Hawking Radiation of Bosons From a Black Hole Having Lorentz Symmetry Breaking

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ABSTRACT

In this thesis, we first start our discussion with understanding the nature of the Schwarzschild like blackhole in bumblebee gravity model. While doing this discussion, we mention about Einstein's field equations, Lorentz symmetry breaking, and after that we continue with the derivation process of the metric. In the sequel, we study the Hawking radiation of Schwarzschild like black hole by introducing several regular coordinate systems.By using the relativistic Hamilton-Jacobi (HJ) method involving the WKB approximation, we calculate the tunneling probabilities of bosons coming and going from the event horizon.We first make those calculations in its naive coordinate, then we continue with the Painleve-Gullstrand (PG) coordinates, Ingoing Eddington-Filkestien (IEF) coordinates, and Kruskal–Szekeres (KS) coordinates, respectively.Then, we derive the associated Hawking temperatures for each coordinate system. We also give the general knowledge about the those regular coordinates. Finally, we discuss the effects of quantum gravity on Hawking radiation and show how the Generalized Uncertainty Principle (GUP) affects the Hawking radiation..

Keywords: Hawking Radiation, Hawking Temperature, Relativistic Hamilton-Jacobi Method, PG Coordinates, IEF Coordinates,KS Coordinates, Generalized Uncertainity Principle

ÖZ

Bu tezde, ilk olarak yaban arısı çekim modelinde Schwarzschild benzeri karadeliğin doğasını anlayarak tartışmamıza başlıyoruz. Bu tartışmayı yaparken Einstein'ın alan denklemlerinden, Lorentz Simetri kırılmasından ve ardından da metriğin türetme işlemine devam ediyoruz. Devamında birkaç düzenli koordinat sistemini tanıtarak Schwarzschild karadeliği benzeri Hawking ışımasını inceliyoruz. WKB yaklaşımını içeren göreli Hamilton-Jacobi (HJ) yöntemini kullanarak, olay ufkundan gelen ve giden bozonların tünelleme olasılıklarını hesaplıyoruz. Önce hesaplamalarımızı naif koordinatlarda, ardından sırasıyla Painleve-Gullstrand (PG) koordinatları, Incoming Eddington-Filkestien (IEF) koordinatları ve Kruskal–Szekeres (KS) koordinatlarında yapıyoruz. Daha sonra her bir koordinat sistemi için ilgili Hawking sıcaklıklarını türetiyoruz. Bu arada bu düzenli koordinatlar hakkında genel bilgiler de veriyoruz. Son olarak, kuantum yerçekiminin Hawking radyasyonu üzerindeki etkilerini tartışıp Genelleştirilmiş Belirsizlik İlkesinin (GUP) Hawking radyasyonunu nasıl etkilediğini gösteriyoruz.

Anahtar Kelimeler: Hawking Radyasyonu, Hawking Sıcaklığı, Relativistik Hamilton-Jacobi Yöntemi, Painleve-Gullstrand Koordinatları, Gelen Eddington-Filkestien Koordinatları, Kruskal-Szekeres Koordinatları, Genelleştirilmiş Belirsizlik İlkesi

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Chapter 1

INTRODUCTION

Hawking radiation is nothing but a black body radiation emitted by black holes due to quantum fluctuations near the event horizon. S. Hawking developed a theoretical argument for the black hole radiation in 1974. Since then, many researchers suggested various methods for studying the Hawking radiation. Some of the notable studies among them are as follows: Damour-Ruffini-Sannan studied the Hawking radiation by using the tortoise coordinate transformation. Parikh and Wilczek (PW) considered the Hawking radiation as a quantum tunneling process with energy conservation. PW method is also known as the null geodesic method. HJ method considers the Hawking radiation as a semi-classical phenomenon with a quantum tunneling of emitted particles. While outgoing particles tunnel across the potential barrier, the imaginary part of the action is found by employing the Feynman prescription and WKB approximation [1]. In this thesis, we shall mainly concentrate on the HJ method.

In the second chapter, we will introduce you the metric function. As a result of the solving the Einstein field equations in the bumblebee gravity field, will give us the spherically symmetric vacuum solution. The vacuum solution in the bumblebee gravity model caused by Lorentz symmetry breaking was derived by R. Casana. [2]. In a brief explanation, the symmetry is equality and fairness of physical laws and the symmetry is necessary of the laws of physics to be the same for all inertial observers is known as Lorentz symmetry.

Lorentz symmetry is one of the pillars of both general relativity and the standard model of particle physics. Motivated by ideas about quantum gravity, unification theories and violations of CPT symmetry. By combining gravity and quantum theory, string theory attempts to unify the four forces of nature simultaneously into one unified theory. [3] Spontaneous Lorentz symmetry breaking can occur when the dynamics of a tensor field cause it to take on a nonzero expectation value in vacuo, thereby providing one or more "preferred directions" in spacetime. Couplings between such fields and spacetime curvature will then affect the dynamics of the metric, leading to interesting gravitational effects. Bailey and Kostelecký developed a post-Newtonian formalism that, under certain conditions concerning the field's couplings and stress-energy, allows for the analysis of gravitational effects in the presence of Lorentz symmetry breaking. [4]

While we talking about Einstein field equations, we would like to mention about the Riemann curvature tensor, Ricci curvature tensor and Ricci scalar. The Riemann curvature tensor ($R^{\alpha}_{\gamma\mu\nu}$) explained that how much curvature exists in any stated region of space. The Ricci tensor ($R_{\mu\nu}$) comes from the need for a curvature tensor with only 2 indexes in Einstein's theory. It is obtained by averaging certain parts of the Riemann curvature tensor. Lastly, the Ricci scalar (sometimes called curvature scalar) (R) the simplest measure of the curvature. It creates a scalar value for every point in space and obtained by taking the average of the Ricci tensor. [4]

In the third chapter, we will study Hawking radiation. To eliminate the singularities, we will examine three different regular coordinates (PG, IEF and KS coordinates) besides its naive coordinate. There are two singular r values where the SBHBGM metric deteriorates: $r = 0$ and $r = 2M\sqrt{1 + l}$. At those positions, one of the components of the metric function diverges but the explication of this divergence is quite different in these two cases. We know that the divergence at the point $r = 0$ is because of the real singularity. As we approach the singularity, the general theory of relativity collapses, and to understand what happened there, we need to turn to quantum gravity theory. Conversely, there is no prominent difference at the surface $r = 2M\sqrt{1 + l}$ and the divergence in the metric can be fixed by choosing appropriate coordinates: this surface is referred to as the event horizon. Many of the puzzling properties of black holes lie in the interpretation of the event horizon.

