Nonlinear Electromagnetics in Flat and Curved Spacetime

Yashar Alizadeh

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Prof. Dr. Elvan Yilmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. Mustafa Halilsoy
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion, it is fully adequate, in scope and quality, as a thesis of the degree of Master of Science in Physics.

Asst. Prof. Dr. S. Habib Mazharimousavi
Supervisor

Examining Committee

1. Prof. Dr. Mustafa Halilsoy
2. Prof. Dr. Özay Gürtuğ
3. Asst. Prof. Dr. S. Habib Mazharimousavi

ABSTRACT

In the framework of the non-linear electrodynamics, we introduce a new Lagrangian with Maxwell limit which admits a regular electric field and electric potential at the origin. In static spherically symmetric spacetime we couple non-minimally the latter Lagrangian with the gravity in 3, 4 and higher dimensions separately to find the black hole solutions. We emphasize in this thesis that this new Lagrangian is easier to be used in some practical cases such as hydrogen atom due to the simple form of the electric potential of a point charge.

Keywords: Born-Infeld, NED, Non-linear Lagrangian, Black hole solution.

ÖZ

Doğrusal olmayan elektrodinamik kapsamında Maxwell limitine sahip, merkezde düzenli elektrik ve potansiyel alan içeren yeni bir Lagrange fonksiyonu sunuluyor. Statik, Küresel simetrik uzayda yerçekimine minimal olmayan şekilde bağlanan 3 ve 4 boyutlu uzaylarda karadelik çözümleri elde ediliyor. Sunduğumuz modelin birçok bakımdan örneğin hidrojen atom model potansiyeli gibi, daha kullanışlı olacağına vurgu yapılıyor.

Anahtar Kelimeler: Born-Infeld, NED, Doğrusal olmayan Lagrange fonksiyonu, Karadelik çözümleri

To My Family

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Chapter 1

INTRODUCTION

Maxwell's standard electromagnetism is a linear theory. This simply means that superposition of inputs corresponds to the superposition of the outputs. Alternatively the EM waves of this theory pass through each other without being affected. If this is not the case the transmission of radio and TV or any other broadcasting devices would not be possible. Transmission of a station at a specified frequency is not interrupted by others and vice versa, simply because the EM waves, or photons in the quantum language, pass through each other. This is the nature of a linear theory. However, in nature we have many examples of source theories that do not behave in this manner. They are classified simply as non-linear theories because their propagating agents affect each other, even more dramatically each wave of a non-linear theory interacts with itself. Most of physical systems are categorized as non-linear and naturally these types of theories are much more intricate than the linear ones. Einstein's theory of general relativity is the best example of non-linear theories which has been tested experimentally and in the linear limit it recovers the Newton's theory. No doubt the best example of a linear gravitational theory is given by Newton's theory in which if ϕ_1 and ϕ_2 are two independent potentials due to different sources, the total potential $\alpha\phi_1 + \beta\phi_2$ with $(\alpha, \beta = constants)$ corresponds to the total source. In brief, the additive nature of potentials, as the solution of Poisson's equation, is the best justification for a theory to be linear. In Einstein's general relativity theory on the other hand the Einstein's equations are non-linear partial differential equations, so that addition of two potentials does not happen to be a potential solution. In simple language consider the classical Laplace equation $\nabla^2\phi = 0$ where ∇^2 is the Laplacian operator while ϕ is a potential. If we have two separate solutions ϕ_1 and ϕ_2 natrually we have $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$ and we obtain $\nabla^2(\phi_1 + \phi_2) = 0$ as a result provided the Laplacian operator ∇^2 does not depend on ϕ_1 and ϕ_2. Precisely this is what happens in a non-linear theory: the Laplacian ∇^2 itself is dependent on ϕs so that the magic linear solution $\phi_1 + \phi_2$ as a solution does not work.

Similar is the Maxwell's electromagnetism in both flat and curved spacetimes. Suppose that the Maxwell's equation $\nabla_\mu F^{\mu\nu} = 0$ has two distinct solutions $F_1^{\mu\nu}$ and $F_2^{\mu\nu}$. Then automatically the superposed solution $\alpha F_1^{\mu\nu} + \beta F_2^{\mu\nu}$, with $\alpha, \beta = constants$, is also a solution because the covariant derivative does not involve any trace of EM field. The non-linear electromagnetics, however, has the form $\nabla_\mu(K(F)F^{\mu\nu}) = 0$ where a weight function $K(F)$ which depends on the EM field tensor enters in the equation and spoils the linearity. Now the addition $\alpha F_1^{\mu\nu} + \beta F_2^{\mu\nu}$ is no more a solution in such a theory. We say that $F^{\mu\nu}$ self-interacts with itself, scatters itself to the extent that it focuses itself to the focal points. Let us add that the quantum theory of linear Maxwell electrodynamics (i.e. Quantum Electrodynamics=QED) is also a non-linear theory. It has experimentally been tested that a photon scatters itself in QED. This takes place in the most abundant H-atom in nature and this scattering modifies the spectra of H-atom, known as the Lamb shift.

The idea of non-linear electrodynamics (NED) is about a century old but it was made popular in 1930s by Born and Infeld[7] which came to be known as the Born-Infeld (BI) theory. The main aim in this formalism was to eliminate the divergences in the physical amplitudes that endangered electromagnetism. In practice, no singularity was observed but the theory gave physical divergences such as $\sim \frac{1}{r}$ as $r \to 0$. This was totally unacceptable. To remedy this problem the non-linear BI theory modified the Coulomb potential by $\frac{1}{r} \to \int \frac{dr}{\sqrt{r^4+1}}$ which was not an easy task at all. But it worked, and the singularity was removed. This turned out to create a new trend in electromagnetism which was to establish a non-linear version of the linear theory and get rid of all singularities. Similar trend was extended to Einstein-Maxwell theory and the non-linear EM amplitudes were exploited to eliminate the diverging gravitational amplitudes as well. This state of art has been partly successful because there are still a number of problems to be overcome. Which NED?, for example. After all, the non-linear extensions of linear Maxwell theory was not unique, there are many ways, even some of them lack a linear Maxwell limit. If we expect that in some limit the NED will converge in the linear Maxwell theory this puts constraints on the adopted NED theory.

An interesting case, unprecedented in a linear electromagnetic theory, for example, is that an NED may admit "run-away solutions" in which the system self-propels itself. This is exactly what we experience in cosmology: the universe self-repulses itself and undergoes accelerated expansion. Although this has been attributed to dark-matter and dark-energy these sources are yet to be seen. An alternative point of view may be that the internal dynamics, by a non-linear mechanism encountered in non-linear theories, does produce the outward repulsion for the universe at large. These all remain to be seen, of course, but the study of non-linear theories has always been much attractive albeit difficult in physics.

