Ulam-Hyers Stabilities of Impulsive Time-Delay Semi Linear Systems with Non-Permutable Matrices

Amal Mohammad Ata Almatarneh

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Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Nazim Mahmudov
Supervisor

Examining Committee

1. Prof. Dr. Huseyin Aktuglu
2. Prof. Dr. Agamirza Bashirov
3. Prof. Dr. Nazim Mahmudov
4. Prof. Dr. Hamza Menken
5. Prof. Dr. Hanlar Residoglu

ABSTRACT

In this work, asymptotic and Ulam-Hyers stabilities in two cases linear and nonlinear time-delay systems of linear impulsive constrains are studied. The linear parts of the impulsive systems are defined by non-permutable matrices. To obtain solution for linear impulsive delay systems with non-permutable matrices in explicit form, current notion of impulsive delayed matrix exponential is presented. Using the representation formula and norm estimation of impulsive delayed matrix exponential, sufficient conditions for the asymptotic and Ulam-Hyers stabilities are obtained.

Keywords: Impulsive Delay Equation; Delayed Matrix Exponential; Stability.

ÖZ

Bu çalışmada, asimtotik ve Ulam-Hyers kararlılıkları iki durumda, lineer ve lineer olmayan lineer impalsif kısıtların zaman gecikmeli sistemleri üzerinde çalışılmuştır. İmpalsif sistemlerin lineer kısımları permutable olmayan matrislerle tanımlanmıştır. Açık formdaki permutable olmayan lineer impalsif gecikmeli sistemlere çözüm bulmak için mevcut impalsif gecikmeli matris üstel kavramı sunulmuştur. Temsil formülü ve impalsif gecikmeli matris üstelinin norm tahmini kullanılarak, aismptotik ve Ulam-Hyers kararlılıkları için yeter koşulları elde edilmiştir.

Anahtar Kelimeler: Impalsif Gecikmeli Denklem; Gecikmeli Matris Ustel; Kararlılık.

DEDICATION

To my Father who is my first teacher...

To my Mother who gives with no limits...

To my siblings, Mahir, Sattam, Samah, Amani, Israa' and Omar...

To my beloved children, who are my best friends that support me in my struggling way, Anas, Fatimah, Aysha, and dear Mua'th...

To my supervisor and great teacher Prof. Dr. Nazım Mahmudov

To every member in the Math Department in Eastern Mediterranean University...

To every person who taught me an idea or drew for me the line that led me to my aim or lit the light that shines my way...

To my faithful friends...

To all the people whom I love and they also share this love with me...

To all I've mentioned I give this work.

ACKNOWLEDGMENT

I remember when I first came to study in Cyprus, all what I was thinking about is to work hard in my studies and to try my best to get high grades. In that time, I didn't expect that I will change my planes and the way I have drawn to my life will go to a different route, which makes studying in that beautiful island an important stage in my life.

From the first Moment I have moved to this marvelous place I was impressed by all the wonderful things that I have experienced, the grate friends and the lovely hearts I faced and their desire to help, their beautiful way of life which is rich of values and good manners that shows respect and passion to others.

Cyprus has become for me and my family as another home, which supports us with love, passion, peace and gives us the feeling that we are still living in our home Jordan where my children have many faithful friends.

Through living and studying here I have gained a lot of valuable meanings that I will do my best to teach them to others. In EMU I have studied and worked for five years as a research assistant where I had the opportunity to work and learn under the supervision of the best professors and teachers who are doing their best to develop the teaching and learning strategies all the time. I had the chance to enjoy my learning days with colorful experience that were very useful for me. EMU and its' educational programs allowed me to meet friends and students from different parts of the world who are rich in culture and knowledge.

I would like to give all the respect for all those who have supported me, my supervisor how is the head of our department, Prof. Dr. Nazım Mahmudov he gave me all the help and support I need, he also put his faith and trust on me. I would like also to trust all my professors and teachers as every one of them has her\his touch in my experience. Finally, I would like to thank all my colleagues who shared with me unforgettable memories which will be curved in my heart forever.

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Chapter 1

INTRODUCTION

Mathematical analysis, handles the integrals and derivatives of arbitrary order. This field is related to many other fields of analysis, such as integral equations, differential equations, function theories, and others. The great number of articles in the related literature and conferences held about mathematical analysis prove that it is a constantly developing field. The related literature includes a wide variety of studies in the form of articles, book chapters, conferences, etc. that include studies about differential equations and all related applications. These studies have been famous to the extend it has become a trend in the studies of the beginning of the previous century.

Nevertheless, the wide amount of study on this field has not resulted in any monographs or books that can thoroughly examine the latest developments and achievements in this field. Having scattered information in different book chapters may be a gap as the new learners in the mathematical field may not be able to find a unified resource that explains in one book that starts with the main concepts' definition and ends with examples that apply the mathematical rules of differential equations problems on the reality of nature. which was even in chemistry rather than a pure mathematical field. Since the fractional calculus is widely used in many different fields, such as engineering, physics, chemistry, it is good to have this book. Nevertheless, mathematical students may not be able to understand the chemistry concepts as well. Hence, there is a need for a book devoted for mathematics students that applies mathematical problems using the fractional calculus.

Furthermore, there are a few thesis studies devoted for examining the mathematical analysis, such as Marke [27], and there are a few historical outline papers published about this topic, such as Davis, Mikolas Ross, XX PREFACE. [27], one reason behind this lack of studies may be because of the rapidly developing fractional integer-differentiation theory. The absence of such a monograph has become crucial for the calculus development. Furthermore, the scarcity of studies is accompanied with some mistakes in the little studies published in this issue, which makes it difficult for novice learners to have a full idea about this theory from one resource. There are many studies in the field of differential equations, in which the studies reach the same results using different methods. Nevertheless, there are little research conducted for the sake of comparing these varied approaches.

The recent developments in both theory and practice of math has mainly depended on many factors. One of the most important factors is derivatives and intergrals of the positive integer order. One more essential factor is the mathematical functions, such as gamma, beta, special functions, and many others. Furthermore, there are other factors that can be considered as a development between theory and practice in the field of mathematics. This includes integer-differential operators, such as singular and non-singular kernels.

This monograph aims to examine the general calculus as well as the general fractional calculus of variable order. It also aims to examine the aforementioned types in relation to other functions. The study also aims to show the rheological and anomalous relaxation models according to their complexity. This study is crucial for many stakeholders including, engineers, chemists, mathematicians, physicians, and scientists.

Both integrals and their derivatives order are normal in analysis, but the fractional order form their peculiar features. There is a need for investigating the modifications of these features, which result from the being in different situations.

The field of differential equations has been a very old field of science, dating back to three centuries. It was very common among the mathematicians' community. Nevertheless, it was not that much popular among the other fields of knowledge, such as science, engineering, etc. This field is considered to be unique since it has the ability to describe the reality of nature better than other fields. Thus, it can be used in fields other than mathematics to solve their problems, as it is the case with engineering. With differential equations, engineers are able to add another dimension of their understanding to the basic nature of their fields. Rather, it is extended to have more variables that can describe the essence results of engineering and science. Hence, the rigorous mathematics is kept to its minimal with the new uses of mathematical calculus presented above.

As it was stated before, the goal of calculus mathematics is to explain the reality of nature in a better way. Suppose we have normal derivative $\frac{d}{dt}$ to represent the rate of accumulation or loss. This can mathematically speaking be gain rate minus loss rate at infinitesimal bounded space. If this infinitesimal space has some traps or has many different sizes, there will be differences that need explanation. It is known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems exhibit the impulse effect.

Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena for several real word problems. In recent years, qualitative properties of the mathematical theory of impulsive differential equations have been developed by a large number of mathematicians; see [1–12].

Since time delay exists in many fields in our society, systems with time delay have received significant attention in recent years. In [4–6], the authors considered the stability of impulsive differential equations with finite delay, and got some results. Systems with infinite delay deserve study because they describe a kind of system present in the real world. For example, in a predator–prey system the predation decreases the average growth rate of the prey species, linearly, with an infinite delay—for the predator cannot hunt prey when the predators are infants, and predators have to mature for a duration of time (which for simplicity in the mathematical analysis has been assumed to be infinite) before they are capable of decreasing the average growth rate of the prey species. And there are some results on systems with infinite delay; see [13,14] and references therein. However, to the best of the authors' knowledge, results for impulsive infinite delay differential equations are rare. Functional Equations Stability was first raised by Ulam at Wisconsin University, who asked the following question: ‘‘Under what conditions does there exist an additive mapping near an approximately additive mapping?'' Answering Ulam's question was first done by Hyers in 1940 through the Banach spaces. Therefore, the stability was since then called Ulam-Hyers Stability.