During the chapter three, we get help some regular coordinates. Here, we want to give some specific informations about them and show you to relations between them. The metric in KS coordinates describes the entire extended black hole spacetime with a single coordinate system.. The main disadvantage of these coordinates is that there is a dependency of both time and space coordinates in the metric. In Eddington–Finkelstein coordinates, as in Schwarzschild coordinates, the metric is independent of the time but it is not include whole the complete spacetime. The Eddington–Finkelstein coordinates and Gullstrand–Painlevé coordinates have some resemblance like they are time-independent, pass through either the black hole or white hole horizons and are not diagonal.

In the last part of this thesis, we will discuss the effects GUP on the Hawking temperature. The GUP is applied indirectly to the gravitational source by correlating the GUP-modified Hawking temperature with a deformation of the background metric models developed to apply the minimum length scale or maximum momentum in various physical systems. One of the higher order GUP approaches gives estimates for minimum length uncertainty. Second, GUP simultaneously estimates the maximum momentum and minimum length uncertainty. One of the higher-order GUP approaches gives predictions for the minimal length uncertainty. A second one predicts a maximum momentum and a minimal length uncertainty, simultaneously. In this chapter , we will additionally apply the GUP correction via the PG coordinates to the metric and we will study the modified Hawking temperature.

At the end of the chapter 4, we will derive the quantum corrected entropy. The thermodynamics of black holes has been successfully established since the discovery of Hawking radiation using quantum field theory in curved geometry.

Chapter 2

SCHWARZSCHILD LIKE BLACK HOLES IN BUMBLEBEE GRAVITY (SBHBGM)

According to extended Einstein field equations;

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.1}$$

where $G_{\mu\nu}$ is Einstein tensor, $T_{\mu\nu}$ is total energy momentum tensor, $g_{\mu\nu}$ is the metric tensor, $\kappa(= 8\pi G_N)$ is a Einstein gravitational constant, $R_{\mu\nu}$ is Ricci curvature tensor, and R is Ricci scalar (curvature scalar). In addition, $T_{\mu\nu}$ is the total source of combining of matter and Bumblebee field. We can easily express as,

$$T_{\mu\nu} = T^{M}_{\mu\nu} + T^{B}_{\mu\nu} \tag{2.2}$$

and $T^{B}_{\mu\nu}$ is;

$$\begin{aligned} T^{B}_{\mu\nu} = {} & -B_{\mu\alpha}B^{\alpha}_{\nu} - \frac{1}{4}B_{\alpha\beta}B^{\beta\alpha}g_{\mu\nu} - Vg_{\mu\nu} + 2V'B_{\mu}B_{\nu} \\ & + \frac{\xi}{k}\Big[\frac{1}{2}B^{\alpha}B^{\beta}R_{\alpha\beta}g_{\mu\nu} - B_{\mu}B^{\alpha}R_{\alpha\nu} - B_{\nu}B^{\alpha}R_{\alpha\mu} \\ & + \frac{1}{2}\nabla_{\alpha}\nabla_{\mu}(B^{\alpha}B_{\nu}) + \frac{1}{2}\nabla_{\alpha}\nabla_{\nu}(B^{\alpha}B_{\mu}) - \frac{1}{2}\nabla^{2}(B_{\mu}B_{\nu}) \\ & - \frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}(B^{\alpha}B^{\beta})\Big] \end{aligned} \tag{2.3}$$

where ξ is coupling constant controling the non-minimal gravity-bumblebee interaction. Here, we choose a potential term (V) that satisfies a non-vanishing vacuum expectation value (VEV) for bumblebee field.

The vacuum solutions of bumblebee field are fixed when $V = V' = 0$. Vacuum means, there is no matter or pressure close to any particles. In other words, there is no source and no time dependency. In order to get extended Einstein field equations, the bumblebee vector should be simple as indicated below;

$$\mathrm{b}\mu = (0, \mathrm{b_r(r)}, 0, 0) \tag{2.4}$$

when the bumblebee field B_μ vanishes, Eq. (1.1) reduces to the Einstein field equations. The vacuum solution in the bumblebee gravity model caused by Lorentz symmetry breaking [2].

There are some discussions about the existence of Lorentz symmetry breaking in the standard model extension, the Lorentz violation occurring in the vector B_μ giving a non-zero vacuum expectation value. These are bumblebee models and are among the only examples of field hypotheses with unrestricted Lorentz and diffeomorphism violation.

Furthermore, the metric signature is (-,+,+,+) and we adopt to the geometrical units. In a bumblebee gravity model, a spherically symmetric vacuum solution is obtained as follows,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + (1 + l)\left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \tag{2.5}$$

or simply,

$$ds^2 = -f dt^2 + \frac{(1 + l)}{f} dr^2 + r^2 d\Omega^2, \tag{2.6}$$

where $d\Omega^2$ is the angular part,

$$d\Omega^2 = d\theta^2 + sin^2\theta d\emptyset^2. \tag{2.7}$$

Here, l is the non-zero Lorentz symmetry breaking parameter. In the case of $l \to 0$ limit, one recovers the well-known Schwarzschild metric. Namely, when l takes zero value in the metric tensor, then SBHBGM reduces to the Schwarzschild black hole. In fact, metric (1) represents a purely radial Lorentz-violating solution outside a spherical body characterizing a modified black hole solution. The metric function and the Hawking temperature [4] are given by.

$$f = 1 - \frac{2M}{r}, \quad r_H = 2M, \quad r_H\text{: } Event\ Horizon$$

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M\sqrt{1+l}}, \tag{2.8}$$

in which κ denotes the surface gravity. As can be seen from above, the non-zero Lorentz symmetry breaking parameter (l) has the effect of reducing the Hawking temperature of a Schwarzschild black hole solution.

Also, the Kretschmann scalar is given by

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = K = \frac{4(12M^2 + 4lMr + l^2r^2)}{r^6(1+l)^2}. \tag{2.9}$$

As can be seen from above, there is a real singularity at $r = 0$.

