Series Approximate Analytical Solution of Fractional Partial Differential Equations

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Eastern Mediterranean University
August 2021
Gazimağusa, North Cyprus

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ABSTRACT

The primary objective of this thesis work is the presentation of a new iterative procedure to achieve both series approximate solution and analytical solution of positive non-integer order partial differential equations. The order of the derivative is considered according to Caputo's assumption. This iterative procedure is called Aboodh transform iterative method.

The Aboodh transform iterative method is a combination of the new iterative method with the Aboodh transform. The new iterative method was introduce as important tool to linearize all the associated nonlinear terms since the Aboodh transform cannot handle the nonlinear terms.

Several examples and cases are examined. The solutions obtained were compared with solutions obtained by other existing methods in literature. Also, the solutions reveals that the Aboodh transform iterative procedure is less computational involving and requires no restrictive assumption, Lagrange multipliers and Adomian polynomial.

The software used to implement the Aboodh transform iterative procedure are LaTex, MATHEMATICA 10.0 and MATLAB R2021.

Keywords: Integral transform, Aboodh transform iterative method, New iterative method, Fractional biological population model, Partial derivative.

ÖZ

Bu tez çalışmasının temel amacı, pozitif ve tamsayı olmayan mertebeden kısmi difernsiyel denklemlerin hem seri yaklaşık çözümünü hem de analitik çözümünü elde etmek için yeni bir iteratif prosedürün sunulmasıdır. Türevin mertebesini belirlemede Caputo'nun varsayımı dikkate alınmıştır. Bu yinelemeli prosedüre "Aboodh dönüşümü yinelemeli yöntemi" denmektedir.

Aboodh dönüşümü yinelemeli yöntemi, Aboodh dönüşümü ile yeni yinelemeli yöntemin bir birleşimidir. Aboodh dönüşümünün doğrusal olmayan terimler için çalışmadığından dolayı, yeni iteratif metod doğrusal olmayan terimlerin doğrusallaştırması özelliğiyle önemli bir araç olarak sunulmaktadır.

Bu tezde, bazı örnekler ve çeşitli vakalar değerlendirilmiştir. Elde edilen sonuçlar, literatürde kullanılan diğer metodlar ile karşılaştırılmıştır. Ayrıca, elde edilen çözümler Adoodh dönüşümü yinelemeli metodun daha az hesap gerektirdiğini ve daha kısıtlı varsayımlar kullanıldığını ortaya koymuştur.

Son olara, Aboodh dönüşümü yinelemeli prosedürünü uygulamak için LaTex, MATHEMATICA 10.0 ve MATLAB R2021 yazılımları kullanılmıştır.

Anahtar Kelimeler: İntegral dönüşüm, Aboodh dönüşümü yinelemeli yöntemi, Yeni yinelemeli yöntem, Kesirli biyolojik popülasyon modeli, Kısmi türev.

This thesis work is dedicated to GOD Almighty the creator of the universe

To Him be the glory, honor and power forever and ever

(Amen).

ACKNOWLEDGMENTS

I appreciate the grace of almighty God and that of his Son Jesus Christ throughout my academic pursuit, counting me worthy of a doctoral degree in Mathematics. All glory belongs to God.

Also, my profound acknowledgement goes to my supervisor Prof. Dr. Nazim I. Mahmudov for allowing me to be myself throughout the research period, your encouragement and support is highly appreciated. My gratitude also goes to my jury members Prof. Dr. Sonuç Zorlu Oğurlu and Assoc. Prof. Dr. Suzan Cival Buranay for their valuable contributions to the success of this research work.

I will like to express my appreciation to Institute of Graduate Studies and Research, also, the Department of Mathematics for granting me scholarship and research assistant position respectively with which I was able to complete my studies.

My appreciation also goes to my parents; Evang. Edward Ojo and Evang. Bimpe Ojo. I am forever grateful for their consistent prayers and the foundational education which I built upon.

I cannot forget the motivation, love and support I enjoyed from my darling wife Omoboyede Ojo during the period of my doctoral studies. Also my Deborah you are the best for your endurance and patience during these period and Daniel who also joined in the pursuit, you all are highly appreciated. I love you all, you all are the best.

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INTRODUCTION

1.1 Motivation

Admittedly, large number of concrete occurrence in engineering and science are generally modelled using positive non-integer order integral and differential equations. Positive non-integer derivative attracts huge interest because it provides better and more realistic models than the integer derivative [1-6]. However, obtaining solutions for this class of differential equations is a major task. Several approaches exists in literature for solving differential equations with fractional order, the defects with most of these methods are that they are complex and they require lots of computational effort.

1.2 Literature Review

Several methods have been proposed in numerious sphere of engineering and science to realize the solutions of fractional differential equation. These methods often appear in research literature. The discussion of some of these methods are presented in this section.

George Adomian developed the Adomian decomposition method in the 1970's [7]. The Adomian decomposition method makes use of Adomian polynomials to decompose the nonlinear term of the fractional differential equation [8]. This method is classified as an iterative technique and provides solutions in a fast converging infinite series form, it is straightforward in principle but requires difficult calculations of Adomian polynomials [9-10].

Variational iteration method was introduced by Shou et al. [11]. This method is capable of constructing correction functional (i.e., correction iteration formula) for the fractional differential equation using the general Lagrange multiplier. This method successfully remove the difficulty of calculating the Adomian polynomials [12].

Homotopy analysis method was introduced by Liao S.J. in his doctoral dissertation [13]. This method utilizes the homotopy approach from topology to present a convergence series form solution for fractional differential equation [14]. However, the method requires an embedding parameter.

The integral transform is also an important tool used to analyze solution of fractional differential equations [15-16]. The use of integral transform for providing solutions of problems in science and engineering can be traced to the work of P.S. Laplace on probability theory presented in the 1780's and the treatise of J.B. Fourier published in 1822, titled " La Thèorie Analytique de La Chaleur" [17]. Since then, various integral transforms with modifications have been developed and established [18-23].

1.3 Fractional Calculus

Fractional calculus emerge as a field from mathematical analysis which consists of partial and ordinary derivatives of non negative fractional order. It has drawn so much interest and attention by virtue of its important application in engineering and science. The development and history of fractional differential operators can be found in [24-26], including most frequently used Riemann-Liouville and Caputo derivatives.

1.3.1 Riemann-Liouville Fractional Order Integral Operator

The Riemann-Liouville fractional order integral operator is defined as [27]:

$$I^{\alpha}Q(\tau)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}Q(s)ds, & \alpha>0.\\ \\ Q(\tau), & \alpha=0.\end{cases} \tag{1.1}$$

Some important properties of Riemann-Liouville fractional order integral operator can be found in [27].

1.3.2 Caputo Fractional Order Derivative Operator

The Caputo fractional order derivative operator is given in [27]:

$$D^{\alpha}Q(\tau)=I^{n-\alpha}\left(\frac{d^n}{d\tau^n}\,Q(\tau)\right),\quad n-1<\alpha\leq n,\, n\in\mathbb{N}. \tag{1.2}$$

Some important properties of Caputo fractional order derivative operator can be found in [27].

1.3.3 Mittag Leffler Function

A special function called Mittag-Leffler function can be described as [28]:

$$E_{\alpha}(\lambda)=\sum_{\kappa=0}^{\infty}\frac{\lambda^{\kappa}}{\Gamma(\kappa\alpha+1)}\,,\ \alpha,\,\lambda\in\mathbb{C},\,\mathbb{R}e(\alpha)\geq 0\,. \tag{1.3}$$

1.4 Outline of this Thesis

The Aboodh transform was conceived by modifying the Laplace transform while the new iterative method was introduced by Daftardar-Gejji and Jafari [29-30].

We derived the series approximate solution and the analytical solution for different types of fractional differential equations that include the spatial diffusion biological population fractional model, fractional gas dynamics equation, fractional Fokker Planck equation, nonhomogenous fractional Kolmogorov equation and the Klein-Gordon type fractional model.

The thesis is arranged in the following format: In Chapter 2, some known results and Aboodh transform iterative method are presented. In Chapter 3, fractional models of some popular equations are evaluated. In Chapter 4, three cases of positive non integer order spatial diffusion biological population equations are considered. Finally, in Chapter 5, some concluding remarks are given.

Chapter 2

ABOODH TRANSFORM ITERATIVE METHOD

2.1 Chapter Overview

This chapter presents the description of Aboodh transform, Aboodh transform iterative method and its convergence analysis.

2.2 Aboodh Transform

The Aboodh transform was defined in 2013 by K.S. Aboodh [21] as a modification of the Laplace transform, which was defined in the time domain $\tau \geq 0$.

Definition 2.1 (Aboodh Transform): Aboodh transform for function Q(τ) of exponential order over the group of function described as [21]:

$$\mathscr{A} = \left\{ Q : \mid Q(\tau) \mid < Me^{k_j|\tau|}, if\ \tau \in (-1)^j \times [0,\infty), j = 1,2; (M,k_1,k_2 > 0) \right\} , \quad (2.1)$$

where the Aboodh transform of Q(τ) is denoted by

$$\mathscr{A}\left[Q(\tau)\right] = \mathscr{H}(\psi) , \quad (2.2)$$

and defined as:

$$\mathscr{A}\left[Q(\tau)\right] = \frac{1}{\psi}\int_0^\infty Q(\tau)e^{-\psi\tau}d\tau = \mathscr{H}(\psi),\ \tau \leq 0,\ k_1 \leq \psi \leq k_2 . \quad (2.3)$$

Definition 2.2 (Inverse Aboodh transform of function Q(τ)): If

$$\mathscr{A}\left[Q(\tau)\right] = \mathscr{H}(\psi) ,$$

then the inverse Aboodh transform of a function Q(τ), $\tau \in (0,\infty)$ is defined as [21]:

$$Q(\tau) = \mathscr{A}^{-1}[\mathscr{H}(\psi)] . \quad (2.4)$$

Theorem 2.1 ([33]): If $\mathscr{H}(\psi)$ is the Aboodh transform of Q(τ) then,

$$\mathscr{A}\left[Q^{'}(\tau)\right] = \psi\mathscr{H}(\psi) - \frac{Q(0)}{\psi}, \tag{2.5}$$

$$\mathscr{A}\left[Q^{''}(\tau)\right] = \psi^2\mathscr{H}(\psi) - \frac{Q^{''}(0)}{\psi} - Q(0), \tag{2.6}$$

$$\mathscr{A}\left[Q^{(n)}(\tau)\right] = \psi^n\mathscr{H}(\psi) - \sum_{s=0}^{n-1}\frac{Q^{(s)}(0)}{\psi^{2-n+s}}. \tag{2.7}$$

Proof. Using Equation (2.3),

$$\mathscr{A}\left[Q^{'}(\tau)\right] = \frac{1}{\psi}\int_0^\infty Q^{'}(\tau)e^{-\psi\tau}d\tau = \mathscr{H}(\psi),$$

applying integration by part, we have:

$$\frac{1}{\psi}\int_0^\infty Q^{'}(\tau)e^{-\psi\tau}d\tau = \psi\mathscr{H}(\psi) - \frac{Q(0)}{\psi}.$$

Also,

$$\mathscr{A}\left[Q^{''}(\tau)\right] = \frac{1}{\psi}\int_0^\infty Q^{''}(\tau)e^{-\psi\tau}d\tau,$$

applying integration by part, we have:

$$\frac{1}{\psi}\int_0^\infty Q^{''}(\tau)e^{-\psi\tau}d\tau = \psi^2\mathscr{H}(\psi) - \frac{Q^{''}(0)}{\psi} - Q(0).$$

For the n-th derivative we show the proof by mathematical induction. Given

$$\mathscr{A}\left[Q^{(n)}(\tau)\right] = \psi^n\mathscr{H}(\psi) - \sum_{s=0}^{n-1}\frac{Q^{(s)}(0)}{\psi^{2-n+s}},\ n \geq 1.$$

Consider when $n = 1$, we have:

$$\mathscr{A}\left[Q^{'}(\tau)\right] = \psi\mathscr{H}(\nu) - \frac{Q(0)}{\psi}.$$

It holds for $n = 1$. Consider when $n = \mathscr{N}$, we get:

$$\mathscr{A}\left[Q^{(\mathscr{N})}(\tau)\right] = \psi^{\mathscr{N}}\mathscr{H}(\psi) - \sum_{s=0}^{\mathscr{N}-1}\frac{Q^{(s)}(0)}{\psi^{2-\mathscr{N}+s}}.$$

It holds for $n = \mathscr{N}$. Now, we show that it equally holds for $\mathscr{N}$ + 1,

$$\mathscr{A}\left[Q^{(\mathscr{N}+1)}(\tau)\right] = \psi^{\mathscr{N}+1}\mathscr{H}(\psi) - \sum_{s=0}^{\mathscr{N}}\frac{Q^{(s)}(0)}{\psi^{2-(\mathscr{N}+1)+s}},$$

note that:

$$\mathscr{A}\left[Q^{(\mathscr{N}+1)}(\tau)\right]=\mathscr{A}\left[(Q^{(\mathscr{N})}(\tau))'\right]$$

$$=\psi\mathscr{A}\left[Q^{(\mathscr{N})}(\tau)\right]-\frac{Q^{(s)}(0)}{\psi}.$$

