Noncommutative Quantum Mechanics

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Physics

Eastern Mediterranean University
September 2021
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

We investigate the effect of Noncommutative Quantum Mechanics for a particle in Earth's gravity using the Hamiltonian in two dimensions. We solve the Schrödinger equation in detail, we then split the Schrödinger equation into x component and y component. For y component, we further solve the equation and obtain a Simple Harmonic Oscillator (SHO) equation. For x component, we come across an imaginary term that makes it a complex equation, after we solve it, we obtain a SHO equation again. The two equations for components of x and y are solved independently, and we obtain their energy levels, normalized stationary states with Hermite polynomials. A perturbation term appears due to noncommutativity in the Schrödinger equation which we dealt with subsequently. Our results contain corrections of which the most important here are the energy levels. Finally, we are able to combine energy levels from both components of x and y, normalized stationary states as our full solutions.

Keywords: Quantum Mechanics, Noncommutativity, Schrödinger equation, Hamiltonian, Harmonic Oscillator, energy levels.

ÖZ

İki boyutlu Hamiltonian'ı kullanarak dünyanın yerçekimindeki bir parçacığın Noncommutative Kuantum Mekanik çerçevesinde araştırdık. İki boyutlu Schrödinger denkleminin çözebilmek için bileşenlerine ayırdık. Bileşenler birbirinden ayrı bir şekilde çözülebilmektedir. y bileşeni bilinen harmonik osilatör denklemine dönüşürken, x bileşeninin denklemi kompleks olup, sanal kısmı yine harmonik osilatör denklemine dönüşmektedir. İki bileşenlerinin denklemlerinin birbirinden bağımız çözülerek sitemin kuantum enerji durumlarını hesapladık. Normalize edilmiş dalga denklemleri Hermit polinom çözümlerini içermektedir. Elde ettiğimiz Schrödinger denkleminin çözümleri noncommutative açısal momentum operatörünü içeren kısmını pertürbasyon olarak ele alarak, enerji seviyelerine düzeltmeler hesaplanmaktadır. Enerji ve birleşenlerden oluşan normalize dalga fonksyonları çözümlerini birleştirerek tek çözüm haline dönüşmektedir.

Anahtar Kelimeler: Kuantum Mekanik, Noncommutative Kuantum Mekaniği, Schrödinger denklemi, Hamiltonian, Harmonik Osilatör, enerji seviyeleri

DEDICATION

This research project is dedicated to;

1. My beloved Mother
2. My late Father, my late brother Abdul and my late Grandmother Haj. Hauwa
3. My family that supported me throughout this period.

ACKNOWLEDGEMENT

First and foremost, all praises are due to Allah (SWA) for life, health, wealth, and ability to carry out this research and the study in general. Peace and blessings be upon our beloved Prophet Muhammad (SAW).

I will like to record my appreciation to my humble and tolerant Supervisor and advisor Prof. Dr. S. Habib Mazharimousavi for his support, honesty, helpfulness, and understanding throughout my research and my stay at EMU. I also thank Prof. Dr. İzzet Sakali, a nice gentleman, Assist. Prof. Mustafa Riza, a Lively and friendly man, Prof. M. Halilsoy, a father to us all, Prof. Omar Mustafa, and all of the Physics department for their support throughout my stay at EMU.

To my mother that is always supportive and encouraging with prayers and resources, I cannot express my gratitude enough, but I will use simple words: I love you Mom. I also thank my uncle, Prof. T.M. Bande, my aunt, Dr. H.M. Bande, my family Ismail, Bashar, Jamila, Hajaru, Hamza, Ibrahim, Amina, Ramatu, Rufai, and all others not mentioned here due to space, I appreciate you all. My friends, Labana, Wali, Junior, Yaya, Dabo, and others, gratitude for your encouragement. My friend and roommate, Ibrahim Sale, I appreciate the support and companionship. lastly, I must thank my course mates Flora and Roza for always being there for me, we achieved a lot together.

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LIST OF ABBREVIATIONS

FT	Field Theory
HO	Harmonic Oscillator
H-atom	Hydrogen Atom
ICHO	Isotropic charged Harmonic Oscillator
NC	Noncommutative
QM	Quantum Mechanics
QED	Quantum Electrodynamics
QFT	Quantum Filed Theory
SHO	Simple Harmonic Oscillator
UMF	Uniform Magnetic Field

Chapter 1

INTRODUCTION

Physics has always been around us since before scientific discoveries. However, in Physics, QM is one of the most successful theories that transformed from "old Quantum theory", i.e., the early attempt to understand microscopic phenomena, to a fully developed QM in the early 1900s. QM is a fundamental theory that describes the physical properties of nature on a very small scale.

Lately, there has been growing interest in the study of noncommutativity in spacetime (Minkowski spacetime) in Physics. This is due to the study of string theory in Physics [See ref 1,2,3,6 and 8]. This kind of research has been conducted thoroughly in different areas of Physics.

An energy-dependent NC QM has been studied in [14]. A model of dynamical NC QM was proposed where the NC strengths, that describe properties of the commutation relations of the coordinates and momenta, respectively, are random functions that depends on energy. For an arbitrary potential, the Schrödinger equation has been derived in a two-dimensional system. The equations found reduce the energy with small limit to the ordinary Quantum mechanical one, whereas the NC effect becomes important for large energy levels. Three cases were studied here thoroughly, where the strengths of the noncommutativity are investigated, by an independent energy-scale, connected to the vacuum Quantum fluctuations, by the energy of the particle, and by a Quantum operator representation. An assumption was made in this research for a random power-law, where the NC strengths parameters, and their algebra have energy dependence. In all the three cases studied here, the Quantum evolutions of free particle and HO have been analyzed respectively. The general solutions of the NC Schrodinger equation and the energy levels were both obtained in detail. For the assumption that noncommutativity to be energy-dependent, it smooths the transition between NC geometry at the Planck length and ordinary commutative Quantum mechanical version. The two approaches; NC and commutative versions of QM are unified. NC effect made corrections in all the results obtained here.

In [10], a Quantum Mechanical system has been studied in a central potential, results show the differences in NC QM and equivalent commutative case. Hamiltonian was used to describe a two-dimensional NC system. The result shows any two-dimensional NC system in a central potential is equivalent to a commutative system. For a two-dimensional system, spectroscopy could be a sensible mechanism for detecting NC corrections in QM [11]. The results obtained reveal that, the connection between the commutative and NC regimes is so sudden (abrupt), i.e., $\theta \to 0$ is not straight, (θ is the NC effect parameter).