Chapter 3

DERIVATION OF HAWKING RADIATION OF SCHWARZSCHILD LIKE BLACK HOLES IN BUMBLEBEE GRAVITY

In this chapter, we are going to analyze the Hawking radiation of four dimensional generic spherically symmetric static metric:

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{3.1}$$

We shall start with the derivation of Hawking Radiation of SBHBGM with Hamilton-Jacobi (HJ) equation without GUP effect in its naive coordinates. The relativistic massive HJ formula reads

$$g_{\mu\nu}\partial_\mu P \partial_\nu P + m^2 = 0. \tag{3.2}$$

Setting $L^2 = g^{\theta\theta}(\partial_\theta P)^2 + g^{\phi\phi}\left(\partial_\phi P\right)^2$, we get;

$$-\frac{1}{f}\left(\frac{\partial P}{\partial t}\right)^2 + \frac{f}{\rho}\left(\frac{\partial P}{\partial r}\right)^2 + L^2 + m^2 = 0, \tag{3.3}$$

where f is a function of r, $\rho = 1 + l$, and $P(t,r) = -\omega t + W(r)$. If we insert the $P(t,r)$ into the main metric (3.1), we get

$$-\frac{\omega^2}{f} + \frac{f}{\rho}\left(\frac{\partial W}{\partial r}\right)^2 + L^2 + m^2 = 0. \tag{3.4}$$

After some manipulations, we obtain

$$\left[\frac{\partial W}{\partial r}\right]_1 = \frac{\sqrt{\omega^2\rho - \rho f(L^2+m^2)}}{f}, \quad \text{and} \quad \left[\frac{\partial W}{\partial r}\right]_2 = -\frac{\sqrt{\omega^2\rho - \rho f(L^2+m^2)}}{f}. \tag{3.5}$$

Thus, one finds W expressions by integration seen in Eq. (3.5):

$$W = \pm\int \frac{\sqrt{\omega^2\rho - \rho f(L^2+m^2)}}{f}\, dr\,. \tag{3.6}$$

Near the horizon; $r \to r_h$

$$W \cong \pm\int \frac{\omega\sqrt{\rho}}{f}\, dr. \tag{3.7}$$

And using the residue theory, we get

$$W_{\pm} = \pm\frac{i\pi\sqrt{\rho}\omega}{f'}|_{r_h} \tag{3.8}$$

where

$$f = 1 - \frac{r}{r_h}, \tag{3.9}$$

and

$$f' = \frac{r_h}{r^2}|_{r=r_h=2M} = \frac{1}{r_h} = \frac{1}{2M}. \tag{3.10}$$

Thus, the solution yields

$$W_{\pm} = \pm 2i\pi\omega M\sqrt{\rho}. \tag{3.11}$$

The out/in tunneling rates are found as

$$\Gamma^{\text{out}} = \exp[-2Im(W_+)] = \exp\left[-2\pi M\omega\sqrt{\rho}\right], \tag{3.12}$$

$$\Gamma^{in} = \exp[-2Im(W_-)] = \exp\left[2\pi M\omega\sqrt{\rho}\right]. \tag{3.13}$$

Using the tunneling probability with the Boltzmann formula

$$P = \frac{\Gamma^{out}}{\Gamma^{in}} = \exp\left[-8\pi M\omega\sqrt{\rho}\right] = \exp\left[-\frac{\omega}{T}\right], \tag{3.14}$$

which corresponds to

$$\exp\left[-\frac{\omega}{T}\right] = \exp\left[-8\pi M\omega\sqrt{1+l}\right]. \quad (3.15)$$

Finally, we obtain the surface temperature of the SBHBGM as the following

$$T = \frac{1}{8\pi M\sqrt{1+l}}, \quad (3.16)$$

which is equal to the Hawking temperature (2.8). In addition to its naive coordinate, we take account of three regular coordinate systems, which are PG, IEF, and KS coordinates. Quantum tunneling computations are shown in detail in all these coordinates in the framework of the HJ method.

3.1 PG Coordinates

In general relativity, PG coordinates are a specific set of coordinates for a solution of the Einstein field equations describing a black hole. The ingoing coordinates are such that the time coordinate follows the proper time of an observer who freely falls radially from rest. There is no coordinate singularity at the event horizon. The outgoing ones are simply the time reverse of ingoing coordinates (the time is the proper time for the observer who reaches infinity without velocity). This provides us to deal with the geometry of black hole both inside and outside of the horizon.

In the literature, PG coordinates are known as the first non-singular coordinate system on the event horizon and allow us to describe time-like or empty world lines crossing the horizon inward. In other words, we use PG coordinates to describe the spacetime on either side of the event horizon of a static BH. In this coordinate system, the general spherically symmetric metric (3) loses its diagonal or static form. Instead, it allows for a cross time-space multiplication that renders the metric form stationary and is no longer symmetrical. Therefore, an observer does not consider the surface of the horizon to be special in any way. In this section, we shall use the PG coordinates as a regular coordinate system in the HJ equation and show how it gives the true Hawking temperature [4].

Let us start with a simple transformation,

$$ds^2 = -\tilde{f}dt^2 + \frac{1+l}{\tilde{f}}d\,\tilde{r}^2 + \tilde{r}^2 d\Omega^2$$

$$dr \rightarrow \sqrt{1+l}\,d\tilde{r}$$

where dr is the new coordinate, and $d\tilde{r}$ represents the old radial coordinate.

Then, we have

$$r = \sqrt{1+l}\,\tilde{r},$$

$$f = 1 - \frac{2M\sqrt{1+l}}{r},$$

and

$$\tilde{f} = 1 - \frac{2M}{\tilde{r}},$$

$$ds^2 = -fd\tilde{t}^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2,$$

$$dt = d\tilde{t} + \frac{\sqrt{1-f}}{f}dr.$$

According to the PG coordinates, the new metric becomes

$$ds^2 = -fdt^2 + 2\sqrt{1-f}dtdr + dr^2 + r^2 d\Omega^2 \qquad (3.17)$$

By the help of the HJ equations, we get

$$-\left[\frac{\partial}{\partial t}P(r,t,\theta,\phi)\right]^2 + 2\sqrt{1-f}\left[\frac{\partial}{\partial r}P(r,t,\theta,\phi)\right]\left[\frac{\partial}{\partial t}P(r,t,\theta,\phi)\right] + f\left[\frac{\partial}{\partial r}P(r,t,\theta,\phi)\right]^2 + L^2 + m^2 = 0 \qquad (3.18)$$

where

$$P(t, r, \theta, \phi) = -\omega t + W_0 \tag{3.19}$$

in which W_0 is the function of r.

The solution of equation (3.18) can be found to be

$$-\omega^2 - 2\sqrt{1-f}\,\omega\left(\frac{\partial}{\partial r}W_0\right) + f\left(\frac{\partial}{\partial r}W_0\right)^2 + L^2 + m^2 = 0 \tag{3.20}$$

From there, we can easily find the roots as

$$\left[\frac{\partial W_0}{\partial r}\right]_1 = \frac{\omega\sqrt{1-f} + \sqrt{\omega^2 - fm^2 - fL^2}}{f} \text{ and } \left[\frac{\partial W_0}{\partial r}\right]_2$$

$$= \frac{\omega\sqrt{1-f} - \sqrt{\omega^2 - fm^2 - fL^2}}{f}$$

Now, to have W_0 we simply apply the integration:

$$W_0 = \int \frac{\sqrt{1-f}\,\omega \pm \sqrt{\omega^2 - f(m^2 + L^2)}}{f} dr\Big|_{r \to r_h = 2M\sqrt{1+l}} \cong \int \frac{(\omega \pm \omega)\sqrt{1-f}}{f} dr,$$

which yields

$$W_0^- = 0, \tag{3.21}$$

and

$$W_0^+ = 2\omega \int \frac{\sqrt{1-f}}{f} dr = 2i\pi\omega r_h = 4i\pi\omega M\sqrt{1+l}. \tag{3.22}$$