The theory of functional differential equations has been attracted by many researchers. Delay phenomena have application in control engineering, biology, medicine, economy and other sciences. Many processes are characterized by quick state changes. The time of event changes are comparatively short with the total duration of the complete process. For the theory of impulsive differential equations, the reader can refer to the monograph of Samoilenko et al. [25] and references theory, automatic engines and engineering therein. On the other hand, phenomena with time delays can be appeared in system systems. Recently, in [3] a concept of delayed matrix exponential is introduced to give a formula of solutions for linear time-delay continuous systems with commutative matrices. In [12], [13] a similar idea is used to find an explicit representation of solutions of linear discrete delay systems. Generally, it is not easy to reformulating of the solution explicitly without knowing impulsive delayed fundamental matrix for impulsive linear time-delay differential equations. In [14] authors adopted the idea of [3], [12], [13] to get the formulation of solutions to linear time-delay continuous impulsive systems. To do so they introduced impulsive delayed matrix concept for commutative matrices. These basic results are very useful to deal with control theory, iterative learning control and stability analysis for time-delay continuous\discrete and impulsive equations; For more details on the recent advances on the stability (Ulam-Hyers) of differential equations, one can see the monographs [22], [23], [24]. However, there is no paper in the literature searching an explicit solution of linear impulsive time-delay differential equations with non-commutative matrices. Due to the double impact of impulses and time-delay, it is an interesting task to get a representation for a solution of a time-delay impulsive differential equation with non-commutative matrices and study some stability concepts for these equations.

Motivated by the above articles, we consider reformulating of solutions of a linear time-delay impulsive differential equation of the way:

$$\begin{cases} y'(t) = Ay(t) + By(t-h) + f(t), t \in [0,T], \mathrm{h} > 0, t \neq t_k \\ \Delta y(t_k) = y(t^+{}_k) - y(t^-{}_k) = C_k y(t_k), k = 1, 2, \ldots, p, \\ y(t) = \varphi(t), -h \le t \le 0, \end{cases} \tag{1}$$

Where are constant matrices $\mathrm{A}, \mathrm{B}, \mathrm{C}_k \in \mathbb{R}^{n\times n}$, $\varphi \in C^1([-h,0], \mathbb{R}^n)$, $f \in C\ ([0,\mathrm{T}), \mathbb{R}^n)$,

and $\{t_k\}_{k=1}^{\infty}$ is a sequence that satisfies $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, for $\mathrm{T} > 0$, and

$y(t_k{}^+) = \lim_{\alpha \to 0^+} y(t_k + \alpha)$, $y(t_k{}^-) = y(t_k)$,

Moreover, we investigate exponential and Ulam-Hyers stabilities of the following semi linear time-delay impulsive differential equation:

$$\begin{cases} y'(t) = Ay(t) + By(t-h) + f(t, y(t)), t \in [0,T], \mathrm{h} > 0, t \neq t_k \\ \Delta y(t_k) = y(t_k) - y(t_k) = C_k y(t_k), k = 1, 2, \ldots, p, \\ y(t) = \varphi(t), -h \le t \le 0, \end{cases} \tag{2}$$

The main contributions are as follows:

• We introduce a novel impulsive delayed matrix exponential function (impulsive delayed exponential) and give its norm estimate. Using this impulsive delayed exponential and the variation of constants method, we give an explicit representation for solutions of impulsive time-delay initial value problems with linear parts defined by no permutable matrices.

• Based on the presentation of solutions and a norm estimate of the impulsive delayed exponential, we obtain sufficient conditions for Ulam-Hyers and asymptotic stabilities.

In the next section (chapter 3), we introduce some, and concepts the impulsive delayed matrix exponential and show that it is the fundamental (Cauchy) matrix for linear time-delay impulsive differential equations. Next, we give explicit formulae for solutions to linear homogeneous/nonhomogeneous time-delay impulsive differential equations via an impulsive delayed matrix exponential. In Chapter 5, we present a norm estimate to the impulsive delayed matrix exponential under the condition that A is an exponentially stable matrix and examine the exponential stability of nonlinear impulsive time-delay system. Chapter is devoted to Ulam-Hyers stability of the system (2). then in chapter 7 we have some results of existence and uniqueness of chapter 8 completed this work by introducing numerical examples.

Chapter 2

PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this thesis.

Schaefer's fixed point theorem:

Let X be a Banach space and $N: X \to X$ be a completely continuous operator. If the set $\varepsilon = \{y \in X : y = \lambda Ny, \text{ for some } \lambda \in (0,\ 1)\}$ is bounded, then N has a fixed point.

Arzela Ascoli theorem

Consider a sequence of real-valued continuous functions $\{f_n : n \in \mathbb{N}\}$ defined on a closed and bounded interval [a, b] of the real line. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $\{f_{nk} : k \in \mathbb{N}\}$ that converges uniformly.

Banach's fixed point theorem:

Let C be a non-empty closed subset of a Banach space X. Then any contraction mapping T from C into itself has a unique fixed point.

Gronwalls inequality:

For $t \geq t_0 \geq 0$, let $x(t) \leq \alpha(t) + \int_{t_0}^{t} g(t,s)x(s)ds + \sum_{t_0<t_k<t} \beta_k(t)x(t_k)$ where, $\beta_k(t), k \in \mathbb{N}$ are non-decreasing functions for $t \geq t_0$, $a \in PC([t_0,\infty), R_+)$ is non-decreasing, and $g(t, s)$ is a continuous nonnegative function for $t, s \geq t_0$ and is non-decreasing with respect to t for any fixed $s \geq t_0$ Then, for, $t \geq t_0$ $x(t) \leq a(t) \prod_{t_0<t_k<t} (1+\beta_k(t)) \exp\left(\int_{t_0}^{t} g(t,s)ds \right)$.

Definition 1. Let A be $n \times n$ matrix, then the matrix exponential is:

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + ... = \sum_{j=0}^{\infty} A^j \frac{t^j}{j!}$$

Lemma 1. Let A be $n \times n$ constant matrix, then $Ae^{At} = e^{At}A$

Proof. $Ae^{At} = A \sum_{j=0}^{\infty} A^j \frac{t^j}{j!} = A\left(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + ... \right)$

$$= IA + A^2\frac{t}{1!} + A^3\frac{t^2}{2!} + A^4\frac{t^3}{3!} + ...$$

$$= \left(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + ... \right).A = e^{At}A .$$

Lemma 2. Let A be $n \times n$ constant matrix. Then, $e^{At}e^{As} = e^{A(t+s)}$

Proof. $e^{At}e^{As} = \sum_{j=0}^{\infty} A^j \frac{t^j}{j!} \sum_{j=0}^{\infty} A^j \frac{s^j}{j!}$

$$= \left(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + ... \right) \times \left(I + A\frac{s}{1!} + A^2\frac{s^2}{2!} + A^3\frac{s^3}{3!} + ... \right)$$

$$= I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + ... + A\frac{s}{1!} + A^2\frac{t^2}{2!}A\frac{s}{1!} + A^3\frac{t^3}{3!}A\frac{s}{1!} + ...$$

$$+ A^2\frac{s^2}{1!} + A^2\frac{s^2}{2!}A\frac{t}{1!} + A^2\frac{s^2}{2!}A^2\frac{t^2}{2!} + ...$$

By reordering the sum

$$= \sum_{j=0}^{\infty} A^j \left(\frac{t^j}{j!} + \frac{t^{j-1}s}{(j-1)!1!} + \frac{t^{j-2}s^2}{(j-2)!2!} + ... + \frac{s^{j-2}t^2}{2!(j-2)!} + \frac{s^{j-1}t}{1!(j-1)!} + \frac{s^j}{j!} \right)$$

$$= \sum_{j=0}^{\infty} \frac{A^n}{j!} \left(\frac{j!t^j}{j!} + \frac{j!t^{j-1}s}{(j-1)!1!} + \frac{j!t^{j-2}s^2}{(j-2)!2!} + ... + \frac{j!s^{j-2}t^2}{2!(j-2)!} + \frac{j!s^{j-1}t}{1!(j-1)!} + \frac{s^j}{j!} \right)$$

$$= \sum_{j=0}^{\infty} \frac{A^j}{j!} \left(C_j^0 t^j + C_j^1 s^{j-1} t + C_j^2 t^{j-2} s^2 + ... + C_j^{j-2} s^{j-2} t^2 + C_j^{j-1} s^{j-1} t + C_j^j s^j \right)$$

$$= \sum_{j=0}^{\infty} \frac{A^j}{j!} (t+s)^j = e^{A(t+s)}$$

where

$$C_j^n = \frac{j!}{(j-n)!n!}$$

Lemma. Let A_1 and A_2 be commutative matrices, (i.e. $A_1 A_2 = A_2 A_1$). Then,

$$A_2 e^{A_1 t} = e^{A_1 t} A_2, \quad t \geq 0$$

Proof. $e^{A_1 t} A_2 = \sum_{j=0}^{\infty} A_1^j \frac{t^j}{j!} A_2 = \left(I + A_1 \frac{t}{1!} + A_1^2 \frac{t^2}{2!} + A_1^3 \frac{t^3}{3!} + ... \right) A_2$

$$= IA_2 + A_1\frac{t}{1!}A_2 + A_1^2\frac{t^2}{2!}A_2 + A_1^3\frac{t^3}{3!}A_2 + \ldots\ldots = A_2 + A_2A_1\frac{t}{1!} + A_2A_1^2\frac{t^2}{2!} + A_2A_1 3\frac{t^3}{3!} + \ldots\ldots$$

$$= A_2 + A_1A_2\frac{t}{1!} + A_1^2A_2\frac{t^2}{2!} + A_1^3A_2\frac{t^3}{3!} + \ldots\ldots = A_2\left(I + A_1\frac{t}{1!} + A_1^2\frac{t^2}{2!} + A_1^3\frac{t^3}{3!} + \ldots\ldots\right)$$

$$= A_2\sum_{j=0}^{\infty} A_1^j\frac{t^j}{\text{j}!} = A_2e^{A_1t}$$

Note: Our work on the general case when the matrices are not commutative.