In [12], the effect of noncommutativity was also studied. This is a Classical mechanics research, where the laws of motion were investigated in a NC state. Firstly, Poisson brackets were redefined in a NC Phase space, both coordinates and momenta components have shown to contain corrections due to noncommutativity. Secondly, based on the newly defined Poisson brackets, Newton's second law of motion has been derived. It shows for a free particle, that the acceleration in commutative space is non zero in NC phase space (i.e., a free particle in commutative space is not a free particle with zero acceleration in NC phase space) because of the noncommutativity of momentum, if it is only the coordinate that is NC, it remains free particle with zero acceleration in NC phase space. The Harmonic oscillator was also studied as another example here. Due to noncommutativity of momenta and coordinates, a damping force appears in the result, though it also depends on the presence of the external field. Linear transformation was also used to solve the equation of motion for the Harmonic oscillator, same result was obtained which proves the correctness of the modified second law of motion found from the redefined Poisson brackets. The effect of noncommutativity was also shown in [9] where the Hamiltonian was considered for Isotropic charged Harmonic oscillator (ICHO) in Uniform Magnetic field (UMF) in a NC phase space. ICHO in UMF was solved both in commutative space and NC space, the results from the two states were compared. The Result from NC phase space indicates ICHO in UMF is seen as a Landau problem. Corresponding exact energies and the eigenfunctions were obtained.

Recently, in [19], the effect of NC QM was investigated in three dimensions on the energy levels of a charged isotropic HO in UMF in the direction of z. The expansion of this study to three dimensions shows to be non-trivial. All the corrections obtained here due to noncommutativity have negative signs, which implies that energy levels in NC state are smaller compared to the commutative ones. Hamiltonian was introduced for the charged isotropic HO in UMF, the momentum and position were both transformed, L_z has been stated explicitly. The perturbative approach has also been introduced to solve the Hamiltonian, and to obtain the energy levels. The results suggest that NC QM can be experimentally studied even in the low energy limit by employing a strong magnetic field to a three-dimension HO.

While some researches show corrections due to noncommutativity in spacetime, some show no corrections at all. For example, a case study in [7] shows no corrections due to the NC effect. A NC multiparticle QM was derived from NC QFT in the nonrelativistic limit. The result shows opposite charged particles have opposite NC effects. As such, there exist no corrections in the H-atom spectrum due to noncommutativity at the tree level.

Research has been carried out in [20] to study QM on NCQ phase space. It is known that the study of NC QM is hugely motivated by the argument of the string theory. A general framework has been developed where NC QM described by noncommutativity matrix parameters for space and momentum turns out to be the same as QM on a suitable transformed quantum phase space. In the literature, results obtained showed the effect of noncommutativity only in space sector, however, there are added terms and corrections due to the NC effect in the momentum sector in quantum phase space as well. In section 2, a general NC QM has been discussed, α-star deformation on the Poisson bracket for classical observables was used to define Heisenberg commutation relation in NC QM. There is an argument that the introduction of a noncommutativity parameter θ on the space sector automatically introduces a noncommutativity parameter β on the momentum sector both in quantum phase spaces since they are linked by the Heisenberg uncertainty relation. A parameter appears in the star-commutation of space and momenta which was ignored because the interest is only in the first-order terms of θ and β. It is easier to use transformed quantum variables instead of quantum variables with star-product. The Jacobian has been stated. It is deduced that the deformation parameter θ in the space sector of Quantum phase space (QPS) has the same magnitude as the deformation parameter β in the momentum sector of the same QPS which depends on the physical system being studied. As stated earlier for noncommutativity parameters, in QPS, the introduction of noncommutativity on the space-like sector as a perturbation automatically introduces an equivalent noncommutativity on the momentum-like sector as a perturbation with opposite sign.

In section 3 of [20], free particle and HO have both been treated. For a free particle, the investigation of the dynamics of a particle on a NC quantum phase space is similar to the investigation of the dynamics of this particle that has a charge q on the ordinary quantum phase space in the influence of magnetic field. An additional term appears in the results which indicates the effect of noncommutativity on the quantum phase space. According to this research, a free particle behaves like the HO with a low frequency which depends on noncommutativity perturbation β on the momentum sector. For the HO with a charge q, the result shows correction in the Hamiltonian, and the energy spectrum has some shifts due to the noncommutative effect in both space and momentum sectors of quantum phase space. A two-particle system has also been studied as another example here on NC quantum phase space. For a two-particle system with their respective masses and charges, it has been considered that their coordinate and momentum operators commute. According to this paper, studying a two-particle system on NC quantum phase space (QPS) is the same as studying it on the ordinary QPS in which transformation is performed. Variables of both momentum and space in the usual quantum phase space obey Heisenberg commutation relations the same way as in the NC quantum phase space without the star-product. The variables have been transformed, and they satisfy the Heisenberg commutation relations in many formats. Also, the Hamiltonian has been transformed in different formats. The Schrödinger equation in NC quantum phase space is solved using separation of variables. There exist some corrections in the energy spectrum due to the noncommutativity effect in the two-particle system as previously shown. The H-atom is the last example here in [20]. Initially, only the electron was considered in a single-particle system in an external Coulomb potential for the H-atom. A previously obtained result in this paper has been used to presume the NC corrections of the Hamiltonian by transforming the potential $(V(x))$. Later on, the H-atom has been considered as a two-particle system i.e., considering both electron and proton as dynamical particles. Treatment of a two-particle system was shown initially in this research. However, for a one-particle system, there are corrections both in the kinetic and potential terms. For the two-particle system, there exist corrections in the Hamiltonian. Also, in addition to the shift in the energy levels at tree level for H-atom, there is an additional term. The corrections and additional terms are due to the effect of noncommutativity in the momentum sector of NC QM, contrary to the belief in the literature that, there exists no NC correction at tree level for H-atom. Following a different approach by this paper, corrections appear in both the Hamiltonian and energy levels.

The Hydrogen Atom spectrum has been reanalyzed in [4] and the result shows that, due to NC space, there are corrections in the spectrum. At first, only the electron has been considered in an external Coulomb field in a one-particle Schrodinger equation, result shows corrections due to noncommutativity in spacetime. But when proton was considered as a dynamical particle, i.e., solving for two-body Schrodinger equation, there exists no change in the spectrum in the NC space, because proton is a composite particle that has a structure. Proton in NC Hydrogen atom is shown essentially behaving as a commutative particle.

