Fourier Series and Integrals

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ABSTRACT

This thesis consists of six chapters. Introduction is in the first chapter. In the second chapter we present a method for solving partial differential equation by use of Fourier series. The method is called separation of variables.

In the third chapter we show that the Fourier series converges under certain reasonable general hypothesis. We give important results like Riemann-Lebesgue Lemma, Dirichlet kernels and three important conditions for the convergence of Fourier series at a point Dini's, Lipchitz and Dirichlet-Jordan conditions.

In the fourth chapter Fourier series are studied in more general point of view, considering functions as elements of abstract inner product space. Bessel's inequality, Parseval's identity, Cesaro summability and Fejer kernels are important results that are given.

In the fifth chapter is set the problem of uniform convergence of Fourier series based on piecewise-smooth functions. In addition it is given Weierstrass approximation theorem and Gibbs phenomenon, the case when the function is not uniformly convergent.

In the last chapter we deal with convergence of Fourier integrals. First we introduce the Fourier integral formula and then give the analogs of Dini's, Lipchitz and Dirichlet-Jordan conditions for Fourier integrals.

Keywords: Dirichlet kernels, Bessel's inequality, Parseval's identity, Cesaro summability, Fejer kernels.

ÖZ

Bu tez altı bölümden oluşmaktadır. Birinci bölüm giriş bölümüdür. İkinci bölümde Fourier serileri kullanarak kısmi türevli denklemin çözüm metodunu sunmaktayız. Bu metoda değişkenlerine ayırma metodu denir.

Üçüncü bölümde genel hipotezler altında Fourier serilerinin yakınsaklığı gösterildi. Riemann Lebesgue Lemma , Dirichlet çekirdekleri gibi önemli sonuçlar verildi ve Fourier serilerinin bir noktada yakınsaması için üç önemli koşul: Dini, Lipshctiz ve Dirichlet-Jordan' dır .

Dördüncü bölümde Fourier serilerinin soyut iç çarpım uzaylarının elemanları olan fonksiyonlar olduğu dikkate alınarak , geniş çapta çalışıldı.Bunlar arasında en önemlileri Bessel eşitsizliği, Parseval özdeşiliği, Cesaro toplanabilirlik ve Fejer çekirdekleridir.

Beşinci bölümde parçalı düzgün fonksiyonlar üzerine Fourier serilerinin düzgün yakınsaması problemi ortaya konulmuştur.Bunun yanı sıra fonksiyon düzgün yakınsak olmadığında Weistrass yaklaşım teoremi ve Gibbs fenomeni verilmiştir.

Son bölümde Fourier integrallerinin yakınsaması ele alınmıştır. Öncelikle Fourier integral formülü ve sonra Fourier integralleri için Dini, Lipschitz ve Dirichlet-Jordan şartlarının benzerleri verilmiştir.

Anahtar Kelimeler: Dirichlet çekirdekleri, Bessel eşitsizliği, Parseval özdeşliği, Cesaro toplanabilirlik, Fejer çekirdekleri.

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LIST OF SYMBOLS

$\mathbb{R}$	the set of real number
(a,b)	an open interval
[a,b]	a closed interval
(a,b)	an open interval
R(a,b)	Riemann integrable functions on (a,b)
$\mathbb{C}$[a,b]	the set of all real-valued and continuous functions defined on the compact interval [a,b]
PC(a,b)	The set of all piecewise continuous functions defined on (a,b)
$PS(a,b)$	The set of all piecewise continuous functions defined on (a,b)
PC(a,b) ∪ PS(a,b)	The set of all piecewise continuous functions or piecewise smooth functions defined on (a,b)
D_m	Dirichlet kernels
F_m	Fejer kernels
σ_n	Cesaro summation

Chapter 1

INTRODUCTION

Just before 1800, french mathematicien Jean Baptise Joseph Fourier made an astonishing invention. In 1807 he presented a paper to the Academy of Science which dealt with the problem of how heat "flows" through metallic rods and plates. In paper Fourier clamed that any function defined on a finite closed interval could be presented as a sum of sine and cosine functions. He proposed that any function $f(x)$ defined over the interval (-π,π) could be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

where a_n and b_n are called constants. Fourier's paper was nevertheless, rejected by Lagrange, Laplace and Legendre who criticised it for lack of rigor. His claims in analysis for series was wrong. The assertation that Fourier bealived that any arbitrary function $f(x)$ can be presented in terms of sine and cosine series over the interval $(-\pi,\pi)$ is pragmatically false. In 1878 Dirichlet (his phd student) showed that a Fourier series is guaranted to converge under the moderately loose conditions that a function has a finite number of finite discontinuities and a finite number of extremas. Today we call those conditions as a Dirichlet conditions. During our work we will deal with conditions wich ensure pointwise and uniform convergence of Fourier Series and Integrals.

This thesis is organized as follows. In chapter two we present a method for solving partial differential equations by use of Fourier series. The method is called separation of variables.In the third chapter we will show that the Fourier series converges under certain reasonable general hypothesis. We will give important results like Riemann-Lebesgue Lemma, Dirichlet kernel and three important conditions for pointwise convergence at a point: Dini's, Lipchitz and Dirichlet-Jordan conditions. In the fourth chapter Fourier series are studied in more general point of view, considering functions as elements of abstract inner product space. Bessel's inequality, Parseval's identity, Cesaro summability and Fejer kernels are important results that are given. In the fifth chapter is set the problem of uniform convergence of Fourier series based on piecewise-smooth functions. In addition it is given Weierstrass approximation theorem and Gibbs phenomenon, the case when the function is not uniformly convergent. In the last chapter we deal with convergence of Fourier integrals. First we introduce the Fourier integral formula and then give the analogs of Dini's, Lipchitz and Dirichlet-Jordan conditions for Fourier integrals.

Chapter 2

SOLUTION OF HEAT EQUATION BY FOURIER METHOD

2.1. Separation of variables

In this section we present one of the simplier methods of solving partial differential equations by the use of Fourier series. This method is called separation of variables (or sometimes the Fourier method). We demosntrate this method by considering the homogeneous heat equation defined on a rod of length 2L with periodic boundary conditions. In mathematical terms we must find a solution $u = u(x,t)$ to the problem

$$\begin{cases} u_t - ku_{xx} = 0, & -L < x < L, \quad 0 < t < \infty \\ u(x,0) = f(x), & -L \leq x \leq L, \\ u(-L,t) = u(L,t), & 0 \leq t < \infty \\ u_x(-L,t) = u_x(L,t) & 0 \leq t < \infty \end{cases}$$

where $k > 0$ is a constant. The common wisdom is that these mathematical equations model (under ideal conditions) the heat flow $u(x,t)$ is the temperature in a ring $2L$, where the initial ($t = 0$) distribution of temperature in the ring is given by the function f. A point on the ring is represented by a point in the interval $[-L, L]$ where the endpoints $x = L$ and $x = -L$ represents the same point in the ring. For this reason the mathematical representation of this problem includes the equations $u(-L,t) = u(L,t)$ and $u_x(-L,t) = u_x(L,t)$. To obtain a good solution to this problem, it is better if we assume that f is a countinous function, $f' \in E$, and f satisfies $f(-L) = f(L)$ and $f'(-L) = f'(L)$. The idea behind the method of separation of variables is first to find all non identically zero solutions of the form $u(x,t) = X(x)T(t)$ to the homogeneous system

$$\begin{cases} u_t - ku_{xx} = 0, & -L < x < L, \quad 0 < t < \infty, \\ u(-L,t) = u(L,t), & 0 \leq t \leq \infty, \\ u_x(-L,t) = u_x(L,t), & 0 \leq t < \infty. \end{cases} \tag{2.1.1}$$

Later we will look for a solution to the equation $u(x,0) = f(x)$ from the linear space generated by these solutions of the system above. Taking into consideration the system and the fact that $u(x,t) = X(x)T(t)$. Then

$$u_t(x,t) = X(t)T'(t), \quad u_{xx}(x,t) = X''(x)T(t).$$

Substituting these forms in the equation we obtain

$$X(x)T'(t) - kX''(x)T(t) = 0$$

and thus

$$X(x)T'(t) = kX''(x)T(t).$$

Dividing both sides of the equation by $kX(x)T(t)$, we obtain

$$\frac{T'(t)}{kT(t)} = \frac{X''(t)}{X(x)}.$$

The expression on the left-hand side is a function of t alone, while the expression on the right-hand side is a function of x. We already know that x and t are independent upon each other, the equation that is given above can hold only if and only if both sides of it is equal to some unknown constant $-\lambda$ for all values of x and t. Thus we may write

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Clearly we obtain one pair of ordinary differential equations with unknown constant λ:

$$X''(x) + \lambda X(x) = 0$$

$$T'(t) + k\lambda T(t) = 0$$

From those two boundary conditions we derive two conditions. From the boundary condition $u(-L,t) = u(L,t)$ it follows that for all $t \geq 0$

$$X(-L)T(t) = X(L)T(t).$$

There exist two possibilities. Either $T(t) = 0$ for all $t \geq 0$, or $X(-L) = X(L)$. After all, the first possibility leads us to the trivial solution for which we are not interested. So we look to the second condition $X(-L) = X(L)$. Similarly we obtain the second condition $X'(-L) = X'(L)$. When we are looking for non trivial solutions of (2.1.1) of the form $u(x,t) = X(x)T(t)$ to the equations for X:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 < x < L, \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{cases} \tag{2.1.2}$$

We can easily check that values of λ for which equation (2.1.2) has non trivial solutions are exactly

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n = 0, 1, 2, ...$$

For $\lambda_0 = 0$ the equation is $X''(x) = 0$ and general solution is

$$X(x) = c_1 x + c_2.$$

From the condition $X(-L) = X(L)$ we obtain $c_1 = 0$, while the condition $X'(-L) = X'(L)$ is always satisfied. This being so, in this case, the constant functions $X(x) = C$ are solutions of (2.1.2). For $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n \geq 1$, the equation is

$$X''(x) + \frac{n^2\pi^2}{L^2} X(x) = 0.$$

General solution has the form of

$$X(x) = c_1 \sin \frac{n\pi}{L} x + c_2 \cos \frac{n\pi}{L} x.$$

Finally, we have two non-trivial linearly independent solutions for all $n \in N$ and $\lambda_n = \frac{n^2\pi^2}{L^2}$

$$X_n(x) = \cos\frac{n\pi x}{L}x, \qquad X_n^* \sin\frac{n\pi x}{L}x.$$

Every other solution is a linear combination of these two solutions. The values λ_n are called the eigenvalues of the problem, and the solutions of X_n and X_n^* are called the eigenfunctions associated with eigenvalue λ_n. We also recall that among the eigenvalues we also have $\lambda_0 = 0$, with associated eigenfunction

$$X_0(x) = 1.$$

Now we consider the second equation $T'(t) + k\lambda T(t) = 0$. We restrict ourself to $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}, n = 0, 1, 2, 3, ...$ For each n there exists non trivial solution

$$T_n(t) = e^{-k\lambda_n t}.$$

Every other solution is a constant multiple therefore. So, finally we can summarize, for each $n \in N$ we have pair of nontrivial solution of (2.1.2) of the form

$$u_n(x,t) = X_n(x)T_n(t) = e^{-k\lambda_n t}\cos\frac{n\pi x}{L},$$
$$u_n^*(x,t) = X_n^*(x)T_n(t) = e^{-k\lambda_n t}\sin\frac{n\pi x}{L}.$$

For $n = 0$ we have the solution

$$u_0(x,t) = X_0(t)T_0(t) = 1.$$

Since the system (2.1.2) is homogeneous every ”infinite linear combination” of the solutions is again a solution (if we asume it converges). So, we have in a sense, an infinity of solutions of the general form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-k\lambda_n t}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right].$$

We must consider the non-honogeneous initial condition $u(x,0) = f(x), \ -L \leq x \leq L$. This condition should determine the two sequences of coefficients $\{a_n\}_{n=0}^{\infty}$ and

$$f(x) = u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-k\lambda_n t}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right].$$

We call it as a Fourier series of f on the interval $[-L, L]$ [2]. Where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 0,1,2,...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \qquad n = 1,2,3,...$$

Chapter 3

CONVERGENCE OF FOURIER SERIES AT A POINT

3.1. Trigonometric Series

Definition 3.1.1 *A series of the form*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called a trigonometric series.

