Self-adjoint Extensions of the Operators and Their Applications in Physics

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ABSTRACT

In this thesis, self-adjoint extensions of some of the operators used in quantum mechanics are studied. First, the necessary mathematical background namely, the vector spaces and Hilbert space are reviewed. Secondly, the theorem of von-Neumann is introduced to determine the self-adjoint extension of the operators. The application of self-adjoint extensions of the momentum and spatial part of the Klein-Gordon equation is investigated. The concept of quantum singularity structure of the negative mass Schwarzchild spacetime is investigated by the wave obeying the Klein-Gordon equation.

Keywords: Self-adjoint extensions, Hilbert-space, von-Neumann Theorem, Klein-Gordon equation.

ÖZ

Bu tezde, kuantum mekaniğinde kullanılan bazı operatörlerün kendi eşlenik uzantıları ele alınmıştır. İlk olarak, gerekli matematiksel altyapı olan, vektör uzayları ve Hilbert uzayı gözden geçirilmiştir. İkinci olarak ise, operatörlerin kendi eşlenik uzantılarını belirlemek için kullanılan von-Neumann teoremi tanıtılmıştır. Daha sonra uygulama olarak Momentum operatörü ve Klein-Gordon denkleminin uzaysal kısmının kendi eşlenik uzantıları incelenmiştir. Son olarak negatif kütle Schwarzschild uzay-zamanın kuantum tekillik yapısı Klein-Gordon denklemine uyan dalgalar için incelenmiştir.

Anahtar Kelimeler: Kendi eşlenik uzantısı, Hilbert uzayı, von-Neumann Teoremi, Klein-Gordon denklemi.

To my grandfather Esat F. Muhtaroğlu who is my idol in my life

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TABLE OF CONTENTS

ABSTRACT iii

ÖZ iv

DEDICATION v

ACKNOWLEDGMENTS vi

1 INTRODUCTION 1

2 THE NECESSARY MATHEMATICAL BACKGROUND 4

2.1 Definition of Metric Space 4

2.2 Application in Euclidean Space 5

2.2.1 Real Line $\boldsymbol{R}$ 5

2.2.2 Euclidean Plane $\boldsymbol{R^2}$ 5

2.2.3 Three-dimensional Euclidean Space $\boldsymbol{R^3}$ 5

2.2.4 Euclidean Space $\boldsymbol{R^n}$, Unitary Space $\boldsymbol{C^n}$, Complex Plane $\boldsymbol{C}$ 5

2.2.5 Space $\boldsymbol{l^p}$, Hilbert Sequence Space $\boldsymbol{l^2}$ 6

2.2.6 Sequence space $\boldsymbol{l}^\infty$ 6

2.2.7 Open Set, Closed Set 7

2.2.7.a Definition of Ball and Sphere 7

2.2.7.b Definition of Open set and Closed set 7

2.2.8 Definition of Continuous mapping 8

2.2.9 Definition of Dense set and Seperable space 8

2.3 Convergence, Cauchy Sequence, Completeness 8

2.3.1 Definition of Convergence of a sequence and limit 8

2.3.2 Cauchy Convergence Criterion 9

2.3.3 Theorem: Cauchy Convergence 9

2.3.4 Definition of Cauchy sequence and Completeness 10
2.3.5 Theorem: Real line and Complex plane 10
2.3.6 Theorem: Convergent sequence 11
2.3.7 Theorem: Closure and Closed Set 11
2.3.8 Theorem: Complete Subspace 12
2.3.9 Theorem: Continuous Mapping 12
2.4 Examples 13
2.4.1 Completeness of $\boldsymbol{R^n}$ and $\boldsymbol{C^n}$ 13
2.4.2 Completeness of $\boldsymbol{l^p}$ 14
2.4.3 Completeness of $\boldsymbol{l}^\infty$ 15
2.4.4 Completeness of $\boldsymbol{C}$ 16
2.5 Normed Spaces, Banach Spaces 16
2.5.1 Introduction 16
2.5.2 Definition of Vector Space 17
2.5.3 Examples 18
2.5.4 Definition of Linear independence and Linear dependence 18
2.5.5 Definition of Finite and Infinite dimensional vector spaces 19
2.5.6 Theorem: Dimension of a subspace 19
2.6 Normed Space, Banach Space 19
2.6.1 Definition of Normed space and Banach space 20
2.6.2 Euclidean space $\boldsymbol{R^n}$ and unitary space $\boldsymbol{C^n}$ 20
2.6.3 Space $\boldsymbol{l^p}$ 20
2.6.4 Space $\boldsymbol{l}^\infty$ 20
2.6.5 Lemma (Translation Invariance) 20
2.7 Further Properties of Normed Spaces 21

2.7.1 Theorem: Subspace of a Banach space 21
2.8 Finite Dimensional Normed Spaces and Subspaces 21
2.8.1 Theorem: Closedness 21
2.8.2 Definition of Equivalent norms 22
2.9 Linear Operators 22
2.9.1 Definition of Linear Operator 22
2.9.2 Definition of Identity operator ($\boldsymbol{I}$) 22
2.9.3 Definition of Zero operator ($\boldsymbol{0}$) 22
2.10 Inner Product Spaces 22
2.10.1 Definition of Inner Product Spaces and Hilbert Spaces 23
2.10.2 Definition of Orthogonality 24
2.10.3 Euclidean space $\boldsymbol{R^n}$ 24
2.10.4 Unitary space $\boldsymbol{C^n}$ 24
2.10.5 Space $\boldsymbol{L^2[a,b]}$ 25
2.10.6 Hilbert sequence space $\boldsymbol{l^2}$ 25
2.10.7 Space $\boldsymbol{l^p}$ 25
2.11 Properties of Inner Product Spaces 26
2.11.1 Lemma (Schwarz Inequality, Triangle Inequality) 26
2.11.2 Lemma (Continuity of Inner Product) 27
2.11.3 Theorem: Subspace 27
2.11.4 Theorem: Riesz Representation 27
2.12 Hilbert-Adjoint Operator 28
2.12.1 Definition of Hilbert-adjoint operator $\boldsymbol{T^*}$ 28
2.12.2 Theorem: Existence 28
2.12.3 Lemma (Zero operator) 29

2.12.4 Theorem: Properties of Hilbert-adjoint operators.................................. 29
2.13 Self-Adjoint, Unitary and Normal Operators.. 31
2.13.1 Definition of Self-adjoint, Unitary and Normal operators..................... 31
2.13.2 Example (Matrices) 31
2.13.3 Theorem: Self-adjointness 32
3 SELF-ADJOINT EXTENSIONS OF THE OPERATORS 33
3.1 Introduction Orphanage Care Problem 33
3.1.1 Definition of Deficiency subspaces 36
3.1.2 Theorem: Criterion for essentially self-adjoint operators........................ 37
4 APPLICATIONS 39
4.1 Momentum operator... 39
4.1.1 The operator ***p*** on the whole real axis 40
4.1.2 The operator ***p*** on the positive semi-axis.. 40
4.1.3 The operator ***p*** on the finite interval 41
4.2 Klein Gordon Fields.. 41
4.2.1 Theorem 44
5 CONCLUSION 50
REFERENCES 52

Chapter 1

INTRODUCTION

In physics operators are known as important tools that upon acting on a physical state they produce another physical states.

In classical mechanics, the Lagrangian formalism is used for determining the dynamics of a system. Here the Lagrangian is written in terms of generalized coordinates q, generalized velocities $\dot{q} = \frac{dq}{dt}$. Alternatively the dynamics of a system can also be determined by the Hamiltonian $H(q,p,t)$ in which p denotes the conjugate momenta $p = \frac{\partial L}{\partial q}$.

Operators in quantum mechanics are extremely important because the whole quantum mechanics is formulated in terms of operators. Any physical quantity which can be measured experimentally is abbreviated as observable, and therefore it should be associated with a self-adjoint linear operator. In quantum mechanics, wave functions vary with space and time $(\psi(r,t))$, or equivalently momentum and time, therefore observables are differantial operators.

In this thesis, the self-adjoint extensions of some of the operators used in quantum mechanics will be investigated. The main goal of the thesis is to understand whether the operator used in the considered problem has self-adjoint extensions or not. This is important because since, the operators are associated with the observables it then provides physically important outcomes.

One of the application arena for self-adjoint extensions is the case that occurs in the potential of a Schrödinger equation that obeys inverse square $\frac{1}{r^2}$ behaviour. To avoid the divergences as $r \to 0$, the concept of self-adjoint extensions are used [1].

In this thesis, we studied the self-adjoint extensions of the momentum and spatial part of the Klein-Gordon equation. First, the momentum operator is considered for three different physical situations specified by the interval of the positions. This problem is considered first in [2]. Secondly, we consider another application arena of the use of self-adjoint extension concept. The application arena is the naked singularities that arose in the relativity theory. This problem is developed by Horowitz and Marolf [3]. One of the remarkable predictions of Einstein's theory of relativity is the occurence of spacetime singularities. If the singularity is covered by horizon(s), this is called a black hole. But if there is no horizon than the spacetime is naked singular.

At the singularity, all the physical quantities diverge. More importantly, all the known laws of physics do not hold at the singularity.

We consider in this thesis, the negative mass Schwarzchild solution. This solution admit naked singularity at $r = 0$. The wave obeying the Klein-Gordon equation will be considered in the negative mass Schwarzchild geometry. This problem is considered in [3], but calculations are not given in detail. The spatial part of the

Klein-Gordon equation will be treated as an operator. This operator is investigated whether it admits self-adjoint extensions or not.

In the analysis of the operators whether they admit self-adjoint extensions or not, the theorem of von-Neumann is used.

In chapter 2, we review the neccesarry mathematical background by stating the metric and vector spaces and finally we give the properties of Hilbert space which is the natural function space of quantum mechanics.

In chapter 3, the theorem of von-Neumann is given. Chapter 4 is devoted for the applications of the momentum and spatial part of the Klein Gordon equation. The thesis is concluded with a conclusion in chapter 5.

Chapter 2

THE NECESSARY MATHEMATICAL BACKGROUND

The concept of "space" is in fact one of the most important tool for describing the motion with physical quantities. Among the others, the 3-dimensional Euclidean space is the simplest one which is used for describing motions in our living world. In fact, the Euclidean space is not a vector space. However, we wish to review its basic properties in order to understand in detail the notion of vector spaces.

At this stage we wish to define metric spaces and related concepts. For this purpose, we will review some of the related topics presented in the book "Introductory Functional Analysis with Applications" written by Erwin Kreyszig [4].

2.1 Definition of Metric Space

A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$. (The symbol $\times$ denotes Cartesian product of sets: $A \times B$ is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence $X \times X$ is the set of all ordered pairs of elements of X) such that for all $x, y, z \in X$ we have:

i. d is real-valued, finite and nonnegative

ii. $d(x, y) = 0$ if and only if $x = y$

iii. $d(x, y) = d(y, x)$ (symmetry)

iv. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

2.2 Application in Euclidean Space

2.2.1 Real Line $\boldsymbol{R}$

This is a 1-dimensional Euclidean space. It defines the set of all real numbers, the metric (distance) defined by

$$d(x,y) = |x - y|. \tag{2.1}$$

2.2.2 Euclidean Plane $\boldsymbol{R^2}$

This is a 2-dimensional Euclidean space. The metric space $\boldsymbol{R^2}$, is called the Euclidean plane, is obtained if we take the set of ordered pairs of real numbers, $x = (\varepsilon_1, \varepsilon_2)$, $y = (\eta_1, \eta_2)$ then, the Euclidean metric defined by

$$d(x,y) = \sqrt{(\varepsilon_1 - \eta_1)^2 + (\varepsilon_2 - \eta_2)^2}. \tag{2.2}$$

Another interpretation of this is that, in 2-dimensional Euclidean space, the shortest distance between two points is a straight line.

