On the $\omega-$Multiple Meixner Polynomials

İlkay Onbaşı

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Eastern Mediterranean University
July 2021
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Sonuç Zorlu Oğurlu
Supervisor

Examining Committee

1. Prof. Dr. Mehmet Küçükaslan
2. Prof. Dr. Mehmet Ali Özarslan
3. Prof. Dr. Derviş Subaşı
4. Prof. Dr. Fatma Taşdelen Yeşildal
5. Prof. Dr. Sonuç Zorlu Oğurlu

ABSTRACT

In this thesis, a new family of discrete MOPs, namely ω-multiple Meixner polynomials, where ω is a positive real number is introduced. For ω-MOPs, orthogonality conditions w.r.t r (with $r > 1$) different Pascal distributions (Negative Binomial distributions) are used. Depending on the selection of the parameters in the Negative Binomial distribution, two kinds of ω-MMPs, namely 1st and 2nd kinds are considered. Some structural properties of ω-MMPs, such as raising operator, Rodrigue's type formula and explicit representation are derived. The generating function for ω-MMPs is obtained and by use of this generating function several consequences for these polynomials are reached. A lowering operator for ω-MMPs which will be helpful for obtaining difference equation is also derived. By combining the lowering operator with the raising operator the difference equation which has the ω-MMPs as a solution are obtained. A third order difference equation for ω-MMPs is given . Also it is shown that for the special case $\omega = 1$, the obtained results coincide with the existing results for MMPs of both kinds. In the last part as an illustrated example for the ω-MMPs of the first kind the special case when $\omega = 1/2$ is considered and for the $1/2$-MMPs of the first kind,the results obtained for the main theorems are stated. For the ω-MMPs of the second kind the special case when $\omega = 5/3$ is studied and for the $5/3$-MMPs of the second kind, the corresponding result obtained for the main theorems are examined.

Keywords: Orthogonal polynomials, Multiple orthogonal polynomials, Rodrigue's type formula, Generating function, Difference equation

ÖZ

Bu tezde, yeni bir kesikli çoklu ortogonal polinom ailesi, yani ω-çoklu Meixner polinomları çalışılmıştır. Burada ω pozitif bir gerçel sayıdır. ω-çoklu Meixner polinomları için r farklı negatif binom dağılımına göre $(r > 1)$ ortogonallik koşulu kullanılmıştır. Seçimine bağlı olarak negatif binom dağılımındaki parametreler, iki tür ω-çoklu Meixner polinomları, yani 1. ve 2. türler dikkate alınmıştır. ω-çoklu Meixner polinomları için bazı yapısal özellikler, örneğin yükseltme operatörü, Rodrigue'nin tür formülü ve açık temsil türetilmiştir. ω-çoklu Meixner polinomları için oluşturma fonksiyonu elde edilmiş ve bu oluşturma fonksiyonunun kullanılmasıyla bu polinomlar için bazı önemli sonuçlara ulaşılmıştır. ω-çoklu Meixner polinomları için Fark denkleminin elde edilmesine de yardımcı olacak bir indirme operatörü türetilmiştir. İndirme operatörünü kaldırma operatörü ile birleştirerek bu polinomlara ait çözümü olacak fark denklemi elde edilmiştir.ω-çoklu Meixner polinomları için üçüncü mertebeden fark denklemi verilmiştir. Ayrıca $\omega = 1$ özel durumu için elde edilen sonuçların, her iki türden genel çoklu Meixner polinomlarına ait mevcut sonuçlarla örtüştüğü gösterilmiştir. Son bölümde birinci türden ω-çoklu Meixner polinomları için $\omega = 1/2$ durumu dikkate alınmış ve elde edilen temel teoremler $1/2$-çoklu Meixner polinomları için uygulama olarak gösterilmiştir. Benzer gösterim, ikinci türden ω-çoklu Meixner polinomları için $\omega = 1/2$ özel durumu ele alınarak ikinci türden $5/3$-çoklu Meixner polinomları için incelenmiştir.

Anahtar Kelimeler: Ortogonal polinomlar, Çoklu ortogonal polinomlar, Rodrigue's tip formülü, Oluşturma fonksiyonu, Fark denklemi

To My Family

ACKNOWLEDGMENTS

First of all I have to thank my research supervisor Prof. Dr.Sonuç Zorlu Oğurlu for her support, patience and continuous guidance throughout the research, and for her corrections in the text. Special thanks are extended to Prof. Dr. Mehmet Ali Özarslan for his advice and support. I must also thank to members of my thesis committee for the time they spent for my thesis work.

I am so grateful to be the part of the department of Mathematics at EMU. I give deep thanks to all the members. My appreciation also goes out to my best friends who have always helped me and believed that I could do it. I also appreciate all the support and encouragement I received from my dear family. They have always been with me and showed their patience throughout my studies.

TABLE OF CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

$(x)_j$	Pochammer Symbol
$(x)_{j,k}$	Pochammer k-Symbol
$\Gamma(\beta)$	Gamma Function
$\Gamma_k(\beta)$	k-GammaFunction
$M_{\vec{n}}^{(\beta,\vec{a})}(x)$	Multiple Meixner Polynomials of the First Kind
$M_{\vec{n}}^{(\vec{\beta};a)}(x)$	Multiple Meixner Polynomials of the Second Kind
CMOP	Continuous multiple orthogonal polynomials
COP	Continuous orthogonal polynomials
DMOP	Discrete multiple orthogonal polynomials
DMP	Discrete orthogonal polynomials
LHS	left hand side
MMP	Multiple Meixner polynomials
MMP-1	Multiple Meixner polynomials of the first kind
MMP-2	Multiple Meixner polynomials of the second kind
MOP	Multiple orthogonal polynomials
OP	Orthogonal polynomials
w.r.t	with respect to
$\omega-$MCP	ω-multiple Charlier polynomials
$\omega-$MMP	$\omega-$Multiple Meixner polynomials
$\omega-$MMP-1	$\omega-$Multiple Meixner polynomials of the first kind
$\omega-$MMP-2	$\omega-$Multiple Meixner polynomials of the second kind

Chapter 1

INTRODUCTION

In this thesis, after an elementary information, in which some important notions and preliminaries for OP and MOP are mentioned, we will concentrate on some new results on the study of $\omega-$MMPs of both kinds, namely $\omega-$MMP-1 and $\omega-$MMP-2.

One feature in the theory of OP is to study them as special functions. The widely used OP can be expressed as terminating hypergeometric series. During the twentieth century researchers have been working on a classification of all such hypergeometric OP and their properties. The very COPs are those which was named after Jacobi, Laguerre, and Hermite. OPs developed in the 19th century from a study of continued fractions by P. L. Chebyshev and was followd by A. A. Markov and T. J. Stieltjes. They appear in many different areas such as Numerical Analysis, Approximation theory, Probability theory, Representation theory of Lie groups, quantum groups, and related subjects, Differential and Integral equations, Combinatorics, Mathematical physics and Number theory. In recent years it has seen a great deal of progress in the area of OPs, which are closely related to many important branches of analysis. They are connected with trigonometric, hypergeometric, Bessel, and elliptic functions, are related to the theory of continued fractions and to important problems of interpolation and mechanical quadrature.

MOPs are generalizations of orthogonal polynomials, which originated from Hermite-Padé approximation in the context of irrationality and transcendence proofs in number theory and further developed in approximation theory. During the past few years, MOPs have also arisen in a natural way in certain models from mathematical physics, including random matrix theory, non-intersecting paths, etc. MOPs are polynomials of one variable which are defined by orthogonality relations with respect to r different weights. recently received renewed interest because tools have become available to investigate their asymptotic behavior. They appear in rational approximation, number theory, random matrices, integrable systems, and geometric function theory. Various families of special multiple orthogonal polynomials have been found, extending the classical orthogonal polynomials but also giving completely new special functions. This notion comes from simultaneous rational approximation, in particular from Hermite-Padé approximation of a system of r functions, and hence has its roots in the nineteenth century.

Let us start with the definition of orthogonality.

The generalisation of the idea of a scalar product of two finite dimensional vectors to an infinite dimensional function space is given below.

$$\int_a^b h(x)g(x)w(x)\,dx,$$

where $w(x) \geq 0$, for $a \leq x \leq b$. In the case when scalar product is zero, we say that the functions $h(x)$ and $g(x)$ are orthogonal.

By use of the above integral definition we are ready to give the definition of orthogonal polynomials. There are two types OPs namely, COPs and DOPs.

For the continuous OPs $P_n(x)$, of order n, the orthogonality relation written as,

$$\int_a^b P_n(x)P_m(x)w(x)\,dx = 0, \quad m \neq n,$$

where $w(x)$ is weight function and $[a,b]$ is interval of orthogonality.

For the COPs, $w(x)$ is continuous or piecewise continuous or integrable function. The Jacobi, Laguerre, and Hermite polynomials are examples for the COP. They form the simplest class of special functions and the theory of these polynomials allows important generalizations. By use of the Rodrigue's formula for these COPs, the integral representations for some important functions such as Bessel and Hypergeometric functions can be obtained.

The polynomial set $P_m(x)$,$(m = 1,2,\dots)$ form a classical orthogonal polynomials set of a continuous variable if $P_m(x)$ (polynomial of degree m) satisfy the hypergeometric type differential equation which has form $\eta(x)P_m''(x) + \rho(x)P_m'(x) + \alpha_m P_m(x) = 0$ with Pearson type equation $\frac{d}{dx}[\eta(x)w(x)] = \rho(x)w(x)$, where $\eta(x)$ and $\rho(x)$ are polynomials of at most second and first degree respectively and α_m is a constant.

The above examples for COPs form a classical orthogonal polynomials set of a continuous variable .

For the DOPs, $P_n(x)$ of order n, the orthogonality relation can be written as,

$$\sum_{i=0}^{\infty} P_n(x_i)P_m(x_i)w_i(x) = 0, \quad m \neq n,$$

where weight function $w(x)$ is a jump function, which means at the point x_i the left and right limit exists but they are not equal. The Hahn, Chebyshev, Meixner, Kravchuk, and Charlier polynomials are examples for the DOP.

The relations for upward and backward difference operators will be given.

The upward difference operator is defined by $\Delta g(z) = g(z+1) - g(z)$ and the backward difference operator is defined by $\nabla g(z) = g(z) - g(z-1)$.

The polynomial set $P_m(x)$, $(m = 1, 2, \dots)$ form a classical orthogonal polynomials set of a discrete variable if $P_m(x)$ (polynomial of degree m) satisfy the hypergeometric type difference equation which has form $\eta(x)\Delta\nabla P_m(x) + \rho(x)\Delta P_m(x) + \alpha_m P_m(x) = 0$ with equation $\Delta[\eta(x)w(x)] = \rho(x)w(x)$, where $\eta(x)$ and $\rho(x)$ are polynomials of at most second and first degree respectively and α_m is a constant.

The above examples for DOPs form a classical orthogonal polynomials set of a discrete variable .

Discrete multiple orthogonal polynomials are useful extension of discrete orthogonal polynomials. The theory of DOPs on a linear lattice were extended to MOP which satisfy orthogonality conditions w.r.t r positive discrete measures, by J. Arvesu, J. Coussement and W. Van Assche [3].

The type II discrete MOP $P_{\overrightarrow{n}}$ of degree $\leq |\overrightarrow{n}|$, corresponding to multi-index $\overrightarrow{n} = (n_1, ..., n_r) \in \mathbb{N}^r$, is a polynomial which satisfies the orthogonality conditions [3]

$$\sum_{x=0}^{\infty} P_{\overrightarrow{n}}(x)(-x)_j w_i(x) = 0, \quad j = 0, 1, \dots n_i - 1, \quad i = 1, \dots, r. \tag{1.1}$$

The orthogonality conditions give us a linear system of $|\overrightarrow{n}| = n_1 + n_2 + ... + n_r$ homogenous equation for the $|\overrightarrow{n}| + 1$ unknown coefficients of polynomials which always has a nontrivial solution. If the given multi-index $\overrightarrow{n}$ is normal, then the corresponding polynomials will be unique polynomials. For the uniqueness of the polynomials, E.M. Nikishin and V.N. Sorokin [15] have introduced a system which is called an AT system.

Let us now give the definition of an AT system and Chebyshev system.

Definition 1.1: (cf. [22]) A system of functions $F = (f_0, f_1, ..., f_n)$ of complex–valued functions defined on a proper interval I is called a Chebyshev system, or Tchebycheff system, or T–system, if the determinant

$$D(f_0, ...f_n; t_0, ...t_n) := det(f_j(t_k); \quad 0 \leq j, \quad k \leq n)$$

does not vanish for any choice of points $t_k; 0 \leq k \leq n$ in I.

Definition 1.2: (cf. [15]) A set of continuous real functions $w_1, w_2, ..., w_r$ defined on $[a,b]$ is called an AT system for the index $n \in \mathbb{Z}_+^r$, $n \neq 0$, if

$$w_1(z), zw_1(z), \ldots, z^{n_1-1}w_1(z),$$

$$.$$

$$.$$

$$.$$

$$w_r(z), zw_r(z), \ldots, z^{n_r-1}w_r(z),$$

is a Chebyshev system of order $|n| - 1$ on $[a,b]$.

In an AT system, all the multi-index $\overrightarrow{n}$ are normal. By using the following two examples it will be easy to show that the weight functions for the DMOPs form an AT system.

Example 1.1: (cf. [3]) The functions

$$w(z)a_1^z, zw(z)a_1^z, \ldots, z^{n_1-1}w(z)a_1^z,$$

$$.$$

$$.$$

$$.$$

$$w(z)a_r^z, zw(z)a_r^z, \ldots, z^{n_r-1}w(z)a_r^z,$$

with all the $a_i > 0, i = 1, ..., r$, different and $w(z)$ a continuous function which has no zeros on $\mathbb{R}^+$, form a Cheybshev system on $\mathbb{R}^+$ for every index $\overrightarrow{n} \in \mathbb{N}^r$.

Example 1.2: (cf. [3]) The functions

$$w(z)\Gamma(x+\beta_1), zw(z)\Gamma(z+\beta_1), \ldots, z^{n_1-1}w(z)\Gamma(z+\beta_1),$$

$$.$$

$$.$$

$$.$$

$$w(z)\Gamma(z+\beta_r), zw(z)\Gamma(z+\beta_r), \ldots, z^{n_r-1}w(z)\Gamma(z+\beta_r),$$

with $\beta_i > 0$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i{\neq}j$ and $w(z)$ a continuous function which has no zeros on $\mathbb{R}^+$, form a Cheybshev system on $\mathbb{R}^+$ for every index $\overrightarrow{n} \in \mathbb{N}^r$. If $\beta_i - \beta_j \notin 0, 1, ..., N-1$ whenever $i{\neq}j$, this gives a Chebyshev system for every $\overrightarrow{n}$ for which $n_i < N+1$, $i = 1, 2, ...r$.

The proof of the above example can be found in [3] and the general form for Example 1.0.2 given in the following example. Before giving the example let check the definiton of ω-gamma function.

Γ_ω is the ω-gamma function given by,

$$\Gamma_\omega(x) = \int_0^\infty t^{x-1} e^{-\frac{t^\omega}{\omega}} dt = \omega^{\frac{x}{\omega}-1}\Gamma(\frac{x}{\omega}).$$

Example 1.3: (cf. [4]) The functions

$$w(z)\Gamma_{\omega}(z+\beta_1), w(z)y(z)\Gamma_{\omega}(z+\beta_1), \ldots, w(z)y(z)^{n_1-1}\Gamma_{\omega}(z+\beta_1),$$

$$.$$

$$.$$

$$.$$

$$w(z)\Gamma_{\omega}(z+\beta_r), w(z)y(z)\Gamma_{\omega}(z+\beta_r), \ldots, w(z)y(z)^{n_r-1}\Gamma_{\omega}(z+\beta_r),$$

with $\beta_i > 0$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$ and $w(z)$ a continuous function which has no zeros on $\mathbb{R}^+$, form a Cheybshev system on $\mathbb{R}^+$ for every index $\overrightarrow{n} \in \mathbb{N}^r$. If $\beta_i - \beta_j \notin 0, 1, ..., N-1$ whenever $i \neq j$, which gives a Chebyshev system for every $\overrightarrow{n}$ for which $n_i < N+1$, $i = 1, 2, ...r$.

The proof of the above example is given in [4].

MMPs are discrete MOPs. There are two kinds of MMP. They are polynomials of one variable satisfying orthogonality conditions w.r.t more than one Negative Binomial distributions.

In [3], J. Arvesu, J. Coussement and W. Van Assche investigated raising operator and Rodrigues formula for the MMP. Also, via Rodrigues formula an explicit formula for these polynomials are obtained by these authors. They investigated these properties for MOPs of discrete variables by extending the COPs of discrete variables.

W. Van Assche in [20] obtained a lowering operator for MMP for the case $r = 2$ and then by combining lowering and raising operator he gave the third order difference equation for these polynomials. Later, D.W. Lee in [10] obtained a lowering operator for the case r and then by combining lowering and raising operator D.W. Lee gave the

$(r+1)$th order difference equation for these polynomials.

F. Ndayiragije and Walter Van Assche in [6] gave generating functions and explicit expressions for the coefficients in the nearest neighbor recurrence relation for MMP.