The terms of this series are periodic functions with period 2π. Hence, if it converges on $(-\pi,\pi)$, then it converges on $\mathbb{R}$. Therefore from now on we will study this series on the interval $[-\pi,\pi]$, taking into consideration that, it produces the same values at $-\pi$ and π.

Definition 3.1.2 *A given function $f(x)$ can be represented, under hypothesis of considerable generality, by an infinite series of the form*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) \tag{3.1.1}$$

Such a series, when the coefficients are determined in the manner to be described below, is called Fourier series.

Since the each term is a periodic function with period 2π, the sum of the series necessarily has the same period. (A function $f(x)$ is said to be periodic if $f(x+a) = f(x)$). If a is a period, any integral multiple of a is also a period $2\pi/n$). On other hand, a Fourier series is sometimes useful for the presentation of a given function in a single interval of length 2π, when the property of periodicity is of no concern except as it results indicidentally from evaluation of the series outside the interval in which the function was orginally defined.

Theorem 3.1.3 *If the series in (3.1.1) converges uniformly to the function f on* $[-\pi,\pi]$ *then $f \in C(-\pi,\pi)$, $f(-\pi) = f(\pi)$ and*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)x\cos nx dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx. \tag{3.1.2}$$

Proof. *The sum of uniformly convergent series of countinous functions is countinuous. Hence, $f \in C(-\pi,\pi)$. Also,*

$$f(\pi) = f(-\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi = \frac{a_0}{2} + \sum_{n=1}^{\infty}(-1)^n a_n$$

Taking any $n \in \mathbb{N}$ so that $n \le m$. If

$$s_m = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos(kx) + b_k \sin(kx)), \tag{3.1.3}$$

than we have

$$\int_{-\pi}^{\pi} s_m \cos nx dx = \int_{-\pi}^{\pi} a_m \cos^2 nx dx = a_n\pi.$$

It is clear that $s_m \to f$ *uniformly,*

$$a_n\pi = \lim_{m\to\infty} \int_{-\pi}^{\pi} s_m(x)\cos nx dx = \int_{-\pi}^{\pi} f(x)\cos nx dx,$$

proving the formula for $a_n, n \in \mathbb{N}$. *The same arguments work for* a_0 *and* b_n , $n \in \mathbb{N}$. ■

Remark 3.1.4 a) *Writing the free constant term of the series in the form of* $\frac{a_0}{2}$ *is for the convinence and is standart notation, the definition of* a_0 *is no part of the general definition of all* a_n *in (3.1.2) (since* $\cos(0) = 1$*).*

b) *In the definition of the Fourier series of* f *we wrote* $\sim$ *and not equality. There is a reason for this. There is no necessity that the series in question converges for all* $x \in [-\pi,\pi]$. *And even if the series converges, it might not converge to the value* $f(x)$. *We need additional conditions on the function* f *to ensure that the series converges to the desired values, and in order to obtain the particular type of onvergence desired (such as uniform or pointwise convergence).*

c) *The Fourier series of* f *is totally determined by the values of the coefficients* a_n *and* b_n *(of which there are a countable number). These coefficients themselves determined by the specific integrals in (3.1.2). If we alter the value of the function* f *at a finite number of points, then the integrals defining* a_n *and* b_n *are unchanged. Thus every two function which differ at a finite number of points have exactly the same Fourier series.*

3.2. Rieman-Lebesgue Lemma

The sufficient conditions for the convergence of Fourier series and integrals, considered in this chapter are based on a result that is called the Riemann-Lebesgue Lemma. In the section we prove this useful result.

Theorem 3.2.1 *(Riemann-Lebesgue Lemma) Let g be absolutely integrable on* $[a,b]$, *either g is Riemann integrable or* $|g|$ *is improperly integrable on* $[a,b]$. *Than*

$$\lim_{\lambda\to\infty}\int_a^b g(x)\sin\lambda x dx = 0,$$

assuming that λ *tends to* ∞ *ever real numbers, not only over integers.*

Proof. *First, assume* $g \in R(a,b)$. *Take any* $\varepsilon > 0$. *There is a partition* $P = \{x_0, x_1, ..., x_n\}$ *of* $[a,b]$ *such that*

$$|S^*(g,P) - S_*(g,P)| < \tfrac{\epsilon}{2},$$

where

$$S^*(g,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \textit{and} \quad S_*(g,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

with $M_i = \sup_{[x_{i-1}-x_i]} g$ *and* $m_i = \inf_{[x_{i-1},x_i]} g$. *On the other hand*

$$\left|\int_{x_{i-1}}^{x_i} \sin\lambda x dx\right| = \frac{|\cos\lambda x_i - \cos\lambda x_{i-1}|}{\lambda} \leq \frac{2}{\lambda}.$$

Hence,

$$\begin{aligned}
\left|\int_a^b g(x)\sin\lambda x dx\right| &\leq \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i} g(x)\sin\lambda x dx\right| \\
&\leq \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} |(g(x)-m_i)\sin\lambda x|\,dx + \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i} m_i \sin\lambda x dx\right| \\
&\leq S^*(g,P) - S_*(g,P) + \frac{2}{\lambda}\sum_{i=1}^{n}|m_i|.
\end{aligned}$$

We obtain that for every

$$\lambda > \frac{4}{\epsilon}\sum_{i=1}^{n}|m_i|,$$

the inequality

$$\left|\int_{-\pi}^{\pi} g(x)\sin\lambda x dx\right| \leq \epsilon,$$

holds. This proves the theorem for $g \in R(a,b)$.

Later on we assume that $|g|$ *is improperly integrable on* $[a,b]$. *Since we have different possibilities it suffices to consider only one case, it means we will cosider when the improperness of* $|g|$ *is due to the point* a, *for every* $a < c < n$, $|g|$ *is unbounded on* $[a,c]$ *and bounded on* $[a,c]$. *So,*

$$|g(x)\sin\lambda x| \leq |g(x)|,$$

the improper integral

$$\int_a^b g(x)\sin\lambda x dx$$

is convergent for all $\lambda > 0$. Take any $\epsilon > 0$. Than there is a c, where $a < c < b$ such that

$$\int_a^c |g(x)|dx < \frac{\epsilon}{2},$$

which implies

$$\left|\int_c^b g(x)\sin\lambda x dx\right| < \int_a^c |g(x)|dx < \frac{\epsilon}{2}.$$

Thus,

$$\left|\int_c^b g(x)\sin\lambda x dx\right| < \epsilon,$$

whenever $\lambda > M$. This complates the proof. ■

Riemann-Lebesgue lemma has a modification to infinite intervals as well.

Theorem 3.2.2 *(Riemann-Lebesgue Lemma) Let g be absolute integrable on $[a, \infty)$. Than*

$$\lim_{\lambda\to\infty}\int_a^\infty g(x)\sin\lambda x dx = 0,$$

assuming that $\lambda \to \infty$ over real numbers.

3.3. Dirichlet Kernels

The function D_m defined by

$$D_m = 1 + 2\sum_{k=1}^{m} \cos kx, \qquad -\infty < x < \infty, \tag{3.2.1}$$

is called Dirichlet kernel. Here m takes values 0,1,2,..., assuming that $D_0(x) = 1$. By use of trigonometric identity

$$2\cos kx \sin\frac{x}{2} = \sin\frac{(2k+1)x}{2} - \sin\frac{(2k-1)x}{2},$$

now from the formula for Dirichlet kernel we can evaluate,

$$\begin{aligned} D_m &= 1 + 2\sum_{k=1}^{m} \cos kx = 1 + \frac{1}{\sin\frac{x}{2}} \sum_{k=1}^{m} 2\cos kx \sin\frac{x}{2} \\ &= 1 + \frac{\sin\frac{(2m+1)x}{2} - \sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{\sin\frac{(2m+1)x}{2}}{\sin\frac{x}{2}}, \end{aligned}$$

whenever $\sin\frac{x}{2} \neq 0$, *where* $x \neq 2\pi n$. Using the continuity of D_m, the values of D_m at $x = 2\pi n$ can be recovered by taking the limit

$$\begin{aligned} \lim_{x\to 2\pi n} \frac{\sin\frac{(2m+1)x}{2}}{\sin\frac{x}{2}} &= \lim_{x\to 2\pi n} \frac{(2m+1)\cos\frac{(2m+1)x}{2}}{\cos\frac{x}{2}}, \\ &= 2m+1 = 1 + 2\sum_{k=1}^{m} \cos 2\pi nk = D_m(2\pi n). \end{aligned}$$

The Dirichlet kernels play a significant rol in studying Fourier series. We can observe the following properties of Dirichlet krnels:

a) D_m is an even function.

b) D_m is a periodic function with the period 2π.

c) $\int\limits_0^\pi D_m dx = \pi.$

Theorem 3.3.1 *Let s_m be the mth partial sum defined in (3.1.3) of an integrable function f of period 2π. Than*

$$s_m(x) = \frac{1}{2\pi}\int\limits_0^\delta f(x-y) + f(x+y))D_m(y)dy. \tag{3.3.2}$$

Proof. *Replacing the Fourier coeficients in (2.1.3), we obtain*

$$\begin{aligned} s_m(x) &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\sum_{k=1}^{m}\int\limits_{-\pi}^{\pi} f(y)(\cos ky \cos kx - \sin ky \sin kx)dy \\ &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(1 + 2\sum_{k=1}^{m}\cos k(y-x)\right)dy = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(y)D_m(y-x)dy. \end{aligned}$$

Since f and D_m are periodic functions with period 2π and D_m is even,

$$\begin{aligned} s_m(x) &= \frac{1}{2\pi}\int\limits_{-\pi-x}^{\pi+x} f(x+y)D_m(y)dy = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x+y)D_m(y)dy \\ &= \frac{1}{2\pi}\int\limits_{-\pi}^{0} f(x+y)D_m(y)dy + \frac{1}{2\pi}\int\limits_{0}^{\pi} f(x+y)D_m(y)dy \\ &= \frac{1}{2\pi}\int\limits_{0}^{\pi} (f(x-y) + f(x+y))\, D_m(y)dy. \end{aligned}$$

Which complates the proof. ■

Theorem 3.3.2 *(Riemann localization theorem) Let $f \in R(-\pi,\pi)$. If*

$$\lim_{m\to\infty}\int_0^{\delta}(f(x-y)+f(x+y))\,D_m(y)dy \tag{3.3.3}$$

exists for some $0<\delta<\pi$, then the Fourier series of f converges at x to this value.