Therefore

$$\mathscr{A}\left[Q^{(\mathscr{N}+1)}(\tau)\right]=\psi^{\mathscr{N}+1}\mathscr{H}(\psi)-\sum_{s=0}^{\mathscr{N}-1}\frac{Q^{(s)}(0)}{\psi^{2-\mathscr{N}+s-1}}-\frac{Q^{(s)}(0)}{\psi}$$

$$=\psi^{\mathscr{N}+1}\mathscr{H}(\psi)-\sum_{s=0}^{\mathscr{N}}\frac{Q^{(s)}(0)}{\psi^{2-(\mathscr{N}+1)+s}}$$

when $n=\mathscr{N}+1$,

$$\mathscr{A}\left[Q^{(n)}(\tau)\right]=\psi^{n}\mathscr{H}(\psi)-\sum_{s=0}^{n-1}\frac{Q^{(s)}(0)}{\psi^{2-n+s}}.$$

□

Lemma 2.1: Aboodh transformation of Caputo fractional derivative of order α is [31]:

$$\mathscr{A}[(D_{\tau}^{\alpha}Q(\tau));\psi]=\psi^{\alpha}\mathscr{A}[Q(\tau)]-\sum_{s=0}^{n-1}\frac{Q^{(s)}(0)}{\psi^{2-\alpha+s}},\ n-1<\alpha\leq n,\ n\in\mathbb{N}\,. \tag{2.8}$$

Lemma 2.2: (Linearity property of Aboodh transform) Assume the Aboodh transform of $Q_1(\tau)$ and $Q_2(\tau)$ are $\mathscr{P}(\psi)$ and $\mathscr{H}(\psi)$ respectively [32]:

$$\mathscr{A}[\gamma_1 Q_1(\tau)+\gamma_2 Q_2(\tau)]=\mathscr{A}[\gamma_1 Q_1(\tau)]+\mathscr{A}[\gamma_2 Q_2(\tau)]$$

$$=\gamma_1\mathscr{P}(\psi)+\gamma_2\mathscr{H}(\psi)\,, \tag{2.9}$$

where γ_1 and γ_2 are arbitrary constants.

See the Aboodh transform Table 2.1 for Aboodh transform of some functions.

Table 2.1: Aboodh Transform Table [31].

Q(τ)	$\mathscr{A}[Q(\tau)] = \mathscr{H}(\psi)$
1	$\dfrac{1}{\psi^2}$
τ	$\dfrac{1}{\psi^3}$
τ^n	$\dfrac{n!}{\psi^{n+2}}\ n=0,1,2,\ldots$
τ^α	$\dfrac{\Gamma(\alpha+1)}{\psi^{\alpha+2}}$

2.3 New Iterative Method

New Iterative Method is a decomposition method introduce by Daftardar-Gejji and Jafari in 2005 [30]. The prominent advantage of the new iterative method is that it can be applied directly with no restrictive assumptions imposed on the problem which make the solution obtained to be more realistic. Also, it conserve computer power, time and memory making it economical without tedious calculations of Adomian polynomials.

2.3.1 Concept of New Iterative Method

Examine the functional equation:

$$Q = f + \mathscr{N}(Q)\,, \tag{2.10}$$

with f as a familiar function and $\mathscr{N}$ is the nonlinear operator from a Banach space $X \to X$. Assume the solution of Equation (2.10) is given in series from as:

$$Q = \sum_{s=0}^{\infty} Q_s\,. \tag{2.11}$$

Here, the nonlinear operator $\mathscr{N}$ is decomposed as:

$$\mathscr{N}\left(\sum_{s=0}^{\infty} Q_s\right) = \mathscr{N}(Q_0) + \sum_{s=1}^{\infty}\left\{\mathscr{N}\left(\sum_{i=0}^{s} Q_i\right) - \mathscr{N}\left(\sum_{i=0}^{s-1} Q_i\right)\right\}, \tag{2.12}$$

using Equations (2.12), (2.11) and (2.10), we have:

$$\sum_{s=0}^{\infty} Q_s = f + \mathscr{N}(Q_0) + \sum_{s=1}^{\infty}\left\{\mathscr{N}\left(\sum_{i=0}^{s} Q_i\right) - \mathscr{N}\left(\sum_{i=0}^{s-1} Q_i\right)\right\}, \tag{2.13}$$

we define here the following recurrence relation:

$$\left.\begin{aligned}
&Q_0 = f, \\
&Q_1 = \mathscr{N}(Q_0), \\
&\vdots \\
&Q_{s+1} = \mathscr{N}(Q_0 + Q_1 + \cdots + Q_s) - \mathscr{N}(Q_0 + Q_1 + \cdots + Q_{s-1}),\ s = 1, 2, \ldots
\end{aligned}\right\} \tag{2.14}$$

Then

$$\mathscr{N}(Q_1 + Q_1 + \cdots + Q_{s+1}) = \mathscr{N}(Q_0 + Q_1 + \cdots + Q_s),\ s = 1, 2, \ldots \tag{2.15}$$

$$Q = \sum_{s=0}^{\infty} Q_s\ .$$

2.4 Basic Idea of Aboodh Transform Iterative Method

Examine the initial value problem of the pattern [33]:

$$D^{\alpha} Q(x,y,\tau) = \mathscr{R}(Q(x,y,\tau)) + \mathscr{N}(Q(x,y,\tau)) + \mathscr{G}(x,y,\tau),\ n-1 < \alpha \leq n\ , \tag{2.16}$$

as well as the initial condition:

$$Q^{(s)}(x,y,0) = Q_s(x,y),\ \ s = 0, 1, 2, \ldots, n-1\ . \tag{2.17}$$

$\mathscr{G}(x,y,\tau)$ is the source function, while the operator $\mathscr{R}$ and operator$\mathscr{N}$ are the linear and nonlinear operators respectively. Applying Aboodh transform on each side of Equation (2.16) and using the initial condition, we get:

$$\mathscr{A}[Q(x,y,\tau)] =$$
$$\frac{1}{\psi^{\alpha}}\left(\sum_{s=0}^{n-1} \frac{Q^{(s)}(x,y,0)}{\psi^{2-\alpha+s}} + \mathscr{A}\left[\mathscr{G}(x,y,\tau)\right]\right) + \frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q(x,y,\tau)) + \mathscr{N}(Q(x,y,\tau))\right]\right)\ . \tag{2.18}$$

Simplifying and applying the inverse Aboodh transform of each side of Equation (2.18), we get:

$$Q(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{Q^{(s)}(x,y,0)}{\psi^{2-\alpha+s}} + \mathscr{A}\left[\mathscr{G}(x,y,\tau)\right]\right)\right] + \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q(x,y,\tau)) + \mathscr{N}(Q(x,y,\tau))\right]\right)\right]. \quad (2.19)$$

The operator $\mathscr{N}$ is decomposed as [30]:

$$\mathscr{N}(Q(x,y,\tau)) = \mathscr{N}\left(\sum_{s=0}^{\infty} Q_s(x,y,\tau)\right)$$

$$= \mathscr{N}(Q_0(x,y,\tau)) + \sum_{s=1}^{\infty}\left\{\mathscr{N}\left(\sum_{i=0}^{s} Q_i(x,y,\tau)\right) - \mathscr{N}\left(\sum_{i=0}^{s-1} Q_i(x,y,\tau)\right)\right\}. \quad (2.20)$$

Also, the operator $\mathscr{R}$ can also be decomposed in similar manner. Next, we define the m-th order approximate series as:

$$\mathscr{Q}^{(m)}(x,y,\tau) = \sum_{\kappa=0}^{m} Q_{\kappa}(x,y,\tau)$$

$$= Q_0(x,y,\tau) + Q_1(x,y,\tau) + Q_2(x,y,\tau) + \cdots + Q_m(x,y,\tau),\ m \in \mathbb{N}. \quad (2.21)$$

Envisage that the solution of Equation (2.16) is given as:

$$Q(x,y,\tau) = \lim_{m\to\infty} \mathscr{Q}^{(m)}(x,y,\tau) = \sum_{\kappa=0}^{m} Q_{\kappa}(x,y,\tau). \quad (2.22)$$

Substituting Equations (2.20) and (2.21) into (2.19) with the application of the linearity property, we get:

$$\sum_{s=0}^{\infty} Q_s(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{Q^{(s)}(x,y,0)}{\psi^{2-\alpha+s}}\right) + \mathscr{A}\left[\mathscr{G}(x,y,\tau)\right]\right]$$
$$+ \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q_0(x,y,\tau)) + \mathscr{N}(Q_0(x,y,\tau))\right]\right)\right]$$
$$+ \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\sum_{s=1}^{\infty}\left(\mathscr{R}(Q_s(x,y,\tau))\right)\right]\right)\right] +$$
$$\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\left(\left\{\mathscr{N}\left(\sum_{i=0}^{s} Q_i(x,y,\tau)\right) - \mathscr{N}\left(\sum_{i=0}^{s-1} Q_i(x,y,\tau)\right)\right\}\right)\right]\right)\right]. \quad (2.23)$$

Now, we create the following iterations

$$Q_0(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\sum_{s=0}^{n-1}\frac{Q^{(s)}(x,y,0)}{\psi^{2-\alpha+s}}\right) + \mathscr{A}\left[\mathscr{G}(x,y,\tau)\right]\right],\ n-1<\alpha\le n \tag{2.24}$$

$$Q_1(x,y,t) = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{R}(Q_0(x,y,\tau)) + \mathscr{N}(Q_0(x,y,\tau))\right],\right)\right] \tag{2.25}$$

$$\vdots$$

$$Q_{s+1}(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\sum_{s=1}^{\infty}\left(\mathscr{R}(Q_s(x,y,\tau))\right)\right]\right)\right] + \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\left(\left\{\mathscr{N}\left(\sum_{i=0}^{s}Q_i(x,y,\tau)\right) - \mathscr{N}\left(\sum_{i=0}^{s-1}Q_i(x,y,\tau)\right)\right\}\right)\right]\right)\right], \tag{2.26}$$

s = 1,2,. . . .

2.5 Convergence of Aboodh Transform Iterative Method

This section presents some known definitions and theorems to establish the convergence of Aboodh transform iterative method.

Definition 2.3 (Normed Space): If X is a vector space, A norm on X over the scalar field F ($\mathbb{R}$ or $\mathbb{C}$) is a function [34], $||.|| : X \to [0,\infty] \subseteq \ \mathbb{R}$, which satisfied the following axiom:

$$\left.\begin{aligned} &||p+q|| \le ||p|| + ||q||,\ \ p,q\in X.\\ &||\beta p|| = |\beta|\ ||p||,\ \ \forall\beta\in F,\ \ p\in X.\\ &||p||\ge 0,\ \ ||p|| = 0\ \ \Rightarrow p = 0.\end{aligned}\right\} \tag{2.27}$$

Definition 2.4 (Metric Space): A metric space is said to be an ordered pair (X, d) where X is a set and d is the distance function known as the metric function on X [34], i.e., function d: $X \times X \to \mathbb{R}$ such that $\forall p,q,r \in X$, the following axioms holds:

$$\left.\begin{aligned} &d(p,q) = 0 \iff p = q, \\ &d(p,q) = d(p,q), \\ &d(p,r) \leq d(p,q) + d(q,r). \end{aligned}\right\} \tag{2.28}$$

The space X is complete if every sequence $\{x_n\}$ is a cauchy sequences in X, $||x_n - x_m|| \to 0$ as $n,m \to \infty$, then there exist $x \in X$ such that $||x_n - x_m|| \to 0$ as $n \to \infty$, that is $\{x_n\}$ is convergent sequence in X.

Definition 2.5 (Banach Space): A complete normed vector space $(X, ||.||)$ is called a Banach Space [34].

Definition 2.6 (Contraction): Assume X be a metric space, $T : X \to X$ is said to be a contraction on X. If a nonnegative real number ρ $(0 \leq \rho < 1) : \forall\ p,q \in X,\ \ d(T_p, T_q)$ $\leq \rho d(p,q)$, [34].