In this thesis some features of the Born-Infeld electrodynamics were studied. In Chapter 2 we focus our attention on their paper published in 1934[M. Born and L. Infeld, Proc. R. Soc. London A 144, 425 (1934)] and try to derive the relations that they obtained through another method. In particular, the electromagnetic field equations are obtained through applying the variational principle considering the variation of the vector potential. The connection between the macroscopic and microscopic fields are also achieved in vector form by means of differential forms. To show the elimination of singularity in this theory, the electrostatic field of a point charge is also obtained. In chapter 3 a new Lagrangian will be introduced and coupled with general relativity. This results in a new metric function in a 4-D, (2+1)-D, and higher dimensional spacetimes. Also a theorem regarding the existence of non-singular metrics with Lagrangians having Maxwell limit will be argued.

Chapter 2

BORN-INFELD ELECTRODYNAMICS

2.1 Modification of Lagrangian and Analogy with Relativistic Mechanics

In 1934 M. Born and L. Infeld[7] introduced a new field theory by replacing the Lagrangian underlying Maxwell's field theory by a modified Lagrangian. Maxwell's field equations can be derived by applying the well known Lagrangian

$$L = \frac{1}{2}(B^2 - E^2)\,, \tag{2.1}$$

or in its covariant form[1]

$$L = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{2.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Minkowski metric tensor is applied to raise and lower indices. The modified Lagrangian was put forward as

$$L = b^2(\sqrt{1 + \frac{1}{b^2}(B^2 - E^2)} - 1). \tag{2.3}$$

where b has the dimension of a field strength and is called *absolute field*. Its value will be discussed in section 2.7. We can show that this Lagrangian has the Maxwell limit if

[1] A different convention for the sign of the Lagrangian is adopted and widely used in contemporary literature: $L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

b approaches zero. We will show that it has the limit as long as we consider the weak field condition later in this chapter. Denoting $F = B^2 - E^2$ we have

$$\lim_{b\to 0} L = \lim_{b\to 0} b^2(\sqrt{1 + \frac{1}{b^2}F} - 1)$$
$$= \lim_{b\to 0} b^2(1 + \frac{1}{2b^2}F - ... - 1)$$
$$= \frac{1}{2}F = \frac{1}{2}(B^2 - E^2). \tag{2.4}$$

One can think of the *principle of finiteness* as the physical idea which lies beneath this modification[7]:

> "...a satisfactory theory should avoid letting physical quantities become infinite."

Applying this requirement to velocity results in an upper limit for velocity, c, as well as alteration of the Newton's action function of a free particle $\frac{1}{2}mv^2$ to its relativistic counterpart

$$L = mc^2(1 - \sqrt{1 - \frac{v^2}{c^2}}) = b^2(1 - \sqrt{1 - \frac{1}{b^2}mv^2}); \qquad b^2 = mc^2.$$

Similarly considering this principle for a field strenght leads to an upper limit for its magnitude which turns out to be of very great order.

2.2 Principle of Invariant Action and Determination of Modified Lagrangian

According to the variational principle of least action

$$\delta \int \mathscr{L} d\tau = 0 \tag{2.5}$$

where $\mathscr{L}$ is the Lagrangian density and $d\tau = dx^0 dx^1 dx^2 dx^3$ is the volume element in four-dimensional spacetime.

$\mathscr{L}$ should be determined in such a way that it satisfies this variational principle. As it was suggested by Born and Infeld, an appropriate expression is $\mathscr{L} = \sqrt{|a_{\mu\nu}|}$ where $a_{\mu\nu}$ is a covariant tensor field which can be separated into a symmetric tensor, $g_{\mu\nu}$, and an antisymmetric tensor, $F_{\mu\nu}$: $a_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu}$, where $F_{\mu\nu}$ is the electromagnetic field tensor and $g_{\mu\nu}$ is the metric tensor. When it comes to the specific case of Minkowski metric tensor the convention $(+,-,-,-)$ should be applied throughout our calculations. Therefore $\mathscr{L}$ can be expressed as

$$\mathscr{L} = \sqrt{-\left|g_{\mu\nu} + F_{\mu\nu}\right|} + A\sqrt{-\left|g_{\mu\nu}\right|} + B\sqrt{\left|F_{\mu\nu}\right|} \tag{2.6}$$

The minus sign is added for $\left|g_{\mu\nu}\right| < 0$.

The next step is to determine the unknown coefficients of the Lagrangian density in general coordinates. Since $F_{\mu\nu}$ is the rotation of a vector potential (according to its definition), the integral of the last term can be converted into a surface integral and hence has no contribution to the integral of $\mathscr{L}$ over spacetime and therefore $B = 0$. To determine A, one can apply the following restrictions:

- considering the calculation of $\mathscr{L}$ in Cartesian coordinates and
- field strength of small values

Applying these conditions will lead us to the case of linear expression 2.2. Knowing the field tensor, $F_{\mu\nu}$, and the metric tensor, $g_{\mu\nu}$, in matrix form, one can easily calculate the value of the determinants in 2.6.

$$-\left|g_{\mu\nu}+F_{\mu\nu}\right| = 1+(F_{23})^2+(F_{13})^2+(F_{12})^2-(F_{14})^2-(F_{24})^2-(F_{34})^2-\left|F_{\mu\nu}\right| \quad (2.7)$$

The last determinant is negligible due to the weak field condition. On the other hand we can have

$$F = (F_{23})^2+(F_{13})^2+(F_{12})^2-(F_{14})^2-(F_{24})^2-(F_{34})^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \quad (2.8)$$

Therefore in Cartesian coordinates

$$\mathscr{L} = \sqrt{1+F}+A \quad (2.9)$$

Expanding the first term of 2.9 in series and neglecting terms of $\mathscr{O}(F^2)$ and smaller results in

$$\mathscr{L} = 1+\frac{1}{2}F+A \quad (2.10)$$

If $A=-1$ the Lagrangian becomes

$$\mathscr{L} = \frac{1}{2}F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = L$$

which is clearly the linear Lagrangian of the Maxwell's field theory. Considering these calculations, $\mathscr{L}$ in Cartesian coordinates becomes

$$L=\sqrt{1+F-G^2}-1 \tag{2.11}$$

where $F=\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $G=F_{23}F_{14}+F_{31}F_{24}+F_{12}F_{34}$. In general coordinates

$$\mathscr{L}=\sqrt{-\left|g_{\mu\nu}+F_{\mu\nu}\right|}-\sqrt{-\left|g_{\mu\nu}\right|}. \tag{2.12}$$

$\left|g_{\mu\nu}+F_{\mu\nu}\right|$ can also be written in terms of F and G. To obtain such an expression for $\mathscr{L}$, G also needs to be expressed in a more compact tensor form. Denoting $\left|g_{\mu\nu}\right|=g$, we will have

$$\left|g_{\mu\nu}+F_{\mu\nu}\right|=g+\phi(g_{\mu\nu},F_{\mu\nu})+\left|F_{\mu\nu}\right|=g(1+\frac{\phi}{g}-\frac{\left|F_{\mu\nu}\right|}{-g}) \tag{2.13}$$