Proof. *First, we divide the integral that is given in (3.3.3) into two integrals, on the intervals $[0,\delta]$ and $[\delta,\pi]$.*

$$\frac{1}{2\pi}\left(\int_0^{\pi}+\int_{\pi}^{\delta}\right)\frac{f(x-y)+f(x+y)}{\sin\frac{y}{2}}\sin\frac{(2m+1)y}{2}dy.$$

Writing the integral on $[\delta,\pi]$ in the form

$$\frac{1}{2\pi}\int_{\delta}^{\pi}\frac{f(x-y)+f(x+y)}{\sin\frac{y}{2}}\sin\frac{(2m+1)y}{2}dy.$$

Here the function

$$f_1(y)=\frac{f(x-y)+f(x+y)}{\sin\frac{y}{2}},$$

is bounded on $[\delta,\pi]$ and hence, belongs to $R(\delta,\pi)$. By Riemann Lebesgue Lemma, the limit of this integral as $m\to\infty$ is zero. Hence the limit of m-th partial sum $s_m(t)$ of the Fourier series of the function is same as its limit if it exists. ■

This theorem is named after Riemann, it was known earlier as Ostogradski and Lobochevski Theorem.

3.4. Dini's condition

Three important theorems for pointwise convergence will be proved in this section. The first one belongs to Dini.

Theorem 3.4.1 *(Dini) Let $f \in R(-\pi,\pi)$. If*

$$\int_0^\delta \left|\frac{f(x-y)+f(x+y)-2s}{y}\right| dy < \infty \tag{3.4.1}$$

for some $0<\delta\leq\pi$ and $s\in\mathbb{R}$, where the integral is proper or improper Riemann integral, than the Fourier series of f converges at x to s.

Proof. *Let s_m be $m-th$ partial sum of the Fourier series of f. From the properties of Dirichlet kernel, we have*

$$\begin{aligned}
s_m - s &= \frac{1}{2\pi}\int_0^\pi (f(x-y)+f(x+y))\,D_m(y)dy - \frac{1}{\pi}\int_0^\pi sD_m(y)dy \\
&= \frac{1}{2\pi}\int_0^\pi (f(x-y+f(x+y)-2s)\,D_m(y)dy \\
&= \frac{1}{\pi}\int_0^\pi \frac{f(x-y)+f(x+y)-2s}{y}\,\frac{\frac{y}{2}}{\sin\frac{y}{2}}\sin\frac{(2m+1)y}{2}dy \\
&= \frac{1}{\pi}\left(\int_0^\delta + \int_\delta^\pi\right)\frac{f(x-y)+f(x+y)-2s}{y}\,\frac{\frac{y}{2}}{\sin\frac{y}{2}}\sin\frac{(2m+1)y}{2}dy.
\end{aligned}$$

Here, $\frac{f(x-y)+f(x+y)-2s}{\sin\frac{y}{2}}$ is properly Riemann integrable on $[0,\pi]$. Hence by Riemann-Lebesgue Lemma the second integral goes to zero as $m\to\infty$. At the same time $\frac{f(x-y)+f(x+y)-2s}{y}$ is absolutely integrable on $[0,\delta]$, and $\frac{f(x-y)+f(x+y)-2s}{y}$ is bounded function on $[0,\delta]$. So the product of absolutely integrable function and bounded function *is absolutely integrable function on* $[0,\pi]$. *So,* $\lim_{m\to\infty} s_m(x) = s$. ■

3.5. Lipschitz Condition

Theorem 3.5.1 *(Lipschitz) Let* $f \in R(-\pi,\pi)$. *If there are numbers* $L \geq 0$, $0 < \alpha \leq 1$ *and* $\sigma > 0$ *such that*

$$|f(x+y) - f(x)| \leq L|y|^{\alpha} \tag{3.5.1}$$

for all $|x-y| < \sigma$, *than the Fourier series of* f *converges at* x *to* $f(x)$.

This sufficient condition is attributed to Lipschitz although this original paper was corrected by Hölder. So, the theorem is called the local Lipschitz condition at x if $\alpha = 1$, and the local Hölder condition at x if $0 < \alpha < 1$. We will verify the Dini's condition for $s = f(x)$. We have

$$\begin{aligned}\left|\frac{f(x-y)+f(x+y)-2f(x)}{y}\right| &\leq \left|\frac{f(x-y)-f(x)}{y}\right| + \left|\frac{f(x+y)-f(x)}{y}\right| \\ &\leq \frac{L|y|^{\alpha}}{y} + \frac{L|y|^{\alpha}}{y} = \frac{2L}{y^{1-\alpha}},\end{aligned}$$

where $0 \leq y \leq \sigma$, *and* $0 < \alpha \leq 1$, in case when $\alpha = 1$, $\int_0^{\sigma} \frac{1}{y^{1-\alpha}} dy$ is proper Reimann integrable, for $0 < \alpha < 1$, $\int_0^{\sigma} \frac{1}{y^{1-\alpha}} dy$ is convergent improper integral. So, in both cases the integral is convergent, hence

$$\int_0^{\sigma} \left|\frac{f(x-y)+f(x+y)-2f(x)}{y}\right| dy < \infty.$$

The Lipchitz condition implies the Dini's condition. But the next condition, due to Dirichlet and Jordan, is incomparable with the Dini's condition. To prove Dirichlet Jordan theorem we need this Lemma.

Definition 3.5.2 *If* $[a,b]$ *is compact interval, a set of points* $P = \{x_0, x_1, x_2, ..., x_n\}$ *is called a partition of* $[a,b]$. *The interval* $[x_{k-1}, x_k]$ *is called the* $k-th$ *subinterval of* p *and we write* $\triangle\, x_k = x_k - x_{k-1}$, *so that*

$$\sum_{k=1}^{n} |\triangle\, f_k| \leq M$$

for all partitions of $[a,b]$, *than* f *is said to be of bounded variation on* $[a,b]$.

Theorem 3.5.3 a) *If* f *is monotonic on* $[a,b]$, *then* f *is of bounded variation on* $[a,b]$,

b) *if* f *is continuous on* $[a,b]$ *and* f' *exists and is bounded, say* $|f'(x)| \leq A$ *for all* x *in* (a,b), *then* f *is of bounded variation on* $[a,b]$,

c) *if* f *is of bounded variation on* $[a,b]$, *say* $\sum |\triangle f_k| \leq M$ *for all partitions of* $[a,b]$, *then* f *is bounded on* $[a,b]$.

In fact

$$|f(x)| \leq |f(a)| + M \qquad for \;\; all \quad x \in [a,b].$$

Proof.

a) Let f be increasing. Than for every partition of $[a,b]$ we have $\triangle f_k \geq 0$ and hence

$$\sum_{k=1}^{n} |\triangle f_k| = \sum_{k=1}^{n} \triangle f_k = \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] = f(b) - f(a),$$

thus f is with bounded variation,

b) applying the mean value theorem,

$$\triangle f_k = f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1}), \qquad t_k \in (x_{k-1}, x_k)$$

this implies

$$\sum_{k=1}^{n} |\triangle f_k| = \sum_{k=1}^{n} \left|f'(t_k)\right| \triangle x_k$$

since f' is bounded, which means that $|f'(x)| \leq A \;\; for \;\; all \;\; x \in (a,b)$

$$\sum_{k=1}^{n} |\triangle f_k| = \sum_{k=1}^{n} \left|f'(t_k)\right| \triangle x_k \leq A \sum_{k=1}^{n} \triangle x_k = A(b-a),$$

c) assume that $x \in (a,b)$. Using the special partition $P = \{a, x, b\}$, we find

$$|f(x) - f(a)| + |f(b) - f(x)| \leq M.$$

This implies $|f(x) - f(a)| \leq M, \;\; f(x) \leq |f(x)| \leq |f(a)| + M$, the same inequality holds if $x = a$ or $x = b$.

■

Definition 3.5.4 *Let f be of bounded variation on $[a,b]$, and let $\sum(P)$ denote the sum*

$$\sum_{k=1}^{n} |\triangle f_k|$$

corresponding to the partition $P = \{x_0, x_1, x_2, ..., x_n\}$ of $[a,b]$. The number $V_f(a,b) =$ $\sup\{\sum(P) : P \in P[a,b]\}$ is called the total variation of f on $[a,b]$.

3.6. Dirichlet-Jordan Lemma

Theorem 3.6.1 *(Dirichlet-Jordan lemma) If $g \in BV$ in (o,σ). Than*

$$\lim_{m\to\infty}\int_0^{\sigma} g(y)\sin\frac{\frac{2m+1}{2}}{y}dy = \frac{\pi}{2}g(0+). \tag{3.6.1}$$

Proof. First note that

$$\lim_{m\to\infty}\int_0^{\sigma} g(y)\sin\frac{2m+1}{2}dy = 0,$$

by Reimann Lebesgue Lemma

$$\int_0^{\sigma} g(y)\frac{\sin\frac{2m+1}{2}}{y}dy = \int_0^{\sigma} g(0+)\frac{\sin\frac{2m+1}{2}}{y}dy + \int_0^{\sigma} [g(y)-g(0+)]\sin\frac{2m+1}{2}dy.$$

$$\begin{aligned}\int_0^{\sigma} g(o+)\frac{\sin\frac{2m+1}{2}}{y}dy &= g(0+)\int_0^{\frac{2m+1}{2}\sigma}\frac{\sin z}{z}dz = g(0+)\,\mathrm{Si}(\frac{2m+1}{2}\sigma)\\ &= g(0+)\frac{\pi}{2} \quad as \quad m\to\infty.\end{aligned}$$

So it remains to show that

$$\int_0^{\sigma} [g(y)-g(0+)]\sin\frac{2m+1}{2}dy = 0.$$

Let $L = \sup_{x\geq 0}|\mathrm{Si}(x)|$, note that according to the fact that $\mathrm{Si}(x)$ is continuous on $(0,\infty)$ and limit of it is finite, than $\mathrm{Si}(x)$ is a bounded function on $(0,\infty)$ So, $0 \leq L \leq \infty$, next $\lim_{y\to 0+} g(y) = g(0+)$ there is $0<\delta<\sigma$ such that

$$|g(y)-g(0+)| < \frac{\varepsilon}{4L}$$

whenever $0<y<\delta$.