Lemma. If $\|B\| \le \alpha e^{\alpha h}$ $,\alpha \in \mathbb{R}^+$ then $\left\|e^{B(t-h)}\right\| \le e^{\alpha h}$ $,t \in \mathbb{R}^+$

Theorem. Delayed matrix exponential is the fundamental matrix of solutions of the matrix differential equation with pure delay

$y'(t) = y(t-h)$, $0 < h < t$ where A is $n\times n$ constant matrix.

Proof. By differentiate of delayed matrix exponential

$$\left(e_h^{At}\right)' = \left(I + A\frac{t}{1!} + A^2\frac{(t-h)^2}{2!} + A^3\frac{(t-h)^3}{3!} + \ldots\ldots + A^k\frac{(t-(k-1)h)^k}{\text{k}!}\right)'$$

$$= \Theta + A\frac{1}{1!} + A^2\frac{(t-h)^1}{1!} + A^3\frac{(t-h)^2}{2!} + \ldots\ldots + A^k\frac{(t-(k-1)h)^{k-1}}{(\text{k}-1)!}$$

$$= A\left(I + A\frac{(t-h)^1}{1!} + A^2\frac{(t-h)^2}{2!} + \ldots\ldots + A^{k-1}\frac{(t-(k-1)h)^{k-1}}{(\text{k}-1)!}\right)$$

$$= Ae_h^{A(t-h)}$$

Chapter 3

IMPULSIVE DELAYED MATRIX EXPONE

Let $J=\left[0,T\right], J_0=\left[0,\ t_1\right]$,....., $J_{p-1}=\left(t_{p-1},\ t_p\right]$,$J_p=\left(t_p,\ T\right]$,...,$t_{p+1}=T$ Further, define $\mathrm{B}=PC^n\left(J,\mathbb{R}^n\right)=\left\{y:J\to\mathbb{R}^n : \mathrm{y}\in C\left(J_m,\mathbb{R}^n\right), \mathrm{m}=0,1,2,.....,p\right\}$, and there exist the left limit $y\left(t_m^-\right)$ and right limit $y\left(t_m^+\right)$. It's clear that B is a Banach space endowed with norm defined by $\|y\|_{PC}=\sup\left\{|y(t)|:t\in J_k, k=1,2,3,...,m\right\}$.

We introduce the spaces:

$C^1(J,\mathbb{R}^n)=\left\{y\in C(J,\mathbb{R}^n), y'\in C(J,\mathbb{R}^n)\right\}$.

$PC^1(J,\mathbb{R}^n)=\left\{y:J\to\mathbb{R}^n : y'\in PC(J,\mathbb{R}^n)\right\}$.

Definition 2. Let $C(J,\mathbb{R})$denotes the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$\|y\|_\infty=\sup\left\{|y(t)|:t\in J\right\}$,

$C^1(J,\mathbb{R}^n)=\left\{y\in C(J,\mathbb{R}^n), y'\in C(J,\mathbb{R}^n)\right\}$

$PC^1(J,\mathbb{R}^n)=\left\{y:J\to\mathbb{R}^n : y\in C(J,\mathbb{R}^n)\right\}$

$y\in C(J,\mathbb{R}^n), y'\in C(J,\mathbb{R}^n)$

and there exist $y(t_k^-)$ and $y(t_k^+)$, $\mathrm{m}=0,1,\ \because;\ p$

with $y(t_k^-)=y(t_k)\}PC^1(J,\mathbb{R}^n)=\left\{y:J\to\mathbb{R}^n : y\in C(J,\mathbb{R}^n)\right\}$

$PC^n\left(J,\mathbb{R}^n\right)=\left\{y:J\to\mathbb{R}^n : \mathrm{y}\in C\left(J_m,\mathbb{R}^n\right), \mathrm{m}=0,1,2,.....,p\right\}$,

$y\left(t_m^-\right), y\left(t_m^+\right)$

Clearly, $PC(J,\mathbb{R})$ is a Banach space with the norm $\|y\|_{PC} = \sup_{t\in J}|y(t)|$.

Moreover, $PC^1(J,\ R) = \{y \in PC(J,\ R) : y'(t)$ is continuous everywhere except for some t_k at which $y'(t_k^+)$ and $y'(t_k^-)$ exist and $y'(t_k^+) = y'(t_k),\ k = 0,...,m\}$ Is a Banach space with the norm $\|y\|_{PC^1} = \max\{\|y\|_{PC}, \|y'\|_{PC}\}$.

Definition 7. A function $y \in C^1([-h,0],\ \mathbb{R}^n) \bigcup PC^1(J,\ \mathbb{R}^n)$ is said to be a solution of (1) if y satisfies the equation $y(t) = \varphi(t)$, $-h \le t \le 0$ and the equation (1) on J.

Definition 8. A function $e_h^B(t):\mathbb{R} \to \mathbb{R}^{n\times n}$ is called delayed matrix exponential if

$$e_h^B(t) = \begin{cases} 0, -\infty < t \le -h, h > 0 \\ I, -h \le t \le 0, \\ I + Bt + B^2\dfrac{(t-h)^2}{2} + ... + B^k\dfrac{(t-(k-1)h)^k}{k!}, (k-1)h \le t \le kh, \end{cases}$$

where $k \in \mathbb{N}, \mathrm{B} \in \mathbb{R}^{n\times n}, 0$ and I are the zero and identity matrices, respectively.

For $\mathrm{k} \ge 0$ we define

$$X_0(t,s) = e^{A(t-s)}, t \ge s$$

$$X_1(t,s+h) = \begin{cases} \int\limits_{s+h}^{t} e^{A(t-r)}BX_0(r-h,s)\,dr, s+h \le t, \\ 0, s+h > t, \end{cases}$$

$$X_k(t,s+kh) = \begin{cases} \int\limits_{s+kh}^{t} e^{A(t-r)}BX_{k-1}(r-h,s+(k-1)h)\,dr, s+kh \le t, \\ 0, s+kh > t, \end{cases}$$

Definition 9. Let $A, B \in \mathbb{R}^{n\times n}$. Delayed perturbation of matrix exponential function

$X_h^{A,B}:\mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$

generated by A, B is defined by

$$X_h^{A,B}(t,s)=\begin{cases}0, -\infty < t-s < 0,\\ I, t=s\\ e^{A(t-s)}+X_1(t,s+h)+....+X_k(t,s+kh), \mathrm{k}\,h \le t-s \le (k+1)h, k=0,1,2,\dots\end{cases} \qquad (4)$$

Lemma 1. Let $X_h^{A,B}(t,s)$ be defined as in equation (4). Then the following holds true:

(i) If $A=0$, then $X_h^{A,B}(t,0)=e_h^B(t-h), kh \le t < (k+1)h$.

(ii) If $B=0$, then $X_h^{A,B}(t,s)=e^{A(t-s)}$.

(iii) If $AB=BA$, then $X_h^{A,B}(t,s)=e^{A(t-s)}e_h^{B1(t-h-s)}, B_1=e^{(-Ah)B}, kh \le t-h \le (k+1)h,$

Proof. (i) If $A=0$, then

$$X_0(t,s)=I\ , X_1(t,s+h)=\int_{s+h}^{t} Bdr = B(t-h-s),$$

$$X_2(t,s+2h)=\int_{s+2h}^{t} B^2(r-2h-s)dr = B^2\frac{(t-2h-s)^2}{2!},$$

$$X_k(t,s+kh)=\int_{s+kh}^{t} B^k(r-kh-s)^{k-1}dr = B^k\frac{(t-kh-s)^k}{\mathrm{k}!}, \mathrm{s}+kh \le t < s+(k+1)h.$$

So,

$$X_h^{A,B}(t,s)=\sum_{j=0}^{k} B^j\frac{(t-jh-s)^j}{\mathrm{j}!}, \mathrm{s}+kh \le t < s+(k+1)h.$$

(ii) If $B=0$, then

$$X_0(t,s)=e^{A(t-s)}, X_k(t,s+kh)=0, k=1,2,....,$$

and

$$X_h^{A,B}(t,s)=e^{A(t-s)}.$$

(iii) By our assumption A and B are commutative, consequently $e^{A(t-s)}B = Be^{A(t-s)}$.