Another research has been carried out on the Hydrogen Atom spectrum and the Lamb shift in NC Quantum Electrodynamics in [5]. Here, the Hamiltonian has been used to describe the H-atom. The Hilbert space was considered and presumed to be the same as in the commutative system, just as in NC field theory. The effect of noncommutativity (θ) was assumed to be small, after studying the H-atom spectrum, it shows that due to the NC effect, even at field theory tree level, there exist some corrections to the Lamb shift transition $(2P_{\frac{1}{2}} \to 2S_{\frac{1}{2}})$. The corrections due to the effect of noncommutativity created a new direction called polarized Lamb shift and there is an open: $2P_{\frac{1}{2}}^{\frac{-1}{2}} \to 2S_{\frac{1}{2}}^{\frac{1}{2}}$. The usual Lamb shift $2P_{\frac{1}{2}} \to 2S_{\frac{1}{2}}$ further split to $2P_{\frac{1}{2}}^{\frac{1}{2}} \to 2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}^{\frac{-1}{2}} \to 2S_{\frac{1}{2}}$. This means that, the effect of the NC parameter increases the widths and split the Lamb shift line by a factor proportional to θ (effect of NC parameter). Results have been presented on the Classical Coulomb potential on NC QM for H-atom, and the Lamb shift corrections were obtained using NC QED. Noncommutativity of spacetime should appear only in a physical system according to this paper.

Dirac and Klein Gordon Oscillators have been studied in NC space [see 16]. Results show that Klein Gordon Oscillator in NC space behaves similar to the dynamics of a particle in commutative space in a constant magnetic field. The Dirac Oscillator in NC space has a new term in the Hamiltonian, which indicates a charged particle with a dipole of electric and magnetic moments. In [13], Klein Gordon Oscillator has been studied again, this time in NC phase space. At first, Klein Gordon Oscillator was discussed in NC space. Later on, Klein Gordon Oscillator was investigated in NC phase space. Results show Klein Gordon Oscillators both in NC space and NC phase space possess similar behavior as dynamics of a particle in commutative space moving in a UMF. After solving the Klein Gordon equation in NC phase space, energy levels were obtained, and an additional term appears due to the NC effect in space-space and momentum-momentum.

In [17], research has been carried out on the Bohr Van Leeuwen theorem in NC space. In this research, the classical Bohr Van Leeuwen theorem was revised. The effect of noncommutativity of space coordinates on the Bohr Van Leeuwen theorem was studied. The result shows that, in general, the Bohr Van Leeuwen theorem is not contented in NC space defined by Symplectic structure. Hence, we need special attention because a classical treatment of the partition function of a charged particle in a magnetic field gives rise to non-zero magnetism. In the end, the discussions here may be expanded to NC phase space where the momenta and the coordinates do not commute according to this research.

In [15], the Landau quantization analog has been studied for a particle with neutral polarization in the influence of equivalent electric and magnetic external fields in the framework of NC QM. The particle possesses electrical and magnetic dipole moments. It interacts with the fields through Aharonov-Casher and He-McKellar-Wilkens effects. This research presented an analysis of the usual Landau quantization for a charged particle that moves in a similar external magnetic field. It also shows a review on Landau-like quantization for magnetic and electric dipoles in the influence of external magnetic and electric fields. A general overview was given on NC QM before investigating the Landau-like effects in NC space and NC phase space. The result shows corrections in the Landau-like energy levels arising due to the NC effect in both space and phase space. Similarly, corrections to the mass and cyclotron frequency in both NC space and NC phase space were detected, likewise as the impact of noncommutativity in the energy levels, the radial wave functions, and the magnetic length. Lastly, it was proved that the commutative result can be recovered within the limit $\theta \to 0$.

In this research project, we will study noncommutative Quantum Mechanics in Minkowski spacetime at the order of the Planck length. It is believed that, at short distances, the concept of spacetime may break down at the order of Planck length

$$l_{\rho=}\sqrt{\frac{G\hbar}{c^2}}. \tag{1.1}$$

$G, \hbar$ and c are the Gravitational constant, reduced Planck constant, and speed of light respectively. Here, when the Heisenberg's Uncertainty Principle was introduced to the system, the idea of space and time collapse from any functional meaning. Some changes in the Physics near the Planck scale such as the spacetime noncommutativity are required.

Noncommutativity is generally associated with the effect of the geometry of space. Noncommutative Quantum Mechanics may be defined as the study of nonvanishing position and momentum commutators, see [19].

From [14], we learn that, the union of Heisenberg's Uncertainty Principle with Eisenstein's theory of General relativity concludes that, at high energy, the usual concept of space and time may lose any functional meaning. In general, we see from literature, at short distances (high energy), that classical geometric ideas and notions are not relevant when describing physical phenomena (measurable variables), therefore, we need to study them in QM. We define the phase space in QM by replacing the classical canonical variables (momenta and coordinates) with their QM counterpart Hermitian Operators, i.e.,

$$P \to \hat{P} \text{ and } X \to \hat{X}. \tag{1.2}$$

such that they satisfy the Heisenberg's commutation relations.

The aim of this research is to modify the Heisenberg's commutation relations and investigate the effect of noncommutativity on some specific Quantum systems.

Chapter 2

COMMUTATIVE AND NONCOMMUTATIVE PHASE SPACES

2.1 Commutative Phase Space

We define commutative phase space by changing the canonical variables, momentum and coordinate with QM Hermitian Operators from the previous Chapter in (1.2) i.e., $P \rightarrow \hat{P}$ and $X \rightarrow \hat{X}$.

General, for two operators $\hat{A}$ and $\hat{B}$, we define their commutator as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \tag{2.1}$$

If $[\hat{A}, \hat{B}] = 0$,we say the operators $\hat{A}$ and $\hat{B}$ do commute (they are Commutative). If the result is non-zero, the operators $\hat{A}$ and $\hat{B}$ do not commute (they are noncommutative).

The German theoretical Physicist, Werner Heisenberg deduced a principle which states that; "*it is impossible to determine or measure both position and momentum of a particle simultaneously with precision*", i.e.,

$$\Delta x \Delta p \geq \hbar/2. \tag{2.2}$$

We present the following commutation relations for momenta and coordinates.

For coordinates, we postulate that

$$[\hat{X},\hat{X}] = \hat{X}\hat{X} - \hat{X}\hat{X} = 0, \tag{2.3}$$

$$[\hat{Y},\hat{Y}] = [\hat{Z},\hat{Z}] = 0, \tag{2.4}$$

and

$$[\hat{X},\hat{Y}] = [\hat{Y},\hat{Z}] = [\hat{X},\hat{Z}] = 0. \tag{2.5}$$

In general, the commutator for coordinates is,

$$[\hat{X}^i,\hat{X}^j] = 0. \tag{2.6}$$

The commutator for momenta is presented as

$$[\hat{P}_x,\hat{P}_x] = [\hat{P}_y,\hat{P}_z] = [\hat{P}_x,\hat{P}_z] = 0. \tag{2.7}$$

In general, the commutator for momenta is

$$[\hat{P}_i,\hat{P}_j] = 0. \tag{2.8}$$

However, commutator of position coordinates and momenta is different

$$[\hat{X},\hat{P}_x] \neq 0. \tag{2.9}$$

Proof;

$$[\hat{X},\hat{P}_x] = \hat{X}\hat{P}_x - \hat{P}_x\hat{X}, \tag{2.10}$$

but we know

$$\hat{X} = \hat{X}, \hat{P}_x = -i\hbar\frac{\partial}{\partial x}, \tag{2.11}$$

substituting (2.11) into (2.10)

$$[\hat{X},\hat{P}_x]\psi = \hat{X}\left(-i\hbar\frac{\partial}{\partial x}\right)\psi - \left(-i\hbar\frac{\partial}{\partial x}\right)\hat{X}\psi, \tag{2.12}$$

we introduced a test function ψ so that $\hat{P}$ can operate on it. An operator acts on a function to give a new function. An operator is also a mathematical object that allow us to represent physical observables in QM.

