta)|\Gamma(\alpha-1)}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}\right.$$

$$\left.+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)$$

$$\left.\times\left(T^{\alpha-2}+|\delta|\frac{\Gamma(\frac{\alpha-2+\rho}{\rho})}{\Gamma(\frac{\alpha-2+\rho q+\rho}{\rho})}\frac{\zeta^{\alpha-2+\rho q}}{\rho^{q}}\right)\right\}$$

$$=\|\Omega\|\Phi\leq d.$$

Therefore, it is clear that $\|H_1x+H_2y\|\leq d$. Hence, $H_1x+H_2y\in B_{d.}$

The next step is related to the compactness and continuity of the operator H_1. The proof is similar to that of Theorem 4.1.2.

Finally, we show that the operator H_2 is contraction. By using assumption (S_1),

$$\|H_2x-H_2y\|\leq\frac{|\beta|}{|\varpi_1(\beta,\varepsilon)|}\,{}^{\rho}I^{q}J^{\alpha}\,|f(s,x(s))-f(s,y(s))|\,(\varepsilon)$$

$$+\frac{|\gamma|}{|\varpi_1(\gamma,\eta|)}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)$$

$$\times\,{}^{\rho}I^{q}J^{\alpha-1}\,|f(s,x(s))-f(s,y(s))|\,(\eta)$$

$$+\frac{|\delta|}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right.$$

$$\left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)$$

$$
\begin{aligned}
&\times\ {}^{\rho}I^{q}J^{\alpha-2}\left|f(s,x(s))-f(s,y(s))\right|(\zeta)\\
&\leq L\|x-y\|\left\{\frac{|\beta|}{|\varpi_1(\beta,\varepsilon)|}\ {}^{\rho}I^{q}J^{\alpha}(1)(\varepsilon)\right.\\
&+\frac{|\gamma|}{|\varpi_1(\gamma,\eta|)}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)\ {}^{\rho}I^{q}J^{\alpha-1}(1)(\eta)\\
&+\frac{|\delta|}{|\varpi_1(\delta,\zeta)|}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right.\\
&\left.\left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)\ {}^{\rho}I^{q}J^{\alpha-2}(1)(\zeta)\right\}\\
&\leq L\left\{\frac{|\beta|}{|\varpi_1(\beta,\varepsilon)|\Gamma(\alpha+1)}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\frac{\varepsilon^{\alpha+\rho q}}{\rho^{q}}\right.\\
&+\frac{|\gamma|}{|\varpi_1(\gamma,\eta|)\Gamma(\alpha)}\left(\frac{|\varpi_2(\beta,\varepsilon)|}{|\varpi_1(\beta,\varepsilon)|}+T\right)\\
&\times\frac{\Gamma(\frac{\alpha-1+\rho}{\rho})}{\Gamma(\frac{\alpha-1+\rho q+\rho}{\rho})}\frac{\eta^{\alpha-1+\rho q}}{\rho^{q}}\\
&+\frac{|\delta|}{|\varpi_1(\delta,\zeta)|\Gamma(\alpha-1)}\left(\frac{|\varpi_3(\beta,\varepsilon)|}{2|\varpi_1(\beta,\varepsilon)|}+\frac{|\varpi_2(\beta,\varepsilon)||\varpi_2(\gamma,\eta)|}{|\varpi_1(\beta,\varepsilon)||\varpi_1(\gamma,\eta)|}\right.\\
&\left.+\frac{|\varpi_2(\gamma,\eta)|T}{|\varpi_1(\gamma,\eta)|}+\frac{T^2}{2}\right)\\
&\left.\times\frac{\Gamma(\frac{\alpha-2+\rho}{\rho})}{\Gamma(\frac{\alpha-2+\rho q+\rho}{\rho})}\frac{\zeta^{\alpha-2+\rho q}}{\rho^{q}}\right\}\|x-y\|\\
&=L\Phi_1\|x-y\|,
\end{aligned}
$$

which means $\|H_2x-H_2y\|\leq L\Phi_1\|x-y\|$. As $L\Phi_1<1$, the operator H_2 is contraction.

For this reason, the problem (4.0.1) has at least one solution on $[0,T]$. ■

4.2 Examples

In this section, some examples are illustrated to show our results.

Example 1. Consider the following fractional differential equation with Katugampola fractional integral conditions

$$\begin{cases} {}^cD^{5/2}x(t) = \frac{\sin^2(\pi t)}{(e^t+10)}\left(\frac{|x(t)|}{|x(t)|+1}+1\right), t \in \left[0, \frac{1}{2}\right], \\ x(\frac{1}{2}) = \frac{1}{2}\, {}^5I^{\frac{1}{3}}x(3/8), x'(\frac{1}{2}) = \frac{1}{2}\, {}^5I^{\frac{1}{3}}x(1/3), \\ x''(\frac{1}{2}) = \frac{1}{2}\, {}^5I^{\frac{1}{3}}x(2/5). \end{cases} \tag{4.2.1}$$

Here , $\alpha = 5/2$, $T = \frac{1}{2}$, $\beta = 1/2$, $\gamma = 1/2$, $\delta = 1/2$, $\varepsilon = 3/8$, $\eta = 1/3$, $\zeta = 2/5$, $\rho = 5$, $q = \frac{1}{3}$, and

$$f(t,x) = \frac{\sin^2(\pi t)}{(e^t+10)}\left(\frac{|x|}{|x|+1}+1\right).$$

Hence, we have $|f(t,x) - f(t,y)| \leq \frac{1}{10}\|x-y\|$. Then, the assumption (S_1) is satisfied with $L = \frac{1}{10}$. By using Matlab program , $\omega_1\left(\frac{1}{2},\frac{3}{8}\right) = 0.9361$, $\omega_1\left(\frac{1}{2},\frac{1}{3}\right) = 0.9475$, $\omega_1\left(\frac{1}{2},\frac{2}{5}\right) = 0.9289$, $\omega_2\left(\frac{1}{2},\frac{3}{8}\right) = 0.4779$, $\omega_2\left(\frac{1}{2},\frac{1}{3}\right) = 0.4838$, $\omega_3\left(\frac{1}{2},\frac{3}{8}\right) = 0.4922$. and $\Phi = 1.2261$ are found. Therefore, $L\Phi = 0.1226 < 1$, which implies that the assumption (S_2) holds true. By using Theorem 4.1.1, the boundary value problem (4.2.1) has a unique solution on $\left[0, \frac{1}{2}\right]$.

Example 2. Consider the following fractional differential equation with Katugampola fractional integral conditions

$$\begin{cases} D^{5/2}x(t) = \left(\frac{t^2+1}{10}\right)\left(\frac{x^2(t)}{|x(t)|+1} + \frac{\sqrt{|x(t)|}}{2(1+\sqrt{|x(t)|})} + \frac{1}{2}\right), t \in \left[0, \frac{1}{2}\right], \\ x(\frac{1}{2}) = \frac{1}{2}\, {}^5I^{\frac{1}{3}}x(3/8), x'(\frac{1}{2}) = \frac{1}{2}\, {}^5I^{\frac{1}{3}}x(1/3), \\ x''(\frac{1}{2}) = \frac{1}{2}\, {}^5I^{\frac{1}{3}}x(2/5). \end{cases} \tag{4.2.2}$$

where $\alpha = 5/2$, $T = \frac{1}{2}$, $\beta = 1/2$, $\gamma = 1/2$, $\delta = 1/2$, $\varepsilon = 3/8$, $\eta = 1/3$, $\zeta = 2/5$, $\rho = 5$, $q = \frac{1}{3}$. Moreover,

$$|f(t,u)| = \left| \left(\frac{t^2+1}{10} \right) \left(\frac{u^2}{|u|+1} + \frac{\sqrt{|u|}}{2(1+\sqrt{|u|})} + \frac{1}{2} \right) \right| \leq \frac{\left(t^2+1\right)(|u|+1)}{10}.$$

By using the assumption (S_3) , it is easy to see that $\Omega(t) = \frac{t^2+1}{10}$ and $\Psi(|u|) = |u|+1$. Moreover, $\|\Omega\| = \frac{1}{8}$ and $\Phi = 1.2261$ which was found in previous example. Now, we need to show that there exist $M > 0$ such that

$$\frac{M}{\Psi(M)\,\|\Omega\|\,\Phi} > 1,$$

and such $M > 0$ exists if

$$1 - \|\Omega\|\,\Phi > 0.$$

By using direct computation $\|\Omega\|\,\Phi = 0.1533 < 1$. That is mean, the assumption (S_4) is satisfied. Hence, by using Theorem 4.1.2, the boundary value problem (4.2.2) has at least one solution on $\left[0, \frac{1}{2}\right]$.

Example 3. Consider the following fractional differential equation with Katugampola fractional integral conditions

$$\begin{cases} {}^{c}D^{5/2}x(t) = \frac{9\sin^2(\pi t)}{(e^t+10)} \left(\frac{|x(t)|}{|x(t)|+1} + 1 \right), t \in \left[0, \frac{1}{2}\right], \\ x(\frac{1}{2}) = \frac{1}{2}\, {}^{5}I^{\frac{1}{3}} x(3/8), x'(\frac{1}{2}) = \frac{1}{2}\, {}^{5}I^{\frac{1}{3}} x(1/3), \\ x''(\frac{1}{2}) = \frac{1}{2}\, {}^{5}I^{\frac{1}{3}} x(2/5). \end{cases} \tag{4.2.3}$$

Here , $\alpha = 5/2, T = \frac{1}{2}, \beta = 1/2, \gamma = 1/2, \delta = 1/2, \varepsilon = 3/8, \eta = 1/3, \zeta = 2/5, \rho = 5, q = \frac{1}{3}$, and

$$f(t,x) = \frac{9\sin^2(\pi t)}{(e^t+10)} \left(\frac{|x|}{|x|+1} + 1 \right).$$

Since $|f(t,x) - f(t,y)| \leq \frac{9}{10}|x-y|$,then, it implies that $L = \frac{9}{10}$ means (S_1) is satisfied but (S_2) which is $L\Phi < 1$ is not satisfied. $[L\Phi = 1.10358 > 1]$. Therefore, we consider (S_5) which is

$$|f(t,x)| \leq \frac{9}{(e^t+10)}\left(\frac{|x|}{|x|+1}+1\right) \leq \frac{18}{(e^t+10)} = \Omega(t).$$

By using (4.1.3), $\Phi_1 = 0.0561$ is found. It is obvious that $L\Phi_1 = 0.05049 < 1$. So, (S_6) is satisfied. Hence, by using Theorem 4.1.3, the boundary value problem (4.2.3) has at least one solution on $\left[0, \frac{1}{2}\right]$.

Chapter 5

ON THE PARAMETRIZATION OF CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATION WITH TWO POINT NONLINEAR BOUNDARY CONDITIONS

We present a new approach of investigation and approximation of solutions of Caputo type fractional differential equation under nonlinear boundary conditions. By using an appropriate parametrization technique, the original problem with nonlinear boundary conditions is reduced to the equivalent parametrized boundary value problem with linear restrictions. In order to study the transformed problem, we construct a numerical-analytic scheme which is succesful in relation to different types two-point and multipoint linear boundary condition and nonlinear boundary conditions. Moreover, we define sufficient conditions of the uniform convergence of the successive approximations. Also, it is indicated that these successive approximations uniformly converge to a parametrized limit function and besides that the relationship of this limit function and exact solution is stated. Finally, an example is presented to illustrate the mentioned theory.

5.1 Statement of Fractional Differential Equation with Nonlinear Boundary Conditions and Identification of Parametrized Boundary Value Problem

In this section, we state the Caputo type fractional differential equation equipped with nonlinear boundary condition and we use vector of parameters to reduce the nonlinear boundary conditions to the linear boundary condition.

Let us consider Caputo type fractional differential equation with nonlinear boundary conditions

$$ {}^{c}D^{\alpha}\, x(t) = h(t, x(t)),\ t \in [0, T], \tag{5.1.1} $$

$$ Ax(0) + Bx(T) + g(x(0), x(T)) = d,\ d \in \mathbb{R}^n, \tag{5.1.2} $$

where ${}^{c}D^{\alpha}$ is the Caputo derivative of order $\alpha \in (0,1]$, the functions $h : [0,T] \times D \to \mathbb{R}$, and $g : D \times D \to \mathbb{R}^n$ are continuous and the set $D \subset \mathbb{R}^n$ is closed and bounded domain. A and B are $n \times n$ matrices, $\det B \neq 0$ and d is a $n-$dimensional vector.

By using appropriate parametrization technique [50], the given problem (5.1.1),(5.1.2) is reduced to certain parametrized two-point boundary conditions. To see that, we introduce the vectors of parameters

$$ \begin{aligned} \omega &:= x(0) = (\omega_1, \omega_2 ... \omega_n)^T, \\ \phi &:= x(T) = (\phi_1, \phi_2, ..., \phi_n)^T, \end{aligned} \tag{5.1.3} $$

$$ d(\omega, \phi) := d - g(\omega, \phi). \tag{5.1.4} $$

By using $(5.1.4)$, the problem $(5.1.1),(5.1.2)$ can be rewritten as follows:

$$^{c}D^{\alpha}\, x(t) = h(t,x(t))$$

$$Ax(0)+Bx(T) = d(\omega,\phi). \tag{5.1.5}$$

5.2 Conditions for Convergence of Successive Approximation

To study the successive approximations, some conditions are needed. In this study, parametrized boundary value problem (5.1.5) is studied under the following conditions:

A) The function $h:[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies the Lipschitz condition:

$$\|h(t,u)-h(t,v)\| \le L\,\|u-v\|, \tag{5.2.1}$$

for all $t\in[0,T]$, $u,v\in D$, where L is a positive constant.

B) Let

$$\kappa(t) = \frac{2t^{\alpha}}{\Gamma(\alpha+1)}\left(1-\frac{t}{T}\right)^{\alpha}.$$

Then , $\kappa(t)$ takes its maximum value at $t=\frac{T}{2}$ and

$$\|\kappa\|_{\infty} = \frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)}.$$

Define,

$$\|h\|_{\infty} = \max_{(t,x)\in[0,T]\times D}\sqrt{h_1^2+h_2^2},$$

and a vector function $\delta: D\times D\to\mathbb{R}^n$ is

$$\delta(\omega,\phi) := \|\kappa\|_{\infty}\|h\|_{\infty} + \left\|\left[B^{-1}d(\omega,\phi)-\left(B^{-1}A+I_n\right)\omega\right]\right\|,$$

where I_n is the $n\times n$ identity matrix and $\omega,\phi\in D$ of the form $(5.1.3)$. δ is the radius of a neighbourhood C of the point $\omega\in D$ is defined as follows:

$$B(\omega,\delta(\omega,\phi)) := \{x \in \mathbb{R}^n : \|x-\omega\| \leq \delta(\omega,\phi) \text{ for all } \phi \in D \subset \mathbb{R}^n\}.$$

the set

$$D_\delta := \{\omega \in D : B(\omega,\delta(\omega,\phi)) \subset D \text{ for all } \phi \in D\}$$

is nonempty.