1.1 Multiple Meixner Polynomials of the First Kind

MMPs of the first kind $M_{\vec{n}}^{(\beta,\vec{a})}(x)$, of degree $|\vec{n}|$, are orthogonal polynomials for the Negative Binomial distributions when $r > 1$. That is,

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{(\beta,\vec{a})}(x)(-x)_j w_i^{\beta}(x) = 0, \quad j = 0,1,\ldots n_i - 1, \quad i = 1,\ldots,r, \tag{1.2}$$

where $\beta > 0$ is the fixed parameter and different values for the parameter $\vec{a} = (a_1,...,a_r)$, ($a_i \neq a_j$ whenever $i \neq j$) where $0 < a < 1$ with multi-index $\vec{n} = (n_1,...,n_r)$ and $(x)_k = x(x+1)(x+2)\ldots(x+k-1)$ is the Pochhammer symbol with $(x)_0 = 1$.

The functions

$$w_i^{\beta}(x) = \begin{cases} \frac{\Gamma(\beta+x)a_i^k}{\Gamma(\beta)\Gamma(x+1)} & \text{if } x \in \mathbb{R}/(\{-1,-2,...\} \cup \{-\beta,-\beta-1,...\}) \\ 0 & \text{if } x \in \{-1,-2,...\} \end{cases}$$

are weight functions of the MMP-1 where $\Gamma(x)$ is the gamma function.

By using Example 1.0.1 it will be easy to see that the weight functions for the MMP-1 form an AT system.

The raising operator for the MMP-1 is given as follows(see [3, equation 4.6, p. 33]),

$$\nabla\left[M_{\vec{n}}^{(\beta,\vec{a})}(x)w_i^{\beta}(x)\right] = \frac{a_i - 1}{a_i(\beta-1)} M_{\vec{n}+\vec{e_i}}^{(\beta-1,\vec{a})}(x) w_i^{\beta-1}(x) \quad i = 1,\ldots,r,$$

where a repeated application of these operators gives Rodrigues formula as [3, equation

4.7, p. 33],

$$M_{\overrightarrow{n}}^{(\beta,\overrightarrow{a})}(x)=(\beta)_{|\overrightarrow{n}|}\left[\prod_{k=1}^{r}\left(\frac{a_k}{a_k-1}\right)^{n_k}\frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta+x)}\right]$$
$$\times\prod_{i=1}^{r}\frac{1}{a_i^x}\nabla^{n_i}\left[\frac{\Gamma(\beta+x+|\overrightarrow{n}|)a_i^x}{\Gamma(x+1)\Gamma(\beta+|\overrightarrow{n}|)}\right].$$

The explicit form for MMP-1 is given as [6, equation 3, p. 3],

$$M_{\overrightarrow{n}}^{(\beta,\overrightarrow{a})}(x)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r}(-x)_{|\overrightarrow{k}|}$$
$$\times\prod_{j=1}^{r}\left[\frac{(a_k)^{n_j-k_j}}{(a_k-1)^{n_j}}(\beta+x)_{|\overrightarrow{n}|-|\overrightarrow{k}|}\right].$$

The generating function for MMP-1 is [6, equation 7, p. 4],

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}M_{\overrightarrow{n}}^{(\beta,\overrightarrow{a})}(x)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}=\left(1-\frac{t_1}{a_1-1}-\ldots-\frac{t_r}{a_r-1}\right)^{x}$$
$$\times\left(1-\frac{t_1a_1}{a_1-1}-\ldots-\frac{t_ra_r}{a_r-1}\right)^{-(\beta+x)}.$$

The lowering operator for MMP-1 has the following form [10, equation 2.5, p. 138],

$$\Delta M_{\overrightarrow{n}}^{(\beta,\overrightarrow{a})}=\sum_{i=1}^{r}n_iM_{\overrightarrow{n}-e_i}^{(\beta+1,\overrightarrow{a})},$$

$(r+1)th$ order difference equation for MMP-1 is given as [10, Theorem 2.5, p.139],

$$L_{a_1}^{(\beta+2-r}L_{a_2}^{(\beta+3-r)}\ldots L_{a_r}^{(\beta+1)}\Delta M_{\overrightarrow{n}}^{(\beta,\overrightarrow{a})}(x)+$$
$$\sum_{i=1}^{r}n_iL_{a_1}^{(\beta+2-r)}\ldots L_{a_{i-1}}^{(\beta+i-r)}L_{a_{i+1}}^{(\beta+(i+1)-r)}\ldots L_{a_r}^{(\beta)}L_{a_i}^{(\beta+1)}M_{\overrightarrow{n}}^{(\beta,\overrightarrow{a})}(x)=0.$$

The third order difference equation for MMP-1 is [10, Corollary 2.6, p.139],

$$x(x-1)\nabla^2\Delta y + x[\beta(a_1+a_2)+(x-1)(a_1+a_2-2)]\nabla\Delta y +$$

$$[(a_1\beta - x(1-a_1))(a_2\beta - x(1-a_1)(1-a_2) - a_1a_2\beta]\Delta y +$$

$$[n_1(1-a_1)+n_2(1-a_2)]x\nabla y +$$

$$(\beta-1)\big[[n_1a_2+n_2a_1-a_1a_2(n_1+n_2)]-(1-a_1)(1-a_2)(n_1+n_2)x\big]y = 0.$$

1.2 Multiple Meixner Polynomials of the Second Kind

The orthogonality conditions for the second kind of MMPs, of degree $|\overrightarrow{n}|$, were defined in (see [3]) by keeping parameter $0 < a < 1$ fixed and by changing the parameter $\overrightarrow{\beta} = (\beta_1, \beta_2, \ldots, \beta_r)$, $\beta_i > 0$ ($\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$) with multi-index $\overrightarrow{n} = (n_1, ..., n_r)$ as follows,

$$\sum_{x=0}^{\infty} M_{\overrightarrow{n}}^{(\overrightarrow{\beta};a)}(x)(-x)_j w^{\beta_i}(x) = 0, \quad j = 0, 1, \ldots n_i - 1, \quad i = 1, \ldots, r. \tag{1.3}$$

The functions

$$w^{\beta_i}(x) = \begin{cases} \frac{\Gamma(\beta_i+x)a^x}{\Gamma(\beta_i)\Gamma(x+1)} & \text{if } x \in \mathbb{R}/(\{-1,-2,...\} \cup \{-\beta_i, -\beta_i - 1, ...\}) \\ 0 & \text{if } x \in \{-1,-2,...\} \end{cases}$$

are weight functions of the MMP-2 where $\Gamma(x)$ is the gamma function.

From Example 1.0.2, it is obvious that the weight functions of the second kind MMPs form an AT system.

The raising operator for MMP-2 is given as [3, equation 4.9, p. 35],

$$\nabla\left[M_{\overrightarrow{n}}^{(\overrightarrow{\beta},a)} w^{\beta_i}(x)\right] = \frac{a-1}{a(\beta_i - 1)} M_{\overrightarrow{n}+\overrightarrow{e_i}}^{(\overrightarrow{\beta}-\overrightarrow{e_i},a)}(x) w_i^{\beta-1}(x) \quad i = 1, \ldots, r,$$

and a repeated application of these operators gives the Rodrigues formula for MMP-

2 [3, equation 4.10, p. 35],

$$M_{\overrightarrow{n}}^{(\overrightarrow{\beta},a)}(x) = \left(\frac{a}{a-1}\right)^{|\overrightarrow{n}|} \prod_{k=1}^{r} (\beta_k)_{n_k} \left[\frac{\Gamma(x+1)}{a^x}\right]$$
$$\prod_{i=1}^{r} \frac{\Gamma(\beta_i)}{\Gamma(\beta_i+x)} \nabla^{n_i} \left[\frac{\Gamma(\beta_i+n_i+x)}{\Gamma(\beta_i+n_i)}\right] \left(\frac{a^x}{\Gamma(x+1)}\right).$$

The explicit form for MMP-2 is [6, equation 5, p. 3],

$$M_{\overrightarrow{n}}^{(\overrightarrow{\beta},a)}(x) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \cdots \sum_{k_r=0}^{n_r} \frac{a^{|\overrightarrow{n}|-|\overrightarrow{k}|}}{(a-1)^{|\overrightarrow{n}|}} \binom{n_1}{k_1} \binom{n_2}{k_2} \cdots \binom{n_r}{k_r} (-x)_{|\overrightarrow{k}|}$$
$$\prod_{j=1}^{r} \left(\beta_j + x - \sum_{i=1}^{j-1} k_i\right)_{n_j-k_j}.$$

The generating function for MMP-2 is [6, equation 9, p. 5],

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\overrightarrow{n}}^{(\overrightarrow{\beta};a)}(x) \frac{t_1^{n_1} \ldots t_r^{n_r}}{n_1! \ldots n_r!} = \left(1 - \frac{1}{a}\left[1 - \prod_{j=1}^{r}(1 + \frac{a}{1-a} t_j)\right]\right)^x$$
$$\prod_{i=1}^{r} \left(1 + \frac{a}{1-a} t_i\right)^{-(x+\beta_i)}.$$

The lowering operator for MMP-2 [10, equation 2.11, p. 142] and [10, equation 2.12, p. 142],

$$\Delta M_{\overrightarrow{n}}^{(\overrightarrow{\beta};a)}(x) = \sum_{i=1}^{r} d_i M_{\overrightarrow{n}-e_i}^{(\overrightarrow{\beta}+e_i;a)}(x),$$

where,

$$d_i = \frac{\prod_{p=1}^{r}(n_p + \beta_p - \beta_i)}{\prod_{q=1,q\neq i}^{r-1}(\beta_i - \beta_q) \prod_{p=i+1}^{r}(\beta_p - \beta_i)}$$
$$\times \sum_{j=1}^{r} \frac{(-1)^{i+j} \prod_{q=1}^{r}(n_j + \beta_j - \beta_q)}{(n_j + \beta_j - \beta_i) \prod_{q=1,q\neq j}^{r-1}(n_q - n_j + \beta_q - \beta_j) \prod_{p=j+1}^{r}(n_j - n_p + \beta_j - \beta_p)}.$$

$(r+1)th$ order difference equation for MMP-2 is given as [10, theorem 2.10, p.145],

$$L_{\beta_1+1}^{a} L_{\beta_2+1}^{a} \ldots L_{\beta_r+1}^{a} [\Delta M_{\overrightarrow{n}}^{(\overrightarrow{\beta};a)}(x)] + \sum_{i=1}^{r} d_i L_{\beta_1+1}^{a} \ldots L_{\beta_{i-1}+1}^{a} L_{\beta_{i+1}+1}^{a} \ldots L_{\beta_r+1}^{a} L_{\beta_i+1}^{a}$$
$$\times [M_{\overrightarrow{n}}^{(\overrightarrow{\beta};a)}(x)] = 0,$$

where the $3rd$ order difference equation for MMP-2 is given as follows [10, corollary 2.11, p.145],

$$
\begin{aligned}
x(x-1)\nabla^2\Delta y &+ [(1-a)(2-2x)+a(1+\beta_1+\beta_2)]x\nabla\Delta y \\
&+ [(a\beta_1-(1-a)x)(a\beta_2-(1-a)x)-x(1-a)]\Delta y \\
&+ (n_1+n_2)(1-a)x\nabla y \\
&+ (1-a)[a(n_1n_2+\beta_2 n_1+\beta_1 n_2)-x(1-a)(n_1+n_2)]y = 0.
\end{aligned}
$$

Chapter 2

ω-CLASSICAL OP OF A DISCRETE VARIABLE

The theory considered this chapter is based on [14]. The properties of classical orthogonal polynomials of a continuous and discrete variable is given. We extend the theory for the classical orthogonal polynomials of a discrete variable to the ω-classical orthogonal polynomials of a discrete variable.

For give the main definition firstly the relations for ω-upward and ω-backward difference operators will be given.

2.1 The $\omega-$Difference Equation

Definition 2.1: The ω-upward difference operator is defined by

$\Delta_\omega g(z) = g(z+\omega) - g(z)$ and the ω-backward difference operator is defined by $\nabla_\omega g(z) = g(z) - g(z-\omega)$.

Definition 2.2: The polynomial set $P_m(x)$, $(m = 1, 2, \ldots)$ form a ω-classical orthogonal polynomials set of a discrete variable if $P_m(x)$ (polynomial of degree m) satisfy the hypergeometric type difference equation which has form $\eta(x)\Delta_\omega\nabla_\omega P_m(x) + \rho(x)\Delta_\omega P_m(x) + \alpha_m P_m(x) = 0$ with equation $\Delta_\omega[\eta(x)w(x)] = \rho(x)w(x)$, where $\eta(x)$ and $\rho(x)$ are polynomials of at most second and first degree respectively and α_m is a constant.

Theorem 2.1: The ω-upward and the ω-backward difference operator have the following properties:

1. $\Delta_\omega g(z) = \nabla_\omega g(z+\omega)$,

2. $\Delta_\omega \nabla_\omega g(z) = g(z+\omega) - 2g(z) + g(z-\omega)$,

3. $\Delta_\omega [g(z)h(z)] = g(z)\Delta_\omega h(z) + h(z+\omega)\Delta_\omega g(z)$

4. $\nabla_\omega [g(z)h(z)] = g(z)\nabla_\omega h(z) + h(z-\omega)\nabla_\omega g(z)$.

Proof. To give the proof for 1 , start with RHS ,

$$\begin{aligned}\nabla_\omega g(z+\omega) &= g(z+\omega) - g(z)\\ &= \Delta_\omega g(z).\end{aligned}$$

To give the proof for 2 , start with LHS ,

$$\begin{aligned}\Delta_\omega \nabla_\omega g(z) &= \Delta_\omega [g(z) - g(z-\omega)]\\ &= g(z+\omega) - g(z) - g(z) + g(z-\omega)\\ &= g(z+\omega) - 2g(z) + g(z-\omega).\end{aligned}$$

To give the proof for 3 , start with LHS ,

$$\Delta_\omega [g(z)h(z)] = g(z+\omega)h(z+\omega) - g(z)h(z).$$

Let add and subtract the term $g(z)h(z+\omega)$ for RHS of the above equation,

$$\begin{aligned}\Delta_\omega [g(z)h(z)] &= g(z+\omega)h(z+\omega) - g(z)h(z) + g(z)h(z+\omega) - g(z)h(z+\omega)\\ &= g(z+\omega)[h(z+\omega) - h(z)] + h(z)[g(z+\omega) - g(z)]\\ &= g(z)\Delta_\omega h(z) + h(z+\omega)\Delta_\omega g(z).\end{aligned}$$

To give the proof for 4 , start with LHS ,

$$\nabla_\omega [g(z)h(z)] = g(z)h(z) - g(z-\omega)h(z-\omega).$$

Let add and subtract the term $g(z)h(z-\omega)$ for RHS of the above equation,

$$\nabla_\omega[g(z)h(z)] = g(z)h(z) - g(z-\omega)h(z-\omega) + g(z)h(z-\omega) - g(z)h(z-\omega)$$

$$= g(z)[h(z) - h(z-\omega)] + h(z-\omega)[g(z) - g(z-\omega)]$$

$$= g(z)\nabla_\omega h(z) + h(z-\omega)\nabla_\omega g(z).$$

□

Theorem 2.2: The ω-summation by parts formula is given by,

$$\sum_{k=0}^{\infty} \triangle_\omega[f(\omega k)]g(\omega k) = -\sum_{k=0}^{\infty} \nabla_\omega[g(\omega k)]f(\omega k) \tag{2.1}$$

where $g(-\omega) = 0$.