Let us insert the equations (3.21) and (3.22) into the following equations:

$$\Gamma^{\text{out}} = \exp[-2Im(W_0^+)] = \exp\left[-8\pi M\omega\sqrt{1+l}\right], \tag{3.23}$$

and

$$\Gamma^{in} = \exp[-2Im(W_0^-)] = \exp[0] = 1. \tag{3.24}$$

The ratio of above the Eqs. (3.23) and (3.24), gives the probability of the radiating particles:

$$P = \frac{\Gamma^{out}}{\Gamma^{in}} = \Gamma^{out} = e^{-\left(\frac{\omega}{T}\right)}.$$

After that, we write the equality here:

$$\exp\left[-\frac{\omega}{T}\right] = \exp\left[-8\pi M\omega\sqrt{1+l}\right].$$

Then, we re-derive the Hawking temperature:

$$T = \frac{1}{8\pi M\sqrt{1+l}}. \tag{3.25}$$

As you noticed, the solutions of the HJ equations in naive coordinates and PG coordinates give the same results [see Eqs. (3.16) and (3.25)].

3.2 IEF Coordinates

IEF coordinates are a pair of coordinate systems for a curved geometry_that adapted for radial null geodesics. Null geodesics are the dour dimensional spacetime path of photons. Namely, these coordinates are aligned with radially moving photons. In these coordinate systems, outwardly moving radial light rays (which each follow a null geodesic) define the fixed 'time' surfaces. One of the important advantage of this coordinate system is, it verifies that the singularity at the Schwarzschild radius is just a coordinate singularity, so it is not a real (physical) singularity. [5]

$$ds^2 = -fdt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2, \tag{3.26}$$

$$v = t + r^* \tag{3.27}$$

$$dv = dt + dr^*, \tag{3.28}$$

r^* is the tortoise coordinate. The tortoise coordinate goes to the minus infinity as r goes to the the Schwarzschild radius. Remember; r^* defined as;

$$r^* = \int \frac{dr}{f},$$

We can easily say that

$$dr^* = \frac{dr}{f}. \tag{3.29}$$

If we insert Eq. (3.29) into Eq. (3.28), then we get

$$dv = dt + \frac{dr}{f}. \tag{3.30}$$

According to Eq. (3.30):

$$dt = dv - \frac{dr}{f}. \tag{3.31}$$

Let us put this new definition of time component into the main metric (3.26):

$$ds^2 = -f\left(dv^2 - \frac{2dvdr}{f} + \frac{dr^2}{f^2}\right) + \frac{1}{f}dr^2 + r^2 d\Omega^2.$$

After some simplifications, we obtain

$$ds^2 = -fdv^2 - \frac{1}{f}dr^2 + 2dvdr + \frac{1}{f}dr^2 + r^2 d\Omega^2,$$

$$\Longrightarrow ds^2 = -fdv^2 + 2dvdr + r^2 d\Omega^2. \tag{3.32}$$

Using the HJ equation with metric (3.32), we find out that

$$2\left[\frac{\partial}{\partial r}P(r,t,\theta,\phi)\right]\left[\frac{\partial}{\partial t}P(r,t,\theta,\phi)\right] + f\left[\frac{\partial}{\partial r}P(r,t,\theta,\phi)\right]^2 + L^2 + m^2 = 0. \tag{3.33}$$

The ansatz for the action can be written as

$$P(t,r,\theta,\phi) = -\omega t + W(r). \tag{3.34}$$

Thus, we get

$$-2\left(\frac{d}{dr}W\right)\omega + f\left(\frac{d}{dr}W\right)^2 + m^2 + L^2 = 0. \tag{3.35}$$

The solution of this second order equation is given by

$$\left[\frac{dW}{dr}\right]_1 = \frac{\omega + \sqrt{\omega^2 - fm^2 - fL^2}}{f} \quad \text{and} \quad \left[\frac{dW}{dr}\right]_2 = \frac{-\omega + \sqrt{\omega^2 - fm^2 - fL^2}}{f},$$

Now we are looking for the solution for W. By integrating the above expressions, we get

$$W = \pm \int \frac{\omega + \sqrt{\omega^2 - f(m^2 + L^2)}}{f} dr|_{r \to r_h = 2M\sqrt{1+l}} \cong \pm \int \frac{\omega + \omega}{f} dr \tag{3.36}$$

where $r_h = 2M\sqrt{1+l}$ is the horizon.

$$W^{-} = 0, \tag{3.37}$$

and

$$W^{+} = 2\omega \int \frac{dr}{f} = \frac{2i\pi\omega}{f'(r_h)} = 4i\pi\omega M\sqrt{1+l}. \tag{3.38}$$

Thus, the tunnelling rates read

$$\Gamma^{\text{out}} = \exp[-2Im(W^{+})] = \exp\left[-8\pi M\omega\sqrt{1+l}\right], \tag{3.39}$$

and

$$\Gamma^{in} = \exp[-2Im(W^{-})] = \exp[0] = 1. \tag{3.40}$$

The tunnelling probability thus becomes

$$P = \frac{\Gamma^{out}}{\Gamma^{in}} = \Gamma^{out} = e^{-\left(\frac{\omega}{T}\right)} \tag{3.41}$$

As a result of the calculations:

$$\exp\left[-\frac{\omega}{T}\right] = \exp\left[-8\pi M\omega\sqrt{1+l}\right], \tag{3.42}$$

we get

$$T = \frac{1}{8\pi M\sqrt{1+l}}. \tag{3.43}$$

3.3 KS Coordinates

KS coordinates are the one of the convenient coordinates for defining a black hole because they are completed geodesically and they do not contain a metric singularity. [6]

In other words, these coordinates envelop the entire spacetime manifold of the extended Schwarzschild solution and they work everywhere except the physical singularity. These coordinates can squeeze infinity into a finite distance so that all spacetime can be visualized on a stamp-like diagram [7]. Let us write the HJ solutions for the KS coordinate form of the SBHBG

$$ds^2 = -\tilde{f}dt^2 + \frac{\rho}{\tilde{f}}dr^2 + \tilde{r}^2 d\Omega^2, \tag{3.44}$$

where $\rho = 1 + l$

$$r = \sqrt{\rho}\tilde{r} \tag{3.45}$$

and

$$\frac{dr}{\sqrt{\rho}} = d\tilde{r}. \tag{3.46}$$

Then substituting Eq. (3.46) to our general metric, we get

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + \frac{r^2}{\rho}d\Omega^2, \tag{3.47}$$

where $R^2 = \frac{r^2}{\rho}$ therefore $R = R(r)$. Meanwhile,

$$\tilde{f} = 1 - \frac{2M}{\tilde{r}} \text{ and } f = 1 - \frac{2M\sqrt{\rho}}{r},$$