C) Let

$$L\|\kappa\|_\infty < 1,$$

where L is a positive constant and satisfies the inequality $(5.2.1)$.

For studying the solution of the parametrized boundary value problem $(5.1.5)$, we consider the sequence of functions $\{x_m\}$ which is defined by the iterative formula as follows:

$$\begin{aligned} x_m(t,\omega,\phi) = \omega + \frac{1}{\Gamma(\alpha)} \Bigg[\int_0^t (t-s)^{\alpha-1} h(s,x_{m-1}(s,\omega,\phi))\,ds \\ - \left(\frac{t}{T}\right)^\alpha \int_0^T (T-s)^{\alpha-1} h(s,x_{m-1}(s,\omega,\phi))\,ds \Bigg] \\ + \left(\frac{t}{T}\right)^\alpha \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right], \end{aligned} \tag{5.2.2}$$

for $t \in [0,T]$, $m = 1,2,3...$ where

$$x_0(t,\omega,\phi) = (x_{01},x_{02},..,x_{0n})^T = \omega \in D_\delta,$$

$$x_m(t,\omega,\phi) = (x_{m,1}(t,z,\phi), x_{m,2}(t,z,\phi) ..., x_{m,n}(t,z,\phi))^T,$$

and ω,ϕ are considered as parameters.In addition, it is easy to see that the sequence of functions $\{x_m\}$ are satisfied linear parametrized boundary conditions $(5.1.5)$ for all $m \geq 1$, $\omega \in D_\delta$, $\phi \in D$.

Now, we prove that the sequence of the functions $(5.2.2)$ is uniformly convergent and show the relationship between this sequence of the functions and the limit function.

Theorem 5.2.1 *Assume that the parametrized boundary value problem* $(5.1.5)$ *satisfy the conditions* (A), (B) *and* (C). *Then, for all fixed* $\phi \in D$ *and* $\omega \in D_\delta$, *the following assertions are true:*

1. *All functions of sequence* $(5.2.2)$ *are continuous and satisfy the parametrized boundary conditions* $(5.1.5)$

$$Ax_m(0,\omega,\phi)+Bx_m(T,\omega,\phi)=d(\omega,\phi),\ m=1,2,3... \tag{5.2.3}$$

2. *The sequence of functions* $(5.2.2)$ *converges uniformly in* $t \in [0,T]$ *as* $m \to \infty$ *to the limit function*

$$x^*(t,\omega,\phi)=\lim_{m\to\infty} x_m(t,\omega,\phi). \tag{5.2.4}$$

3. *The limit function* x^* *satisfies the initial conditions*

$$x^*(0,\omega,\phi)=\omega$$

and

$$Ax^*(0,\omega,\phi)+Bx^*(T,\omega,\phi)=d(\omega,\phi)$$

4. *The limit function* $(5.2.4)$ *is the unique continuous solution of the integral equation*

$$\begin{aligned} x(t) := \omega + \frac{1}{\Gamma(\alpha)} & \left[\int_0^t (t-s)^{\alpha-1} h(s,x(s))\,ds \right. \\ & \left. - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} h(s,x(s))\,ds \right] \\ & + \left(\frac{t}{T}\right)^{\alpha} \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A+I_n\right)\omega\right], \end{aligned} \tag{5.2.5}$$

or $x(t)$ is the unique solution on $[0,T]$ of the Cauchy problem:

$$^{c}D^{\alpha}\, x(t) = h(t,x(t)) + \,^{\alpha}\Omega\,(\omega,\phi)\,,\; x(0) = \omega \tag{5.2.6}$$

where

$$^{\alpha}\Omega\,(\omega,\phi) = -\frac{\alpha}{T^{\alpha}}\left[\int_{0}^{T}(T-s)^{\alpha-1}\,h\,(s,x^{*}(t,\omega,\phi))\,ds\right.$$
$$\left. -\Gamma\,(\alpha)\left[B^{-1}d(\omega,\phi)-\left(B^{-1}A+I_n\right)\omega\right]\right]. \tag{5.2.7}$$

5. *Error estimation:*

$$\|x^{*}(t,\omega,\phi)-x_m(t,\omega,\phi)\| \leq (L\,\|\kappa\|_{\infty})^{m}\,(\|h\|_{\infty}\,\|\kappa\|_{\infty}$$
$$+\left\|\left[B^{-1}d(\omega,\phi)-\left(B^{-1}A+I_n\right)\omega\right]\right\|)\,\frac{1}{1-L\,\|\kappa\|_{\infty}}. \tag{5.2.8}$$

Proof.

1. Continuity of the sequence $\{x_m\}$ defined by $(5.2.2)$ follows directly from the construction of sequence and by direct computation, it is easy to show that the sequence $\{x_m\}$ satisfies the parametrized boundary conditions $(5.1.5)$.

2. We prove that the sequence of functions is a Cauchy sequence in the Banach space $C\,([a,b]\,,\mathbb{R}^{n})$. Therefore, we first need to show that $x_m(t,\omega,\phi)\in D$ for all $(t,\omega,\phi)\in[0,T]\times D_{\delta}\times D$, $m\in\mathbb{N}$. We start from the equation $(5.2.2)$. When $m=1$, the equation $(5.2.2)$ is obtained:

$$x_1(t,\omega,\phi) = \omega + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} h(s, x_0(s,\omega,\phi))\,ds \right.$$
$$\left. - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} h(s, x_0(s,\omega,\phi))\,ds \right]$$
$$+ \left(\frac{t}{T}\right)^{\alpha} \left[B^{-1} d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \tag{5.2.9}$$

The equation (5.2.9) can be written as follows:

$$\|x_1(t,\omega,\phi) - \omega\| \le \frac{1}{\Gamma(\alpha)}\left[\int_0^t \left|(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^{\alpha}(T-s)^{\alpha-1}\right| \|h(s,\omega)\|\,ds \right.$$
$$\left. + \int_t^T \left|\left(\frac{t}{T}\right)^{\alpha}(T-s)^{\alpha-1}\right| \|h(s,\omega)\|\,ds\right]$$
$$+ \left|\left(\frac{t}{T}\right)^{\alpha}\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right| := I_1 + I_2 + I_3 \tag{5.2.10}$$

We start from the estimation of I_1 :

$$I_1 \le \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^{\alpha}(T-s)^{\alpha-1}\right| \|h\|_\infty\,ds$$
$$= \left(\frac{t}{T}\right)^{\alpha}\frac{(T-t)^{\alpha}}{\Gamma(\alpha+1)}\|h\|_\infty\,, \tag{5.2.11}$$

where the expression under the absolute value is nonnegative

$$\frac{1}{(t-s)^{1-\alpha}} \ge \left(\frac{t}{T}\right)^{\alpha}\frac{1}{(t-s)^{1-\alpha}} \ge \left(\frac{t}{T}\right)^{\alpha}\frac{1}{(T-s)^{1-\alpha}}.$$

Then, we estimate I_2 and I_3:

$$I_2 \le \frac{1}{\Gamma(\alpha)}\int_t^T \left|\left(\frac{t}{T}\right)^{\alpha}(T-s)^{\alpha-1}\right| \|h(s,\omega)\|\,ds$$
$$= \left(\frac{t}{T}\right)^{\alpha}\frac{(T-t)^{\alpha}}{\Gamma(\alpha+1)}\|h\|_\infty \tag{5.2.12}$$

and

$$I_3 = \left(\frac{t}{T}\right)^{\alpha} \left\| B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega \right\|. \tag{5.2.13}$$

Subsituting $(5.2.11), (5.2.12)$ and $(5.2.13)$ into the relation $(5.2.10)$ and we obtain the following result:

$$\begin{aligned}
\|x_1(t,\omega,\phi) - \omega\| &\leq \frac{2t^{\alpha}}{\Gamma(\alpha+1)}\left(1 - \frac{t}{T}\right)^{\alpha} \|h\|_{\infty} \\
&+ \left(\frac{t}{T}\right)^{\alpha} \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\| \\
&\leq \frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)} \|h\|_{\infty} + \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\| \\
&= \|\kappa\|_{\infty} \|h\|_{\infty} + \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\| = \delta(\omega,\phi).
\end{aligned} \tag{5.2.14}$$

Thus,

$$x_1(t,\omega,\phi) \in D \text{ for } (t,\omega,\phi) \in [0,T] \times D_{\delta} \times D.$$

By induction, it can be shown that all functions $x_m(t,\omega,\phi)$ defined by $(5.2.2)$ also belong to D for all $m = 1,2,3,...$ $t \in [0,T]$, $\omega \in D_{\delta}$, $\phi \in D$. To show that, we start with the difference between x_{m+1} and x_m:

$$\begin{aligned}
x_{m+1}(t,\omega,\phi) - x_m(t,\omega,\phi) = \frac{1}{\Gamma(\alpha)} \Bigg(&\int_0^t (t-s)^{\alpha-1} \left[h\left(s, x_m(s,\omega,\phi)\right)\right. \\
&\left. - h\left(s, x_{m-1}(s,\omega,\phi)\right)\right] ds - \int_0^T \left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1} \\
&\times \left[h\left(s, x_m(s,\omega,\phi)\right) - h\left(s, x_{m-1}(s,\omega,\phi)\right)\right] ds \Bigg)
\end{aligned} \tag{5.2.15}$$

for $m = 1,2,...$

Here, we denote the difference $(5.2.15)$ by $r_m(t,\omega,\phi)$ as follows:

$$r_m(t,\omega,\phi) := \|x_m(t,\omega,\phi) - x_{m-1}(t,\omega,\phi)\|, \text{ for all } m = 1,2,3,... \tag{5.2.16}$$

We rewrite the inequality (5.2.14), by using (5.2.16) for $m = 1$. Then, we obtain:

$$\begin{aligned} r_1(t,\omega,\phi) &= \|x_1(t,\omega,\phi) - \omega\| \\ &\leq \|\kappa\|_\infty \|h\|_\infty + \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\|. \end{aligned} \tag{5.2.17}$$

Taking into account the Lipshitz condition (A) and the relation (5.2.17) for $m = 2$ into the equation (5.2.16), we get:

$$\begin{aligned} r_2(t,\omega,\phi) &\leq \frac{L}{\Gamma(\alpha)} \left(\int_0^t \left[(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1} \right] \right. \\ &\quad \left. + \int_t^T \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1} \right) \|x_1(s,\omega,\phi) - \omega\|\, ds \\ &= \frac{L}{\Gamma(\alpha)} \left(\int_0^t \left[(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1} \right] \right. \\ &\quad \left. + \int_t^T \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1} ds \right) r_1(s,\omega,\phi) \\ &\leq \frac{2Lt^\alpha}{\Gamma(\alpha+1)} \left(1 - \frac{t}{T}\right)^\alpha \left[\|\kappa\|_\infty \|h\|_\infty + \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\| \right] \\ &\leq L\|\kappa\|_\infty^2 \|h\|_\infty + L\|\kappa\|_\infty \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\|. \end{aligned}$$

Hence,

$$r_2(t,\omega,\phi) \leq L\|\kappa\|_\infty \left[\|\kappa\|_\infty \|h\|_\infty + \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\| \right]. \tag{5.2.18}$$

Therefore, by using mathematical induction we obtain the following inequality:

$$\begin{aligned} r_{m+1}(t,\omega,\phi) &\leq (L\|\kappa\|_\infty)^m \left[\|\kappa\|_\infty \|h\|_\infty + \left\| \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right] \right\| \right] \\ & m = 0,1,2.. \end{aligned} \tag{5.2.19}$$

In view of $(5.2.19)$ and by using triangular inequality we get:

$$
\begin{aligned}
&\left\|x_{m+j}(t,\omega,\phi)-x_m(t,\omega,\phi)\right\|\\
&\leq \left\|x_{m+j}(t,\omega,\phi)-x_{m+j-1}(t,\omega,\phi)\right\|+\left\|x_{m+j-1}(t,\omega,\phi)-x_{m+j-2}(t,\omega,\phi)\right\|\\
&+\ldots+\left\|x_{m+1}(t,\omega,\phi)-x_m(t,\omega,\phi)\right\|\\
&=r_{m+j}(t,\omega,\phi)+r_{m+j-1}(t,\omega,\phi)+\ldots+r_{m+1}(t,\omega,\phi)\\
&=\sum_{i=1}^{j} r_{m+i}(t,\omega,\phi)\\
&\leq (L\|\kappa\|_\infty)^m\left(\|h\|_\infty\|\kappa\|_\infty+\left\|\left[B^{-1}d(\omega,\phi)-\left(B^{-1}A+I_n\right)\omega\right]\right\|\right)\sum_{i=1}^{\infty}L^{i-1}\|\kappa\|_\infty^{i-1}.
\end{aligned}
\tag{5.2.20}
$$

From the assumption (C), it follows:

$$\lim_{m\to 0}(|L|\,\|\kappa\|_\infty)^m=0.$$

Hence, by $(5.2.20)$, $\{x_m\}$ is Cauchy sequence and uniformly converges on $[0,T]\times D_\delta\times D$ to a certain limit x^*.

3. Taking limit in $(5.2.3)$ as $m\to\infty$,we see that x^* satisfies the boundary conditions directly.

4. By using contradiction, the uniqueness of solution is shown. Assume that there are two limit functions such as $x_1^*(t,\omega,\phi)$ and $x_2^*(t,\omega,\phi)$. Then, the difference between x_1^* and x_2^* is estimated as follows:

$$
\begin{aligned}
\|x_1^*(t,\omega,\phi)-x_2^*(t,\omega,\phi)\| &\leq \frac{L}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\|x_1^*(s,\omega,\phi)-x_2^*(s,\omega,\phi)\|\,ds\right.\\
&\left.+\int_0^T (T-s)^{\alpha-1}\|x_1^*(s,\omega,\phi,\psi)-x_2^*(s,\omega,\phi)\|\,ds\right]\\
&\leq L\|\kappa\|_\infty\|x_1^*-\ x_2^*\|_\infty
\end{aligned}
$$

Thus,

$$\|x_1^* - x_2^*\|_\infty \leq L\|\kappa\|_\infty \|x_1^* - x_2^*\|_\infty$$

It can be written as:

$$(1 - L\|\kappa\|_\infty)\|x_1^* - x_2^*\|_\infty \leq 0$$

So, $\|x_1^* - x_2^*\| = 0 \Longrightarrow x_1^* - x_2^* = 0 \Longrightarrow x_1^* = x_2^*$.