Proof. To give the proof for ω-summation by parts formula start with the following expression,

$$\sum_{k=m}^{l} \Big[\triangle_\omega[f(\omega k)]g(\omega k) + \nabla_\omega[g(\omega k)]f(\omega k)\Big].$$

From the definitions of the ω-upward and the ω-backward difference operators the above equation get the following form,

$$\sum_{k=m}^{l} \Big[\triangle_\omega[f(\omega k)]g(\omega k) + \nabla_\omega[g(\omega k)]f(\omega k)\Big] = \sum_{k=m}^{l} \Big[f(\omega k+\omega)g(\omega k) - f(\omega k)g(\omega k-\omega)\Big].$$

Let write the terms for the RHS,

$$\begin{aligned}\sum_{k=m}^{l} \Big[\triangle_\omega[f(\omega k)]g(\omega k) + \nabla_\omega[g(\omega k)]f(\omega k)\Big] &= \Big[f\big(\omega(m+1)\big)g(\omega m) - f(\omega m)g\big(\omega(m-1)\big)\Big] \\ &+ \Big[f\big(\omega(m+2)\big)g(\omega(m+1)) - f\big(\omega(m+1)\big)g(\omega m)\Big] \\ &+ \Big[f\big(\omega(m+3)\big)g(\omega(m+2)) - f\big(\omega(m+2)\big)g\big(\omega(m+1)\big)\Big]\ldots \\ &+ \Big[f(\omega l)g(\omega(l-1)) - f\big(\omega(l-1)\big)g\big(\omega(l-2)\big)\Big] \\ &+ \Big[f\big(\omega(l+1)\big)g(\omega l) - f(\omega l)g\big(\omega(l-1)\big)\Big]\end{aligned}$$

At the end we obtain that,

$$\sum_{k=m}^{l} \Big[\triangle_\omega[f(\omega k)]g(\omega k) + \nabla_\omega[g(\omega k)]f(\omega k)\Big] = f\big(\omega(l+1)\big)g(\omega l) - f(\omega m)g\big(\omega(m-1)\big).$$

By choosing $m \to 0, \quad l \to \infty$ where $g(-\omega) = 0$ we get the desired result for the theorem. □

For a polynomial $g_n(z)$ of degree n the expressions $\Delta_\omega g_n(z)$ and $\nabla_\omega g_n(z)$ are polynomials of degree $n-1$. Thus ,

$$\nabla_\omega^n g_n(x) = \Delta_\omega^n g_n(x) = g_n^{(n)}(x).$$

Theorem 2.3: Let $V_1(x) = \Delta_\omega y(x)$ and ω be a positive real number. $V_1(x)$ satisfies the following $\omega-$difference equation,

$$\eta(x)\Delta_\omega\nabla_\omega y(x) + \rho(x)\Delta_\omega y(x) + \alpha y(x) = 0. \tag{2.2}$$

Proof. Since $V_1(x) = \Delta_\omega y(x)$, we can rewrite (2.2) again as,

$$\eta(x)\nabla_\omega V_1(x) + \rho(x)V_1(x) + \alpha y(x) = 0.$$

By applying Δ_ω to both sides of above equation we obtain,

$$\Delta_\omega\left[\eta(x)\nabla_\omega V_1(x)\right] + \Delta_\omega\left[\rho(x)V_1(x)\right] + \alpha\Delta_\omega y(x) = 0.$$

By using property 3 from the Theorem 2.0.1 we get,

$$\eta(x)\Delta_\omega\nabla_\omega V_1(x) + \Delta_\omega V_1(x)\left[\Delta_\omega\eta(x) + \rho(x+\omega)\right] + V_1(x)\left[\Delta_\omega\rho(x) + \alpha\right] = 0.$$

By getting,

$$\rho_1^\omega(x) = \Delta_\omega\eta(x) + \rho(x+\omega) \tag{2.3}$$

and

$$\alpha_1^\omega = \Delta_\omega\rho(x) + \alpha. \tag{2.4}$$

We desired the result,

$$\eta(x)\Delta_\omega\nabla_\omega V_1(x) + \rho_1^\omega(x)\Delta_\omega V_1(x) + \alpha_1^\omega V_1(x) = 0. \tag{2.5}$$

□

Theorem 2.4: Let $V_n(x) = \Delta_\omega^n y(x)$ and ω be a positive real number. $V_n(x)$ satisfies the following ω-difference equation of hypergeometric type,

$$\eta(x)\Delta_\omega\nabla_\omega V_n(x) + \rho_n^\omega(x)\Delta_\omega V_n(x) + \alpha_n^\omega V_n(x) = 0, \tag{2.6}$$

where

$$\rho_n^\omega(x) = \Delta_\omega\eta(x) + \rho_{n-1}^\omega(x+\omega), \tag{2.7}$$

$$\alpha_n^\omega = \Delta_\omega\rho_{n-1}^\omega(x) + \alpha_{n-1}^\omega. \tag{2.8}$$

Proof. In a similar way with the Theorem 2.0.3, by setting $V_2(x) = \Delta_\omega V_1(x)$ in (2.5) we get,

$$\eta(x)\nabla_\omega V_2(x) + \rho_1^\omega(x)V_2(x) + \alpha_1^\omega V_1(x) = 0.$$

By applying Δ_ω to both sides of the above equation we obtain,

$$\Delta_\omega\left[\eta(x)\nabla_\omega V_2(x)\right] + \Delta_\omega\left[\rho_1^\omega(x)V_2(x)\right] + \alpha_1^\omega\Delta_\omega V_1(x) = 0.$$

By using property 3 from the Theorem 2.0.1 we get,

$$\eta(x)\Delta_\omega\nabla_\omega V_2(x) + \Delta_\omega V_2(x)\left[\Delta_\omega\eta(x) + \rho_1^\omega(x+\omega)\right] + V_2(x)\left[\Delta_\omega\rho_1^\omega(x) + \alpha_1^\omega\right] = 0.$$

By getting,

$$\rho_2^\omega(x) = \Delta_\omega\eta(x) + \rho_1^\omega(x+\omega),$$

and

$$\alpha_2^\omega = \Delta_\omega\rho_1^\omega(x) + \alpha_1^\omega,$$

we get the result for $V_2(x) = \Delta_\omega V_1(x)$. By iterating n times we desired the result for the theorem. □

Theorem 2.5: The explicit expression for $\rho_n^\omega(x)$ is given by,

$$\rho_n^\omega(x) = \rho(x+\omega n) + \eta(x) + \eta(x+\omega n). \tag{2.9}$$

Proof. From (2.7) by using the definition of the ω-upward difference operator we get,

$$\rho_n^{\omega}(x)+\eta(x)=\eta(x+\omega)+\rho_{n-1}^{\omega}(x+\omega). \tag{2.10}$$

By iterating above equation we desired the result for the theorem. □

Theorem 2.6: The explicit expression for α_n^{ω} is given by,

$$\alpha_n^{\omega}=\alpha^{\omega}+n\rho'(x)+\frac{1}{2}n(n-1)\eta''(x). \tag{2.11}$$

Proof. For give the proof of the theorem first let show that $\Delta_\omega\rho(x)$ and $\Delta_\omega^2\eta(x)$ are independent of x.

By applying Δ_ω to eguation (2.7) we obtain,

$$\Delta_\omega\rho_n^{\omega}(x)=\Delta_\omega^2\eta(x)+\Delta_\omega\rho_{n-1}^{\omega}(x+\omega).$$

The above equation get the following form by iterating n times.

$$\Delta_\omega\rho_n^{\omega}(x)=n\Delta_\omega^2\eta(x)+\Delta_\omega\rho(x),$$

which is equivalent with,

$$\Delta_\omega\rho_n^{\omega}(x)=n\eta''(x)+\rho'(x). \tag{2.12}$$

For equation (2.8) we use (2.12) for $\Delta_\omega\rho_{n-1}^{\omega}(x)$ and we obtain,

$$\alpha_n^{\omega}=\alpha_{n-1}^{\omega}+\Delta_\omega\rho_{n-1}(x)+\Delta_\omega\rho_{n-2}(x).$$

After iterating above equation n times we get,

$$\alpha_n^{\omega}=\alpha_0^{\omega}+\Delta_\omega\rho_{n-1}(x)+\Delta_\omega\rho_{n-2}(x)+\cdots+\Delta_\omega\rho_1(x)+\Delta_\omega\rho(x),$$

where $\alpha_0^{\omega}=\alpha$. For the above equation we use (2.12)to obtain the result of the theorem.

□

Theorem 2.7: For the difference equation,

$$\eta(x)\Delta_\omega\nabla_\omega y(x)+\rho(x)\Delta_\omega y(x)+\alpha y(x)=0,$$

the self-adjoint form is given by,

$$\Delta_\omega\left[\sigma(x)\eta(x)\nabla_\omega y(x)\right]+\alpha\sigma(x)y(x)=0. \tag{2.13}$$

Where $\sigma(x)$ satisfies the following ω-Pearson equation,

$$\Delta_\omega\left[\sigma(x)\eta(x)\right]=\sigma(x)\rho(x). \tag{2.14}$$

Proof. Start with multiplying (2.2) with $\sigma(x)$ to obtain,

$$\sigma(x)\eta(x)\Delta_\omega\nabla_\omega y(x)+\sigma(x)\rho(x)\Delta_\omega y(x)+\alpha\sigma(x)y(x)=0.$$

By use of ω-Pearson equation (2.14) in the above equation we get,

$$\sigma(x)\eta(x)\Delta_\omega\nabla_\omega y(x)+\Delta_\omega\left[\sigma(x)\eta(x)\right]\rho(x)\Delta_\omega y(x)+\alpha\sigma(x)y(x)=0.$$

From Theorem 2.0.1 we use the properties 1. and 3. to obtain the self-adjoint form (2.13). □

Theorem 2.8: For the difference equation,

$$\eta(x)\Delta_\omega\nabla_\omega V_n(x)+\rho_n^\omega(x)\Delta_\omega V_n(x)+\alpha_n^\omega V_n(x)=0,$$

the self-adjoint form is given by,

$$\Delta_\omega\left[\sigma_n(x)\eta(x)\nabla_\omega V_n(x)\right]+\alpha_n^\omega\sigma_n(x)V_n(x)=0. \tag{2.15}$$

Where $\sigma_n(x)$ satisfies the following ω-Pearson equation,

$$\Delta_\omega\left[\sigma_n(x)\eta(x)\right]=\sigma_n(x)\rho_n^\omega(x). \tag{2.16}$$

Proof. Start with multiplying (2.6) with $\sigma_n(x)$ to obtain,

$$\sigma_n(x)\eta(x)\Delta_\omega\nabla_\omega V_n(x)+\sigma_n(x)\rho_n^\omega(x)\Delta_\omega V_n(x)+\alpha_n^\omega\sigma_n(x)V_n(x)=0.$$

By use of ω-Pearson equation (2.16) in the above equation we get,

$$\sigma_n(x)\eta(x)\Delta_\omega\nabla_\omega V_n(x)+\Delta_\omega\left[\sigma_n(x)\eta(x)\right]\Delta_\omega V_n(x)+\alpha_n^\omega\sigma_n(x)V_n(x)=0.$$

From Theorem 2.0.1 we use the properties 1. and 3. to obtain the self-adjoint form

(2.15). $\square$

Theorem 2.9: The connection formula between the functions $\sigma(x)$ and $\sigma_n(x)$ is given by,

$$\sigma_n(x) = \sigma(x+n\omega)\prod_{k=1}^{n}\eta(x+k\omega) \tag{2.17}$$

Proof. For give the proof of the theorem start with the equation (2.16) and write in the form,

$$\sigma_n(x+\omega)\eta(x+\omega) - \sigma_n(x)\eta(x) = \sigma_n(x)\rho_n^{\omega}(x),$$

which is equivalent with,

$$\frac{\sigma_n(x+\omega)\eta(x+\omega)}{\sigma_n(x)} = \rho_n^{\omega}(x) + \eta(x). \tag{2.18}$$

By use of the equation (2.10) and iterating the above equation we obtain,

$$\frac{\sigma_n(x+\omega)}{\sigma_{n-1}(x+2\omega)\eta(x+2\omega)} = \frac{\sigma_n(x)}{\sigma_{n-1}(x+\omega)\eta(x+\omega)} = c_n(x), \tag{2.19}$$

where $c_n(x)$ is any function of period 1. By taking $c_n(x)=1$ we get,

$$\sigma_n(x) = \sigma_{n-1}(x+\omega)\eta(x+\omega). \tag{2.20}$$

Since $\sigma_0(x)=\sigma(x)$, iterating above equation n times we get the equation (2.17). $\square$

2.2 Finite $\omega-$Difference Analogs of Polynomials of $\omega-$Hypergeometric Type and of Their Derivatives and the $\omega-$Rodrigues Formula

We are ready to construct a theory of $\omega-$classical orthogonal polynomials of a discrete variable. By getting $n=m$ in the $\omega-$difference equation of hyprergeometric type we obtain that,

$$\eta(x)\Delta_\omega\nabla_\omega V_m(x) + \rho_m^{\omega}(x)\Delta_\omega V_m(x) + \alpha_m^{\omega} V_m(x) = 0,$$

has a particular solution $V_m(x) = $ constant if $\alpha_m^{\omega} = 0$.

Since $V_m(x) = \Delta_\omega^m y(x)$, this means that if

$$\alpha = \alpha_m^\omega = -m\rho'(x) - \frac{1}{2}m(m-1)\eta''(x) \tag{2.21}$$

there is a particular solution $y = y_m(x)$ of the $\omega-$difference equation of hyprergeometric type, which is a polynomial of degree m, provided that $\alpha_n^\omega \neq 0$ for $n = 0, 2, \ldots, m-1$.

We can rewrite the equation (2.6) for $V_n(x)$ in the following form,

$$V_n(x) = \frac{-1}{\alpha_n^\omega}\Bigg[\eta(x)\nabla_\omega V_{n+1}(x) + \rho_n^\omega(x)V_{n+1}(x)\Bigg].$$

From the above equation it is clear that if $V_{n+1}(x)$ is a polynomial, then $V_n(x)$ is also polynomial if $\alpha_n^\omega \neq 0$.

Theorem 2.10: The $\omega-$Rodrigues formula for the $\omega-$discrete polynomials is given by,

$$\sigma_n(x)V_n(x) = \frac{A_m}{A_n}\nabla_\omega^{n-m}\Bigg[\sigma_m(x)V_m(x)\Bigg], \tag{2.22}$$

where

$$A_n = (-1)^n\prod_{k=0}^{n-1}\alpha_k^\omega, \quad A_0 = 1. \tag{2.23}$$

Proof. The self-adjoint form (2.15) written in the following form,

$$\sigma_n(x)V_n(x) = \frac{-1}{\alpha_n^\omega}\Delta_\omega\big[\sigma_n(x)\eta(x)\nabla_\omega V_n(x)\big].$$

From Theorem 2.0.1 by use of propety 1 and $\Delta V_n(x) = V_{n+1}(x)$ we obtain,

$$\sigma_n(x)V_n(x) = \frac{-1}{\alpha_n^\omega}\nabla_\omega\big[\sigma_n(x+\omega)\eta(x+\omega)V_{n+1}(x)\big].$$

By use of equation (2.20) the above equation get the following form,

$$\sigma_n(x)V_n(x) = \frac{-1}{\alpha_n^\omega}\nabla_\omega\big[\sigma_{n+1}(x)V_{n+1}(x)\big].$$

For $n < m$ we obtain,

$$\sigma_n(x)V_n(x) = \frac{-1}{\alpha_n^\omega}\nabla_\omega\left[\sigma_{n+1}(x)V_{n+1}(x)\right]$$
$$= \left(\frac{-1}{\alpha_n^\omega}\right)\left(\frac{-1}{\alpha_{n+1}^\omega}\right)\nabla_\omega^2\left[\sigma_{n+2}(x)V_{n+2}(x)\right] = \ldots$$
$$= \frac{A_m}{A_n}\nabla_\omega^{n-m}\Bigg[\sigma_m(x)V_m(x)\Bigg].$$

□

Theorem 2.11: The second $\omega-$Rodrigues formula for the $\omega-$discrete polynomials is given by,

$$\Delta_\omega^m y_n(x) = \frac{A_{mn}B_n}{\sigma_m(x)}\nabla_\omega^{n-m}\left[\sigma_n(x)\right], \tag{2.24}$$

where

$$A_{mn} = \frac{n!}{(n-m)!}\prod_{k=0}^{n-1}\left[\rho'(x) - \frac{n+k-1}{2}\eta''(x)\right] \quad A_{0n} = 1, \quad n \leq m, \tag{2.25}$$

$$B_n = \frac{\Delta_\omega^n y_n(x)}{\sigma(x)} = \frac{y_n^{(n)}(x)}{\sigma(x)}\nabla_\omega^n\left[\sigma_n(x)\right]. \tag{2.26}$$

Proof. The self-adjoint form (2.15) written for the $\Delta_\omega^m y_n(x)$ as,

$$\sigma_m(x)\Delta_\omega^m y_n(x) = \frac{-1}{\alpha_{mn}^\omega}\Delta_\omega\left[\sigma_m(x)\eta(x)\nabla_\omega\Delta_\omega^m y_n(x)\right].$$

From Theorem 2.0.1 by use of propety 1 and $\nabla_\omega\Delta_\omega^m y_n(x) = \Delta_\omega^{m+1}y_n(x)$ we obtain,

$$\sigma_m(x)\Delta_\omega^m y_n(x) = \frac{-1}{\alpha_{mn}^\omega}\nabla_\omega\left[\sigma_m(x+\omega)\eta(x+\omega)\Delta_\omega^{m+1}y_n(x)\right].$$

By use of equation (2.20) the above equation get the following form,

$$\sigma_m(x)\Delta_\omega^m y_n(x) = \frac{-1}{\alpha_{mn}^\omega}\nabla_\omega\left[\sigma_{m+1}(x)\Delta_\omega^{m+1}y_n(x)\right].$$

After iterating the above equation we obtain the following form,

$$\sigma_m(x)\Delta_\omega^m y_n(x) = \frac{-1}{\alpha_{mn}^\omega}\nabla_\omega\left[\sigma_{m+1}(x)\Delta_\omega^{m+1}y_n(x)\right]$$
$$= \left(\frac{-1}{\alpha_{mn}^\omega}\right)\left(\frac{-1}{\alpha_{(m+1)n}^\omega}\right)\nabla_\omega^2\left[\sigma_{m+2}(x)\Delta_\omega^{m+2}y_n(x)\right] = \ldots$$
$$= \left(\frac{-1}{\alpha_{mn}^\omega}\right)\left(\frac{-1}{\alpha_{(m+1)n}^\omega}\right)\cdots\left(\frac{-1}{\alpha_{(n-1)n}^\omega}\right)\nabla_\omega^{n-m}\Bigg[\sigma_n(x)\Delta_\omega^n y_n(x)\Bigg],$$

where $\Delta_\omega^n y_n(x) = $ constant whence we obtain the equation (2.24). □

Corollary 2.1: The explicit expression for $y_n(x)$ is,

$$y_n(x) = \frac{B_n}{\sigma(x)} \nabla_\omega^n \left[\sigma_n(x)\right]. \tag{2.27}$$

Proof. By setting $m = 0$ in equation (2.24) we get the result. □

Corollary 2.2: The explicit expression for $y_n(x)$ is also given by,

$$y_n(x) = \frac{B_n}{\sigma(x)} \nabla_\omega^n \left[\sigma(x) \prod_{k=0}^{n-1} \eta(x - k\omega)\right]. \tag{2.28}$$

Proof. For the equation (2.27) from Theorem 2.0.1 by use of property 1and relation (2.17) we obtain the result for $y_n(x)$. □

Thus the polynomial solutions of ω-difference equation (2.2) are determined by (2.27) up to the normalizing factor B_n. These solutions correspond to the values $\alpha = \alpha_n^\omega$ from (2.21).