After simplifying, equation (2.12) becomes

$$[\hat{X},\hat{P}_x] = i\hbar. \tag{2.13}$$

(2.13) shows position coordinate and corresponding momentum (momentum in the $x-$axis) are noncommutative. This is the same for other coordinates with their corresponding momenta as well

$$[\hat{X},\hat{P}_x] = [\hat{Y},\hat{P}_y] = [\hat{Z},\hat{P}_z] = i\hbar. \tag{2.14}$$

Now, commutator of coordinates with noncorresponding momenta is different as we see here

$$[\hat{X},\hat{P}_y] = \hat{X}\hat{P}_y - \hat{P}_y\hat{X}, \tag{2.15}$$

we know that

$$\hat{X} = \hat{X}, \hat{P}_y = -i\hbar\frac{\partial}{\partial y}, \tag{2.16}$$

(2.16) into (2.15)

$$[\hat{X},\hat{P}_y]\psi = \hat{X}\left(-i\hbar\frac{\partial}{\partial y}\right)\psi - \left(-i\hbar\frac{\partial}{\partial y}\right)\hat{X}\psi, \tag{2.17}$$

after simplification, one finds

$$[\hat{X},\hat{P}_y] = 0. \tag{2.18}$$

(2.18) clearly shows coordinate commutes with noncorresponding momentum. This is the same for all coordinates and noncorresponding momenta

$$[\hat{X},\hat{P}_y] = [\hat{X},\hat{P}_z] = [\hat{Y},\hat{P}_z] = [\hat{Z},\hat{P}_x] = 0. \tag{2.19}$$

In general, the commutator of coordinates and momenta is given by;

$$[\hat{X}^i,\hat{P}_j] = \delta^i_j i\hbar. \tag{2.20}$$

where δ^i_j is the Kronecker delta.

The above commutations for momenta and coordinates indicate that; when momentum is in the direction of the corresponding coordinate i.e., (2.14) the operators do not commute with each other, this implies that their values cannot be found simultaneously with precision. However, when momentum is not in the corresponding direction of the position coordinate i.e., (2.19) the operators do commute, which implies that their values can be measured with accuracy simultaneously.

The general commutation relations in Commutative phase space from (2.6) (2.8) and (2.20) are summarized here

$$[\hat{X}^i, \hat{X}^j] = 0. \tag{2.21}$$

$$[\hat{P}_i, \hat{P}_j] = 0. \tag{2.22}$$

$$\left[\hat{X}^i, \hat{P}_j\right] = \delta^i_j i\hbar. \tag{2.23}$$

2.2 Noncommutative (NC) Phase Space

From [19], noncommutative Quantum Mechanics may be defined as the study of nonvanishing commutator of position and momentum.

When operators do not commute, i.e., their values can't be measured simultaneously with precision, then, they are noncommutative. (2.14) shows operators not commuting. when momentum is in the direction of the corresponding position i.e., (2.14) the operators do not commute with each other.

To deal with noncommutativity in phase space, we employ the Weyl Moyal star product, thereby changing the standard product of the fields by the star product

$$(f * g)(x,p) = \exp\left(\frac{i}{2\alpha^2}\theta_{ij}\partial_i^x\partial_j^x + \frac{i}{2\alpha^2}\eta_{ij}\partial_i^p\partial_j^p\right)f(x)g(x). \tag{2.24}$$

However, instead of the star product, we employ the Bopp's shift i.e., we transform our coordinates and momenta as thus

$$\hat{X}^a \to X^a - \frac{1}{2}\theta^{ab}P_b. \tag{2.25}$$

$$\hat{P}_a \to P_a + \frac{1}{2}\eta_{ab}X^b. \qquad (2.26)$$

Such that, they satisfy the commutation relations in NC phase space

$$\left[\hat{X}^a, \hat{X}^b\right] = i\hbar\theta^{ab}, \theta^{ab} = \theta\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \qquad (2.27)$$

$$\left[\hat{P}_a, \hat{P}_b\right] = i\hbar\eta_{ab}, \eta_{ab} = \eta\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \qquad (2.28)$$

and

$$\left[\hat{X}^i, \hat{P}_j\right] = i\hbar\gamma^i_j. \qquad (2.29)$$

<u>PROOF:</u> Using transformations from (2.25) and (2.26) we shall prove (2.27), (2,28) and (2,29).

A. (2.27)

$$\left[\hat{X}^a, \hat{X}^b\right] = \left[X^a - \frac{1}{2}\theta^{ai}P_i, X^b - \frac{1}{2}\theta^{bj}P_j\right], \qquad (2.30)$$

now, we apply the commutator rule to (2.29)

$$\left[\hat{X}^a, \hat{X}^b\right] = [X^a, X^b] - \frac{1}{2}\theta^{bj}[\ X^a, P_j] - \frac{1}{2}\theta^{ai}[P_i, X^b] + \frac{1}{4}\theta^{ai}\theta^{bj}[P_i, P_j], \qquad (2.31)$$

further simplifications and summing up from i to i and j to j in 2nd and 3rd terms respectively, we've

$$\left[\hat{X}^a, \hat{X}^b\right] = i\hbar\theta^{ab}. \qquad (2.32)$$

B. (2.28)

$$\left[\hat{P}_a, \hat{P}_b\right] = \left[P_a + \frac{1}{2}\eta_{aj}X^j, P_b + \frac{1}{2}\eta_{bi}X^i\right], \qquad (2.33)$$

applying the commutator rule

$$\left[\hat{P}_a, \hat{P}_b\right] = [P_a, P_b] + \frac{1}{2}\eta_{bi}\left[P_{a,}X^i\right] + \frac{1}{2}\eta_{aj}\left[X^j, P_b\right]$$
$$+ \frac{1}{4}\eta_{aj}\eta_{bi}\left[X^j, X^i\right], \tag{2.34}$$

summing up from i to i and j to j in 2nd and 3rd terms respectively and simplifying the equation, we have

$$\left[\hat{P}_a, \hat{P}_b\right] = i\hbar\eta_{ab}. \tag{2.35}$$

C. (2.29)

$$\left[\hat{X}^i, \hat{P}_j\right] = \left[X^i - \frac{1}{2}\theta^{ik}P_k, P_j + \frac{1}{2}\eta_{jk}X^l\right], \tag{2.36}$$

when one applies the commutator rule to (2.36), we have

$$\left[\hat{X}^i, \hat{P}_j\right] = \left[X^i, P_j\right] + \frac{1}{2}\eta_{jl}\left[X^i, X^l\right] - \frac{1}{2}\theta^{ik}[P_k, P_j]$$
$$- \frac{1}{4}\theta^{ik}\eta_{jl}[P_k, X^l], \tag{2.37}$$

simplifying (2.37), we have

$$\left[\hat{X}^i, \hat{P}_j\right] = i\hbar\gamma_j^i, \gamma_j^i = \left(\delta_j^i + \delta_j^k \frac{1}{4}\theta^{ik}\eta_{jk}\right). \tag{2.38}$$