5. Passing to $j \to \infty$ in (5.2.20) we get:

$$\|x_1^*(t,\omega,\phi) - x_2^*(t,\omega,\phi)\|$$
$$\leq (L\|\kappa\|_\infty)^m \left(\|h\|_\infty \|\kappa\|_\infty + \left\|\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right\|\right) \sum_{i=1}^{\infty} L^{i-1} \|\kappa\|_\infty^{i-1}$$
$$= (L\|\kappa\|_\infty)^m \left(\|h\|_\infty \|\kappa\|_\infty + \left\|\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right\|\right) \frac{1}{1 - L\|\kappa\|_\infty}.$$

■

Remark 5.2.2 *If* $A = I_n$, $B = -I_n$, $g(x(0), x(T)) = 0$, $d = 0$, *boundary condition* (5.1.2) *becomes* $x(0) = x(T)$. *Note that, this problem was studied in [50].*

5.3 Relationship between the Limit Function and the Solution of the Nonlinear Boundary-Value Problem

We consider the following equation

$$^{c}D^{\alpha}\, x(t) = h(t,x) + \psi,\ t \in [0,T] \qquad (5.3.1)$$

and

$$x(0) = \omega, \qquad (5.3.2)$$

where $\psi = col\,(\psi_1 ... \psi_n)$ is the parameter of control.

Theorem 5.3.1 *Let* $\omega \in D_\delta$, $\phi \in D$ *be arbitrarily defined vectors. Suppose that all conditions of Theorem 5.2.1 are satisfied. The solution* $x = (t, \omega, \phi, \psi)$ *of the initial-value problem* (5.3.1),(5.3.2) *satisfies the boundary conditions* (5.1.5) *if and only* *if* $x = (t,\omega,\phi,\psi)$ *coincides with the limit function* $x^* = x^*(t,\omega,\phi,\psi)$ *of sequence* $(5.2.2)$. *Moreover,*

$$\psi = \psi_{\omega,\phi} = -\frac{\alpha}{T^\alpha}\left[\int_0^T (T-s)^{\alpha-1} h(s,x^*(t,\omega,\phi))\,ds\right.$$

$$\left. -\Gamma(\alpha)\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right]. \tag{5.3.3}$$

Proof. Sufficiency: The proof is similar to the proof of theorem in [31].

Necessity: We fix an arbitrary value $\overline{\psi} \in \mathbb{R}^n$ and assume that:

$$^cD^\alpha x(t) = h(t,x) + \overline{\psi},\ t \in [0,T] \tag{5.3.4}$$

with initial condition $x(0) = \omega$. The solution $\overline{x} = \overline{x}(t)$ of the problem (5.3.4) satisfying the two-point boundary conditions (5.1.5) as follows:

$$A\overline{x}(0) + B\overline{x}(T) = d(\omega,\phi).$$

Besides that, $\overline{x}$ is a solution of the following integral equation:

$$\overline{x}(t) = \omega + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,\overline{x}(s))\,ds + \frac{t^\alpha \overline{\psi}}{\Gamma(\alpha+1)}. \tag{5.3.5}$$

When $t = T$ in $(5.3.5)$, we get the following equation:

$$\overline{x}(T) = \omega + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} h\left(s,\overline{x}(s)\right)ds + \frac{T^\alpha \overline{\psi}}{\Gamma(\alpha+1)}. \tag{5.3.6}$$

Also,

$$\overline{x}(0) = \omega.$$

From the boundary conditions (5.1.5), we have:

$$\overline{x}(T) = B^{-1}\left[d(\omega,\phi) - A\omega\right]. \tag{5.3.7}$$

By using (6.2.12) and (5.3.7), we obtain:

$$\overline{\psi} = -\frac{\alpha}{T^{\alpha}} \left[\int_0^T (T-s)^{\alpha-1} h(s,\overline{x}(s))\, ds \right.$$
$$\left. -\Gamma(\alpha) \left[B^{-1} d(\omega,\phi) - \left(B^{-1}A + I_n \right) \omega \right] \right]. \tag{5.3.8}$$

Then, substituting (5.3.8) into the (5.3.5), we have:

$$\overline{x}(t) := \omega + \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} h(s,\overline{x}(s))\, ds \right.$$
$$\left. - \left(\frac{t}{T} \right)^{\alpha} \int_0^T (T-s)^{\alpha-1} h(s,\overline{x}(s))\, ds \right]$$
$$+ \left(\frac{t}{T} \right)^{\alpha} \left[B^{-1} d(\omega,\phi) - \left(B^{-1}A + I_n \right) \omega \right],$$

Moreover, the limit function x^* is a solution of the (5.3.1), (5.3.2) for $\psi = \psi_{\omega,\phi}$ of the form (5.3.3) and satisfies the boundary conditions (5.1.5).

$$x^*(t,\omega,\phi,\psi) = \omega + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, x^*(t,\omega,\phi,\psi))\, ds + \frac{t^{\alpha}\psi}{\Gamma(\alpha+1)}. \tag{5.3.9}$$

Similarly, we start with the solution $x^*(T,\omega,\phi,\psi)$ of the integral equation:

$$x^*(T,\omega,\phi,\psi) = \omega + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s, x^*(T,\omega,\phi,\psi))\, ds + \frac{T^{\alpha}\psi}{\Gamma(\alpha+1)}. \tag{5.3.10}$$

Then, the limit function x^* satisfies the following boundary conditions:

$$Ax^*(0,\omega,\phi,\psi) + Bx^*(T,\omega,\phi,\psi) = d(\omega,\phi) \tag{5.3.11}$$

with the initial condition

$$x^*(0,\omega,\phi,\psi) = \omega.$$

From (5.3.11), we obtain:

$$x^*(T,\omega,\phi,\psi) = B^{-1}\left[d(\omega,\phi) - A\omega\right]. \tag{5.3.12}$$

By using relations (5.3.10) and (5.3.12) we get:

$$\psi_{\omega,\phi} = -\frac{\alpha}{T^\alpha}\left[\int_0^T (T-s)^{\alpha-1} h(s,x^*(s,\omega,\phi,\psi))\,ds \right.$$
$$\left. -\Gamma(\alpha)\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right]. \tag{5.3.13}$$

After substituting relation (5.3.13) into (5.3.9), we have:

$$x^*(t,\omega,\phi,\psi) := \omega + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} h(s,x^*(s,\omega,\phi,\psi))\,ds \right.$$
$$\left. - \left(\frac{t}{T}\right)^\alpha \int_0^T (T-s)^{\alpha-1} h(s,x^*(s,\omega,\phi,\psi))\,ds \right]$$
$$+ \left(\frac{t}{T}\right)^\alpha \left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right].$$

Taking the difference between $\bar{x}$ and x^* , we get:

$$x^*(t,\omega,\phi,\psi) - \bar{x}(t) = \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\left[h(s,x^*(s,\omega,\phi,\psi)) - h\left(s,\bar{x}(s)\right)\right]ds \right.$$
$$\left. - \left(\frac{t}{T}\right)^\alpha \int_0^T (T-s)^{\alpha-1}\left[h(s,x^*(s,\omega,\phi,\psi)) - h\left(s,\bar{x}(s)\right)\right]ds\right].$$

Thus, we have the following inequalities:

$$\left\|x^*(t,\omega,\phi,\psi) - \bar{x}(t)\right\| \le \frac{L}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\left\|x^*(s,\omega,\phi,\psi) - \bar{x}(s)\right\|ds \right.$$
$$\left. + \int_0^T (T-s)^{\alpha-1}\left\|x^*(s,\omega,\phi,\psi) - \bar{x}(s)\right\|ds\right]$$
$$\le L\,\|\kappa\|_\infty \left\|x^* - \bar{x}\right\|_\infty.$$

Thus,

$$\left\|x^{*}-\bar{x}\right\|_{\infty} \leq L\|\kappa\|_{\infty}\left\|x^{*}-\bar{x}\right\|_{\infty}.$$

It can be written:

$$\left(1-L\|\kappa\|_{\infty}\right)\left\|x^{*}-\bar{x}\right\|_{\infty} \leq 0$$

So, we have:

$$\left\|x^{*}-\bar{x}\right\|_{\infty}=0 \Longrightarrow x^{*}-\bar{x}=0 \Longrightarrow x^{*}=\bar{x}.$$

This means that the function $\bar{x}$ coincides with x^{*} . Also, by using (5.3.8) and (5.3.13), we obtain $\psi_{\omega,\phi}=\overline{\psi}$. The theorem is proved. ■

Theorem 5.3.2 *Assume that the conditons* (A) *,*(B) *and* (C) *are satisfied for the Caputo type fractional differential equation* $(5.1.1)$ *with nonlinear boundary conditions* $(5.1.2)$*. Then,* $\left(x^{*}\left(\cdot,\omega^{*},\phi^{*}\right),\phi^{*}\right)$ *is a solution of the parametrized boundary-value problem* $(5.1.1),(5.1.5)$ *if and only if* $\omega^{*}=\left(\omega_{1}^{*},\omega_{2}^{*},....,\omega_{n}^{*}\right)$ *and* $\phi^{*}=\left(\phi_{1}^{*},\phi_{2}^{*},....,\phi_{n}^{*}\right)$ *satisfy the system of determining algebraic or transcendental equations*

$$\Omega(\omega,\phi)=-\frac{\alpha}{T^{\alpha}}\left[\int_{0}^{T}(T-s)^{\alpha-1}h\left(s,x^{*}(s,\omega,\phi)\right)ds\right.$$
$$\left.-\Gamma(\alpha)\left[B^{-1}d(\omega,\phi)-\left(B^{-1}A+I_{n}\right)\omega\right]\right]=0, \tag{5.3.14}$$

$$x^{*}(T,\omega,\phi)=\phi. \tag{5.3.15}$$

Proof. The result is obtained from the Theorem 5.3.1 and by observing that the differential equation (5.2.6) coincides with (5.1.1) if and only if the couple (ω^{*},ϕ^{*}) satisfies the equation

$$\Omega\left(\omega^{*},\phi^{*}\right)=0.$$

■

The following assertion indicates the determining system of equation $(5.3.14),(5.3.15)$ shows all possible solution of the Caputo type differential equation $(5.1.1)$ with non-linear boundary conditions $(5.1.2)$.

Remark 5.3.3 *Assume that all conditions of Theorem 5.2.1 are satisfied and there exist vectors* $\omega \in D_\delta$ *and* $\phi \in D$ *satisfying the system of determining equations* $(5.3.14),(5.3.15)$. *Then, the Caputo type differential equation* $(5.1.1)$ *with nonlinear boundary conditions* $(5.1.2)$ *have the solution* $x(\cdot)$ *such that*

$$x(0) = \omega,$$

$$x(T) = \phi.$$

Also, this solution has the following form

$$x(t) = x^*(t, \omega, \phi),\ t \in [0, T], \tag{5.3.16}$$

where x^* *is the limit function of sequence* $(5.2.2)$. *Conversely, if the Caputo type differential equation* $(5.1.1)$ *with nonlinear boundary conditions* $(5.1.2)$ *has a solution* $x(\cdot)$, *this solution necessarily has the form* $(5.3.16)$ *and the system of determining equations* $(5.3.14),(5.3.15)$ *is satisfied for*

$$\omega = x(0),$$

$$\phi = x(T).$$

Remark 5.3.4 *For some $m \geq 1$, a function $\Omega_m : D \times D \to \mathbb{R}^n$ is defined by the formula*

$$\Omega_m(\omega,\phi) := -\frac{\alpha}{T^\alpha}\left[\int_0^T (T-s)^{\alpha-1} h(s,x_m(t,\omega,\phi))\,ds \right.$$
$$\left. -\Gamma(\alpha)\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right],$$

where ω and ϕ are given by (5.1.3) *.To study the solvability of the parametrized boundary-value problem* (5.1.5) *, we consider the approximate determining system of algebraic equations of the form*

$$\Omega_m(\omega,\phi) = -\frac{\alpha}{T^\alpha}\left[\int_0^T (T-s)^{\alpha-1} h(s,x_m(t,\omega,\phi))\,ds \right.$$
$$\left. -\Gamma(\alpha)\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right] = 0, \tag{5.3.17}$$

$$x_m(T,\omega,\phi) = \phi, \tag{5.3.18}$$

where x_m is the vector function specified by the recurrence relation (5.2.2) .

5.4 Example

Motivated by [50] , we consider a system of Caputo type fractional differential equation

$$\begin{aligned} {}^cD^\alpha x_1 &= x_2 = h_1(t,x_1,x_2) \\ {}^cD^\alpha x_2 &= -\frac{1}{2}x_2^2 - \frac{1}{2}x_1 + \frac{t}{8}x_2 + \frac{t^{1-\alpha}}{4\Gamma(2-\alpha)} + \frac{2t^{\alpha+1}+1}{16\Gamma(2+\alpha)} = h_2(t,x_1,x_2) \end{aligned} \tag{5.4.1}$$

with nonlinear boundary conditions

$$\begin{aligned} x_1(0) + x_1\left(\frac{1}{2}\right) - \left[x_2\left(\frac{1}{2}\right)\right]^2 &= \frac{2^{\alpha+1}+1}{2^\alpha\, 8\,\Gamma(\alpha+2)} - \frac{1}{64}, \\ x_2(0) + x_1\left(\frac{1}{2}\right) - x_2\left(\frac{1}{2}\right) &= \frac{2^\alpha+1}{2^\alpha 8\,\Gamma(\alpha+2)} - \frac{1}{8}. \end{aligned} \tag{5.4.2}$$

The pair of functions

$$x_1^* = \frac{2t^{\alpha+1}+1}{8\Gamma(\alpha+2)},$$
$$x_2^* = \frac{t}{4}$$

are the exact solution of the Caputo type fractional differential equation (5.4.1) with nonlinear boundary conditions (5.4.2). Then, the nonlinear boundary conditions can be shown by the form of matrix vectors as follows:

$$Ax(0)+Bx\left(\frac{1}{2}\right)+g\left(x(0),x\left(\frac{1}{2}\right)\right)=d, \tag{5.4.3}$$

where

$$A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B=\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$
$$d=\begin{pmatrix} \frac{2^{\alpha+1}+1}{2^{\alpha}\,8\,\Gamma(\alpha+2)}-\frac{1}{64} \\ \frac{2^{\alpha}+1}{2^{\alpha}8\,\Gamma(\alpha+2)}-\frac{1}{8} \end{pmatrix}, g\left(x(0),x\left(\frac{1}{2}\right)\right)=\begin{pmatrix} -[x_2\left(\frac{1}{2}\right)]^2 \\ 0 \end{pmatrix}.$$

Here, $\det(B)=-1\neq 0$.