Theorem 2.2 ([17]): If $\mathscr{A}[Q(\tau)]$ and $\mathscr{A}[||Q(\tau)||]$ exist where $\mathscr{A}\,[Q(\tau)]$ is the Aboodh transform for any function Q in the Banach space X, then:

$$||\mathscr{A}[Q(\tau)]|| \leq \mathscr{A}[||Q(\tau)||].$$

Proof. Given that the Aboodh transform of Q(τ) is:

$$\mathscr{A}[Q(\tau)] = \frac{1}{\psi}\int_0^\infty Q(\tau)e^{-\psi\tau}d\tau,$$

we have

$$\begin{aligned} ||\mathscr{A}[Q(\tau)]|| &= ||\frac{1}{\psi}\int_0^\infty Q(\tau)e^{-\psi\tau}d\tau|| \\ &\leq \frac{1}{\psi}\int_0^\infty ||Q(\tau)e^{-\psi\tau}||d\tau \end{aligned}$$

$$= \frac{1}{\psi}\int_0^\infty ||Q(\tau)||e^{-\psi\tau}d\tau,$$

$$= \mathscr{A}[||Q(\tau)||],$$

$\Rightarrow ||\mathscr{A}[Q(\tau)]|| \leq \mathscr{A}[||Q(\tau)||].$ □

Theorem 2.3 ([29]): Given the fractional initial value problem below:

$$D_\tau^\alpha Q(x,y,\tau) = \mathscr{R}(Q(x,y,\tau)) + \mathscr{N}(Q(x,y,\tau)) + \mathscr{G}(x,y,\tau),\ n-1 < \alpha \leq n\ ,$$

as well as the initial condition:

$$Q^{(s)}(x,y,0) = Q_s(x,y),\ \ s = 0,1,2,\ldots,n-1\ .$$

where $\mathscr{G}(x,y,\tau)$ is the source function, operator $\mathscr{R}$ and operator $\mathscr{N}$ are the linear and nonlinear operators form the Banach space X to itself. $Q(x,y,\tau)$ is analytic about τ, the solution of the initial value problem can be written in series pattern:

$$Q(x,y,\tau) = \sum_{s=0}^{\infty} Q_s(x,y,\tau),$$

with the consecutive recurrence relation:

$$Q_0(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\sum_{s=0}^{n-1}\frac{Q^{(s)}(x,y,0)}{\psi^{2-\alpha+s}}\right) + \mathscr{A}\left[\mathscr{G}(x,y,\tau)\right]\right],\ n-1 < \alpha \leq n,$$

$$Q_1(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{R}(Q_0(x,y,\tau)) + \mathscr{N}(Q_0(x,y,\tau))\right]\right)\right],$$

$$\vdots$$

$$Q_{s+1}(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\sum_{s=1}^{\infty}\left(\mathscr{R}(Q_s(x,y,\tau))\right)\right]\right)\right]$$

$$+\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\left(\left(\left\{\mathscr{N}\left(\sum_{i=0}^{s}Q_i(x,y,\tau)\right) - \mathscr{N}\left(\sum_{i=0}^{s-1}Q_i(x,y,\tau)\right)\right\}\right)\right]\right)\right],$$

$s = 1,2,\ldots.$

If the operators $\mathscr{R}$ and $\mathscr{N}$ are contractions, then the series:

$$\sum_{s=0}^{\infty} Q_s(x,y,\tau)$$

converges absolutely if $\tau^{\alpha} < \dfrac{\Gamma(\alpha+1)}{\rho}$.

Proof. From Equation (2.16)

$$Q(x,y,\tau) = \sum_{s=0}^{\infty} Q_s,$$

the operator $\mathscr{N}$ is decomposed as:

$$\mathscr{N}(Q(x,y,\tau)) = \mathscr{N}(Q_0) + \sum_{s=1}^{\infty}\left\{\mathscr{N}\left(\sum_{i=0}^{s} Q_i\right) - \mathscr{N}\left(\sum_{i=0}^{s-1} Q_i\right)\right\}.$$

Similarly, the operator $\mathscr{R}$ can be decomposed in like manner. Let Ω be an operator, we set:

$$\Omega(Q(x,y,\tau)) = \Omega(Q_0) + \sum_{s=1}^{\infty}\left\{\Omega\left(\sum_{i=0}^{s} Q_i\right) - \Omega\left(\sum_{i=0}^{s-1} Q_i\right)\right\},$$

we generate the following recurrence relation:

$$Q_0(x,y,\tau) = Q_0\ ,$$

$$Q_1(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\mathscr{A}\left[\Omega(Q_0)\right]\right)\right]\ ,$$

$$\vdots$$

$$Q_{s+1}(x,y,\tau) = \mathscr{A}^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\mathscr{A}\left[\sum_{s=1}^{\infty}\left(\left\{\Omega\left(\sum_{i=0}^{s} Q_i\right) - \Omega\left(\sum_{i=0}^{s-1} Q_i\right)\right\}\right)\right]\right)\right],$$

$s = 1,2,\ldots.$

Since the operator $\mathscr{R}$ and $\mathscr{N}$ are contractions, Ω is also contraction and there is a constant $0 < \rho < 1$, such that:

$$||\Omega(w_i) - \Omega(\omega_j)|| \leq \rho||\omega_i - \omega_j||,\ \ \forall \omega_i, \omega_j \in X$$

||.|| is a norm on X. Finally, we give the estimate of Q_{s+1}

$$||Q_{s+1}(x,y,\tau)|| = \left\| \left(\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}} \left(\mathscr{A} \left[\left(\left\{ \Omega \left(\sum_{i=0}^{s} Q_i \right) - \Omega \left(\sum_{i=0}^{s-1} Q_i \right) \right\} \right) \right] \right) \right] \right) \right\|,$$

$$= \left\| \left(\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}} \left(\mathscr{A} \left[\left(\left\{ \Omega \left(Q_0 + Q_1 + \cdots + Q_s \right) - \Omega \left(Q_0 + Q_1 + \cdots + Q_{s-1} \right) \right\} \right) \right] \right) \right] \right) \right\|,$$

$$\leq \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}} \left(\mathscr{A} \left[\left\| \left(\Omega \left(Q_0 + Q_1 + \cdots + Q_s \right) - \Omega \left(Q_0 + Q_1 + \cdots + Q_{s-1} \right) \right) \right\| \right] \right) \right],$$

$$\leq \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}} \left(\mathscr{A} \left[\rho \left\| \left(\Omega \left(Q_0 + Q_1 + \cdots + Q_s \right) - \Omega \left(Q_0 + Q_1 + \cdots + Q_{s-1} \right) \right) \right\| \right] \right) \right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}} \left(\mathscr{A} \left[\rho \left\| \left(Q_s \right) \right\| \right] \right) \right],$$

$$= \rho \left\| \left(Q_s \right) \right\| \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}} \left(\mathscr{A} \left[1 \right] \right) \right],$$

where $\mathscr{A}[1] = \dfrac{1}{\psi^2}$,

$$= \rho \left\| \left(Q_s \right) \right\| \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha+2}} \right],$$

$$= \rho \left\| \left(Q_s \right) \right\| \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}.$$

Then $||Q_{s+1}|| \ \leq \ \dfrac{\rho\tau^{\alpha}}{\Gamma(\alpha+1)} \, ||Q_s||,$

$$\Rightarrow \frac{||Q_{s+1}||}{||Q_s||} \leq \frac{\rho\tau^{\alpha}}{\Gamma(\alpha+1)}, \quad s = 1, 2, \ldots$$

By the ratio test, the series is convergent if:

$$\frac{||Q_{s+1}||}{||Q_s||} < 1 \ ,$$

$$\Rightarrow \frac{\rho\tau^{\alpha}}{\Gamma(\alpha+1)} < 1,$$

$$\Rightarrow \tau^{\alpha} < \frac{\Gamma(\alpha+1)}{\rho}.$$

□

Chapter 3

ABOODH TRANSFORM ITERATIVE METHOD FOR APPROXIMATE SERIES ANALYTICAL SOLUTION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

3.1 Chapter Overview

The present chapter is concerned with the application of Aboodh transform iterative method described in [35]. This method utilized the new iterative method discussed in Chapter 2 to decompose the nonlinear term. The solutions shows that Aboodh transform iterative method gives comparable solutions to the existing methods with less computational effort.

3.2 Problem Statement

This particular chapter consider closely the following type problems:

$$D_\tau^\alpha Q(x,\tau) = \mathscr{L}(Q(x,\tau)) + \mathscr{N}(Q(x,\tau)) + \mathscr{G}(x,\tau), \ \ n-1 < \alpha \leq n,$$

as well as the initial condition:

$$Q^{(r)}(x,0) = Q_r(x), \ r = 0,1,2,\ldots$$

D^α symbolizes the Caputo derivative while $\mathscr{L}$, $\mathscr{N}$ and $\mathscr{G}(x,\tau)$ denotes the linear operator, nonlinear operator and known function. For the existence and uniqueness for this problem see [36].

3.3 Descriptive Examples

The current section examine four descriptive examples to demonstrate the efficiency of Aboodh transform Iterative method.

3.3.1 Example 1

The fractional gas dynamics equation is described as [37]:

$$D_\tau^\alpha Q+\frac{1}{2}(Q^2)_x-Q(1-Q)=0,\ 0<\alpha\leq 1, \tag{3.1}$$

as well as the initial condition considered as:

$$Q_0(x)=e^{-x}, \tag{3.2}$$

using Equation (3.1), we set:

$$\left.\begin{aligned}&\mathscr{L}(Q(x,\tau))=-Q,\\&\mathscr{N}(Q(x,\tau))=\frac{1}{2}(Q^2)_x+Q^2,\\&Q(x,0)=e^{-x}.\end{aligned}\right\} \tag{3.3}$$

Now, using the iterative procedure described in Chapter 2, we have:

$$\begin{aligned}Q_0&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\psi^{2-\alpha+r}}\right)\right]\\&=\mathscr{A}^{-1}\left[\frac{Q(x,0)}{\psi^2}\right]\\&=e^{-x}.\end{aligned} \tag{3.4}$$

$$\begin{aligned}Q_1&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{L}(Q_0(x,\tau))+\mathscr{N}(Q_0(x,\tau))\right]\right)\right],\\&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[Q_0-\left\{\left(\frac{1}{2}(Q_0^2)_x+Q_0^2\right)\right\}\right]\right)\right],\\&=\frac{e^{-x}\tau^\alpha}{\Gamma(\alpha+1)}.\end{aligned} \tag{3.5}$$

$$Q_2=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{L}(Q_1(x,\tau))+\left\{\mathscr{N}(Q_0(x,\tau)+Q_1(x,\tau))-\mathscr{N}(Q_0(x,\tau))\right\}\right]\right)\right],$$

$$=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[Q_1-\left\{\left(\frac{1}{2}((Q_0+Q_1)^2)_x+(Q_0+Q_1)^2\right)+\left(\frac{1}{2}(Q_0)_x^2+Q_0^2\right)\right\}\right]\right)\right],$$

$$=\frac{e^{-x}\tau^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{3.6}$$

$$\vdots$$

$$Q_\kappa = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{L}\left(Q_{k-1}(x,\tau)\right)\right]\right)\right] +$$
$$\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\left\{\mathscr{N}\left(\sum_{j=0}^{r-1} Q_j(x,\tau)\right) - \mathscr{N}\left(\sum_{j=0}^{r=2} Q_j(x,\tau)\right)\right\}\right]\right)\right],$$
$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[Q_{k-1}\right]\right)\right],$$
$$= \frac{e^{-x}\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}. \tag{3.7}$$

We get the κ-th order approximate series as:

$$\mathscr{Q}^{(\kappa)}(x,\tau) = \sum_{m=0}^{\kappa} Q_m(x,\tau)$$
$$= Q_0(x,\tau) + Q_1(x,\tau) + Q_2(x,\tau) + \cdots + Q_k(x,\tau),\ \kappa \in \mathbb{N}\,.$$
$$= e^{-x} + \frac{e^{-x}\tau^\alpha}{\Gamma(\alpha+1)} + \frac{e^{-x}\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{e^{-x}\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}$$
$$= e^{-x}\left(1 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} + \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}\right)$$
$$= e^{-x}\sum_{m=0}^{\kappa} \frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{3.8}$$

The approximate series solution come near to the exact solution as $\kappa \to \infty$,

$$Q(x,\tau) = \lim_{\kappa\to\infty} \mathscr{Q}^{(\kappa)}(x,\tau)$$
$$= e^{-x}\lim_{\kappa\to\infty}\sum_{m=0}^{\kappa} \frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}$$
$$= e^{-x}E_\alpha(\tau^\alpha). \tag{3.9}$$

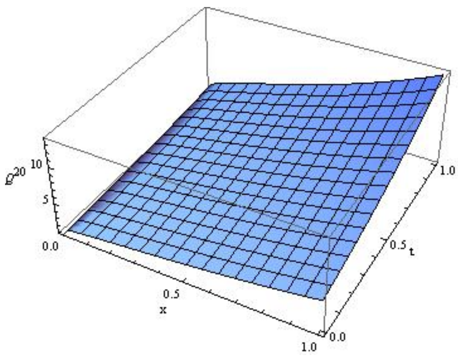
If $\alpha = 1$, then the exact solution of Equation (3.1) is :

$$Q_e(x,\tau) = e^{-x}E_1(\tau),$$
$$= e^{\tau-x}, \tag{3.10}$$

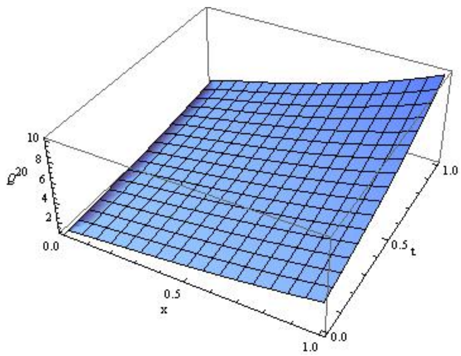
which is same solution obtained in [38] using the homotopy analysis method merged with the Laplace transform method. Also by applying the fractional reduced differential transform in [39]. The solution realized grows exponentially with time. We compare the absolute difference $E_{dif} = |Q(0.25,\tau) - \mathscr{Q}^{(10)}(0.25,\tau)|$ between the series approximate solution and the close form solution for various values of α and obtained the absolute error $E_{abs} = |Q_e(0.25,\tau) - \mathscr{Q}^{(10)}(0.25,\tau)|$ at $\alpha = 1$, and τ = 0.25, 0.50, 0.75, 1.00 when x = 0.25 as display in Table 3.1, while Figure 3.1 is surface plot for $\alpha = 0.5, 0.7, 0.9$ and 1. with the choice of Q_0 unchanged.