Calculating this determinant and obtaining the right-hand side of 2.13 is straightforward and the following expressions will be found

$$\frac{\phi}{g}=F=\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$$

and

$$G^2=\frac{\left|F_{\mu\nu}\right|}{-g}=\frac{(F_{23}F_{14}+F_{31}F_{24}+F_{12}F_{34})^2}{-g}.$$

We write $G = \frac{1}{4} F_{\mu\nu} {}^{*}F^{\mu\nu}$ where ${}^{*}F^{\mu\nu}$ is the dual of the field tensor $F_{\mu\nu}$ and is defined[2]

$$ {}^{*}F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} \tag{2.14} $$

where $\varepsilon^{\mu\nu\kappa\lambda}$ is the Levi-Civita symbol. Ultimately

$$ \left| g_{\mu\nu} + F_{\mu\nu} \right| = g(1 + F - G^2) \tag{2.15} $$

and the Lagrangian density in general coordinates is

$$ \mathscr{L} = \sqrt{-g}(\sqrt{1 + F - G^2} - 1). \tag{2.16} $$

2.3 Field Equations

In this section we find the field equations in tensor form and then express them by 2-form field equations. To find the homogeneous set of equations we start from the identity

$$ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{2.17} $$

$\sqrt{-g}$ needs to be included in case we consider the field equations in a general coordinate system. Using 2.17 and 2.14 we have

$$ \partial_\nu \sqrt{-g} {}^{*}F^{\mu\nu} = 0 \tag{2.18} $$

[2] In their paper (1934), Born and Infeld defined[7] the dual tensor as ${}^{*}F^{\mu\nu} = j^{\mu\nu\kappa\lambda} F_{\kappa\lambda}$ where $j^{\mu\nu\kappa\lambda}$ has the value $\pm \frac{1}{2\sqrt{-g}}$ or zero. The sign rule is similar to the Levi-Civita symbol.

Equation 2.18 gives the homogeneous set of Maxwell's equations.

To obtain the inhomogeneous set of *sourceless* field equations we can apply the variational principle [3] . Introducing the BI Lagrangian as $\mathscr{L} = \sqrt{-g}L(F,G)$ and considering the variation of the vector potential, **A**, we will have

$$\delta\mathscr{L} = \sqrt{-g}\delta L = \sqrt{-g}(\frac{\partial L}{\partial F}\delta F + \frac{\partial L}{\partial G}\delta G). \tag{2.19}$$

Then we should find the variations of F and G. Writing F and G in the forms below

$$F = \frac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2.20}$$

and

$$G = \frac{1}{4}{}^*F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2.21}$$

we have then

$$\delta F = F^{\mu\nu}\delta(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2.22}$$

and

$$\delta G = \frac{1}{2}{}^*F^{\mu\nu}\delta(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2.23}$$

[3] Originally, In [7] these equations are stated to be obtained by defining the antisymmetric tensor $p^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial F_{\mu\nu}}$.

and subsequently

$$\delta I = \int \delta \mathscr{L} d^4x = 2\int \sqrt{-g}(\frac{\partial L}{\partial F}F^{\mu\nu} + \frac{1}{2}\frac{\partial L}{\partial G}{}^*F^{\mu\nu})\partial_\mu(\delta A_\nu)d^4x = 0 \tag{2.24}$$

we define an antisymmetric tensor $p^{\mu\nu}$

$$p^{\mu\nu} = 2\frac{\partial L}{\partial F}F^{\mu\nu} + \frac{\partial L}{\partial G}{}^*F^{\mu\nu} \tag{2.25}$$

and through integrating by parts we obtain

$$\delta I = -\int \partial_\mu(\sqrt{-g}p^{\mu\nu})\delta A_\nu d^4x = 0 \tag{2.26}$$

Since equation 2.26 is valid for any arbitrary δA_ν, so

$$\partial_\mu(\sqrt{-g}p^{\mu\nu}) = 0 \tag{2.27}$$

represents the inhomogeneous set of field equations. In fact these equations give the fields in media with electric permittivity and magnetic permeability.

These equations can also be expressed by means of differential forms. The homogeneous set of equations will be shown as

$$dF = 0 \tag{2.28}$$

where $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$ is the 2-form electromagnetic field. And the inhomogeneous set is

$$d(2^*FL_F + FL_G) = 0 \tag{2.29}$$

where $L_F = \frac{\partial L}{\partial F}$ and $L_G = \frac{\partial L}{\partial G}$.

2.4 Energy-Momentum Tensor and Conservation Law

In this section we achieve canonical and symmetric energy-momentum tensor[4] . The canonical energy-momentum tensor[19] for the free non-linear electromagnetic Lagrangian is

$$T_{\alpha\beta} = \frac{\partial L}{\partial(\partial^{\alpha} A_{\lambda})}\partial_{\beta} A_{\lambda} - g_{\alpha\beta} L \tag{2.30}$$

while

$$\frac{\partial L}{\partial(\partial^{\alpha} A_{\lambda})} = \frac{\partial L}{\partial F}\frac{\partial F}{\partial(\partial^{\alpha} A_{\lambda})} + \frac{\partial L}{\partial G}\frac{\partial G}{\partial(\partial^{\alpha} A_{\lambda})} \tag{2.31}$$

We have

$$\frac{\partial F}{\partial(\partial^{\alpha} A^{\lambda})} = \frac{1}{2} g_{\mu\rho} g_{\nu\sigma}[2\delta^{\rho}_{\alpha}\delta^{\sigma}_{\lambda} F^{\mu\nu} + 2\delta^{\mu}_{\alpha}\delta^{\nu}_{\lambda} F^{\rho\sigma}] = 2F_{\alpha\lambda} \tag{2.32}$$

therefore

$$\frac{\partial F}{\partial(\partial^{\alpha} A_{\lambda})} = 2g^{\lambda\mu} F_{\alpha\mu}. \tag{2.33}$$

[4]In [7] the symmetrized energy-momentum tensor was obtained through $-2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} = \sqrt{-g}T_{\mu\nu}$.