Theorem 2.12: For the polynomials $y_n(x)$ and their differences $\Delta_\omega y_n(x)$ the relation given as,

$$\Delta_\omega y_n(x) = -\frac{\alpha_n^\omega B_n}{\overline{B}_{n-1}(x)} \overline{y}_{n-1}(x). \tag{2.29}$$

Proof. According to (2.17) we have,

$$\begin{aligned}
[\sigma_1(x)]_{n-1} &= \sigma_1\left(x + (n-1)\,\omega\right) \prod_{k=1}^{n-1} \eta\left(x + k\omega\right) \\
&= \sigma\left(x + n\omega\right) \prod_{k=1}^{n-1} \eta\left(x + k\omega\right) \\
&= \sigma_n\left(x\right).
\end{aligned}$$

By getting $m = 1$ in the equation (2.24) we obtain,

$$\begin{aligned}
\Delta_\omega y_n(x) &= -\frac{\alpha_n^\omega B_n}{\sigma_1(x)} \nabla_\omega^{n-1} [\sigma_1(x)]_{n-1} \\
&= -\frac{\alpha_n^\omega B_n \overline{B}_{n-1}}{\overline{B}_{n-1} \sigma_1(x)} \nabla_\omega^{n-1} [\sigma_1(x)]_{n-1} \\
\Delta_\omega y_n(x) &= -\frac{\alpha_n^\omega B_n}{\overline{B}_{n-1}(x)} \overline{y}_{n-1}(x).
\end{aligned} \tag{2.30}$$

Here $\overline{y}_n(x)$ is the polynomial obtained by replacing $\sigma(x)$ with $\sigma_1(x)$ in the equation for $y_n(x)$ and $\overline{B}_n$ is the normalizing constant in the $\omega-$Rodrigues formula for $\overline{y}_n(x)$. Hence we obtain the result of the theorem. □

Theorem 2.13: The linear relation that connects the difference $\nabla_\omega y_n(x)$ with $y_n(x)$ and $y_{n+1}(x)$ is given by,

$$\eta(x) \nabla_\omega y_n(x) = \frac{1}{n\alpha_n^\omega \rho_n'} \left[\rho_n(x) y_n(x) - \frac{B_n}{B_{n+1}} y_{n+1}(x) \right]. \tag{2.31}$$

Proof. From $\omega-$Rodrigues formula we have,

$$\begin{aligned}
y_{n+1}(x) &= \frac{B_{n+1}}{\sigma(x)} \nabla_\omega^{n+1} [\sigma_{n+1}(x)] \\
&= \frac{B_{n+1}}{\sigma(x)} \nabla_\omega^{n} [\Delta_\omega \sigma_{n+1}(x-\omega)].
\end{aligned}$$

By using the equality,

$$\begin{aligned}
\Delta_\omega \sigma_{n+1}(x-\omega) &= \Delta_\omega [\sigma_n(x) \eta(x)] \\
&= \rho_n(x) \sigma_n(x),
\end{aligned}$$

and by using the properties from Theorem 2.0.1 we get,

$$\begin{aligned}
y_{n+1}(x) &= \frac{B_{n+1}}{\sigma(x)} \nabla_\omega^{n} [\rho_n(x) \sigma_n(x)] \\
&= \frac{B_{n+1}}{\sigma(x)} \left[\rho_n(x) \nabla_\omega^n \sigma_n(x) + n\rho_n'(x) \nabla_\omega^{n-1} \sigma_n(x-\omega) \right].
\end{aligned}$$

Since

$$\nabla_\omega y_n(x) = \frac{-\alpha_n^\omega B_n}{\sigma(x)\eta(x)} \nabla_\omega^{n-1} [\sigma_n(x-\omega)],$$

we get the result of the theorem. □

2.3 Orthogonality Propety

The polynomial solutions $y_n(x)$ have the orthogonality property under certain restrictions of $\omega-$ difference equation (2.2). The polynomials $y_n(x)$ is called $\omega-$classical orthogoal polynomials of a discrete variable.

Theorem 2.14: For the polynomial $y_n(x)$ there is a following orthogonality relation,

$$\sum_{x_i=a}^{b-1} y_n(\omega x_i) y_m(\omega x_i) \sigma(\omega x_i) = \delta_{mn} d_n^2, \tag{2.32}$$

under the boundary conditions

$$\sigma(\omega x)\eta(\omega x)x^k|_{x=a,b} = 0 \quad (k = 0,1,2\ldots). \tag{2.33}$$

The function δ_{mn} is kronocker delta and d_n^2 is the norm.

Proof. Write the $\omega-$ difference equation in the self-adjoint form for the polynomials $y_n(x)$ and $y_m(x)$.

$$\Delta_\omega\left[\sigma(\omega x)\eta(\omega x)\nabla_\omega y_n(\omega x)\right] + \alpha_n \sigma(\omega x) y_n(\omega x) = 0,$$

$$\Delta_\omega\left[\sigma(\omega x)\eta(\omega x)\nabla_\omega y_m(\omega x)\right] + \alpha_m \sigma(\omega x) y_m(\omega x) = 0.$$

After multiplying the first equation with $y_m(x)$ and the second equation with $y_n(x)$ and subtracting them we obtain,

$$\Delta_\omega\left[\sigma(\omega x)\eta(\omega x)\nabla_\omega y_n(\omega x)\right] y_m(x) - \Delta_\omega\left[\sigma(\omega x)\eta(\omega x)\nabla_\omega y_m(\omega x)\right] y_n(x) =$$
$$(\alpha_n - \alpha_m)\sigma(\omega x) y_m(\omega x).$$

For the LHS of the above equation from the Theorem 2.0.1 by using property 3 we get,

$$\Delta_\omega\left[\left[\sigma(\omega x)\eta(\omega x)\right]\left[y_m(\omega x)\nabla_\omega y_n(\omega x) - y_n(\omega x)\nabla_\omega y_m(\omega x)\right]\right] = (\alpha_n - \alpha_m)\sigma(\omega x) y_m(\omega x).$$

Let $x = x_i$, $x_{i+1} = x_{i+1}$ and take sum over for the values $a \leq x_i \leq b-1$ we get,

$$\sum_{x_i=a}^{b-1} \Delta_\omega \left[[\sigma(\omega x_i)\eta(\omega x_i)] [y_m(\omega x_i)\nabla_\omega y_n(\omega x_i) - y_n(\omega x_i)\nabla_\omega y_m(\omega x_i)] \right] = \sum_{x_i=a}^{b-1} (\alpha_n - \alpha_m)\sigma(\omega x_i) y_m(\omega x_i).$$

For the LHS after applying the $\omega-$forward operator the equation get the following form,

$$\sigma(\omega x_i)\eta(\omega x_i) [y_m(\omega x_i)\nabla_\omega y_n(\omega x_i) - y_n(\omega x_i)\nabla_\omega y_m(\omega x_i)] |_a^b = \sum_{x_i=a}^{b-1} (\alpha_n - \alpha_m)\sigma(\omega x_i) y_m(\omega x_i).$$

Since the expression $y_m(\omega x_i)\nabla_\omega y_n(\omega x_i) - y_n(\omega x_i)\nabla_\omega y_m(\omega x_i)$ is a polynomial in x, the polynomial solutions of $\omega-$difference equation on $[a, b-1]$ with weight function $\sigma(x)$. □

By applying the same procedure the orthogonality realtion for the polynomials $\Delta_\omega^k y_n(x) = V_{kn}(x)$ can be given.

Theorem 2.15: For the polynomial $\Delta_\omega^k y_n(x) = V_{kn}(x)$ there is a following orthogonality relation,

$$\sum_{x_i=a}^{b-k-1} V_{kn}(\omega x_i) V_{km}(\omega x_i) \sigma_k(\omega x_i) = \delta_{mn} d_{kn}^2, \tag{2.34}$$

under the boundary conditions

$$\sigma_l(\omega x)\eta(\omega x) x^k |_{x=a,b-l} = 0 \quad (k = 0, 1, 2 \ldots). \tag{2.35}$$

Chapter 3

ON SOME FAMILIES ω-MULTIPLE ORTHOGONAL POLYNOMIALS

3.1 $\omega-$Multiple Orthogonal Polynomials

This chapter is based on [3], we extend the theory of multiple orthogonal polynomials to the ω-MOPs.

Definition 3.1: The polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ which satisfies the following orthogonality conditions called as ω-MOP.

$$\sum_{x=0}^{\infty} P_{\vec{n}}(\omega x)(-\omega x)_{j,\omega} w_i(\omega x) = 0, \quad j = 0, 1, \ldots n_i - 1, \quad i = 1, \ldots, r, \tag{3.1}$$

where $(c)_{n,\omega} = c(c+\omega)(c+2\omega)...(c+(n-1)\omega)$ is the Pochhammer ω symbol for $c \in \mathbb{C}$ and $n \in \mathbb{N}$ and $\omega > 0$.

From the properties of the Pochhammer ω symbol we know that, $(c)_{n,\omega} = \omega^n \left(\frac{c}{\omega}\right)_n$, which imply that (3.1) is equivalent with (1.1).

3.1.1 ω-Multiple Meixner Polynomials of the First Kind

Definition 3.2: The monic discrete $\omega-$MMP of the first kind, corresponding to the multi-index $\vec{n} = (n_1, ..., n_r)$, the fixed parameter $\beta > 0$ and the parameter $\vec{a} = (a_1, ..., a_r)$, ($a_i \neq a_j$ whenever $i \neq j$), is the unique polynomial of degree $|\vec{n}|$ which satisfies the orthogonality conditions,

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{(\omega;\beta;\vec{a})}(\omega x)(-\omega x)_{j,\omega} w_i^{(\omega;\beta)}(\omega x) = 0, \quad j = 0, 1, \ldots n_i - 1, \quad i = 1, 2, \ldots, r. \tag{3.2}$$

The functions,

$$w_i^{(\omega;\beta)}(x) = \begin{cases} \frac{\Gamma_\omega(\beta+x)a_i^x}{\Gamma_\omega(\beta)\Gamma_\omega(x+\omega)} & \text{if } x \in \mathbb{R}/(\{-1,-2,...\} \cup \{-\beta,-\beta-1,...\}) \\ 0 & \text{if } x \in \{-1,-2,...\}\ . \end{cases} \tag{3.3}$$

are weight functions for ω-MMP-1 where $0 < a_i < 1$, for $i =, 1, 2 \ldots, r$, with all $a_i's$ different.

It can easily be seen from Example 1.0.1 that the weight functions form an AT system thus the corresponding polynomials are unique. When $\omega = 1$, the given orthogonality conditions coincide with the orthogonality conditions in (1.2).

3.1.2 ω-Multiple Meixner Polynomials of the Second Kind

Definition 3.3: The monic discrete $\omega-$MMP of the second kind, corresponding to the multi-index $\overrightarrow{n} = (n_1, ..., n_r)$ and the parameters $\overrightarrow{\beta} = (\beta_1, ..., \beta_r)$, $\beta_i > 0$, $(\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j)$,$0 < a < 1$, is the unique polynomial of degree $|\overrightarrow{n}|$ which satisfies the orthogonality conditions,

$$\sum_{x=0}^{\infty} M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta},a)}(\omega x)(-\omega x)_{j,\omega} w^{(\omega;\beta_i)}(\omega x) = 0, \quad j = 0, 1, \ldots n_i - 1, \quad i = 1, 2, \ldots, r. \tag{3.4}$$

The functions,

$$w^{(\omega;\beta_i)}(x) = \begin{cases} \frac{\Gamma_\omega(\beta_i+x)a^x}{\Gamma_\omega(\beta_i)\Gamma_\omega(x+\omega)} & \text{if } x \in \mathbb{R}/(\{-1,-2,...\} \cup \{-\beta_i,-\beta_i-1,...\}) \\ 0 & \text{if } x \in \{-1,-2,...\}\ . \end{cases} \tag{3.5}$$

are weight functions for ω-MMP-2 for $i = 1, 2, \ldots, r$ and Γ_ω is the ω-gamma function.

From Example 1.0.3 by choosing $z(x) = x$, the corresponding polynomials are unique since the weight functions form an AT system. When $\omega = 1$ the given orthogonality conditions coincide with the conditions given in (1.3).

3.2 ω-Rodrigues Formula for ω-Multiple Meixner Polynomials

In this section, the raising operators will be obtained and by use of raising operators ω-Rodrigues formulas and explicit expressions will be derived for both types ω-MMPs.

3.2.1 ω-Multiple Meixner Polynomials of the First Kind

Theorem 3.1: ω-MMPs of the first kind have the following raising operator ,

$$\nabla_\omega\left[M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)w_i^{(\omega;\beta)}(x)\right]=\frac{a_i^\omega-1}{a_i^\omega(\beta-\omega)}M_{\vec{n}+\vec{e_i}}^{(\omega;\beta-\omega;\vec{a})}(x)w_i^{(\omega;\beta-\omega)}(x). \tag{3.6}$$

Proof. By using the product rule for the ω-bacward operator $\nabla_\omega[f(x)g(x)]=f(x)\nabla_\omega g(x)+g(x-\omega)\nabla_\omega f(x)$, we obtain

$$\nabla_\omega\left[M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)w_i^{(\omega;\beta)}(x)\right]=\frac{a_i^\omega-1}{a_i^\omega(\beta-\omega)}(x)w_i^{(\omega;\beta-\omega)}(x)P_{\vec{n}+\vec{e_i}}.$$

Using ω- summation by parts formula,

$$\sum_{k=0}^{\infty}\triangle_\omega[f(\omega k)]g(\omega k)=-\sum_{k=0}^{\infty}\nabla_\omega[g(\omega k)]f(\omega k) where \quad g(-\omega)=0,$$

and the orthogonality conditions

$$\sum_{k=0}^{\infty}P_{\vec{n}+\vec{e_i}}(\omega x)(-\omega x)_{j,\omega}w_i^{(\omega;\beta-\omega)}(\omega x)=0,$$

we reach the result with $P_{\vec{n}+\vec{e_i}}=M_{\vec{n}+\vec{e_i}}^{(\omega;\beta-\omega;\vec{a})}$, which was guaranteed from the uniqness of the orthogonal polynomials. □

Theorem 3.2: Rodrigues formula for $\omega-$MMPs of the first kind is given by,

$$\begin{aligned}M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)=(\beta)_{|\vec{n}|,\omega}&\left[\prod_{k=1}^{r}\left(\frac{a_k^\omega}{a_k^\omega-1}\right)^{n_k}\frac{\Gamma_\omega(\beta)\Gamma_\omega(x+\omega)}{\Gamma_\omega(\beta+x)}\right]\\&\prod_{i=1}^{r}\frac{1}{a_i^x}\nabla_\omega^{n_i}\left[\frac{\Gamma_\omega(\beta+x+|\vec{n}|\omega)a_i^x}{\Gamma_\omega(x+\omega)\Gamma_\omega(\beta+|\vec{n}|\omega)}\right].\end{aligned} \tag{3.7}$$

Proof. Replacing $\vec{n}$ by $\vec{n}-\vec{e_i}$ and β by $\beta+\omega$ in the raising operator formula, we obtain

$$w_i^{(\omega;\beta)}(x)M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)=\frac{\beta a_i^\omega}{a_i^\omega-1}\nabla_\omega\left[w_i^{(\omega;\beta+\omega)}(x)M_{\vec{n}-\vec{e_i}}^{(\omega;\beta+\omega;\vec{a})}(x)\right].$$

Then for $r=2$ the multi-index will be $\vec{n}=(n_1,n_2)$. For $i=1$ we have,

$$w_1^{(\omega;\beta)}(x)M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x)=\frac{\beta a_1^{\omega}}{a_1^{\omega}-1}\nabla_{\omega}\left[w_1^{(\omega;\beta+\omega)}(x)M_{n_1-1,n_2}^{(\omega;\beta+\omega;a_1,a_2)}(x)\right],$$

and iterating it n_1 times we get,

$$w_1^{((\omega;\beta)}(x)M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x)=(\beta)_{n_1,\omega}\left(\frac{a_1^{\omega}}{a_1^{\omega}-1}\right)^{n_1}$$
$$\nabla_{\omega}^{n_1}\left[w_1^{(\omega;\beta+n_1\omega)}(x)M_{0,n_2}^{(\omega;\beta+n_1\omega;a_1,a_2)}(x)\right].$$

For $i=2$ we have,

$$w_2^{(\omega;\beta)}(x)M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x)=\frac{\beta a_2^{\omega}}{a_2^{\omega}-1}\nabla_{\omega}\left[w_2^{(\omega;\beta+\omega)}(x)M_{n_1,n_2-1}^{(\omega;\beta+\omega;a_1,a_2)}(x)\right].$$

and iterating it n_2 times we get,

$$w_2^{(\omega;\beta)}(x)M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x)=(\beta)_{n_2,\omega}\left(\frac{a_2^{\omega}}{a_2^{\omega}-1}\right)^{n_2}$$
$$\nabla_{\omega}^{n_2}\left[w_1^{(\omega;\beta+n_2\omega)}(x)M_{n_1,0}^{(\omega;\beta+n_2\omega;a_1,a_2)}(x)\right].$$

By combining these two equations, we obtain the expression for $M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x)$ and if we continue the iteration for r, we derive the Rodrigues formula for $\omega-$MMP-1. □