We conclude this Chapter with the commutation relations in NC phase space from (2.32), (2.35) and (2.38).

$$\left[\hat{X}^a, \hat{X}^b\right] = i\hbar\theta^{ab}, \tag{2.39}$$

$$\left[\hat{P}_a, \hat{P}_b\right] = i\hbar\eta_{ab}, \tag{2.40}$$

and

$$\left[\hat{X}^i, \hat{P}_j\right] = i\hbar\gamma_j^i, \gamma_j^i = \left(\delta_j^i + \delta_j^k \frac{1}{4}\theta^{ik}\eta_{jk}\right). \tag{2.41}$$

θ^{ik} and η_{jk} are antisymmetric tensors, they are noncommutativity parameters for coordinates and momenta respectively.

In the next chapter, we are going to use the Schrödinger equation in two-dimensions to solve for a Quantum particle inside Earth’s gravity. We will transform both the coordinate and momentum in the Schrödinger equation. We will solve for the energy levels and the normalized stationary states with the Hermite polynomials.

Chapter 3

PARTICLE IN EARTH'S GRAVITY

We consider a Quantum particle inside Earth's gravity. Using the Hamiltonian, we describe the particle in a two-dimensional system

$$\hat{H} = \hat{T} + \hat{V}. \tag{3.1}$$

According to the Classical mechanics, the kinetic energy $\hat{T}$ and Potential energy $\hat{V}$ are defined as

$$\hat{T} = \frac{1}{2}mv^2, \hat{V} = \begin{cases} mg\hat{y}, & y \geq y_o \\ \infty, & y < y_o \end{cases}, \tag{3.2}$$

where y_o is a constant. The Hamiltonian takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + mg\hat{y}. \tag{3.3}$$

We present the momentum $\hat{P}$ in two-dimension, $\hat{P} = \hat{P}_x + \hat{P}_y$, which yields

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_x^2 + \hat{P}_y^2\right) + mg\hat{y}. \tag{3.4}$$

The corresponding time independent Schrödinger equation is written as

$$\hat{H}\phi = E\phi, \tag{3.5}$$

Where E represent the particle's energy.

Now, using the Bopp's shift, we transform $\hat{P}_x, \hat{P}_y$ and $\hat{y}$ as we studied in chapter 2.

For $\hat{P}_x$, one writes

$$\hat{P}_x = \alpha P_x + \frac{1}{2\alpha\hbar}\eta_{12}y, \tag{3.6}$$

where η_{ab} (NC effect for momentum) is the anti-symmetric tensor. See [18] for tensor reading, where

$$\eta_{12} = \eta_1 = \eta, \tag{3.7}$$

which implies that

$$\hat{P}_x = \alpha P_x + \frac{1}{2\alpha\hbar}\eta y. \tag{3.8}$$

For $\hat{P}_y$, we apply

$$\hat{P}_y = \alpha P_y + \frac{1}{2\alpha\hbar}\eta_{21} x, \tag{3.9}$$

where η_{21}, is given by

$$\eta_{21} = -\eta, \tag{3.10}$$

which results in

$$\hat{P}_y = \alpha P_y - \frac{1}{2\alpha\hbar}\eta x. \tag{3.11}$$

For $\hat{y}$, the transformation implies

$$\hat{y} = \alpha y - \frac{1}{2\alpha\hbar}\theta_{21} P_x, \tag{3.12}$$

where θ_{ij}, (NC effect of the position) is the anti-symmetric tensor introduced before, such that,

$$\theta_{21} = -\theta, \tag{3.13}$$

and therefore, we have for $\hat{y}$

$$\hat{y} = \alpha y + \frac{1}{2\alpha\hbar}\theta P_x. \tag{3.14}$$

From (3.5), our Schrödinger equation takes the form

$$\frac{1}{2}\left(\hat{P}_x^2 + \hat{P}_y^2\right)\phi + mg\hat{y}\phi = E\phi. \tag{3.15}$$

Now, we substitute (3.8), (3.11) and (3.14) into (3.15) and the Schrödinger equation takes the form

$$\frac{1}{2m}\left(\left(\alpha P_x + \frac{1}{2\alpha\hbar}\eta y\right)^2 + \left(\alpha P_y - \frac{1}{2\alpha\hbar}\eta x\right)^2\right)\phi$$
$$+ mg\left(\alpha y + \frac{1}{2\alpha\hbar}\theta P_x\right)\phi = E\phi. \tag{3.16}$$

One simplifies (3.16) as

$$\frac{1}{2m}\left[\alpha^2\left(P_x^2 + P_y^2\right) + \frac{\eta}{\hbar}\left(yP_x - xP_y\right) + \eta^2\frac{(x^2+y^2)}{4\alpha^2\hbar^2}\right.$$
$$\left. + mg\left(\alpha y + \frac{1}{2\alpha\hbar}\theta P_x\right)\right]\phi = E\phi. \tag{3.17}$$

However, we know that

$$xP_y + yP_x = L_z, \tag{3.18}$$

so (3.17) yields

$$\left(\frac{\alpha^2}{2m}\left(P_x^2 + P_y^2\right) - \frac{\eta}{2m\hbar}L_z + \eta^2\frac{(x^2+y^2)}{8m\alpha^2\hbar^2}\right.$$
$$\left. + mg\left(\alpha y + \frac{1}{2\alpha\hbar}\theta P_x\right)\right)\phi = E\phi. \tag{3.19}$$

We recall the definition of the operators

$$P_x = -i\hbar\partial_x, P_x^2 = -\hbar^2\partial_x^2, P_y^2 = -\hbar^2\partial_y^2. \tag{3.20}$$