To obtain the second term, we calculate

$$\frac{\partial G}{\partial(\partial^{\alpha}A^{\lambda})}=\frac{1}{8}\varepsilon^{\mu\nu\kappa\eta}g_{\mu\rho}g_{\nu\sigma}g_{\kappa\tau}g_{\eta\gamma}[2\delta^{\rho}_{\alpha}\delta^{\sigma}_{\lambda}F^{\tau\gamma}+2\delta^{\tau}_{\alpha}\delta^{\gamma}_{\lambda}F^{\rho\sigma}]=^{*}F_{\alpha\lambda} \tag{2.34}$$

and

$$\frac{\partial G}{\partial(\partial^{\alpha}A_{\lambda})}=g^{\lambda\mu*}F_{\alpha\mu} \tag{2.35}$$

Subsequently

$$T_{\alpha\beta}=g^{\lambda\mu}[2L_F F_{\alpha\mu}+L_G{}^{*}F_{\alpha\mu}]\partial_{\beta}A_{\lambda}-g_{\alpha\beta}L \tag{2.36}$$

and using 2.25

$$T_{\alpha\beta}=g^{\lambda\mu}p_{\alpha\mu}\partial_{\beta}A_{\lambda}-g_{\alpha\beta}L \tag{2.37}$$

The above expression for the (canonical) energy-momentum tensor needs to be symmetrized. We replace $\partial_{\beta}A_{\lambda}$ by $-F_{\lambda\beta}+\partial_{\lambda}A_{\beta}$ and then

$$T_{\alpha\beta}=g^{\lambda\mu}p_{\alpha\mu}F_{\beta\lambda}-g_{\alpha\beta}L+g^{\lambda\mu}p_{\alpha\mu}\partial_{\lambda}A_{\beta} \tag{2.38}$$

Denoting the last term by $T'_{\alpha\beta}$ the symmetrized energy-momentum tensor will be

$$\Theta_{\alpha\beta}=T_{\alpha\beta}-T'_{\alpha\beta}=g^{\lambda\mu}p_{\alpha\mu}F_{\beta\lambda}-g_{\alpha\beta}L \tag{2.39}$$

and the mixed form will be

$$\Theta^{\alpha}_{\beta} = p^{\alpha\lambda}F_{\beta\lambda} - \delta^{\alpha}_{\beta}L \tag{2.40}$$

To finalize this section we will achieve the conservation law through multiplying 2.17 by $p^{\alpha\lambda}$. In Cartesian coordinates

$$p^{\alpha\lambda}(\partial_{\beta}F_{\alpha\lambda} + \partial_{\alpha}F_{\lambda\beta} + \partial_{\lambda}F_{\beta\alpha}) = 0 \tag{2.41}$$

and due to 2.27 in the last two terms $p^{\alpha\lambda}$ can be taken into differentiation and using the fact $p^{\alpha\lambda} = \frac{\partial L}{\partial F_{\alpha\lambda}}$ in the first term, we get

$$-2\partial_{\alpha}(p^{\alpha\lambda}F_{\beta\lambda}) + 2\frac{\partial L}{\partial x^{\beta}} = 0 \tag{2.42}$$

and by means of 2.40

$$\partial_{\alpha}\Theta^{\alpha}_{\beta} = 0 \tag{2.43}$$

In general coordinates

$$\partial_{\alpha}(\sqrt{-g}\Theta^{\alpha}_{\beta}) - \frac{1}{2}\sqrt{-g}\Theta^{\mu\nu}\partial_{\beta}g_{\mu\nu} = 0 \tag{2.44}$$

Equation below is used to obtain the conservation law in general coordinates

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}. \tag{2.45}$$

2.5 Field Equations in Vector Space

Field equations 2.28 and 2.29 can be expressed in vector form. In achieving these equations the relations between **D** and **H** with **E** and **B** will be revealed. The 2-form field and its dual are

$$F = -E_x dt\wedge dx - E_y dt\wedge dy - E_z dt\wedge dz + B_z dx\wedge dy + B_y dz\wedge dx + B_x dy\wedge dz \quad (2.46)$$

and

$$^{*}F = B_x dt\wedge dx + B_y dt\wedge dy + B_z dt\wedge dz + E_z dx\wedge dy + E_y dz\wedge dx + E_x dy\wedge dz \quad (2.47)$$

Starting with 2.28 and replacing F by 2.46 we can have

$$(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t})dt\wedge dx\wedge dy + (\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t})dt\wedge dz\wedge dx$$

$$+(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t})dt\wedge dy\wedge dz + (\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z})dx\wedge dy\wedge dz = 0 \quad (2.48)$$

A couple of Maxwell's equations emerge from equation 2.48

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \qquad ; \qquad \nabla.\mathbf{B} = 0 \quad (2.49)$$

The other two equations will be found through 2.29. Prior to the use of that equation we need to calculate L_F and L_G.

$$L_F = \frac{\partial L}{\partial F} = \frac{1}{2\sqrt{1+F-G^2}} \qquad ; \qquad L_G = \frac{\partial L}{\partial G} = \frac{-G}{\sqrt{1+F-G^2}} \quad (2.50)$$

Therefore

$$[\frac{\partial}{\partial x}(\frac{E_x+GB_x}{\sqrt{1+F-G^2}})+\frac{\partial}{\partial y}(\frac{E_y+GB_y}{\sqrt{1+F-G^2}})+\frac{\partial}{\partial z}(\frac{E_z+GB_z}{\sqrt{1+F-G^2}})]dx\wedge dy\wedge dz$$

$$+[\frac{\partial}{\partial y}(\frac{B_z-GE_z}{\sqrt{1+F-G^2}})-\frac{\partial}{\partial z}(\frac{B_y-GE_y}{\sqrt{1+F-G^2}})-\frac{\partial}{\partial t}(\frac{E_x+GB_x}{\sqrt{1+F-G^2}})]dy\wedge dt\wedge dz$$

$$+[\frac{\partial}{\partial z}(\frac{B_x-GE_x}{\sqrt{1+F-G^2}})-\frac{\partial}{\partial x}(\frac{B_z-GE_z}{\sqrt{1+F-G^2}})-\frac{\partial}{\partial t}(\frac{E_y+GB_y}{\sqrt{1+F-G^2}})]dz\wedge dt\wedge dx$$

$$+[\frac{\partial}{\partial x}(\frac{B_y-GE_y}{\sqrt{1+F-G^2}})-\frac{\partial}{\partial y}(\frac{B_x-GE_x}{\sqrt{1+F-G^2}})-\frac{\partial}{\partial t}(\frac{E_z+GB_z}{\sqrt{1+F-G^2}})]dx\wedge dt\wedge dy=0 \tag{2.51}$$

The first bracket is known to be

$$\nabla.\mathbf{D}=0 \tag{2.52}$$

and the last three terms represent

$$\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=0 \tag{2.53}$$

where **D** and **H** have relations with **E** and **B** in the forms below

$$\mathbf{D}=\frac{\mathbf{E}+G\mathbf{B}}{\sqrt{1+F-G^2}} \tag{2.54}$$

and

$$\mathbf{H} = \frac{\mathbf{B} - G\mathbf{E}}{\sqrt{1 + F - G^2}} \tag{2.55}$$

On this basis one can understand the meaning of non-linearity. In addition, these expressions for **D** and **H** can be obtained by means of the Lagrangian derivatives with respect to **E** and **B** respectively.