Theorem 3.3: The explicit form for $\omega-$MMPs of the first kind is given by

$$M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r}(-x)_{|\vec{k}|,\omega}$$
$$\prod_{j=1}^{r}\left[\frac{(a_j^{\omega})^{n_j-k_j}}{(a_j^{\omega}-1)^{n_j}}(\beta+x)_{|\vec{n}|-|\vec{k}|,\omega}\right]. \tag{3.8}$$

Proof. The explicit form obtained from the Rodrigues formula by using the Leibniz rule for the $\omega-$backward operator,

$$\nabla_{\omega}^{n}f(x)=\sum_{i=1}^{n}(-1)^{i}\binom{n}{i}f(x-i\omega).$$

□

3.2.2 ω-Multiple Meixner Polynomials of the Second Kind

Theorem 3.4: ω-MMPs of the second kind have the following raising operator ,

$$\nabla_{\omega}\left[M_{\vec{n}}^{(\omega;\vec{\beta},a)}w^{(\omega;\beta_i)}(x)\right]=\frac{a^{\omega}-1}{a^{\omega}(\beta_i-\omega)}M_{\vec{n}+\vec{e_i}}^{(\omega;\vec{\beta}-\omega\vec{e_i},a)}(x)w^{(\omega;\beta_i-\omega)}(x). \tag{3.9}$$

Proof. By using the product rule for the ω-backward operator, $\nabla_\omega[f(x)g(x)] = f(x)\nabla_\omega g(x) + g(x-\omega)\nabla_\omega f(x)$, we get,

$$\nabla_\omega\left[M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta},a)} w^{(\omega;\beta_i)}(x)\right] = \frac{a^\omega - 1}{a^\omega(\beta_i - \omega)} w^{(\omega;\beta_i-\omega)}(x) P_{\overrightarrow{n}+\overrightarrow{e_i}}(x)$$

is given in the following equation.

$$\sum_{k=0}^{\infty} \triangle_\omega[f(\omega k)]g(\omega k) = -\sum_{k=0}^{\infty} \nabla_\omega[g(\omega k)]f(\omega k) \quad where \quad g(-\omega) = 0,$$

By using ω- summation by parts and the orthogonality conditions we obtain,

$$\sum_{k=0}^{\infty} P_{\overrightarrow{n}+\overrightarrow{e_i}}(\omega x)(-\omega x)_{j,\omega} w^{\beta_i-\omega}(\omega x) = 0,$$

whence we obtain the result since we are guaranteed that $P_{\overrightarrow{n}+\overrightarrow{e_i}}(\omega x) = M_{\overrightarrow{n}+\overrightarrow{e_i}}^{(\omega;\overrightarrow{\beta}-\omega\overrightarrow{e_i},a)}(\omega x)$ from the uniqness of the $\omega-$MMP-2. □

Theorem 3.5: Rodrigues formula for $\omega-$MMPs of the second kind is given by

$$\begin{aligned} M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta},a)}(x) = \left(\frac{a^\omega}{a^\omega - 1}\right)^{|\overrightarrow{n}|} \prod_{k=1}^{r} (\beta_k)_{n_k,\omega} \left[\frac{\Gamma_\omega(x+\omega)}{a^x}\right] \prod_{i=1}^{r} \frac{\Gamma_\omega(\beta_i)}{\Gamma_\omega(\beta_i+x)} \\ \times \nabla_\omega^{n_i}\left[\frac{\Gamma_\omega(\beta_i+n_i\omega+x)}{\Gamma_\omega(\beta_i+n_i\omega)}\right]\left(\frac{a^x}{\Gamma_\omega(x+\omega)}\right). \end{aligned} \tag{3.10}$$

Proof. In the case when $r = 2$, we have $i = 1, 2$.

For $i = 1$, The raising operator have the following form,

$$M_{n_1,n_2}^{(\omega;\beta_1,\beta_2;a)}(x) = \frac{1}{w^{(\omega;\beta_1)}(x)}\left[\frac{a^\omega \beta_1}{a^\omega - 1}\right]\nabla_\omega[w^{(\omega;\beta_1+\omega)}(x) M_{n_1-1,n_2}^{(\omega;\beta_1+\omega,\beta_2;a)}(x)].$$

If we iterate the above equation n_1 times with changing β_1 we obtain,

$$M_{n_1,n_2}^{(\omega;\beta_1,\beta_2;a)}(x) = \frac{(\beta_1)_{n_1,\omega}}{\omega^{(\omega;\beta_1)}}\left[\frac{a^\omega}{a^\omega - 1}\right]^{n_1}\nabla_\omega^{n_1}[w^{(\omega;\beta_1+n_1\omega)}(x) M_{0,n_2}^{(\omega;\beta_1+n_1\omega,\beta_2;a)}(x)].$$

For $i = 2$ the raising operator have the following form,

$$M_{n_1,n_2}^{(\omega;\beta_1,\beta_2;a)}(x) = \frac{1}{w^{(\omega;\beta_2)}(x)}\left[\frac{a^\omega \beta_2}{a^\omega - 1}\right]\nabla_\omega[w^{(\omega;\beta_2+\omega)}(x) M_{n_1,n_2-1}^{(\omega;\beta_1,\beta_2+\omega;a)}(x)].$$

If we iterate that equation n_2 times with changing β_2 we obtain,

$$M_{n_1,n_2}^{(\omega;\beta_1,\beta_2;a)}(x) = \frac{(\beta_2)_{n_2,\omega}}{\omega^{(\omega;\beta_2)}}\left[\frac{a^\omega}{a^\omega - 1}\right]^{n_2}\nabla_\omega^{n_2}[w^{(\omega;\beta_2+n_2\omega)}(x) M_{n_1,0}^{(\omega;\beta_1,\beta_2+n_2\omega;a)}(x)].$$

Now, by combining these two equations we derive the Rodrigues formula for $M_{n_1,n_2}^{(\omega;\beta_1,\beta_2;a)}(x)$ and if we continue to the iteration for r we obtain the Rodrigues formula for $\omega-$MMP-2. $\square$

Theorem 3.6: The explicit form for $\omega-$MMPs of the second kind is given by,

$$\begin{aligned} M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta},a)}(x) = & \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}\frac{a^{\omega|\overrightarrow{n}|-\omega|\overrightarrow{k}|}}{(a^{\omega}-1)^{|\overrightarrow{n}|}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r}(-x)_{|\overrightarrow{k}|,\omega} \\ & \times\prod_{j=1}^{r}\left(\beta_j+x-\sum_{i=1}^{j-1}k_i\omega\right)_{n_j-k_j,\omega}. \end{aligned} \tag{3.11}$$

Proof. By use of the Rodrigues formula and the Leibniz rule for the $\omega-$backward operator, we obtain the explicit form.by using

$$\nabla_{\omega}^{n}f(x)=\sum_{i=1}^{n}(-1)^{i}\binom{n}{i}f(x-i\omega).$$

$\square$

Chapter 4

SOME PROPERTIES OF ω-MULTIPLE MEIXNER POLYNOMIALS

4.1 Generating Functions

ω-MMPs have a multivariate generating function with r variables.The following Lemma will be useful for the proof of the theorem given for the generating function.

Lemma 4.1: [6, lemma 1, p. 4] The generating function for the multinomial coefficients is given as follows,

$$\sum_{m_1=0}^{\infty}\dots\sum_{m_r=0}^{\infty}\frac{(-x)_{|\overrightarrow{m}|}}{m_1!\dots m_r!}t_1^{m_1}\dots t_r^{m_r}=(1-t_1\dots-t_r)^x.$$

This series converges absolutely and uniformly for $|t_1|+\dots+|t_r|<1$ when $x\notin\mathbb{N}$ and contains a finite number of terms if $x\in\mathbb{N}$.

4.1.1 ω-Multiple Meixner Polynomials of the First Kind

Theorem 4.1: Let ω be a positive real number. The generating function for the ω-MMPs of the first kind is given as follows,

$$\begin{aligned}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\dots\sum_{n_r=0}^{\infty}M_{\overrightarrow{n}}^{(\omega;\beta;\overrightarrow{a})}(x)\frac{t_1^{n_1}\dots t_r^{n_r}}{n_1!\dots n_r!}&=\left(1-\frac{\omega t_1}{a_1^{\omega}-1}-\dots-\frac{\omega t_r}{a_r^{\omega}-1}\right)^{\frac{x}{\omega}}\\&\times\left(1-\frac{\omega t_1a_1^{\omega}}{a_1^{\omega}-1}-\dots-\frac{\omega t_ra_r^{\omega}}{a_r^{\omega}-1}\right)^{\frac{-(\beta+x)}{\omega}}.\end{aligned}\tag{4.1}$$

Proof. Replacing $M_{\overrightarrow{n}}^{(\omega;\beta;\overrightarrow{a})}(x)$ with the explicit form in LHS of (4.1) we obtain,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r}(-x)_{|\overrightarrow{k}|,\omega}$$
$$\times\prod_{j=1}^{r}\left[\frac{(a_k^{\omega})^{n_j-k_j}}{(a_k^{\omega}-1)^{n_j}}(\beta+x)_{|\overrightarrow{n}|-|\overrightarrow{k}|,\omega}\right]\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}.$$

We now change the order of summation to get,

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\sum_{n_1=k_1}^{\infty}\cdots\sum_{n_r=k_r}^{\infty}\frac{(-x)_{|\overrightarrow{k}|,\omega}}{k_1!(n_1-k_1)!\ldots k_r!(n_r-k_r)!}\frac{a_1^{\omega(n_1-k_1)}\ldots a_r^{\omega(n_r-k_r)}}{(a_1^{\omega}-1)^{n_1}\ldots(a_r^{\omega}-1)^{n_r}}$$
$$\times x t_1^{n_1}\ldots t_r^{n_r}(\beta+x)_{|\overrightarrow{n}|-|\overrightarrow{k}|,\omega}.$$

Bext, we set $m_i=n_i-k_i$ and put the factors in m_i and k_i together to obtain ,

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{(-x)_{|\overrightarrow{k}|,\omega}}{k_1!\ldots k_r!}\left(\frac{t_1}{a_1^{\omega}-1}\right)^{k_1}\cdots\left(\frac{t_r}{a_r^{\omega}-1}\right)^{k_r}$$
$$\times\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(\beta+x)_{|\overrightarrow{m}|,\omega}}{m_1!\ldots m_r!}\left(\frac{t_1a_1^{\omega}}{a_1^{\omega}-1}\right)^{m_1}\cdots\left(\frac{t_ra_r^{\omega}}{a_r^{\omega}-1}\right)^{m_r}.$$

Using the relation between pochhammer symbol and ω-pochhammer symbol , the above equation becomes,

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{\left(\frac{-x}{\omega}\right)_{|\overrightarrow{k}|}}{k_1!\ldots k_r!}\left(\frac{\omega t_1}{a_1^{\omega}-1}\right)^{k_1}\cdots\left(\frac{\omega t_r}{a_r^{\omega}-1}\right)^{k_r}$$
$$\times\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{\left(\frac{\beta+x}{\omega}\right)_{|\overrightarrow{m}|}}{m_1!\ldots m_r!}\left(\frac{\omega t_1a_1^{\omega}}{a_1^{\omega}-1}\right)^{m_1}\cdots\left(\frac{\omega t_ra_r^{\omega}}{a_r^{\omega}-1}\right)^{m_r}.$$

Finally, using Lemma 3.1.1, we obtain the generating function for ω-MMP-1, which is the desired result. □

4.1.2 ω-Multiple Meixner Polynomials of the Second Kind

Theorem 4.2: Let ω be a positive real number. The generating function for the ω-MMPs of the second kind is given as,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}=\left(1-\frac{1}{\omega a^{\omega}}\left[1-\prod_{j=1}^{r}(1+\frac{\omega a^{\omega}}{1-a^{\omega}}t_j)\right]\right)^{\frac{x}{\omega}}$$
$$\times\prod_{i=1}^{r}\left(1+\frac{\omega a^{\omega}}{1-a^{\omega}}t_i\right)^{-\frac{x+\beta_i}{\omega}}. \tag{4.2}$$

Proof. Let us denote the LHS of (4.2) by H, then by replacing $M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)$ with explicit form (3.11) we obtain,

$$H=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\frac{a^{(|\overrightarrow{n}|-|\overrightarrow{k}|)\omega}}{(a^{\omega}-1)^{|\overrightarrow{n}|}}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}(-x)_{|\overrightarrow{k}|,\omega}(\beta_1+x)_{n_1-k_1,\omega}$$
$$\times(\beta_2+x-k_1\omega)_{n_2-k_2,\omega}\ldots(\beta_r+x-k_1\omega-\ldots k_{r-1}\omega)_{n_r-k_r,\omega}\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}.$$

Changing the order of the summation in the above equation gives us,

$$H=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{n_1=k_1}^{\infty}\cdots\sum_{n_r=k_r}^{\infty}\frac{a^{(|\overrightarrow{n}|-|\overrightarrow{k}|)\omega}}{(a^{\omega}-1)^{|\overrightarrow{n}|}}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}(-x)_{|\overrightarrow{k}|,\omega}(\beta_1+x)_{n_1-k_1,\omega}$$
$$\times(\beta_2+x-k_1\omega)_{n_2-k_2,\omega}\ldots(\beta_r+x-k_1\omega-\ldots k_{r-1}\omega)_{n_r-k_r,\omega}\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}.$$

Now, by setting $m_i=n_i-k_i$, putting the factors in m_i and k_i together and then using the relation between pochhammer symbol and ω-pochhammer symbol the above equation reaches the following form,

$$H=\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{(\frac{-x}{\omega})_{|\overrightarrow{k}|}}{k_1!..k_r!}\left(\frac{\omega t_1}{a^{\omega}-1}\right)^{k_1}\cdots\left(\frac{\omega t_r}{a^{\omega}-1}\right)^{k_r}\sum_{m_1=0}^{\infty}\frac{(\frac{\beta_1+x}{\omega})_{m_1}}{m_1!}\left(\frac{\omega a^{\omega}t_1}{a^{\omega}-1}\right)^{m_1}$$
$$\times\sum_{m_2=0}^{\infty}\frac{(\frac{\beta_2+x}{\omega}-k_1)_{m_2}}{m_2!}\left(\frac{\omega a^{\omega}t_2}{a^{\omega}-1}\right)^{m_2}\cdots\sum_{m_r=0}^{\infty}\frac{(\frac{\beta_r+x}{\omega}-k_1\ldots k_{r-1})_{m_r}}{m_r!}\left(\frac{\omega a^{\omega}t_r}{a^{\omega}-1}\right)^{m_r}.$$

Let us now use the Binomial theorem for each sum over m_i

$$\sum_{k}^{\infty}\frac{(a)_k}{k!}x^k=(1-x)^{-a},\quad |x|<1,$$

to obtain

$$H=\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{(\frac{-x}{\omega})_{|\overrightarrow{k}|}}{k_1!..k_r!}\left(\frac{\omega t_1}{a^{\omega}-1}\left(1+\frac{\omega t_2a^{\omega}}{1-a^{\omega}}\right)\cdots\left(1+\frac{\omega t_ra^{\omega}}{1-a^{\omega}}\right)\right)^{k_1}$$
$$\times\left(\frac{\omega t_2}{a^{\omega}-1}\left(1+\frac{\omega t_3a^{\omega}}{1-a^{\omega}}\right)\cdots\left(1+\frac{\omega t_ra^{\omega}}{1-a^{\omega}}\right)\right)^{k_2}\left(\frac{\omega t_{r-1}}{a^{\omega}-1}\left(1+\frac{\omega t_ra^{\omega}}{1-a^{\omega}}\right)\right)^{k_{r-1}}$$
$$\times\left(\frac{\omega t_r}{a^{\omega}-1}\right)^{k_r}\prod_{i=1}^{r}\left(1+\frac{\omega t_ia^{\omega}}{1-a^{\omega}}\right)^{-\frac{\beta_i+x}{\omega}}.$$

Letting $c=\frac{\omega a^{\omega}}{1-a^{\omega}}$ and

$$s_j=t_j(1+ct_{j+1})\ldots(1+ct_t),\quad 1\le j\le r,$$

implies the following,

$$H=\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{(\frac{-x}{\omega})_{|\overrightarrow{k}|}}{k_1!..k_r!}s_1^{k_1}\ldots s_r^{k_r}\left(\frac{-c}{\omega a^{\omega}}\right)^{|\overrightarrow{k}|}\prod_{i=1}^{r}\left(1+\frac{\omega a^{\omega}t_i}{1-a^{\omega}}\right)^{-\frac{\beta_i+x}{\omega}}.$$

In the next step, we use Lemma 3.1.1 to reach

$$H=\left[1+\frac{c}{\omega a^{\omega}}(s_1+\cdots+s_r)\right]^{x/\omega}\prod_{i=1}^{r}\left(1+\frac{\omega a^{\omega}t_i}{1-a^{\omega}}\right)^{-\frac{\beta_i+x}{\omega}}.$$

Finally, after some simple calculations it easy to see that,

$$1+\frac{c}{\omega a^{\omega}}(s_1+\cdots+s_r)=1-\frac{1}{\omega a^{\omega}}\left[1-\prod_{j=1}^{r}\left(1+\frac{\omega a^{\omega}}{1-a^{\omega}}t_j\right)\right],$$

whence we obtain the desired proof for the generating function of ω-MMP-2. $\square$

4.2 Connection and Addition Formulas

Generating function will be used to establish the connection formula and the addition formula for ω-MMPs.