Using this property, we obtain

$$X_0(t,s) = e^{A(t-s)},\ X_1(t,s+h) = \int_{s+h}^{t} e^{A(t-r)}Be^{A(r-h-s)}dr = e^{A(t-s)}Be^{-Ah}(t-h-s),$$

$$X_2(t,s+2h) = \int_{s+2h}^{t} e^{A(t-r)}Be^{A(r-2h-s)}B(t-2h-s)dr = e^{A(t-r)}B^2e^{-A2h}\frac{(t-2h-s)^2}{2!}$$

$$.\ X_k(t,s+kh) = e^{A(t-s)}B^ke^{-Akh}\frac{(t-kh-s)^k}{\mathrm{k}!}, \mathrm{s}+kh \le t < s+(k+1)h$$

It follows that

$$X_h^{A,B}(t,\mathrm{s}) = \sum_{j=0}^{k} X_j(t,s+jh) = \sum_{j=0}^{k} e^{A(t-s)}B^je^{-Ajh}e'\frac{(t-jh-s)^j}{\mathrm{j}!}$$

$$= e^{A(t-s)}e_h^{B1(t-h-s)}, s+kh \le t < s+(k+1)h$$

Lemma is proved.

Lemma 2. For all $t,s \in \mathbb{R}$, we have

$$\frac{\partial}{\partial t}X_h^{A,B}(t,s) = AX_h^{A,B}(t,s) + BX_h^{A,B}(t-h,s).$$

Proof. The proof is based on the following formula:

$$\frac{\partial}{\partial t}\int_{s+jh}^{t} e^{A(t-r)}BX_{j-1}(r-h,s+(j-1)h)dr$$

$$= A\int_{s+jh}^{t} e^{A(t-r)}BX_{j-1}(r-h,s+(j-1)h)dr + BX_{j-1}(t-h,s+(j-1)h).$$

Indeed, for $kh \le t-s < (k+1)h$, we have

$$\frac{\partial}{\partial t}X_h^{A,B}(t,s) = \frac{\partial}{\partial t}\sum_{j=0}^{k} X_j(t,s+jh)$$

$$= \sum_{j=0}^{k} \frac{\partial}{\partial t} \int_{s+jh}^{t} e^{A(t-r)} B X_{j-1}\left(r-h, s+(j-1)h\right) dr$$

$$= \sum_{j=0}^{k} \left[A \int_{s+jh}^{t} e^{A(t-r)} B X_{j-1}\left(r-h, s+(j-1)h\right) dr + B X_{j-1}\left(t-h, s+(j-1)h\right) \right]$$

$$= A \sum_{j=0}^{k} X_{j}\left(t, s+jh\right) + B \sum_{j=0}^{k} X_{j-1}\left(t-h, s+(j-1)h\right)$$

$$= A \sum_{j=0}^{k} X_{j}\left(t, s+jh\right) + B \sum_{j=0}^{k} X_{j-1}\left(t-h, s+jh\right).$$

The proof is finished.

Now, we introduce an impulsive analogue $Y_h^{A,B,C}(t,s)$ of the delayed matrix exponential $X_h^{A,B}(t,\mathrm{s})$.Since in equation (1) impulse has the linear form $\Delta y(t_k) = C_k y(t_k)$ impulsive Cauchy matrix has to contain the matrices A, B, C_k, that is why we introduce the following impulsive delayed matrix exponential function.

Definition 10. Let $A, B, C_k \in \mathbb{R}^{n\times n}$ be constant matrices. Impulsive delayed matrix exponential function $Y_h^{A,B}(t,\mathrm{s})$ is defined by

$$Y_h^{A,B,C}(t,\mathrm{s}) = \begin{cases} 0, t < s, \\ I, t = s \\ X_h^{A,B}(t,\mathrm{s}) + \sum\limits_{s<t_k<t} X_h^{A,B}(t,t_k) C_k Y_h^{A,B,C}(t_k,\mathrm{s}) \end{cases} \tag{5}$$

It should be emphasized that in commutative case that is, if A, B, C_k are commutative matrices impulsive delayed matrix exponential function was introduced in (11).

Definition 11. If A, B, C_k are commutative matrices, then impulsive delayed matrix exponential function is defined as follows:

$V(t,s)=e^{A(t-s)}X(t,s+h)$

$$X(t,s+h)=e_h^{B_1(t-h-s)}+\sum_{s<t_k<t}C_k e_h^{B_1(t-h-t_k)}X(t_k,s+h), B_1=e^{-Ah}. \quad (6)$$

Lemma 3. Let $Y_h^{A,B}(t,s)$ be defined by (5). If A,B,C_k are commutative, then $Y_h^{A,B,C}(t,s)=V(t,s)$.

Proof. Since $AB=BA$ then by Lemma (1). We have $X_h^{A,B}(t,s)=e^{A(t-s)}e_h^{B_1(t-h-s)}$.Thus

$$Y_h^{A,B,C}(t,s)=X_h^{A,B}(t,s)+\sum_{s<t_k<t}X_h^{A,B}(t,t_k)C_kY_h^{A,B,C}(t_k,s)$$

$$=e^{A(t-s)}e_h^{B_1(t-h-s)}+\sum_{s<t_k<t}e^{A(t-t_k)}e_h^{B_1(t-h-t_k)}C_kY_h^{A,B,C}(t_k,s)$$

$$=e^{A(t-s)}\left(e_h^{B_1(t-h-s)}+\sum_{s<t_k<t}C_k e_h^{B_1(t-h-t_k)}e^{A(s-t_k)}Y_h^{A,B,C}(t_k,s)\right)$$

$$=e^{A(t-s)}\left(e_h^{B_1(t-h-s)}+\sum_{s<t_k<t}C_k e_h^{B_1(t-h-t_k)}X(t_k,s+h)\right)$$

$$=e^{A(t-s)}X(t,s+h)=V(t,s).$$

Lemma 9. Impulsive delayed matrix exponential function $Y_h^{A,B,C}(t,s)$ satisfies

$$\frac{\partial}{\partial t}Y_h^{A,B,C}(t,s)=AY_h^{A,B,C}(t,s)+BY_h^{A,B,C}(t-h,s), t\neq t_k \quad (7)$$

$$Y_h^{A,B,C}(t_k^{+},s)=Y_h^{A,B,C}(t_k,s)+C_kY_h^{A,B,C}(t_k,s) \quad (8)$$

$$\frac{\partial}{\partial t}Y_h^{A,B,C}(t_k^{+},s)=\frac{\partial}{\partial t}Y_h^{A,B,C}(t_k,s)+AC_kY_h^{A,B,C}(t_k,s) \quad (9)$$

Proof.

Step 1: We verify that $Y_h^{A,B,C}(t,\mathrm{s})$ satisfies the differential equation (7).

$$\frac{\partial}{\partial t}Y_h^{A,B,C}(t,\mathrm{s})=\frac{\partial}{\partial t}X_h^{A,B}(t,\mathrm{s})+\sum_{s<t_k<t}\frac{\partial}{\partial t}X_h^{A,B}(t,t_k)C_kY_h^{A,B,C}(t_k,\mathrm{s})+\sum_{s<t_k<t}\frac{\partial}{\partial t}X_h^{A,B}(t,t_k)\phi_k$$

$$=AX_h^{A,B}(t,\mathrm{s})+\sum_{s<t_k<t}AX_h^{A,B}(t,t_k)C_kY_h^{A,B,C}(t_k,\mathrm{s})+\sum_{s<t_k<t}AX_h^{A,B}(t,t_k)\phi_k$$

$$.=AY_h^{A,B,C}(t,\mathrm{s})+BY_h^{A,B,\mathrm{C}}(t-h,\mathrm{s})$$

Step 2: We verify the equality (8). Note that $X_h^{A,B}(t^+,\mathrm{s})=X_h^{A,B}(t,\mathrm{s})$. Then

$$Y_h^{A,B,C}(t_m{}^+,\mathrm{s})=X_h^{A,B}(t_m{}^+,\mathrm{s})+\sum_{s<t_k<t}X_h^{A,B}(t_m{}^+,\mathrm{s})C_kY_h^{A,B,C}(t_k,\mathrm{s})$$

$$=X_h^{A,B}(t_m{}^-,\mathrm{s})+\sum_{s<t_k<t}X_h^{A,B}(t_m{}^+,\mathrm{t}_k)C_kY_h^{A,B,C}(t_k,\mathrm{s})+X_h^{A,B}(t_m{}^+,t_m)C_mY_h^{A,B,C}(t_m,\mathrm{s})$$

$$=Y_h^{A,B,C}(t_m{}^-,\mathrm{s})+C_mY_h^{A,B,C}(t_m,\mathrm{s})$$

Step 3: The proof of (9) is similar to that of (8).

This ends the proof.