Rewrite the general metric in the new form:

$$ds^2 = -f\left(dt^2 + \frac{1}{f^2}dr^2\right) + R^2 d\Omega^2 \tag{3.48}$$

Let us set

$$u = t - r^*, \tag{3.49}$$

and

$$v = t + r^*, \tag{3.50}$$

where u, v are light-cone coordinates and r^* is known as the tortoise coordinate. For the outer region of the black hole, it is found to be;

$$r^* = \int \frac{dr}{f}. \tag{3.51}$$

Differential forms of Eqs. (3.49) and (3.50),

$$du = dt - dr^*, \tag{3.52}$$

$$dv = dt + dr^*. \tag{3.53}$$

From there,

$$dudv = dt^2 - d(r^*)^2 = dt^2 - \frac{dr^2}{f^2}. \tag{3.54}$$

Therefore, we get

$$ds^2 = -fdudv + R^2 d\Omega^2. \tag{3.55}$$

Defining

$$U = -e^{-\mathrm{K}u} \tag{3.56}$$

$$V = e^{\mathrm{K}v}, \tag{3.57}$$

and differentiating them, we obtain

$$dU = -\mathrm{K}e^{\kappa u}du = -\mathrm{K}Udu, \tag{3.58}$$

and

$$dV = -\mathrm{K}e^{\kappa v}dv = -\mathrm{KV}dv. \tag{3.59}$$

Thus, we have

$$dUdV = -\mathrm{K}^2 UVdudv. \tag{3.60}$$

By inserting Eq. (3.60) into the metric (3.55), we find

$$ds^2 = -f\left(\frac{dUdV}{-\mathrm{K}^2 UV}\right) + R^2 d\Omega^2, \tag{3.61}$$

or

$$ds^2 = \frac{f}{\mathrm{K}^2 UV} dUdV + R^2 d\Omega^2 \tag{3.62}$$

On the other hand,

$$UV = -e^{-\mathrm{K}(V-U)} = -e^{2\mathrm{K}r^*}, \tag{3.63}$$

in which

$$r^* = \int \frac{dr}{f} = r + r_h \ln(r - r_h) + c = r + r_h \ln\left(\frac{r}{r_h} - 1\right) + r_h ln r_h + c, \tag{3.64}$$

where $r_h ln r_h$ and c are ignorable terms. Thus, we have

$$r^* = r + \ln\left(\frac{r}{r_h} - 1\right)^{r_h} \tag{3.65}$$

Meanwhile,

$$\mathrm{K} = \frac{f'(r_h)}{2}. \tag{3.66}$$

Let us recall f and f' one more time:

$$f = 1 - \frac{r_h}{r} \tag{3.67}$$

and

$$f' = \frac{1}{r_h}. \tag{3.68}$$

Thus, Eq. (3.66) recasts in

$$\mathrm{K} = \frac{f'(r_h)}{2} = \frac{1}{2r_h}. \tag{3.69}$$

Inverse of Eq. (3.69) gives r_h:

$$r_h = \frac{1}{2\mathrm{K}}. \tag{3.70}$$

Thus Eq. (3.63) yields

$$UV = -\exp\left[2Kr + 2Kr_h\, ln\left(\frac{r}{r_h} - 1\right)\right], \tag{3.71}$$

$$UV = -e^{2\mathrm{K}r}\left(\frac{r}{r_h} - 1\right) = -\frac{r}{r_h} e^{2\mathrm{K}r}\left(1 - \frac{r_h}{r}\right) = -\frac{fr}{r_h} e^{2\mathrm{K}r}. \tag{3.72}$$

From there we can write the first component of Eq. (3.62) as follows

$$\frac{f}{\mathrm{K}^2 UV} = \frac{f}{\mathrm{K}^2\left(-\frac{fr}{r_h}\right) e^{2\mathrm{K}r}} \tag{3.73}$$

After some simplifications and substitutions, I want to introduce,

$$-£ = -\frac{4r_h^3 e^{-2\mathrm{K}r}}{r} = -\frac{4r_h^3 e^{-\frac{r}{r_h}}}{r} \tag{3.74}$$

Now our metric become,

$$ds^2 = -£dUdV + R^2 d\Omega^2 \tag{3.75}$$

Setting,

$$\mathrm{T} = \frac{1}{2}(V + U) \tag{3.76}$$

$$d\mathrm{T} = \frac{1}{2}(dV + dU) \tag{3.77}$$

$$\Re = \frac{1}{2}(V - U) \tag{3.78}$$

$$d\Re = \frac{1}{2}(dV - dU) \tag{3.79}$$

$$d\mathrm{T}^2 - d\Re^2 = dUdV \tag{3.80}$$

Therefore;

$$ds^2 = -£(d\mathrm{T}^2 - d\Re^2) + R^2 d\Omega^2 \tag{3.81}$$

$$\mathrm{T} = \frac{1}{2}(U + V) = \frac{1}{2}\left[e^{\mathrm{K}(t+r^*)} - e^{-\mathrm{K}(t-r^*)}\right] = \frac{1}{2}e^{\mathrm{K}r^*}\left(e^{\mathrm{K}t} - e^{-\mathrm{K}t}\right) \tag{3.82}$$

$$= e^{\mathrm{K}r^*}\sinh(Kt)$$

where;

$$\mathrm{K}r^* = \frac{r}{2r_h} + \frac{1}{2}\ln\left(\frac{r}{r_h} - 1\right) \tag{3.83}$$

$$e^{\mathrm{K}r^*} = e^{r/2r_h}\sqrt{\frac{r}{r_h} - 1} \tag{3.84}$$

Then;

$$\mathrm{T} = e^{r/2r_h}\sqrt{\frac{r}{r_h} - 1}\,\sinh(Kt) \tag{3.85}$$

And similarly;

$$\mathfrak{R} = \frac{1}{2}(V - U) \tag{3.86}$$

$$\mathfrak{R} = \frac{1}{2}(U - V) = \frac{1}{2}\left[e^{\mathrm{K}(t+r^*)} + e^{-\mathrm{K}(t-r^*)}\right] = \frac{1}{2}e^{\mathrm{K}r^*}\left(e^{\mathrm{K}t} + e^{-\mathrm{K}t}\right) \tag{3.87}$$

$$= e^{\mathrm{K}r^*}\cosh(Kt)$$

$$\mathfrak{R} = e^{r/2r_h}\sqrt{\frac{r}{r_h} - 1}\cosh(Kt) \tag{3.88}$$

One can set the Killing vector in the KS coordinates as follows,

$$\partial\tau = \mathcal{N}(\mathfrak{R}\partial_{\mathrm{T}} + \mathrm{T}\partial_{\mathfrak{R}}) \tag{3.89}$$