4.2.1 ω-Multiple Meixner Polynomials of the First Kind

Theorem 4.3: Let ω be a positive real number. ω-MMPs of the first kind $M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)$ and $M_{\vec{n}-\vec{k}}^{(\omega;\beta+\gamma;\vec{a})}(x)$ satisfy the following connection formula,

$$\begin{aligned}M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)=&\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}(-\gamma)_{|\vec{k}|,\omega}\\&\times\left(\frac{a_1^{\omega}}{a_1^{\omega-1}}\right)^{k_1}\cdots\left(\frac{a_r^{\omega}}{a_r^{\omega-1}}\right)^{k_r}M_{\vec{n}-\vec{k}}^{(\omega;\beta+\gamma;\vec{a})}(x).\end{aligned}\tag{4.3}$$

Proof. Replacing β by $\beta-\gamma+\gamma$ in the generating function (4.1) we obtain,

$$\begin{aligned}\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}=&\left(1-\frac{\omega t_1}{a_1^{\omega}-1}-\ldots-\frac{\omega t_r}{a_r^{\omega}-1}\right)^{\frac{x}{\omega}}\\&\times\left(1-\frac{\omega t_1a_1^{\omega}}{a_1^{\omega}-1}-\ldots-\frac{\omega t_ra_r^{\omega}}{a_r^{\omega}-1}\right)^{\frac{-(\beta+\gamma+x)}{\omega}}\\&\times\left(1-\frac{\omega t_1a_1^{\omega}}{a_1^{\omega}-1}-\ldots-\frac{\omega t_ra_r^{\omega}}{a_r^{\omega}-1}\right)^{\frac{\gamma}{\omega}}.\end{aligned}$$

From generating function (4.1) and Lemma 3.1.1, the above equation gets the following form,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} M_{\overrightarrow{n}}^{(\omega;\beta;\overrightarrow{a})}(x)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} M_{\overrightarrow{n}}^{(\beta+\gamma,\overrightarrow{a})}(x)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}$$
$$\times\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{\left(\frac{-\gamma}{\omega}\right)_{|\overrightarrow{k}|}}{k_1!\ldots k_r!}\left(\frac{\omega a_1^{\omega}}{a_1^{\omega}-1}\right)^{k_1}\cdots\left(\frac{\omega a_r^{\omega}}{a_r^{\omega}-1}\right)^{k_r}$$
$$\times\, t_1^{k_1}\ldots t_r^{k_r}.$$

Changing the order of summations and using the relation between pochhammer symbol and ω-pochhammer symbol we get,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} M_{\overrightarrow{n}}^{(\omega;\beta;\overrightarrow{a})}(x)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}$$
$$\times M_{\overrightarrow{n}-\overrightarrow{k}}^{(\omega;\beta+\gamma;\overrightarrow{a})}(x)(-\gamma)_{|\overrightarrow{k}|,\omega}\left(\frac{a_1^{\omega}}{a_1^{\omega-1}}\right)^{k_1}\cdots$$
$$\times\left(\frac{a_r^{\omega}}{a_r^{\omega-1}}\right)^{k_r}\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}.$$

Finally, comparing the coefficients of $\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}$, appearing on both sides of the above equation,we obtained the desired result. □

Theorem 4.4: Let ω be a positive real number. The addition fomula for ω-MMPs of the first kind is given by,

$$M_{\overrightarrow{n}}^{(\omega;\beta+\gamma;\overrightarrow{a})}(x+y) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\omega;\beta;\overrightarrow{a})}(x)M_{\overrightarrow{k}}^{(\omega;\gamma;\overrightarrow{a})}(y). \tag{4.4}$$

Proof. Firstly, we change x with $x+y$ and β with $\beta+\gamma$ in the generating function (4.1), and then change the order of summations we obtain,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} M_{\overrightarrow{n}}^{(\beta+\gamma,\overrightarrow{a})}(x+y)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\frac{t_1^{k_1}\ldots t_r^{k_r}}{k_1!\ldots k_r!}M_{\overrightarrow{k}}^{(\gamma,\overrightarrow{a})}(y)$$
$$*\,\frac{t_1^{n_1-k_1}\ldots t_r^{n_r-k_r}}{(n_1-k_1)!\ldots(n_r-k_r)!}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\beta,\overrightarrow{a})}(x).$$

Finally, comparing the coefficients of $\frac{t_1^{n_1}\ldots t_r^{nr}}{n_1!\ldots n_r!}$, appearing on both sides of the above equation we get the result. □

Corollary 4.1: MMPs of the first kind satisfy the following connection and addition formulas respectively,

$$M_{\overrightarrow{n}}^{(\beta,\overrightarrow{a})}(x)=\sum_{k_1=0}^{n_1}...\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}...\binom{n_r}{k_r}a_1^{k_1}\dots a_r^{k_r}(-\gamma)_{|\overrightarrow{k}|}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\beta+\gamma,\overrightarrow{a})}(x),$$

$$M_{\overrightarrow{n}}^{(\beta+\gamma,\overrightarrow{a})}(x+y)=\sum_{k_1=0}^{n_1}...\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}...\binom{n_r}{k_r}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\beta,\overrightarrow{a})}(x)M_{\overrightarrow{k}}^{(\gamma,\overrightarrow{a})}(y).$$

4.2.2 ω-Multiple Meixner Polynomials of the Second Kind

Theorem 4.5: Let ω be a positive real number. ω-MMPs of the second kind $M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)$ and $M_{\overrightarrow{n}-\overrightarrow{k}}^{(\omega;\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x)$ satisfy the following connection formula,

$$\begin{aligned}M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)=&\sum_{k_1=0}^{n_1}...\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}...\binom{n_r}{k_r}(-\gamma_1)_{k_1,\omega}\dots(-\gamma_r)_{k_r,\omega}\\&\left(\frac{a^\omega}{a^\omega-1}\right)^{|\overrightarrow{k}|}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\omega;\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x).\end{aligned}\tag{4.5}$$

Proof. Let us replace $\overrightarrow{\beta}$ with $\overrightarrow{\beta}+\overrightarrow{\gamma}-\overrightarrow{\gamma}$ in the generating function given by (4.2) to obtain,

$$\begin{aligned}\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}=&\left(1-\frac{1}{\omega a^\omega}\left[1-\prod_{j=1}^{r}(1+\frac{\omega a^\omega}{1-a^\omega}t_j)\right]\right)^{\frac{x}{\omega}}\\&\times\prod_{i=1}^{r}\left(1+\frac{\omega a^\omega}{1-a^\omega}t_i\right)^{-\frac{x}{\omega}}\prod_{i=1}^{r}\left(1+\frac{\omega a^\omega}{1-a^\omega}t_i\right)^{-\frac{\overrightarrow{\beta}+\overrightarrow{\gamma}}{\omega}}\\&\times\prod_{i=1}^{r}\left(1+\frac{\omega a^\omega}{1-a^\omega}t_i\right)^{\frac{\overrightarrow{\gamma}}{\omega}}.\end{aligned}$$

From the generating function (4.2) and Lemma 3.1.1 the above equation get the following form,

$$\begin{aligned}\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}=&\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x)\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}\\&\times\sum_{k_1=0}^{\infty}\frac{(\frac{-\gamma_1}{\omega})_{k_1}(\frac{\omega a^\omega}{a^\omega-1})^{k_1}}{k_1!}t_1^{k_1}\dots\frac{(\frac{-\gamma_r}{\omega})_{k_r}(\frac{\omega a^\omega}{a^\omega-1})^{k_r}}{k_r!}t_r^{k_r}.\end{aligned}$$

By changing the order of summations, then using the relation between pochhammer symbol and $\omega-$pochhammer symbol we obtain,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!} = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}$$
$$\times M_{\overrightarrow{n}-\overrightarrow{k}}^{(\omega;\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x)(-\gamma_1)_{k_1,\omega}\cdots(-\gamma_r)_{k_r,\omega}$$
$$\times\left(\frac{a^{\omega}}{a^{\omega}-1}\right)^{|\overrightarrow{k}|}\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}.$$

Finally, comparing the coefficients of $\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}$ appearing on both sides of the above equation, the result for the connection formula for the ω-MMP-2 is obtained. □

Theorem 4.6: Let ω be a positive real number. The addition formula for ω-MMPs of the second kind is,

$$M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x+y) = \sum_{k_1=0}^{n_1}...\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}...\binom{n_r}{k_r}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\omega;\overrightarrow{\beta};a)}(x)M_{\overrightarrow{k}}^{(\omega;\overrightarrow{\gamma};a)}(y). \tag{4.6}$$

Proof. By changing x with $x+y$ and $\overrightarrow{\beta}$ with $\overrightarrow{\beta}+\overrightarrow{\gamma}$ in the generating function (4.2) and then changing the order of summations we get,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x+y)\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!} = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}$$
$$\times M_{\overrightarrow{n}-\overrightarrow{k}}^{(\omega;\overrightarrow{\beta};a)}(x)M_{\overrightarrow{k}}^{(\omega;\overrightarrow{\gamma};a)}(y)\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}.$$

Finally, comparing the coefficients of $\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}$ appearing on both sides of the equation the result is obtained for the addition formula. □

Corollary 4.2: MMPs of the second kind satisfy the following connection and addition formulas respectively,

$$M_{\overrightarrow{n}}^{(\overrightarrow{\beta};a)}(x) = \sum_{k_1=0}^{n_1}...\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}...\binom{n_r}{k_r}(-\gamma_1)_{k_1}\dots(-\gamma_r)_{k_r}a^{|\overrightarrow{k}|}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x),$$
$$M_{\overrightarrow{n}}^{(\overrightarrow{\beta}+\overrightarrow{\gamma};a)}(x+y) = \sum_{k_1=0}^{n_1}...\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}...\binom{n_r}{k_r}M_{\overrightarrow{n}-\overrightarrow{k}}^{(\overrightarrow{\beta};a)}(x)M_{\overrightarrow{k}}^{(\overrightarrow{\gamma};a)}(y).$$

4.3 Limit Relations

By use of (3.1) the ω- MCPs is defined in [17], which has the following explicit form.

$$C_{\vec{n}}^{\vec{c}}(x)=(-c_1^{\omega})^{n_1}\dots(-c_r^{\omega})^{n_r}\sum_{k_1=0}^{n_1}\dots\sum_{k_r=0}^{n_r}\frac{(-n_1)_{k_1}\dots(-n_r)_{k_r}}{k_1!\dots k_r!}\left(\frac{-x}{\omega}\right)_{k_1+k_2+\dots+k_r}$$
$$\left[\left(\frac{-1}{c_1}\right)^{\omega}\omega\right]^{k_1}\dots\left[\left(\frac{-1}{c_r}\right)^{\omega}\omega\right]^{k_r} \tag{4.7}$$

In this section, the limit relations between ω-MMPs and ω- MCPs is given.

4.3.1 ω-Multiple Meixner Polynomials of the First Kind

Theorem 4.7: The ω- MCPs are a limit case of ω-MMPs of the first kind,

$$\lim_{\beta\to\infty} M_{\vec{n}}^{(\omega,\beta,\vec{a})}(x)=C_{\vec{n}}^{\vec{c}}(x). \tag{4.8}$$

Proof. By use of relation between pochammer symbol with ω-pochammer symbol, the explicit equation of ω-MMP-1 (3.8) get the following form,

$$M_{\vec{n}}^{(\omega,\beta,\vec{a})}(x)=\sum_{k_1=0}^{n_1}\dots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\dots\binom{n_r}{k_r}\left(\frac{\omega a_1^{\omega}}{a_1^{\omega}-1}\right)^{n_1}\dots\left(\frac{\omega a_r^{\omega}}{a_r^{\omega}-1}\right)^{n_r}$$
$$\times\left(\frac{1}{a_1^{\omega}}\right)^{k_1}\dots\left(\frac{1}{a_r^{\omega}}\right)^{k_r}\times\left(\frac{-x}{\omega}\right)_{|\vec{k}|}\left(\frac{\beta+x}{\omega}\right)_{|\vec{n}|-|\vec{k}|}$$

Setting $a_1^{\omega}=\frac{c_1^{\omega}}{\beta}\dots a_r^{\omega}=\frac{c_r^{\omega}}{\beta}$ and getting limit when $\beta\to\infty$ we obtain the desired result.

□

4.3.2 ω-Multiple Meixner Polynomials of the Second Kind

Theorem 4.8: The ω- MCPs are a limit case of ω-MMPs of the second kind,

$$\lim_{\beta\to\infty} M_{\vec{n}}^{(\omega,\vec{\beta};a)}(x)=C_{\vec{n}}^{\vec{c}}(x). \tag{4.9}$$

Proof. By use of relation between pochammer symbol with ω-pochammer symbol, the explicit equation of ω-MMP-2 (3.11) get the following form,

$$M_{\vec{n}}^{(\omega,\vec{\beta};a)}(x)=\sum_{k_1=0}^{n_1}\dots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\dots\binom{n_r}{k_r}\left[\frac{a^{\omega n_1}}{a^{\omega k_1}}\dots\frac{a^{\omega n_r}}{a^{\omega k_r}}\right]\left[\frac{1}{(a^{\omega}-1)^{n_1}}\dots\frac{1}{(a^{\omega}-1)^{n_r}}\right]$$
$$\times\,\omega^{n_1+n_2\dots+n_r}\left(\frac{-x}{\omega}\right)_{|\vec{k}|}\left(\frac{\beta_1+x}{\omega}\right)_{|\vec{n_1}|-|\vec{k_1}|}\left(\frac{\beta_2+x-k_1\omega}{\omega}\right)_{|\vec{n_2}|-|\vec{k_2}|}\dots$$
$$\times\left(\frac{\beta_r+x-k_1\omega\dots-k_{r-1}\omega}{\omega}\right)_{|\vec{n_r}|-|\vec{k_r}|}$$

Setting $a^{\omega}=\frac{1}{\beta}$, $\beta_1=c_1^{\omega}\beta$, $\beta_2=c_2^{\omega}\beta,\dots,\beta_r=c_r^{\omega}\beta$ and getting limit when

$\beta \to \infty$ we obtain the desired result. $\square$

Chapter 5

ALGEBRAIC PROPERTIES OF ω-MULTIPLE MEIXNER POLYNOMIALS

5.1 Raising and Lowering Operators

5.1.1 ω-Multiple Meixner Polynomials of the First Kind

Theorem 5.1: Let ω be a positive real number. The raising operator for ω-MMPs of the first kind is given as,

$$L_{a_i}^{(\beta)}[M_{\vec{n}}^{(\omega;\beta;\vec{a})}] = -M_{\vec{n}+e_i}^{(\omega;\beta-\omega;\vec{a})} \tag{5.1}$$

where $L_{a_i}^{(\beta)}[.]$ is defined by

$$L_{a_i}^{(\beta)}[y] = \frac{x}{1-a_i^{\omega}}\nabla_{\omega}y - \left[\frac{a_i^{\omega}(\beta-\omega)}{1-a_i^{\omega}} - x\right]y.$$

Proof. Proof follows directly from the Rodrigues formula (3.7) for the ω-MMP-1. □

Theorem 5.2: Let ω be a positive real number. The lowering operator for ω-MMPs of the first kind is,

$$\Delta_{\omega}M_{\vec{n}}^{(\omega;\beta;\vec{a})} = \sum_{i=1}^{r}\omega n_i M_{\vec{n}-e_i}^{(\omega;\beta+\omega;\vec{a})}. \tag{5.2}$$

In particular, for $r=2$

$$\Delta_{\omega}M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)} = \omega n_1 M_{n_1-1,n_2}^{(\omega;\beta+\omega;a_1,a_2)} + \omega n_2 M_{n_1,n_2-1}^{(\omega;\beta+\omega;a_1,a_2)}.$$

Proof. Replacing β with $\beta-\omega$ and applying operator Δ_{ω} to both sides of the generating function (4.1) for the case $r=2$ we obtain,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Delta_\omega M_{n_1,n_2}^{(\omega;\beta-\omega;a_1,a_2)}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}=(\omega t_1+\omega t_2)\left(1-\frac{\omega t_1}{a_1^\omega-1}-\frac{\omega t_2}{a_2^\omega-1}\right)^{\frac{x}{\omega}}$$
$$\times\left(1-\frac{\omega t_1 a_1^\omega}{a_1^\omega-1}-\frac{\omega t_2 a_2^\omega}{a_2^\omega-1}\right)^{\frac{-(\beta+x)}{\omega}}.$$

By use of the generating function (4.1), the above equation get the following form,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Delta_\omega M_{n_1,n_2}^{(\omega;\beta-\omega;a_1,a_2)}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}=\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}M_{n_1,n_2}^{(\beta,a_1,a_2)}\frac{t_1^{n_1+1}t_2^{n_2}}{n_1!n_2!}$$
$$+\omega\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}M_{n_1,n_2}^{(\beta,a_1,a_2)}\frac{t_1^{n_1}t_2^{n_2+1}}{n_1!n_2!}.$$

Changing β with $\beta+\omega$ and then replacing n_1 with n_1-1 and n_2 with n_2-1 in the right side of equation we obtain,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Delta_\omega M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\omega n_1 M_{n_1-1,n_2}^{(\omega;\beta+\omega;a_1,a_2)}\right.$$
$$\left.+\omega n_2 M_{n_1,n_2-1}^{(\omega;\beta+\omega;a_1,a_2)}\right]\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}.$$

Finally, comparing the coefficients of $\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}$, appearing on both sides of the above equation the desired result is obtained for the case $r=2$. □

Remark 5.1: In the case $\omega=1$, the lowering operator for the ω-MMPs of the first kind coincides with the lowering operator for the MMPs of the first kind which is given in [10]. For the proof of the ω type we consider a different approach, where the generating function plays an important role and the proof becomes simpler when compared with the corresponding proof when $\omega=1$. [10, Theorem 2.4, p. 138]

5.1.2 ω-Multiple Meixner Polynomials of the Second Kind

Theorem 5.3: Let ω be a positive real number.The raising operator for ω-MMPs of the second kind is given as

$$L_{\beta_i}^{a}[M_{\vec{n}}^{(\omega;\vec{\beta}\,a)}(x)]=-M_{\vec{n}+e_i}^{(\omega;\vec{\beta}-\omega e_i;a)}(x),\tag{5.3}$$

where $L_{\beta_i}^{a}[.]$ is defined by

$$L^a_{\beta_i}[y] = \frac{x}{1-a^\omega}\nabla_\omega y - \left[\frac{a^\omega(\beta_i-\omega)}{1-a^\omega} - x\right] y.$$

Proof. The proof follows from the Rodrigues formula (3.10) for the ω-MMP-2. $\square$

We now need to give the following three lemmas before the proof of the main theorem.