2.2.3 Three-dimensional Euclidean Space $\boldsymbol{R^3}$

This metric space consists of the set of ordered triples of real numbers $x = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $y = (\eta_1, \eta_2, \eta_3)$ then the Euclidean metric defined by

$$d(x,y) = \sqrt{(\varepsilon_1 - \eta_1)^2 + (\varepsilon_2 - \eta_2)^2 + (\varepsilon_3 - \eta_3)^2}. \tag{2.3}$$

2.2.4 Euclidean Space $\boldsymbol{R^n}$, Unitary Space $\boldsymbol{C^n}$, Complex Plane $\boldsymbol{C}$

The previous examples are special cases of n-dimensional Euclidean space $\boldsymbol{R^n}$. This space is obtained if the set of all ordered n-tuples of real numbers, written $x = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, $y = (\eta_1, \eta_2, \dots, \eta_n)$ etc, and the Euclidean metric defined by

$$d(x,y) = \sqrt{(\varepsilon_1 - \eta_1)^2 + (\varepsilon_2 - \eta_2)^2 + \cdots + (\varepsilon_n - \eta_n)^2}. \tag{2.4}$$

Unitary space $\boldsymbol{C^n}$ is defined by the space of all ordered n-tuples of complex numbers with the metric defined by

$$d(x,y) = \sqrt{|\varepsilon_1 - \eta_1|^2 + |\varepsilon_2 - \eta_2|^2 + \cdots + |\varepsilon_n - \eta_n|^2}. \tag{2.5}$$

The complex plane $\boldsymbol{C}$ is defined when $n = 1$. The usual metric defined by

$$d(x,y) = |x-y|. \qquad (2.6)$$

2.2.5 Space l^p, Hilbert Sequence Space l^2

Let $p \geq 1$ be a fixed real number. By definition each element in the space l^p is a sequence $x = (\varepsilon_j) = (\varepsilon_1, \varepsilon_2, \dots)$ of numbers such that $|\varepsilon_1|^p + |\varepsilon_2|^p + \cdots$ converges; thus

$$\sum_{j=1}^{\infty} |\varepsilon_j|^p < \infty \qquad (2.7)$$

and the metric is defined by

$$d(x,y) = \left(\sum_{j=1}^{\infty} |\varepsilon_j - \eta_j|^p\right)^{\frac{1}{p}} \qquad (2.8)$$

where $y = (\eta_j)$ and $\sum_{j=1}^{\infty} |\eta_j|^p < \infty$. The real and complex space l^p is obtained if one takes real sequence or complex sequence respectively, subject to the condition that Eq. (2.7) is satisfied. In the case $p = 2$ we have the famous Hilbert sequence space l^2 with metric defined by

$$d(x,y) = \sqrt{\sum_{j=1}^{\infty} |\varepsilon_j - \eta_j|^2}. \qquad (2.9)$$

2.2.6 Sequence space l^∞

This example shows how the general concept of a metric space is.

As a set X, the set of all bounded sequences of complex numbers is taken; then every element of X is a complex sequence, $X = (\varepsilon_1, \varepsilon_2, \dots)$ briefly $X = (\varepsilon_j)$ such that for all $j = 1,2, \dots$ we have,

$$|\varepsilon_j| \leq c_x \qquad (2.10)$$

where c_x is real number. c_x does not depend on j, but depends on x. The metric defined by

$$d(x,y) = sup_{j \in N} |\varepsilon_i - \eta_j| \qquad (2.11)$$

$y = (\eta_j) \in X$ and $N = \{1,2, \dots\}$

sup means the least upper bound.(Supremum)

A subset E of the real line $\boldsymbol{R}$ is bounded above if E has an upper bound, that is, if there is a $b \in \boldsymbol{R}$ such that $x \leq b$ for all $x \in E$. Then if $E \neq \emptyset$, there exists the supremum of E (or least upper bound of E), written

$$supE,$$

that is, the upper bound of E such that $\sup E \leq b$ for every upper bound b of E. Also

$$supC \leq supE$$

for every nonempty subset $C \subset E$.

Each element of X is a sequence so that $\boldsymbol{l}^{\infty}$ is a sequence space.

2.2.7 Open Set, Closed Set

There are auxiliary concepts which play a role for connection of metric spaces. We wish to consider some types of subset in a given metric space $X = (X, d)$.

2.2.7.a Definition of Ball and Sphere

Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of sets:

i. $B(x_0; r) = \{x \in X |\ d(x, x_0) < r\}$ (open ball)

ii. $\tilde{B}(x_0; r) = \{x \in X |\ d(x, x_0) \leq r\}$ (closed ball)

iii. $S(x_0; r) = \{x \in X |\ d(x, x_0) = r\}$ (sphere)

x_0 is center, and r is the radius. Moreover, definition means

$$S(x_0; r) = \tilde{B}(x_0, r) - B(x_0, r). \tag{2.12}$$

2.2.7.b Definition of Open set and Closed set

If a subset M of a metric space X contains a ball about each of its points, it is an open set. In addition of that, K is a subset of X, and if K's complement is open in X, then X is said to be closed, $K^c = X - K$ is open.

Consequently; an open ball is an open set and a closed ball is a closed set.

2.2.8 Definition of Continuous mapping

Assume that $X = (X, d)$ and $Y = (Y, d)$ are metric spaces.

A mapping $T: X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\widetilde{d}\,(Tx, Tx_0) < \varepsilon \qquad \text{for all satisfying } d(x, x_0) < \delta. \tag{2.13}$$

If T is continuous at every point of X, this implies that T is continuous.

Note that, continuous mapping is characterized in terms of open sets.

2.2.9 Definition of Dense set and Seperable space

If M is a subset of a metric space X, then M is called dense in X if $\overline{M} = X$, where $^{-}$ denotes the closure of M that represents the smallest closed set containing M.

If X has a countable subset which is dense in X, X is said to be seperable.

Countable subset is a set which has finitely many elements or if we can associate positive integers with the elements of M.

2.3 Convergence, Cauchy Sequence, Completeness

In order to discuss the concept of convergence of the sequence of real numbers, the metric on $\boldsymbol{R}$ is very useful. Similarly, to be able to discuss the convergence of the sequence of complex numbers, the metric on the complex plane $\boldsymbol{C}$ must be used. Hence, in an arbitrary metric space $X = (X, d)$, we may consider a sequence (X_n) of elements x_1, x_2 ... of $X$and use the metric d to define convergence.

2.3.1 Definition of Convergence of a Sequence and Limit

A sequence (X_n) in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \epsilon X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0, \tag{2.14}$$

x is called the limit of (x_n) and we write

$$\lim_{n\to\infty} x_n = x \qquad or \qquad x_n \to x. \tag{2.15}$$

We say that (x_n) converges to x. If (x) is not convergent, it is said to be divergent.

It will be usefull to recall that a sequence (x_n) of real or complex numbers converges on the real line $\boldsymbol{R}$ or in the complex plane $\boldsymbol{C}$, respectively, if and only if it satisfies the Cauchy convergence criterion.

2.3.2 Cauchy Convergence Criterion

A number a is called a limit point of a (real or complex) sequence of numbers (X_n) if for every given $\varepsilon > 0$ we have

$$|x_n - a| < \varepsilon \tag{2.16}$$

for infinitely many n.

A (real or complex) sequence (x_n) is said to be convergent, if there is a number x such that, for every given $\varepsilon > 0$, the following condition holds;

$$|x_n - x| < \varepsilon \tag{2.17}$$

for all but finitely many n. This x is called the limit of the sequence (x_n).

2.3.3 Theorem: Cauchy Convergence

A (real or complex) sequence (x_n) is convergent if and only if for every $\varepsilon > 0$ there is an N such that

$$|x_m - x_n| < \varepsilon \qquad \text{for all } \; m, n > N \, . \tag{2.18}$$

Proof:

(a) If (x_n) converges and c is its limit, then for every given $\varepsilon > 0$ there is an N (depending on ε) such that

$$|x_n - c| < \frac{\varepsilon}{2} \qquad \text{for every } \; n > N$$

so that by the triangle inequality for $m, n > N$ we obtain

$$|x_m - x_n| \leq |x - c| + |c - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(b) Conversly, suppose that the statement involving Eq. (2.18) holds. Given $\varepsilon > 0$, we can choose an $n = k > N$ in Eq. (2.18) and see that every x_m with $m > N$

lies in the disk D of radius $\in$ about x_k. Since there is a disk which contains D as well as the finitely many $x_n \notin D$, the sequence (x_n) is bounded. By the Bolzano-Weierstrass theorem (The Bolzano-Weierstrass theorem states that a bounded sequence (x_n) has at least one limit point) it has limit point a. Since Eq. (2.18) holds for every $\varepsilon > 0$, an $\varepsilon > 0$ being given, there is an N^* such that $|x_m - a| < \frac{\varepsilon}{2}$, by the triangle inequality we have for all $m > N^*$

$$|x_m - a| \leq |x_m - x_n| + |x_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which shows that (x_m) is convergent with the limit a.

Now, the concept of the completeness of a metric space must be defined for future analysis in this context. The quantity $|x_m - x_n|$ is the distance $d(x_m, x_n)$ from x_m to x_n on the real line $\boldsymbol{R}$ or in the complex plane $\boldsymbol{C}$. Therefore, the inequality of the Cauchy criterion may be written in the following form,

$$d(x_m, x_n) < \varepsilon \quad (m, n > N). \tag{2.19}$$

Recall that $\{x_n\}$ is called Cauchy sequence if the condition of Cauchy criterion is satisfied. This simply means that the Cauchy sequence converges on real line $\boldsymbol{R}$ or in complex plane $\boldsymbol{C}$. However, in some cases, the Cauchy sequence may not converge and violates the completeness phenomena of the space.

2.3.4 Definition of Cauchy sequence and Completeness

A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy if for every $\varepsilon > 0$ there is an $N = N(\epsilon)$ such that

$$d(x_m, x_n) \qquad \text{for every } < \varepsilon \;\; m, n > N \tag{2.20}$$

The space X is said to be complete if every Cauchy sequence in X converges.

The Cauchy convergence criterion in terms of completeness implies the following:

2.3.5 Theorem: Real line and Complex plane

The real line and the complex plane are complete metric spaces.

Complete and incomplete metric spaces are important in applications. For example, in complete metric spaces the geodesic equation which describes the future time evolution of the particle is also complete and possesses no divergences. However, if the metric space is incomplete, the geodesic equations are also incomplete, hence, it is designated as the singularity which is a very important subject in physics.

2.3.6 Theorem: Convergent sequence

Every convergent sequence in a metric space is a Cauchy sequence.

Proof:

If $x_n \to x$, then for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$d(x_n, x) < \frac{\varepsilon}{2} \qquad \text{for all} \qquad n > N$$

Hence by the triangle inequality we obtain for $m, n > N$

$$d(x_m, x_n) \le d(x_m, x) + d(x, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

this shows that (x_n) is Cauchy.

2.3.7 Theorem: Closure and Closed Set

Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure.

Then:

(a) $x \in \bar{M}$ *if and only if there is a sequence* (x_n) *in* M *such that* $x_n \to x$.

(b) M *is closed if and only if the situation* $x_n \in M$, $x_n \to x$ *implies that* $x \in M$.

Proof:

(a) Let $x \in \bar{M}$. If $x \in M$, a sequence of that type is $(x, x, \ldots)$. If $x \notin M$, it is a point of accumulation of M. Hence for each $n = 1, 2, \ldots$ the ball $B(x; \frac{1}{n})$ contains an $x_n \in M$, and $x_n \to x$ because $\frac{1}{n} \to 0$ as $n \to \infty$.

Conversly, if (x_n) is in M and $x_n \to x$, then $x \in M$ or every neighborhood of x contains points $x_n \neq x$, so that x is a point of accumulation of M. Hence $x \in \bar{M}$, by the definition of the closure.

(b) M is closed if and only if $M = \bar{M}$, so that Thm. (2.3.7b) follows from (a).

2.3.8 Theorem: Complete Subspace

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

Proof:

Let M be complete. By Thm. (2.3.7a), for every $x \in \bar{M}$ there is a sequence (x_n) in M which converges to x. Since (x_n) is Cauchy by Thm. (2.3.4) and M is complete, (x_n) converges in M, the limit being unique. Hence $x \in M$. This proves that M is closed because $x \in \bar{M}$ was arbitrary.

Conversly, let M be closed and (x_n) Cauchy in M. Then $x_n \to x \in X$, which implies $x \in \bar{M}$ by Thm. (2.3.7a), and $x \in M$ since $M = \bar{M}$ by assumption. Hence the arbitrary Cauchy sequence (x_n) converges in M, which proves completeness of M.