$$L = \sqrt{1 + F - G^2} - 1 \qquad ; \qquad F = \frac{1}{b^2}(B^2 - E^2) \qquad ; \qquad G = \frac{1}{b^2}(\mathbf{B}.\mathbf{E})$$

$$\mathbf{H} = b^2 \frac{\partial L}{\partial \mathbf{B}} \qquad ; \qquad \mathbf{D} = -b^2 \frac{\partial L}{\partial \mathbf{E}}. \tag{2.56}$$

2.6 Static Field of a Point Charge

In this section we consider the electrostatic field of a point charge for which $\mathbf{B} = \mathbf{H} = 0$ and **E** and **D** are time independent. From 2.49 we have

$$\nabla \times \mathbf{E} = 0 \tag{2.57}$$

and therefore

$$\mathbf{E} = -\nabla \Phi. \tag{2.58}$$

For the case of spherical symmetry 2.52 becomes

$$\frac{d}{dr}(r^2 D_r) = 0 \tag{2.59}$$

which has the solution

$$D_r = \frac{e}{r^2} \tag{2.60}$$

and through 2.54 we have

$$D_r = \frac{E_r}{\sqrt{1 - \frac{1}{b^2}E_r^2}} \tag{2.61}$$

Therefore

$$E_r = \frac{e}{r_0^2\sqrt{1+(\frac{r}{r_0})^4}} \qquad ; \qquad r_0 = \sqrt{\frac{e}{b}} \tag{2.62}$$

One can clearly see that D_r is singular at $r = 0$ whereas E_r is finite everywhere. We can also find the potential of a point charge. Replacing E_r by $-\Phi'(r)$ in 2.61 we have

$$\frac{e}{r^2} = \frac{-\Phi'(r)}{\sqrt{1 - \frac{1}{b^2}\Phi'^2(r)}} \tag{2.63}$$

This leads us to

$$\Phi(r) = \frac{e}{r_0^2} f(\frac{r}{r_0}) \tag{2.64}$$

$$f(x) = \int_x^\infty \frac{dy}{\sqrt{1+y^4}} \tag{2.65}$$

This is the potential of a point charge e. Substituting $x = \tan\frac{\beta}{2}$ integral 2.65 becomes

$$f(x) = \frac{1}{2}\int_\beta^\pi \frac{d\beta}{\sqrt{1-\frac{1}{2}sin^2(\beta)}} = f(0) - \frac{1}{2}F(\beta, \frac{1}{\sqrt{2}}) \tag{2.66}$$

where $\beta(x) = 2\arctan x$. $F(\beta, \frac{1}{\sqrt{2}})$ is the elliptic function of the first kind[1] and

$$f(0) = F(\frac{\pi}{2}, \frac{1}{\sqrt{2}}) = 1.8541 \tag{2.67}$$

At $r = 0$

$$\Phi(0) = \frac{e}{r_0^2}(1.8541). \tag{2.68}$$

2.7 On the Absolute Field Constant

To conclude this chapter we briefly discuss the value of the absolute field constant b. In order to obtain its value we consider the electrostatic case of an electron. We can find the energy density of its field provided we either have the Hamiltonian or the time component of the energy-momentum tensor (We already found this tensor). One can show that

$$T^{00} = 4\pi U = \mathbf{D}.\mathbf{E} + b^2 L = b^2 H \tag{2.69}$$

Where U is the energy density. The total energy is the volume integral of U and its value is

$$E = \int U dv = 1.2361 \frac{e^2}{r_0} \tag{2.70}$$

On the other hand $E = m_0 c^2$, hence

$$r_0 = 2.28 \times 10^{-13} cm$$

r_0 can be considered as the radius of electron and

$$b = \frac{e}{r_0^2} = 9.18 \times 10^{15} e.s.u.$$

Other attempts to obtain a value for the absolute field constant are worth mentioning here. H. Carley et. al.[10] found the value of the absolute field constant (denoted by β in their paper) through studying the effect of the Born-Infeld-based potential on the spectrum of the hydrogen atom. In their work they expressed β in terms of α, the fine structure constant ($\approx 1/137.036$). In a similar attempt S. H. Mazharimousavi and M. Halilsoy[23] found the BI parameter (formerly called absolute field constant) by inserting a Morse-type potential in the Schrödinger's equation. Table 2.1 shows the obtained values for the BI parameter.

Table 2.1:Born-Infeld Parameter

	$\alpha\beta$
Ref.[24]	1.65820
Ref.[10]	1.83297
Born's proposal	1.2361

Chapter 3

A NEW MODEL OF NON-LINEAR ELECTRODYNAMICS

3.1 Introduction

In this chapter a non-linear field theory will be developed based on a Lagrangian proposed by S. H. Mazharimousavi in the form

$$L = -\frac{2}{\alpha^4}\ln\left(1-\alpha^2\sqrt{|F|+\alpha^2 G^2}\right) - \frac{2\sqrt{|F|+\alpha^2 G^2}}{\alpha^2} \tag{3.1}$$

where $F = F_{\mu\nu}F^{\mu\nu}$ and $G = F_{\mu\nu}{}^{*}F^{\mu\nu}$ and α plays a similar role to that of the Born's parameter added to imply the Maxwell limit as it approaches zero. This fact will be elaborated later in this section. The main objective of the chapter is to consider general relativity minimally coupled with non-linear electrodynamics with the above Lagrangian. In particular, we scrutinize the case $G = 0$ when we have the electric charge in a static spherically symmetric spacetime. A singularity-free field will be the outcome of applying the field equation 2.29 to this case.

In the first instance, we consider the problem of coupling in 4-dimensional spacetime. Our study will be followed in $2+1$ dimensions as well as higher dimensions. The following examines the above-mentioned Lagrangian having Maxwell limit while α approaches zero. Recalling the Lagrangian underlying the Maxwell's equations, G

must be zero in order to show that limit. Thus the Lagrangian becomes

$$L = -\frac{2}{\alpha^4}\ln(1-\alpha^2\sqrt{|F|}) - \frac{2\sqrt{|F|}}{\alpha^2} \tag{3.2}$$

We can expand the first term like the Taylor expansion of $\ln(1-x)$. Thereby

$$\lim_{\alpha\to 0} L = \lim_{\alpha\to 0}[-\frac{2}{\alpha^4}(-\alpha^2\sqrt{|F|} - \frac{1}{2}\alpha^4|F| - ...) - \frac{2\sqrt{|F|}}{\alpha^2}] = |F| = -F. \tag{3.3}$$

3.2 Electrostatic Spherically Symmetric Field of a Point Charge

The electric field of a point charge at rest is given by the 2-from field

$$F = -E(r)dt\wedge dr \tag{3.4}$$

and its dual is

$$^*F = E(r)r^2\sin\theta d\theta\wedge d\phi \tag{3.5}$$

These together show the field in 4-dimensional spherically symmetric spacetime. On the other hand, derivative of the Lagrangian with respect to F, the electromagnetic invariant, is

$$L_F = \frac{F}{|F|(1-\alpha^2\sqrt{|F|})} \tag{3.6}$$

where $F = -2E^2(r)$. Now we consider equation 2.29.