Take $0<\eta<\delta$, then

$$\int_0^{\sigma} [g(y)-g(0+)]\frac{\sin\frac{2m+1}{2}}{y}dy,$$

$$\left(\int_0^{\eta} + \int_{\eta}^{\sigma}\right)[g(y)-g(0+)]\frac{\sin\frac{2m+1}{2}}{y}dy,$$

here

$$\int_{\eta}^{\sigma} [g(y)-g(0+)]\frac{\sin\frac{2m+1}{2}}{y}dy \to 0 \quad as \quad m\to\infty.$$

Hence , there is $M>0$ such that for every $\lambda > M$,

$$\left|\int_{\eta}^{\sigma} [g(y)-g(0+)]\frac{\sin\lambda}{y}dy\right| < \frac{\varepsilon}{2}. \tag{3.6.2}$$

On the other hand by Reimann Lebesgue Lemma the other integral can be written as

$$\begin{aligned}\int_0^\eta [g(y)-g(0+)]\frac{\sin\lambda}{y}dy &= \int_0^\eta g(y)-g(0+))d\,\mathrm{Si}(\lambda y)\\ &= (g(0)+g(0+))(\mathrm{Si}(\lambda c)-\mathrm{Si}(0))+\\ &\quad +(g(\eta)-g(0+))(\mathrm{Si}(\lambda\eta)-\mathrm{Si}(\lambda c)),\end{aligned}$$

where $0 \le c \le \eta$. Here c depends on λ and as well as on $g(0)$ if we make $g(0)$ free in the interval $(-\infty, g(0+))$. Such a freedom does not demage the increasing property of g and does not change the value of the integral in (3.6.1). Taking c corresponding to $g(0) = g(0+)$. Then

$$\int_0^\eta [g(y)-g(0+)]\frac{\sin\lambda}{y}dy = (g(\eta)-g(0+))(\mathrm{Si}(\lambda\eta)-\mathrm{Si}(\lambda c)).$$

Here $g(\eta)-g(0+) < \frac{\varepsilon}{4L}$, since $0 < \eta < \delta$

$$\left|\int_0^\eta [g(y)-g(0+)]\frac{\sin\lambda}{y}dy\right| < \frac{\varepsilon}{4L}|\mathrm{Si}(\lambda\eta)-\mathrm{Si}(\lambda c)| \le \frac{\varepsilon}{4L}2L = \frac{\varepsilon}{2}, \qquad (3.6.3)$$

independently on λ. Hence, from (3.6.2) and (3.6.3) yield that for every $\lambda > M$,

$$\left|\int_0^\sigma [g(y)-g(0+)]\frac{\sin\lambda y}{y}dy\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,$$

which complates the proof. ■

Theorem 3.6.2 *(Dirichlet-Jordan) Let $f \in R(-\pi,\pi)$. If f has a bounded variation on some onterval $[x-\sigma, x+\sigma]$, than its Fourier series at x converges to $(f(x+)+f(x-))/2$.*

Proof. *By Riemann localization lemma, it suffices to evaluate the limit in (3.3.3)*

$$\frac{1}{\pi}\int_0^{\sigma}(f(x-y)+f(x+y))\frac{\frac{y}{2}}{\sin\frac{y}{2}}\frac{\sin\frac{(2m+1)y}{2}}{y}dy.$$

Here $f_1(y) = f(x-y)+f(x+y)$ *has a bounded variation on* $[0,\sigma]$ *under fixed x. Also,* $f_2(y) = \frac{y/2}{\sin(y/2)}$ *is increasing on* $[0,\sigma]$ *assuming that* $f_2(0) = \lim_{y\to 0+} f_2(y) = 1$. *So, the product of an increasing function and function of bounded variation is also of bounded variation on* $[0,\sigma]$. *Than by Dirichlet-Jordan Lemma, limit in (3.3.3) exists and equals to*

$$\frac{\pi}{2\pi}g(0+) = \frac{1}{2}f_1(0+)f_2(0+) = \frac{f(x-)+f(x+)}{2}.$$

This complates the proof. ■

Chapter 4

FOURIER SERIES IN INNER PRODUCT SPACES

4.1. Linear and Inner Product Spaces

In this chpter we will examine Fourier series from more general point of view considering functions as elements of abstract inner product spaces.

Definition 4.1.1 *A vector space E is called an inner product space if the real number $\langle p, q\rangle$, called the inner product of p and q, is assigned to each $p, q \in E$ such that the following axioms hold:*

a) *(nonnegativity)* $\forall\ p \in E,\ \langle p, p\rangle \geq 0$;

b) *(nondegeneracy)* $\langle p, p\rangle = 0 \Leftrightarrow p = 0$;

c) *(symmetry)* $\forall p, q \in E,\ \langle p, q\rangle = \langle q, p\rangle$;

d) *(additivity)* $\forall p, q, r \in E, \langle p + q, r\rangle = \langle p, r\rangle + \langle q, r\rangle$;

e) *(homogeneity)* $\forall p, q \in E$ *and* $\forall a \in \mathbb{R}, \langle ap, q\rangle = a\langle p, q\rangle$.

Every inner product space E can be converted to normed space with the norm

$$\|p\| = \sqrt{\langle p, p\rangle}.$$

Convergence with respect to this norm is called convergence in E. The axioms of norm can be verified by use of axioms of inner product. A verification of triangle inequality needs an additional fact as stated below.

Theorem 4.1.2 *(Cauchy-Schwarz inequality) Let E be an inner product space, then for every* $p,q \in E$, $|\langle p,q\rangle| \leq ||p||\,||q||$.

Theorem 4.1.3 *(Triangle inequality) Let E be an inner product space. Than for every* $p,q \in E$, $||p+q|| \leq ||p|| + ||q||$.

Theorem 4.1.4 *(Continuity of inner product) Let E be a inner product space. Assume that the sequence* $\{p_n\}$ *converges to* p *in* E. *Than for every* $q \in E$, $\lim_{n\to\infty} \langle p_n,q\rangle = \langle p,q\rangle$.

Proof. *Since* $p_n \to p$, *this means* $||p_n - p|| \to 0$. *Than* $|\langle p_n,q\rangle - \langle p,q\rangle| = |\langle p_n - p, q\rangle| \leq ||p_n - p|| \cdot ||q|| \to 0$. ■

Definition 4.1.5 *An inner product space that can be converted into a Banach space in the above mentioned way is called a Hilbert space.*

Example 4.1.6 *One can verify that for* $f,g \in C(a,b)$ *the function defined by*

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx, \tag{4.1.1}$$

satisfies the axioms (a)-(e) of the definition. This makes $C(a,b)$ *an inner product* *space, which becomes a normed space with the norm*

$$\|f\| = \left(\int_a^b f(x)^2 dx \right)^{1/2}. \tag{4.1.2}$$

This space will be denoted by $\tilde{C}(a,b)$ in order to make a distinction of norms. The convergence in $\tilde{C}(a,b)$ is called a mean square convergence (sometimes, mean convergence). The space $\tilde{C}(a,b)$ is neither Hilbert or Banach space.

In normed spaces, hence, in inner product spaces, it is possible to define infinite series in a very similar way as numerical series. A series $\sum_{i=1}^{\infty} p_i$ is said to converge if the sequence of partial sums $s_n = \sum_{i=1}^{\infty} p_i$ converges as $n \to \infty$. If the numerical series $\sum_{i=1}^{\infty} \|p_i\|$ converges, than $\sum_{i=1}^{\infty} p_i$ is said to converge absolutely. In a normed space absolute convergence does not yet imply convergence, but in a Banach space absolute convergence implies convergence. Thus the series of the form $\sum_{i=1}^{\infty} a_i p_i$, where $a_1, a_2, ... \in \mathbb{R}$ and $p_1, p_2, ..$ are vectors, has sense in normed spaces.

Another important concept in an inner product space E is orthogonality. Two vectors $p, q \in E$ are said to be orthogonal if $\langle p, q \rangle = 0$. This fact we write like $p \perp q$. A sequence $\{p_i\}$ (finite or infinite) of nonzero terms E is said to be orthogonal system, if $p_i \perp p_j$ for every $i \neq j$. If, additionally, all p_i are units vectors, than $\{p_i\}$ can be made orthonormal by normalizing its vectors, i.e., by changing p_i by $e_i = \frac{p_i}{\|p_i\|}$.

Example 4.1.7 *In the inner product spaces $\tilde{C}(-\pi,\pi)$ and $R(-\pi,\pi)$, the functions*

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin 2x, \frac{1}{\sqrt{\pi}} \cos 2x, ... \tag{4.1.3}$$

form an orthonormal system. This follows from the trigonometric integrals.

Theorem 4.1.8 *(Orthogonal projection) Let* $\{e_1, e_2, ..., e_n\}$ *be a finite orthonormal system in an inner product space E. For fixed* $x \in E$. *The function*

$$f(a_1, a_2, ..., a_n) = \left\| x - \sum_{i=1}^{n} a_i e_i \right\|^2, \quad a_1, ..., a_n \in \mathbb{R}$$

takes its minimal value at $a_1 = \langle x, a_1 \rangle, ..., \langle x, a_n \rangle$ *and*

$$\min f = \|x\|^2 - \sum_{i=1}^{n} \langle x, e_1 \rangle^2 .$$

Proof. *One can evaluate and find that*

$$f(\langle x, e_1 \rangle, ..., \langle x, e_n \rangle) = \|x\|^2 + \sum_{i=1}^{n} \langle p, e_1 \rangle (\langle p, e_i \rangle - 2 \langle p, e_i \rangle) = \|p\|^2 - \sum_{i=1}^{n} \langle p, e_i \rangle^2 .$$

■

4.2. Bessel's Inequality

Now we consider an inner product space E and a countably infinite orthonormal system $\{e_i\}$ in E. Taking first n of them we see that

$$x_n = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i,$$

is the best approximation of $x \in E$ by linear combinations of $e_1, e_2, ...e_n$. Motivated by this , we can associate with $x \in E$ the series

$$x \sim \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i. \tag{4.2.1}$$

It can be observe that the series in (4.2.1) match with the Fourier series of $f \in \tilde{C}(-\pi, \pi)$ with respect to the orthonormal system.

Theorem 4.2.1 *(Bessel's inequality) Let $\{e_i\}$ be countable infinite orthonormal system in inner product space E. Then for every $x \in E$,*

$$\sum_{i=1}^{\infty} \langle x, e_i \rangle^2 \leq \|x\|^2.$$

Proof. *From orthogonal projection we know that f is a nonnegative function. Hence*

$$\sum_{i=1}^{n} \langle x, e_i \rangle^2 \leq \|x\|^2$$

for every n. Taking the limit in both sides and moving n to infinity, we obtain the Bessel's inequality. ■

Corollary 4.2.2 *Let the Fourier series of $f \in R(-\pi, \pi)$ be given by (3.1.1), then*

$$\frac{a_0^2}{2} + \sum_{i=1}^{n} \left(a_n^2 + b_n^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx.$$

Proof. Let $f(x)$ be geven by (3.1.1), and

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)).$$

Taking

$$\int_{-\pi}^{\pi}(f(x) - f_N(x))^2 dx = \int_{-\pi}^{\pi}(f^2(x) - 2f(x)f_N(x) + f_N^2 x))dx.$$

Hence, easy calculations give us

$$\int_{-\pi}^{\pi} f_N^2(x)dx = \pi(\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)).$$

Therefore

$$\int_{-\pi}^{\pi}(f(x) - f_N(x))^2 dx = \int_{-\pi}^{\pi} f^2(x)dx - \pi(\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)).$$

since

$$\int_{-\pi}^{\pi}(f(x) - f_N(x))^2 dx \geq 0,$$

then

$$\pi(\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)) \leq \int_{-\pi}^{\pi} f^2(x)dx,$$

for any $N > 1$. Finally

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx.$$

■

If we have equality

$$\sum_{n=1}^{n} \langle x, e_i \rangle^2 = \|x\|^2 \tag{4.2.2}$$

then we say Parseval's identity holds for x.