Therefore;

$$\xi^{\mu} = [\mathcal{N}\mathfrak{R}, +\aleph\mathrm{T}, 0,0]$$

Meanwhile, it worth noting that in naïve coordinates, the normalization condition reads:

$$\xi^{\mu} = [1,0,0,0] \quad \longrightarrow \quad g_{\mu\nu}\xi^{\mu}\xi^{\nu} = -1,$$

whence

$$-f(r) = -1,$$

which means that

$$1 - \frac{r_h}{r} = 1 \tag{3.90}$$

And we find the location of observer who will measure the Hawking temperature:

$$r \to \infty.$$

In fact, this is the true location in naïve coordinates that one can measure the Hawking temperature as $\frac{\mathrm{K}}{2\pi}$. In the KS coordinate system, if we repeat the same procedure:

$$g_{\mu\nu}\xi^{\mu}\xi^{\nu} = -1,$$

$$-£\mathcal{N}^2\Re^2 + £\mathcal{N}^2\mathrm{T}^2 = -1 \tag{3.91}$$

and knowing that

$$\Re^2 - \mathrm{T}^2 = e^{\frac{r}{r_h}}\left(\frac{r}{r_h} - 1\right) \tag{3.92}$$

we get

$$\mathcal{N} = \pm\frac{1}{\sqrt{£}\sqrt{\Re^2 - \mathrm{T}^2}}. \tag{3.93}$$

We must select (-) sign to get normalization constant:

$$\mathcal{N} = \frac{\sqrt{r}}{2r_h\sqrt{r - r_h}},$$

$$\lim_{r\to\infty} \mathcal{N} = \frac{1}{2r_h} = \mathrm{K} \tag{3.93}$$

Since the mass and angular parts do not alter the quantum tunnelling, without loss of generality, one consider (1+1)-dimensional form of the KS metric:

$$ds^2 = -£(d\mathrm{T}^2 - d\Re^2)$$

HJ equation now becomes

$$g^{\mu\nu}\partial_\mu P\partial_\nu P = 0 \tag{3.94}$$

$$-\frac{1}{£}(\partial_\mathrm{T} P - \partial_\Re P) = 0$$

Therefore;

$$\partial_\mathrm{T} P - \partial_\Re P = 0$$

Setting ansatz $P = \rho(y)$ where $y = \Re - \mathrm{T}$

Energy $E = -\partial_{\tilde{\mathrm{T}}} P = -\mathcal{N}(\Re\partial_\mathrm{T} P + \mathrm{T}\partial_\Re P) = -\mathcal{N}\left(\Re\frac{\partial\rho}{\partial y}\frac{\partial y}{\partial \mathrm{T}} + \mathrm{T}\frac{\partial\rho}{\partial y}\frac{\partial y}{\partial\Re}\right)$

$$E = -\mathcal{N}\left(-\Re\frac{\partial\rho}{\partial y} + \mathrm{T}\frac{\partial\rho}{\partial y}\right) = +\mathcal{N}\frac{\partial\rho}{\partial y}(\Re - \tau) = \mathcal{N}y\frac{\partial\rho}{\partial y} \tag{3.95}$$

Then;

$$\rho = \int \frac{E}{\mathcal{N}y}\, dy \tag{3.96}$$

When $y = 0$ $(\Re = \mathrm{T})$ there is a pole at horizon (develops a divergence at horizon: singularity)

$$\rho = i\pi \frac{E}{\mathcal{N}} = i\pi \frac{E}{\mathrm{K}}$$

$$\Gamma = \exp[-2ImP] = \exp[-2Im\rho] = \exp\left[\frac{2\pi E}{K}\right] = \exp[\beta E] \tag{3.97}$$

where $e^{\beta E}$ is the Boltzmann formula. Thus, we get

$$\beta = \frac{2\pi}{\mathrm{K}} = \frac{1}{T},$$

And

$$T = \frac{\mathrm{K}}{2\pi} = \frac{1}{8\pi M\sqrt{1+l}}. \tag{3.98}$$

Chapter 4

GUP MODIFIED HAWKING RADIATION OF SBHBGM

In this chapter, we are going to discuss about how GUP affects the Hawking temperatures. Most approaches to quantum gravity theories predict the existence of the minimal length.In the theoretical framework, the minimal length can be achieved in different ways [8]. One way to realize the minimal length is utilizing the GUP [9].

Let us start with the general formula of the modified HJ equation, which is stated below,

$$g^{0j}(\partial_0 S)(\partial_j S) + [g^{kk}(\partial_k S)^2 + m^2] \times \left\{1 - 2\beta \left[g^{jj}(\partial_j S)^2 + m^2\right]\right\} = 0 \quad (4.1)$$

where $j = 0,1,2,3$ and $k = 1,2,3 \equiv r, \theta, \phi$.

For the scalar particles the HJ equation becomes:

$$g^{00}(\partial_0 S)^2 + [g^{kk}(\partial_k S)^2 + m^2] \times \left\{1 - 2\beta \left[g^{jj}(\partial_j S)^2 + m^2\right]\right\} = 0 \quad (4.2)$$

Let us use this form in our general metric,

$$ds^2 = -f dt^2 + \frac{1+l}{f} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2 \quad (4.3)$$

Thus metric (4.3) turns into

$$
\frac{2\left[\left(\frac{\partial}{\partial r}P(t,r,\theta,\phi)\right)^2 f+(L^2+m^2)(1+l)\right]^2\beta}{(1+l)^2}
$$

$$
\begin{aligned}
&+\left(\frac{fl}{(1+l)^2}+\frac{f}{(1+l)^2}\right)\left(\frac{\partial}{\partial r}P(t,r,\theta,\phi)\right)^2\\
&+\left(-\frac{2l}{f(1+l)^2}-\frac{1}{f(1+l)^2}\right.\\
&\left.-\frac{l^2}{f(1+l)^2}\right)\left(\frac{\partial}{\partial r}P(t,r,\theta,\phi)\right)^2+L^2+m^2=0
\end{aligned}
\tag{4.4}
$$

where

$$
P(t,r,\theta,\phi)=-\omega t+W=-\omega t+W_0+\beta W_1 \tag{4.5}
$$

by which W is the function of r. Let us write Eq. (4.4) in a more simple way;

$$
\begin{aligned}
&-\frac{2\beta\left(\frac{d}{dr}W\right)^4 f^2}{(1+l)^2}-\frac{f(4\beta L^2+4\beta m^2-1)\left(\frac{d}{dr}W\right)^2}{1+l}-2(L^2+m^2)^2\beta\\
&+m^2+\frac{-\omega^2+fL^2}{f}=0
\end{aligned}
\tag{4.6}
$$

where W includes W_0 and W_1 .