Then, new parameters are introduced as follows:

$$x(0)=\omega:=\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},$$
$$x\left(\frac{1}{2}\right)=\phi:=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{5.4.4}$$

By using the parameters in (5.4.4), the nonlinear boundary condition (5.4.2) can be written in the following form:

$$Ax(0)+Bx\left(\frac{1}{2}\right)=d-g(\omega,\phi).$$

Thus,

$$d(\omega,\phi) = d - g(\omega,\phi) = \begin{pmatrix} \frac{2^{\alpha+1}+1}{2^{\alpha}\,8\,\Gamma(\alpha+2)} - \frac{1}{64} + \phi_2^2 \\ \frac{2^{\alpha}+1}{2^{\alpha}8\,\Gamma(\alpha+2)} - \frac{1}{8} \end{pmatrix}. \tag{5.4.5}$$

By using (5.4.5), the nonlinear boundary conditions (5.4.2) are transformed to the linear conditions as follows:

$$Ax(0) + Bx\left(\frac{1}{2}\right) = d(\omega,\phi). \tag{5.4.6}$$

The stated conditions of convergence of successive approximations (A), (B) and (C) are checked. First, we begin by defining domain D as follows:

$$D = \left\{(x_1,x_2) : |x_1| \leq 1, |x_2| \leq \frac{3}{4}\right\}. \tag{5.4.7}$$

Then, the condition (A) which is related with Lipschitz condition is satisfied as follows:

$$L = \max(0,1,1/2,7/8).$$

Thus

$$L = 1.$$

Then,

$$\|\kappa\|_\infty = 0.2143,$$

and

$$\|h\|_\infty \leq 1.6207$$

are obtained for $\alpha = 0.9$. The vector $\delta(\omega,\phi)$ is stated as follows:

$$\begin{aligned}\delta(\omega,\phi) &:= \|\kappa\|_\infty \|h\|_\infty + \left\|\left[B^{-1}d(\omega,\phi) - \left(B^{-1}A + I_n\right)\omega\right]\right\| \\ &\leq 0.3473 + \sqrt{0.0565 + 2\phi_2^4 - 6\phi_2^2(-0.111833 + \omega_1) - 0.9866\omega_1 + 5\omega_1^2}\end{aligned}$$

So, the conditions (B) and (C) are satisfied. Thus, it is verified that all needed conditions are fulfilled. Hence, it can be proceeded with procedure of numerical-analytic scheme described above. Therefore, the sequence of approximate solutions are constructed. For the Caputo type boundary value problem (5.4.1) , (5.4.6) the successive approximations (5.2.2) have the following form:

$$
\begin{aligned}
x_{m,1}(t,\omega,\phi) := \omega_1 + \frac{1}{\Gamma(\alpha)} \Bigg[& \int_0^t (t-s)^{\alpha-1} h_1(s, x_{m-1,1}(s,\omega,\phi), x_{m-1,1}(s,\omega,\phi))\, ds \\
& - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} h_1(s, x_{m-1,1}(s,\omega,\phi), x_{m-1,1}(s,\omega,\phi))\, ds \Bigg] \\
& + \left(\frac{t}{T}\right)^{\alpha} \left[\frac{2^{\alpha+1}+1}{2^{\alpha}\, 8\, \Gamma(\alpha+2)} - \frac{1}{64} + \phi_2^2 - 2\omega_1 \right],
\end{aligned}
$$

$$
\begin{aligned}
x_{m,2}(t,\omega,\phi) := \omega_2 + \frac{1}{\Gamma(\alpha)} \Bigg[& \int_0^t (t-s)^{\alpha-1} h_2(s, x_{m-1,1}(s,\omega,\phi), x_{m-1,1}(s,\omega,\phi))\, ds \\
& - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} h_2(s, x_{m-1,1}(s,\omega,\phi), x_{m-1,1}(s,\omega,\phi))\, ds \Bigg] \\
& + \left(\frac{t}{T}\right)^{\alpha} \left[\frac{1}{8\Gamma(\alpha+2)} + \frac{7}{64} + \phi_2^2 - \omega_1 \right], \text{ where } \alpha = 0.9.
\end{aligned}
$$

After that, by using Mathematica, we get following results:

Iteration 1: We start from the approximate system of algebraic equations (5.3.17) and (5.3.18) for $m = 1$. Then, the approximate system has the following solutions:

$$\omega_1 = \omega_{11} = 0.0656973365195, \tag{5.4.8}$$

$$\omega_2 = \omega_{12} = -0.00219529679272, \tag{5.4.9}$$

$$\phi_1 = \phi_{11} = 0.179133148137, \tag{5.4.10}$$

$$\phi_2 = \phi_{12} = 0.239437851344. \tag{5.4.11}$$

After substituting (5.4.8), (5.4.9), (5.4.10),(5.4.11) into the equations of $x_{1,1}$ and $x_{1,2}$, we obtain $x_{1,1}(t)$ and $x_{1,2}(t)$. Figure 5.1 shows the graphic of $x_{1,1}(t)$ and $x_1(t)$. On the other hand, Figure 5.2 indicates the graphic of $x_{1,2}(t)$ and $x_2(t)$.

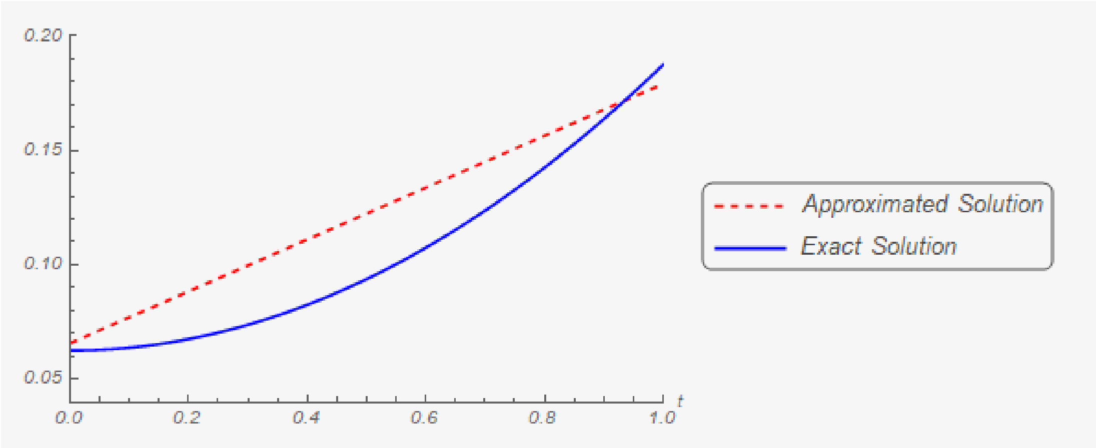

Figure 5.1: The first component of the exact solution and its first approximation

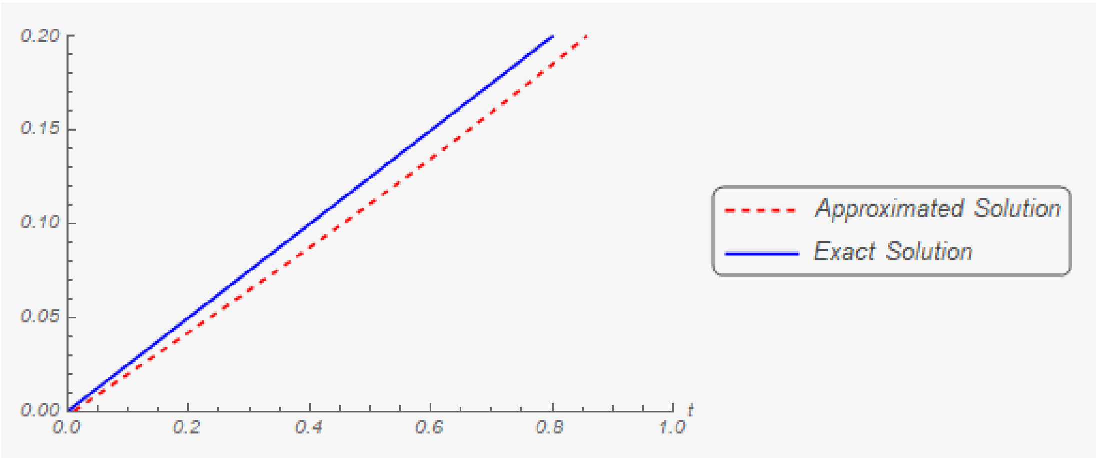

Figure 5.2: The second component of the exact solution and its first approximation

Also, for the first iteration, the maximum deviations of the exact solution are given as follows:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{1,1}(t)\right| \leq 0.02893,$$

$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{1,2}(t)\right| \leq 0.01547.$$

Similarly, we use equations (5.3.17) and (5.3.18) to find the unknown parameters for each iteration. Moreover, for each iteration the solutions of approximate systems are relatively same with the solutions (5.4.8), (5.4.9), (5.4.10), (5.4.11) .

Therefore, for the the next iterations, components of exact and approximate solutions are shown by figures and with their maximum errors.

Iteration 50: The graphs of the first and second components of the exact and approximate (in the fifth iteration) solutions are shown on Figure 5.3, Figure 5.4 respectively.

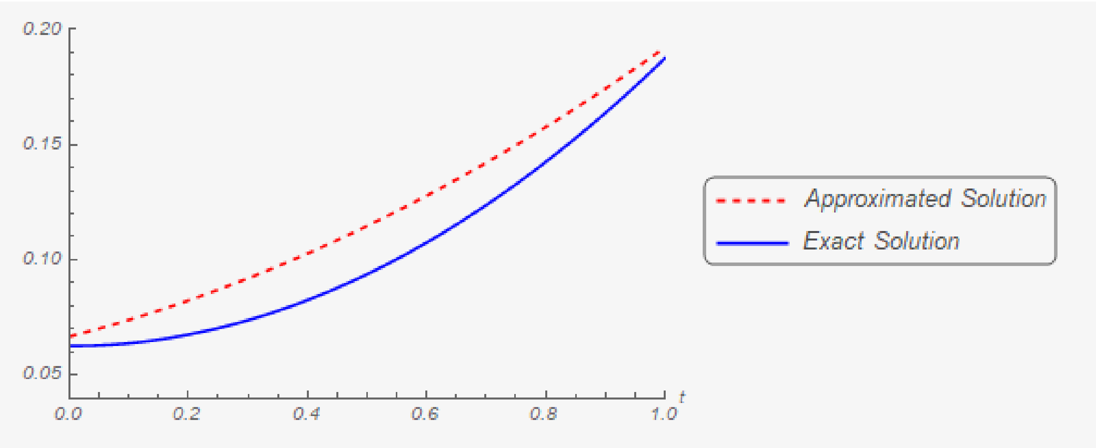

Figure 5.3: The first component of the exact solution and its fifth approximation

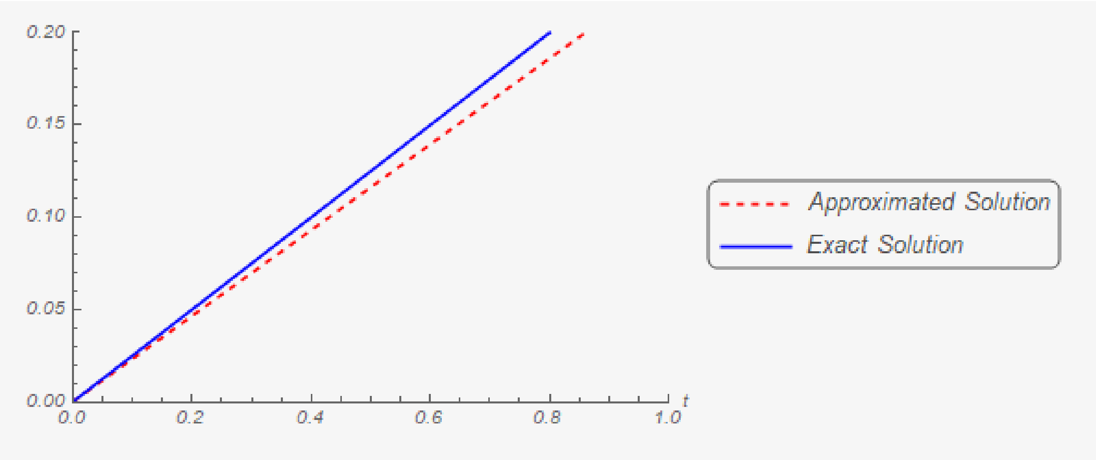

Figure 5.4: The second component of the exact solution and its fifth approximation

The following inequalities are related with the maximum deviation of the exact solution with its fifth approximations:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{50,1}(t)\right| \le 0.02096,$$

$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{50,2}(t)\right| \le 0.01744.$$

Iteration 100: The graphs of the first and second components of the exact and approximate (in the hunderedth iteration) solutions are shown on Figure 5.5, Figure 5.6 respectively.

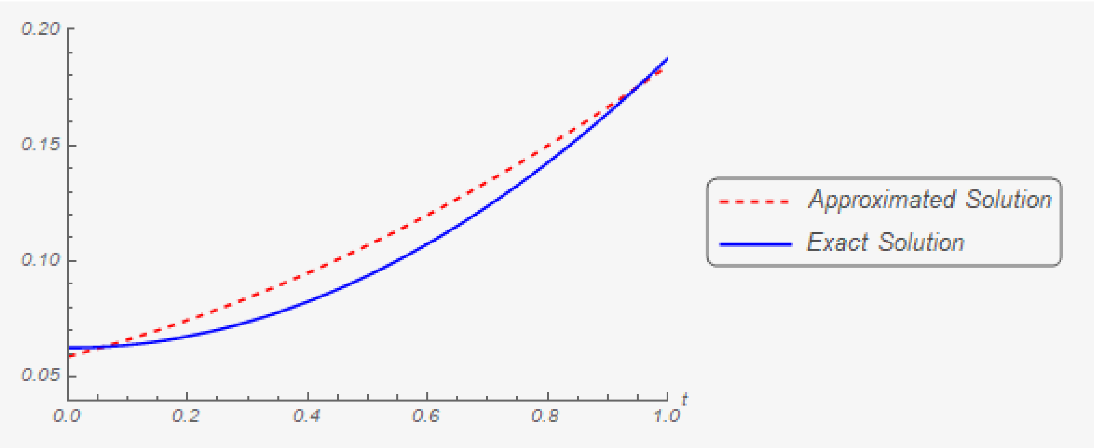

Figure 5.5: The first component of the exact solution and its hunderedth approximation

For the hunderedth approximation the maximum deviations of the exact solution are given as follows:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{100,1}(t)\right| \le 0.01311,$$

$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{100,2}(t)\right| \le 0.01471.$$

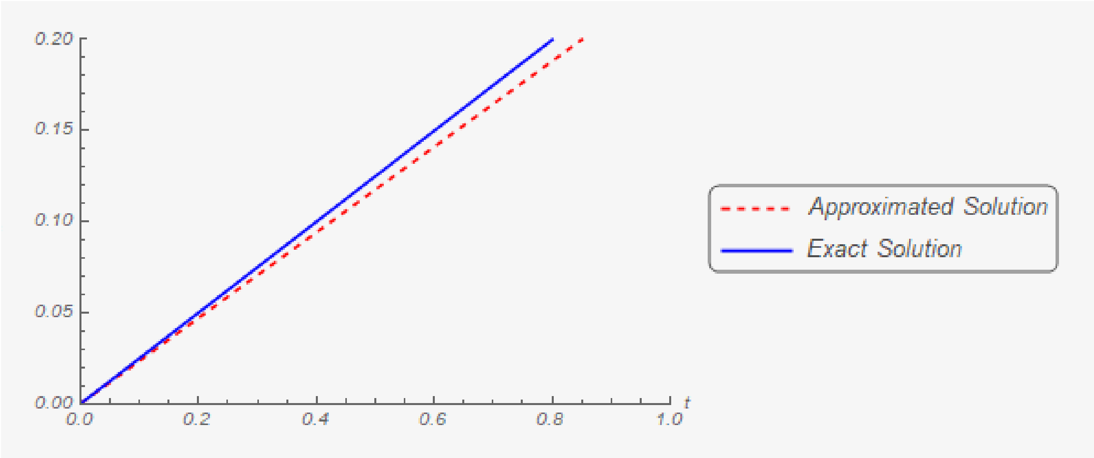

Figure 5.6: The second component of the exact solution and its hunderedth approximation

Iteration 150: The graphs of the first and second components of the exact and approximate (in the one hundredth and fifth iteration) solutions are shown on Figure 5.7, Figure 5.8 respectively.