Theorem 4.2.3 *If f and g are piecewise continuous functions of period 2π, with Fourier coefficients a_n, b_n and α_n, β_n respectively, then*

$$\frac{1}{\pi}\int_{0}^{2\pi} f(x)g(x)dx = \frac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n\alpha_n + b_n\beta_n. \tag{4.2.3}$$

Proof. *Since the Fourier series expansion of piecewise continuous function of period 2π converges in the mean to function. So the Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx))$$

converge in the mean to g. Multiplying each term of this series by $f(x)/\pi$ and integrating over the interval $(0, 2\pi)$; the resulting series will converge to $\frac{1}{\pi}\int_{0}^{2\pi} f(x)g(x)dx$. After easy calculations the series obtained is preciesly the right side of (4.2.2). This complates the proof. ■

Corollary 4.2.4 *(Parseval's Identity). If f is piecewise continuouse, of period 2π,* *then*

$$\frac{1}{\pi}\int_{0}^{2\pi}(f(x))^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2). \tag{4.2.4}$$

Proof. The proof follows from the above theorem, taking $f = g$. ■

Parseval's identity (4.2.4) is an infinite-dimensional, the square of the length of the vector is the sum of the squares of the scalar components of the vector along the coordinate axes. As it is expressed, it appears rather more complicated than was the corresponding formula given in (4.2.2) the reason is that the functions $\sin nx, \cos nx$, are not normalized, i.e., $||\sin nx||$ and $||\cos nx||$ are not equal to unity.

Theorem 4.2.5 *Let $\{e_i\}$ be countable infinite orthonormal system in inner product space E. Than $x \in E$ is represented by its Fourier series*

$$x = \sum_{i=1}^{\infty}\langle x, e_i\rangle\, e_i \tag{4.2.5}$$

if and only if the Parseval's identity holds for x.

Proof. *The proof is based on the equality*

$$\left\| x - \sum_{n=1}^{n}\langle x, e_i\rangle\, e_i \right\|^2 = ||x||^2 - \sum_{i=1}^{n}\langle x, e_i\rangle^2 .$$

If the Parseval's identity holds for x, then the right hand side converges to 0 as n goes to ∞. Hence, the left hand side also converges to 0, proving that the partial sum of the Fourier series of x converges to x in E. Conversely, if x is presented by its Fourier series, then the left hand side goes to 0. This means that the right hand side also *converges to* 0*, i.e., the Parseval's identity holds for* x. ■

Another imoprtant issue related to orthonormal systems is that of completeness.

Definition 4.2.6 *Let* $\{e_i\}$ *be an infinite orthonormal system in inner product space* E*. We say the system is complate in* E *if* $x \in E$ *the Parseval's identity holds with respct to this orthonormal system.*

Later on we will prove that the orthonormal system in the inner product space $\widetilde{C}(-\pi,\pi)$ is complate and obtain the Fourier series of every continuous function converges to it in mean square sense.

4.3. Cesàro Summability and Fejér's Theorem

As motivation for our future work in this section we will consider the following result

Example 4.3.1 *Let us consider the series*

$$1 - 1 + 1 - 1 + \ldots \tag{4.3.1}$$

The sequence of partial sums of (4.3.1) is

$$1, 0, 1, 0, \ldots$$

which we know that it does not converge. Therefore, by definition., (3.2.1) is a divergent series. On the other hand if we set s to be equal to (3.2.1),

$$s = 1 - 1 + 1 - 1 + ...$$
$$= 1 - (1 - 1 + 1 - 1 + ...) = 1 - s$$

where $s = 1/2$.

Now we will introduce two new definitions of "sum".

Given any series

$$u_1 + u_2 + u_3 + ... \tag{4.3.2}$$

with partial sums

$$s_n = u_1 + u_2 + u_3 + ... + u_n, \tag{4.3.3}$$

the $n-th$ arithmetic mean of these partial sums is defined

$$\sigma_n = \frac{s_1 + s_2 + s_3 + ...s_n}{n}, \qquad (4.3.3)$$

which is the avarage of the first n partial sums of (4.3.2).

Let us consider a less trivial example. Consider the series of functions

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + ... \tag{4.3.5}$$

This series diverge for all x.The $n+1st$ partial sum is

$$\frac{1}{2}+\sum_{m=1}^{n}\cos mx=\frac{\sin(2n+1)(\frac{x}{2})}{2\sin(\frac{x}{2})}.$$

Therefore the arithmetic mean is

$$\frac{1}{n}\sum_{k=o}^{n-1}\frac{\sin(2k+1)(\frac{x}{2})}{2\sin(\frac{x}{2})}=\frac{1}{2n\sin(\frac{x}{2})}\sum_{k=o}^{n-1}\sin(k+\frac{1}{2})x,$$

which we can write in closed form like

$$\sigma_n(x)=\frac{\sin^2 n(x/2)}{2n\sin^2(x/2)}. \tag{4.3.6}$$

So, if x is in the interval $(0,2\pi)$, the numerator increases. Therefore $\sigma_n(x)$ tends to zero. It follows that the Cesaro sum of (4.3.5) is zero for every x in the interval $(0,2\pi)$. Observe however that when $x=0$ or $x=\pi$, the nth arithmetic has the value (obtained from (4.3.6) from the limit convention, obtained directly from (4.3.5)), $\sigma_n(x)=n^2/2n=n/2$. Therefore the Cesaro sum of (4.3.5) does not exists when x is an integral multipleof 2π.Although (4.3.5)is not a Fourier Series.[1]

If the sequence of the arithmetic means $\sigma_1,\sigma_2,\sigma_3,...$ converges to σ, we say that σ in the Cessaro sum of the series (4.3.3).

In 1904 Hungarian mathematicient Leopold Fejer used Cesaro summability to Fourier series and achived a great succes. He proved that the Fourier series of every continuos function on $[-\pi,\pi]$ converges in the mean square sense to the same function. So, now we will concentrate on this issue:

The function F_m, defined by

$$F_m(x) = \frac{1}{m+1}\sum_{k=0}^{m} D_k(x), \quad -\infty < x < \infty,$$

is called Fejer kernel, where D_k is Dirichlet kernel and is defined in (3.3.1), $m = 0, 1, 2, 3, \ldots$. For $m = 0$, we have $F_0(x) = D_0(x) = 1$. One can evaluate the closed formula for F_m, starting

$$\begin{aligned} F_m &= \frac{1}{m+1}\sum_{k=0}^{m} D_k(x) \\ &= \frac{1}{(m+1)}\sum_{k=0}^{m} \frac{\sin\frac{(2k+1)x}{2}}{\sin\frac{x}{2}} \\ &= \frac{1}{2(m+1)\sin^2\frac{x}{2}}\sum_{k=0}^{m} \sin\frac{(2k+1)x}{2}\sin\frac{x}{2}, \end{aligned}$$

by use of trigonometric identies

$$\begin{aligned} 2\sin\frac{(2k+1)x}{2}\sin\frac{x}{2} &= \cos kx - \cos(k+1)x \quad \textit{and} \\ \sin^2\frac{(m+1)x}{2} &= \frac{1-\cos(m+1)x}{2}, \end{aligned}$$

Finally we get the Formula for Fejer kernel

$$F_m = \frac{1-\cos(m+1)x}{2(m+1)\sin^2\frac{x}{2}} = \frac{1}{m+1}\frac{\sin^2\frac{(m+1)x}{2}}{\sin^2\frac{x}{2}}$$

whenever $x \neq 2\pi n$. The value of F_m at $x = 2\pi n$ can be recovered as well.

The following properties of Fejer kernels hold:

a) F_m is a nonnegative function;

b) F_m is periodic function with the period 2π;

c) F_m id an even function;

d) $\int\limits_0^\pi F_m(x)dx = \pi$.

Theorem 4.3.2 *Let s_m be the mth partial sum defined in (3.1.3) of an integrable function f of period 2π. Define*

$$\sigma_m = \frac{1}{m+1}\sum_{k=0}^{m} s_k(x). \tag{4.3.7}$$

Then

$$\sigma_m = \frac{1}{2\pi}\int\limits_0^\pi (f(x-y)+f(x+y))\sum_{k=0}^{m} D_k(y)dy.$$

Proof. *Starting from*

$$\sigma_m = \frac{1}{m+1}\sum_{k=0}^{m} s_k(x)$$

and from (2.2.2), we have

$$\begin{aligned}\sigma_m &= \frac{1}{2\pi(m+1)}\int\limits_0^\pi ((f(x-y)+f(x+y))\sum_{k=0}^{m} D_k(y)dy\\ &= \frac{1}{2\pi}\int\limits_0^\pi ((f(x-y)+f(x+y))\frac{1}{m+1}\sum_{k=0}^{m} D_k(y)dy \qquad (4.3.7)\\ &= \frac{1}{2\pi}\int\limits_0^\pi ((f(x-y)+f(x+y))F_m(y)dy. \qquad (4.3.8)\end{aligned}$$

Which complates the proof. ∎

Theorem 4.3.3 *(Fejer's theorem) Let f be a continuous function on $[-\pi,\pi]$ and $f(-\pi) = f(\pi)$ and let σ_m be Cesaro sum defined by (4.3.7). Than σ_m converges to f uniformly on $[-\pi,\pi]$ as m goes to ∞.*

Proof. *Since f is continuouse function on $[-\pi,\pi]$, than f is uniformly continuouse on $[-\pi,\pi]$. The periodic extension of f to $\mathbb{R}$ with the period 2π is also uniformly continuous, since $f(-\pi) = f(\pi)$. Take any $x \in [-\pi,\pi]$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x+y) - f(x)| < \varepsilon/2$ whenever $|y| < \delta$. By the previous theorem and the properties of Fejer kernels,*

$$\sigma_m - f(x) = \frac{1}{2\pi}\int_0^\pi (f(x-y) + f(x+y) - 2f(x))F_m(y)dy.$$

Taking the absolute value, we have

$$|\sigma_m - f(x)| \leq \frac{1}{2\pi}\int_0^\pi |f(x-y) + f(x+y) - 2f(x)|\, F_m(y)dy$$

writing the integral in right hand side as the sum of two integrals on $[0,\delta]$ and $[\delta,\pi]$

$$|\sigma_m - f(x)| \leq \frac{1}{2\pi}\left(\int_0^\delta + \int_\delta^\pi\right)|f(x-y) + f(x+y) - 2f(x)|\, F_m(y)dy,$$

the first integral can be estimated as

$$\frac{1}{2\pi}\int_0^\delta |f(x-y) + f(x+y) - 2f(x)|\, F_m(y)dy \leq \frac{1}{2\pi}\frac{2\varepsilon}{2}\int_0^\pi F_m(y)dy \leq \frac{\varepsilon}{2}.$$

