Real spectra for non-Hermitian Dirac equation in 1+1 dimensions with a most general coupling

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March 23, 2009

Abstract

The most general combination of couplings of fermions with external potentials in 1+1 dimensions, must include vector, scalar and pseudoscalar potentials. We consider such a mixing of potentials in a PT-symmetric time-independent Dirac equation. The Dirac equation is mapped into an effective PT-symmetric Schrödinger equation. Despite the non-hermiticity of the effective potential, we find real energies for the fermion.

PACS numbers: 03.65.Ge, 03.65.Pm, 03.65.Fd

1 Introduction

In the last ten years the, so-called, PT symmetric systems introduced in the seminal paper by Bender and Boettcher [2] have attracted very much attention. In fact, there are many works devoted to develop and understand this new kind of situation [1, 19]. They consist of non-Hermitian Hamiltonians with real eigenvalues, which however exhibit parity and time-reversal symmetries. The one-dimensional time-independent Schrödinger equation is invariant under space-time inversion. In addition, there exist other classes of Hamiltonians with real spectra without being PT symmetric, as can be seen, for instance, in the references [5, 6]; and systems where the PT symmetry is spontaneously broken with complex energy eigenvalues [7]. The problem of non-Hermitian time-dependent interactions which does not exhibit PT symmetry but still admits real energies is considered in the references [9]-[12].

More recently, some problems of relativistic fermions interacting with non-Hermitian potentials of scalar and vector natures have been reported in the literature [1],[19]-[29]. In general, it has been shown that for some configurations of those non-Hermitian potentials, the Dirac equation admits real energies. In 1+1 dimensions the Dirac problem is easily mapped into a Sturm-Liouville problem or, in other words, into a time-independent Schödinger equation with real or complex potentials whose bound-state solutions present real energy eigenvalues. Some authors [13]-[18] have investigated the Klein-Gordon equation and Dirac equation in the context of PT symmetry and pseudo-Hermiticity. Mustafa [13] studied the exact energies for Klein-Gordon particle and Dirac particle in the generalized complex Coulomb potential. Znojil [14] analyzed the Klein-Gordon equation presenting a pseudo-Hermiticity. Egrifes et al [15] investigated the bound states of the Klein-Gordon and Dirac equations for the one-dimensional

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generalized Hulthén potential within the framework of PT-symmetric quantum mechanics. In [17], the authors have investigated the bound states for the PT-symmetric versions of the Rosen-Morse and Scarf II potentials in the Klein-Gordon equation with equally mixed vector and scalar potentials. Sinha and Roy [18] investigated the one-dimensional solvable Dirac equation with non-Hermitian scalar and pseudoscalar couplings, possessing real energy spectra.

The coupling with scalar potentials in the Klein-Gordon and Dirac equations can be seen as a position-dependent effective mass. In the relativistic ambiance, the ordering ambiguity of the mass and momentum operators, which is present in the non-relativistic one, should disappear. Nevertheless, there are difficulties to define consistently fermions and bosons, whenever one takes into account space-time dependent masses. This happens due to the fact that physical particles in quantum field theory must belong to an irreducible representation of the Poincarè algebra [30, 31]. One should be able to find generators specifying the particle properties, usually its mass and helicity. However, it is quite hard to accomplish with this task in the case of spatially dependent masses. Thus, one should keep in mind that all of these usually thought as relativistic equations for position-dependent masses should be taken as effective equations. In this regard, Alhaidari [32] studied the exact solution of the three-dimensional Dirac equation for a charged particle with spherically symmetric singular mass distribution in the Coulomb field. Vakarchuk [33] investigated the exact solution of the Dirac equation for a particle whose potential energy and mass are inversely proportional to the distance from the force center. In [26], the authors considered the smooth step mass distribution and solved approximately the one-dimensional Dirac equation with the spatially dependent mass for the generalized Hulthén potential. In [34], the authors investigated the exact solution of the one-dimensional Klein-Gordon equation with the spatially dependent mass for the inversely linear potential. Therefore, it is of considerable interest to investigate the solution of the effective mass Klein-Gordon and Dirac equations with non-Hermitian complex potentials with real energy spectra.

One interesting problem that has been tackled in this context is the Dirac equation in 1+1 dimensions in the presence of a convenient complex vector potential plus a real scalar potential [1], wherein the scalar potential plays the role of a position dependent mass. Here, we will show that one can realize a more general system of massless fermions in two dimensions interacting with a mixing of complex vector, scalar and pseudoscalar potentials. Although complex, the scalar and pseudoscalar potentials are responsible to open a mass gap for the fermions. Such a scenario might be important for some condensed matter systems, where the electrical conduction is essentially one-dimensional. The scalar and pseudoscalar potentials can be thought as defects in the lattice and the electrons are also subject to a background potential of vector nature due to the ions in the lattice.