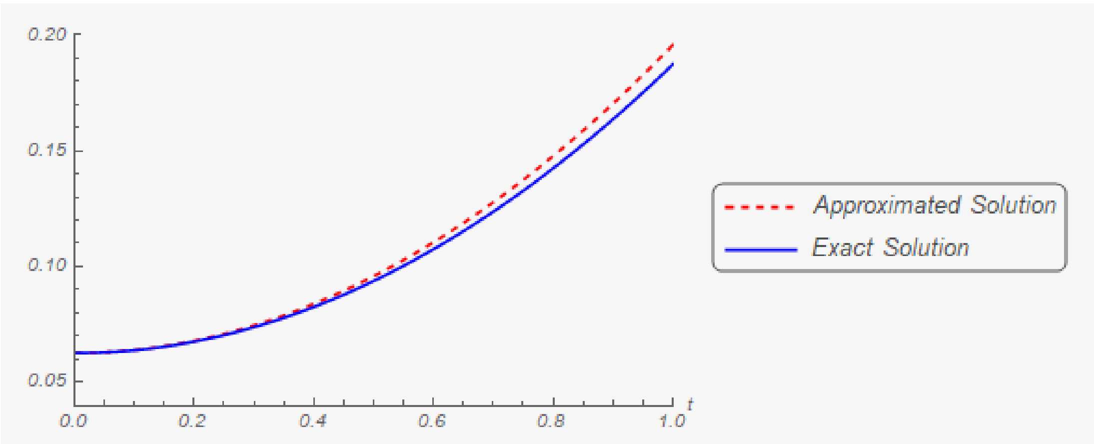

Figure 5.7: The first component of the exact solution and its one hundred and fifth approximation

The following inequalities are related with the maximum deviations of the exact solution with its one hundred and fifth approximations:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{150,1}(t)\right| \leq 0.008333,$$
$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{150,2}(t)\right| \leq 0.01077.$$

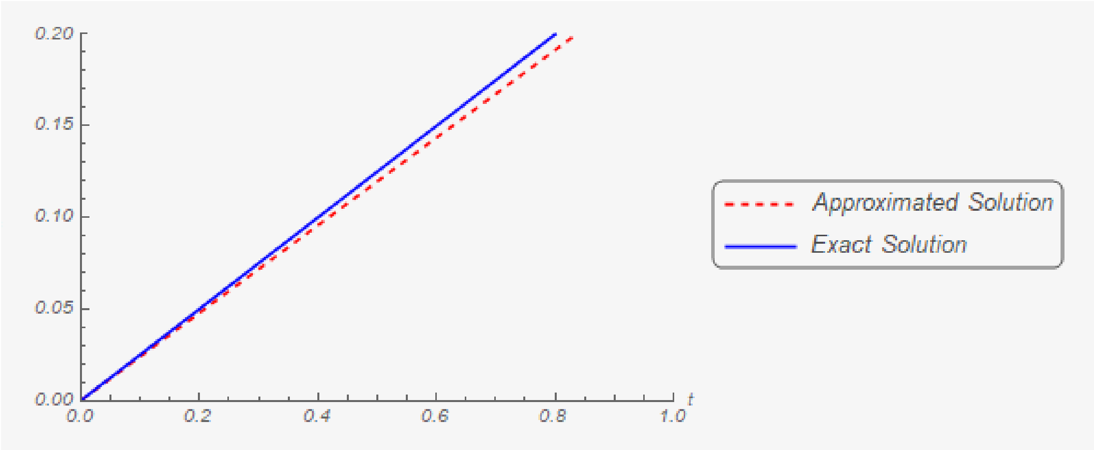

Figure 5.8: The second component of the exact solution and its one hundred and fifth approximation

Iteration 200: The graphs of the first and second components of the exact and approximate (in the two hundredth iteration) solutions are shown on Figure 5.9, Figure 5.10, respectively.

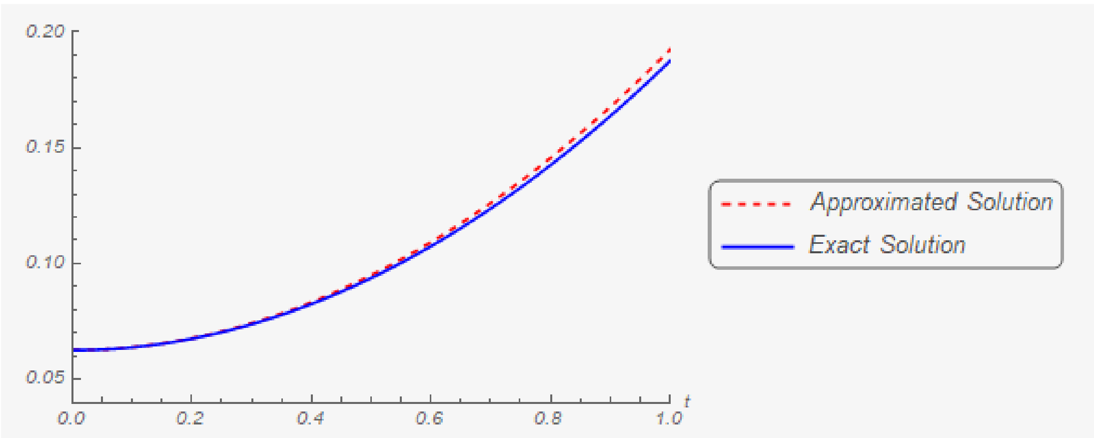

Figure 5.9: The first component of the exact solution and its two hundredth approximation

For the two hundredth iteration, the maximum deviations of the exact solution are shown as follows:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{200,1}(t)\right| \leq 0.00487,$$

$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{200,2}(t)\right| \leq 0.006097.$$

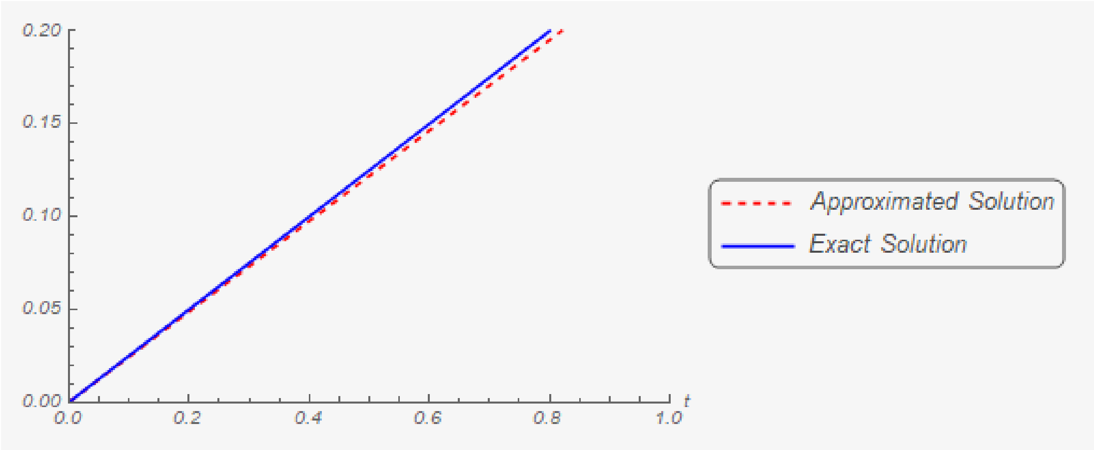

Figure 5.10: The second component of the exact solution and its two hundredth approximation

Iteration 250: The graphs of the first and second components of the exact and approximate (in the two hundredth and fifth iteration) solutions are shown on Figure 5.11, Figure 5.12, respectively.

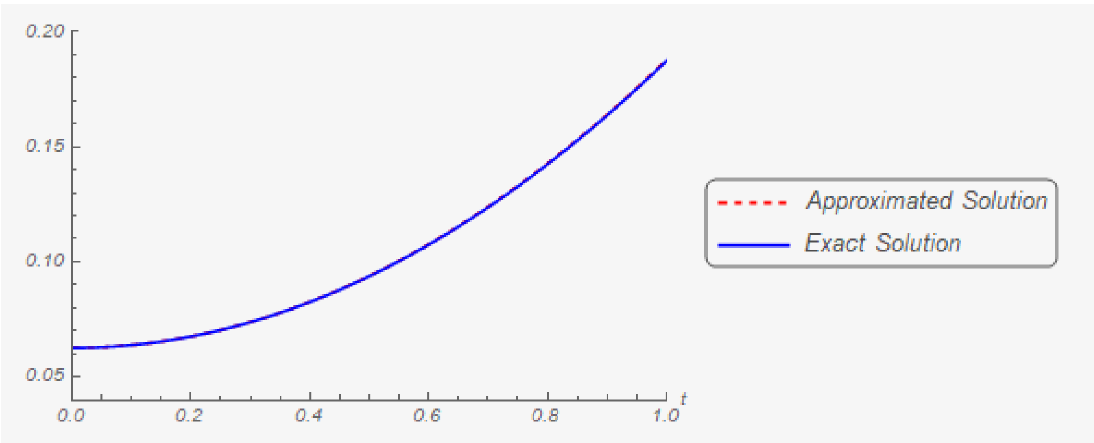

Figure 5.11: The first component of the exact solution and its two hundredth and fifth approximation

The following inequalities are related with the maximum deviations of the exact solution with its two hundredth and fifth approximations:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{250,1}(t)\right| \leq 0.0004704,$$

$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{250,2}(t)\right| \leq 0.0006233.$$

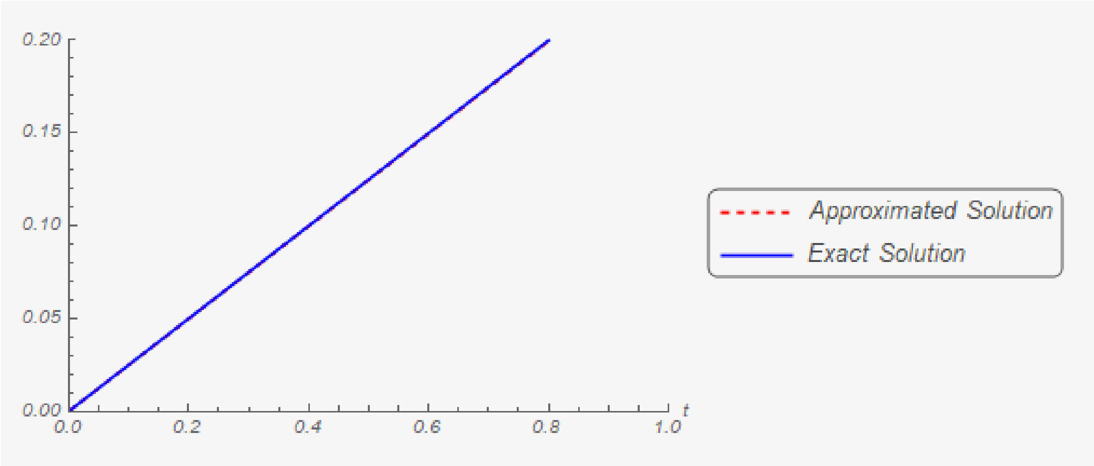

Figure 5.12: The second component of the exact solution and its two hundredth and fifth approximation

Iteration 300: The graphs of the first and second components of the exact and approximate (in the three hundredth iteration) solutions are shown on Figure 5.13, Figure 5.14, respectively.

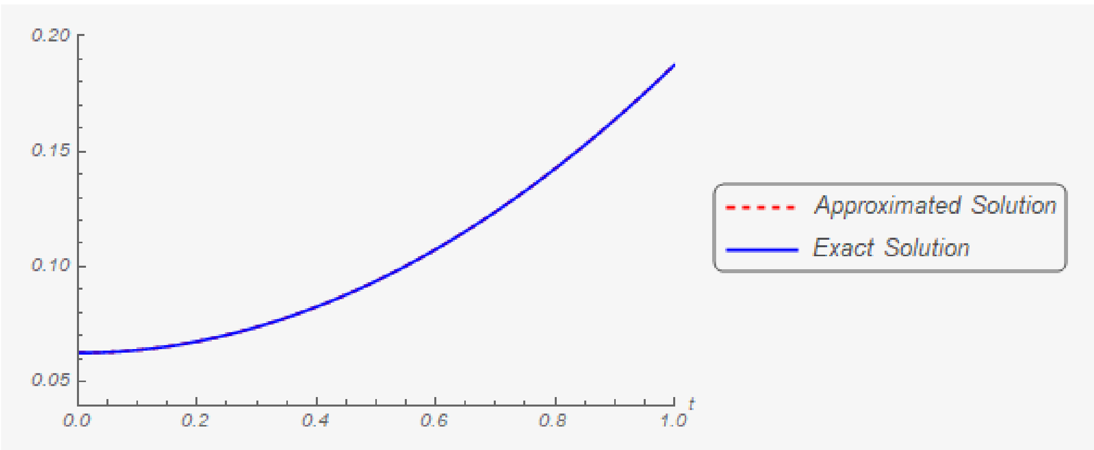

Figure 5.13: The first component of the exact solution and its three hundredth approximation

The following inequalities are related with the maximum deviations of the exact solution with its three hundredth approximations.