$$d(\frac{-E(r)\ r^2\ \sin\theta}{1-\alpha^2\sqrt{2E^2(r)}}\ d\theta\wedge d\phi) = 0 \tag{3.7}$$

Thus

$$\frac{-E(r)\, r^2}{1-\sqrt{2}\alpha^2|E(r)|} = Q \qquad ; \qquad Q = constant. \tag{3.8}$$

And having $|E(r)| = -E(r)$ we get

$$|E(r)| = \frac{Q}{r^2+\sqrt{2}\alpha^2 Q}. \tag{3.9}$$

Since $|E| > 0$, therefore $Q > 0$. Thus we can write $E(r)$ in the form below

$$E(r) = \pm\frac{Q}{r^2+\sqrt{2}\alpha^2 Q} \qquad ; \qquad Q > 0 \tag{3.10}$$

or considering $a^2 = \sqrt{2}\alpha^2 Q$, we have

$$|E(r)| = \frac{Q}{r^2+a^2}. \tag{3.11}$$

It is clear that $E(r)$ is finite everywhere. The electric potential is obtained

$$V - V_{ref} = -\int E(r)dr = -\int \frac{Q}{r^2+a^2}dr = \frac{Q\pi}{2a} - \frac{Q}{a}\arctan(\frac{r}{a})$$

3.2.1 The Energy-Momentum Tensor

Enroute to our goal of coupling GR with NED(Non-linear Electrodynamics) we need to obtain the energy-momentum tensor of the source in question. The energy-momentum tensor is given by

$$T_\mu^\nu = \frac{1}{2}(L\delta_\mu^\nu - 4F_{\mu\lambda}F^{\nu\lambda}L_F) \tag{3.12}$$

Substituting L and L_F from 3.2 and 3.6, respectively, we get

$$T_\mu^\nu = \left(-\frac{1}{\alpha^4}\ln(1-\alpha^2\sqrt{2E^2}) - \frac{\sqrt{2E^2}}{\alpha^2}\right)\delta_\mu^\nu + \frac{2F_{\mu\lambda}F^{\nu\lambda}}{1-\alpha^2\sqrt{2E^2}} \tag{3.13}$$

Knowing $F_{rt} = -E(r)$ it becomes

$$T_t^t = T_r^r = -\frac{1}{\alpha^4}\ln(1-\alpha^2\sqrt{2E^2}) - \frac{\sqrt{2E^2}}{\alpha^2(1-\alpha^2\sqrt{2E^2})} \tag{3.14}$$

and

$$T_\theta^\theta = T_\phi^\phi = -\frac{1}{\alpha^4}\ln(1-\alpha^2\sqrt{2E^2}) - \frac{\sqrt{2E^2}}{\alpha^2} \tag{3.15}$$

Replacing $|E(r)|$ by 3.11 one can get

$$T_t^t = T_r^r = -\frac{1}{\alpha^4}[\ln(\frac{r^2}{r^2+a^2}) + \frac{a^2}{r^2}] \tag{3.16}$$

and

$$T_\theta^\theta = T_\phi^\phi = -\frac{1}{\alpha^4}[\ln(\frac{r^2}{r^2+a^2}) + \frac{a^2}{r^2+a^2}] \tag{3.17}$$

3.2.2 Metric Function of a Static Spherically Symmetric 4-D Spacetime

We now consider the gravitational field equation with the obtained energy-momentum tensor (3.16, 3.17) of an electric point charge.

$$G_\mu^\nu + \frac{1}{3}\Lambda\delta_\mu^\nu = T_\mu^\nu \tag{3.18}$$

where $G_\mu^\nu = R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu$ is the Einstein tensor and Λ is the cosmological constant. The following are the components of the Einstein tensor.

$$G_t^t = G_r^r = \frac{(\frac{df}{dr})r - 1 + f(r)}{r^2} \qquad and \qquad G_\theta^\theta = G_\phi^\phi = \frac{1}{2}\ \frac{2(\frac{df}{dr}) + r(\frac{d^2f}{dr^2})}{r} \tag{3.19}$$

$f(r)$ is the required metric function in the line element

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{3.20}$$

Substituting 3.16, 3.17, and 3.19 in 3.18, we get

$$equation - I: \quad \frac{1}{r}\frac{df}{dr} + \frac{1}{r^2}f(r) + \frac{1}{\alpha^4}\ln(\frac{r^2}{r^2+a^2}) + (\frac{a^2}{\alpha^4} - 1)\frac{1}{r^2} + \Lambda = 0 \tag{3.21}$$

$$equation - II: \quad \frac{1}{2}\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \frac{1}{\alpha^4}\ln(\frac{r^2}{r^2+a^2}) + \frac{1}{\alpha^4}(\frac{a^2}{r^2+a^2}) + \Lambda = 0 \tag{3.22}$$

One can multiply equation-I by r^2 and consider the first two terms as $\frac{d}{dr}(rf)$ and find the metric function.

$$f(r) = 1 - \frac{2Q^2}{3a^2} - \frac{2M - \frac{2Q^2\pi}{3a}}{r} - \frac{1}{3}\Lambda r^2 + \frac{2Q^2r^2}{3a^4}\ln(1+\frac{a^2}{r^2}) - \frac{4Q^2}{3ar}\arctan(\frac{r}{a}) \tag{3.23}$$

limit of the metric as a approaches zero is[5]

$$\lim_{a\to 0} f(r) = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2 + \frac{Q^2}{r^2} \tag{3.24}$$

[5]Reissner-Nordström metric: $ds^2 = B(r)dt^2 - B^{-1}(r)dr^2 - r^2 d\Omega^2$ where $B(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}$

and when a approaches infinity we expect the metric function approaches the Schwarzschild limit.

$$\lim_{a\to\infty} f(r) = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2$$

3.3 Point Charge in (2+1)-dimensional Static, Spherically Symmetric Spacetime

Now we consider the electric field of a point charge in a (2+1)-dimensional spacetime. As usual we aim to find the metric function while we expect to obtain a singularity-free electric field.