Chapter 4

REPRESENTATION OF SOLOUTION

In this part of our work, we are looked for an explicit formula for the solutions of the linear impulsive inhomogeneous delay system fostering the traditional ways to find solution of a linear ordinary differential equations.

At the beginning, two explicit formulae of solutions to linear impulsive homogeneous delay system are driven.

Theorem 2. Let $\varphi \in C^1([-h,0],\mathbb{R}^n)$. Then the solution of the initial value problem (1) with $f = 0$ has the form

$$y(t) = Y_h^{A,B,C}(t,-h)\varphi(-h) + \int_{-h}^{0} Y_h^{A,B,C}(t,\mathrm{s})\left[\varphi'(s) - A\varphi(s)\right]ds, t \geq -h \tag{10}$$

$$y(t) = Y_h^{A,B,C}(t,0)\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,\mathrm{s+h})B\varphi(s)ds, \mathrm{t} \geq 0 \tag{11}$$

Proof. To prove the formula (10), we looked for the solution in the form

$$y(t) = Y_h^{A,B,C}(t,-h)\,g(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,\mathrm{s})\,g(s)ds, t \geq 0 \tag{12}$$

Where $g(t):[-h,0] \to \mathbb{R}^n$ is a continuous differentiable function and furthermore, condition $y(t) = \varphi(t), -h \leq t \leq 0$ should be hold

$$y(t) = Y_h^{A,B,C}(t,-h)\,g(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,\mathrm{s})\,g(s)ds = \varphi(t)\,, -h \leq t \leq 0$$

If $t=-h$, we have

$$Y_h^{A,B,C}(-h,-h)g(0)+\int_{-h}^{0}Y_h^{A,B,C}(-h,\mathrm{s})g(s)ds=g(0)=\varphi(-h)$$

Thus $g(0)=\varphi(-h)$.On the interval $-h\le t\le 0$, one can easily derive that

$$\varphi(t)=Y_h^{A,B,C}(t,-h)\varphi(-h)+\left(\int_{-h}^{t}+\int_{t}^{0}\right)Y_h^{A,B,C}(t,\mathrm{s})g(s)ds$$

$$=e^{A(t-h)}\varphi(-h)+\int_{-h}^{0}e^{A(t-s)}g(s)ds\,.$$

Differentiating the above equality, we have

$$\varphi'(t)=Ae^{A(t-h)}\varphi(-h)+A\int_{-h}^{0}e^{A(t-s)}g(s)ds+g(t)\,.$$

$$=A\varphi(t)+g(t)\,.$$

Therefore,

$$g(t)=\varphi'(t)-A\varphi(t)\,.$$

Next, we prove equivalence of (10) and (11). To do this, we use the integration by parts formula

$$\int_{-h}^{0}Y_h^{A,B,C}(\mathrm{t},\mathrm{s})\varphi(s)ds=Y_h^{A,B,C}(t,\mathrm{s})\varphi(s)\Big|_{s=-h}^{s=0}-\int_{-h}^{0}\frac{\partial}{\partial s}Y_h^{A,B,C}(\mathrm{t},\mathrm{s})\varphi(s)ds$$

$$=Y_h^{A,B,C}(\mathrm{t},0)\varphi(0)-Y_h^{A,B,C}(t,-h)\varphi(-h)$$

$$+\int_{-h}^{0}Y_h^{A,B,C}(\mathrm{t},\mathrm{s})A\varphi(s)ds+\int_{-h}^{0}Y_h^{A,B,C}(\mathrm{t},\mathrm{s}+h)B\varphi(s)ds$$

Thus, we obtain

$$y(t)=Y_h^{A,B,C}(\mathrm{t},-h)\varphi(-h)+\int_{-h}^{0}Y_h^{A,B,C}(\mathrm{t},\mathrm{s})\left[\varphi'(s)-\varphi(s)\right]ds$$

$$=Y_h^{A,B,C}(\mathrm{t},0)\varphi(0)+\int_{-h}^{0}Y_h^{A,B,C}(\mathrm{t},\mathrm{s}+h)B\varphi(s)\,ds\,.$$

Next, we have a formula of solutions to linear impulsive a nonhomogeneous delay system with zero initial condition.

Theorem 3. The solution $y_p(t)$ of (8) satisfying zero initial condition, has a form

$$y_p(t)=\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(\text{t},\text{s})\, f(s)ds+\int_{t_k}^{t} Y_h^{A,B,C}(\text{t},\text{s}) f(s)ds, t\geq 0 \tag{13}$$

Proof. We are trying to find the solution $y_p(t)$ in the form

$$y_p(t)=\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(\text{t},\text{s})\, g(s)ds+\int_{t_k}^{t} Y_h^{A,B,C}(\text{t},\text{s}) g_k(s)ds, t\geq 0$$

Where $g_j(s)$, $j=0,1,2,.....,k$ are unknown vector functions. The proof can be done by many steps:

Step 1: $0<t\leq t_1$. In this case, we have

$$y_p(t)=\int_0^t Y_h^{A,B,C}(\text{t},\text{s}) g_0(s)ds$$

We differentiate y_p and use the property $Y_h^{A,B,C}(t-h,s)=\Theta, t-h<s$ to obtain

$$y_p'(t)=A\int_0^t Y_h^{A,B,C}(\text{t},\text{s}) g_0(s)ds+B\int_0^t Y_h^{A,B,C}(\text{t}-h,\text{s}) g_0(s)ds+g_0(t)$$

$$=A\int_0^t Y_h^{A,B,C}(\text{t},\text{s}) g_0(s)ds+B\left(\int_0^{t-h}+\int_{t-h}^{t}\right) Y_h^{A,B,C}(t-h,\text{s})\, g(s)ds+g_0(t)$$

$$=A\int_0^t Y_h^{A,B,C}(\text{t},\text{s}) g_0(s)ds+B\int_0^{t-h} Y_h^{A,B,C}(t-h,\text{s})\, g(s)ds+g_0(t)$$

$$=Ay_p(t)+By_p(t-h)+f(t)$$

It follows that $g_0(t)=f(t)$.

Step 2: $t_1 < t \le t_2$. In this case

$$y_p(t) = \int_0^{t_1} Y_h^{A,B,C}(t,s) f(s)\,ds + \int_{t_1}^{t} Y_h^{A,B,C}(t,s) g_1(s)\,ds \text{ .}$$

We differentiate $y_p(t)$ again to obtain

$$y_p'(t) = \int_0^{t_1} \left[AY_h^{A,B,C}(t,s) + BY_h^{A,B,C}(t-h,s) \right] f(s)\,ds$$

$$+\int_{t_1}^{t} \left[AY_h^{A,B,C}(t,s) + BY_h^{A,B,C}(t-h,s) \right] g_1(s)\,ds + g_1(t)$$

$$= Ay_p(t) + By_p(t-h) + f(t).$$

Which implies that $g_1(t) = f(t)$.

Step 3: Suppose that $g_{k-1}(t) = f(t)$ holds on the subintervals $(t_{k-1}, t_k]$, $k = 2,3,....$ then for any $t_k < t < t_{k-1}$, we have

$$y_p(t) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s) f(s)\,ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s) g_k(s)\,ds \text{ .}$$

We differentiate $y_p(t)$ again to obtain

$$y_p'(t) = Ay_p(t) + B\left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j-1}} Y_h^{A,B,C}(t-h,s) f(s)\,ds \right]$$

$$+\int_{t_k}^{t} Y_h^{A,B,C}(t-h,s) g_k(s)\,ds + g_k(t)$$

$$= Ay_p(t) + By_p(t-h) + f(t)$$

It follows that $g_k(t) = f(t)$.

By the mathematical induction, we have $g_k(t) = f(t)$, $k = 0,1,2,....$ thus, the formula (13) is obtained.

Theorems 2 and 3 will obtain the following representation formula.

Theorem 4. Let $\varphi \in C^1([-h,0],\mathbb{R}^n)$, $f \in C([0,T],\mathbb{R}^n)$ Then the solution of the initial value problem (1) has the form

$$y(t)=\begin{cases}\varphi(t), -h \le t \le 0\\ Y_h^{A,B,C}(t,0)\varphi(0)+\int_{-h}^{0} Y_h^{A,B,C}(t,s+h)B\varphi(s)ds\\ +\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s)f(s)ds+\int_{t_k}^{t} Y_h^{A,B,C}(t,s)g_k(s)ds, t \ge 0\end{cases}$$

Where k is the number of points t_j in the interval $(0,t)$.

Chapter 5

ULAM-HYERS STABILITY

In this section, we will discuss Ulam-Hyers stability (2). In Ulam-Hyers stability, we compare the solution of given differential equation with the solution of some differential inequality. The solution of differential equation is Ulam-Hyers stable if it stays close to a solution of some differential inequality related to the original equation. Ulam-Hyers stability does not imply the asymptotic stability, in general.