Lemma 5.1: Let ω be a positive real number. We have the folowing relation for ω-MMPs $M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)_{|\vec{n}|=0}^{\infty}$ of the second kind ,

$$\sum_{x=0}^{\infty} M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)\nabla_\omega[(-x)_{n_k-1,\omega}W_{\omega,k}(x)] = \frac{a^\omega-1}{a^\omega}\sum_{x=0}^{\infty} M^{(\vec{\beta};a)}_{\vec{n}}(x)(-x)_{n_k-1,\omega}W_{\omega,k}(x), \tag{5.4}$$

and for $i = 1,2\dots,r$,

$$\sum_{x=0}^{\infty} M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)(-x)_{n_k-1,\omega}W_{\omega,k}(x) = \frac{a^\omega(\beta_k+\omega n_k-\beta_i)}{1-a^\omega} \times \sum_{x=0}^{\infty} M^{(\omega;\vec{\beta}+\omega e_i;a)}_{\vec{n}-e_i}(x)(-x)_{n_k-1,\omega}W_{\omega,k}(x), \tag{5.5}$$

where $W_{\omega,k}(x) := w^{(\omega;\beta_k+\omega)}(x)$.

Proof. By using the product rule for the ω-backward operator it can easily be seen that,

$$\begin{aligned}\nabla_\omega[(-x)_{n_k-1,\omega}W_{\omega,k}(x)] &= \nabla_\omega[(-x)_{n_k-1,\omega}]W_{\omega,k}(x) + (-x+\omega)_{n_k-1,\omega}\nabla_\omega[W_{\omega,k}(x)]\\ &= \delta(x)w^{(\omega;\beta_k)}(x),\end{aligned}$$

where $\delta(x) = \frac{1-a^\omega}{a^\omega}(-x)_{n_k,\omega} + \dots$.

From the orthogonality conditions, it is known that,

$\sum_{x=0}^{\infty} M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)(-x)_{n_k-1,\omega}w^{(\omega;\beta_k)}(x) = 0$. Thus we obtain the result for (5.4) as ,

$$\begin{aligned}\sum_{x=0}^{\infty} M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)\nabla_\omega[(-x)_{n_k-1,\omega}W_{\omega,k}(x)] &= \frac{1-a^\omega}{a^\omega}\sum_{x=0}^{\infty} M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)(-x)_{n_k,\omega}w^{(\omega;\beta_k)}(x)\\ &= \frac{1-a^\omega}{a^\omega}\sum_{x=0}^{\infty} M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)(-x)_{n_k-1,\omega}(-x-\beta_k)\\ &\quad\times w^{(\omega;\beta_k)}(x)\\ &= \frac{a^\omega-1}{a^\omega}\sum_{x=0}^{\infty} M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)(-x)_{n_k-1,\omega}W_{\omega,k}(x).\end{aligned}$$

To prove (5.5), we use the relation $M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x) = -L^a_{\beta_i+\omega}[M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)]$ to obtain,

$$\begin{aligned}
&\sum_{x=0}^{\infty} M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)(-x)_{n_k-1,\omega}W_{\omega,k}(x)\\
&\quad= -\sum_{x=0}^{\infty} L^a_{\beta_i+\omega}[M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)](-x)_{n_k-1,\omega}W_{\omega,k}(x)\\
&\quad= \sum_{x=0}^{\infty} M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)\left[\frac{1}{1-a^{\omega}}\Delta_{\omega}[x(-x)_{n_k-1}W_{\omega,k}(x)-\right.\\
&\quad\times\left.\left(\frac{a^{\omega}\beta_i}{1-a^{\omega}}-x\right)(-x)_{n_k-1}W_{\omega,k}(x)\right]\\
&\quad= \frac{a^{\omega}(\beta_k+\omega n_k-\beta_i)}{1-a^{\omega}}\sum_{x=0}^{\infty} M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)(-x)_{n_k-1}W_{\omega,k}(x)\\
&\quad-\frac{\omega a^{\omega}(n_k-1)(\omega(n_k-1)+\beta_k)}{1-a^{\omega}}\sum_{x=0}^{\infty} M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)(-x)_{n_k-2}(\beta_k+x)w^{(\omega;\beta_k)}(x).
\end{aligned}$$

From the orthogonality conditions we have,

$$\sum_{x=0}^{\infty} M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)(-x)_{n_k-2}(\beta_k+x)w^{(\omega;\beta_k)}(x) = 0.$$

This gives the desired result (5.5). □

Lemma 5.2: [10, lemma 2.8, p. 141] Let $C = [c_{ij}]_{i,j=1}^r$ be a $r\times r$ matrix. If $c_{ip}-c_{iq} = \varepsilon_{pq}c_{ip}c_{iq}$ and

$c_{pi}-c_{qi} = \delta_{pq}c_{pi}c_{qi}$, $i,p,q = 1,\dots,r$, then the determinant $|C|$ of C is

$$|C| = \left(\prod_{i=1}^{r}\prod_{j=1}^{r} c_{ij}\right)\left(\prod_{q=1}^{r-1}\prod_{p=q+1}^{r}\varepsilon_{pq}\delta_{pq}\right). \tag{5.6}$$

Proof. See [10] for the proof of lemma. □

Lemma 5.3: Let ω be a positive real number. For the ω-MMP $M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)_{|\vec{n}|=0}^{\infty}$ of the second kind, we have

$$\sum_{x=0}^{\infty} M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)(-x)_{n_k-1,\omega}W_{\omega,k}(x) = c_{ki}\sum_{x=0}^{\infty} M_{\vec{n}-\vec{e}}^{(\omega;\vec{\beta}+\vec{e};a)}(x)(-x)_{n_k-1,\omega}W_{\omega,k}(x), \tag{5.7}$$

where,

$$c_{ki} = \left(\frac{a^{\omega}}{1-a^{\omega}}\right)^{r-1}\left(\frac{1}{\omega n_k+\beta_k-\beta_i}\right)\prod_{j=1}^{r}(\omega n_k+\beta_k-\beta_j). \tag{5.8}$$

Proof. If we apply Lemma 4.1.1 on the indices β_k+1, a and $\overrightarrow{n}-\overrightarrow{e_i}$, we can obtain inductively the proof of the relation. $\square$

Theorem 5.4: Let ω be a positive real number. For the ω-MMP $M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)_{|\overrightarrow{n}|=0}^{\infty}$ of the second kind, we have the following lowering operator,

$$\Delta_{\omega}M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x) = \sum_{i=1}^{r} d_i M_{\overrightarrow{n}-e_i}^{(\omega;\overrightarrow{\beta}+\omega e_i;a)}(x), \tag{5.9}$$

where,

$$d_i = \frac{\prod_{p=1}^{r}(\omega n_p+\beta_p-\beta_i)}{\prod_{q=1,q\neq i}^{r-1}(\beta_i-\beta_q)\prod_{p=i+1}^{r}(\beta_p-\beta_i)} \times \sum_{j=1}^{r}\frac{(-1)^{i+j}\prod_{q=1}^{r}(\omega n_j+\beta_j-\beta_q)}{(\omega n_j+\beta_j-\beta_i)\prod_{q=1,q\neq j}^{r-1}(\omega n_q-\omega n_j+\beta_q-\beta_j)\prod_{p=j+1}^{r}(\omega n_{-}\omega n_p+\beta_j-\beta_p)}. \tag{5.10}$$

Proof. Let V be the space of polynomial Θ such that $deg(\Theta) \leq |n|-1$ and

$$\sum_{x=0}^{\infty}\Theta(\omega x)(-\omega x)_{k,\omega}w^{(\omega;\beta_j+\omega)}(\omega x) = 0 \quad 0 \leq k \leq n_j-2 \quad j=1,2,\ldots,r.$$

By use of orthogonality conditions and

$$\nabla_{\omega}w^{(\omega;\beta_j+\omega)} = \left(\beta_j - \frac{1-a^{\omega}}{a^{\omega}}x\right)w^{(\omega;\beta_j)},$$

it is obvious that $\Delta_{\omega}M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x) \in V$ and $M_{\overrightarrow{n}-\overrightarrow{e_i}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e_i};a)}(x) \in V$. Let us assume that,

$$\sum_{i=1}^{r} d_i M_{\overrightarrow{n}-\overrightarrow{e_i}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e_i};a)}(x) = 0,$$

where $d_i's$ are constants.

Multiplying above equation with $(-x)_{n_k-1,\omega}w^{(\omega;\beta_k+\omega)}(x)$ and then taking summation over x we get,

$$\sum_{i=1}^{r} d_i \sum_{x=0}^{\infty} M_{\overrightarrow{n}-\overrightarrow{e_i}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e_i};a)}(x)(-x)_{n_k-1,\omega}w^{(\omega;\beta_k+\omega)}(x) = 0.$$

Using the equivalent relation given in Lemma 4.1.3, the above equation becomes,

$$\sum_{i=1}^{r} c_{ki} d_i \sum_{x_0}^{\infty} M_{\vec{n}-\vec{e}}^{(\omega;\vec{\beta}+\vec{e};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x) = 0.$$

Since, $\sum_{x_0}^{\infty} M_{\vec{n}-\vec{e}}^{(\omega;\vec{\beta}+\omega\vec{e};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x) \neq 0$ we obtain,

$$\sum_{i=1}^{r} c_{ki} d_i = 0,$$

which is equivalent to

$$Cd = 0,$$

where c_{ki}'s are given in equation (5.8), $d = [d_1, d_2, \ldots, d_r]^T$ and $C = [c_{ki}^r]_{k,i=1}^r$. From Lemma 4.1.4 we know that $|C| \neq 0$ so that $d_i = 0$ for $i = 1, 2, \ldots, r$. Hence $\{M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)\}_{i=1}^r$ is linearly independent in V. $\Delta_\omega M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)$ can be represented as a linear combination of $\{M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x)\}_{i=1}^r$ since the dimension of V is at most r. This is clear from the information that any polynomial can be written with n coefficients and $(|n| - r)$ linear conditions are imposed on V

Let

$$\Delta_\omega M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x) = \sum_{i=1}^{r} d_i M_{\vec{n}-\vec{e_i}}^{(\omega;\vec{\beta}+\omega\vec{e_i};a)}(x).$$

After mutiplying the above equation with $(-x)_{n_k-1,\omega} w^{(\omega;\beta_k+\omega)}(x)$, taking summation on x and then using equation (5.7) we get,

$$\sum_{x=0}^{\infty} \Delta_\omega M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)(-x)_{n_k-1,\omega} w^{(\omega;\beta_k+\omega)}(x) = \sum_{i=1}^{r} c_{ki} d_i \sum_{x=0}^{\infty} M_{\vec{n}-\vec{e}}^{(\omega;\vec{\beta}+\omega\vec{e};a)}(x) \times (-x)_{n_k-1,\omega} w^{(\omega;\beta_k+\omega)}(x).$$

This equation can easily be rearranged as follows,

$$\sum_{i=1}^{r} c_{ki} d_i = \frac{\sum_{x=0}^{\infty} \Delta_\omega M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)(-x)_{n_k-1,\omega} w^{(\omega;\beta_k+\omega)}(x)}{\sum_{x=0}^{\infty} M_{\vec{n}-\vec{e}}^{(\omega;\vec{\beta}+\omega\vec{e};a)}(x)(-x)_{n_k-1,\omega} w^{(\omega;\beta_k+\omega)}(x)}.$$

The above equation can also be represented in the form,

$$Cd = (y_1, y_2, \ldots, y_r)^T, \tag{5.11}$$

where

$$y_k = \frac{\sum_{x=0}^{\infty} \Delta_\omega M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)(-x)_{n_k-1,\omega} w^{(\omega;\beta_k+\omega)}(x)}{\sum_{x=0}^{\infty} M_{\overrightarrow{n}-\overrightarrow{e}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e};a)}(x)(-x)_{n_k-1,\omega} w^{(\omega;\beta_k+\omega)}(x)}.$$

Using summation by part and the properties given in Lemma 4.1.1, the numerator of the above expression be written as,

$$\begin{aligned}
\sum_{x=0}^{\infty} \Delta_\omega M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x) &= -\sum_{x=0}^{\infty} M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)\nabla_\omega[(-x)_{n_k-1,\omega} W_{\omega,k}(x)] \\
&= \frac{1-a^\omega}{a^\omega} \sum_{x=0}^{\infty} M_{\overrightarrow{n}}^{(\omega;\overrightarrow{\beta};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x) \\
&= \omega n_k \sum_{x=0}^{\infty} M_{\overrightarrow{n}-\overrightarrow{e_k}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e_k};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x) \\
&= \omega n_k c_{kk} \sum_{x=0}^{\infty} M_{\overrightarrow{n}-\overrightarrow{e}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x).
\end{aligned}$$

Thus we obtain,

$$\begin{aligned}
y_k &= \frac{\omega n_k c_{kk} \sum_{x=0}^{\infty} M_{\overrightarrow{n}-\overrightarrow{e}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x)}{\sum_{x=0}^{\infty} M_{\overrightarrow{n}-\overrightarrow{e}}^{(\omega;\overrightarrow{\beta}+\omega\overrightarrow{e};a)}(x)(-x)_{n_k-1,\omega} W_{\omega,k}(x)} \\
&= \omega n_k c_{kk}.
\end{aligned}$$

We can represent (5.11) in the matrix form as follows,

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1r} \\ c_{21} & c_{22} & \dots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \dots & c_{rr} \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{bmatrix} = \begin{bmatrix} \omega n_1 c_{11} \\ \omega n_2 c_{22} \\ \vdots \\ \omega n_r c_{rr} \end{bmatrix}.$$

Let us multiply the above system from left with the matrix,

$$\begin{bmatrix} \frac{1}{\omega n_1 c_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{\omega n_2 c_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\omega n_r c_{rr}} \end{bmatrix},$$

to obtain,

$$\begin{bmatrix} \frac{1}{\omega n_1} & \frac{c_{12}}{\omega n_1 c_{11}} & \cdots & \frac{c_{1r}}{\omega n_1 c_{11}} \\ \frac{c_{21}}{\omega n_2 c_{22}} & \frac{1}{\omega n_2} & \cdots & \frac{c_{2r}}{\omega n_2 c_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_{r1}}{\omega n_r c_{rr}} & \frac{c_{r2}}{\omega n_r c_{rr}} & \cdots & \frac{1}{\omega n_r} \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Now denote the coefficient matrix as $\overline{C}$ and use (5.8) for c_{ki}'s.To find the unknowns we use the Cramer's rule where $d_i = \frac{det\overline{C}_i}{det\overline{C}}$ for $i = 1, 2, \ldots, r.$

From Lemma 4.1.2 , we have

$$det\overline{C} = \frac{\prod_{q=1}^{r-1}\prod_{p=q+1}^{r}(\beta_p - \beta_q)(\omega n_q - \omega n_p + \beta_q - \beta_p)}{\prod_{q=1}^{r}\prod_{p=1}^{r}(\omega n_p + \beta_p - \beta_q)}.$$

The determinant of the augmented matrix is,

$$\begin{vmatrix} \frac{1}{\omega n_1} & \frac{1}{\omega n_1+\beta_1-\beta_2} & \cdots & 1 & \cdots & \frac{1}{\omega n_1+\beta_1-\beta_r} \\ \frac{1}{\omega n_2+\beta_2-\beta_1} & \frac{1}{\omega n_2} & \cdots & 1 & \cdots & \frac{1}{\omega n_2+\beta_2-\beta_r} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{1}{\omega n_r+\beta_r-\beta_1} & \frac{1}{\omega n_r+\beta_r-\beta_2} & \cdots & 1 & \cdots & \frac{1}{\omega n_r} \end{vmatrix} = \sum_{j=1}^{r}(-1)^{i+j}|M_{ji}|,$$

where M_{ji} is the (j,i)-minor of $\overline{C}$ and from Lemma 4.1.2 we have,

$$|M_{ji}| = \frac{\prod_{q=1,q\neq i}^{r-1}\prod_{p=q+1,p\neq i}^{r}(\beta_p - \beta_q)\prod_{q=1,q\neq j}^{r-1}\prod_{p=q+1,p\neq j}^{r}(\omega n_q - \omega n_p + \beta_q - \beta_p)}{\prod_{q=1,q\neq i}^{r}\prod_{p=1,p\neq j}^{r}(\omega n_p + \beta_p - \beta_q)}.$$

Thus we obtain $d_i = \frac{1}{|\overline{C}|}\sum_{j=1}^{r}(-1)^{i+j}|M_{ji}|$, which is equation (5.10). $\square$

5.2 Difference Equations

5.2.1 ω-Multiple Meixner Polynomials of the First Kind

Theorem 5.5: Let ω be a positive real number.The difference equation for the ω-Multiple Meixner Polynomials $M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)_{|\vec{n}|=0}^{\infty}$ is given by ,

$$\begin{aligned} &L_{a_1}^{(\beta+2\omega-r\omega)}L_{a_2}^{(\beta+3\omega-r\omega)}\ldots L_{a_r}^{(\beta+\omega)}\Delta_\omega M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x) \\ &\quad + \sum_{i=1}^{r}\omega n_i L_{a_1}^{(\beta+2\omega-r\omega)}\ldots L_{a_{i-1}}^{(\beta+i\omega-r\omega)}L_{a_{i+1}}^{(\beta+(i+1)\omega-r\omega)}\ldots \\ &\quad * L_{a_r}^{(\beta)}L_{a_i}^{(\beta+\omega)}M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x) = 0. \end{aligned} \tag{5.12}$$

Proof. Since $L_{a_k}^{(n)}L_{a_m}^{(n+\omega)} = L_{a_m}^{(n)}L_{a_k}^{(n+\omega)}$ for $n, a_k, a_m \in \mathbb{R}$, we obtain that

$$L_{a_1}^{(\beta+2\omega-r\omega)}L_{a_2}^{(\beta+3\omega-r\omega)}\dots L_{a_r}^{(\beta+\omega)} = L_{a_1}^{(\beta+2\omega-r\omega)}\dots L_{a_{i-1}}^{(\beta+i\omega-r\omega)} \times L_{a_{i+1}}^{(\beta+(i+1)\omega-r\omega)}\dots L_{a_r}^{(\beta)}L_{a_i}^{(\beta+\omega)},$$

for $i = 1\dots r$.