3.3.1 Electric Field and the Exact Electric Potential

The 2-form field and its dual in 3 dimensions are

$$F = E(r)dt \wedge dr \qquad ; \qquad {}^*F = E(r)rd\theta \tag{3.25}$$

and $F = F_{\mu\nu}F^{\mu\nu} = -2E^2(r)$. Using 2.29 and 3.6 one can get

$$d(-\frac{rE(r)}{1-\alpha^2\sqrt{2}|E|}d\theta) = 0 \tag{3.26}$$

which results in

$$\frac{r|E|}{1-\alpha^2\sqrt{2}|E|} = Q \tag{3.27}$$

Therefore

$$|E| = \frac{Q}{r + \alpha^2\sqrt{2}Q} = \frac{Q}{r+a} \qquad ; \qquad a = \sqrt{2}\alpha^2 Q \tag{3.28}$$

which is finite at $r = 0$. The electric potential can be found through E:

$$V - V_{ref} = -\int E dr = -\int \frac{Q}{r+a} dr = Q \ln\left(\frac{1}{r+a}\right) \tag{3.29}$$

3.3.2 Energy-Momentum Tensor and the Metric Function

The static, circularly symmetric line element is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\phi^2 \tag{3.30}$$

The energy-momentum tensor is found through 3.14 and 3.15

$$T_t^t = T_r^r = -\frac{1}{\alpha^4}\left[\ln\left(\frac{r}{r+a}\right) + \frac{a}{r}\right] \tag{3.31}$$

$$T_\phi^\phi = -\frac{1}{\alpha^4}\left[\ln\left(\frac{r}{r+a}\right) + \frac{a}{r+a}\right] \tag{3.32}$$

The non-zero components of G_μ^ν are given by[17]

$$G_\mu^\nu = diag\left[\frac{f'}{2r}, \frac{f'}{2r}, \frac{f''}{2}\right] \tag{3.33}$$

This renders the metric function via 3.18.

$$\frac{f'}{2r}+\frac{\Lambda}{3}=-\frac{1}{\alpha^4}[\ln(\frac{r}{r+a})+\frac{a}{r}]$$

thus

$$f(r)=-\frac{1}{\alpha^4}\{-a^2\ln(\frac{a}{r+a})+\ln(\frac{r}{r+a})r^2+ar-a^2\}-\frac{\Lambda}{3}r^2+C \tag{3.34}$$

Now we can examine the limits of $f(r)$ as α approaches zero and infinity.

$$\lim_{\alpha\to 0} f(r)=\lim_{\alpha\to 0}\{Q^2+2Q^2[\ln(-\frac{\sqrt{2}Q}{r})+2\ln\alpha]-\frac{\Lambda}{3}r^2+C+\mathscr{O}(\alpha)\} \tag{3.35}$$

Taking $C=-2Q^2[\ln(-\sqrt{2}Q)+2\ln(\alpha)]$ and $M=-Q^2$ it becomes

$$\lim_{\alpha\to 0} f(r)=-M-Q^2\ln(r^2)+\frac{r^2}{l^2} \tag{3.36}$$

which is called BTZ metric (It is a black hole metric for (2+1)-dimensional spacetime with a negative cosmological constant.) with $\frac{1}{l^2}=-\frac{\Lambda}{3}$. For α approaches infinity one can find

$$\lim_{\alpha\to\infty} f(r)=-M+\frac{r^2}{l^2} \tag{3.37}$$

which happens when we solve 3.18 for $T_\mu^\nu=0$ with the components of G_μ^ν given earlier.

3.4 Electrostatic Field of a Point Charge in Higher Dimensions

Following our study on coupling the electromagnetic field with gravitation we now consider the problem in d-dimensional spacetime. The same procedure as we did in previous sections will be applied except for the metric function that we will find the integral solution instead of an exact one.

3.4.1 Electric Field of a Point Charge

The 2-form electromagnetic field and its dual are given by

$$F = E(r)dt \wedge dr \tag{3.38}$$

and

$$^{*}F = E(r)r^{d-2}\phi(\theta_i)d\theta_1 \wedge d\theta_2 ... d\theta_{d-2} \tag{3.39}$$

Knowing the electromagnetic invariant $F = -2E^2(r)$, from 3.6 and 2.29 we have

$$d\left(\frac{-E(r)r^{d-2}}{1-\alpha^2\sqrt{2}|E(r)|}\phi(\theta_i)d\theta_1 \wedge d\theta_2 ... d\theta_{d-2}\right) = 0 \tag{3.40}$$

therefore

$$\frac{|E|r^{d-2}}{1-\alpha^2\sqrt{2}|E|} = C \qquad ; \qquad (C = constant) \tag{3.41}$$

and

$$|E| = \frac{C}{r^{d-2}+C\alpha^2\sqrt{2}} \tag{3.42}$$

The Maxwell limit implies $C = Q$ hence

$$|E| = \frac{Q}{r^{d-2} + a^{d-2}} \qquad ; \qquad a^{d-2} = Q\sqrt{2}\alpha^2 \tag{3.43}$$

3.4.2 The Energy-Momentum Tensor

Using 3.14 and 3.15 the energy-momentum tensor components become

$$T_t^t = T_r^r = -\frac{1}{\alpha^4}[\ln(\frac{r^{d-2}}{r^{d-2} + a^{d-2}}) + (\frac{a}{r})^{d-2}] \tag{3.44}$$

and

$$T_{\theta_i}^{\theta_i} = -\frac{1}{\alpha^4}[\ln(\frac{r^{d-2}}{r^{d-2} + a^{d-2}}) + \frac{a^{d-2}}{r^{d-2} + a^{d-2}}] \tag{3.45}$$

3.4.3 The Metric Function

The line element of a spherically symmetric d-dimensional spacetime is given by[22]

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega_{d-2}^2 \tag{3.46}$$

where

$$d\Omega_{d-2}^2 = d\theta_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} sin^2\theta_j d\theta_i^2$$

with

$$0 \leq \theta_{d-2} \leq 2\pi \quad , \quad 0 \leq \theta_i \leq \pi \quad , \quad 1 \leq i \leq d-3$$

The Ricci scalar in d dimensions is given by[22]

$$R = -A'' - 2(d-2)\frac{A'}{r} - (d-2)(d-3)\frac{(A-1)}{r^2} \tag{3.47}$$

and the Ricci tensor components are

$$R_t^t = R_r^r = -\frac{1}{2}A'' - \frac{(d-2)}{2}\frac{A'}{r} \tag{3.48}$$

and

$$R_{\theta_i}^{\theta_i} = -\frac{A'}{r} - (d-3)\frac{(A-1)}{r^2} \tag{3.49}$$

Therefore the components of Einstein tensor are

$$G_\mu^\nu = R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu$$

$$G_t^t = G_r^r = \frac{1}{2}(d-2)\frac{A'}{r} + \frac{1}{2}(d-2)(d-3)\frac{(A-1)}{r^2} \tag{3.50}$$

and

$$G_{\theta_i}^{\theta_i} = \frac{1}{2}A'' + (d-3)\frac{A'}{r} + \frac{1}{2}(d-3)(d-4)\frac{(A-1)}{r^2} \tag{3.51}$$