For the second integral, we consider $F_m(y)$ on $[\delta,\pi]$, letting $M = \max_{[-\pi,\pi]}|f|$, we *have*

$$\frac{1}{2\pi}\int_{\delta}^{\pi}|f(x-y)+f(x+y)-2f(x)|F_m(y)dy \le \frac{2M}{\pi}\int_{0}^{\pi}F_m(y)dy$$

Here $F_m(y)$ *converges uniformly to* 0 *since*

$$0 \le F_m(y) = \frac{1}{m+1}\frac{\sin^2\frac{(m+1)y}{2}}{\sin^2\frac{y}{2}} \le \frac{1}{(m+1)\sin^2\frac{\delta}{2}}, \quad \delta \le y \le \pi$$

Select $N \in \mathbb{N}$ *independent on* $x \in [-\pi,\pi]$, *such that*

$$\max_{[\delta.\pi]} F_m < \frac{\varepsilon}{4M},$$

for evry $m > N$

$$\frac{1}{2\pi}\int_{\delta}^{\pi}|f(x-y)+f(x+y)-2f(x)|F_m(y)dy \le \frac{2M}{\pi}\frac{(\pi-\delta)}{4M} < \frac{\varepsilon}{2},$$

finally

$$|\sigma_m(x)-f(x)| \le \frac{1}{2\pi}\int_{0}^{\pi}|f(x-y)+f(x+y)-2f(x)|F_m(y)dy < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This proves that σ_m *converges uniformly to* f *on* $[-\pi,\pi]$ *as* $m \to \infty$. ■

Corollary 4.3.4 *Let* f *be continuous function on* $[-\pi,\pi]$ *and* $f(-\pi) = f(\pi)$, *and let* σ_m *be defined as (4.3.1). Then* σ_m *converges to* f *in mean square sense.*

Proof. *The proof of this corollary follows from the fact that uniforme convergence* *implies mean square convergence. Indeed*

$$\begin{aligned} \|\sigma_m(x) - f(x)\|_{\tilde{C}} &= \sqrt{\int_{-\pi}^{\pi} (\sigma_m(x) - f(x))^2 \, dx} \\ &\le \sqrt{\left(\int_0^{\delta} \sup_{x \in [-\pi, \pi]} |\sigma_m - f(x)| \right)^2 dx} \\ &= \sup_{x \in [-\pi.\pi]} |\sigma_m(x) - f(x)| \sqrt{\int_{-\pi}^{\pi} dx} = \sqrt{2\pi} \|\sigma_m(x) - f(x)\|_C \end{aligned}$$

So,

$$\|\sigma_m(x) - f(x)\|_C \to 0 \ \ \textit{implies} \ \ \|\sigma_m(x) - f(x)\|_{\tilde{C}} \to 0$$

■

The method of summation we have been discussing is called Cesaro's method or method of the first arithmetic mean. If the arithmetic means do not converge, one might try taking the avarages of the first $2, 3, 4, ..., n$ arithmetic means, and seeing if this sequence converge. Now we will turn to another method, known as Abel's method or the method of convergence factors.

us suppos we are given a series

$$u_0 + u_1 + u_2 + ... \tag{3.2.8}$$

whose terms may be numbers or functions. Now we form a new series

$$u_0 + u_1 r + u_2 r^2 + u_3 r^3 + ... \tag{3.2.9}$$

If it should happen that (3.2.9) converges when r is in the interval $0 \leq r < 1$, and tends to a finite limit when $r \to 1$, then we call this limit the Abel sum of the series (3.2.8). As a simple example, let us sum (3.2.1) by the method of convergence factors. Multiplying the $n+1st$ term r^n, we obtain the series

$$1 - r + r^2 - r^3 + ... \tag{3.2.10}$$

which converges in the interval $(-1, 1)$ to $\frac{1}{1+r}$.Although the series does not converge at $r = 1$, the limiting value of $\frac{1}{(1+r)}$ as $r \to 1$ is $\frac{1}{2}$. Therefore Abel's sum of (3.2.10) is $\frac{1}{2}$.

As a less trivial example let us find the Abel sum of (3.2.5). As in the proceeding example, we form the series containing the convergence factors r^n, which in this case gives

$$\frac{1}{2}+\sum_{n=1}^{\infty} r^n \cos nx. \tag{3.2.11}$$

To write this in closed form, we observe that (3.2.11) is real part of the complex series

$$\frac{1}{2}+z+z^2+z^3+\ldots \qquad (z=re^{ix})$$

which converges for $|z|<1$ and has sum

$$\frac{1}{2}+\frac{z}{1+z}=\frac{1+z}{2(1-z)} \tag{3.2.12}$$

By a simple algebraic calculation, the real part of (3.2.12) is

$$\frac{1-r^2}{2(1-2r\cos x+r^2)} \tag{3.2.13}$$

and therefore, in interval $0 \le r < 1$, (3.2.11) converges,

$$\frac{1}{2}+\sum_{n=1}^{\infty} r^n \cos nx=\frac{1-r^2}{2(1-2r\cos x+r^2)}$$

as $r \to \infty$, this tends to zero, provided that x is in the interior of the interval (0.2π). Therefore (3.2.11) is Abel summable to zero in the interior of this interval. At the endpoints $x = 0$ and $x = 2\pi$,the series does not have an Abel sum.[1]

4.4. Complex Fourier Series

In this section we introduce a very important orthonormal system whose elements are complex valued. Here the inner product space is slightly different and is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx. \tag{4.4.1}$$

The set of all functions $\{e^{inx}\}_{n=-\infty}^{\infty}$ form an orthonormal system with respect to (4.4.1). For each $f \in E$ the appropriate series with this orthonormal system is goven

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{4.4.2}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n = 0, \pm 1, \pm, 2, \pm, ...$$

For $n = 1, 2, , ...$

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx + i\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx \\ &= \frac{a_n - ib_n}{2}, \end{aligned} \tag{4.4.3}$$

and

$$\begin{aligned} c_{-n} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx + i\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ &= \frac{a_n + ib_n}{2} \end{aligned} \tag{4.4.4}$$

from (4.4.3) and (4.4.4)

$$a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}).$$

Theorem 4.4.1 *Let* $\{\varphi_1, \varphi_2, ...\}$ *be orthonormal on E, and suppose that*

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \varphi_n(x).$$

Than

a) The series $\sum |c_n|^2$ *converges and satisfies the inequality*

$$\sum_{n=0}^{\infty} |c_n|^2 \leq ||f||^2 \quad (Bessel's\ inequality).$$

b)The equation

$$\sum_{n=0}^{\infty} |c_n|^2 \leq ||f||^2 \quad (Pars\mathrm{eval}'s\ identity),$$

holds, if and only if we also have

$$\lim_{n\to\infty} |f - S_n| = 0,$$

where $\{S_n\}$ is the sequence of partial sums defined by

$$S_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x).$$

Proof. a) Let

$$t_n(x) = \sum_{k=0}^{n} b_k \varphi_k(x) \quad \textit{and} \quad S_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x),$$

and

$$|f - t_n|^2 = \|f\|^2 - \sum_{k=0}^{n} |c_k|^2 + \sum_{k=0}^{n} |b_k - c_k|^2, \tag{4.4.5}$$

we take $b_k = c_k$ in (4.4.5) and observe that the left member is nonnegative

$$\sum_{k=0}^{n} |c_k|^2 \leq \|f\|^2.$$

b) To prove b), again we set $b_k = c_k$,

$$\|f - S_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} |c_k|^2,$$

letting $n \to \infty$, we have ■

Chapter 5

UNIFORM CONVERGENCE OF FOURIER SERIES

5.1. Piecewise Continuouse and Piecewise Smooth Functions

In this chapter we will set the problem of uniform convergence of Fourier series based on piecewise smooth functions, since the analogs of Dini, Lipchitz and Dirichlet-Jordan conditions for uniform convergence of Fourier series are known. By Theorem 3.1.3, a uniformly convergent Fourier series has a continuouse sum with the equal values at $-\pi$ and π. Therefore, our problem will be functions $f \in C(-\pi,\pi) \cap PS(-\pi,\pi)$ satisfying $f(-\pi) = f(\pi)$. A function $f : [-\pi,\pi] \to \mathbb{R}$ is said to be piecewise continuos if there is a partition $-\pi = x_0 < x_1 < ... < x_n = \pi$ such that f is continuose on every interval (x_{i-1}, x_i) and has one sided limits at points $x_0, x_1, ..., x_n$. The collection of all piecewise continuouse functions on $[-\pi,\pi]$ are denoted by $PC(-\pi,\pi)$. A function $f : [-\pi,\pi] \to \mathbb{R}$ is said t be piecewise smooth if $f \in PC(-\pi,\pi)$, there is a partition $-\pi = x_0 < x_1 < ... < x_n = \pi$ such that f is continuosly differentiable on each (x_{i-1}, x_i) and at points $x_0, x_1, ..., x_n$, f' has one sided limits (and they are finite), The collection of all piecewise smooth functions is denoted by $PS(-\pi,\pi)$.

Clearly piecewise functions has bounded variation. Indeed if $x_{i-1} \le a < b \le x_i$ than by mean value theorem

$$f(b) - f(a) = f'(c)(b - a)$$

which demonstrates that on every $[x_{i-1}, x_i]$, f has bounded variation. Than on the interval $[-\pi,\pi]$, f has also bounded variation

Lemma 5.1.1 *Let $f \in PC(-\pi,\pi) \cap PS(-\pi,\pi)$ satisfying $f(-\pi) = f(\pi)$, and let a_n and b_n be Fourier coefficients of f and let α_n and β_n be the Fourier coefficients of f'. Than $\alpha_0 = 0$, $\alpha_n = nb_n$ and $\beta_n = -na_n$, for $n \in \mathbb{N}$. Furthermore, the series $\sum_{n=1}^{\infty} \sqrt{\alpha_n^2 + \beta_n^2}$ converges.*

Proof. *From the fundamental theorem of calculus,*

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x)dx = \frac{1}{\pi}(f(\pi) - f(-\pi)) = 0.$$

From integration by parts and the fact that $f(-\pi) = f(\pi)$,

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx dx = \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = nb_n$$

and

$$\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx = -\frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = -na_n.$$

for all $n \in \mathbb{N}$, by Bessel's inequality, the sequence of partial sums of the Fourier series $\sum\limits_{n=1}^{\infty} \alpha_n^2 + \beta_n^2$ is bounded and increasing, therefore it converges. ■

Theorem 5.1.2 *Let $f \in C(-\pi,\pi) \cap PS(-\pi,\pi)$ satisfying $f(-\pi) = f(\pi)$. Than the Fourier series of f converges absolutely and uniformly on $[-\pi,\pi]$ to f.*

Proof. Let the Fourier series be given like (3.3.1), than

$$|a_n \cos nx + b_n \sin nx| \le |a_n| + |b_n|$$

since $f \in C(-\pi,\pi) \cap PS(-\pi,\pi)$ and $f(-\pi) = f(\pi)$ the series $\sqrt{2(a_n^2 + b_n^2)}$ converges. So the Fourier series of f converges absolutely and uniformly on $[-\pi,\pi]$. After all the sum of Fourier series of f equals to f. ■

5.2. Term by Term Integration and Differentiation

Theorem 5.2.1 *(Term by term integration) Let the Fourier series of* $f \in PC(-\pi,\pi)$ *be given by (3.1.1). Than*

$$\int_0^x f(y)dy = \frac{a_0}{2}x + \sum_{n=1}^{\infty}\left(a_n \int_0^x \cos ny dy + b_n \int_0^x \sin ny dy\right),$$

where the convergence is absolute and uniform on $[-\pi,\pi]$.