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{300,1}(t)\right| \leq 0.00007809,$$
$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{300,2}(t)\right| \leq 0.00006241.$$

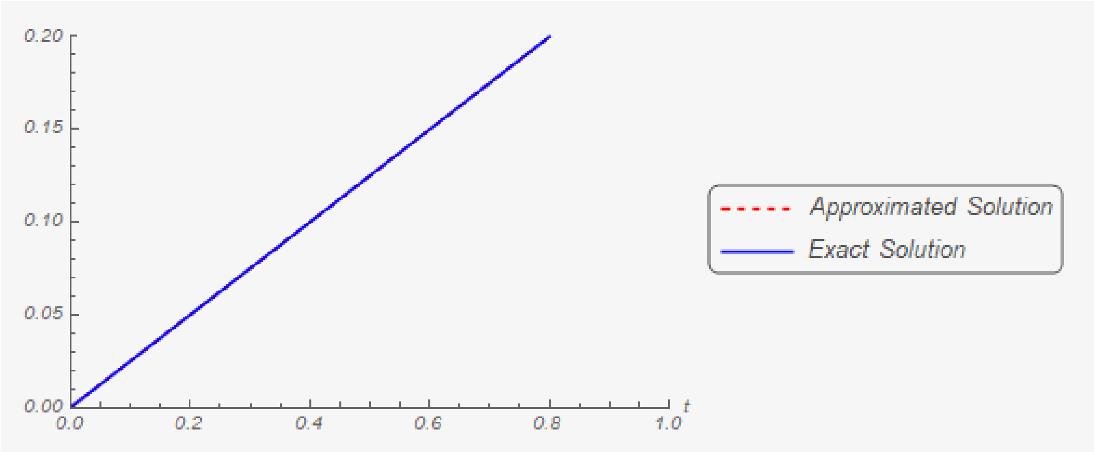

Figure 5.14: The second component of the exact solution and its three hundredth approximation

Iteration 364: The graphs of the first and second components of the exact and approximate (in thethree hundred and sixty fourth iteration) solutions are shown on Figure 5.15, Figure 5.16, respectively.

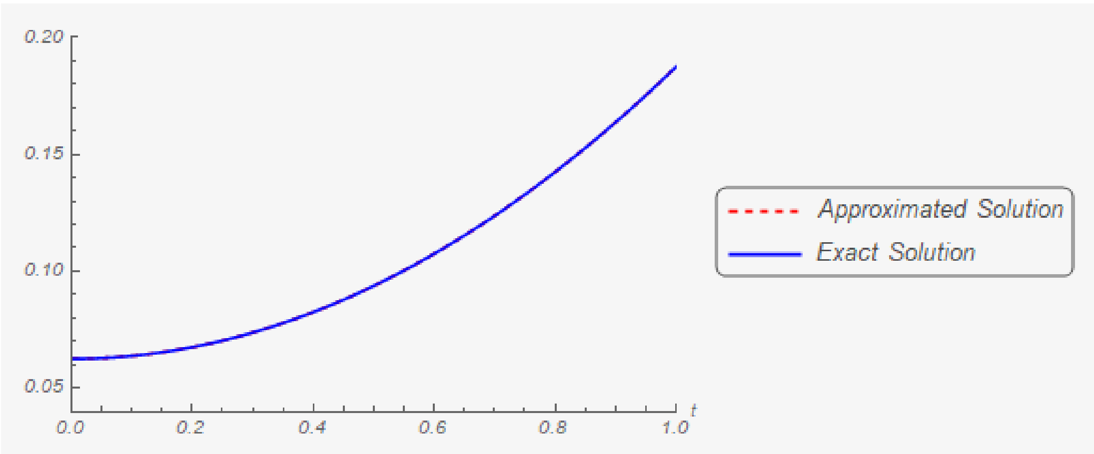

Figure 5.15: The first component of the exact solution and its three hundred and sixty fourth approximation

The following inequalities are related with the maximum deviations of the exact solution with its three hundred and sixty fourth approximations:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{364,1}(t)\right| \leq 1.209 \times 10^{-6},$$

$$\max_{t\in[0,1]} \left|x_2^*(t) - x_{364,2}(t)\right| \leq 5.813 \times 10^{-6}.$$

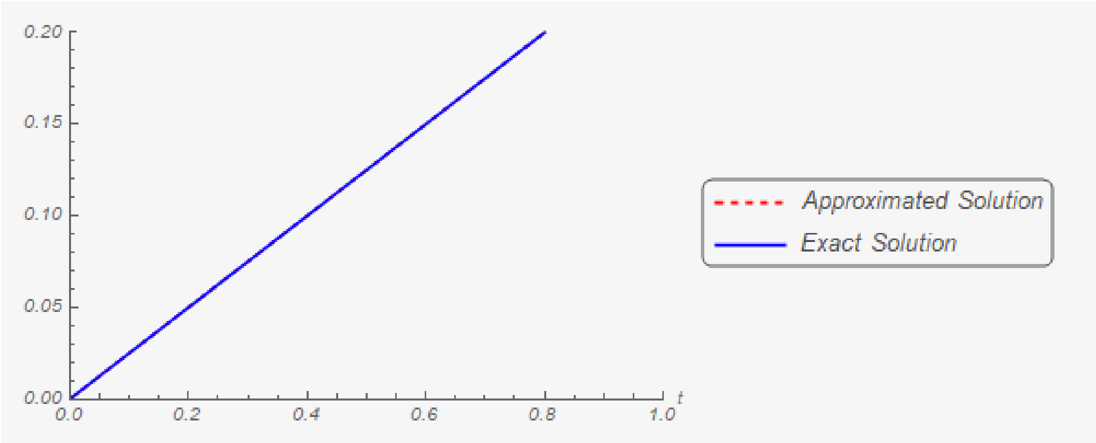

Figure 5.16: The second component of the exact solution and its three hundred and sixty fourth approximation

Lastly, we compare the results of exact and approximate solutions for Iteration 1 and Iteration 364, with errors in the below Table 5.1, Table 5.2, Table 5.3, Table 5.4 . Here, $t = 0.1$ is selected as a time step. Moreover, in this study, the numerical method has order of accuracy 2.

Table 5.1 shows the values of exact solution x_1 and approximate solution $x_{1,1}$ with errors. The values of exact solution x_2 and approximate solution $x_{1,2}$ are depicted in Table 5.2 with errors (for Iteration 1). In Table 5.3, the values of exact solution x_1 and approximate solution $x_{364,1}$ are given with errors. Similarly, Table 5.4 presents the values of exact solution x_2 and approximate solution $x_{364,2}$ (for Iteration 364).

Table 5.1: First component of exact solutions with approximate solutions

t	Exact Solution	Approximate Solution	Error	Relative Errors
0.1	0.06375	0.077040	0.01329	0.20850
0.2	0.0675	0.088380	0.02088	0.30930
0.3	0.07375	0.099730	0.02598	0.35230
0.4	0.0825	0.11110	0.02857	0.34670
0.5	0.09375	0.12240	0.02867	0.30560
0.6	0.1075	0.13380	0.02626	0.24470
0.7	0.1238	0.14510	0.02135	0.17201
0.8	0.1425	0.15640	0.01395	0.09750
0.9	0.1638	0.16780	0.00404	0.02440
1	0.1875	0.17910	0.008367	0.04480

Table 5.2: Second component of exact solutions with approximate solutions

t	Exact Solution	Approximate Solution	Error	Relative Errors
0.1	0.025	0.01992	0.005082	0.2032
0.2	0.05	0.04215	0.007847	0.1570
0.3	0.075	0.06464	0.01036	0.1381
0.4	0.1	0.08749	0.01251	0.1251
0.5	0.125	0.1108	0.01416	0.1136
0.6	0.15	0.1348	0.01518	0.1013
0.7	0.175	0.1595	0.01546	0.0886
0.8	0.2	0.1851	0.01487	0.0745
0.9	0.225	0.2117	0.01328	0.0591
1	0.25	0.2394	0.01056	0.0424

Table 5.3: First component of exact solutions with approximate solutions

t	Exact Solution	Approximate Solution	Error	Relative Errors
0.1	0.06375	0.06375	1.563e-08	2.418e-07
0.2	0.0675	0.0675	6.25e-08	9.2593e-07
0.3	0.07375	0.07375	1.406e-07	1.9064e-06
0.4	0.0825	0.0825	0.00000025	3.0303e-06
0.5	0.09375	0.09375	3.906E-07	4.1664e-06
0.6	0.1075	0.1075	5.625e-07	5.2326e-06
0.7	0.1238	0.1238	7.656e-07	6.1842e-06
0.8	0.1425	0.1425	0.000001	7.0175e-06
0.9	0.1638	0.1638	0.000001266	7.7289e-06
1	0.1875	0.1875	0.000001563	8.3360e-06

Table 5.4: Second component of exact solutions with approximate solutions

t	Exact Solution	Approximate Solution	Error	Relative Errors
0.1	0.025	0.025	0.000000625	2.5000e-05
0.2	0.05	0.05	0.00000125	2.5000e-05
0.3	0.075	0.075	0.000001875	2.5000e-05
0.4	0.1	0.1	0.0000025	2.5000e-05
0.5	0.125	0.125	0.000003125	2.5000e-05
0.6	0.15	0.15	0.00000375	2.5000e-05
0.7	0.175	0.175	0.000004375	2.5000e-05
0.8	0.2	0.2	0.000005	2.5000e-05
0.9	0.225	0.225	0.000005625	2.5000e-05
1	0.25	0.25	0.00000625	2.5000e-05

Chapter 6

EXISTENCE, UNIQUENESS AND STABILITY OF CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

In this chapter, we study existence, uniqueness and Ulam-Hyers stability results for the coupled fixed points of operators on complete metrix space. We consider the operators on the parametrized Caputo type boundary value problem. The main touch is based on Perov type fixed point theorem and Ulam-Hyers stability.

6.1 Statement of the Problem

We consider Caputo type fractional differential eqautions with nonlinear boundary conditions as follows:

$$\begin{aligned} {}_o^cD_t^p x(t) &= f(t,x(t),y(t)), \\ Ax(0)+Bx(T)+h_1(x(0),x(T)) &= \beta_1,\ \beta_1 \in \mathbb{R}^n,\ \det(B) \neq 0, \end{aligned} \tag{6.1.1}$$

and

$$\begin{aligned} {}_o^cD_t^q y(t) &= g(t,x(t),y(t)), \\ Cy(0)+Ey(T)+h_2(y(0),y(T)) &= \beta_2,\ \beta_2 \in \mathbb{R}^n,\ \det(E) \neq 0, \end{aligned} \tag{6.1.2}$$

where ${}_o^cD_t^p$ and ${}_o^cD_t^q$ are Caputo derivative of order $p,q \in (0,1]$ for any $t \in [0,T]$. The functions $f : G_f \to \mathbb{R}^n$, $g : G_g \to \mathbb{R}^n$ are continuous functions, $n \geq 2$, $G_f := [0,T] \times D_f \times D_g$ $G_g := [0,T] \times D_f \times D_g$ and $D_f, D_g \subset \mathbb{R}^n$ are closed and bounded domains.

A, B, C and E are given $n-$dimensional matrices, and β_1, β_2 are given $n-$dimensional vectors.

We introduce the vectors of parameters:

$$\begin{aligned} w &:= x(0) = (w_1, w_2 ... w_n)^T, \\ \phi &:= x(T) = (\phi_1, \phi_2, ..., \phi_n)^T, \end{aligned} \tag{6.1.3}$$

and

$$\begin{aligned} z &:= y(0) = (z_1, z_2 ... z_n)^T, \\ \lambda &:= y(T) = (\lambda_1, \lambda_2, ..., \lambda_n)^T. \end{aligned} \tag{6.1.4}$$

Then, by using $(6.1.3)$,$(6.1.4)$, nonlinear boundary conditions $(6.1.1)$, $(6.1.2)$ can be rewritten as follows:

$$Ax(0) + Bx(T) = \beta_1 - h_1(w, \phi) := \beta_1(w, \phi). \tag{6.1.5}$$

and

$$Cy(0) + Ey(T) = \beta_2 - h_2(z, \lambda) := \beta_2(z, \lambda) \tag{6.1.6}$$

Therefore, we have Caputo type fractional differential equations with linear boundary conditions depending on parameters (w, ϕ) and (z, λ) :

$$\begin{aligned} {}_o^cD_t^p x(t) &= f(t, x(t), y(t)) \\ Ax(0) + Bx(T) &= \beta_1(w, \phi) \end{aligned} \tag{6.1.7}$$

and

$$\begin{aligned} {}_o^cD_t^q y(t) &= g(t, x(t), y(t)) \\ Cy(0) + Ey(T) &= \beta_2(z, \lambda) \end{aligned} \tag{6.1.8}$$

Solutions of the given paremetrized boundary value problems $(6.1.7)$, $(6.1.8)$ are given respectively :

$$\begin{aligned} x(t) &:= w+\frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s,x(s),y(s))\,ds \right. \\ &\left. -\left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s,x(s),y(s))\,ds\right] \\ &+\left(\frac{t}{T}\right)^p \left[B^{-1}\beta_1(w,\phi)-\left(B^{-1}A+I_n\right)w\right], \end{aligned}$$

and

$$\begin{aligned} y(t) \;:\; &= z+\frac{1}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1} g(s,x(s),y(s))\,ds \right. \\ &\left. -\left(\frac{t}{T}\right)^q \int_0^T (T-s)^{q-1} g(s,x(s),y(s))\,ds\right] \\ &+\left(\frac{t}{T}\right)^q \left[E^{-1}\beta_2(z,\lambda)-\left(E^{-1}C+I_n\right)z\right]. \end{aligned}$$

6.2 Existence, Uniqueness and Stability Result:

In this section, we prove the main theorem which depends on Perov's fixed point theorem and Ulam-Hyers stable. The following definitions and assumptions are needed for proving the main theorem.