2.3.9 Theorem: Continuous Mapping

A mapping $A\colon X \to Y$ of a metric space (X, d) into a metric space (Y, d) is continuous at a point $x_0 \in X$ if and only if

$$x_n \to x_0 \qquad implies \qquad Ax_n \to x_0. \tag{2.21}$$

Proof:

Assume A to be continuous at x_0. Then for a given $\varepsilon > 0$ there is $\delta > 0$ such that

$$d(x, x_0) < \delta \qquad \text{implies} \qquad \tilde{d}(Ax, Ax_0) < \varepsilon$$

Let $x_n \to x_0$. Then there is an N such that for all $n > N$ we have

$$d(x_n, x_0) < \delta.$$

Hence, for all $n > N$,

$$\tilde{d}(Ax_n, Ax_0) < \varepsilon.$$

By definition this means that $Ax_n \to Ax_0$.

Conversly, we assume that

$$x_n \to x_0 \quad \text{implies} \quad Ax_n \to Ax_0$$

and prove that then A is continuous at x_0. Suppose this is false. Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is an $x \neq x_0$ satisfying $d(x, x_0) < \delta$ but $\tilde{d}(Ax, Ax_0) \geq \varepsilon$.

In particular, for $\delta = \frac{1}{n}$ there is an x_n satisfying $d(x_m, x_n) < \frac{1}{n}$ but $\tilde{d}(Ax_n, Ax_0) \geq \varepsilon$.

Clearly $x_n \to x_0$ but (Ax_n) does not converge to Ax_0. This contradicts $Ax_n \to Ax_0$ and proves the theorem.

2.4 Examples

2.4.1 Completeness of $\boldsymbol{R^n}$ and $\boldsymbol{C^n}$

Euclidean space $\boldsymbol{R^n}$ and unitary space $\boldsymbol{C^n}$ are complete.

Proof:

Consider $\boldsymbol{R^n}$. The Euclidean metric on $\boldsymbol{R^n}$ is defined by

$$d(x, y) = \left(\sum_{j=1}^{n} (\varepsilon_j - \eta_j)^2\right)^{\frac{1}{2}}$$

where $x = (\varepsilon_j)$ and $y = (\eta_j)$. We consider any Cauchy sequence (x_m) in $\boldsymbol{R^n}$, writing $x_m = (\varepsilon_1{}^m, \dots, \varepsilon_n{}^m$. Since (x_m) is Cauchy, for every $\varepsilon > 0$ there is an N such that

$$d(x_m, x_r) = \left(\sum_{j=1}^{n} (\varepsilon_j{}^{(m)} - \varepsilon_j{}^{(r)})^2\right)^{\frac{1}{2}} < \varepsilon.$$

Squaring, we have for $m, r > N$ and $j = 1, \dots, , n$

$$(\varepsilon_j{}^m - \varepsilon_j{}^r)^2 < \varepsilon^2 \quad \text{and} \quad |\varepsilon_j{}^m - \varepsilon_j{}^r| < \varepsilon.$$

This shows that for each fixed j, $(1 \leq j \leq n)$, the sequence $(\varepsilon_j{}^{(1)}, \varepsilon_j{}^{(2)}, \dots)$ is a Cauchy sequence of real numbers. It converges by Thm. (2.3.3) say, $\varepsilon_j{}^{(m)} \to \varepsilon_j$ as $m \to \infty$. Using these n limits, we define $x = (\varepsilon_1, \dots, \varepsilon_n)$. Clearly, $x \in R^n$. For $r \to \infty$

$$d(x_m, x) \leq \varepsilon.$$

This shows that x is the limit of (x_m) and proves completeness of $\boldsymbol{R}^n$ because (x_m) was an arbitrary Cauchy sequence. Completeness of $\boldsymbol{C}^n$, follows from Thm. (2.3.3) by the same procedure.

2.4.2 Completeness of $\boldsymbol{l^p}$

The space $\boldsymbol{l^p}$ is complete; here p is fixed and $1 \leq p \leq +\infty$.

Proof:

Let (x_n) be any Cauchy sequence in the space $\boldsymbol{l^p}$, where $x_m = (\varepsilon_1{}^{(m)}, \varepsilon_2{}^{(m)}, \dots)$. Then for every $\varepsilon > 0$ there is an N such that for all $m, n > N$,

$$d(x_m, x_n) = \left(\sum_{j=1}^{\infty} |\varepsilon_j{}^{(m)} - \varepsilon_j{}^{(n)}|^p\right)^{\frac{1}{p}} < \varepsilon.$$

It follows that for every $j = 1,2 \dots$ we have

$$\left|\varepsilon_j{}^{(m)} - \varepsilon_j{}^{(n)}\right| < \varepsilon \;\; (m, n > N).$$

For a fixed j we see that $(\varepsilon_j{}^{(1)} - \varepsilon_j{}^{(2)})$ is a Cauchy sequence of numbers. It converges since $\boldsymbol{R}$ and $\boldsymbol{C}$ are complete.

With limits, $\xi_j{}^{(m)} \to \xi_j$ and $m \to \infty$, we define $x = (\xi_1, \xi_2, \dots)$ and show that $x \in l^p$ and $x_m \to x$.

$$\sum_{j=1}^{k} |\xi_j{}^{(m)} - \xi_j{}^{(n)}|^p < \varepsilon^p \qquad \text{for all } m, n > N, (k = 1,2 \dots).$$

Let $n \to \infty$;

$$\sum_{j=1}^{k} |\xi_j{}^{(m)} - \xi_j{}^{(n)}|^p \leq \varepsilon^p \qquad \text{for } m > N, (k = 1,2 \dots).$$

Let $k \to \infty$;

$$\sum_{j=1}^{k} \left|\xi_j{}^{(m)} - \xi_j{}^{(n)}\right|^p \leq \varepsilon^p \qquad \text{for } m > N.$$

This represents $[d(x_m, x)]^p$. Since (x_m) was an arbitrary Cauchy sequence in $\boldsymbol{l^p}$, this provides completeness of l^p, where $1 \leq p \leq \infty$.

2.4.3 Completeness of $\boldsymbol{l^\infty}$

The space $\boldsymbol{l^\infty}$ is complete.

Proof:

Let $x_m = \left(\xi_1{}^{(m)}, \xi_2{}^{(m)}, \dots\right)$ and (x_m) is any Cauchy sequence in the space $\boldsymbol{l^\infty}$. Then the metric on l^∞ is defined by

$$d(x, y) = sup_j\left|\xi_j - \eta_j\right| \qquad \text{where } x = (\xi_j) \text{ and } y = (\eta_j).$$

(x_m) is Cauchy, for any $\varepsilon > 0$ there is an N such that for all $m, n > N$,

$$d(x_m, x_n) = sup_j|\xi_j{}^{(m)} - \xi_j{}^{(n)}| < \varepsilon$$

$$\left|\xi_j{}^{(m)} - \xi_j{}^{(n)}\right| < \varepsilon \qquad m > N \quad \text{for every fixed } j.$$

From Thm. (2.3.3), if we convert $\varepsilon_j \to \xi_j$ as $m \to \infty$, the sequence becomes $\left(\xi_1{}^{(1)}, \xi_2{}^{(2)}, \dots\right)$. Using these infinitely many limits ξ_1, ξ_2 ... we define $x = (\xi_1, \xi_2 \dots)$ and show that $x \in l^\infty$ and $x_m \to x$. Now $n \to \infty$, we have

$$\left|\xi_j{}^{(m)} - \xi_j\right| \leq \varepsilon \qquad m > N$$

for $x_m = (\xi_j{}^{(m)}) \in l^\infty$, $|\xi_j{}^{(m)}| \leq k_m$ is for all j. k_m is a real number.

If we use the triangle inequality

$$\left|\xi_j\right| \leq \left|\xi_j - \xi_j{}^{(m)}\right| + \left|\xi_j{}^{(m)}\right| \leq \varepsilon + k_m \quad m > N.$$

This inequality is valid for all j. (ξ_m) is a bounded sequence of numbers. This implies that $x = (\xi_j) \in l^\infty$.Then

$$d(x_m, x) = sup_j|\xi_j{}^{(m)} - \xi_j| \leq \varepsilon.$$

It shows $x_m \to x$. x_m was an arbitrary Cauchy sequence, depend on that $\boldsymbol{l^\infty}$ is complete.

2.4.4 Completeness of $\boldsymbol{C}$

If a space consists of all convergent sequences $x = (\xi_j)$ of complex number with the metric which is induced from the space $\boldsymbol{l}^{\infty}$, the space is complete.

Proof:

C is a subspace of $\boldsymbol{l}^{\infty}$, and $\boldsymbol{C}$ is closed in $\boldsymbol{l}^{\infty}$. We consider any $x = (\xi_j) \in \overline{C}$, the closure of $\boldsymbol{C}$.

Let $x_n = (\xi_j{}^{(n)}) \in C$ such that $x_n \to x$. So given any $\varepsilon > 0$, there is an N such that for $n \geq N$ and all j we have

$$|\xi_j{}^{(n)} - \xi_j| \leq d(x_n, x) < \frac{\varepsilon}{3}, \quad \text{for } n = N \text{ and all } j.$$

The terms of $\xi_j{}^{(n)}$ form a convergent sequence when $x_n \in C$.

Then

$$|\xi_j{}^{(N)} - \xi_k{}^{(N)}| < \frac{\varepsilon}{3} \qquad\qquad (j, k \geq N).$$

Now triangle inequality holds for all j

$$|\xi_j - \xi_k| \leq \left|\xi_j - \xi_j{}^{(N)}\right| + \left|\xi_j{}^{(N)} - \xi_k{}^{(N)}\right| + \left|\xi_k{}^{(N)} - \xi_K\right| < \varepsilon \qquad k \geq N$$

It means $x = (\xi_j)$ is convergent. Since $x \in \overline{C}$ was arbitrary, this proves closedness of $\boldsymbol{C}$ in $\boldsymbol{l}^{\infty}$, and completeness of $\boldsymbol{C}$.

2.5 Normed Spaces, Banach Spaces

2.5.1 Introduction

A normed space is a vector space which is releated with a metric defined by norm.

A Banach space is complete metric space. An operator is a mapping from a normed space (X) to another normed space (Y). A functional is a mapping from a normed space (X) to a scalar field $(\boldsymbol{R}\ or\ \boldsymbol{C})$. If they take the vector space structure, and are continuous, they are called bounded linear operators (functionals). If and only if an operator is bounded, it is continuous.

$B(X, Y)$ is the set of all bounded linear operators from a normed space (X) to another normed space (Y).

2.5.2 Definition of Vector Space

A vector space(or linear space) over a field $\boldsymbol{K}$ is a nonempty set X of elements x, y together with two algebraic operations. $x, y\ldots$ called vectors.

These operations are called vector addition and multiplication.

i. Vector addition:

It is commutative and associative for all vectors.

$$x + y = y + x \tag{2.22}$$

$$x + (y + z) = (x + y) + z\ . \tag{2.23}$$

There exists a zero vector (0). Moreover, there exists a vector $(-x)$ for every vector.

$$x + 0 = x \tag{2.24}$$

$$x + (-x) = 0. \tag{2.25}$$

ii. Multiplication by scalars:

For all vectors x, y and scalars α, β;

$$\alpha(\beta x) = (\alpha\beta)x \tag{2.26}$$

$$1x = x \tag{2.27}$$

$$\alpha(x + y) = \alpha x + \alpha y \tag{2.28}$$

$$(\alpha + \beta)x = \alpha x + \beta x. \qquad \text{(Distributive laws)} \tag{2.29}$$

For addition is a mapping $\boldsymbol{X}\mathbf{x}\boldsymbol{X} \rightarrow \boldsymbol{X}$

For multiplication is a mapping $\boldsymbol{K}\mathbf{x}\boldsymbol{X} \rightarrow \boldsymbol{X}$

$\boldsymbol{K}$ is a scalar field of the vector space X. If $\boldsymbol{K} = \boldsymbol{R}$ (the field of real numbers), $\boldsymbol{X}$ is called a real vector space. If $\boldsymbol{K} = \boldsymbol{C}$ (the field of complex numbers), $\boldsymbol{X}$ is called complex vector space. We can denote the zero vector by θ.

$$0x = \theta \tag{2.30}$$

$$a\theta = \theta \tag{2.31}$$

$$(-1)x = -x. \tag{2.32}$$

2.5.3 Examples

a) Space $\boldsymbol{R^n}$(Euclidean space)

This is a real vector space with the two algebraic operations;

$$x + y = |\xi_1 + \eta_1 \dots + \xi_n + \eta_n| \tag{2.33}$$

$$\alpha x = (\alpha\xi_1, \dots, \alpha\xi_n). \tag{2.34}$$

b) Space $\boldsymbol{C^n}$

This is a complex vector space with the algebraic operations, $\alpha \in C$.

c) Space $\boldsymbol{l^2}$

It's a vector space with the algebraic operations as usual in connection with sequences;

$$(\xi_1, \xi_2, \dots) + (\eta_1, \eta_2, \dots) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \dots) \tag{2.35}$$

$$\alpha((\xi_1, \xi_2, \dots) = (\alpha\xi_1 + \alpha\xi_2, \dots) \tag{2.36}$$

$x = (\xi_j) \in l^2$ and $y = (\eta_j) \in l^2$ implies $x + y \in l^2$.