Table 3.1: Comparison of error difference for diverse values of α

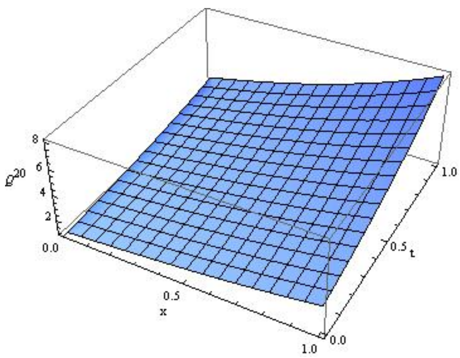
τ	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	E_{abs} when $\alpha = 1.0$
0.25	1.5×10^{-6}	9.2×10^{-10}	3.1×10^{-13}	4.8×10^{-15}
0.50	8.3×10^{-5}	2.0×10^{-7}	3.0×10^{-10}	9.9×10^{-12}
0.75	8.4×10^{-4}	4.9×10^{-6}	1.7×10^{-8}	8.8×10^{-10}
1.00	4.4×10^{-3}	4.7×10^{-5}	3.1×10^{-7}	2.1×10^{-8}



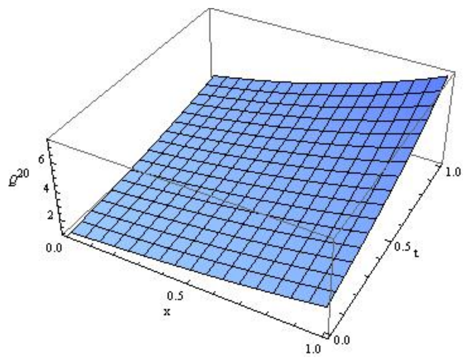
(a) $\alpha = 0.5$



(b) $\alpha = 0.7$



(c) $\alpha = 0.9$



(d) $\alpha = 1$

Figure 3.1: The surface plot for various values of α

3.3.2 Example 2

Fractional Fokker-Planck equation is expressed as [40]:

$$D_{\tau}^{\alpha}Q+\left(\frac{4}{x}Q^2\right)_x-\left(\frac{x}{3}Q\right)_x-(Q^2)_{xx}=0,\ \ 0<\alpha\leq 1, \tag{3.11}$$

and the initial condition described as:

$$Q_0(x)=x^2, \tag{3.12}$$

using Equation (3.11), we set:

$$\left.\begin{aligned}&\mathscr{L}(Q(x,\tau))=-\left(\frac{x}{3}Q\right)_x,\\&\mathscr{N}(Q(x,\tau))=-(Q^2)_{xx}+\left(\frac{4}{x}Q^2\right)_x,\\&Q(x,0)=x^2.\end{aligned}\right\} \tag{3.13}$$

Now, with the use of iterative method illustrated in the previous chapter, we have:

$$\begin{aligned}Q_0&=\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\psi^{2-\alpha+r}}\right)\right]\\&=\mathscr{A}^{-1}\left[\frac{Q(x,0)}{\psi^2}\right]\\&=x^2.\end{aligned} \tag{3.14}$$

$$\begin{aligned}Q_1&=\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{L}(Q_0(x,\tau))+\mathscr{N}(Q_0(x,\tau))\right]\right)\right],\\&=\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\left(\frac{x}{3}Q_0\right)_x+\left\{(Q_0^2)_{xx}-\left(\frac{4}{x}Q_0^2\right)_x\right\}\right]\right)\right],\\&=\frac{x^2\tau^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned} \tag{3.15}$$

$$\begin{aligned}Q_2&=\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{L}(Q_1(x,\tau))+\left\{\mathscr{N}(Q_0(x,\tau)+Q_1(x,\tau))-\mathscr{N}(Q_0(x,\tau))\right\}\right]\right)\right],\\&=\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\left(\tfrac{x}{3}Q_1\right)_x-\left\{((Q_0+Q_1)^2)_{xx}-\left(\tfrac{4}{x}(Q_0+Q_1)^2\right)_x-(Q_0^2)_{xx}+\left(\frac{4}{x}Q_0^2\right)_x\right\}\right]\right)\right],\\&=\frac{x^2\tau^{2\alpha}}{\Gamma(2\alpha+1)}.\end{aligned} \tag{3.16}$$

$$\vdots$$

$$Q_\kappa = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{L}\left(Q_{k-1}(x,\tau)\right)\right]\right)\right] +$$

$$\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\left\{\mathscr{N}\left(\sum_{j=0}^{r-1} Q_j(x,\tau)\right) - \mathscr{N}\left(\sum_{j=0}^{r=2} Q_j(x,\tau)\right)\right\}\right]\right)\right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\left(\frac{x}{3}Q_{k-1}\right)_x\right]\right)\right],$$

$$= \frac{x^2\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}. \tag{3.17}$$

We get the κ-th order approximate series as:

$$\mathscr{Q}^{(\kappa)}(x,\tau) = \sum_{m=0}^{\kappa} Q_m(x,\tau)$$

$$= Q_0(x,\tau) + Q_1(x,\tau) + Q_2(x,\tau) + \cdots + Q_k(x,\tau),\ \kappa \in \mathbb{N}\,.$$

$$= x^2 + \frac{x^2\tau^\alpha}{\Gamma(\alpha+1)} + \frac{x^2\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{x^2\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}$$

$$= x^2\left(1 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} + \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}\right)$$

$$= x^2\sum_{m=0}^{\kappa}\frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{3.18}$$

The approximate series solution get close to the exact solution as $\kappa \to \infty$,

$$Q(x,\tau) = \lim_{\kappa\to\infty}\mathscr{Q}^{(\kappa)}(x,\tau)$$

$$= x^2\lim_{\kappa\to\infty}\sum_{m=0}^{\kappa}\frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$= x^2 E_\alpha(\tau^\alpha). \tag{3.19}$$

If $\alpha = 1$, then the exact solution of Equation (3.11) is :

$$Q_e(x,\tau) = x^2 E_1(\tau),$$

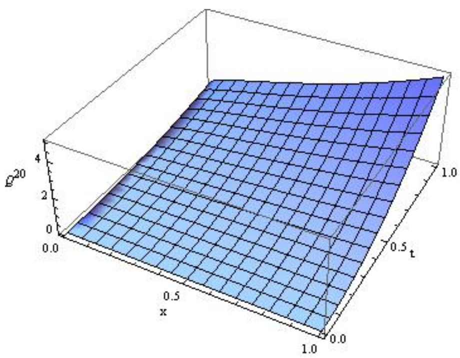
$$= x^2 e^\tau. \tag{3.20}$$

which is same solution obtained in [40] using homotopy perturbation Sumudu transform method.

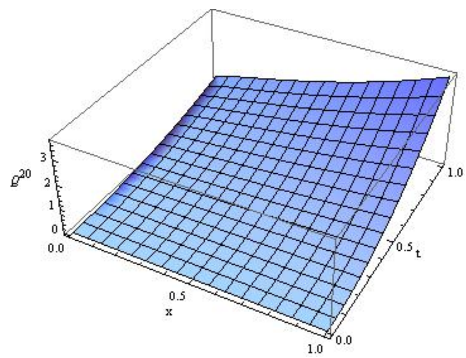
We compare the absolute difference $E_{dif} = |Q(0.25,\tau) - \mathscr{Q}^{(10)}(0.25,\tau)|$ between the series approximate solution and the close form solution for various values of α and obtained the absolute error $E_{abs} = |Q_e(0.25,\tau) - \mathscr{Q}^{(10)}(0.25,\tau)|$ at $\alpha = 1$, and τ = 0.25, 0.50, 0.75, 1.00 with x = 0.25 in Table 3.2. Also, Figure 3.2 displays the surface plot for α = 0.5, 0.7, 0.9 and 1 with the choice of Q_0 unchanged.

Table 3.2: Comparison of error difference for different α

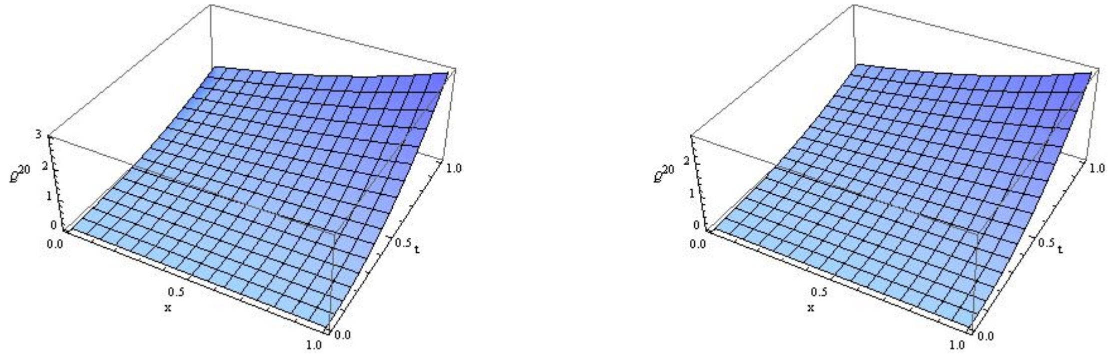
τ	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	E_{abs} when $\alpha = 1.0$
0.25	1.3×10^{-7}	7.4×10^{-11}	2.5×10^{-14}	3.9×10^{-16}
0.50	6.7×10^{-6}	1.6×10^{-8}	2.4×10^{-10}	8.0×10^{-13}
0.75	6.8×10^{-5}	3.9×10^{-7}	1.4×10^{-9}	7.1×10^{-11}
1.00	3.6×10^{-4}	3.8×10^{-6}	2.5×10^{-8}	1.7×10^{-9}



(a) $\alpha = 0.5$



(b) $\alpha = 0.7$



(a) $\alpha = 0.9$ (b) $\alpha = 1$

Figure 3.3: The surface plot with diverse values of α

3.3.3 Example 3

The non-homogeneous time-fractional Kolmogorov equation represented as [41]:

$$D_\tau^\alpha Q + x^2 e^\tau Q_{xx} - (x-1)Q_x = x\tau, \ 0 < \alpha \leq 1, \tag{3.21}$$

as well as the initial condition express as:

$$Q_0(x) = x + 1, \tag{3.22}$$

from Equation (3.21), we set:

$$\left.\begin{aligned} &\mathscr{L}(Q(x,\tau)) = -x^2 e^\tau Q_{xx} + (x+1)Q_x \\ &\mathscr{N}(Q(x,\tau)) = 0, \\ &\mathscr{G}(Q(x,\tau)) = x\tau. \end{aligned}\right\} \tag{3.23}$$