When we substitute the operators (3.20) into (3.19), the Schrödinger equation becomes

$$\left(-\frac{\hbar^2\alpha^2}{2m}\left(\partial_x^2 + \partial_y^2\right) - \frac{\eta}{2m\hbar}L_z + \eta^2\frac{(x^2+y^2)}{8m\alpha^2\hbar^2}\right.$$
$$\left. + mg\left(\alpha y - \frac{i\theta}{2\alpha}\partial_x\right)\right)\phi = E\phi. \tag{3.21}$$

Now, we let

$$\frac{\eta^2}{8m\alpha^2\hbar^2} = \frac{1}{2}m\omega^2, \tag{3.22}$$

or equivalently

$$\omega = \frac{\eta}{2m\alpha\hbar}, \tag{3.22$'$}$$

upon which (3.21) becomes

$$\left(-\frac{\hbar^2\alpha^2}{2m}\left(\partial_x^2 + \partial_y^2\right) - \frac{\eta}{2m\hbar}L_z + \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2y^2 + mg\alpha y - \frac{img\theta}{2\alpha}\partial_x\right)\phi = E\phi. \tag{3.23}$$

To simplify (3.23) we take

$$\frac{1}{2}m\omega^2y^2 + mg\alpha y = \frac{1}{2}m\omega^2\left(y^2 + \frac{2g\alpha}{\omega^2}y + (\frac{g\alpha}{\omega^2})^2 - (\frac{g\alpha}{\omega^2})^2\right), \tag{3.24}$$

so that

$$\frac{1}{2}m\omega^2y^2 + mg\alpha y = \frac{1}{2}m\omega^2(y + \frac{g\alpha}{\omega^2})^2 - \frac{1}{2}m\left(\frac{g\alpha}{\omega}\right)^2. \tag{3.25}$$

We replace $\left(y + \frac{g\alpha}{\omega^2}\right)$ with $\tilde{y}$ so that

$$\frac{1}{2}m\omega^2y^2 + mg\alpha y = \frac{1}{2}m\omega^2\tilde{y}^2 - \frac{1}{2}m(\frac{g\alpha}{\omega})^2. \tag{3.26}$$

Coming back to our equation (3.23), the Schrödinger equation takes the form

$$\left(-\frac{\hbar^2\alpha^2}{2m}\left(\partial_x^2 + \partial_y^2\right) + \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2\tilde{y}^2 - \frac{img\theta}{2\alpha}\partial_x\right)\phi - \frac{\eta}{2m\hbar}L_z\phi = (E + \frac{1}{2}m\left(\frac{g\alpha}{\omega}\right)^2)\phi. \tag{3.27}$$

We note that, $\tilde{y} = \left(y + \frac{g\alpha}{\omega^2}\right)$ which implies $\frac{d}{dy} = \frac{d}{d\tilde{y}}\frac{d\tilde{y}}{dy} = \frac{d}{d\tilde{y}}$, and consequently $\left(\frac{d}{d\tilde{y}}\right)^2 = \left(\frac{d}{dy}\right)^2$.

The Schrödinger equation (3.27) may be written as

$$\left(H_0 - \frac{\eta}{2m\hbar}L_z\right)\phi = \tilde{E}\phi, \tag{3.28}$$

in which

$$L_z = \left(xP_y - yP_x\right), \quad (3.29)$$

$$H_0 = -\frac{\hbar^2\alpha^2}{2m}\left(\partial_x^2 + \partial_{\tilde{y}}^2\right) + \frac{1}{2}m\omega^2(x^2 + \tilde{y}^2) - \frac{imgθ}{2\alpha}\partial_x, \quad (3.30)$$

and

$$\tilde{E} = E + \frac{1}{2}m(\frac{g\alpha}{\omega})^2. \quad (3.31)$$

The term $-\frac{\eta}{2m\hbar}L_z$ doesn't commute with H_o i.e.,

$$\left[H_0, -\frac{\eta}{2m\hbar}L_z\right] \neq 0, \quad (3.32)$$

which forces us to consider it as a perturbation term. Therefore, first we solve the Schrödinger equation for H_0 i.e.,

$$H_0\phi_0 = \tilde{E}_0\phi_0, \quad (3.33)$$

and then we apply the time independent perturbation theory to obtain the effect of $-\frac{\eta}{2m\hbar}L_z$ on the energy spectrum as well as the eigenfunctions.

The main Schrödinger equation is given by

$$\left(-\frac{\hbar^2\alpha^2}{2m}\left(\partial_x^2 + \partial_{\tilde{y}}^2\right) + \frac{1}{2}m\omega^2(x^2 + \tilde{y}^2) - \frac{imgθ}{2\alpha}\partial_x\right)\phi_0 = \tilde{E}_0\phi_0, \quad (3.34)$$

which after some manipulation reads as

$$\left(-\left(\partial_x^2 + \partial_{\tilde{y}}^2\right) + \frac{m^2\omega^2}{\hbar^2\alpha^2}(x^2 + \tilde{y}^2) - \frac{im^2gθ}{\hbar^2\alpha^3}\partial_x\right)\phi_o = \frac{2m\tilde{E}_0}{\hbar^2\alpha^2}\phi_0. \quad (3.35)$$

For simplicity, we set $(\frac{m\omega}{\alpha\hbar})^2 = \xi^2$, $\frac{m^2gθ}{\hbar^2\alpha^3} = \delta$, and $\frac{2m\breve{E}_0}{\hbar^2\alpha^2} = \tilde{\varepsilon}_0$.

This implies (3.35) shapes out to be

$$(-(\partial_x^2 + \partial_{\tilde{y}}^2) + \xi^2(x^2 + \tilde{y}^2) - i\delta\partial_x)\phi_0 = \tilde{\varepsilon}_0\phi_0. \quad (3.36)$$

Applying the separation method, we introduce

$$\phi_0(x, \tilde{y}\,) = X(x)Y(\tilde{y}\,), \quad (3.37)$$

upon which (3.36) reduces to

$$-\frac{1}{X(x)}\frac{d^2}{dx^2}X-\frac{1}{Y(\tilde{y}\,)}\frac{d^2}{d\tilde{y}^{\,2}}Y+\xi^2(x^2+\tilde{y}^2)-i\frac{\delta}{X}\frac{d}{dx}X=\tilde{\varepsilon}_0, \tag{3.38}$$

or after the separation

$$-\frac{1}{X}\frac{d^2}{dx^2}X+\xi^2x^2-i\frac{\delta}{X}\frac{d}{dx}X=\tilde{\varepsilon}_{0x}, \tag{3.39}$$

and

$$-\frac{1}{Y}\frac{d^2}{d\tilde{y}^{\,2}}Y+\xi^2\tilde{y}^2=\tilde{\varepsilon}_{0y}, \tag{3.40}$$

where the energy

$$\tilde{\varepsilon}_0=\tilde{\varepsilon}_{0x}+\tilde{\varepsilon}_{0y}. \tag{3.41}$$

Equation (3.40) may be written as

$$-Y''(\tilde{y})+\xi^2\tilde{y}^2Y=\tilde{\varepsilon}_{0y}Y, \tag{3.42}$$

which is the equation of the Simple Harmonic Oscillator (SHO) in QM. However, the standard SHO is represented by the following Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n(x)+\frac{1}{2}m\Omega^2x^2\psi_n(x)=E_n\psi_n(x), \tag{3.43}$$

such that, the energy eigenvalues are given by

$$E_n=\left(n+\frac{1}{2}\right)\hbar\Omega, \tag{3.44}$$

and the energy eigenfunctions read as

$$\psi_n(x)=\frac{1}{\sqrt{2^nn!}}\left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\Omega x^2}{2\hbar}}H_n\left(\sqrt{\frac{m\Omega}{\hbar}}\,x\right), \tag{3.45}$$

$n\ =\ 0,1,2,3$ ....