A subspace of a vector space X is a nonempty subset Y of X such that for all $y_1, y_2 \in Y$ and all scalars α, β we have $\alpha y_1 + \beta y_2 \in Y$.

A linear combination of vectors $X_1, \dots, X_m$ of a vector space $\boldsymbol{X}$;

$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$ where $\alpha_1, \alpha_2, \dots, \alpha_m$ are any scalars.

2.5.4 Definition of Linear independence and Linear dependence

For M is a set of vectors $x_1, \dots, x_r$ $(r \geq 1)$ in a vector space $\boldsymbol{X}$, then linear independence and dependence are defined by;

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = 0 \qquad (\alpha_1, \dots, \alpha_r \text{ are scalars}) \tag{2.37}$$

If this is the only r-tuple of scalars for last equation holds, the set M is said to be linearly independent. If M is not linearly independent, M is linearly dependent. If every nonempty finite subset of M is linearly independent, an arbitrary subset M of X is said to be linearly independent.

2.5.5 Definition of Finite and Infinite dimensional vector spaces

If there is a positive integer n such that X contains a linearly independent set of n vectors where as any set of $n+1$ or more vectors of X is linearly dependent, then vector space X is finite dimensional. n is dimension of $X.(dimX = n)$

X is not finite, means, X is finite.

2.5.6 Theorem: Dimension of a subspace

Let X be an n-dimensional vector space. Then any proper subspace Y of X has dimension less than n.

Proof:

If $n = 0$, then $X = \{0\}$ (no proper subspace)

If $dimY = 0$, then $Y = \{0\}$, and $X \neq Y$ implies $dimX \geq 1$.

$$dimY \leq dimX = n.$$

If $dimY$ were n, then Y would have a basis of n elements so that $X = Y$. Consequently any linear independent set of vectors in Y must have fewer than n elements.

$$dimY < n.$$

2.6 Normed Space, Banach Space

For a relation between algebraic and geometric properties of X, we define on X a metric d in a special way.

First of all, we introduce a norm which uses the algebraic operations of vector space. Then we obtain a metric with using norm. This leads to the normed space.

2.6.1 Definition of Normed space and Banach space

A normed space $\boldsymbol{X}$ is a vector space with a norm defined on it. A Banach space is a complete normed space. A norm ($||x||$) on a X is a real-valued function on $\boldsymbol{X}$ whose value at a $x \in X$.

i. $||x|| \geq 0$

ii. $||x|| \leftrightarrow x = 0$

iii. $||\alpha x|| = ||\alpha|| \; ||x||$

iv. $||x + y|| \leq ||x|| + ||y||$ (Triangle inequality)

x and y are arbitrary vectors in $\boldsymbol{X}$. α is any scalar. A metric d on X is denoted by;

$$d(x, y) = ||x - y|| \quad (x, y \in X) \text{ (metric induced by norm)}. \tag{2.38}$$

2.6.2 Euclidean space $\boldsymbol{R^n}$ and unitary space $\boldsymbol{C^n}$

They are Banach space with norm defined by

$$||x|| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{\frac{1}{2}} = \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}. \tag{2.39}$$

The metric is denoted by

$$d(x, y) = ||x - y|| = \sqrt{|\xi_1 - \eta_1|^2 + \cdots + |\xi_n - \eta_n|^2}. \tag{2.40}$$

2.6.3 Space $\boldsymbol{l^p}$

It's Banach space with norm and metric

$$||x|| = \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{\frac{1}{p}} \quad \text{and} \quad d(x, y) = ||x - y|| = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{\frac{1}{p}}. \tag{2.41}$$

2.6.4 Space $\boldsymbol{l^\infty}$

It's Banach space with norm

$$||x|| = sup_j |\xi_j|. \tag{2.42}$$

2.6.5 Lemma (Translation Invariance)

A metric d induced by a norm on a normed space $\boldsymbol{X}$ satisfies

$d(x+\alpha, y+\alpha) = d(x,y)$ and $d(\alpha x, \alpha y) = |\alpha| d(x,y)$ for all $x, y\alpha \in X$ and every scaler α.

Proof:

$$d(x+\alpha, y+\alpha) = \left||x+\alpha-(y+\alpha)|\right| = \left||x-y|\right| = d(x,y)$$

$$d(\alpha x, \alpha y) = \left||\alpha x - \alpha y|\right| = |\alpha| \left||x-y|\right| = |\alpha|\, d(x,y).$$

2.7 Further Properties of Normed Spaces

A subspace Y of a normed space $\boldsymbol{X}$ is a subspace of X considered as a vector space.

A subspace Y of a Banach space $\boldsymbol{X}$ is a subspace of X considered as a normed space.

2.7.1 Theorem: Subspace of a Banach space

A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

i. (x_n) is a sequence in a normed space x. (x_n) is convergent, if x contains an x ;

$$\lim_{n\to\infty}\left||x_n - x|\right| = 0. \qquad (2.43)$$

ii. (x_n) is Cauchy if for every $\varepsilon > 0$ there is N;

$$||x_m - x|| < \varepsilon \qquad \text{for all } m, n > N. \qquad (2.44)$$

We can associate with (x_k) the sequence (s_n) of partial sums

$$s_n = x_1 + x_2 + \cdots + x_n \quad n = 1, 2, \ldots \qquad (2.45)$$

If s_n is convergent, then $s_n \to s$ and $||s_n - s|| \to 0$.

The infinite series $s = \sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots$ is said convergent. s is the sum of the series.

2.8 Finite Dimensional Normed Spaces and Subspaces

2.8.1 Theorem: Closedness

Every finite dimensional subspace Y of a normed space X is closed in X.

2.8.2 Definition of Equivalent norms

A norm $||.||$ on a vector space X is said to be equivalent to a norm $||.||_0$ on X, if there are positive numbers a and b for all $x \in X$

$$\alpha\left||X|\right|_0 \leq \left||X|\right| \leq b||X||_0. \quad (2.46)$$

2.9 Linear Operators

2.9.1 Definition of Linear Operator

Assume that A is a linear operator

i. The domain $\mathcal{D}(A)$ of T is a vector space. The range $R(A)$ lies in a vector space over the same field.

ii. For all $x, y \in \mathcal{D}(A)$ and scalars α

$$T(x + y) = Tx + Ty \quad (2.47)$$

$$T(\alpha x) = \alpha Tx\,. \quad (2.48)$$

The null space of A $(N(A))$ is the set of all $x \in \mathcal{D}(A)$ such that $Ax = 0$.

2.9.2 Definition of Identity operator (I)

$I: X \rightarrow X$ is defined by $Ix = x$ for all $x \in X$.

2.9.3 Definition of Zero operator (0)

$0: X \rightarrow Y$ is defined by $0x = 0$ for all $x \in X$.

2.10 Inner Product Spaces

We can add and multiply vectors by scalars in a normed spaces. The length of a vector generalizes by norm. However, what is still missing in a general normed space, is an analogue of the familiar dot product.

$$a.b = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \quad (2.49)$$

$$|a| = \sqrt{a.a}. \quad (2.50)$$

The case for orthogonality;

$$a.b = 0. \quad (2.51)$$

The dot product and orthogonality can be generalized to arbitrary vector spaces. It leads to inner product spaces. Moreover, Hilbert spaces are complete inner product spaces.

Hilbert spaces is known to be the natural function spaces of Quantum mechanics. It is the space of square integrable ($L^2(R)$) complex-valued functions on $\boldsymbol{R}$, that is, of all functions $f: R \rightarrow K$ for which

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty. \tag{2.52}$$

Definition of inner product;

$$X =< x, y >. \tag{2.53}$$

Norm of a vector X;

$$||X|| =< x, x >^{\frac{1}{2}} \quad . \tag{2.54}$$

Orthogonality condition for vectors x and y;

$$< x, y > = 0. \tag{2.55}$$

If H is Hilbert space, then;

i. H's representations are a direct sum of a closed subspaces. It's a orthogonal complement.

ii. H has orthonormal sets and sequences.

iii. The Riesz representation is bounded linear functionals by inner products.

iv. A^* is a Hilbert-adjoint operator of a bounded linear operator A.

2.10.1 Definition of Inner Product Spaces and Hilbert Spaces

An inner product on $\boldsymbol{X}$ is a mapping of XxX into the scalar field $\boldsymbol{K}$.

For all vectors x, y, z and scalars α we have;

i. $< x + y, z > = < x, z > + < y, z >$

ii. $< \alpha x, y > = \alpha < x, y >$

iii. $< x, y > = < \overline{y, x} >$

iv. $< x, x \geq 0$, $< x, x > = 0$ when $x = 0$

v. Norm of x, $||x|| = \sqrt{< x, x >}$ (≥ 0)

vi. Metric on X, $d(x, y) = ||x - y|| = \sqrt{< x - y, x - y >}$.

Inner product spaces are normed spaces, Hilbert spaces are Banach spaces.

vii. If $\boldsymbol{X}$ is a real vector space, $< x, y > = < y, x >$ (symmetry)

viii. A norm on an inner product space satisfies;

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2). \quad \text{(parallelogram equality)} \qquad (2.56)$$

So all normed spaces are not inner product spaces.

2.10.2 Definition of Orthogonality

$$x, y \in X \quad \text{and} \quad < x, y > = 0. \qquad (2.57)$$

We also say that x and y are orthogonal and they are perpendicular to each other.

2.10.3 Euclidean space $\boldsymbol{R^n}$

The space $\boldsymbol{R^n}$ is a Hilbert space with inner product

$$< x, y > = \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_n\eta_n \qquad (2.58)$$

$$x = (\xi_j) = (\xi_1, \ldots, \xi_n) \quad \text{and} \quad y = (\eta_j) = (\eta_1, \ldots, \eta_n). \qquad (2.59)$$

Norm becomes;

$$||x|| = < x, x >^{\frac{1}{2}} = (\xi_1{}^2 + \cdots + \xi_n{}^2)^{\frac{1}{2}}. \qquad (2.60)$$

Euclidean metric defined by;

$$d(x, y) = ||x - y|| = < x - y, x - y >^{\frac{1}{2}}$$

$$= [(\xi_1 - \eta_1)^2 + \cdots + (\xi_n - \eta_n)^2]^{\frac{1}{2}}. \qquad (2.61)$$

2.10.4 Unitary space $\boldsymbol{C^n}$

The space $\boldsymbol{C^n}$ is given by

$$< x, y > = \xi_1\bar{\eta}_1 + \cdots + \xi_n\bar{\eta}_n. \qquad (2.62)$$

2.10.5 Space $L^2[a,b]$

The vector space of all continuous real-valued functions on $[a,b]$ forms a normed space $\boldsymbol{X}$. The norm is defined by

$$||x|| = \left(\int_a^b x(t)^2 dt\right)^{\frac{1}{2}}. \tag{2.63}$$

Inner product is defined by

$$< x, y > = \int_a^b x(t)y(t)dt. \tag{2.64}$$

When we keep $t \in [a,b]$ real, we consider complex-valued functions. For these functions, a complex vector space is formed. Inner product becomes;

$$< x, y > = \int_a^b x(t)\overline{y(t)}dt \qquad . \tag{2.65}$$

So for norm of X,

$$||x|| = < x, x > = \int_a^b x(t)\overline{x(t)}dt \qquad \text{and} \qquad x(t)\overline{x(t)} = |x(t)|^2. \tag{2.66}$$

Here $\overline{x(t)}$ is the complex conjugate of $x(t)$. Finally;

$$||x|| = (\int_a^b |x(t)|^2 dt)^{\frac{1}{2}}. \tag{2.67}$$

2.10.6 Hilbert sequence space l^2

For this space inner product and norm is defined by

$$< x, y > = \sum_{j=1}^{\infty} \xi_j \overline{\eta_j} \qquad ||x|| = < x, x >^{\frac{1}{2}} = \left(\sum_{j=1}^{\infty} |\xi_j|^2\right)^{\frac{1}{2}}. \tag{2.68}$$

2.10.7 Space l^p

The space $\boldsymbol{l^p}$ with $p \neq 2$ is not an inner product space, hence not a Hilbert space.