Now applying $L_{a_1}^{(\beta+2\omega-r\omega)}L_{a_2}^{(\beta+3\omega-r\omega)}\dots L_{a_r}^{(\beta+\omega)}$ to the lowering operator (5.2) and using the raising operator (5.1), we get the result. □

Theorem 5.6: Let ω be a positive real number.The third order difference equation for the ω-MMPs $M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)_{n_1+n_2=0}^{\infty}$ of the first kind is given as,

$$\begin{aligned}
&x(x-\omega)\nabla_{\omega}^2\Delta_{\omega}y + x[\beta(a_1^{\omega}+a_2^{\omega}) + (x-\omega)(a_1^{\omega}+a_2^{\omega}-2)]\nabla_{\omega}\Delta_{\omega}y \\
&+[a_1^{\omega}\beta - x(1-a_1^{\omega})(a_2\omega\beta - x\omega(1-a_1^{\omega})(1-a_2^{\omega}) - a_1^{\omega}a_2^{\omega}\beta\omega]\Delta_{\omega}y + \\
&[n_1(1-a_1^{\omega}) + n_2(1-a_2^{\omega})]\omega x\nabla_{\omega}y + \\
&\omega(\beta-\omega)\big[[n_1a_2^{\omega} + n_2a_1^{\omega} - a_1^{\omega}a_2^{\omega}(n_1+n_2)] - (1-a_1^{\omega})(1-a_2^{\omega})(n_1+n_2)x\big]y = 0
\end{aligned} \tag{5.13}$$

Proof. Considering the case for $r = 2$ in Theorem 4.2.1, we have

$$L_{a_1}^{(\beta)}L_{a_2}^{(\beta+\omega)}\Delta_{\omega}y + \omega n_1 L_{a_2}^{(\beta)}y + \omega n_2 L_{a_1}^{(\beta)}y = 0,$$

where $y = M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x)$, which gives the proof. □

5.2.2 ω-Multiple Meixner Polynomials of the Second Kind

Theorem 5.7: Let ω be a positive real number.The difference equation for the ω-MMPs $M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)_{|\vec{n}|=0}^{\infty}$ of the second kind is given as,

$$\begin{aligned}
&L_{\beta_1+\omega}^{a}L_{\beta_2+\omega}^{a}\cdots L_{\beta_r+\omega}^{a}[\Delta_{\omega}M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)] \\
&\qquad + \sum_{i=1}^{r} d_i L_{\beta_1+\omega}^{a}\cdots L_{\beta_{i-1}+\omega}^{a}L_{\beta_{i+1}+\omega}^{a}\cdots L_{\beta_r+\omega}^{a}L_{\beta_i+\omega}^{a}[M_{\vec{n}}^{(\omega;\vec{\beta};a)}(x)] = 0,
\end{aligned} \tag{5.14}$$

where d_i's are the constants in equation (5.10).

Proof. Since $L^a_{\beta_l+\omega}L^a_{\beta_k+\omega}=L^a_{\beta_k+\omega}L^a_{\beta_l+\omega}$ for $\beta_l,\beta_k\in\mathbb{R}$, we obtain that,

$$L^a_{\beta_1+\omega}L^a_{\beta_2+\omega}\dots L^a_{\beta_r+\omega}=L^a_{\beta_1+\omega}\dots L^a_{\beta_{i-1}+\omega}L^a_{\beta_{i+1}+\omega}\dots L^a_{\beta_r+\omega}L^a_{\beta_i+\omega}\quad for\quad i=1,2\dots,r.$$

Now applying $L^a_{\beta_1+\omega}L^a_{\beta_2+\omega}\dots L^a_{\beta_r+\omega}$ to lowering operator (5.9) and using raising operator (5.3), we obtain the result. □

Theorem 5.8: Let ω be a positive real number. The third order difference equation for the ω-MMPs $M^{(\omega;\vec{\beta};a)}_{\vec{n}}(x)_{n_1+n_2=0}^{\infty}$ of the second kind is given as,

$$\begin{aligned}x(x-\omega)\nabla^2_\omega\Delta_\omega y&+[(1-a^\omega)(1+\omega-2x)+a^\omega(1+\beta_1+\beta_2)]x\nabla_\omega\Delta_\omega y\\&+[(a^\omega\beta_1-(1-a^\omega)x)(a^\omega\beta_2-(1-a^\omega)x)-x\omega(1-a^\omega)]\Delta_\omega y\\&+(1-a^\omega)[\omega a^\omega(\omega n_1n_2+\beta_2n_1+\beta_1n_2)-x(1-a^\omega)(\omega n_1+\omega n_2)]y\\&+\omega(n_1+n_2)(1-a^\omega)x\nabla_\omega y=0.\end{aligned}\tag{5.15}$$

Proof. From Theorem 4.2.3 we consider the case for $r=2$,

$$L_{\beta_1+\omega}L_{\beta_2+\omega}\Delta_\omega y+d_1L_{\beta_2+\omega}y+d_2L_{\beta_1+\omega}y=0,$$

where $y=M^{(\omega;\beta_1,\beta_2;a)}_{n_1,n_2}(x)$ which gives the proof of the theorem. □

Chapter 6

SOME RESULTS ON $\omega-$ MULTIPLE MEIXNER POLYNOMIALS

As we mentioned before for the case when $\omega = 1$, $\omega-$ MMPs reduces the known MMPs. For the other values of ω we have new classes for MMPs where ω is positive real number.

6.1 1/2-Multiple Meixner Polynomials of the first Kind

In this section we exhibit the case $\omega = 1/2$ and state some relations for 1/2-MMPs of the first kind such as weight functions, orthogonality conditions, explicit form, generating function and third order difference equation.

1/2-MMPs of the first kind have the following weight functions,

$$w_i^{(1/2;\beta)}(x) = \frac{\Gamma_{1/2}(\beta+x)a_i^x}{\Gamma_{1/2}(\beta)\Gamma_{1/2}(x+1/2)} = \frac{\Gamma(2(\beta+x))a_i^x}{\Gamma(2\beta)\Gamma(2(x+1/2))} \qquad i = 1,2,\ldots,r.$$

By using these weight functions in (3.2), the orthogonality conditions for 1/2-MMP-1 can be written as,

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{(1/2;\beta;\vec{a})}\left(\frac{x}{2}\right)(-x)_j \frac{\Gamma(2\beta+x)a_i^{x/2}}{\Gamma(2\beta)\Gamma(x+1)} = 0, \quad j = 0,1,\ldots n_i - 1.$$

The explicit form for 1/2-MMP-1 can easily be obtained from (3.8) as follows:

$$M_{\vec{n}}^{(1/2;\beta;\vec{a})}(x) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}$$
$$\times\prod_{j=1}^{r}\left[\frac{(a_j^{1/2})^{n_j-k_j}}{(2a_j^{1/2}-2)^{n_j}}(-2x)_{|\vec{k}|}(2\beta+2x)_{|\vec{n}|-|\vec{k}|}\right].$$

1/2-MMP-1 have the following generating function,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} M_{\vec{n}}^{(1/2;\beta;\vec{a})}(x)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} = \left(1-\frac{t_1}{2(a_1^{1/2}-1)}-\cdots-\frac{t_r}{2(a_r^{1/2}-1)}\right)^{2x} \times\left(1-\frac{t_1a_1^{1/2}}{2(a_1^{1/2}-1)}-\cdots-\frac{t_ra_r^{1/2}}{2(a_r^{1/2}-1)}\right)^{-2(\beta+x)}.$$

1/2-MMP-1 satisfy the following third order difference equation,

$$\begin{aligned}
&x(x-1/2)\nabla_{1/2}^2\Delta_{1/2}y+x[\beta(a_1^{1/2}+a_2^{1/2})+(x-1/2)(a_1^{1/2}+a_2^{1/2}-2)]\nabla_{1/2}\Delta_{1/2}y+\\
&[a_1^{1/2}\beta-x(1-a_1^{1/2})((a_2\beta)/2-[x(1-a_1^{1/2})(1-a_2^{1/2})]/2-[a_1^{1/2}a_1^{1/2}\beta]/2]\Delta_{1/2}y+\\
&[n_1(1-a_1^{1/2})+n_2(1-a_2^{1/2})]x/2\nabla_{1/2}y+\\
&(\beta-1/2)/2\big[[n_1a_2^{1/2}+n_2a_1^{1/2}-a_1^{1/2}a_2^{1/2}(n_1+n_2)]-\\
&(1-a_1^{1/2})(1-a_2^{1/2})(n_1+n_2)x\big]y=0,
\end{aligned}$$

where $\Delta_{1/2}f(x)=f(x+1/2)-f(x)$ and $\nabla_{1/2}f(x)=f(x)-f(x-1/2)$.

6.2 5/3-Multiple Meixner Polynomials of the Second Kind

In this section, we get $\omega=5/3$ and we obtain some relations for $5/3-$MMPs of the second kind such as weight function, orthogonality conditions, explicit form, generating function and third order difference equation.

$5/3-$MMPs of the second kind have the following weight functions,

$$w^{(5/3;\beta_i)}(x)=\frac{\Gamma_{5/3}(\beta_i+x)a^x}{\Gamma_{5/3}(\beta_i)\Gamma_{5/3}(x+5/3)}=\frac{\Gamma\big(3/5(\beta_i+x)\big)a^x}{\Gamma(3\beta_i/5)\Gamma(3x/5+1)}\quad for\quad i=1,2,\ldots,r.$$

By using these weight functions in (3.4) we get the orthogonality conditions for $5/3-$MMP-2,

$$\sum_{x=0}^{\infty}M_{\vec{n}}^{(5/3;\vec{\beta},a)}(3x/5)(-x)_j\frac{\Gamma(3\beta_i/5+x)a^{5x/3}}{\Gamma(3\beta_i/5)\Gamma(x+1)}=0,\quad j=0,1,\ldots n_i-1.$$

The explicit form for $5/3-$MMP-2 easily seen from (3.11),

$$M_{\overrightarrow{n}}^{(5/3;\overrightarrow{\beta},a)}(x)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}...\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}\left(\frac{5a^{5/3}}{3(a^{5/3}-1)}\right)^{|\overrightarrow{n}|}\left(\frac{1}{a^{5|\overrightarrow{k}|/3}}\right)$$
$$\times(-3x/5)_{|\overrightarrow{k}|}\prod_{j=1}^{r}\left(3(\beta_j+x)/5-\sum_{i=1}^{j-1}k_i\right)_{n_j-k_j}.$$

5/3−MMP-2 have the following generating function,

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}M_{\overrightarrow{n}}^{(5/3;\overrightarrow{\beta};a)}(x)\frac{t_1^{n_1}...t_r^{n_r}}{n_1!...n_r!}=\left(1-\frac{3}{5a^{5/3}}\left[1-\prod_{j=1}^{r}\left(1+\frac{5a^{3/5}}{3(1-a^{3/5})}t_j\right)\right]\right)^{3x/5}$$
$$\times\prod_{i=1}^{r}\left(1+\frac{5a^{3/5}}{3(1-a^{3/5})}t_i\right)^{-\frac{3(x+\beta_i)}{5}}.$$

5/3−MMP-2 satisfy the following third order difference equation,

$$x(x-5/3)\nabla_{5/3}^2\Delta_{5/3}y+\left[(1-a^{5/3})(1+5/3-2x)+a^{5/3}(1+\beta_1+\beta_2)\right]x\nabla_{5/3}\Delta_{5/3}y+$$
$$\left[(a^{5/3}\beta_1-(1-a^{5/3})x)(a^{5/3}\beta_2-(1-a^{5/3})x)-5x(1-a^{5/3})/3\right]\Delta_{5/3}y+$$
$$(1-a^{5/3})\left[5a^{5/3}(5n_1n_2/3+\beta_2n_1+\beta_1n_2)/3-x(1-a^{5/3})(5(n_1+n_2)/3)\right]y+$$
$$\frac{5(n_1+n_2)(1-a^{5/3})}{3}x\nabla_{5/3}y=0,$$

where $\Delta_{5/3}f(x)=f(x+5/3)-f(x)$ and $\nabla_{5/3}f(x)=f(x)-f(x-5/3)$.

REFERENCES

[1] Atakishiyev, N. M., Rahman, M., & Suslov, S. K. (1995). On classical orthogonal polynomials. *Constructive Approximation*, 11(2), 181-226.

[2] Aptekarev, A. I. (1998). Multiple orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 99(1-2), 423-447.

[3] Arvesú, J., Coussement, J., & Van Assche, W. (2003). Some discrete multiple orthogonal polynomials. *Journal of computational and applied mathematics*, 153(1-2), 19-45.

[4] Ramírez Aberasturis, A. M. (2018). On algebraic properties of some q-multiple orthogonal polynomials.

[5] Chihara, T. S. (2011). *An introduction to orthogonal polynomials.* Courier Corporation.

[6] Ndayiragije, F., & Van Assche, W. (2013). Multiple Meixner polynomials and non-Hermitian oscillator Hamiltonians. *Journal of Physics A: Mathematical and Theoretical*, 46(50), 505201.

[7] Szegö, G. (1955). A. Erdélyi, Higher transcendental functions. Bulletin of the American Mathematical Society, 61(6), 593-593.

[8] Doman, B. G. S. (2015). *The classical orthogonal polynomials.* World Scientific.

[9] Koekoek, R., & Swarttouw, R. F. (1994). *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue.*

[10] Lee, D. W. (2008). Difference equations for discrete classical multiple orthogonal polynomials. *Journal of Approximation Theory*, 150(2), 132-152.

[11] Miki, H., Tsujimoto, S., Vinet, L., & Zhedanov, A. (2012). An algebraic model for the multiple Meixner polynomials of the first kind. *Journal of Physics A: Mathematical and Theoretical*, 45(32), 325205.

[12] Mubeen, S., & Rehman, A. (2014). A note on k-Gamma function and Pochhammer k-symbol. *Journal of Informatics and Mathematical Sciences*, 6(2), 93-107.

[13] Ndayiragije, F. (2012). Multiple Charlier and Meixner polynomials.

[14] Nikiforov, A. F., Uvarov, V. B., & Suslov, S. K. (1991). Classical orthogonal polynomials of a discrete variable. *In Classical orthogonal polynomials of a discrete variable* (pp. 18-54). Springer, Berlin, Heidelberg.

[15] Nikishin, E. M., & Sorokin, V. N. (1991). *Rational approximations and orthogonality*. Amer Mathematical Society.

[16] Sadjang, P. N., Koepf, W., & Foupouagnigni, M. (2015). On moments of classical orthogonal polynomials. *Journal of Mathematical Analysis and Applications*, 424(1), 122-151.

[17] Özarslan, M. A., & Baran, G. (2021). On the ω-multiple Charlier polynomials. *Advances in Difference Equations*, 2021(1), 1-16.

[18] Spicer, P. E. (2006). *On orthogonal polynomials and related discrete integrable systems* (Doctoral dissertation, University of Leeds).

[19] Van Assche, W., & Coussement, E. (2001). Some classical multiple orthogonal polynomials. *arXiv preprint math/0103131.*

[20] Van Assche, W. (2004). Difference equations for multiple Charlier and Meixner polynomials. *In Proceedings of the Sixth International Conference on Difference Equations, CRC Press, Boca Raton, FL* (pp. 549-557).

[21] Oğurlu, S. Z., & Elidemir, İ. (2020). On the ω-multiple Meixner polynomials of the first kind. *Journal of Inequalities and Applications*, 2020(1), 1-14.

[22] Zalik, R. A. (2011). Some properties of Chebyshev systems. *J. Comput. Analysis and Appl*, 13, 20-26.