The set of gravitational field equations reads

$$\frac{1}{2}(d-2)\frac{A'}{r} + \frac{1}{2}(d-2)(d-3)\frac{(A-1)}{r^2} + \frac{1}{3}\Lambda$$

$$= -\frac{1}{\alpha^4}[\ln(\frac{r^{d-2}}{r^{d-2}+a^{d-2}}) + (\frac{a}{r})^{d-2}] \tag{3.52}$$

and

$$\frac{1}{2}A'' + (d-3)\frac{A'}{r} + \frac{1}{2}(d-3)(d-4)\frac{(A-1)}{r^2} + \frac{1}{3}\Lambda$$

$$= -\frac{1}{\alpha^4}[\ln(\frac{r^{d-2}}{r^{d-2}+a^{d-2}}) + \frac{a^{d-2}}{r^{d-2}+a^{d-2}}] \tag{3.53}$$

Solving the Einstein field equations with the aforementioned energy-momentum tensor and the Einstein tensor leads us to the following metric

$$A(r) = (d^2-5d+6)[\frac{1}{(d-2)r^{d-3}}\int r^{d-4}dr] - \frac{2}{3(d-2)}\frac{\Lambda}{r^{d-3}}\int r^{d-2}dr$$

$$-\frac{1}{\alpha^4}\frac{2a^{d-2}}{(d-2)}\frac{1}{r^{d-4}} + \frac{C}{r^{d-3}} - (\frac{1}{\alpha^4}\frac{2}{(d-2)}\frac{1}{r^{d-3}})\int r^{d-2}\ln(\frac{r^{d-2}}{r^{d-2}+a^{d-2}})dr \tag{3.54}$$

so

$$A(r) = 1 - (\frac{1}{\alpha^4}\frac{2}{(d-2)}\frac{1}{r^{d-3}})\int r^{d-2}\ln(\frac{r^{d-2}}{r^{d-2}+a^{d-2}})dr$$

$$-\frac{1}{\alpha^4}\frac{2a^{d-2}}{(d-2)}\frac{1}{r^{d-4}} - \frac{2\Lambda r^2}{3(d-1)(d-2)} + \frac{C}{r^{d-3}} \tag{3.55}$$

3.5 Existence of the Globally Regular Metrics-NED with Maxwell Limit

In this section we discuss a theorem on the relation between the existence of a metric with a regular center and a NED Lagrangian $L(F)$ having Maxwell asymptotic at weak field limit $(F \to 0)$. The problem arises in coupling general relativity to NED with the aforesaid limiting condition on the Lagrangian underlying it. Being valid, the theorem does not permit a nonsingular metric to come to light under the condition of this theorem with electric point charge. Here follows the statement of the theorem and its proof.[8, 9]

Theorem. *The Lagrangian $L(F), (F = F_{\mu\nu}F^{\mu\nu})$, with Maxwell asymptotic at small F, i.e. $L \sim F$ and $L_F = \frac{dL}{dF} \to constant$, coupled to R, the scalar curvature, does not lead us to a static, spherically symmetric metric with a regular pole and a nonzero electric charge.*

Proof. The Ricci tensor for such a metric is diagonal, hence the invariant $R_{\mu\nu}R^{\mu\nu} = R^{\nu}_{\mu}R^{\mu}_{\nu}$ is a sum of squares; therefore each R^{μ}_{μ} is finite at a regular point so does the energy-momentum tensor, T^{ν}_{μ}. The Latter follows from the field equation.

$$-G^{\nu}_{\mu} = T^{\nu}_{\mu} = -2L_F F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{2}\delta^{\nu}_{\mu}L,$$

For an electric point charge, this implies that

$$-2E^2(r)L_F < \infty$$

Replacing $-2E^2$ by F, the EM invariant of the case in question, one gets

$$FL_F < \infty \qquad ; \qquad (L_F \to cons. \quad ; \quad F \to 0)$$

On the other hand

$$d(^*FL_F) = 0$$

thus

$$^*FL_F = C$$

and

$$r^2E(r)L_F = C$$

One can consider $C = Q$ hereafter, so

$$E(r)L_F = \frac{Q}{r^2}$$

and

$$E^2(r)L_F^2 = \frac{Q^2}{r^4}$$

Multiplying both sides by -2 we have

$$FL_F^2 = -2\frac{Q^2}{r^4}$$

Now let $r \to 0$ and consequently $FL_F^2 \to \infty$. With Fapproaching zero, the last conclusion implies that $L_F \to \infty$ which is a non-Maxwell behavior at small F. $\square$

Starting with the assumption that we can have a metric with a regular pole in the presence of an electric point charge, we come to a non-Maxwell feature while we considered the Maxwell limit of any suggested Lagrangian $L(F)$ coupled to R. This apparent contradiction leads us to reach the conclusion that with a nonzero electric charge we cannot have a metric with a regular pole while $L \sim F$; $F \to 0$. The same theorem is valid when the electric charge and magnetic charge are both present.

Chapter 4

CONCLUSION

The non-linear electrodynamics is one of the highly non-linear theories in physics which has attracted for almost 3-quarters of a century many physicists in different fields such as String Theory, High Energy Physics, Classical and Quantum Gravity and General Relativity. The premier purpose for considering this non-linear theory was to remove the singularity in the Maxwell theory of electrodynamics. As the standard Maxwell theory yields an electric field $\mathbf{E} = \frac{Q}{r^2}\hat{\mathbf{r}}$ for a point charge Q located at the origin the natural question would be "what happens at the origin?". The BI non-linear theory has changed the picture of the problem by introducing a new Lagrangian which is known as BI-Lagrangian. Based on this theory the electric field of the same charge at the same point is given by $\mathbf{E} = \frac{Q}{\sqrt{r^4+\beta^4}}\hat{\mathbf{r}}$ in which β is a constant to be found in experiments. In this theory it is very clear that the origin is no longer a distinct point and the electric field at the origin is regular. The other applications for this theory have been found very recently after that a similar Lagrangian has been used in string theory. Nowadays hundreds of papers are published in non-linear electrodynamics in flat or curved space by using the BI Lagrangian or some other forms of Lagrangians.

In this thesis first of all in Chapter 2 we have studied the historical paper of Born and Infeld and with much details we have shown that how the non-linear electrodynamics has been developed. In this line we repeated all the way gone by the first constructors of the theory and in some parts we added other contributions too. After that in Chapter 3 we have introduced a new non-linear electrodynamic theory with a new Lagrangian 3.1. The main difference between this new theory and the one introduced by Born and Infeld turns back to the form of the electric field and the electric potential of a point charge at a distance r from the charge. Actually in our theory the electric field of a point charge located at the origin at a point of distance r from the charge is given by $\mathbf{E} = \frac{Q}{r^2+a^2}\hat{\mathbf{r}}$ in which a is a free parameter to be found empirically. The form of the electric potential also reads

$$\phi(r) = \frac{Q\pi}{2a} - \frac{Q}{a}\arctan(\frac{r}{a}).$$

In contrast with the BI theory the closed form of the latter fields are simpler and more feasible to be applied for the cases such as hydrogen atom. This theory has been developed in 3+1 dimensions in a large number of details and also a black hole solution based on this Lagrangian coupled minimally with gravity has been found. After 4 dimensions we extended our work in lower and higher dimensions too.

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