Proof:

The norm of $\boldsymbol{l^p}$ with $p \neq 2$ cannot be obtained from an inner product. It means that the norm does not satisfy the parallelogram equality.

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).$$

Let take $x = (1,1,0,0 \dots) \in l^p$ and $y = (1,-1,0,0 \dots) \in l^p$

$$||x|| = ||y|| = 2^{\frac{1}{p}}$$

$$||x+y|| = ||x-y|| = 2$$

$\boldsymbol{l^p}$ is complete. Hence l^p with $p \neq 2$ is a Banach space which is not a Hilbert space.

2.11 Properties of Inner Product Spaces

2.11.1 Lemma (Schwarz Inequality, Triangle Inequality)

First of all we have an equation;

$$||\alpha x||^2 = <\alpha x, \alpha x> = \alpha\bar{\alpha} <x, x> = |\alpha|^2 ||x||^2. \quad (2.69)$$

An inner product and corresponding norm satisfy the Schwarz inequality and the triangle inequality.

a) $|<x,y>| \leq ||x||\,||y||.$ (Schwarz Inequality) (2.70)

The meaning of equality sign is $\{x, y\}$ is a linearly dependent set.

Proof:

If $y = 0$, then $<x, 0> = 0$. Let $y \neq 0$ for every scalar α ;

$$0 \leq ||x - \alpha y||^2 = <x - \alpha y, x - \alpha y>.$$

$$= <x,x> - \bar{\alpha} <x,y> - \alpha[<y,x> - \bar{\alpha} <y,y>]$$

If we choose $\bar{\alpha} = \frac{<y,x>}{<y,y>}$, the expression in the brackets [ ] is zero.

$$0 \leq <x,x> - \frac{<y,x>}{<y,y>} <x,y> = ||x||^2 - \frac{|<x,y>|^2}{||y||^2}$$

where $<y,x> = \overline{<x,y>}$.

b) $||x+y|| \leq ||x|| + ||y||.$ (Triangle inequality) (2.71)

The equality sign means $y = 0$ or $x = cy$ (c real and ≥ 0)

Proof:

We have

$$||x+y||^2 = <x+y, x+y> = ||x||^2 + <x,y> + <y,x> + ||y||^2 .$$

By the Schwarz Inequality,

$|< x, y >| = |< y, x >| \leq ||x|| \, ||y||.$

And by the triangle equality,

$$||x + y||^2 \leq ||x||^2 + 2|< x, y >| + ||y||^2$$

$$\leq ||x||^2 + 2||x||||y|| + ||y||^2$$

$$= (|(|x|)| + |(|y|)|)^2 \ .$$

2.11.2 Lemma (Continuity of Inner Product)

If an inner product space, $x_n \to x$ and $y_n \to y$, then $< x_n, y_n > \to < x, y >$.

Proof:

$$|< x_n, y_n >| - < x, y > | = |< x_n, y_n >| - < x_n, y > + < x_n, y > - < x, y > |$$

$$\leq |< x_n, y_n - y >| + | < x_n - x, y > |.$$

$$\leq ||x_n||||y_n - y|| + ||x_n - x||||y|| \to 0$$

since $y_n - y \to 0$ and $x_n - x \to 0$ as $n \to \infty$.

2.11.3 Theorem: Subspace

Let Y be a subspace of a Hilbert space H. Then

a) *Y is complete if and only if Y is closed in H.*

b) *If Y is finite dimensional, then Y is complete.*

c) *If H is seperable, so is Y.*

Every subset of a seperable inner product space is seperable.

2.11.4 Theorem: Riesz Representation

Let H_1, H_2 be Hilbert spaces and $h : H_1 \mathrm{x} H_2 \to K$ a bounded sesquilinear form. Then h has a representation $h(x, y) = < Sx, y >$ where $S: H_1 \to H_2$ is a bounded linear operator. S is uniquely determined by h has norm

$$||S|| = ||h||. \tag{2.72}$$

2.12 Hilbert-Adjoint Operator

2.12.1 Definition of Hilbert-adjoint operator T^*

H_1 and H_2 are Hilbert spaces, where $A: H_1 \to H_2$ be a bounded linear operator. Then A^* is the Hilbert-adjoint operator of A.

$$A^*: H_2 \to H_1$$

$< Ax, y > = < x, A^*y >$ for all $x \in H_1$ and $Y \in H_2$.

2.12.2 Theorem: Existence

The Hilbert adjoint operator exists, is unique and a bounded linear operator with norm

$$||A^*|| = ||A||. \tag{2.73}$$

Proof:

A sesquilinear form on $H_2 \mathrm{x} \boldsymbol{H_1}$ is defined by

$$h(y, x) =< y, Tx >.$$

The inner product is sesquilinear and A is linear. Conjugate linearity of the form is

$$h(y, \alpha x_1 + \beta x_2) =< y, A(\alpha x_1 + \beta x_2) >$$

$$< y, \alpha A x_1 + \beta A x_2 > = \bar{\alpha} < y, Ax_1 > + \bar{\beta} < y, Ax_2 >$$

$$= \bar{\alpha} h(y, x_1) + \bar{\beta} h(y, x_2).$$

In fact h is bounded. From the Schwarz inequality,

$$|h(y, x)| = |< y, Ax >| \leq ||y|| ||Ax|| \leq ||A|| ||x|| ||y||$$

$$||h|| \leq ||A|| \qquad \text{and} \qquad ||h|| \geq ||A||$$

$$||h|| = sub_{\substack{x \neq 0 \\ y \neq 0}} \frac{|< y, Ax >|}{||y|| ||x||} \geq \frac{|< Ax, Ax >|}{||Ax|| ||x||} = ||A||$$

Finally;

$$||h|| = ||A||.$$

From Riesz representation for h;

$$h(x, y) = < A^* y, x >.$$

$A^*: H_2 \to H_1$ is a uniquely determined bounded linear operator with norm

$$||A^*|| = ||h|| = ||A||.$$

2.12.3 Lemma (Zero operator)

Let X and Y be inner product spaces and $Q: X \to Y$ a bounded linear operator.

Then:

a) $Q = 0$ if and only if $< Qx, y >= 0$ for all $x \in X$ and $y \in Y$.

b) If $Q: x \to X$, where X is complex, and $< Qx, x >= 0$ for all $x \in X$, then $Q = 0$.

Proof:

a) $Q = 0$ means $Qx = 0$ for all X.

$$< Qx, y > = < 0, y > = 0 < w, y > = 0.$$

b) $< Qv, v > = 0$ for every $v = \alpha x + y \ \in X$

$$< Q(\alpha x + y), \alpha x + y > = |\alpha|^2 < Qx, x > \ + \alpha < Qy, y > \ + \alpha < Qx, y >$$
$$+ \bar{\alpha} < Qy, x >.$$

By assumption, the first two terms are zero. Let $\alpha = 1$;

$$< Qx, y > \ + < Qy, x > = 0.$$

Let $\alpha = i$ and $\bar{\alpha} = -i$;

$$< Qx, y > \ - < Qy, x > = 0.$$

It is essential that X be complex.

2.12.4 Theorem: Properties of Hilbert-adjoint operators

Let H_1 ,H_2 be Hilbert spaces, $S: H_1 \to H_2$ and $A: H_1 \to H_2$ bounded linear operators and α any scalar. Then we have;

a) $< A^* y, x > = < y, Ax >$ $\quad (x \in H_1, y \in H_2)$

b) $(S + A)^* = \ S^* + A^*$

c) $(\alpha A)^* = \bar{\alpha}A^*$

d) $(A^*)^* = T$

e) $||T^*T|| = ||AA^*|| = ||A||^2$

f) $A^*A = 0 \quad \rightarrow \quad A = 0$

g) $(SA)^* = A^*S^*$ (assuming $H_2 = H_1$)

Proof:

a) $< Ax, y > = < x, A^*y >$ so

$$< A^*y, x > = \overline{< x, A^*y}> = \overline{< Ax, y >} =< y, Tx >$$

b) $< Ax, y > \quad = < x, A^*y >$ so $< x, (S + A)^*y > = < (S + T)x, y >$

$= < Sx, y > + < Ax, y > = < x, S^*y > +< x, A^*y >$

$= < x, (S^* + A^*)y >.$

Finally $(S + A)^*y = (S^* + A^*)y$ for all y.

c) $Q = (\alpha A)^* - \bar{\alpha}A^*$

$$< (\alpha A)^*y, x > = < y, (\alpha A)x > = < y, \alpha(Ax) >$$

$$= \bar{\alpha} < y, Ax > = \bar{\alpha} < A^*y, x > = < \bar{\alpha}A^*y, x >.$$

d) $(A^*)^*$ is written $A^{**} = A$ for all $x \in H_1$.

$< (A^*)^*x, y > = < x, A^*y > = < Ax, y >$ and $Q = (A^*)^* - A$.

e) $A^*A: H_1 \rightarrow H_1$, but $AA^*: H_2 \rightarrow H_2$. By the Schwarz inequality

$$||Ax||^2 = < Ax, Ax > = << A^*Ax, x > \leq ||A^*Ax||||x|| \leq ||A^*A|| \, ||x||^2$$

$$||A||^2 \leq ||A^*A|| \leq ||A^*|| \, ||\mathrm{A}|| = ||A||^{\,2}$$

replace A by A^*

$$||A^{**}A^*|| = ||A^*||^2 = ||A||^2.$$

f) $(6e)$ obtain $(6f)$

g) $< x, (SA)^*y > = < (SA)x, y > = < Ax, S^*y > = < x, A^*S^*y >.$

2.13 Self-Adjoint, Unitary and Normal Operators

2.13.1 Definition of Self-adjoint, Unitary and Normal operators

A is a bounded linear operator where $H \to H$ on a Hilbert space H then;

A is self-adjoint or Hermitian if $A^* = A$ [10].

A is unitary if A is bijective and $A = A^{-1}$.

A is normal if $AA^* = A^*A$.

When A is self-adjoint; $< Ax, y > = < x, A^*y >$ becomes $< Ax, y > = < x, Ay >$.

If A is self-adjoint or unitary, A is normal. However, a normal operator need not be self-adjoint or unitary.

2.13.2 Example (Matrices)

If a basis for $\boldsymbol{C^n}$ is given and a linear operator on $\boldsymbol{C^n}$ is represented by a certain matrix, then its Hilbert-adjoint operator is represented by the complex conjugate transpose of that matrix.

The inner product defined by

$< x, y > = x^T\bar{y}$, where x and y are written as column vectors.

T means transposition;

$$x^T = (\xi_1, \dots, \xi_n) . \tag{2.74}$$

Let T: $\boldsymbol{C^n} \to \boldsymbol{C^n}$ be a linear operator.

C and C^* are represented by two n-rowed square matrices, say, C and B.

$$(Bx)^T = x^T B^T \tag{2.75}$$

$$< Ax, y > = (Cx)^T\bar{y} = x^T C^T \bar{y} \text{ and} < x, A^*y > = x^T \bar{B}\bar{y}. \tag{2.76}$$

Consequently,

$$C^T = \bar{B} \; and \; B = \bar{C}^T. \tag{2.77}$$

i. For Representing matrices;

Hermitian if A is self-adjoint(Hermitian), Unitary if A is unitary,

Normal if A is normal.

ii. For a linear operator A: $\boldsymbol{R^n} \rightarrow \boldsymbol{R^n}$, representing matrices are;

Real symmetric if A is self-adjoint,

Orthogonal is A is unitary.

iii. For a square matrix $C = (a_{jk})$;

Hermitian if $\bar{C}^T = C$ (hence $\bar{a}_{kj} = a_{jk}$)

Skew-Hermitian if $\bar{C}^T = -C$ (hence $\bar{a}_{kj} = -a_{jk}$)

Unitary if $\bar{C}^T = C^{-1}$

Normal if $C\bar{C}^T = \bar{C}^T C$.