Both of them are the functions of r. After that,

$$
-\frac{2f^2\beta^5\left(\frac{d}{dr}W_1\right)^4}{(1+l)^2}-\frac{8f^2\left(\frac{d}{dr}W_0\right)\beta^4\left(\frac{d}{dr}W_1\right)^3}{(1+l)^2}
$$

$$
+\left[-\frac{12f^2\left(\frac{d}{dr}W_0\right)^2\left(\frac{d}{dr}W_1\right)^2}{(1+l)^2}\right.
$$

$$
\left.+\left(-\frac{4fL^2}{1+l}-\frac{4fm^2}{1+l}\right)\left(\frac{d}{dr}W_1\right)^2\right]\beta^3
$$

$$
+\left[-\frac{8f^2\left(\frac{d}{dr}W_0\right)^3\left(\frac{d}{dr}W_1\right)}{(1+l)^2}\right.
$$

$$
\left.+\left(-\frac{8fL^2}{1+l}-\frac{8fm^2}{1+l}\right)\left(\frac{d}{dr}W_1\right)\left(\frac{d}{dr}W_0\right)+\frac{f\left(\frac{d}{dr}W_1\right)^2}{1+l}\right]\beta^2 \tag{4.7}
$$

$$
+\left[-\frac{2f^2\left(\frac{d}{dr}W_0\right)^4}{(1+l)^2}+\left(-\frac{4fL^2}{1+l}-\frac{4fm^2}{1+l}\right)\left(\frac{d}{dr}W_0\right)^2\right.
$$

$$
\left.+\frac{2f\left(\frac{d}{dr}W_0\right)\left(\frac{d}{dr}W_1\right)}{1+l}-4m^2L^2-2L^4-2m^4\right]\beta+L^2
$$

$$
+\frac{f\left(\frac{d}{dr}W_0\right)^2}{1+l}+m^2-\frac{\omega^2}{f}=0
$$

All the β (β^5, β^4, β^3, β^2) terms except β approach to zero. That is why in this study, we ignore them.

So, the leading terms that we are dealing with are as follows

$$\left[-\frac{2f^2\left(\frac{d}{dr}W_0\right)^4}{(1+l)^2}+\left(-\frac{4fL^2}{1+l}-\frac{4fm^2}{1+l}\right)\left(\frac{d}{dr}W_0\right)^2+\frac{2f\left(\frac{d}{dr}W_0\right)\left(\frac{d}{dr}W_1\right)}{1+l}\right.$$

$$\left.-4m^2L^2-2L^4-2m^4\right]\beta+L^2+\frac{f\left(\frac{d}{dr}W_0\right)^2}{1+l}+m^2-\frac{\omega^2}{f} \tag{4.8}$$

$$=0$$

Let us find W_0:

$$L^2+\frac{f\left(\frac{d}{dr}W_0\right)^2}{1+l}+m^2-\frac{\omega^2}{f}=0 \tag{4.9}$$

Solution of Eq. (4.9), we get two roots,

$$\left[\frac{d}{dr}W_0\right]_1=\sqrt{\frac{-fm^2l+fL^2l-\omega^2l-\omega^2+fm^2+fL^2}{f^2}} \tag{4.10}$$

and

$$\left[\frac{d}{dr}W_0\right]_2=-\sqrt{\frac{-fm^2l+fL^2l-\omega^2l-\omega^2+fm^2+fL^2}{f^2}} \tag{4.11}$$

When $r\rightarrow r_h$ then $f(r_h)\cong 0$

$$W_0=\pm\int\frac{\omega\sqrt{1+l}}{f}\,dr=\pm\frac{i\pi\omega\sqrt{1+l}}{f'} \tag{4.12}$$

While $f=1-\frac{2M}{r}$ and $f'(r=r_h)=\frac{1}{2M}$, W_0 is,

$$W_0=\pm\frac{i\pi\omega\sqrt{1+l}}{f'}=\pm 2i\pi\omega M\sqrt{1+l} \tag{4.13}$$

Now, lets find W_1 by using the equation below,

$$-\frac{2f^2\left(\frac{d}{dr}W_0\right)^4}{(1+l)^2}+\left(-\frac{4fL^2}{1+l}-\frac{4fm^2}{1+l}\right)\left(\frac{d}{dr}W_0\right)^2+\frac{2f\left(\frac{d}{dr}W_0\right)\left(\frac{d}{dr}W_1\right)}{1+l}$$

$$-4m^2L^2-2L^4-2m^4=0 \tag{4.14}$$

From there,

$$\frac{d}{dr}W = \omega^4 l + \omega^4/(\sqrt{1+l}\sqrt{-\frac{-\omega^2 + fm^2 + fL^2}{f^2}}\ f^3 \qquad (4.15)$$

When $r \to r_h$ then $f(r_h) \cong 0$, integrate the equation 4.15,

$$W_1 = \pm\int \frac{\omega^4(1+l)}{\sqrt{1+l}\sqrt{\omega^2 - f(m^2+L^2)}f^2}\,dr$$

$$= \pm\int \frac{\omega^3\sqrt{1+l}}{f^2}\,dr \cong \pm i\pi\omega^3(2r_h)\sqrt{1+l}$$

$$W_1 = \pm 4i\pi\omega^3 M\sqrt{1+l} \qquad (4.16)$$

Combining Eqs. (4.13) and (4.16), we can find W,

$$W = W_0 + \beta W_1$$

$$W = \pm 2i\pi\omega M\sqrt{1+l} \pm 4\beta i\pi\omega^3 M\sqrt{1+l} = \pm 2i\pi\omega M\sqrt{1+l}(1+2\beta\omega^2) \qquad (4.17)$$

Now we can calculate the tunnelling rate then the probabability as,

$$\Gamma^{\mathrm{out}} = \exp[-2Im(W^+)] = \exp\left[-4\pi M\omega\sqrt{1+l}(1+2\beta\omega^2)\right] \qquad (4.18)$$

$$\Gamma^{in} = \exp[-2Im(W^-)] = \exp\left[4\pi M\omega\sqrt{1+l}(1+2\beta\omega^2)\right] = 1 \qquad (4.19)$$

$$P = \frac{\Gamma^{out}}{\Gamma^{in}} = e^{-\left(\frac{\omega}{T}\right)} \qquad (4.20)$$