In (3.45), $H_n(t)$ are the Hermite polynomials defined by

$$H_n(t)=(-1)^ne^{t^2}\frac{d^n}{dt^n}(e^{-t^2}). \tag{3.46}$$

Comparing (3.42) and (3.43), we set $\frac{\hbar^2}{2m} = 1, \frac{1}{2}m\Omega^2 = \xi^2$and $E_n = \tilde{\varepsilon}_{0y}$, which yields $m = \frac{\hbar^2}{2}$, and $\Omega = \frac{2\xi}{\hbar}$.

and

$$(\tilde{\varepsilon}_{0y})_n = (2n+1)\xi. \tag{3.47}$$

Also, from (3.45) we get

$$Y_n(\tilde{y}\,) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\xi}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi\tilde{y}^{\,2}} H_n\left(\sqrt{\xi}\tilde{y}\,\right). \tag{3.48}$$

Let's add that setting $y_o = \frac{g\alpha}{\omega^2}$ in (3.2) which helps to satisfy the boundary condition $Y_n(0) = 0$. Therefore, only the odd solution of (3.47) are acceptable i.e., $n = 1,3,5, ...$ which reveals $n = 2\tilde{n}\ + 1$, $\tilde{n} = 0,1,2, ...$

$$(\tilde{\varepsilon}_{0y})_{\tilde{n}} = (4\tilde{n}+3)\xi, \tag{3.49}$$

and

$$Y_{\tilde{n}}(\tilde{y}\,) = \frac{1}{\sqrt{2^{2\tilde{n}\,+1}(2\tilde{n}\ +1)!}}\left(\frac{\xi}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi\tilde{y}^{\,2}} H_{2\tilde{n}\,+1}\left(\sqrt{\xi}\tilde{y}\,\right). \tag{3.50}$$

After we solved the y-component of the Schrödinger equation, we go back to solve the x-component as well.

Equation (3.39) may be written as

$$-X''(x) + \xi^2 x^2 X - i\delta X'(x) = \tilde{\xi}_{ox} X(x), \tag{3.51}$$

we apply the ansatz

$$X(x) = h(x)U(x), \tag{3.52}$$

into (3.51) to obtain

$$-(h''U + 2h'U' + hU'') + \xi^2 x^2 hU - i\delta(h'U + hU') = \tilde{\varepsilon}_{0x} hU, \tag{3.53}$$

we divide (3.55) by h and factorize by derivatives of U

$$-U'' + \left(\frac{-2h'}{h} - i\delta\frac{h}{h}\right)U' + U\left(\frac{-h''}{h} + \xi^2 x^2 - i\delta\frac{h'}{h}\right) = \tilde{\varepsilon}_{0x}U, \qquad (3.54)$$

we also set coefficient of U' to zero

$$\frac{2h'}{h} + i\delta = 0, \qquad (3.55)$$

we find h to be

$$h = e^{\frac{-i\delta}{2}x}, \qquad (3.56)$$

and therefore (3.54) becomes

$$-U'' + \left(-\left(\frac{-i\delta}{2}\right)^2 + \xi^2 x^2 - i\delta\left(\frac{-i\delta}{2}\right)\right)U = \tilde{\varepsilon}_{0x}U. \qquad (3.57)$$

We simplify (3.57) to get

$$-U'' + \xi^2 x^2 U = (\tilde{\varepsilon}_{ox} + \frac{\delta^2}{4})U. \qquad (3.58)$$

As can be seen from (3.58), it is again a SHO equation with $\frac{\hbar^2}{2m} = 1, \frac{1}{2}m\Omega^2 = \xi^2$ and $E_n = \tilde{\varepsilon}_{0x} + \frac{\delta^2}{4}$.

Therefore, the solution can be written as $\left(\text{note that}, m = \frac{\hbar^2}{2} \text{ and } \Omega = \frac{2\xi}{\hbar}\right)$

$$(\tilde{\varepsilon}_{0x})_{n'} + \frac{\delta^2}{4} = \left(n' + \frac{1}{2}\right)2\xi, \qquad (3.59)$$

(3.59) reduce to

$$(\tilde{\varepsilon}_{0x})_{n'} = (2n' + 1)\xi - \frac{\delta^2}{4}, \qquad (3.60)$$

and

$$X_{n'}(x) = \frac{1}{\sqrt{2^{n'}n'!}}\left(\frac{\xi}{\pi}\right)^{\frac{1}{4}} e^{\frac{-\xi}{2}x^2} H_{n'}\left(\sqrt{\xi}x\right). \qquad (3.61)$$

$n' = 0,1,2,3, \ldots.$

Finally, the full solution in x-direction is given by

$$X_{n'}(x) = e^{\frac{-i\delta}{2}x}U_{n'}(x). \tag{3.62}$$

Next, we write the full solution for $\phi_0(x,\tilde{y})$ which is the multiplication of $X_{n'}(x)$ and $Y_{\tilde{n}}(\tilde{y})$, given by

$$\phi_{0n'\tilde{n}}(x,\tilde{y}) = e^{\frac{-i\delta}{2}x}U_{n'}(x)Y_{\tilde{n}}(\tilde{y}), \tag{3.63}$$

with the energy given by

$$\tilde{\varepsilon}_{0n'\tilde{n}} = (\tilde{\varepsilon}_{0x})_{n'} + (\tilde{\varepsilon}_{0\tilde{y}})_{\tilde{n}} = ((2n'+1)\xi - \frac{\delta^2}{4}) + ((4\tilde{n}+3)\xi). \tag{3.64}$$