Now, using the iterative procedure described in the previous chapter, we have:

$$\begin{aligned} Q_0 &= \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\psi^{2-\alpha+r}} + \mathscr{A}[\mathscr{G}(Q(x,\tau))]\right)\right] \\ &= \mathscr{A}^{-1}\left[\frac{Q(x,0)}{\psi^2} + \frac{x}{\psi^{3+\alpha}}\right] \\ &= (x+1) + \frac{x\tau^{\alpha+1}}{\Gamma(\alpha+2)}. \end{aligned} \tag{3.24}$$

$$Q_1 = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{L}(Q_0(x,\tau)) + \mathscr{N}(Q_0(x,\tau))\right]\right)\right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[-x^2e^{\tau}(Q_0)_{xx}+(x+1)(Q_0)_x\right]\right)\right],$$

$$= (x+1)\left(\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{\tau^{2\alpha+1}}{\Gamma(2\alpha+2)}\right). \tag{3.25}$$

$$Q_2 = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{L}(Q_1(x,\tau))+\{\mathscr{N}(Q_0(x,\tau)+Q_1(x,\tau))-\mathscr{N}(Q_0(x,\tau))\}\right]\right)\right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[-x^2e^{\tau}(Q_1)_{xx}+(x+1)(Q_1)_x\right]\right)\right],$$

$$= (x+1)\left(\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\tau^{3\alpha+1}}{\Gamma(3\alpha+2)}\right). \tag{3.26}$$

$$\vdots$$

$$Q_k = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{L}\left(Q_{k-1}(x,\tau)\right)+\left\{\mathscr{N}\left(\textstyle\sum_{j=0}^{r-1}Q_j(x,\tau)\right)-\mathscr{N}\left(\textstyle\sum_{j=0}^{r=2}Q_j(x,\tau)\right)\right\}\right]\right)\right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[(x+1)(Q_{\kappa-1})_x\right]\right)\right],$$

$$= (x+1)\left(\frac{\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}+\frac{\tau^{(\kappa+1)\alpha+1}}{\Gamma((\kappa+1)\alpha+2)}\right). \tag{3.27}$$

We get the κ-th order approximate series as:

$$\mathscr{Q}^{(\kappa)}(x,\tau) = \sum_{m=0}^{\kappa} Q_m(x,\tau)$$

$$= Q_0(x,\tau)+Q_1(x,\tau)+Q_2(x,\tau)+\cdots+Q_k(x,\tau),\ \kappa\in\mathbb{N}\,.$$

$$= (x+1)+((x+1)-1)\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}+(x+1)\left(\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tau^{(2\alpha+1}}{\Gamma(2\alpha+2)}\right)+$$

$$(x+1)\left(\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\tau^{3\alpha+1}}{\Gamma(3\alpha+2)}\right)+\cdots+(x+1)\left(\frac{\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}+\frac{\tau^{(\kappa+1)\alpha+1}}{\Gamma((\kappa+1)\alpha+2)}\right)$$

$$= \frac{-\tau^{\alpha+1}}{\Gamma(\alpha+2)}+(x+1)\left(\sum_{m=0}^{\kappa}\frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}+\sum_{m=0}^{\kappa}\frac{\tau^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)}\right). \tag{3.28}$$

The approximate series solution approaches to the exact solution as $\kappa\to\infty$,

$$Q(x,\tau) = \lim_{\kappa\to\infty}\mathscr{Q}^{(\kappa)}(x,\tau)$$

$$= \frac{-\tau^{\alpha+1}}{\Gamma(\alpha+2)} + (x+1) \lim_{\kappa\to\infty} \left(\sum_{m=0}^{\kappa} \frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)} + \sum_{m=0}^{\kappa} \frac{\tau^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)} \right)$$

$$= \frac{-\tau^{\alpha+1}}{\Gamma(\alpha+2)} + (x+1) \left(E_\alpha(\tau^\alpha) + \lim_{\kappa\to\infty} \sum_{m=0}^{\kappa} \frac{\tau^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)} \right). \tag{3.29}$$

If $\alpha = 1$, then the exact solution of Equation (3.21) is :

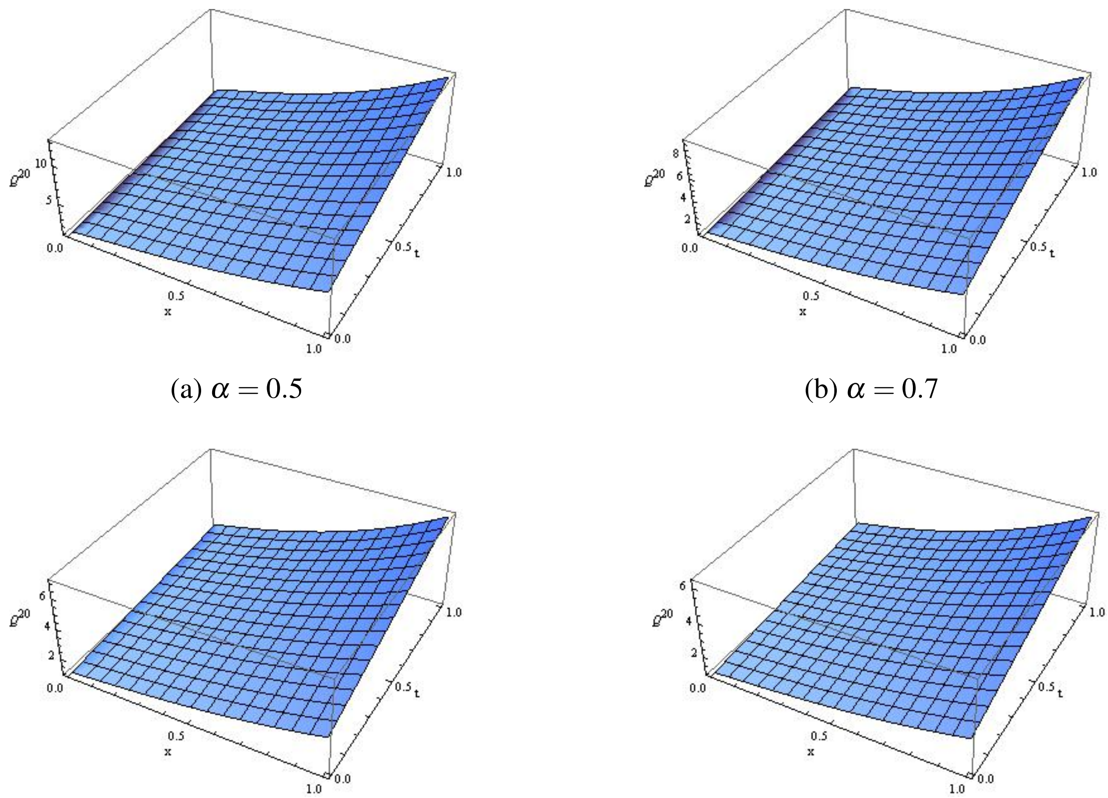
$$Q_e(x,\tau) = \frac{-\tau^2}{2} + (x+1) \left(E_1(\tau) + \lim_{\kappa\to\infty} \sum_{m=0}^{\kappa} \frac{\tau^{(m+1)+1}}{\Gamma((m+1)+2)} \right),$$

$$= \frac{-\tau^2}{2} + (x+1)(2e^\tau - \tau - 1). \tag{3.30}$$

We compare the absolute difference $E_{dif} = |Q(0.25,\tau) - \mathscr{Q}^{(10)}(0.25,\tau)|$ between the series approximate solution and the close form solution for various values of α and obtain the absolute error $E_{abs} = |Q_e(0.25,\tau) - \mathscr{Q}^{(10)}(0.25,\tau)|$ at $\alpha = 1$, and τ = 0.25, 0.50, 0.75, 1.00 when x = 0.25 in Table 3.3. Figure 3.3 displays the surface plot when $\alpha = 0.5, 0.7, 0.9$ and 1. with the choice of Q_0 unchanged.

Table 3.3: Comparison of error difference for distinct values of α.

τ	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	E_{abs} when $\alpha = 1.0$
0.25	2.6×10^{-6}	1.5×10^{-9}	4.9×10^{-13}	8.0×10^{-15}
0.50	1.3×10^{-4}	3.3×10^{-7}	4.9×10^{-10}	1.6×10^{-11}
0.75	1.4×10^{-3}	7.7×10^{-6}	2.8×10^{-8}	1.4×10^{-9}
1.00	7.9×10^{-3}	7.5×10^{-5}	4.9×10^{-7}	3.4×10^{-8}



(a) $\alpha = 0.5$ (b) $\alpha = 0.7$

(c) $\alpha = 0.9$ (d) $\alpha = 1$

Figure 3.4: The surface plot with distinctive values of α.

3.3.4 Example 4

Time-fractional Klein Gordon type equation is represented as [35]:

$$D_{\tau}^{\alpha} Q = Q_{xx} - Q + 2cos(x),\ 1 < \alpha \leq 2, \tag{3.31}$$

and the initial conditions given as:

$$Q_0(x) = cos(x),\ Q_0^{'} = 1, \tag{3.32}$$

from Equation (3.31), we set:

$$\left.\begin{aligned} &\mathscr{L}(Q(x,\tau)) = Q_{xx} - Q \\ &\mathscr{N}(Q(x,\tau)) = 0, \\ &\mathscr{G}(Q(x,\tau)) = 2cos(x). \end{aligned}\right\} \tag{3.33}$$

Now, using the iterative procedure described in the previous chapter, we have:

$$Q_0 = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\psi^{2-\alpha+r}} + \mathscr{A}[\mathscr{G}(Q(x,\tau))]\right)\right], n = 2$$

$$= \mathscr{A}^{-1}\left[\frac{Q^{'}(x,0)}{\psi^{2}}+\frac{Q^{(0)}(x,0)}{\psi^{3}}+\frac{2cos(x)}{\psi^{2+\alpha}}\right]$$

$$= cos(x)+\tau+2cos(x)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.34}$$

$$Q_1 = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{L}(Q_0(x,\tau))+\mathscr{N}(Q_0(x,\tau))\right]\right)\right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[(Q_0)_{xx}-Q_0\right]\right)\right],$$

$$= -2cos(x)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}-4cos(x)\frac{\tau^{(2\alpha)}}{\Gamma(2\alpha+1)}-\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{3.35}$$

$$Q_2 = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{L}(Q_1(x,\tau))+\{\mathscr{N}(Q_0(x,\tau)+Q_1(x,\tau))-\mathscr{N}(Q_0(x,\tau))\}\right]\right)\right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[(Q_1)_{xx}-Q_1\right]\right)\right],$$

$$= 4cos(x)\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+8cos(x)\frac{\tau^{(}3\alpha)}{\Gamma(3\alpha+1)}+\frac{\tau^{2\alpha+1}}{\Gamma(2\alpha+2)}. \tag{3.36}$$

$$\vdots$$

$$Q_\kappa = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{L}\left(Q_{k-1}(x,\tau)\right)\right]\right)\right]+$$
$$\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\left\{\mathscr{N}\left(\sum_{j=0}^{r-1}Q_j(x,\tau)\right)-\mathscr{N}\left(\sum_{j=0}^{r=2}Q_j(x,\tau)\right)\right\}\right]\right)\right],$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[(Q_{\kappa-1})_{xx}-Q_{\kappa-1}\right]\right)\right],$$

$$= (-1)^{\kappa}2^{\kappa}cos(x)\frac{\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}+(-1)^{\kappa}2^{\kappa+1}\frac{\tau^{(\kappa+1)\alpha}}{\Gamma((\kappa+1)\alpha+1)}+(-1)^{\kappa}\frac{\tau^{\kappa\alpha+1}}{\Gamma(\kappa\alpha+2)}. \tag{3.37}$$

We get the κ-th order approximate series as:

$$\mathscr{Q}^{(\kappa)}(x,\tau) = \sum_{m=0}^{\kappa}Q_m(x,\tau)$$

$$= Q_0(x,\tau)+Q_1(x,\tau)+Q_2(x,\tau)+\cdots+Q_\kappa(x,\tau),\ \kappa\in\mathbb{N}\,.$$

$$= cos(x) + \tau - \frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{\tau^{2\alpha}+1}{\Gamma(2\alpha+2)} - \frac{\tau^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots + (-1)^{\kappa}\frac{\tau^{\kappa\alpha+1}}{\Gamma(\kappa\alpha+2)}$$

$$= cos(x) + \sum_{m=0}^{\kappa}(-1)^{\kappa}\frac{\tau^{m\alpha+1}}{\Gamma(m\alpha+2)}. \tag{3.38}$$

The approximate series solution come near to the exact solution as $\kappa \to \infty$,

$$Q(x,\tau) = \lim_{\kappa\to\infty} \mathscr{Q}^{(\kappa)}(x,\tau)$$

$$= cos(x) + \lim_{\kappa\to\infty}\sum_{m=0}^{\kappa}(-1)^{\kappa}\frac{\tau^{m\alpha+1}}{\Gamma(m\alpha+2)}$$

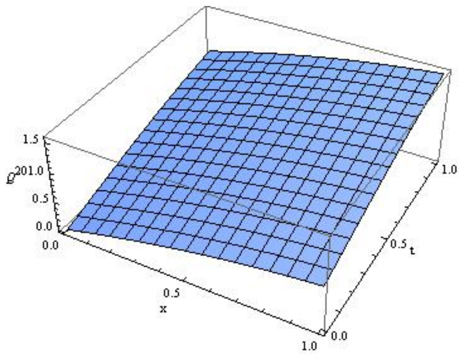
If $\alpha = 2$, then the exact solution of Equation (3.31) is :

$$Q_e(x,\tau) = cos(x) + sin(\tau). \tag{3.39}$$

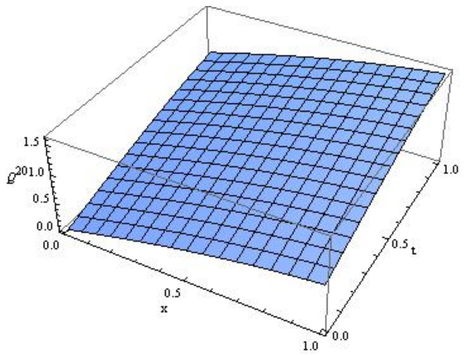
We compare the absolute difference between the obtained solutions for various values of α and obtained the absolute error $E_{abs} = |Q_e(0.1,\tau) - \mathscr{Q}^{(10)}(0.1,\tau)|$ at τ = 0.01, 0.02, 0.03, 0.04 when $\alpha = 2$, with $x = 0.1$ in Table 3.4. Figure 3.4 displays the surface plot when $\alpha = 1.4, 1.6, 1.8$ and 2. with the choice of Q_0 unchanged.