2.13.3 Theorem: Self-adjointness

Let $T: H \rightarrow H$ be a bounded linear operator on a Hilbert-space H. Then:

a) If A is self-adjoint, $< Ax, x >$ is real for all $x \in H$.

b) *If H is complex and $< Ax, x >$ is real for all $x \in H$, the operator A is self-adjoint.*

Proof:

a) If A is self-adjoint

$$\overline{< Ax, x >} = < x, Ax > = < Ax, x > \text{ for all } x.$$

Complex conjugate is equal to itself so that it is real.

b) If $< Ax, x >$ is real for all x, then

$$< Ax, x > = \overline{< Ax, x >} = \overline{< x, A^*x >} = < A^*x, x >$$

$$0 = < Ax, x > - < A^*x, x > = < (A - A^*)x, x >.$$

$A - A^* = 0$ so H is complex.

Chapter 3

SELF-ADJOINT EXTENSOINS OF THE OPERATORS

3.1 Introduction

In quantum mechanics, one of the important questions is to count the how many self-adjoint extensions of the operator admit. In order to determine the number of self-adjoint extensions of the operators, the best well-known reliable method which is introduced by von-Neumann is used. In this method the concept of deficiency indices is used, which is related with an ordered pair of positive integers (n_+, n_-). This mathematical review is introduced on a paper written by Ishibashi and Hosoya [5]. Let us start with some essential definitions.

Consider a Hilbert space H which represents by inner product $<.,.>$. An operator on H is a pair: a linear mapping $A: H \to H$ and its domain of definition $\mathcal{D}(A)$. The pair $(A, \mathcal{D}(A))$ can be written as A. If an operator A with $\mathcal{D}(A)$ densly defined (which means that any vector $v \in H$ can be approximated by vectors in $\mathcal{D}(A)$ as closely as possible) in H satisfies

$$< \phi, A\psi > = < A\phi, \psi > \qquad \forall\, \phi, \psi \in \mathcal{D}(A) \tag{3.1}$$

if this is the case then A is called symmetric or Hermitian ($A = A^\dagger$).

An operator A' is an extension of A if $\mathcal{D}(A) \subset \mathcal{D}(A')$ and $A'\psi = A\psi$, $\forall\, \psi \in \mathcal{D}(A)$. Extensions are obtained by relaxing the boundary conditions of the $\mathcal{D}(A)$. Consider sequences $\{\psi_n\} \subset \mathcal{D}(A)$ such that there exit limits

$$\lim_{n\to\infty} \psi_n =: \xi \in H \qquad and \quad \lim_{n\to\infty} A\psi_n =: \eta \in H. \tag{3.2}$$

If, for every such sequence, $\xi \in \mathcal{D}(A)$ and $A\xi = \eta$, then $(A, \mathcal{D}(A))$ is said to be closed. If nonclosed operator A has a closed extension then it is called closable. Furthermore, closure is defined if every closable operator has smallest extension.

Assume that $(A, \mathcal{D}(A))$ is a symmetric operator. Let us denote $\mathcal{D}(A^*)$ to be the set of all $\phi \in H$ for which there exist a $\chi \in H$ such that

$$(\phi, A\psi) = (\chi, \psi), \qquad \forall \psi \in \mathcal{D}(A). \tag{3.3}$$

Then, since $\mathcal{D}(A)$ is dense, χ is uniquely defined by $\phi \in \mathcal{D}(A^*)$ and Eq. (3.1). An operator $(A^*, \mathcal{D}(A^*))$ defined by $A^*\phi = \chi$ for every $\phi \in \mathcal{D}(A^*)$ is called the adjoint of $(A, \mathcal{D}(A))$. When the case is, A^* is a proper extension of A, then $\mathcal{D}(A^*)$ can be larger than $\mathcal{D}(A)$. If $\left(A^*, \mathcal{D}(A)\right) = (A, \mathcal{D}(A))$, then $(A, \mathcal{D}(A))$ is said to be self-adjoint.

Now let us consider the following examples which illustrate extensions of the symmetric operators to self-adjoint ones.

Let the Hilbert space $H = L^2(0,1)$, the set of square integrable functions in the interval $[0,1]$. Consider the momentum operator p;

$$p\psi = -i\frac{d}{dx}\psi \tag{3.4}$$

with

$$\mathcal{D}(p) = \{\psi | \psi(0) = \psi(1) = 0,\ \psi \in AC \subset [0,1]\}, \tag{3.5}$$

where $AC \subset [0,1]$ denotes the set of absolutely continuous functions on $[0,1]$, and their derivatives are in $L^2(0,1)$.

Let us first show that the momentum operator p is a symmetric operator

$$< \phi, p\psi > = < p\phi, \psi > \forall \phi, \psi \in \mathcal{D}(p). \tag{3.6}$$

Proof:

$$< \phi, p\psi > = \int_0^1 \phi^* p\psi dx = -\int_0^1 \phi^* i \frac{d}{dx}\psi dx = -i\int_0^1 \phi^* \frac{d\psi}{dx} dx$$

$$\frac{d}{dx}(\phi^*\psi) = \psi\frac{d\phi^*}{dx} + \phi^*\frac{d\psi}{dx} \rightarrow \phi^*\frac{d\psi}{dx} = \frac{d}{dx}(\phi^*\psi) - \psi\frac{d\phi^*}{dx}$$

$$< \phi, p\psi > = -i\int_0^1 \left[\frac{d}{dx}(\phi^*\psi) - \psi\frac{d\phi^*}{dx}\right] dx = -i\int_0^1 \frac{d}{dx}(\phi^*\psi)dx + i\int_0^1 \psi\frac{d\phi^*}{dx} dx$$

$$< \phi, p\psi > = -i(\phi^*\psi)|_0^1 + \int_0^1 \psi i \frac{d}{dx}\phi^* dx = \int_0^1 \psi\left(-i\frac{d}{dx}\phi\right)^* dx = < p\phi, \psi >.$$

However, if $\phi(x) = e^{ikx} \in \mathcal{D}(p^*)$ hence,

$$p^*\phi = \chi = -i\frac{d}{dx}\left(e^{ikx}\right) = -i^2ke^{ikx} = ke^{ikx} \in H. \tag{3.7}$$

Note that $p^*\phi = \chi = ke^{ikx} \notin \mathcal{D}(p)$ so that $\mathcal{D}(p^*) \supset \mathcal{D}(p)$ and is not self-adjoint.

Next, take up an operator p_α with same action as p in $D(p)$ with the domain

$$\mathcal{D}(p_\alpha) = \{\psi | \psi(0) = e^{i\alpha}\psi(1),\ \psi \in AC \subset [0,1]\}, \tag{3.8}$$

in which α is a real number. Obviously, this is an extension of p. For $\phi \in D(p_\alpha{}^*)$, there is $\chi \in H$, which is defined by

$$\forall\psi \in D(p_\alpha), (\phi, p_\alpha\psi) = (\chi, \psi),\ p_\alpha{}^*\phi = \chi. \tag{3.9}$$

Namely since,

$< \phi, p_\alpha\psi > = < p_\alpha{}^*\phi, \psi >$, which reads as

$$\int_0^1 \phi^*\left(-i\frac{d\psi}{dx}\right) dx = \int_0^1 dx (p_\alpha{}^*\phi)^*\psi \tag{3.10}$$

The Eq. (3.10) can be verified, if we integrate the LHS by the method of integration by parts;

$$LHS = \int_0^1 \phi^*\left(-i\frac{d\psi}{dx}\right) dx \tag{3.11}$$

Let take derivatives

$$u = \phi^* \rightarrow \; du = \frac{d\phi^*}{dx} dx$$

$$dv = -i\frac{d\psi}{dx} dx \rightarrow v = -i\psi.$$

$$\int_0^1 \phi^* \left(-i\frac{d\psi}{dx}\right) dx = -i\psi\phi^*|_0^1 + \int_0^1 \psi i \frac{d\phi^*}{dx} dx. \tag{3.12}$$

Note that the term $-i\psi\phi^*$ should vanish in order to satisfy Eq. (3.10). Therefore, $-i\psi\phi^*|_0^1 = -i\{\psi(1)\phi^*(1) - \psi(0)\phi^*(0)\} = 0$. Hence

$$\int_0^1 \phi^* \left(-i\frac{d\psi}{dx}\right) dx = \int_0^1 \psi i \frac{d}{dx}\phi^* dx \tag{3.13}$$

$$=\int_0^1 \psi(p_\alpha \phi^*) dx \tag{3.14}$$

$$=\int_0^1 \psi({p_\alpha}^* \phi)^* dx. \tag{3.15}$$

Recall from the imposed condition that

$$\psi(1)\phi^*(1) = \psi(0)\phi^*(0). \tag{3.16}$$

Using the boundary condition for ψ, in the above equation,

$$\psi(1)\phi^*(1) = e^{i\alpha}\psi(1)\phi^*(0). \tag{3.17}$$

$$\phi^*(1) = e^{i\alpha}\phi^*(0) = \left[e^{-i\alpha}\phi(0)\right]^* \tag{3.18}$$

hence, $\phi(1) = e^{-i\alpha}\phi(0) \rightarrow \phi(0)=e^{i\alpha}\phi(1)$.

This result implies that

$$D({p_\alpha}^*) = D(p_\alpha) \tag{3.19}$$

Hence, p_α is self-adjoint. Since α is arbitrary, it shows that p has infinitely many different self-adjoint extensions.

3.1.1 Definition of Deficiency subspaces

In order to count how many self-adjoint extensions of an operator has, the concept of deficiency indices is used. The deficiency subspaces $N_\pm$ is defined by

$$N_{+} = \{\psi \in D(A^{*})\ ,\ A^{*}\psi = Z_{+}\psi,\ ImZ_{+} > 0\} \tag{3.20}$$

$$N_{-} = \{\psi \in D(A^{*}),\ A^{*}\psi = Z_{-}\psi,\ ImZ_{-} < 0\} \tag{3.21}$$

The dimensions n_{+}, n_{-} are the deficiency indices of the operator A and will be denoted by the ordered pair (n_{+}, n_{-}). The ordered pair are not depend on the choice of $Z_{+}(Z_{-})$. They depends only on whether Z lies in the upper(lower) half complex plane.

Generally we consider $Z_{+} = i\lambda$ and $Z_{-} = -i\lambda$, where λ is an arbitrary positive constant and it is required for dimensional reasons. The deficiency indices are found by counting the number of solutions of $A^{*}\psi = Z\psi$; (for $\lambda = 1$),

$$A^{*}\psi \mp i\psi = 0 \tag{3.22}$$

that belong to the Hilbert space H. If solutions do not satisfy the square integrability condition (i.e $n_{+} = n_{-} = 0$), the operator A has a unique self-adjoint extension and it is self-adjoint. As a result, the operator A has a unique self-adjoint extension if and only if, the solutions which satisfy Eq. (3.22), don't belong to the Hilbert-space.

3.1.2 Theorem: Criteria for essentially self-adjoint operators

For an operator A with deficiency indices (n_{+}, n_{-}), *there are three possibilities:*

i. If $n_{+} = n_{-} = 0$ then A is self-adjoint. (necessary and sufficient condition)

ii. If $n_{+} = n_{-} = n \geq 1$, then A has infinitely many self-adjoint extensions and is parametrized by a unitart $n \times n$ matrix. (n^2 real parameters)

iii. If $n_{+} \neq n_{-}$; A has no self-adjoint extensions.

How can we decide the deficiency indices n_{+}, n_{-}?

We check the square integrability conditions for functions. In mathematics, a square-integrable function, also called a quadratically integrable function, is a real- or complex-valued measurable function for which the integral of the square of the absolute value is finite. Thus, if

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx \ < \infty, \tag{3.23}$$

then ψ is quadratically integrable on the real line $(-\infty, +\infty)$. The quadratically integrable functions form an inner product space namely the Hilbert space.

Chapter 4

APPLICATIONS

In this section, we are concerned with the self-adjoint extensions of the operators. The momentum and the Klein-Gordon operators will be considered as an application.

4.1 Momentum operator

The theorem of von-Neumann will be considered for momentum operator for three different intervals which describe three different physical situations.

Let us consider the Hilbert space is $H = L^2(a, b)$. The interval $[a, b]$ will take different values for each physical situation. The one dimensional momentum operator p is given by

$$p = -i\hbar \frac{\partial}{\partial x}. \tag{4.1}$$

Therefore, the operator A in the theorem will be replaced by the momentum operator p for the present problem.