From there we find the GUP modified Hawking temperature of SBHBGM:

$$\exp\left[-\frac{\omega}{T}\right] = \exp\left[-8\pi M\omega\sqrt{1+l}(1+2\beta\omega^2)\right]$$

$$T = \frac{1}{8\pi\omega M\sqrt{1+l}(1+2\beta\omega^2)} \qquad (4.21)$$

So;

$$T = T_H(1 - 2\beta\omega^2) \tag{4.22}$$

As you know T_H is the ordinary Hawking temperature (2.8):

$$T = \frac{1}{8\pi M\sqrt{1+l}}$$

If if we insert $\frac{dW_0}{dr}$ into Eq. (4.8) and then we get,

$$\begin{aligned}\frac{8\omega^2L^2}{f} - 8\omega^2L^2 - 8\omega^2m^2 - 2\omega^4 - \frac{16\omega^4}{f^2} + \frac{16\omega^4}{f} \\ + 2\left[\frac{d}{dr}W_1\right]\sqrt{\omega^2 - fm^2 - fL^2} + \frac{8\omega^2m^2}{f} \\ - \frac{16\sqrt{1-f}\,\omega^3\sqrt{\omega^2 - fm^2 - fL^2}}{f^2} \\ + \frac{8\sqrt{1-f}\,\omega^3\sqrt{\omega^2 - fm^2 - fL^2}}{f} = 0\end{aligned} \tag{4.23}$$

From there;

$$\frac{dW_1}{dr} = \frac{\omega^2[(4L^2 + 4m^2 + \omega^2)f^2 + (-8\omega^2 - 4m^2 - 4L^2)f + 8\omega^2]}{\sqrt{\omega^2 - fm^2 - fL^2}\,f^2} - \frac{4\omega^3\sqrt{1-f}(-2+f)}{f^2}$$

When $r \to r_h = 2M\sqrt{1+l}$, $f(r_h) = 0$

$$W_1 = \int \frac{2\omega^3}{f^2}\,dr = 4\omega^3 i\pi r_h = 8i\pi\omega^3 M\sqrt{1+l} \tag{4.24}$$

Then;

$$W = W_0 + \beta W_1 = 4i\pi\omega M\sqrt{1+l}[1 + 2\beta\omega^2] \tag{4.25}$$

$$\Gamma^{\text{out}} = \exp[-2Im(W^+)] = \exp\left[-8\pi M\omega\sqrt{1+l}(1 + 2\beta\omega^2)\right] \tag{4.26}$$

$$\Gamma^{in} = \exp[-2Im(W^-)] = \exp[0] = 1 \tag{4.27}$$

$$P = \frac{\Gamma^{out}}{\Gamma^{in}} = \Gamma^{out} = e^{-\left(\frac{\omega}{T}\right)} \tag{4.28}$$

From there;

$$\exp\left[-\frac{\omega}{T}\right] = \exp\left[-8\pi M\omega\sqrt{1+l}(1+2\beta\omega^2)\right] \tag{4.29}$$

The modified Hawking temperature is,

$$T = \frac{1}{8\pi M\sqrt{1+l}(1+2\beta\omega^2)} \tag{4.30}$$

Here is the relation between regular Hawking temperature and modified one,

$$T = T_H(1-2\beta\omega^2) \tag{4.31}$$

Finally, I would like to discuss entropy .The Bekenstein-Hawking entropy is given by;

$$S_{BH} = \frac{A_H}{4} \tag{4.32}$$

$$A_H = 4\pi {R_H}^2 = \frac{4\pi {r_h}^2}{1+l} = 16\pi M^2 \tag{4.33}$$

Lets look at the first law of the thermodynamics,

$$dE = T_H dS_{BH} \tag{4.34}$$

$$T_H = \frac{\kappa}{2\pi} = \frac{f'}{4\pi} = \frac{r_H}{4\pi r^2}_{r=r_H} = \frac{1}{4\pi r_H} = \frac{1}{8\pi M\sqrt{1+l}} \tag{4.35}$$

where $f = 1 - \frac{r_H}{r}$ and $f' = \frac{r_H}{r^2}$

$$dE = \frac{1}{8\pi M\sqrt{1+l}} 8\pi M dM = \frac{dM}{\sqrt{1+l}} \tag{4.36}$$

By integrating Eq. (4.36), E is found by;

$$E = \frac{M}{\sqrt{1+l}} \tag{4.37}$$

where E is energy and M is mass.

After combining Eqs. (4.3) and (4.37), we get

$$S_{BH} = 4\pi E^2(1+l) = 4\pi M^2 \tag{4.38}$$

In string theory and loop quantum gravity, the quantum corrected entropy (S_{QG}) is introduced as follows,

$$S_{QG} = S_{BH} + \alpha \ln(4S_{BH}) \tag{4.39}$$

$$\begin{aligned} \Delta S_{QG} &= S_{QG}(E-\omega) - S_{QG}(E) \\ &= 4\pi(E-\omega)^2(1+l) + \alpha \ln[16\pi(E-\omega)^2(1+l)] \\ &- 4\pi E^2(1+l) - \alpha\ln(16\pi E^2(1+l)) \end{aligned} \tag{4.40}$$

If we expand equation 4.40 to the Taylor series, we get,

$$-\left[8\pi E(1+l) + \frac{2\alpha}{E}\right]\omega = -\left[\frac{1}{T_H} + \frac{2\alpha}{E}\right]\omega \tag{4.41}$$

The modified tunneling rate is given by,

$$\Gamma_{QG} \sim e^{\Delta S_{QG}} = e^{-\omega/T_{QG}} \tag{4.42}$$

We can write the T_{QG},

$$T_{QG} = T_H\left(1 + \frac{2\alpha T_H}{E}\right)^{-1} \tag{4.43}$$

As you see when $\alpha \to 0$, $T_{QG} \to T_H$

$$T_{QG} = \frac{T_H}{1 + \frac{\alpha(1+l)}{\pi r_H^2}} \tag{4.44}$$

Chapter 5

CONCLUSION

In this thesis by using the relativistic HJ equation we have studied the Hawking radiation in the SBHBGM. In addition to its naive coordinates, we work with three different regular coordinate systems which are PG coordinates, IEF coordinates and KS coordinates. It has been shown in detail that the computed horizon temperatures via the HJ method exactly matches with the standard Hawking temperature. I have to remind that during this discussion, without loss of generality, we neglected the angular dependency of the HJ equation. We have introduced the Lorentz symmetry breaking and we observed that the non-zero Lorentz symmetry breaking parameter (l) has the effect of reducing the Hawking temperature of the Schwarzschild black hole radiation.

In the final chapter, we have been discussed how Hawking temperature is affected by the GUP. In the continuation the tunnelling rate and modified Hawking temperature are calculated. The results indicate that the corrected Hawking temperature depends on the black hole’s mass.

In the light of the information that I have learned in this thesis, I plan to apply and develop this methodology for various black holes and for different particles having non-zero spin.

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