Table 3.4: Comparison of error difference for various α.

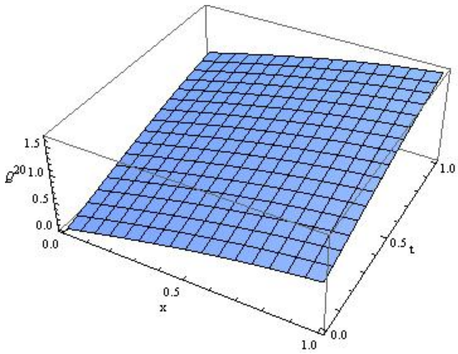
τ	$\alpha = 1.95$	$\alpha = 1.97$	$\alpha = 1.99$	E_{abs} when $\alpha = 2.00$
0.01	5.7×10^{-8}	3.2×10^{-8}	1.0×10^{-8}	0
0.02	3.9×10^{-7}	2.2×10^{-7}	7.1×10^{-8}	0
0.03	1.4×10^{-3}	6.9×10^{-7}	2.1×10^{-7}	0
0.04	2.7×10^{-6}	1.5×10^{-6}	4.9×10^{-7}	0



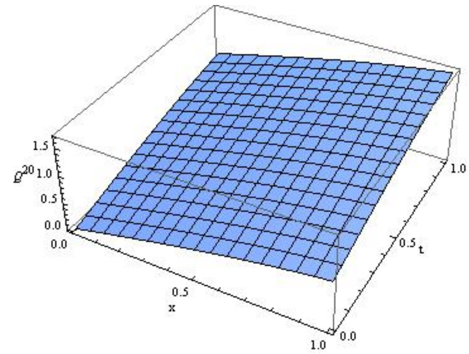
(a) $\alpha = 1.4$



(b) $\alpha = 1.6$



(c) $\alpha = 1.8$



(d) $\alpha = 2$

Figure 3.5: The surface plot for distinct values of α.

Chapter 4

ABOODH TRANSFORM ITERATIVE TECHNIQUE FOR SOLVING SPATIAL DIFFUSION BIOLOGICAL POPULATION MODEL WITH FRACTIONAL ORDER

4.1 Chapter Overview

The significant objective here is to solve the time fractional biological population model with fractional order using the Aboodh transform iterative method [42]. The solution is given in a fast convergent series form. Three descriptive cases are examined, the numerical solutions are closely observed with distinctive values of alpha in comparison with the exact solution. The leverage of Aboodh transform iterative method over other methods is the elegant style of implementation couple with ease.

4.2 Problem Statement

Spatial diffusion biological population model with fractional order is of the form:

$$D_\tau^\alpha Q = (Q^2)_{xx} + (Q^2)_{yy} + \mathscr{G}(Q),\ \tau > 0,\ x, y \in \mathbb{R},\ 0 < \alpha \leq 1\ , \tag{4.1}$$

$$\mathscr{G}(Q) = fQ^q(1 - rQ^p)\ , \tag{4.2}$$

as well as the initial condition:

$$Q(x, y, 0) = Q_0(x, y)\ , \tag{4.3}$$

where Q imply the population density and $\mathscr{G}(Q)$ denote the supply of population by reason of deaths and births, f, q, p, r are real numbers, D^α denotes the differential operator in Caputo sense, for Hölder estimates and its solution see [43], the constitutive equations are given as

$$\mathscr{G}(Q) = CQ\,, \tag{4.4}$$

C is a constant, Malthusian Law [44]:

$$\mathscr{G}(Q) = C_1 Q - C_2 Q\,, \tag{4.5}$$

for positive constant C_1 and C_2, Verhulst Law [43].

$$\mathscr{G}(Q) = CQ^{\theta},\ (C > 0,\ 0 < \theta \leq 1)\,, \tag{4.6}$$

C is a constant, Porous media [45, 46].

4.3 Descriptive Cases

In this section, we considered three cases to demonstrate the efficiency of Aboodh transform Iterative method.

4.3.1 Case 1

If $f = \frac{1}{4},\ q = 1,\ r = 0$, (Malthusian Law [44]) and $Q_0(x) = x^{1/2}$ one dimensional time-fractional biological population model is derived and Equation (4.1) becomes:

$$D_{\tau}^{\alpha} Q = (Q^2)_{xx} + \frac{Q}{4},\ \ \tau > 0,\ \ 0 < \alpha \leq 1\,, \tag{4.7}$$

from Case (1) we set:

$$\left.\begin{aligned} &\mathscr{R}(Q(x,\tau)) = \frac{Q}{4}, \\ &\mathscr{N}(Q(x,\tau)) = (Q^2)_{xx}, \\ &Q(x,0) = x^{1/2}\,. \end{aligned}\right\} \tag{4.8}$$

Using the iterative procedure described in Chapter 2,

$$\begin{aligned} Q_0 &= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{Q^{(s)}(x,0)}{\psi^{2-\alpha+s}}\right)\right] \\ &= \mathscr{A}^{-1}\left[\frac{Q(x,0)}{\psi^2}\right] \end{aligned} \tag{4.9}$$

$$= x^{1/2}\,.$$

$$\begin{aligned}
Q_1 &= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q_0(x,\tau)) + \mathscr{N}(Q_0(x,\tau))\right]\right)\right] \\
&= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\frac{Q_0}{4} + (Q_0^2)_{xx}\right]\right)\right] \qquad (4.10) \\
&= x^{1/2}\frac{\frac{1}{4}\tau^{\alpha}}{\Gamma(\alpha+1)}\,.
\end{aligned}$$

$$\begin{aligned}
Q_2 &= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q_1(x,\tau)) + \{\mathscr{N}(Q_0(x,\tau)+Q_1(x,\tau)) - \mathscr{N}(Q_0(x,\tau))\}\right]\right)\right] \\
&= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\frac{Q_1}{4} + \left\{((Q_0+Q_1)^2)_{xx} - (Q_0^2)_{xx}\right\}\right]\right)\right] \qquad (4.11) \\
&= \frac{x^{1/2}\left(\frac{1}{4}\tau^{\alpha}\right)^2}{\Gamma(2\alpha+1)}\,.
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
Q_m &= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q_{m-1}(x,\tau))\right]\right)\right] + \\
&\qquad \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\left\{\mathscr{N}\left(\sum_{i=0}^{m-1} Q_i(x,\tau)\right) - \mathscr{N}\left(\sum_{i=0}^{m-2} Q_i(x,\tau)\right)\right\}\right]\right)\right] \\
&= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\frac{Q_{m-1}(x,\tau)}{4}\right]\right)\right] \qquad (4.12) \\
&= \frac{x^{1/2}\left(\frac{1}{4}\tau^{\alpha}\right)^m}{\Gamma(m\alpha+1)}\,.
\end{aligned}$$

The m-th order approximate series is derived as:

$$\begin{aligned}
\mathscr{Q}^{(m)}(x,\tau) &= \sum_{\kappa=0}^{m} Q_{\kappa}(x,\tau) = Q_0(x,\tau) + Q_1(x,\tau) + Q_2(x,\tau) + \cdots + Q_m(x,\tau) \\
&= x^{1/2} + \frac{x^{1/2}\frac{1}{4}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{1/2}\left(\frac{1}{4}\tau^{\alpha}\right)^2}{\Gamma(2\alpha+1)} + \cdots + \frac{x^{1/2}\left(\frac{1}{4}\tau^{\alpha}\right)^m}{\Gamma(m\alpha+1)}
\end{aligned}$$

$$= x^{1/2}\left(\frac{\frac{1}{4}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{\left(\frac{1}{4}\tau^{\alpha}\right)^{2}}{\Gamma(2\alpha+1)} + \cdots + \frac{\left(\frac{1}{4}\tau^{\alpha}\right)^{m}}{\Gamma(m\alpha+1)}\right) \tag{4.13}$$

$$= x^{1/2}\sum_{\kappa=0}^{m}\frac{\left(\frac{1}{4}\tau^{\alpha}\right)^{\kappa}}{\Gamma(\kappa\alpha+1)} .$$

So, the m-th order approximate series solution come closer to the exact solution as $m \to \infty$,

$$Q(x,\tau) = \lim_{m\to\infty}\mathscr{Q}^{(m)}(x,\tau)$$

$$= x^{1/2}\lim_{m\to\infty}\sum_{\kappa=0}^{m}\frac{\left(\frac{1}{4}\tau^{\alpha}\right)^{\kappa}}{\Gamma(\kappa\alpha+1)} \tag{4.14}$$

$$= x^{1/2}E_{\alpha}\left(\frac{1}{4}\tau^{\alpha}\right), \tag{4.15}$$

taking $\alpha = 1$, the exact solution to Equation (4.7) is:

$$Q_e(x,\tau) = x^{1/2}E_1\left(\frac{1}{4}\tau\right)$$

$$= x^{1/2}\exp\left(\frac{\tau}{4}\right) . \tag{4.16}$$

We compare the absolute difference $E_{dif} = |Q(x,\tau) - \mathscr{Q}^{(10)}(x,\tau)|$ between the series approximate solution and the close form solution for various values of α and obtained the absolute error $E_{abs} = |Q_e(x,\tau) - \mathscr{Q}^{(10)}(x,\tau)|$ when $\alpha = 1$ in Table 4.1. Figure 4.1 unveil the comparison plot of the exact solution with the approximate solution while Figure 4.2 displays the shape of the population density when $\alpha = 0.25, 0.5, 0.75$ and 1. with the choice of Q_0 unchanged.

Table 4.1: Comparison of error difference for Case 1 when $\tau = 0.2$.

x	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	E_{abs} when $\alpha = 1.00$
0.1	5.4×10^{-3}	6.2×10^{-7}	2.7×10^{-11}	4.4×10^{-16}
0.5	6.2×10^{-3}	7.1×10^{-7}	3.1×10^{-11}	6.7×10^{-16}
0.9	7.1×10^{-3}	8.2×10^{-4}	3.5×10^{-11}	6.7×10^{-16}

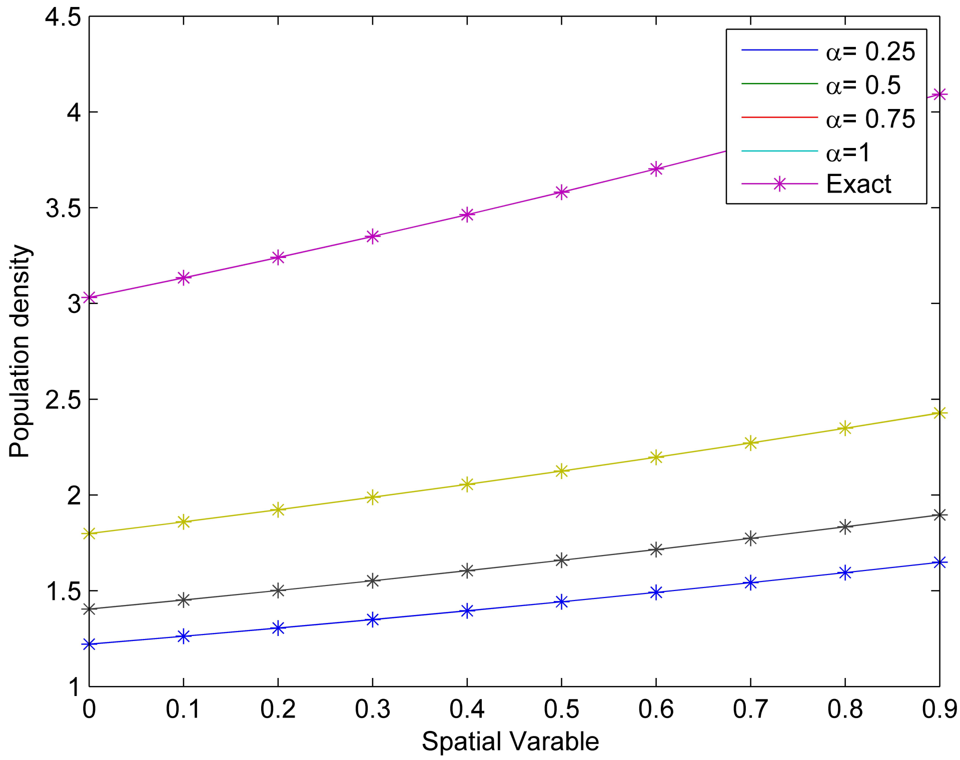
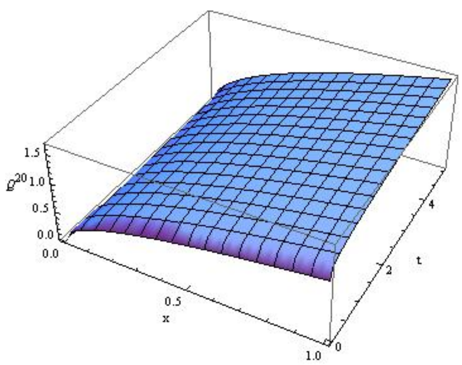
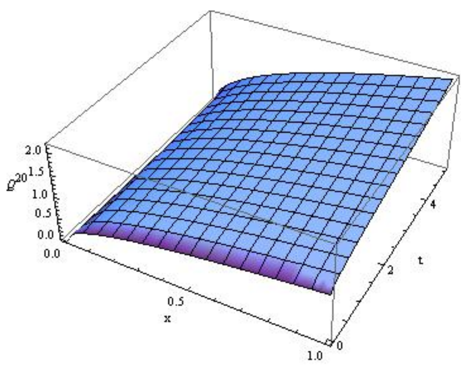