First of all, the functions f,g in $(6.1.7)$ and $(6.1.8)$ are satisfied the followings:

$$\begin{aligned} \|f\|_\infty &= \max_{(t,x,y)\in[0,T]\times D_f\times D_g} \|f(t,x,y)\| \\ \|g\|_\infty &= \max_{(t,x,y)\in[0,T]\times D_f\times D_g} \|g(t,x,y)\| \end{aligned}$$

for all $t\in[0,T]$, $(x,y)\in D_f\times D_g$. Also,

$$A_p(t)=\frac{2t^p}{\Gamma(p+1)}\left(1-\frac{t}{T}\right)^p$$

and

$$A_q(t) = \frac{2t^q}{\Gamma(q+1)}\left(1-\frac{t}{T}\right)^q$$

are defined. Then , $A_p(t)$ and $A_q(t)$ take their maximum values at $t=\frac{T}{2}$ and

$$\begin{aligned}\left\|A_p\right\|_\infty &= \frac{T^p}{2^{2p-1}\Gamma(p+1)},\\ \left\|A_q\right\|_\infty &= \frac{T^q}{2^{2q-1}\Gamma(q+1)}.\end{aligned}$$

Then, parametrized boundary value problems (6.1.7) and (6.1.8) will be studied under the following conditions:

A) Functions f,g in (6.1.7) and (6.1.8) satisfy the Lipschitz type conditions

$$\begin{aligned}|f(t,x_1,x_2)-f(t,y_1,y_2)| &\leq k_1|x_1-y_1|+k_2|x_2-y_2|\\ |g(t,x_1,x_2)-g(t,y_1,y_2)| &\leq k_3|x_1-y_1|+k_4|x_2-y_2|\end{aligned}$$

for all $k_i>0$, $i=1,2,3,4$ and $x_1,x_2\in D_f$, $y_1,y_2\in D_g$.

B) The spectral radius $r(K)$ of the matrix $K=\begin{pmatrix}K_1 & K_2\\ K_3 & K_4\end{pmatrix}$ is $r(K)<1$ where $k_i\left\|A_p\right\|_\infty = K_i>0$ when $i=1,2$ and $k_i\left\|A_q\right\|_\infty = K_i>0$ when $i=3,4$.

We define operators $(T_1,T_2): C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^n)\to C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^n)$ which depend on w,ϕ,z,λ as follows:

$$\begin{aligned}T_1(x,y)(t) \;:\; &= w+\frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s,x(s),y(s))\,ds\right.\\ &\left.-\left(\frac{t}{T}\right)^p\int_0^T (T-s)^{p-1} f(s,x(s),y(s))\,ds\right]\\ &+\left(\frac{t}{T}\right)^p\left[B^{-1}\beta_1(w,\phi)-\left(B^{-1}A+I_n\right)w\right] \qquad (6.2.1)\end{aligned}$$

and

$$
\begin{aligned}
T_2(x,y)(t) \quad : \quad &= z + \frac{1}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} g(s,x(s),y(s))\, ds \right. \\
&\left. - \left(\frac{t}{T}\right)^q \int_0^T (T-s)^{q-1} g(s,x(s),y(s))\, ds \right] \\
&+ \left(\frac{t}{T}\right)^q \left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right) z \right]. \qquad (6.2.2)
\end{aligned}
$$

The operators T_1 and T_2 can be considered as a system of operatorial equations such as

$$
\begin{aligned}
x &= T_1(x,y) \\
y &= T_2(x,y)
\end{aligned}
$$

where $(T_1,T_2) : C([0,T],\mathbb{R}^n) \times C([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R}^n) \times C([0,T],\mathbb{R}^n)$.

Let X be a nonempty set. A mapping $\dot{d} : X \times X \to \mathbb{R}^n$ is called a vector-valued metric on X if the following properties are satisfied:

d1) $d(x,y) \geq 0$ for all $x,y \in X$; if $d(x,y) = 0$, then $x = y$.

d2) $d(x,y) = d(y,x)$ for all $x,y \in X$.

d3) $d(x,y) \leq d(x,y) + d(z,y)$ for all $x,y,z \in X$.

The following theorems are also used for the proof of the main result. Following theorem is about the classical result in matrix analysis. (see [14, 54, 58])

Theorem 6.2.1 *Let $A \in M_{mm}$. The following assertions are equivalents:*

i) *A is convergent towards zero*

ii) $A^n \to 0$ *as* $n \to \infty$

iii) *The eigenvalues of A are in the open unit disc,i.e* $|\lambda| < 1$, *for every* $\lambda \in \mathbb{C}$ *with* $\det(A - \lambda I) = 0$

iv) *The matrix* $(I-A)$ *is nonsingular and*

$$(I-A)^{-1} = I + A + \ldots + A^n + \ldots$$

v) *The matrix* $(I-A)$ *is nonsingular and* $((I-A))^{-1}$ *has nonnegative elements*

vi) $A^n q \to 0$ *and* $qA^n \to 0$ *as* $n \to \infty$, *for each* $q \in \mathbb{R}^m$.

Theorem 6.2.2 *[47] (Perov) Let* (X,d) *be a complex generalized metric space and the operator* $f : X \to X$ *with the property that there exists a matrix* $A \in M_{mm}$ *such that* $d(f(x), f(y)) \leq Ad(x,y)$ *for all* $x,y \in X$. *If* A *is a matrix convergent towards zero, then:*

i) $Fix(f) = \{x^*\}$

ii) *the sequence of successive approximations* $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$ *is convergent and has the limit* x^*, *for all* $x_0 \in X$;

iii) *one has the following estimation*

$$d(x_n, x^*) \leq A^n (I-A)^{-1} d(x_0, x_1)$$

iv) *if* $g : X \to X$ *is an operator such that there exist* $y^* \in Fix(g)$ *and* $\eta \in \left(\mathbb{R}^m_+\right)^*$ *with* $d(f(x), g(x)) \leq \eta$, *for each* $x \in X$, *then*

$$d(x^*, y^*) \leq (I-A)^{-1} \eta$$

v) *if* $g : X \to X$ *is an operator and there exists* $\eta \in \left(\mathbb{R}^m_+\right)$ *such that* $d(f(x), g(x)) \leq \eta$, *for all* $x \in X$*,then for the sequence* $y_n := g^n(x_0)$ *we have the following estimation*

$$d(y_n, x^*) \leq (I-A)^{-1} \eta + A^n (I-A)^{-1} d(x_0, x_1).$$

Definition 6.2.3 *Let (X,d) be a metric space and let $T_1,T_2 : X\times X \to X$ be two operators. Then the operatorial equations system*

$$\begin{aligned} x &= T_1(x,y) \\ y &= T_2(x,y) \end{aligned} \qquad (6.2.3)$$

is said to be Ulam-Hyers stable if there exist $c_1,c_2,c_3,c_4>0$ such that for each $\theta_1,\theta_2>0$ and each pair $(u^*,v^*)\in X\times X$ such that

$$\begin{aligned} d(u^*,T_1(u^*,v^*)) &\le \theta_1 \\ d(v^*,T_2(u^*,v^*)) &\le \theta_2 \end{aligned}$$

there exists a solution $(x^*,y^*)\in X\times X$ of $(6.2.3)$ such that

$$\begin{aligned} d(u^*,x^*) &\le c_1\theta_1+c_2\theta_2 \\ d(v^*,y^*) &\le c_3\theta_1+c_4\theta_2 \end{aligned}$$

Theorem 6.2.4 *Let $(X,d)=(C([0,T],\mathbb{R}^n),\|x-y\|_\infty)$ and $T_1,T_2 : X\times X\to X$ be two operators. Suppose that*

$$K := \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$$

satisfies an inequality $r(K)<1$ which is defined in the assumption (D). Then,

i) *for all $(x,y),(u,v)\in X\times X$, the following inequalities are satisfied:*

$$\begin{aligned} \|T_1(x,y)-T_1(u,v)\|_\infty &\le K_1\|x-u\|_\infty+K_2\|y-v\|_\infty \\ \|T_2(x,y)-T_2(u,v)\|_\infty &\le K_3\|x-u\|_\infty+K_4\|y-v\|_\infty . \end{aligned}$$

ii) *there exists a unique element $(x^*,y^*)\in X\times X$ such that*

$$
\begin{aligned}
x^* &= T_1(x^*, y^*) \\
y^* &= T_2(x^*, y^*)
\end{aligned}
$$

iii) *the sequence* $\left(T_1^n(x,y), T_2^n(x,y)\right)_{n\in N}$ *converges to* (x^*, y^*) *as* $n \to \infty$*, where*

$$
\begin{aligned}
T_1^{n+1}(x,y) &: = T_1^n\left(T_1(x,y), T_2(x,y)\right) \\
T_2^{n+1}(x,y) &: = T_2^n\left(T_1(x,y), T_2(x,y)\right)
\end{aligned}
$$

for all $n \in N$.

iv) *we have the following estimation:*

$$
\begin{aligned}
\begin{pmatrix} \left\|T_1^{n+j}(x,y) - T_1^n(x,y)\right\|_\infty \\ \left\|T_2^{n+j}(x,y) - T_2^n(x,y)\right\|_\infty \end{pmatrix} &\leq K^n (I-K)^{-1} \begin{pmatrix} \|T_1 - w\|_\infty \\ \|T_2 - z\|_\infty \end{pmatrix} \\
&\leq K^n (I-K)^{-1} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}. \qquad (6.2.4)
\end{aligned}
$$

where

$$
N_1 \geq \left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A + I_n\right)w\right]\right|, \qquad (6.2.5)
$$

$$
N_2 \geq \left\|A_q\right\|_\infty \|g\|_\infty \left| + \left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right)z\right]\right|. \qquad (6.2.6)
$$

v) *the operatorial equation system*

$$
\begin{aligned}
x &= T_1(x,y) \\
y &= T_2(x,y) \qquad (6.2.7)
\end{aligned}
$$

is Ulam-Hyers stable.

Proof.

i) By using the operators (6.2.1),(6.2.2) and considering (6.2.8),(6.2.9), we have:

$$\begin{aligned} |T_1(x,y)(t) - T_1(u,v)(t)| &\leq \left| \frac{1}{\Gamma(p)} \left[\int_0^t \left((t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1} \right) ds \right.\right. \\ &\quad \left.\left. + \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} ds \right] \right| \| f(s,x(s),y(s)) \\ &\quad - f(s,u(s),v(s)) \|_\infty \\ &\leq \|A_p\|_\infty \left(k_1 \|x-u\|_\infty + k_2 \|y-v\|_\infty \right) \\ &\leq K_1 \|x-u\|_\infty + K_2 \|y-v\|_\infty . \end{aligned}$$

where

$$\begin{aligned} (t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1} &= (t-s)^{p-1} \left(1 - \left(\frac{t}{T}\right)^p \left(\frac{t-s}{T-s}\right)^{1-p} \right) \\ &\geq (t-s)^{p-1} \left(1 - \left(\frac{t}{T}\right)^p \left(\frac{t}{T}\right)^{1-p} \right) \\ &= (t-s)^{p-1} \left(1 - \frac{t}{T} \right) \geq 0 \qquad (6.2.8) \end{aligned}$$

and

$$\begin{aligned} |T_2(x,y)(t) - T_2(u,v)(t)| &\leq \left| \frac{1}{\Gamma(q)} \left[\int_0^t \left((t-s)^{q-1} - \left(\frac{t}{T}\right)^q (T-s)^{q-1} \right) ds \right.\right. \\ &\quad \left.\left. + \left(\frac{t}{T}\right)^q \int_t^T (T-s)^{q-1} ds \right| \| f(s,x(s),y(s)) \right. \\ &\quad - f(s,u(s),v(s)) \|_\infty \\ &\leq \|A_p\|_\infty \left(k_3 \|x-u\|_\infty + k_4 \|y-v\|_\infty \right) \\ &\leq K_3 \|x-u\|_\infty + K_4 \|y-v\|_\infty . \end{aligned}$$

where

$$(t-s)^{q-1} - \left(\frac{t}{T}\right)^q (T-s)^{q-1} \geq 0 \qquad (6.2.9)$$

Therefore, we get:

$$\begin{aligned}
\|T_1(x,y)-T_1(u,v)\|_\infty &\leq K_1\|x-u\|_\infty + K_2\|y-v\|_\infty \\
\|T_2(x,y)-T_2(u,v)\|_\infty &\leq K_3\|x-u\|_\infty + K_4\|y-v\|_\infty .
\end{aligned}$$

where $k_i\left\|A_p\right\|_\infty = K_i > 0$ when $i=1,2$ and $k_i\left\|A_q\right\|_\infty = K_i > 0$ when $i=3,4$.

ii) We denote $Z := X\times X$ and consider $\widetilde{d}: Z\times Z \to \mathbb{R}^2_+$ as follows:

$$\widetilde{d}\left((x,y),(u,v)\right) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix} = \begin{pmatrix} \|x-u\|_\infty \\ \|y-v\|_\infty \end{pmatrix}. \tag{6.2.10}$$

By using (6.2.10) ,

$$\begin{aligned}
\widetilde{d}\left(T(x,y),T(u,v)\right) &= \left(\begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix} \quad \begin{pmatrix} T_1(u,v) \\ T_2(u,v) \end{pmatrix} \right) \\
&= \begin{pmatrix} \|T_1(x,y)-T_1(u,v)\|_\infty \\ \|T_2(x,y)-T_2(u,v)\|_\infty \end{pmatrix} \\
&\leq \begin{pmatrix} K_1\|x-u\|_\infty + K_2\|y-v\|_\infty \\ K_3\|x-u\|_\infty + K_4\|y-v\|_\infty \end{pmatrix} \\
&= \begin{pmatrix} K_1 & K_2 \\ \\ K_3 & K_4 \end{pmatrix} \begin{pmatrix} \|x-u\|_\infty \\ \|y-v\|_\infty \end{pmatrix} \\
&= K\widetilde{d}\left((x,y),(u,v)\right).
\end{aligned} \tag{6.2.11}$$

Here, if $(x,y) := l$, $(u,v) := m$, then (6.2.11) can be rewritten as follows:

$$\widetilde{d}\left(T(l),T(m)\right) \leq K\,\widetilde{d}\left((l),(m)\right),$$

where the matrix K is convergent to zero. According to the conclusion (i) of Perov's theorem,

$$Fix(T) = \{l^*\}$$

which means

$$T(x^*,y^*) = (x^*,y^*). \tag{6.2.12}$$

Moreover, $(6.2.12)$ can be rewritten in the following form:

$$\begin{aligned} x^* &= T_1(x^*,y^*) \\ y^* &= T_2(x^*,y^*). \end{aligned}$$

iii) We define operators T_1^n and T_2^n as follows:

$$\begin{aligned} T_1^n(x,y)(t) \quad : &= w + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f\left(s, T_1^{n-1}(x,y)(s), T_2^{n-1}(x,y)(s)\right) ds \right. \\ & \left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f\left(s, T_1^{n-1}(x,y)(s), T_2^{n-1}(x,y)(s)\right) ds \right] \\ & + \left(\frac{t}{T}\right)^p \left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A + I_n\right) w \right] \end{aligned} \tag{6.2.13}$$

and

$$\begin{aligned} T_2^n(x,y)(t) \quad : &= z + \frac{1}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} g\left(s, T_1^{n-1}(x,y)(s), T_2^{n-1}(x,y)(s)\right) ds \right. \\ & \left. - \left(\frac{t}{T}\right)^q \int_0^T (T-s)^{q-1} g\left(s, T_1^{n-1}(x,y)(s), T_2^{n-1}(x,y)(s)\right) ds \right] \\ & + \left(\frac{t}{T}\right)^q \left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right) z \right]. \end{aligned} \tag{6.2.14}$$