Hence we have

$$p^*\psi_\pm \mp i\lambda\psi_\pm = 0 \tag{4.2}$$

Assume that $\lambda = \frac{\hbar}{c}$,

$$-i\hbar \frac{\partial}{\partial x}\psi_\pm = \pm i\frac{\hbar}{c}\psi_\pm \tag{4.3}$$

$$\frac{\partial}{\partial x}\psi_\pm = \mp\frac{1}{c}\psi_\pm \tag{4.4}$$

$$\int \frac{\partial\psi_\pm}{\psi_\pm} = \mp \int \frac{dx}{c} \tag{4.5}$$

$$\ln\psi_{\pm} = \mp\frac{x}{c} + C_{\pm} \tag{4.6}$$

$$\psi_{\pm}(x) = C_{\pm}e^{\mp\frac{x}{c}}. \tag{4.7}$$

Now, we have to the investigate the behaviour of this function for three different intervals $[a, b]$.

First, the whole real axis is considered in the interval $-\infty < x < \infty$. Then in the second case, the positive axis whose range is $0 < x < \infty$ will be taken into consideration. Finally, the range will be taken $0 < x < L$ on a finite interval.

4.1.1 The operator p on the whole real axis

This simply means that the interval will be

$$-\infty < x < \infty.$$

The condition for a self-adjointness is explained in the previous section. It is required to determine the deficiency indices (n_+, n_-) which requires to use the inner-product Eq. (3.23).

$$||\psi_{\pm}||^2 = \int_{-\infty}^{\infty}\psi\psi^* dx = \int_{-\infty}^{\infty}|\psi|^2 \tag{4.8}$$

$$= \int_{-\infty}^{\infty} {C_{\pm}}^2 e^{\mp\frac{2x}{c}}dx = \mp {C_{\pm}}^2\frac{c}{2}e^{\mp\frac{2x}{c}}|_{-\infty}^{+\infty} \rightarrow \infty. \tag{4.9}$$

None of the functions $\psi_{\pm}$ belongs to the Hilbert space and therefore the deficiency indices $(n_+ = 0,\ n_- = 0)$. Therefore, the momentum operator p in the considered interval is self-adjoint.

4.1.2 The operator p on the positive semi-axis

In this case the interval is

$$0 \leq x < \infty.$$

$$||\psi_+||^2 = \int_0^{\infty}\psi^* dx = {C_+}^2\int_0^{\infty} e^{-\frac{2x}{c}}dx = \frac{-c{C_+}^2}{2}e^{-\frac{2x}{\hbar}}|_0^{+\infty} < \infty \tag{4.10}$$

$$||\psi_-||^2 = \int_0^{\infty}\psi^* dx = {C_-}^2\int_0^{\infty} e^{+\frac{2x}{c}}dx = \frac{c{C_-}^2}{2}e^{+\frac{2x}{c}}|_0^{+\infty} \rightarrow \infty \tag{4.11}$$

Calculations has revealed that only ψ_+ belongs to the Hilbert space and therefore the deficiency indices are $(1,0)$. According to the von-Neumann theorem, the momentum operator p has no self-adjoint extension.

4.1.3 The operator p on the finite interval

In this case interval is

$$0 \leq x < L.$$

$$||\psi_+||^2 = \int_0^L \psi^* dx = {C_+}^2 \int_0^L e^{-\frac{2x}{c}} dx = \frac{-c{C_+}^2}{2} e^{-\frac{2x}{c}}|_0^L < \infty \qquad (4.12)$$

$$||\psi_-||^2 = \int_0^L \psi^* dx = {C_-}^2 \int_0^L e^{+\frac{2x}{c}} dx = \frac{+c{C_-}^2}{2} e^{+\frac{2x}{c}}|_0^L < \infty \qquad (4.13)$$

Calculations has revealed that both of $\psi_+$and ψ_- belong to the Hilbert space and therefore the deficiency indices are $(1,1)$. According to the von-Neumann theorem, the momentum operator p has many self-adjoint extensions.

4.2 Klein Gordon Fields

As a second example we consider the spatial part of the Klein-Gordon massless wave equation in a curved geometry. The massive Klein-Gordon wave equation is given by

$$(\square - M)\psi = 0 \qquad (4.14)$$

where M stands for the mass. Since our focus on massless wave, without loss of generality we take it as $m = 0$. The symbol $\square$ stands for the dalambertian operator defined by

$$\square = \nabla^\mu \nabla_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu\right) \qquad (4.15)$$

in which, $g^{\mu\nu}$ is the metric tensor in contravariant form and g is the determinant of the metric.

As a curved geometry we consider the following metric

$$ds^2 = -\left[1 - \frac{2m}{r}\right] dt^2 + \frac{dr^2}{\left[1-\frac{2m}{r}\right]} + r^2(d\theta^2 + \sin\theta^2 \, d\varphi^2), \qquad (4.16)$$

this metric is known as the negative mass Schwarzchild solution. Schwarzchild solution for $m > 0$ is known as the static black-hole solution in general relativity. Black holes are known to be the mysterious objects predicted by Einstein's theory of relativity that even light can not escape from its horizon located at $r = 2m$. This spherically symmetric black hole has a central curvature singularity at $r = 0$. In classical general relativity singularities are defined as the geodesics incompleteness for the timelike and null geodesics. Particles are following timelike, while the photons are following null geodesics. However, this singularity in the case of black holes is hidden by horizon. This picture changes completely if the mass is negative. In this case, there is no black hole and therefore the singularity at $r = 0$ becomes naked. This type of singularities are called naked singularities.

As an application of self-adjoint extension of an operators, we wish to consider the propagation of quantum fields obeying the Klein-Gordon equation to see whether or not the quantum field falls into the singularity or not. This way of analysing the singularities helps us to understand whether the classical singularity is quantum mechanically regular or not. In other words, the singularity will be analysed in quantum mechanical point of view. To achieve this goal, the notion of self-adjoint extension of the spatial part of the Klein-Gordon operator will be used, and we will try to count the number of self-adjoint extension with the help of von-Neumann theorem.

In order to understand better the concept of self-adjoint extension of an operator, let us first investigate, the Klein-Gordon equation for a free particle.

We know that Schrödinger equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi \tag{4.17}$$

where $V(r)$ is an effective potential and, since we are interested with free particles, we take the potential $V(r) = 0$.

Then Eq. (4.17) becomes,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi. \tag{4.18}$$

The right hand side of Eq. (4.18) is known as the Hamiltonian of the system that can be written as,

$$H = -\frac{\hbar^2}{2m}\nabla^2 = \frac{p^2}{2m} \tag{4.19}$$

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{4.20}$$

where p is momentum.

$\psi(\vec{r},0)$ is the wave corresponding to the initial state and, $\psi(\vec{r},t)$ is the wave corresponding to some later time. $T(t)$ is the temporal part of wave function.

$$\psi(\vec{r},t) = \psi(\vec{r},0)T(t). \tag{4.21}$$

If we write this equation into Eq. (4.20), we find

$$TH\psi(\vec{r},0) = i\hbar\psi(\vec{r},0)\frac{dT}{dt} \tag{4.22}$$

$$\frac{1}{\psi(\vec{r},0)}H\psi(\vec{r},0) = \frac{i\hbar}{T}\frac{dT}{dt} = E \tag{4.23}$$

where E is a constant.

Finally we have two equations

$$\begin{cases} H\psi(\vec{r},0) = E\psi(\vec{r},0) \\ \quad i\hbar\frac{\partial T}{\partial t} = ET \end{cases} \tag{4.24}$$

If we solve second part of Eq. (4.24),

$$\frac{dT}{T} = \frac{E}{i\hbar}dt \tag{4.25}$$

$$\ln T = -\frac{iE}{\hbar}(t) + cons \tag{4.26}$$

We find solution when $t_0 = 0$

$$T = e^{-\frac{iEt}{\hbar}}. \tag{4.27}$$

4.2.1 Theorem:

If U_a is the eigenfunction of the operator A, and a is the eigenvalue of the operator. Then,

$$\begin{gathered} AU_a = aU_a \\ f(A)U_a = f(a)U_a \end{gathered} \tag{4.28}$$

Proof:

If $AU_a = aU_a$, then $f(A)U_a = f(a)U_a$ is satisfied.

Let prove,

$$f(A)U_a = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n U_a$$

We expand the Taylor series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (A^n U_a)$$

$$A^n U_a = A^{n-1} A U_a = A^{n-1} a U_a$$

$$a A^{n-2} A U_a = \cdots = a^n U_a$$

Finally,

$\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} a^n\right) U_a = f(a)U_a.$

If we apply the theorem to the first part of the Eq. (4.24) then,

$$H\psi_E(\vec{r},0) = E\psi_E(\vec{r},0)$$

$$f(H)\psi_E = f(E)\psi_E$$

$$\psi(\vec{r},t) = e^{-\frac{iEt}{\hbar}}\psi_E(\vec{r},0) = e^{-\frac{iHt}{\hbar}}\psi_E(\vec{r},0)$$

$$\psi_E(\vec{r},t) = e^{-\frac{iHt}{\hbar}}\psi_E(\vec{r},0). \tag{4.29}$$

The above result can be written in terms of operator A such that,

$$\psi_E(\vec{r},t) = e^{-i\sqrt{A_E}t}\psi_E(\vec{r},0). \tag{4.30}$$

If the operator A is not essentially self-adjoint, the future time evolution of the wave function is ambigous. The reason is that, we do not know which extension of the operator A_E is used. This physically implies that the future time evolution of the wave can not be predicted. Hence, the classically singular spacetime remains quantum singular as well.

But, if the operator A_E has a unique self-adjoint extension, the future time evolution of the wave can be predicted with the given initial condition $\psi(\vec{r},0)$. Then we say that the classically singular spacetime is quantum mechanically regular.

Our aim in this thesis to investigate whether the spatial part of the Klein-Gordon operator admit the unique self-adjoint extensions or not. For the massless case the Klein-Gordon operator is

$$\Box\psi = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\right]\psi = 0. \tag{4.31}$$

The covariant form of the negative mass Schwarzchild solution is given by

$$g_{\mu\nu} = \begin{Bmatrix} -f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin\theta^2 \end{Bmatrix} \tag{4.32}$$

and its contravariant form is given by

$$g^{\mu\nu} = \begin{Bmatrix} \frac{-1}{f(r)} & 0 & 0 & 0 \\ 0 & f(r) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin\theta^2} \end{Bmatrix} \tag{4.33}$$

where $f(r) = 1 + \frac{2|m|}{r}$ and the determinant g is calculated and found as

$$g = -r^4 \sin\theta^2. \tag{4.34}$$

In order to find the spatial part of the Klein-Gordon equation, we write Eq. (4.31) for the metric in the following form

$$\frac{\partial^2\psi}{\partial t^2} = -A\psi \tag{4.35}$$

where A is the spatial part of the Klein-Gordon operator.