We recall that $\tilde{\varepsilon}_0 = \frac{2m}{\hbar^2\alpha^2}\tilde{E}_0$, where m represent the mass of the particle and $\tilde{E}_0$ is the energy of the main Hamiltonian i.e., energy of the unperturbed Hamiltonian H_0. Therefore, one writes

$$(\tilde{E}_0)_{n'\tilde{n}} = \frac{\hbar^2\alpha^2}{2m}(\tilde{\varepsilon}_0)_{n'\tilde{n}} = \frac{\hbar^2\alpha^2}{2m}\left((2n'+4\tilde{n}+4)\xi - \frac{\delta^2}{4}\right), \tag{3.65}$$

and the full eigenfunctions $\phi_{0n'\tilde{n}}$ are given by

$$= \frac{1}{\sqrt{2^{n'}n'!}\sqrt{2^{2\tilde{n}+1}(2\tilde{n}+1)!}}\left(\frac{\xi}{\pi}\right)^{\frac{1}{2}} e^{\frac{-i\delta}{2}x}e^{-\frac{1}{2}\xi(\tilde{y}^2+x^2)}H_{n'}\left(\sqrt{\xi}x\right)H_{2\tilde{n}+1}\left(\sqrt{\xi}\tilde{y}\right). \tag{3.66}$$

Our next move is to find the effect of the ignored term $H_1 = -\frac{\eta}{2m\hbar}L_z$ on our energy spectrum as a small perturbation.

The first order correction to the energy eigenvalues is simply given by

$$\Delta E^{(1)}_{n'\tilde{n}} =< \phi_{0n'\tilde{n}}|H_1|\phi_{0n'\tilde{n}} >. \tag{3.67}$$

In this regard one obtains

$$\Delta E^{(1)}_{n'\tilde{n}} =< \phi_{0n'\tilde{n}}| - \frac{\eta}{2m\hbar}\left(xP_y - yP_x\right)|\phi_{0n'\tilde{n}} >, \tag{3.68}$$

we then have,

$$\Delta E^{(1)}_{n'\tilde{n}} = -\frac{\eta}{2m\hbar}\left(<\phi_{0n'\tilde{n}}\middle|xP_y\middle|\phi_{0n'\tilde{n}}> - <\phi_{0n'\tilde{n}}|yP_x|\phi_{0n'\tilde{n}}>\right). \quad (3.69)$$

We see that, since both x and P_x play as odd functions, the integral over the symmetric limit vanishes, and consequently there is no correction on the energy of the system up to the first order.

Therefore, in terms of the initial parameters, we find the energy spectrum to be

$$\tilde{E}_{n'\tilde{n}} = \frac{\hbar^2\alpha^2}{2m}\left(2(n'+2\tilde{n}+2)\left(\frac{m\omega}{\alpha\hbar}\right) - \frac{\delta^2}{4}\right), \quad (3.70)$$

where $\omega = \frac{\eta}{2m\alpha\hbar}$ and $\delta = \frac{m^2 g\theta}{\hbar^2\alpha^3}$ upon which

$$\tilde{E}_{n'\tilde{n}} = \frac{\hbar^2\alpha^2}{2m}\left(2(n'+2\tilde{n}+2)\left(\frac{\eta}{2\hbar^2\alpha^2}\right) - \frac{1}{4}\left(\frac{m^2 g\theta}{\hbar^2\alpha^3}\right)^2\right), \quad (3.71)$$

which reduce to

$$\tilde{E}_{n'\tilde{n}} = \frac{\eta}{2m}(n'+2\tilde{n}+2) - \frac{1}{8}\frac{m^3 g^2\theta^2}{\hbar^2\alpha^4}. \quad (3.72)$$

From (3.31) we evaluate our final energy i.e., energy of the particle

$$E = \tilde{E} - \frac{1}{2}m\left(\frac{g\alpha}{\omega}\right)^2. \quad (3.73)$$

thus, (note that $\omega = \frac{\eta}{2m\alpha\hbar}$)

$$E = 2m(n'+2\tilde{n}+2) - 2m^3 g^2\left(\frac{\theta^2}{16\hbar^2\alpha^4} + \frac{\hbar^2\alpha^4}{\eta^2}\right) \quad (3.74)$$

We conclude this Chapter having found our energy levels for both x and y components of the Schrodinger equation. Also, the normalized stationary states with their Hermite polynomials in both x-ccomponent and y-component have been compiled. There're corrections due to the effect of noncommutativity in both x and y components of the Schrödinger equation. We harmonized the energy eigenvalues and the eigenfunctions into single equations as our full solutions. In the coming chapter, we shall discuss more on those corrections and some additional term we have found due to noncommutativity.

Chapter 4

DISCUSSION AND CONCLUSION

We see from literature how the NC effect modifies equations by bringing in some corrections or introducing a whole new term(s). We know that at high energy i.e., short distance, Classical geometrical notions and concepts aren't suited to describe physical phenomena, we therefore transform our Classical variables; coordinates and momenta into QM Hermitian operators.

We have used the transformation of coordinates and momenta to satisfy the Heisenberg commutation relations in the commutative phase space section. Also, we have used the transformations from (2.20) and (2.21) to satisfy the Heisenberg commutation relations in the NC phase space.

In Chapter 3, we introduced the Hamiltonian in two-dimension for a particle in Earth's gravity. We transformed the coordinates and momenta in the Schrödinger equation which led us to series of Mathematical manipulations, calculations, and findings. We then split the Schrödinger equation into x and y components. For y component, we solved the equation into becoming a Simple Harmonic Oscillator SHO equation. We solved it and obtained the energy levels and the eigenfunctions with its Hermite Polynomials.

Nevertheless, in the x-component, the equation is a little bit complicated, with the appearance of an imaginary term makes it a complex equation. We applied the separation method and solved it to become a Simple Harmonic Oscillator equation. The energy levels have been obtained so also the eigenfunction with the Hermite Polynomials.

Both in the x and y components, the energy levels have corrections. In y-component, the energy levels have slight corrections after we substituted back the values of $\hbar$ and Ω, it got shifted by the multiple of 2ξ and $n = 2\tilde{n} + 1$.

However, in the x-component, after substituting back the values of $\hbar$ and Ω, the energy levels also shifted by a multiple of 2ξ, and a new term appeared $\left(-\frac{\delta^2}{4}\right)$, which means the energy levels in NC phase space for x-component are smaller by $\left(-\frac{\delta^2}{4}\right)$ compared to the commutative ones.

A perturbation term appeared in the Schrödinger equation due to noncommutativity effect. However, it doesn’t commute with the main Hamiltonian. We applied time independent perturbation theory to obtain the effect of the perturbation term on the energy spectrum as well as the eigenfunction. The integral over the symmetric limit vanishes due to odd functions in coordinate and corresponding momentum, and consequently there exist no correction on the energy of the system up to the first order.

Finally, we combined energy eigenvalues from both components of x and y, so also the normalized stationary states (eigenfunctions) back into single results as our full solutions. All these corrections in eigenfunctions and eigenvalues are due to the effect of noncommutativity in spacetime.

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