For problem (2), for some $\varepsilon < 0$ we focus on the following inequalities:

$$\left\| y'(t) - Ay(t) - By(t-h) - f\left(t, y(t)\right) \right\| \leq \varepsilon,\ 0 \leq \mathrm{t} \leq \mathrm{T} \tag{14}$$

$$\left\| \Delta y\left(t_k\right) - C_k y\left(t_k\right) \right\| \leq \varepsilon,\ k = 1, \ldots, \mathrm{P}.$$

Definition 12. Equation (2) is Ulam-Hyers stable on $\left[-h, \mathrm{T}\right]$ if for every $y \in PC\left(\left[-h, \mathrm{T}\right] \cap \mathbb{R}^n\right) \cap PC^1\left(\left[0, \mathrm{T}\right] \cap \mathbb{R}^n\right)$ satisfying (14), there exists a solution $x \in PC\left(\left[-h, \mathrm{T}\right] \cap \mathbb{R}^n\right) \cap PC^1\left(\left[0, \mathrm{T}\right] \cap \mathbb{R}^n\right)$ of (2) with $\left\| y - x \right\|_\infty \leq L\varepsilon$, for all $t \in \left[-h, \mathrm{T}\right]$.

Proposition 1. A function $y \in PC^1\left(\left[0, \mathrm{T}\right] \cap \mathbb{R}^n\right)$ satisfies (10) if and only if there is a function $\phi \in PC\left(\left[-h, \mathrm{T}\right] \cap \mathbb{R}^n\right)$ and a sequence g_k depending on y such that $\left\| \phi \right\| \leq \varepsilon$ for all $t \in \left[-h, \mathrm{T}\right]$, $\left\| g_k \right\| \leq \varepsilon$ j for all $k = 1, \ldots, \mathrm{P}$,

$$y'(t) = Ay(t) - By(t-h) - f(t, y(t)) + \phi(t) \quad 0 \le \mathrm{t} \le \mathrm{T}$$

$$\Delta y(t_k) = C_k y(t) + g_k,\ k = 1, \dots, \mathrm{P} \qquad (11)$$

Lemma 5. For $s < t$, we have.

$$\left\| X_h^{A,B}(t, s) \right\| \le e^{(\|A\| + \|B\|)(t-s)}.$$

Proof. For k = 1, we have

$$\left\| X_1(t, s+h) \right\| = \|B\| \int_{s+h}^{t} \left\| e^{A(t-r)} \right\| \left\| e^{A(r-h-s)} \right\| dr \le \|B\| \int_{s+h}^{t} e^{\|A\|(t-r)} e^{\|A\|(r-h-s)} dr$$

$$\le \|B\| \int_{s+h}^{t} e^{\|A\|(t-h-s)} dr = \|B\| e^{\|A\|(t-h-s)} (t-h-s).$$

For k = 2, we get

$$\left\| X_2(t, s+2h) \right\| \le \int_{s+2h}^{t} \left\| e^{A(t-r)} \right\| \|B\| \left\| X_1(r-h, s+h) \right\| dr$$

$$\le \int_{s+2h}^{t} \|B\| e^{\|A\|(t-r)} \|B\| e^{\|A\|(r-2h-s)} (r-2h-s) dr$$

$$\le \|B\|^2 e^{\|A\|(t-2h-s)} \int_{s+2h}^{t} (r-2h-s) dr$$

$$= \|B\|^2 e^{\|A\|(t-2h-s)} \frac{(t-2h-s)^2}{2}.$$

By the mathematical induction assuming

$$\left\| X_{k-1}(t, s+(k-1)h) \right\| \le \|B\|^{k-1} e^{\|A\|(t-(k-1)h-s)} \frac{(t-(k-1)h-s)^k}{\mathrm{k!}}$$

One can get

$$\left\| X_k(t, s+kh) \right\| \le \int_{s+kh}^{t} \left\| e^{A(t-r)} \right\| \|B\| \left\| X_{k-1}(r-h, s+(k-1)h) \right\| dr$$

$$\le \int_{s+kh}^{t} e^{\|A\|(t-r)} \|B\| \|B\|^{k-1} e^{\|A\|(r-(k-1)h-s)} \frac{(r-kh-s)^{k-1}}{(k-1)!} dr$$

$$\le \|B\|^k e^{\|A\|(t-kh-s)} \frac{(t-kh-s)k}{k!}$$

Thus, for $s+kh \le t < s+(k+1)h$ we get

$$\left\| X_h^{A,B}(t,s) \right\| \le \sum_{j=0}^{k} \left\| X_j(t, s+jh) \right\|$$

$$\le \sum_{j=0}^{k} \|B\|^j e^{\|A\|(t-jh-s)} \frac{(t-jh-s)^j}{\text{j}!}$$

$$= e^{\|A\|(t-kh-s)} \sum_{j=0}^{k} \|B\|^j \frac{(t-jh-s)^j}{j!}$$

$$\le e^{(\|A\|+\|B\|)(t-s)}$$

The impulsive delayed matrix exponential $Y_h^{A,B,C}(t,\text{s})$ for the problem in the proposition is defined as follows:

$$Y_h^{A,B,C}(t,s) := \begin{cases} \vartheta, t < s, \\ I, t = s \\ X_h^{A,B}(t,s) + \sum\limits_{s < t_k < t} X_h^{A,B}(t,t_k)\left(C_k Y_h^{A,B,C}(t_k,s) + g_k\right) \end{cases}$$

Lemma 6. For $s < t$, we have the following estimation:

$$\left\| Y_h^{A,B,\mathrm{C}}(t,s) \right\| \le \prod_{s<t_k<t} \left(1 + \|g_k\| + \|C_k\|\right) e^{(\|A\|+\|B\|)(t-s)} \tag{15}$$

Proof. Our proof is based on the mathematical induction. We may assume that $t_m < s < t_{m+1}$ and $t_{m+n} < s < t < t_{m+n+1}$ for some natural number n.

(i) $t_m < s < t < t_{m+1}$.

By Lemma 5

$$Y_h^{A,B,C}(t,s) = X_h^{A,B}(t,s)$$

$$\left\|Y_h^{A,B,C}(t,s)\right\| \le e^{(\|A\|+\|B\|)(t-s)}$$

(ii) $t_{m+1} < t < t_{m+2}$ **:**Then

$$Y_h^{A,B,C}(t,s) = X_h^{A,B}(t,s) + X_h^{A,B}(t,t_{m+1})\left(C_{m+1}Y_h^{A,B,C}(t_{m-1},s) + g_{m+1}\right)$$

$$\left\|Y_h^{A,B,C}(t,s)\right\| \le e^{(\|A\|+\|B\|)(t-s)}$$

$$+e^{(\|A\|+\|B\|)(t-t_{m+1})}\left(\left\|C_{m+1}\right\|e^{(\|A\|+\|B\|)(t_{m+1}-S)} + \left\|g_{m+1}\right\|\right)$$

$$\le \left(1+\left\|C_{m+1}\right\|\right)e^{(\|A\|+\|B\|)(t-s)} + \left\|g_{m+1}\right\|e^{(\|A\|+\|B\|)(t-t_{m+1})}$$

$$\le \left(1+\left\|g_{m+1}\right\|+\left\|C_{m+1}\right\|\right)e^{(\|A\|+\|B\|)(t-s)}$$

(iii) For $t_{m+2} < t < t_{m+3}$, we have

$$Y_h^{A,B,C}(t,s) = X_h^{A,B}(t,s) + X_h^{A,B}(t,t_{m+1})\left(C_{m+1}Y_h^{A,B,C}(t_{m+1},s)+g_{m+1}\right) + X_h^{A,B}(t,t_{m+2})\left(C_{m+2}Y_h^{A,B,C}(t_{m+2},s)+g_{m+2}\right)$$

Consequently

$$\left\|Y_h^{A,B}(t,s)\right\| \le e^{(\|A\|+\|B\|)(t-s)}e + e^{(\|A\|+\|B\|)(t-t_{m+1})}\left(\left\|C_{m+1}\right\|e^{(\|A\|+\|B\|)(t_{m+1}-s)} + \left\|g_{m+1}\right\|\right)$$

$$+e^{(\|A\|+\|B\|)(t-t_{m+2})}\left(\left\|C_{m+2}\right\|\left(1+\left\|g_{m+1}\right\|+\left\|C_{m+1}\right\|\right)e^{(\|A\|+\|B\|)(t_{m+2}-s)} + \left\|g_{m+1}\right\|\right)$$

$$\le e^{(\|A\|+\|B\|)(t-s)}\left(1+\left\|C_{m+1}\right\|+\left\|g_{m+1}\right\|\right) + e^{(\|A\|+\|B\|)(t-s)}\left(\left\|C_{m+2}\right\|\left(1+\left\|g_{m+1}\right\|+\left\|C_{m+1}\right\|\right)+\left\|g_{m+2}\right\|\right)$$

$$\le e^{(\|A\|+\|B\|)(t-s)}\left(1+\left\|g_{m+1}\right\|+\left\|C_{m+1}\right\|\right)\times\left(1+\left\|C_{m+2}\right\|+\left\|g_{m+2}\right\|\right)$$