Our purpose in this work is to show that the Dirac equation in a two-dimensional world can still have real discrete energy spectrum and supports fermion bound-states when a convenient mixing of complex vector, scalar and pseudoscalar potentials is considered. We call attention to the transformation of the potentials under parity in order to have the PT-symmetry in the Dirac equation. Particular configurations of those potentials are worked out in some detail. The approach here is the mapping of the PT-symmetric Dirac problem into a naturally PT-symmetric effective Schrödinger equation.

2 The time-independent Dirac equation in 1+1 dimensions

We consider here the 1+1 dimensional time-independent Dirac equation for a massless fermion under the action of a general potential $\mathcal{V}$. It is written as

$$H\Psi(x) = E\Psi(x)\,, \tag{1}$$

$$H = c\,\alpha\,p + \mathcal{V}\,, \tag{2}$$

where E is the energy of the fermion, c is the velocity of light and p is the momentum operator. α and β are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1$, $\{\alpha, \beta\} = 0$. The positive definite function $|\Psi|^2 = \Psi^\dagger\Psi$, satisfying a continuity equation, is interpreted as a position probability density. This interpretation is completely satisfactory for single-particle states.

We set $\mathcal{V}$ to be

$$\mathcal{V} = \beta\,M(x) + \beta\,\gamma_5\,P(x) + V(x) + \beta\,\alpha\mathcal{A}(x) \tag{3}$$

where $M(x)$ is a scalar potential, $P(x)$ a pseudoscalar potential and $V(x)$ is the time-component of a Lorentzian 2-vector potential, whose space component is $\mathcal{A}(x)$. The space component of the 2-vector potential can be eliminated by a gauge transformation without affecting the physics. Once we have only four linearly independent 2×2 matrices, the structure of coupling in $\mathcal{V}$ is the most general one can consider in the time-independent Dirac equation in one space dimension.

In terms of the potentials the Hamiltonian (2) becomes

$$H = c\,\alpha\,p + V(x) + \beta\,M(x) + \beta\,\gamma^5\,P(x)\,, \tag{4}$$

where $\gamma^5 = -i\alpha$. An explicit expression for the α and β matrices can be chosen from the Pauli matrices that satisfy the same algebra. We use $\beta = \sigma_1$, $\alpha = \sigma_3$, and thus $\beta\gamma^5 = -\sigma_2$. The equation (1) can be decomposed in two coupled first-order differential equations, for the upper, $\psi_+(x)$, and lower, $\psi_-(x)$, components of the spinor $\Psi(x)$. In a simplified notation and by using the natural system of units $\hbar = c = 1$, we have

$$\begin{aligned} -i\psi'_+ + M\psi_- + V\psi_+ + iP\psi_- &= E\psi_+\ , \\ +\,i\psi'_- + M\psi_+ + V\psi_- - iP\psi_+ &= E\psi_-\ . \end{aligned} \tag{5}$$

where the prime stands for the derivative with respect to x.

In the case that all the potentials are real functions, the Dirac equation is Hermitian and invariant under space-reversal (parity) transformation. We recall that the parity transformation is an improper Lorentz transformation and that the spinor in one frame is constructed from the spinor in the other frame by means of the relation $\tilde{\Psi}(\tilde{x}, t) = S\Psi(x, t) = e^{i\delta}\beta\Psi(x, t)$, with $\tilde{x} = -x$ and δ an overall constant phase factor. Moreover, under parity transformation, $M(x)$ and $V(x)$ do not change and $P(x)$ changes its sign. The matrices must transform as $S^{-1}\beta S = \beta$ and $S^{-1}\alpha S = \alpha$.

The same transformations could be used even in the case that the potentials are complex, but if we want the Dirac equation invariant under the combination of parity and time-reversal transformations this issue becomes trickier with a non-Hermitian Hamiltonian. This is because the time-reversal transformation implies that $\mathcal{T}(i)\mathcal{T}^{-1} = -i$ and that the potentials in the Hamiltonian are complex. Thus, althougt the time-reversal does not change each part of the potentials, since they are time-independent, it changes the relative sign between the real and the imaginary parts of the potentials; as a consequence, the imaginary part of each one of the potentials must change under parity in the reversed form of its real part, in order to have the Dirac equation invariant under the combination of parity and time-reversal transformations. In summary, in order to have PT-symmetry even when the potentials are complex, the imaginary part of the vector and scalar potentials must change their signs under parity, whereas the imaginary part of the pseudoscalar potential does not change. The spinor in the time-reversed system is obtained from the spinor $\tilde{\Psi}(\tilde{x},t)$ by means of the following the transformation $\tilde{\Psi}_{\mathcal{T}}(\tilde{x},\tilde{t}) = \mathcal{T}\tilde{\Psi}(\tilde{x},t) = T\tilde{\Psi}^{*}(\tilde{x},t)$, with $\tilde{t}=-t$ and T a square matrix such that $T^{-1}\beta^{*}T=\beta$ and $T^{-1}\alpha^{*}T=-\alpha$. Then T must commute with β and anti-commute with α, that is $T\equiv\beta$.

All the examples of potentials we are going to deal with follow the transformation rule given above.