Eq. (4.31) can be written explicitly as

$$\Box\psi = \frac{1}{\sqrt{-g}}\left\{\frac{\partial}{\partial t}\left[\sqrt{-g}g^{tt}\frac{\partial\psi}{\partial t}\right] + \frac{\partial}{\partial r}\left[\sqrt{-g}g^{rr}\frac{\partial\psi}{\partial r}\right]\right\}$$

$$+\frac{1}{\sqrt{-g}}\left\{\frac{\partial}{\partial\theta}\left[\sqrt{-g}g^{\theta\theta}\frac{\partial\psi}{\partial\theta}\right] + \frac{\partial}{\partial\varphi}\left[\sqrt{-g}g^{\varphi\varphi}\frac{\partial\psi}{\partial\varphi}\right]\right\}=0 \tag{4.36}$$

Substituting the related functions we have,

$$\frac{1}{r^2\sin\theta}\left\{\partial_t[r^2\sin\theta\frac{-1}{f(r)}\partial_t]\psi + \partial_r[r^2\sin\theta\, f(r)\partial_r]\psi\right\}$$

$$+\frac{1}{r^2\sin\theta}\left\{\partial_\theta\left[r^2\sin\theta\frac{1}{r^2}\partial_\theta\right]\psi + \partial_\varphi[r^2\sin\theta\frac{1}{r^2\sin\theta^2}\partial_\varphi]\psi\right\} = 0 \tag{4.37}$$

$$\frac{1}{r^2\sin\theta}\left\{r^2\sin\theta\frac{-1}{f(r)}\partial_{tt}\psi + \sin\theta\,\partial_r[r^2f(r)\partial_r]\psi\right\}$$

$$+\frac{1}{r^2\sin\theta}\left\{\left[\cos\theta\frac{\partial\psi}{\partial\theta} + \sin\theta\frac{\partial^2\psi}{\partial\theta^2}\right] + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right\}=0 \tag{4.38}$$

$$\frac{1}{r^2\sin\theta}\left\{\frac{-r^2\sin\theta}{f(r)}\frac{\partial^2\psi}{\partial t^2} + \sin\theta\frac{\partial}{\partial r}[r^2f(r)]\frac{\partial\psi}{\partial r}\right\}$$

$$+\frac{1}{r^2\sin\theta}\left\{r^2f(r)\frac{\partial^2\psi}{\partial r^2}\sin\theta + \cos\theta\frac{\partial\psi}{\partial\theta} + \sin\theta\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right\} = 0 \tag{4.39}$$

rewriting the above equation as

$$\frac{-1}{f(r)}\frac{\partial^2\psi}{\partial t^2} + \frac{2f(r)}{r}\frac{\partial\psi}{\partial r} + f(r)'\frac{\partial\psi}{\partial r} + f(r)\frac{\partial^2\psi}{\partial r^2}$$

$$+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2\sin\theta^2}\frac{\partial^2\psi}{\partial\varphi^2} = 0 \tag{4.40}$$

Seperating the temporal part, we have

$$\frac{\partial^2\psi}{\partial t^2} = \frac{2f(r)^2}{r}\frac{\partial\psi}{\partial r} + f(r)f(r)'\frac{\partial\psi}{\partial r} + f(r)^2\frac{\partial^2\psi}{\partial r^2} + \frac{f(r)\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} + \frac{f(r)}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{f(r)}{r^2\sin\theta^2}\frac{\partial^2\psi}{\partial\varphi^2} \tag{4.41}$$

This equation can be simplified further to have,

$$\frac{\partial^2\psi}{\partial t^2} = \left[\frac{2f(r)^2}{r}\frac{\partial}{\partial r} + f(r)f(r)'\frac{\partial}{\partial r} + f(r)^2\frac{\partial^2}{\partial r^2}\right]\psi$$
$$+\left[\frac{f(r)\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{f(r)}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{f(r)}{r^2\sin\theta^2}\frac{\partial^2}{\partial\varphi^2}\right]\psi \tag{4.42}$$

If this equation is compared with Eq. (4.35) then, one can easily read the spatial operator A to be

$$A = -f^2(r)\frac{\partial^2}{\partial r^2} - f(r)\left[\frac{2f(r)}{r} + \frac{\partial f(r)}{\partial r}\right]\frac{\partial}{\partial r} - \frac{f(r)}{r^2}\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right] - \frac{f(r)}{r^2\sin\theta^2}\frac{\partial^2}{\partial\varphi^2} \tag{4.43}$$

The next step is to investigate this operator by using the theorem of von-Neumann explained in Chapter 3. The key point in the theorem is to apply the Kernel

$$(A^* \mp i)\psi = 0 \tag{4.44}$$

where A^* is the extension of the operator A. According to the theorem if the spatial part of the Klein-Gordon equation has a unique self-adjoint extension then the solution to the Eq. (4.44) must not belong to the Hilbert space. In other words, solutions must not satisfy the square integrability conditions that is,

$$\left||\psi|\right| = \int \psi\psi^* dx \to \infty. \tag{4.45}$$

Since the singularity is at $r = 0$, our aim will be look for a seperable solution to the equation $A^*\psi \mp i\psi = 0$, and analyse whether the radial part of the operator and its solution is essentially self-adjoint or not.

We assume a separable solution in the form of

$$\psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi). \tag{4.46}$$

$$\left[\frac{2f(r)^2}{r} + f(r)f(r)'\right]Y(\theta,\varphi)\frac{\partial R}{\partial r} + f(r)^2 Y(\theta,\varphi)\frac{\partial^2 R}{\partial r^2} + \frac{\cos\theta f(r)R}{r^2\sin\theta^2}\frac{\partial Y(\theta,\varphi)}{\partial\theta}$$
$$+\frac{f(r)}{r^2}R\frac{\partial^2 Y(\theta,\varphi)}{\partial\theta^2} + \frac{f(r)R}{r^2\sin\theta^2}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2} \mp i\psi = 0 \tag{4.47}$$

$$Y(\theta,\varphi)\left[\left(\frac{2f(r)^2}{r}+f(r)f(r)'\right)\frac{\partial R}{\partial r}+f(r)^2\frac{\partial^2 R}{\partial r^2}\right]$$

$$+\frac{f(r)}{r^2}R\left[\frac{\cos\theta}{\sin\theta}\frac{\partial Y(\theta,\varphi)}{\partial\theta}+\frac{\partial^2 Y(\theta,\varphi)}{\partial\theta^2}+\frac{1}{\sin\theta^2}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2}\right]\mp i\psi=0 \tag{4.48}$$

We know $f(r)=1+\frac{2m}{r}$ then $f(r)'=\frac{-2m}{r^2}$;

$$\left\{\frac{1}{R}\left[\left(\frac{2f(r)^2}{r}+f(r)f(r)'\right)R'+f(r)^2R''\right]+\frac{f}{r^2}\frac{1}{Y(\theta,\varphi)}\left[\frac{\cos\theta}{\sin\theta}\frac{\partial Y(\theta,\varphi)}{\partial\theta}+\frac{\partial^2 Y(\theta,\varphi)}{\partial\theta^2}+\right.\right.$$

$$\left.\left.\frac{1}{\sin\theta^2}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2}\right]\right\}\mp i=0 \tag{4.49}$$

$$\left\{f^2\frac{\partial^2 R}{\partial r^2}+\left(\frac{2f^2}{r}+ff'\right)\frac{\partial R}{\partial r}+R\left(\frac{-fl(l+1)}{r^2}\right)\right\}\mp iR=0 \tag{4.50}$$

Finally, the radial part of Klein-gordon equation

$$\frac{\partial^2 R}{\partial r^2}+\left(\frac{2}{r}+\frac{f'}{f}\right)\frac{\partial R}{\partial r}-\left(\frac{l(l+1)}{fr^2}\pm\frac{i}{f^2}\right)R=0 \tag{4.51}$$

$$\frac{\partial^2 R}{\partial r^2}+\left(\frac{2}{r}-\frac{2m}{r^2}\frac{1}{1+\frac{2m}{r}}\right)\frac{\partial R}{\partial r}-\left(\frac{l(l+1)}{r^2+2mr}\pm\frac{i}{(1+\frac{2m}{r})^2}\right)R=0 \tag{4.52}$$

$$\frac{\partial^2 R}{\partial r^2}+\left(\frac{2}{r}-\frac{2m}{r^2+2mr}\right)\frac{\partial R}{\partial r}-\left(\frac{l(l+1)}{r^2+2mr}\pm\frac{ir^2}{(r+2m)^2}\right)R=0 \tag{4.53}$$

where $l(l+1)$ is a seperability constant. Since the singularity is at $r=0$, we need to find the behaviour of the metric near $r\to 0$, which leads, $f(r)=1+\frac{2m}{r}\cong\frac{2m}{r}$, then the metric becomes

$$ds^2\cong-(\frac{2m}{r})dt^2+\frac{dr^2}{(\frac{2m}{r})}+r^2(d\theta^2+\sin\theta^2\,d\varphi^2). \tag{4.54}$$

According to the limiting values and assuming for $l=0$ case which corresponds to the S-wave, the Eq. (4.53) simplies to

$$\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}=0 \tag{4.55}$$

whose solution is given by

$$R(r)=C_1\ln r+C_2 \tag{4.56}$$

In which C_1 and C_2 are integration constants. The next step is to investigate the square integrability of the solution near $r = 0$. The squared norm for the metric Eq. (4.54) is given by,

$$||R||^2 = \int r^2 \frac{|R|^2}{f} dr. \tag{4.57}$$

$$||R||^2 = \int_0^{cons} r^2 \frac{|C_1 \ln r|^2}{f} dr + \int_0^{cons} r^2 \frac{|C_2|^2}{f} dr \tag{4.58}$$

where $f = \frac{2m}{r}$.

$$||R||^2 = \frac{{C_1}^2}{2m} \int_0^{cons} r^3 (\ln r)^2 \, dr + \frac{{C_2}^2}{2m} \int_0^{cons} r^3 dr \tag{4.59}$$

$$||R||^2 = \frac{{C_1}^2}{2m} \left[\frac{1}{32} r^4 8(\ln(r))^2 - \frac{4\ln(r) r^4}{32} + \frac{r^4}{32}\right] + \frac{{C_2}^2}{2m}\left[\frac{r^4}{4}\right] \tag{4.60}$$

$$||R||^2 = r^4 \left[\frac{{C_1}^2}{8m} \ln(r)^2 - \frac{{C_1}^2 \ln(r)}{16m} + \frac{{C_1}^2}{64m} + \frac{{C_2}^2}{8m}\right] \tag{4.61}$$

The above integral as $r \to 0$ is finite. That is to say $\int r^2 \frac{|R|^2}{f} dr < \infty$.

This implies that the solution $R(r)$ is square integrable and hence the spatial operator of the Klein-Gordon equation has an extension.

According to the von-Neumann defficiency indices $n = 1$. The physical meaning of this result is that, the classical naked singularity of a negative mass Schwarzchild solution is quantum mechanically singular as well.

The use of the concept of the extensions of the self-adjoint operators in analysing the singularities is used succesfully in 3 dimensional [6,7] geometries as well.

Chapter 5

CONCLUSION

In this thesis, application of self-adjoint extensions of some of the operators are investigated. In the analysis, the theorem proposed by von-Neumann is used.

After giving a review of mathematical background of the topic in chapter 2, we describe the theorem of von-Neumann in chapter 3. The main idea of the theorem is to investigate operator with the Kernel $A^*\psi \mp i\lambda\psi = 0$, and counting the number of solutions that belongs to the Hilbert space which is a function space of L^2. If the squared-norm of the solution do not belong to Hilbert space (i.e $\int_{-\infty}^{\infty} \psi\psi^* dx \to \infty$) then the deficiency indices $n_+ = n_- = 0$. According to the theorem this means that, the operator A is self-adjoint and possesses unique extension. However, if $n_+ = n_- = n \geq 1$, then the operator A has infinitely many self-adjoint extensions. This particular case is verified if the squared-norm, $\int_{-\infty}^{\infty} \psi\psi^* dx < \infty$.

First, the momentum operator is considered for three different physical situations. Three different physical situations are obtained by bounding the interval of the position(x). The results obtained for this problem are; The momentum operator p, has a unique self-adjoint extension on the whole real axis (i.e $-\infty \leq x < \infty$). If the position interval is bounded on the positive semi-axis (i.e $0 \leq x < \infty$), then the dediciency indices are $(1,0)$ and therefore the momentum operator p has no self-adjoint extension. As a third example the momentum operator p is considered on a finite interval (i.e $0 \leq x \leq L$). Analysis has revealed that the deficiency indices

$(1,1)$. From the von-Neumann theorem this result indicates that momentum operator p has infinitely many self-adjoint extensions.

Finally, we consider the Klein-Gordon equation in the background of negative mass Schwarchild spacetime. This spacetime admits naked singularity at $r = 0$. Classically, this spacetime is singular. However, it is of interest whether this spacetime remains regular against the propagation of waves obeying the Klein-Gordon equation. Hence, the Klein-Gordon equation is written by seperating the temporal part (i.e $\frac{\partial^2 \psi}{\partial t^2} = -A\psi$) and the spatial operator A is considered.

As was given in the thesis, the solution of Schrödinger equation for free particles is given by $\psi_E(\vec{r}, t) = e^{-i\sqrt{A_E}t}\psi_E(\vec{r}, 0)$. Here A_E denotes the spatial part of Klein-Gordon equation. If we prove that the spatial operator A_E has a unique self-adjoint extension, then the future time evolution of the wave can be predicted. Our analysis has revealed that the deficiency indice is $n = 1$. Hence, the spatial part of the Klein-Gordon equation has infinitely many self-adjoint extension. The conclusion is that, the classically singular spacetime remains quantum mechanically singular as well.

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