We may use the mathematical induction on n to get

Lemma 7. Every $y \in PC([-h,T],\mathbb{R}^n)$ that satisfies (14) also satisfies the following inequality

$$\left\| y(t) - Y_h^{A,B,C}(t,0)\varphi(0) - \int_{-h}^{0} Y_h^{A,B,C}(t,s)B\varphi(s)ds - \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s)f(s)ds - \int_{t_k}^{t} Y_h^{A,B,C}(t,s)f(s,y(t))ds \right\| \le c_\varepsilon$$

for all $t \in [0,T]$, where

$$c := \left(\frac{1}{\|A\|+\|B\|} \prod_{0<t_k<T} \left(1+\|g_k\|+\|C_k\|\right)\left(e^{(\|A\|+\|B\|)T}-1\right) + \sum_{j=0}^{k-1} e^{(\|A\|+\|B\|)(t-t_j)} \right) \tag{16}$$

Proof. $y \in PC([-h,T],\mathbb{R}^n)$ Satisfies (14), then by Proposition 1, we have

$\|\phi\|_{PC} \le \varepsilon$ For all $t \in [0,T]$, $\|g_k\| \le \varepsilon$ for all $k = 1,2,\ldots,\text{p};$

$y'(t) = Ay(t) + By(t-h) + f(t,y(t)) + \varphi(t)$, $0 \le t \le T;$

$\Delta y(t_k) = C_k\, y(t_k) + g_k \qquad\qquad k = 1,2,\ldots,\text{p};$

Then, by Theorem 3, we have the following representation formula for the above problem $y(t) = Y_h^{A,B,C}(t,0)\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)B\varphi(s)ds + \sum_{j=0}^{k} X_h^{A,B}(t,t_j)g_j$

$$+\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} Y_h^{A,B}(t,s)\left[f(s,y(s))+\varphi(s)\right]ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s)\left[f(s,y(s))+\varphi(s)\right]ds,$$

It follows that

$$\left\| y(t) - Y_h^{A,B,C}(t,0)\varphi(0) - \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)B\varphi(s)ds - \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} Y_h^{A,B}(t,s)\left[f(s,y(s))+\varphi(s)\right]ds - \int_{t_k}^{t} Y_h^{A,B,C}(t,s)\left[f(s,y(s))+\varphi(s)\right]ds, \right\|$$

$$\le \int_{0}^{t} \left\|Y_h^{A,B,C}(t,s)\right\| \left\|\varphi(s)\right\| ds + \sum_{j=0}^{k=1} X_h^{A,B,C}(t,t_j)\left\|g_j\right\|$$

$$\le \int_0^t \prod_{s<t_k<t} \left(1+\|g_k\|+\|C_k\|\right) e^{(\|A\|+\|B\|)(t-s)} ds \|\phi\|_{PC} + \sum_{j=0}^{k-1} e^{(\|A\|+\|B\|)(t-t_j)} \|g_j\|$$

$$\le \left(\frac{1}{\|A\|+\|B\|} \prod_{s<t_k<T} \left(1+\|g_k\|+\|C_k\|\right)\left(e^{(\|A\|+\|B\|)T}-1\right) + \sum_{j=0}^{k-1} e^{(\|A\|+\|B\|)(t-t_j)} \|g_j\| \right)\varepsilon$$

Now, we are able to present our second main result on Ulam-Hyers stability.

Theorem 5. If $f:[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the Lipchitz condition: there exists $L_f \ge 0$ such that for all $(t,y_1)(t,y_2)\in[0,T]\times\mathbb{R}^n$

$\|f(t,y_1)-f(t,y_2)\| \le L_f \|y_1-y_2\|$.

Then

- the equation (2) has a unique solution $y \in PC\left([-h,T]\cap\mathbb{R}^n\right)\cap PC^1\left([0,T]\cap\mathbb{R}^n\right)$;
- the equation (2) is stable in Ulam-Hyers sense.

Proof. We define

$$\Pi y(t) = Y_h^{A,B,C}(t,0)\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)B\varphi(s)ds + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} Y_h^{A,B}(t,s) f(s,y(s))ds + \int_{t_k}^{t} Y_h^{A,B}(t,s) f(s,y(s))ds,$$

On the space $PC\left([-h,T]\cap\mathbb{R}^n\right)$. We will apply the contraction mapping theorem to show that Π has a unique fixed point. At first glance, it seems natural to use the supermom norm. But the choice of supermom norm only leads us to a local solution defined in the subinterval of $[-h,T]$. The idea is to use the weighted supermom norm

$\|y\|_{\delta} : \sup\left\{e^{-\delta t}\|y(t)\| : -h \le t \le T\right\}$.

On $PC\left([-h,\mathrm{T}]\cap\mathbb{R}^n\right)$Observe that $PC^1\left([-\mathrm{h},\mathrm{T}]\cap\mathbb{R}^n\right)$ is a Banach space with this norm since it is equivalent to supermom norm

(i) We show that Π is a contraction on $PC\left([-h,\mathrm{T}]\cap\mathbb{R}^n\right)$.Indeed, for

$x, y \in PC\left([-h,\mathrm{T}]\cap\mathbb{R}^n\right)$

We have

$$e^{-\delta t}\|\Pi x(t)-\Pi y(t)\|$$

$$\leq e^{-\delta t}\int_0^t\left\|Y_h^{A,B,C}(t,s)\right\|e^{\delta s}e^{-\delta s}\|f(s,x(s))-f(s,y(s))\|ds$$

$$\leq e^{(-\delta t)}\int_0^t e^{(+\delta s)}\left\|Y_h^{A,B,C}(t,s)\right\|dsL_f\|x-y\|_\delta$$

$$\leq \prod_{s<t_k<T}\left(1+\|C_k\|\right)\int_0^t e^{(\|A\|+\|B\|-\delta)(t-s)}dsL_f\|x-y\|_\delta$$

$$=\frac{1}{\delta-\|A\|-\|B\|}\prod_{s<t_k<T}\left(1+\|C_k\|\right)\left(1-e^{(\|A\|+\|B\|-\delta)T}\right)\|x-y\|_\delta \qquad (17)$$

Taking supermom over $[0,\mathrm{T}]$ we get

$$\|\Pi x-\Pi y\|_\delta \leq \frac{1}{\delta-\|A\|-\|B\|}\prod_{s<t_k<T}\left(1+\|C_k\|\right)\left(1-e^{(\|A\|+\|B\|-\delta)T}\right)\|x-y\|_\delta .$$

We can choose $\delta>\|A\|+\|B\|$ so that the coefficient of $\|x-y\|_\delta$ become strictly less than one. Hence Π is a contractive operator and by the Banach contraction principle P is a unique fixed point in $PC\left([-h,\mathrm{T}]\cap\mathbb{R}^n\right)$and the equation (2) has a unique solution.

(ii) Let $y\in PC\left([-h,\mathrm{T}]\cap\mathbb{R}^n\right)$be a solution (14), and let x be a unique solution of (2).

We see that

$\|y(t)-x(t)\|=0$ for $-h\leq t\leq 0$. For $t\in[0,T]$ we have

$\|y(t)-x(t)\|=\|y(t)-\Pi x(t)\|\leq\|y(t)-\Pi y(t)\|+\leq\|\Pi y(t)-\Pi x(t)\|$.

Now we use Lemma 7 and inequality (17) to get

$$e^{-\delta t}\left\|y(t)-x(t)\right\| \leq C\varepsilon + \frac{1}{\delta-\|A\|-\|B\|}\prod_{s<t_k<T}\left(1+\left\|C_k\right\|\right)\left(1-e^{\left(\|A\|+\|B\|-\delta\right)T}\right)\|x-y\|_{\delta}.$$

Where C is defined by (16). Consequently,

$$\|x-y\|_{\delta} \leq \frac{C}{1-\frac{1}{\delta-\|A\|-\|B\|}\prod_{s<t_k<T}\left(1+\left\|C_k\right\|\right)\left(1-e^{\left(\|A\|+\|B\|-\delta\right)T}\right)}\varepsilon .$$

Hence equation (2) is Ulam-Hyers stable.

Chapter 6

EXISTENCE

Existence results

The following result depends on the Schaefer's fixed point theorem. For getting the optimal results,

We have the following assumptions:

(H1) The function $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

(H2) There exists a constant $M_f > 0$ such that

$\left\| f(\mathrm{t}, \mathrm{y}) \right\| \leq M_f \left(1 + \|y\|\right)$, for $\mathrm{t} \in J$ and $y \in \mathbb{R}^n$.

Theorem 6. If the assumptions (H1) and (H2) are satisfied, then problem (2) has at least one solution.

Proof. Let Π that showed in Theorem 5. The Schaefer's fixed-point theorem will be used to Illustrate that Π has a fixed point. We should divide the proof into four steps**.**

Step 1. Π is continuous.