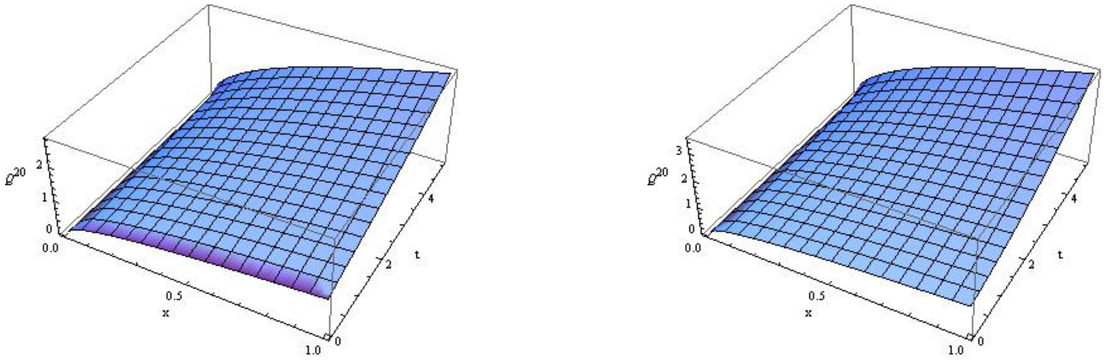
Figure 4.1: Comparison plot for Case (1).



(a) $\alpha = 0.25$



(b) $\alpha = 0.5$



(a) $\alpha = 0.75$ (b) $\alpha = 1$

Figure 4.3: 3-D plot for Case (1) with distinct values of α.

4.3.2 Case 2

If $f = 1$, $r = 0$ (Porous media [47]) and $Q_0(x,y) = (Sin x \times Sinh y)^{1/2}$, Equation (4.1) becomes:

$$D_\tau^\alpha Q = (Q^2)_{xx} + (Q^2)_{yy} + Q, \;\; \tau > 0, \;\; 0 < \alpha \leq 1 \;, \tag{4.17}$$

from Case (2), we set:

$$\left.\begin{aligned} &\mathscr{R}(Q(x,y,\tau)) = Q, \\ &\mathscr{N}(Q(x,y,\tau)) = (Q^2)_{xx} + (Q^2)_{yy}, \\ &Q(x,y,0) = (Sin\,(x) \times Sinh\,(y))^{1/2}. \end{aligned}\right\} \tag{4.18}$$

Using the Aboodh transform iterative procedure,

$$\begin{aligned} Q_0 = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\sum_{s=0}^{n-1}\frac{Q^{(s)}(x,y,0)}{\psi^{2-\alpha+s}}\right)\right] \\ = \mathscr{A}^{-1}\left[\frac{1}{\psi^2}\,Q(x,y,0)\right] \\ = (Sin\,(x) \times Sinh\,(y))^{1/2}\;. \end{aligned} \tag{4.19}$$

$$Q_1 = \mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{R}(Q_0(x,y,0)) + \mathscr{N}(Q_0(x,y,0))\right]\right)\right]$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[Q_0+(Q_0^2)_{xx}+(Q_0^2)_{yy}\right]\right)\right] \tag{4.20}$$

$$= (Sin\,(x)\,\times\,Sinh\,(y))^{1/2}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\;.$$

$Q_2 =$

$$\mathscr{A}^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q_0(x,y,\tau))+\{\mathscr{N}(Q_0(x,y,\tau)+Q_1(x,y,\tau))-\mathscr{N}(Q_0(x,y,\tau))\}\right]\right)\right]$$

$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[Q_1+\left\{((Q_0+Q_1)^2)_{xx}+((Q_0+Q_1)^2)_{yy}-((Q_0^2)_{xx}+(Q_0^2)_{yy})\right\}\right]\right)\right] \tag{4.21}$$

$$= (Sin\,(x)\,\times\,Sinh\,(y))^{1/2}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\;.$$

$$\vdots$$

$$Q_m = \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q_{m-1}(x,y,\tau))\right]\right)\right]+$$
$$\mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\left\{\mathscr{N}\left(\sum_{i=0}^{m-1}Q_i(x,y,\tau)\right)-\mathscr{N}\left(\sum_{i=0}^{m-2}Q_i(x,y,\tau)\right)\right\}\right]\right)\right]$$
$$= \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[Q_{m-1}(x,y,\tau)\right]\right)\right] \tag{4.22}$$
$$= (Sin\,(x)\,\times Sinh\,(y))^{1/2}\frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}\;.$$

The m-th order approximate series is derived as:

$$\mathscr{Q}^{(m)}(x,y,\tau) = \sum_{\kappa=0}^{m} Q_{\kappa}(x,y,\tau) = Q_0(x,y,\tau)+Q_1(x,y,\tau)+Q_2(x,y,\tau)+\cdots+Q_m(x,y,\tau)$$

$$= (Sin\,(x)\,\times\,Sinh\,(y))^{1/2}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots+\frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}\right) \tag{4.23}$$

$$= (Sin\,(x)\,\times\,Sinh\,(y))^{1/2}\sum_{\kappa=0}^{m}\frac{\tau^{\kappa\alpha}}{\Gamma(\kappa\alpha+1)}\;.$$

So, the m-th order approximate series solution come close to the exact solution as

$m \to \infty$,

$$Q(x,y,\tau) = \lim_{m\to\infty} \mathscr{Q}^{(m)}(x,y,\tau)$$

$$= (Sin\,(x) \times Sinh\,(y))^{1/2} \lim_{m\to\infty} \sum_{\kappa=0}^{m} \frac{\tau^{\kappa\alpha}}{\Gamma(k\alpha+1)} \tag{4.24}$$

$$= (Sin\,(x) \times Sinh\,(y))^{1/2}\, E_\alpha(\tau^\alpha).$$

Taking $\alpha = 1$, the exact solution of Equation (4.17) is:

$$Q_e(x,y,\tau) = (Sin\,(x) \times Sinh\,(y))^{1/2}\, E_1(\tau)$$

$$= (Sin\,(x) \times Sinh\,(y))^{1/2} \exp(\tau)\,, \tag{4.25}$$

which tally with the solution in [47]. We compare the absolute difference between the series approximate solution and close form solution for $\alpha = 0.5$ and obtained the absolute error similarity of the present method with the method in [48] when $\tau = 0.2$ and $\alpha = 1$. The similarity plot of the exact and approximate solutions is presented in Figure 4.3 while Figure 4.4 is the surface shape of the population density when $\alpha = 0.25, 0.5, 0.75$ and 1.

Table 4.2: Similarity of the error investigation for Case (2).

x	y	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 0.5$ in [48]	$\alpha = 1.0$ in [48]
0.1	0.2	8.6×10^{-8}	5.6×10^{-17}	8.6×10^{-8}	5.6×10^{-17}
0.1	0.6	1.5×10^{-7}	1.1×10^{-16}	1.5×10^{-7}	1.1×10^{-16}
0.1	1	2.1×10^{-7}	1.1×10^{-16}	2.1×10^{-7}	1.1×10^{-16}
0.5	0.2	1.9×10^{-7}	1.7×10^{-16}	1.9×10^{-7}	1.7×10^{-16}
0.5	0.6	3.3×10^{-8}	2.2×10^{-17}	3.3×10^{-8}	2.2×10^{-17}
0.5	1	4.5×10^{-7}	3.3×10^{-16}	4.5×10^{-7}	3.3×10^{-16}
0.9	0.2	2.4×10^{-7}	1.7×10^{-16}	2.4×10^{-7}	1.7×10^{-16}
0.9	0.6	4.3×10^{-7}	3.3×10^{-16}	4.3×10^{-7}	3.3×10^{-16}
0.9	1	5.8×10^{-7}	4.4×10^{-16}	5.8×10^{-7}	4.4×10^{-16}

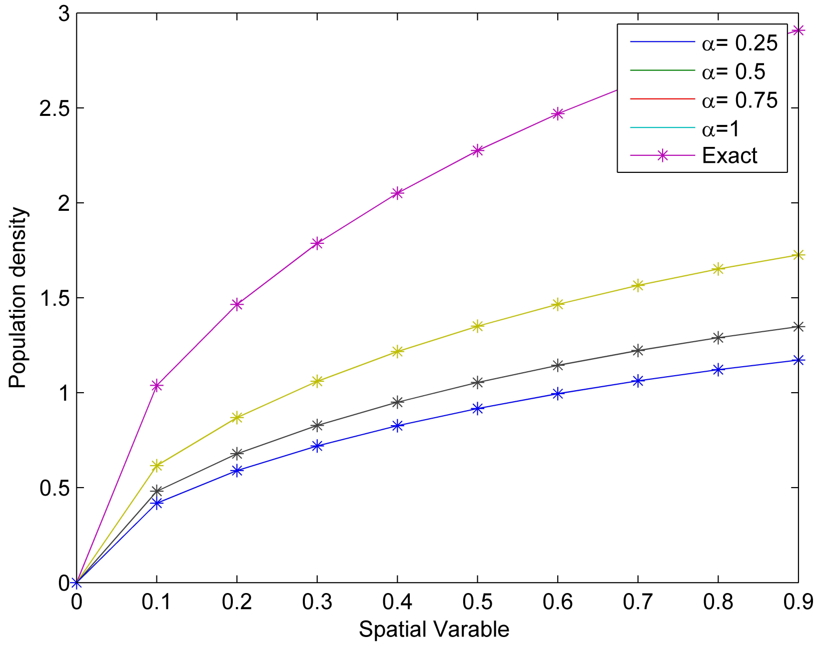
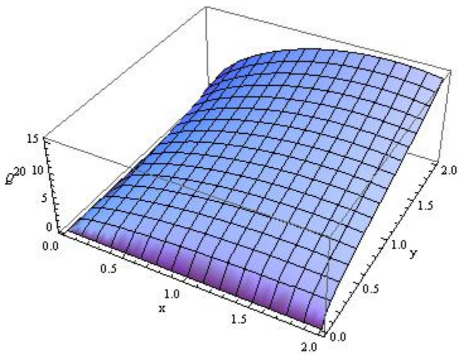
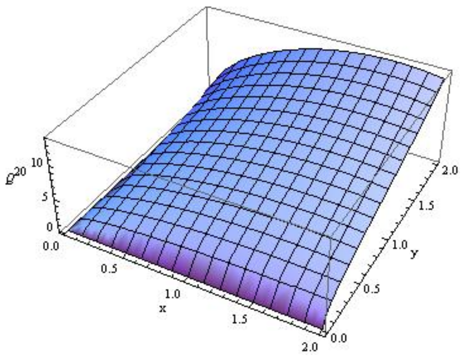


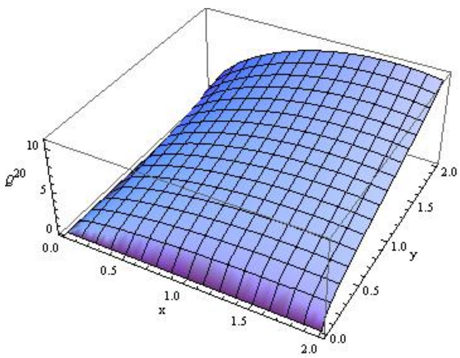
Figure 4.4: Similarity plot of Case (2).