Proof. Let

$$F(x) = \int_0^x \left(f(y) - \frac{a_0}{2}\right)dy, \qquad -\pi \le x \le \pi.$$

$F \in C(-\pi,\pi) \cap PS(-\pi,\pi)$, since $F \in C(-\pi,\pi)$ and $F' \in PS(-\pi,\pi)$. Next we see whether $F(-\pi) = F(\pi)$,

$$\begin{aligned} F(\pi) &= \int_0^{\pi} \left(f(y) - \frac{a_0}{2} \right) dy = \frac{1}{2} \left(\int_0^{\pi} f(y)dy - \int_{-\pi}^{0} f(y)dy \right) \\ &= \int_0^{\pi} \left(f(y) - \frac{a_0}{2} \right) dy = F(-\pi). \end{aligned}$$

By the previous theorem, Fourier series of F converges absolutely and uniformly to F, so we can write

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx).$$

And since $a_n = nB_n$ and $b_n = -nA_n$. Letting $x = 0$, we find

$$\frac{A_0}{2} = -\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Hence,

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \frac{a_n \sin nx + b_n(1 - \cos nx)}{n} \\ &= \sum_{n=1}^{\infty} \frac{a_n \sin nx}{n} + \frac{b_n(1 - \cos nx)}{n} \\ &= \sum_{n=1}^{\infty} \left(a_n \int_0^{\pi} \cos nx dx + b_n \int_0^{\pi} \sin nx dx \right), \end{aligned}$$

which proves the theorem. ■

Remark 5.2.2 *Theorem of term by term integration is valid even if the Fourier series* *of f is divergent. On other hand, the term by term differentiation of Fourier series requires stronger conditions.*

Theorem 5.2.3 *(Term by term differentiation) Let $f \in C(-\pi,\pi) \cap PS(-\pi,\pi)$ satisfying $f(-\pi) = f(\pi)$. Than,*

$$f'(x) \sim \sum_{n=1}^{\infty} (a_n (\cos nx)' + b_n (\sin nx)'),$$

where the series converges absolutely and uniformly on $[-\pi,\pi]$ to f' if $f' \in C(-\pi,\pi) \cap PS(-\pi,\pi)$ and $f'(-\pi) = f'(\pi)$.

Proof. Let

$$f'(x) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos(nx) + \beta_n \sin(nx))$$

Since $\alpha_0 = 0$, $\alpha_n = nb_n$ and $\beta_n = -na_n$ for all $n \in \mathbb{N}$. And hence,

$$f'(x) \sim \sum_{n=1}^{\infty} (nb_n \cos(nx) - na_n \sin(nx)) = \sum_{n=1}^{\infty} (a_n (\cos nx)' + b_n (\sin nx)').$$

$f' \in C(-\pi,\pi) \cap PS(-\pi,\pi)$ and $f'(-\pi) = f'(\pi)$, than Fourier series of f' converges uniformly and absolutely to f'. ■

5.3. Weierstrass Approximation Theorem

In this section we present a theorem due to Weierstrass , which states that although not all continuous functions can be presented by their Fourier series, all of them can be approximated by trigonometric polinomials even in uniform sense.

Theorem 5.3.1 *Trigonometric polynomials can be approximated uniformly by polynomials in any interval of finite length.*

Proof. *First we should state that a trigonometric polynomial is a linear combination of functions of the form* $A_n \cos nx$ *and* $B_n \sin nx$*. Trigonometric functions* $\cos nx$ *and* $\sin nx$ *have power series expansions that converges for all x. This means that every trigonometric plynomial has a power series expansion that converges for all x. So the partial sums of such a power series converge uniformly in any interval of finite length. Each of these partial sums is a polynomial. It follows that any trigonometric polynomial can be approximated uniformly by a polynomial in such an interval.* ■

Theorem 5.3.2 *[1] Every continuous function can be approximated uniformly by a piecewise smooth continuous function in any closed interval of finite length.*

Proof. *(Outline of proof) Every continuous function* f*, defined in the interval* $a \leq x \leq b$*, can be approximated by a broken line function. To see this, we subdivide the interval into n parts, which for convenience can be taken equal in length:* $a = x_0 < x_1 < ... < x_n = b$*. Then, we construct a broken-line function* W_n *by joining the successive points* $(x_0, f(x_0)), (x_1, f(x_1)), ..., (x_n, f(x_n))$ *with line segments; the resulting graph defines* W_n *in the interval. From the continuity of* f*, it is obvious that* $W_n(x) \to f(x)$ *for every x in the interval. It is somewhat less obvious, but nonetheless true, that* W_n *converges uniformly to* f *in the interval.* ■

Theorem 5.3.3 *[1] Every continuous function having period* 2π *can be approximated uniformly by trigonometric polynomials(in any interval).*

Proof. *Let $0 \leq x \leq 2\pi$. All the functions W_n constructed in this manner have the property $W_n(0) = W_n(2\pi)$ in this case. Let the slopes of the n line segments be $k_1, k_2, ..., k_n$, and let $K = \max|k_i|$. Now from the mean value theorem we have*

$$\left|W_n(x') - W_n(x'')\right| \leq K\left|x' - x''\right|.$$

The period 2π extension of W_n also has this property, and the class of such functions uniformly approximate f (which also has period 2π by hypothesis) in any interval. So W_n can be approximated by trignometric polynomials. ■

Theorem 5.3.4 *(Weierstrass Theorem) Every $f \in C(-\pi,\pi)$ with $f(-\pi) = f(\pi)$ can be approximated uniformly by trigonometric polynomials of the form*

$$\sigma_n(x) = \alpha_{n,0} + \sum_{k=1}^{n} (a_{n,k} \cos kx + b_{n,k} \sin kx).$$

Proof. *Let $f \in C(-\pi,\pi)$. Then f is uniformly continuous, that is ($\forall\varepsilon > 0$, $\exists\delta_\varepsilon > 0$ $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$), select partitition of $-\pi = x_0 < x_1 < ... < x_n = \pi$ so that $\max(x_i - x_{i-1}, i = 1, ..., n) < \delta$. Now consider a piecewise linear function $\varphi(x)$ on $[-\pi,\pi]$ which is obrained stright by stright joining the points*

$$(x_0, f(x_0)),\ (x_1, f(x_1)),\ ...,\ (x_n, f(x_n))$$

Clearly, $\varphi(x)$ is piecewise smooth, continuous and $\varphi(-\pi) = \varphi(\pi)$. Therefore, φ converges uniformly to φ. Let $T_n(x)$, n-th partial sum of the Fourier series of $\varphi(x)$. Than *there exists N such that* $\forall n > N$

$$\left|\varphi_n(x) - T_n(x)\right| < \frac{\varepsilon}{2}, \qquad -\pi \leq x \leq \pi$$

also

$$|\varphi(x) - f(x)| < \frac{\varepsilon}{2}$$

Combining these two inequalities, we obtain

$$|f(x) - Tn(x)| < \varepsilon, \qquad -\pi \leq x \leq \pi.$$

■

5.4. Gibbs Phenomenon

In section 4.1 we state the sufficient conditions under which the Fourier series converges uniformly on $[-\pi,\pi]$ to its function . Uniform convergence is the best possible convergence. However does not always hold. In this section we will consider the situations where the function is not uniform convergent.

Let

$$-\pi = d_1 < d_2 < \ldots < d_n = \pi$$

denote the jump points of f in $[-\pi,\pi]$, where f is 2π peiodic function satisfying the conditions of Dirichlet's Theorem. We proved that under certain conditions the Fourier series of f converges uniformly on every subinterval of $[-\pi,\pi]$ which does not contain any of these points. But the points $d_k, 1 \leq k \leq n$, something rather odd occurs which is called the ”Gibbs phenomenon”. This phenomenon was noted by physicist A.Michelson at the end of the nineteenth century. He built a ”machine” which could calculate some initial Fourier coefficients of a graphically given function f. He noticed that the praphs of ”*good*” functions (those functions satisfying the conditions in Theorem 3.1.3) the graphs of the partial sum of Fourier series where close to the function f. Bur, for $f(x) = sgn(x)$ the graph of partial sums estimates a large error in the neighbourhood of $x = 0$ and $x = \pm\pi$ independent of the number of termes in partial sum. It was discovered first by Wilbraham in 1848, but later studied in detail by Gibbs in 1898.

To see this issue, we consider this example.Let

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0, \\ 0, & x = \pi, \\ 1, & 0 < x \leq \pi. \end{cases}$$

This is a piecewise smooth and odd function. By the definition, its Fourier series has the form

$$\sum_{n=1}^{\infty} b_n \sin nx,$$

and it converges to $f(x)$ pointwise, but not uniformly.

Calculations shows that

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx = \frac{2}{n\pi}(\cos 0 - \cos n\pi) = \frac{2(1-(-1)^n)}{n\pi} = \begin{cases} \frac{4}{n\pi}, & n \ is \ odd, \\ 0, & n \ is \ even. \end{cases}$$

So,

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

Denote by s_{n-1} the $(2n-1)st$ partial sum of Fourier series of f :

$$s_{2n-1}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

$s'_{2n-1}(x)$ is odd and we can restrict our self to study positive values of x. We are interested in the local maximum of $s_{2n-1}(x)$. Taking the derivative

$$s'_{2n-1}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \cos(2k-1)x.$$

Or

$$\begin{aligned} s'_{2n-1}(x) &= \frac{4}{\pi \sin x}\sum_{n=1}^{\infty} \sin x \cos(2k-1)x \\ &= \frac{2}{\pi \sin x}\sum_{n=1}^{\infty} (\sin 2kx - \sin 2(k-1)x) \\ &= \frac{2\sin 2nx}{\pi \sin x}, \end{aligned}$$

which implies that the equation $s'_{2n-1}(x) = 0$ has solution if $x = \pi m/2n$. The solution which is more close to 0 if $x = \pi/2n$. The second derivative

$$s''_{2n-1}(x) = \frac{2(2n\cos 2nx \sin x - \sin 2nx \cos x)}{\pi \sin^2 x}.$$

Hence, at $x = \pi/2n$,

$$s''_{2n-1}(\pi/2n) = -\frac{4}{\pi \sin^2 \frac{\pi}{2n}} < 0.$$

For $x = \pi/2n$, $s_{2n-1}(x)$ takes its local maximum, so we have to estimate

$$s_{2n-1}(\pi/2n) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2k-1)\frac{\pi}{2n}}{2k-1}.$$

Later we observe that the Riemann sum $S(g,\rho)$ of the function $g(x) = \frac{\sin x}{x}$ for the partition

$$\rho = \{0, \pi/n, 2\pi/n, ..., (n-1)\pi/n, \pi\}$$

of the interval $[0,\pi]$ with tags selected to be the center of each partition interval equals to

$$S(g,\rho) = \frac{\pi}{n} \sum_{n=1}^{\infty} \frac{\sin(2k-1)\frac{\pi}{2n}}{\frac{(2k-1)\pi}{2n}} = \frac{\pi}{2} s_{2n-1}(\pi/2n).$$

Hence

$$\zeta = \lim_{n\to\infty} s_{2n-1}(\pi/2n) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x} dx = \frac{2}{\pi} \mathrm{Si}(\pi),$$

which exceed the value $f(0+) = 1$.