In addition, for each $(x,y) \in X \times X$, we have:

$$T^n(x,y)(t) = (T_1^n(x,y), T_2^n(x,y)) \to (x^*,y^*)$$

as $n \to \infty$, where

$$\begin{aligned} T_1^0(x,y) \quad : &= w, \ T_2^0(x,y) := z, \\ T_1^1(x,y) \quad : &= T_1(x,y), \ T_2^1(x,y) := T_2(x,y), \\ T_1^2(x,y) \quad : &= T_1\left(T_1(x,y), T_2(x,y)\right), \\ T_2^2(x,y) \quad : &= T_2\left(T_1(x,y), T_2(x,y)\right). \end{aligned} \tag{6.2.15}$$

Thus, (6.2.15) can be generalized in the following form:

$$T_1^{n+1}(x,y) \quad : \quad = T_1^n\left(T_1(x,y),T_2(x,y)\right)$$

$$T_2^{n+1}(x,y) \quad : \quad = T_2^n\left(T_1(x,y),T_2(x,y)\right).$$

iv) To prove the inequality given in (6.2.4), at first, we put $n=1$ into the equation (6.2.13).Then, we have:

$$\begin{aligned} T_1(x,y)(t) = w + \frac{1}{\Gamma(p)} &\left[\int_0^t (t-s)^{p-1} f(s,w,z)\,ds \right. \\ &\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s,w,z)\,ds\right] \\ &+ \left(\frac{t}{T}\right)^p \left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A+I_n\right)w\right] \end{aligned} \tag{6.2.16}$$

So, (6.2.16) can be estimated as follows:

$$\begin{aligned} |T_1(x,y)(t) - w| &\le \frac{1}{\Gamma(p)}\left[\int_0^t \left|(t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1}\right| \|f\|_\infty\,ds \right. \\ &\quad \left. + \int_t^T \left|\left(\frac{t}{T}\right)^p (T-s)^{p-1}\right| \|f\|_\infty\,ds\right] \\ &\quad + \left|\left(\frac{t}{T}\right)^p \left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A+I_n\right)w\right]\right| \\ &\le \frac{2t^p}{\Gamma(p+1)}\left(1-\frac{t}{T}\right)^p \|f\|_\infty \\ &\quad + \left(\frac{t}{T}\right)^p \left|\left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A+I_n\right)w\right]\right| \\ &\le \frac{T^p}{2^{2p-1}\Gamma(p+1)}\|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A+I_n\right)w\right]\right| \\ &= \left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A+I_n\right)w\right]\right| \\ &\le N_1 \end{aligned} \tag{6.2.17}$$

where N_1 is defined in (6.2.5). Then, we consider the difference (6.2.17) in general,

$$
\begin{aligned}
&\left|T_1^{n+1}(x,y)(t) - T_1^n(x,y)(t)\right| \\
\leq \; &\left|\frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1}\, ds \right.\right. \\
&\left.\left. + \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1}\, ds\right]\right| \left\| f\left(s, T_1^n(x,y), T_2^n(x,y)\right) \right. \\
&\left. - f(s, T_1^{n-1}(x,y), T_2^{n-1}(x,y))\right\|_\infty \qquad (6.2.18)
\end{aligned}
$$

for $n = 1, 2, \ldots$

Now, we consider $(6.2.14)$ for $n = 1$.

$$
\begin{aligned}
|T_2(x,y)(t) - z| &\leq \frac{1}{\Gamma(q)}\left[\int_0^t \left|(t-s)^{q-1} - \left(\frac{t}{T}\right)^q (T-s)^{q-1}\right| \|g(s,w,z)\|_\infty\, ds \right. \\
&\left. + \int_t^T \left|\left(\frac{t}{T}\right)^q (T-s)^{q-1}\right| \|g(s,w,z)\|_\infty\, ds\right] \\
&+ \left|\left(\frac{t}{T}\right)^q \left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right) z\right]\right| \\
&\leq \frac{2t^q}{\Gamma(q+1)}\left(1 - \frac{t}{T}\right)^q \|g\|_\infty \\
&+ \left(\frac{t}{T}\right)^q \left|\left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right) z\right]\right| \\
&\leq \frac{T^q}{2^{2q-1}\Gamma(q+1)} \|g\|_\infty + \left|\left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right) z\right]\right| \\
&= \left\|A_q\right\|_\infty \|g\|_\infty + \left|\left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right) z\right]\right| \\
&\leq N_2 \qquad (6.2.19)
\end{aligned}
$$

where N_2 is defined in $(6.2.6)$.

Therefore, the difference between T_2^{n+1} and T_2^n can be generalized as follows:

$$\begin{aligned}
& \left|T_2^{n+1}(x,y)(t) - T_2^n(x,y)(t)\right| \\
\leq & \left|\frac{1}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1} - \left(\frac{t}{T}\right)^q (T-s)^{q-1}\,ds \right.\right. \\
& \left.\left. + \left(\frac{t}{T}\right)^q \int_t^T (T-s)^q\,ds\right]\right| \left\|g\left(s, T_1^n(x,y), T_2^n(x,y)\right)\right. \\
& \left. -g(s, T_1^{n-1}(x,y), T_2^{n-1}(x,y))\right\|_\infty \qquad (6.2.20)
\end{aligned}$$

More generally, the difference (6.2.18) is denoted by $r_n(t,w,\phi,z,\lambda)$ such as

$$r_n(t,w,\phi,z,\lambda) := \left|T_1^n(x,y)(t) - T_1^{n-1}(x,y)(t)\right|, \text{ for all } n = 2,3,... \quad (6.2.21)$$

and the difference (6.2.20) is indicated by $\Omega_n(t,w,\phi,z,\lambda)$ in the following form:

$$\Omega_n(t,w,\phi,z,\lambda) := \left|T_2^n(x,y)(t) - T_2^{n-1}(x,y)(t)\right|, \text{ for all } n = 2,3,... \quad (6.2.22)$$

So, when $n = 2$ in (6.2.21), we have:

$$\begin{aligned}
r_2(t,w,\phi,z,\lambda) \leq & \frac{1}{\Gamma(p)}\left(\int_0^t \left|\left[(t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1}\right]\right|\right. \\
& \left. + \int_t^T \left|\left(\frac{t}{T}\right)^p (T-s)^{p-1}\right|\right) \left\|f\left(s, T_1(x,y), T_2(x,y)\right) - f(s,w,z)\right\|_\infty ds \\
& \leq \frac{1}{\Gamma(p)}\left(\int_0^t \left[(t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1}\right]\right. \\
& \left. + \int_t^T \left(\frac{t}{T}\right)^p (T-s)^{p-1}\,ds\right) \left(k_1 \left\|T_1 - w\right\|_\infty + k_2 \left\|T_2 - z\right\|_\infty\right) \\
& \leq k_1 \frac{2t^p}{\Gamma(p+1)}\left(1 - \frac{t}{T}\right)^p \left[\left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi) - \left(B^{-1}A + I_n\right)w\right]\right|\right] \\
& + k_2 \frac{2t^p}{\Gamma(p+1)}\left(1 - \frac{t}{T}\right)^p \left[\left\|A_q\right\|_\infty \|g\|_\infty \left| + \left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right)z\right]\right|\right] \\
& \leq k_1 \left\|A_p\right\|_\infty \left[\left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(\omega,\phi) - \left(B^{-1}A + I_n\right)w\right]\right|\right] \\
& + k_2 \left\|A_p\right\|_\infty \left[\left\|A_q\right\|_\infty \|g\|_\infty \left| + \left[E^{-1}\beta_2(z,\lambda) - \left(E^{-1}C + I_n\right)z\right]\right|\right]
\end{aligned}$$

Now, $n=2$ in (6.2.22) and we have:

$$\Omega_2(t,w,\phi,z,\lambda) \leq \frac{1}{\Gamma(q)}\left(\int_0^t \left|\left[(t-s)^{q-1}-\left(\frac{t}{T}\right)^q (T-s)^{q-1}\right]\right| + \int_t^T \left|\left(\frac{t}{T}\right)^q (T-s)^{q-1}\right|\right) \|g(s,T_1(x,y),T_2(x,y))-g(s,w,z)\|\,ds$$

$$\leq \frac{1}{\Gamma(q)}\left(\int_0^t \left[(t-s)^{q-1}-\left(\frac{t}{T}\right)^q (T-s)^{q-1}\right] + \int_t^T \left(\frac{t}{T}\right)^q (T-s)^{q-1}\,ds\right)(k_3\|T_1-w\|+k_4\|T_2-z\|)$$

$$\leq k_3\frac{2t^q}{\Gamma(q+1)}\left(1-\frac{t}{T}\right)^q\left[\left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi)-\left(B^{-1}A+I_n\right)w\right]\right|\right]$$

$$+k_4\frac{2t^q}{\Gamma(q+1)}\left(1-\frac{t}{T}\right)^q\left[\left\|A_q\right\|_\infty \|g\|_\infty \right| + \left[E^{-1}\beta_2(z,\lambda)-\left(E^{-1}C+I_n\right)z\right]\left|\right]$$

$$\leq k_3\left\|A_q\right\|_\infty\left[\left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi)-\left(B^{-1}A+I_n\right)w\right]\right|\right]$$

$$+k_4\left\|A_q\right\|_\infty\left[\left\|A_q\right\|_\infty \|g\|_\infty \right| + \left[E^{-1}\beta_2(z,\lambda)-\left(E^{-1}C+I_n\right)z\right]\left|\right]$$

Therefore, by using the mathematical induction we obtain the following equations:

$$r_{n+1}(t,w,\phi,z,\lambda) \leq \left(k_1\left\|A_p\right\|_\infty\right)^n\left[\left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi)-\left(B^{-1}A+I_n\right)w\right]\right|\right]$$

$$+\left(k_2\left\|A_p\right\|_\infty\right)^n\left[\left\|A_q\right\|_\infty \|g\|_\infty \right| + \left[E^{-1}\beta_2(z,\lambda)-\left(E^{-1}C+I_n\right)z\right]\left|\right]$$

$$\leq K_1^n N_1 + K_2^n N_2$$

$$n=1,2.. \tag{6.2.23}$$

$$\Omega_{n+1}(t,w,\phi,z,\lambda) \leq \left(k_3\left\|A_q\right\|_\infty\right)^n\left[\left\|A_p\right\|_\infty \|f\|_\infty + \left|\left[B^{-1}\beta_1(w,\phi)-\left(B^{-1}A+I_n\right)w\right]\right|\right]$$

$$+\left(k_4\left\|A_q\right\|_\infty\right)^n\left[\left\|A_q\right\|_\infty \|g\|_\infty \right| + \left[E^{-1}\beta_2(z,\lambda)-\left(E^{-1}C+I_n\right)z\right]\left|\right]$$

$$\leq K_3^n N_1 + K_4^n N_2$$

$$n=1,2.. \tag{6.2.24}$$

In view of (6.2.23) and (6.2.24) we have:

$$\begin{pmatrix} \left\| T_1^{n+j}(x,y) - T_1^n(x,y) \right\|_\infty \\ \left\| T_2^{n+j}(x,y) - T_2^n(x,y) \right\|_\infty \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{j} r_{n+i}(t,w,\phi,z,\lambda) \\ \sum_{i=1}^{j} \Omega_{n+i}(t,w,\phi,z,\lambda) \end{pmatrix}$$
$$\leq K^n \sum_{i=0}^{j-1} K^i \begin{pmatrix} \|T_1 - w\|_\infty \\ \|T_2 - z\|_\infty \end{pmatrix}$$
$$\leq K^n \sum_{i=0}^{j-1} K^i \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

Then, we have

$$\begin{pmatrix} \left\| T_1^{n+j}(x,y) - T_1^n(x,y) \right\|_\infty \\ \left\| T_2^{n+j}(x,y) - T_2^n(x,y) \right\|_\infty \end{pmatrix} \leq K^n (I-K)^{-1} \begin{pmatrix} \|T_1 - w\|_\infty \\ \|T_2 - z\|_\infty \end{pmatrix}$$
$$\leq K^n (I-K)^{-1} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

v) Let $\theta_1, \theta_2 > 0$ and $(u^*, v^*) \in X \times X$ such that

$$\|u^* - T_1(u^*, v^*)\|_\infty \leq \theta_1$$

$$\|v^* - T_2(u^*, v^*)\|_\infty \leq \theta_2.$$

Then,

$$\begin{aligned} \widetilde{d}((u^*,v^*),(x^*,y^*)) &\leq \widetilde{d}((u^*,v^*),T(x^*,y^*)) \\ &\leq \widetilde{d}((u^*,v^*),T(u^*,v^*)) + \widetilde{d}(T(u^*,v^*),T(x^*,y^*)) \\ &= \begin{pmatrix} d(u^*,T_1(u^*,v^*)) \\ d(v^*,T_2(u^*,v^*)) \end{pmatrix} + \begin{pmatrix} d(T_1(u^*,v^*),T_1(x^*,y^*)) \\ d(T_2(u^*,v^*),T_2(x^*,y^*)) \end{pmatrix} \\ &= \begin{pmatrix} \|u - T_1(u^*,v^*)\|_\infty + \|T_1(u^*,v^*) - T_1(x^*,y^*)\|_\infty \\ \|v^* - T_2(u^*,v^*)\|_\infty + \|T_2(u^*,v^*) - T_2(x^*,y^*)\|_\infty \end{pmatrix} \\ &\leq \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + K\widetilde{d}((u^*,v^*),(x^*,y^*)) \end{aligned}$$

Therefore, there exist

$$\widetilde{d}((u^*,v^*),(x^*,y^*)) \leq (I-K)^{-1}\theta$$

which is equivalent to

$$\widetilde{d}((u^*,v^*),(x^*,y^*)) = \begin{pmatrix} \|u^*-x^*\|_\infty \\ \|v^*-y^*\|_\infty \end{pmatrix} \leq (I-K)^{-1}\theta \tag{6.2.25}$$

Then, in $(6.2.25)$, the matrix $(I-K)^{-1}$ can be denoted by

$$(I-K)^{-1} := \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}.$$

Thus, we obtain the system $(6.2.7)$ is Ulam-Hyers stable where

$$\begin{aligned} \|u^*-x^*\|_\infty &\leq c_1\theta_1 + c_2\theta_2 \\ \|v^*-y^*\|_\infty &\leq c_3\theta_1 + c_4\theta_2. \end{aligned}$$

■

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