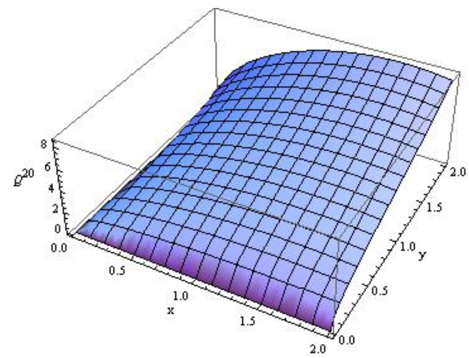
(a) $\alpha = 0.25$



(b) $\alpha = 0.5$



(c) $\alpha = 0.75$



(d) $\alpha = 1$

Figure 4.5: 3-D plot for Case (2) with distinct values for α.

4.3.3 Case 3

If $q=p=1$, (Verhulst Law [44]) and $Q_0(x,y)=\exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)$, the time-fractional biological population model Equation (4.1) becomes:

$$D_\tau^\alpha Q=(Q^2)_{xx}+(Q^2)_{yy}+fQ(1-rQ),\ \tau>0,\ 0<\alpha\leq 1\ , \tag{4.26}$$

from case 3, we set:

$$\left.\begin{aligned}&\mathscr{R}(Q(x,y,\tau))=fQ,\\&\mathscr{N}(Q(x,y,\tau))=(Q^2)_{xx}+(Q^2)_{yy}-frQ^2,\\&Q(x,y,0)=\exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right).\end{aligned}\right\} \tag{4.27}$$

Using the Aboodh transform iterative procedure ,

$$\begin{aligned}Q_0&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\sum_{s=0}^{n-1}\frac{Q^{(s)}(x,y,0)}{\psi^{2-\alpha+s}}\right)\right]\\&=\mathscr{A}^{-1}\left[\frac{Q(x,y,0)}{\psi^2}\right]\\&=\exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\ .\end{aligned} \tag{4.28}$$

$$\begin{aligned}Q_1&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{R}(Q_0(x,y,0))+\mathscr{N}(Q_0(x,y,0))\right]\right)\right]\\&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[fQ_0+(Q_0^2)_{xx}+(Q_0^2)_{yy}-frQ_0^2\right]\right)\right]\\&=\exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\frac{f\tau^\alpha}{\Gamma(\alpha+1)}\ .\end{aligned} \tag{4.29}$$

$$\begin{aligned}Q_2&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[\mathscr{R}(Q_1(x,y,0))+\{\mathscr{N}(Q_0(x,y,\tau)+Q_1(x,y,\tau))-N(Q_0(x,y,\tau))\}\right]\right)\right]\\&=\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[fQ_1+((Q_0+Q_1)^2)_{xx}+((Q_0+Q_1)^2)_{yy}-fr(Q_0+Q_1)^2\right]\right)\right]\\&\quad-\mathscr{A}^{-1}\left[\frac{1}{\psi^\alpha}\left(\mathscr{A}\left[((Q_0^2)_{xx}-(Q_0^2)_{yy}+frQ_0^2)\right]\right)\right]\end{aligned} \tag{4.30}$$

$$= \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\frac{(f\tau^{\alpha})^2}{\Gamma(2\alpha+1)}.$$

$$\vdots$$

$$\begin{aligned} Q_m = {} & \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\mathscr{R}(Q_{m-1}(x,y,\tau))\right]\right)\right] + \\ & \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[\left\{\mathscr{N}\left(\sum_{i=0}^{m-1} Q_i(x,y,\tau)\right) - \mathscr{N}\left(\sum_{i=0}^{m-2} Q_i(x,y,\tau)\right)\right\}\right]\right)\right] \\ = {} & \mathscr{A}^{-1}\left[\frac{1}{\psi^{\alpha}}\left(\mathscr{A}\left[fQ_{m-1}(x,y,\tau)\right]\right)\right] \\ = {} & \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\frac{(f\tau^{\alpha})^m}{\Gamma(m\alpha+1)}. \end{aligned} \tag{4.31}$$

The m-th order approximate series is derived as:

$$\mathscr{Q}^{(m)}(x,y,\tau) = \sum_{\kappa=0}^{m} Q_{\kappa}(x,y,\tau) = Q_0(x,y,\tau) + Q_1(x,y,\tau) + Q_2(x,y,\tau) + \cdots + Q_m(x,y,\tau)$$

$$\begin{aligned} & = \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right) + \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\frac{f\tau^{\alpha}}{\Gamma(\alpha+1)} + \\ & \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\frac{(f\tau^{\alpha})^2}{\Gamma(2\alpha+1)} + \ldots + \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\frac{(f\tau^{\alpha})^m}{\Gamma(m\alpha+1)} \\ & = \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\left(1 + \frac{f\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{(f\tau^{\alpha})^2}{\Gamma(2\alpha+1)} + \cdots + \frac{(f\tau^{\alpha})^m}{\Gamma(m\alpha+1)}\right) \\ & = \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\sum_{\kappa=0}^{m}\frac{(f\tau^{\alpha})^{\kappa}}{\Gamma(\kappa\alpha+1)}. \end{aligned} \tag{4.32}$$

So, the m-th order series approximate solution come near to the exact solution as $m \to \infty$,

$$\begin{aligned} Q(x,y,\tau) & = \lim_{m\to\infty}\mathscr{Q}^{(m)}(x,y,\tau) \\ & = \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right)\lim_{m\to\infty}\sum_{k=0}^{m}\frac{(f\tau^{\alpha})^{k}}{\Gamma(k\alpha+1)} \end{aligned} \tag{4.33}$$

$$= \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right) E_\alpha(f\tau^\alpha).$$

Taking $\alpha = 1$, the exact solution to Equation (4.26) is:

$$Q_e(x,y,\tau) = \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y)\right) E_1(f\tau)$$

$$= \exp\left(\frac{1}{2}\left(\frac{fr}{2}\right)^{\frac{1}{2}}(x+y) + f\tau\right), \tag{4.34}$$

which is the exact solution to the biological population model in [15] when $f = 1$. We compare the absolute difference $E_{dif} = |Q(x,y,\tau) - \mathscr{Q}^{(10)}(x,y,\tau)|$ between the series approximate solution and the close form solution for various values of α and obtained the absolution error $E_{abs} = |Q_e(x,y,\tau) - \mathscr{Q}^{(10)}(x,y,\tau)|$ when $\alpha = 1$ in Table 4.3. Figure 4.5 reveals the comparison of the exact solution with the approximate solution while Figure 4.6 displays the surface shape of the population density when $\alpha = 0.25, 0.5, 0.75$ and 1. with $\tau = 46$, $f = 0.02$, $r = 0.2$.

Table 4.3: Comparison of error difference for Case (3) when $\tau = 0.2$.

x	y	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.0$
0.1	0.2	6.4×10^{-11}	1.1×10^{-14}	0	0
0.1	0.6	6.9×10^{-11}	1.2×10^{-14}	0	0
0.1	1	7.3×10^{-11}	1.3×10^{-14}	0	0
0.5	0.2	6.9×10^{-11}	1.2×10^{-14}	0	0
0.5	0.6	7.3×10^{-11}	1.3×10^{-14}	0	0
0.5	1	7.8×10^{-11}	1.4×10^{-14}	0	0
0.9	0.2	7.3×10^{-11}	1.2×10^{-14}	0	0
0.9	0.6	7.8×10^{-11}	1.4×10^{-14}	0	0
0.9	1	8.3×10^{-11}	1.4×10^{-14}	0	0

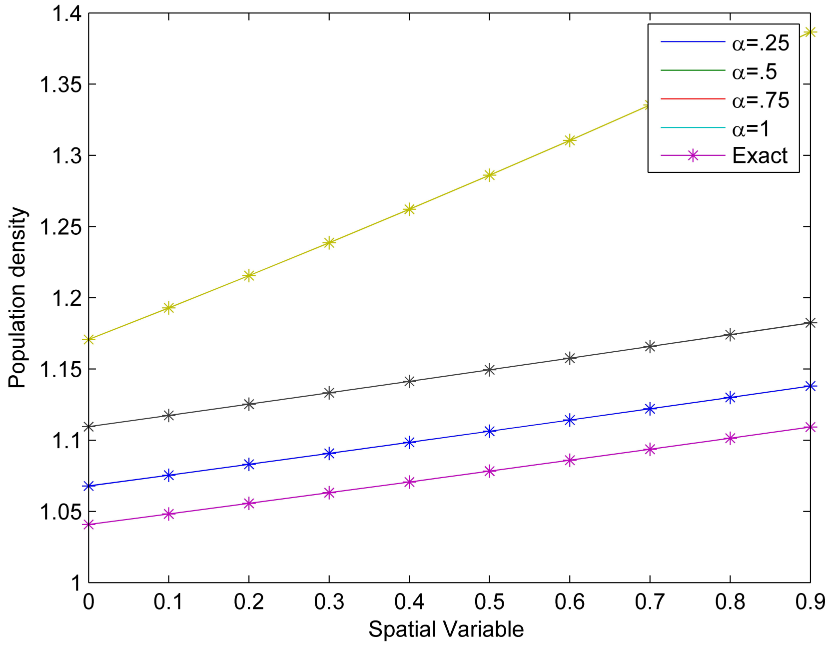
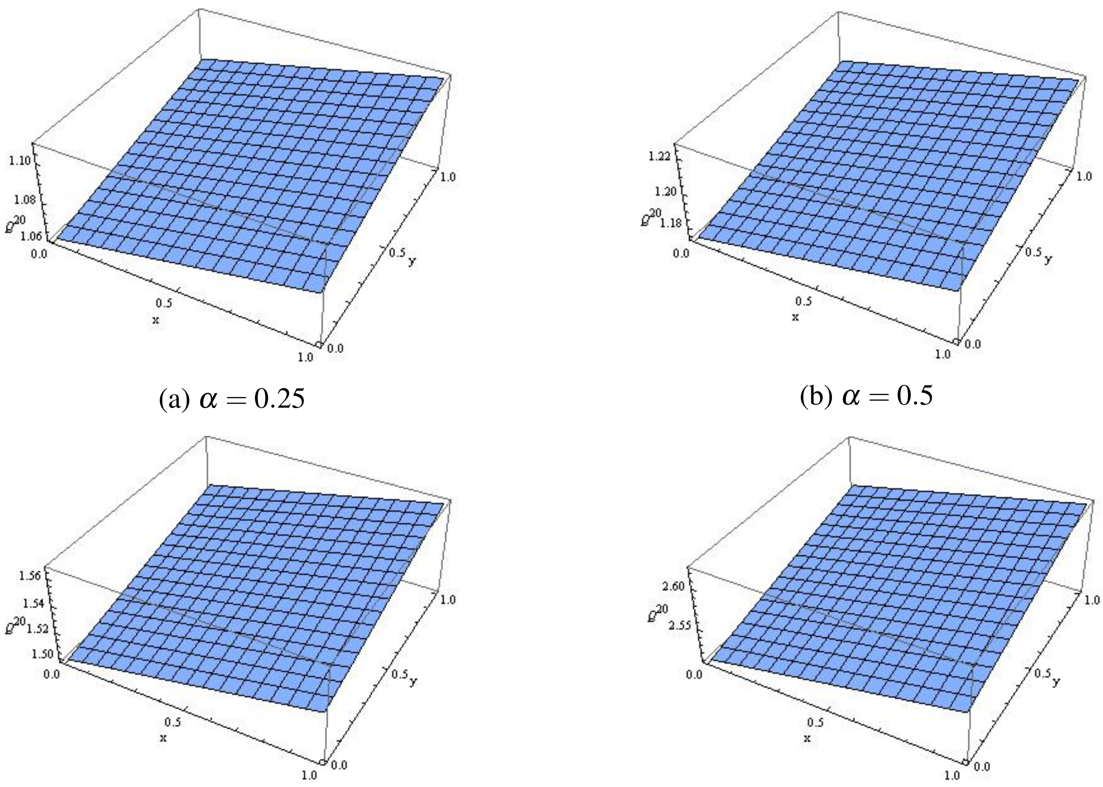


Figure 4.6: Comparison plot of the exact and approximate solutions for Case (3).


(a) $\alpha = 0.25$ (b) $\alpha = 0.5$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

Figure 4.7: 3-D plot for Case (3) with various values of α.

Chapter 5

CONCLUSION

In this thesis work, we have presented and applied the Aboodh transform iterative method to achieve series approximate analytical solution of gas dynamics equation with fractional order, time-fractional Fokker-Planck equation, fractional Kolmogorov equation, Klein-Gordon type equation with fractional order and spatial diffusion biological population model with the positive non-integer order. We decompose the nonlinear terms using the new iterative method. The method is easy to implement without the requirement for restrictive assumptions, Largrange multipliers and Adomian's polynomials, we have provided several examples to support these claim. It is possible to apply Aboodh transform iterative method to obtain series approximate analytical solution of some fluid dynamics problems and further expand the method to solve boundary value problems.

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