Take a sequence $\{y_n\} \subset \mathrm{B}$, such that y_n converges to $y \in \mathrm{B}$ as $n \to \infty$. Then for $t \in J_m$ we have $\left\| \left(\Pi y_n\right)(t) - (\Pi y)(t) \right\|$

$$\leq \int_{t_m}^{t} \left\| Y_h^{A,B,\mathrm{C}}(t, \mathrm{s}) \right\| \left\| f\left(s, \mathrm{y}_n(s)\right) - f\left(s, \mathrm{y}(s)\right) \right\| ds$$

$$\leq \int_0^T \left\| Y_h^{A,B,C}(\mathrm{T},\mathrm{s}) \right\| \left\| f(s, \mathrm{y}_n(s)) - f(s, \mathrm{y}(s)) \right\| ds$$

As a result of Lebesgue dominated convergent theorem, the right-hand side of the previous Inequality goes to zero as $n \to \infty$,

Hence,

$$\left\| (\Pi y_n)(t) - (\Pi y)(t) \right\| \to 0 \qquad \text{when} \qquad n \to \infty$$

Which means that.

$$\left\| \Pi y_n(t) - \Pi y\ (t) \right\|_{PC} \to 0 \qquad \text{when} \qquad n \to \infty$$

Thus Π is a continuous function on J.

Step 2. Π takes bounded sets into bounded sets in P.

Let r_1 be a positive number, and there is a constant $r_2 \geq 0$

Then, for all $y \in B_{r_1} := \{ y \in B : \|y\|_{PC} \leq r_1 \}$ **.**

We have $\|\Pi y\|_{PC} \leq r_2$ **.**For $t \in J_m$ $, m = 0,1,2,...., p,$

And

$$\left\| \Pi y(t) \right\| \leq \left\| Y_h^{A,B}(t,0) \right\| \left\| \varphi(0) \right\| + \int_{-h}^{0} \left\| Y_h^{A,B}(t, \mathrm{s}+\mathrm{h}) \right\| \left\| B \right\| \left\| \varphi(s) \right\| ds \ + \int_{t_m}^{t} \left\| Y_h^{A,B,C}(t,\mathrm{s}) \right\| \left\| f(s, \mathrm{y}(s)) \right\| ds$$

$$\leq C_0 + \left\| Y_h^{A,B,C}(\mathrm{T},0) \right\| M_f T \left(1 + \|y\|_{PC} \right)$$

$$\leq C_0 + \left\| Y_h^{A,B,C}(\mathrm{T},0) \right\| M_f T (1 + r_1) := r_2 .$$

Which implies that $\|\Pi y\|_{PC} \leq r_2$.

Step 3. Π Maps bounded set into equicontinuous set of P.

Let $t_1, t_2 \in J_m$ **,** $\mathrm{m} = 0,1,2,....., \mathrm{p}$ **,** with $t_1 < t_2$ and B_{r_1} be a ball as in the second step. Then for $y \in \mathrm{B}$

We have

$$\left\|(\Pi y)(t_2)-(\Pi y)(t_1)\right\| \le \left\|Y_h^{A,B,C}(t_2,0)-Y_h^{A,B,C}(t_1,0)\right\|\left\|\varphi(0)\right\|$$

$$+\int_{-h}^{0}\left\|Y_h^{A,B,C}(t_2,s+h)-Y_h^{A,B,C}(t_1,s+h)\right\|\left\|B\right\|\left\|\varphi(s)\right\|ds+\int_{t_m}^{t_1}\left\|Y_h^{A,B,C}(t_2,s)-Y_h^{A,B,C}(t_1,s)\right\|\left\|f(s,y(s))\right\|ds$$

$$+\int_{t_1}^{t_2}\left\|Y_h^{A,B,C}(t_2,s)\right\|\left\|f(s,y(s))\right\|ds$$

We see that the right hand side of the previous inequality goes to zero as $t_2 \to t_1$, since $Y_h^{A,B,C}(t,s)$ is continuous in $t \in J_m$ and f is bounded on B_{r_1}.

Π is completely continuous by the previous steps and by Arzela–Ascoli Theorem.

Step 4. A priori bound.

The last step illustrates this definition:

$W=\{y \in B : y=\lambda\Pi(y) \text{ for some } 0<\lambda<1\}$ is bounded.

Let $y \in W$, the for some $0<\lambda<1$, $y=\lambda\Pi(y)$.

Therefore for $t \in J_m$ as in Step 2, we have

$$\left\|y(t)\right\| \le \lambda\left\|\Pi(y(t))\right\| \le C_0+\left|Y_h^{A,B,C}(T,0)\right|M_f T+\left\|Y_h^{A,B,C}(T,0)\right\|M_f\int_0^t\left\|y(s)\right\|ds$$

Gronwell's inequality yields

$$\left\|y(t)\right\| \le C_0+\left|Y_h^{A,B,C}(T,0)\right|M_f T\exp\left(\left\|Y_h^{A,B,C}(T,0)\right\|M_f T\right)<\infty$$

Then the set W is bounded.

So, by the Schaefer's fixed-point result, we deduce that Π has a fixed point which means the solution of the suggested problem (2).

Chapter 7

ILLUSTRATIVE EXAMPLES

In this section we introduce two numerical examples.

Example 1. Consider the linear problem (1):

$$\begin{cases} y'(t) = Ay(t) + By(t-h) + f(t, y(t)), t \in [0,T], \mathrm{h} > 0, t \neq t_k \\ \Delta y(t_k) = y(t_k) - y(t_k) = C_k y(t_k), k = 1, 2, ..., p, \\ y(t) = \varphi(t), -h \leq t \leq 0, \end{cases}$$

Where $\mathrm{A}, \mathrm{B}, \mathrm{C}_k \in \mathbb{R}^{n\times n,}$ are constant matrices, $\varphi \in C^1([-h,0], \mathbb{R}^n)$, $f \in C\ ([0,\infty), \mathbb{R}^n)$, $\{t_k\}_{k=1}^{\infty}$

satisfies $0 = t_0 < t_1 < ..... < t_p < t_{p+1} = T$.

This problem satisfies the conditions of Theorem 4 and this linear impulsive system is stable in Ulam-Hyers sense.

Example 2. Consider (2) with $h = 0.2$

$$A = \begin{pmatrix} -3.3 & 1 \\ 0 & -0.3 \end{pmatrix} . B = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.6 \end{pmatrix} . C_j = \begin{pmatrix} 1.2 & 0.5 \\ 0.2 & 1.2 \end{pmatrix}, j = 1, 2$$

$$\varphi(t) = \begin{pmatrix} e^{-3} \\ e^{-4} \end{pmatrix} \quad f(x(t)) = \begin{pmatrix} 0.25 \sin x_1 \\ 0.25 \sin x_2 \end{pmatrix}$$

Where $[x]$ is the biggest integer less than real x.

And $AB \neq BA.$ $\mathrm{j} = 1, 2, ...$

$\mathrm{AC}_j \neq \mathrm{C}_j A$.

$BC_j \neq C_j B.$

Obviously, f satisfies the Lipchitz condition $L_f = 0.25 > 0$, the conditions of Theorem 4 are satisfied and equation (2) has a uniqueness solution in $PC[-h,1] \cap PC^1[0,1]$ which is Ulam-Hyers stable on $[-h,1]$.

Example 3. Consider the below fractional problem

$$\begin{cases} y'(t) = \begin{pmatrix} -60 & 0 \\ 0 & -5.5 \end{pmatrix} y(t) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} y(t-h) + f(t, y(t)), \\ \Delta y(t_k) = y(t^+{}_k) - y(t^-{}_k) = \begin{pmatrix} 2 + \frac{1}{k} & 0 \\ 1 & 2 \end{pmatrix} y(t_k), k = 1, 2, \ldots\ldots, 4, \quad t \in [0,1], \mathrm{h} = 0.2 > 0, t \neq t_k \\ y(t) = \begin{pmatrix} e^{-3} \\ e^{-4} \end{pmatrix}, -h \leq t \leq 0, \end{cases}$$

Obviously, A, B and C_J are mutually non-commutative

$AB \neq BA$.

$AC_J \neq C_J A$

$BC_J \neq C_J B$, j=1, 2

Assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is any continuous function satisfying (H_2).

Then, by Theorem 6 the equation (2) has at least one solution on $[-h,1]$.

Chapter 8

CONCLUSION

The major contribution of our work is to establish an impulsive delayed matrix exponential for non-permutable matrices and use it to construct explicit results to solve the problem of impulsive delay systems that they have linear portions determined by non-permutable matrices. We give a sufficient for asymptotic stability of impulsive delay systems. And, Banach fixed point method is applied to present existence, uniqueness, and Ulam-Hyers stability of the impulsive delay system. The study on representation and stability of delay differential impulsive systems has prospective for coming times study on fractional impulsive delay systems, on fractional multiple delay impulsive problems, or on a delayed nonlinear problem.

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