The Gibbs phenomenon is valid for every piecewise smooth function. If a function is $PS(-\pi,\pi)$ has a discontinuity at $c \in [-\pi,\pi)$ with the jump $d = |f(c+) - f(c-)| > 0$ and $a = \frac{1}{2}(f(c+) - f(c-))$, then

$$\limsup_{n\to\infty} s_n(c) = a + \frac{d}{\pi}\operatorname{Si}(\pi) \text{ and } \liminf_{n\to\infty} s_n(c) = a - \frac{d}{\pi}\operatorname{Si}(\pi),$$

where $\operatorname{Si}(\pi)$ is called a Wilbraham-Gibbs constant.

Chapter 6

FOURIER INTEGRALS

6.1. A Fourier Integral Formula

In this chapter we will deal with the convergence of infinite integrals. The concepts of infinite series have their counterparts in the theory of infinite integrals. The word ”infinite” here refers to the length of the interval over which we are integrating. Those integrals are called ”improper integrals of the first kind” to differ them from integrals of unbounded functions which are known as ”improper integrals of the second kind”. Let us consider a function f which is integrable over (a, x) for all values of $x \geq a$. We then define the integral

$$\int_a^\infty f(x)dx = \lim_{x\to\infty} \int_a^x f(t)dt.$$

If this limit exists, the integral on the left is said to converge and is assigned the value of the limit, in the same way that we assign a number to an infinite series if its sequence of partial sums of f converges.

Now we assume that f is a function on $[-l, l]$, where $l > 0$. Then $g(x) = f(lx/\pi)$ defines a function g on $[-\pi, \pi]$. Assuming that the Foourier series of g exists, we can use the inverse substitution and create a series for f on $[-l,l]$ in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right),$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \qquad \text{and} \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx.$$

Now, we consider

$$r_1 = \frac{\pi}{l},\ r_2 = \frac{2\pi}{l}, ..., r_n = \frac{n\pi}{l}, ...$$

where the values of r are in $[0,\infty)$. Letting $\triangle r = r_{n+1} - r_n = \frac{\pi}{l}$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos r_n x + b_n \sin r_n x),$$

where

$$a_n = \frac{\triangle r}{\pi}\int_{-l}^{l} f(x)\cos r_n x dx \qquad \text{and} \qquad b_n = \frac{\triangle r}{\pi}\int_{-l}^{l} f(x)\sin r_n x dx.$$

So

$$f(x) \sim \frac{1}{2l}\int_{-l}^{l} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{\infty} \triangle r \int_{-l}^{l} f(y)(\cos r_n y \cos r_n x + \sin r_n y \sin r_n x)dy,$$

In the case if the improper integrals are convergent we can move l to ∞. Then the first term in the right hand side approches 0 and the second term transforms to an integral.

Thus, we obtain

$$f(x) \sim \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(y)(\cos ry \cos rx + \sin ry \sin rx) dy dr$$

or in other form

$$f(x) \sim \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(y) \cos r(y-x) dy dr. \tag{6.1.1}$$

This formula is called Fourier integral of f. The right hand side can be interpreted as

$$\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(y) \cos r(y-x) dy dr$$

if the improper integral

$$\int_{-\infty}^{\infty} f(y) \cos r(y-x) dy \tag{6.1.2}$$

converges for $r \geq 0$, and $x \in \mathbb{R}$.

6.2. Uniform Convergence of Fourier Integrals

There are different conditions on convergence of Fourier integrals. We will state the analogs of Dini's, Lipchitz and Dirichlet-Jordan conditions. But first we need some helpful results .

Theorem 6.2.1 *Let f be absolutely integrable on $\mathbb{R}$.Then the improper integral in*

(6.1.1) converges absolutely and uniformly for $r \geq 0$ *and* $x \in \mathbb{R}$ *and for* $\lambda > 0$,

$$\int_0^{\lambda}\int_{-\infty}^{\infty} f(y)\cos r(y-x)dydr = \int_0^{\infty}(f(x-y)+f(x+y))\frac{\sin\lambda y}{y}dy. \qquad (6.2.1)$$

Proof. *Easily we find that*

$$|f(y)\cos r(y-x)| \leq |f(y)|,$$

which means that (6.1.2) converges absolutely and uniformly for $r \geq 0$ *and* $x \in \mathbb{R}$.*Therefore*

$$\begin{aligned}\int_0^{\lambda}\int_{-\infty}^{\infty} f(y)\cos r(y-x)dydr &= \int_{-\infty}^{\infty}\int_0^{\lambda} f(y)\cos r(y-x)dydr \\ &= \int_{-\infty}^{\infty} f(y)\frac{\sin\lambda(y-x)}{y-x}dy \\ &= \int_{-\infty}^{\infty} f(x+y)\frac{\sin\lambda y}{y}dy,\end{aligned}$$

where

$$\begin{aligned}\int_{-\infty}^{\infty} f(x+y)\frac{\sin\lambda y}{y}dy &= \int_{-\infty}^{0} f(x+y)\frac{\sin\lambda y}{y}dy + \int_0^{\infty} f(x+y)\frac{\sin\lambda y}{y}dy \\ &= \int_0^{\infty}(f(x-y)+f(x+y))\frac{\sin\lambda y}{y}dy.\end{aligned}$$

■

Theorem 6.2.2 *Let* f *be absolutley integrable on* $\mathbb{R}$. *Then*

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{\sigma}^{\infty}(f(x-y)+f(x+y))\frac{\sin\lambda y}{y}dy = 0,$$

for $\sigma > 0$. Additionally, if

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^{\sigma}(f(x-y)+f(x+y))\,\frac{\sin\lambda y}{y}dy, \tag{6.2.2}$$

exists for some $\sigma > 0$, than the Fourier integral of f at x converges to this value.

Proof. *For fixed $x \in \mathbb{R}$,*

$$g(y) = \frac{f(x-y)+f(x+y)}{y}$$

is absolutely integrable on $[\sigma,\infty)$ since

$$|g(y)| \leq \frac{1}{\sigma}\left(|f(x-y)|+|f(x+y)|\right).$$

Hence, by Riemann-Lebesgue Lemma, we obtain

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^{\infty}\frac{(f(x-y)+f(x+y))}{y}\sin\lambda y dy = 0.$$

Using this fact in (6.2.1), we obtain

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^{\lambda}\int_{-\infty}^{\infty}f(y)\cos r(y-x)dydr &= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^{\infty}\frac{(f(x-y)+f(x+y))}{y}\sin\lambda y dy\\ &= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^{\sigma}\frac{(f(x-y)+f(x+y))}{y}\sin\lambda y dy.\end{aligned}$$

This proves the theorem. ∎

Theorem 6.2.3 *(Dini) Let f be abolutely integrable on $\mathbb{R}$. If*

$$\int_0^\sigma \left|\frac{f(x-y)+f(x+y)-2s}{y}\right| dy < \infty$$

for some $\sigma > 0$ and $s \in \mathbb{R}$, where the integral is a proper or improper Riemann integral, then the Fourier integral of f converges at x to s.

Proof. *From the fact that*

$$\begin{aligned} & \frac{1}{\pi}\int_0^\lambda \int_{-\infty}^{\infty} f(y)\cos r(y-x)\,dy\,dr - s \\ = & \frac{1}{\pi}\int_0^\infty (f(x-y)+f(x+y))\frac{\sin \lambda y}{y}dy - s \\ = & \frac{1}{\pi}\int_0^\infty \frac{f(x-y)+f(x+y)-2s}{y}\sin \lambda y\,dy. \end{aligned} \qquad (6.2.3)$$

Later, we write the integral as

$$\frac{1}{\pi}\left(\int_0^\sigma + \int_\sigma^\infty\right)\frac{f(x-y)+f(x+y)-2s}{y}\sin \lambda y\,dy$$

The first factor is absolutely integrable on $[0,\sigma]$ by Dini's condition, the second factor is Riemann integrable on $(0,\sigma)$. Hence their product is absolutely integrable on $[0,\sigma]$. Hence

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\sigma \frac{f(x-y)+f(x+y)-2s}{y}\sin \lambda y\,dy = 0 \qquad (6.2.4)$$

On the other hand

$$\frac{1}{\pi}\int_{\sigma}^{\infty}\frac{f(x-y)+f(x+y)-2s}{y}\sin\lambda y dy$$
$$\frac{1}{\pi}\int_{\sigma}^{\infty}\frac{f(x-y)+f(x+y)}{y}\sin\lambda y dy-\frac{2s}{\pi}\int_{\sigma}^{\infty}\frac{\sin\lambda y}{y}dy.$$

By the previous theorem, the first term converges to 0 *as* $\lambda\to\infty$, *and the second term can be written as*

$$\frac{2s}{\pi}\int_{\sigma}^{\infty}\frac{\sin\lambda y}{y}dy=\frac{2s}{\pi}\int_{\lambda\sigma}^{\infty}\frac{\sin y}{y}dy,$$

it is clear that it converges to 0 *as* $\lambda\to\infty$. *Thus*

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{\sigma}^{\infty}\frac{f(x-y)+f(x+y)-2s}{y}\sin\lambda y dy=0. \tag{6.2.5}$$

From (6.2.3), (6.2.4), (6.2.5), we obtain

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{0}^{\lambda}\int_{-\infty}^{\infty}f(y)\cos r(y-x)dydr=s$$

■

Theorem 6.2.4 *(Lipchitz) Let* f *be absolutely integrable on* $\mathbb{R}$. *If there are constants* $L\geq 0,\ 0\leq\alpha\leq 1$ *and* $\sigma>0$ *such that*

$$|f(x+y)-f(x)|\leq L|\alpha|^{\alpha}$$

for all $|x-y|<\sigma$, *then the Fourier integral of* f *converges at* x *to* $f(x)$

Proof. The proof is identically as in Chapter 3. ■

Theorem 6.2.5 *(Dirichlet-Jordan) Let f be absolutely integrable on $\mathbb{R}$. If f has a bounded variation on some interval $[x-\sigma, x+\sigma]$, than its Fourier integral at x converges to*

$$\frac{f(x-)+f(x+)}{2}.$$

Proof. *It suffices to evaluate the limit in (6.2.2). Since f has a bounded variation on $[x-\sigma, x+\sigma]$, the function*

$$g(y) = f(x-y)+f(x+y)$$

has a bounded variation on $[0,\sigma]$, hence by (2.5.1),

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^{\sigma}\frac{f(x-y)+f(x+y)}{y}\sin\lambda y dy = \frac{f(x-)+f(x+)}{2},$$

this proves the theorem. ■

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