3 The effective PT symmetric problem

Whenever one considers only the coupling either with the scalar or the pseudoscalar potential, the differential equations can be uncoupled in such a way that both components of the spinor satisfy second-order differential equations, similar to each other and to the Schrödinger equation. By including the coupling with the vector potential this is no longer possible. Although, it is possible to show that one of the components obeys a kind of Schrödinger equation, and the other component is given in terms of the previous one, as was done in the references in [1]. From now on, we are going to follow that approach. By applying the space derivative in the first of the equations (5) we have

$$-i\psi_{+}'' + A_{+}\psi_{-}' + A_{+}'\psi_{-} = B\psi_{+}' + B'\psi_{+} \tag{6}$$

where we have defined $A_{\pm}\equiv A_{\pm}(x)=M(x)\pm iP(x)$ and $B\equiv B(x)=E-V(x)$. By substituting in the above equation the expressions for ψ_{-}' and ψ_{-} taken from equations (5), we obtain the following equation for the upper component

$$-i\psi_{+}'' + i\frac{A_{+}'}{A_{+}}\psi_{+}' + \left[\frac{BA_{+}'}{A_{+}} - B' + i(A_{+}A_{-} - B^{2})\right]\psi_{+} = 0, \tag{7}$$

and the equation obeyed by the lower component can be rewritten as

$$\psi_{-} = \frac{1}{A_{+}}\left(i\psi_{+}' + B\psi_{+}\right). \tag{8}$$

We notice that by means of the redefinition of the upper component

$$\psi_{+}(x) \equiv \sqrt{A_{+}}\chi(x)\ , \tag{9}$$

the first-derivative term can be eliminated and χ satisfies the differential equation

$$-i\chi'' + \left\{\frac{BA_{+}'}{A_{+}} - B' + i\left(A_{+}A_{-} - B^{2}\right) + i\left[\frac{3}{4}\left(\frac{A_{+}'}{A_{+}}\right)^{2} - \frac{1}{2}\frac{A_{+}''}{A_{+}}\right]\right\}\chi = 0, \tag{10}$$

which, in its turn, can be written as a Schrödinger, equation

$$-\chi'' + V_{eff}\chi = E^{2}\chi, \tag{11}$$

where

$$V_{eff} = E\left(2V + i\frac{A_{+}'}{A_{+}}\right) - iV' + i\left(\frac{A_{+}'}{A_{+}}\right)V + \left[A_{+}A_{-} - V^{2} + \frac{3}{4}\left(\frac{A_{+}'}{A_{+}}\right)^{2} - \frac{1}{2}\left(\frac{A_{+}''}{A_{+}}\right)\right]. \tag{12}$$

Equations (7)-(12) constitute the essential set of equations we are going to deal with throughout this work. Our intention is to find convenient configurations of the potentials for which the equation (11) presents normalizable solutions and positive eigenvalues or, in other words, bound-state solutions corresponding to a discrete energy spectrum, with $E^2 \geq 0$. Naturally, the solutions for the lower component that satisfies equation (8) must also be normalizable.

In the literature we find several studies of the Dirac equation in a two-dimensional world with convenient mixing of scalar plus vector potentials and pseudoscalar plus vector potentials. As far as we know, most of them assume an intrinsic relation between the scalar and vector potential, or between the pseudoscalar and the vector potential. A typical case is the one which deals with a constraint of the type $M(x) = \kappa V(x)$, where $|\kappa| > 0$ and $M(x)$ is a binding potential [35]. There are some cases in four dimensions for which such convenient relation leads to analytical and exact solutions for the Dirac equation. Here, we are not going to escape from a constraint among the potentials. On the contrary, we follow the same procedure carried out in the references [1] by constraining the vector potential to obey the more general relation

$$V = \frac{i}{2}\left(\frac{A_+'}{A_+}\right) + \widetilde{V} = \frac{i}{2}\left(\frac{M' + iP'}{M + iP}\right) + \widetilde{V}. \tag{13}$$

With this relation the Schrödinger-like equation (11) is given by

$$-\chi'' + \left[M^2 + P^2 - \widetilde{V}^2 + 2E\widetilde{V} \; - i\widetilde{V}'\right]\chi = E^2\chi \; . \tag{14}$$

The effective potential written explicitly in terms of the real and imaginary parts of the potentials

$$\begin{aligned} V_{eff}(x) \;\; &= \;\; \left(M_r^2 - M_i^2 + P_r^2 - P_i^2 - \widetilde{V_r}^2 + \widetilde{V_i}^2 + 2E\widetilde{V_r} \; + \widetilde{V_i}'\right) + \\ &\quad +2i\left(M_r M_i + P_r P_i - \tilde{V}_r\tilde{V}_i + 2E\tilde{V}_i - \tilde{V}_r'\right) \end{aligned} \tag{15}$$

is complex, but the effective Schrödinger equation exhibits PT symmetry if M_r, $\tilde{V}_r$ and P_i remains invariant, whereas M_i, $\tilde{V}_i$ and P_r changes their signs under parity transformation, as was stated in the second section. Naturally we are thinking of the cases for which the energy in the Dirac problem is real (positive or negative).

4 Real spectrum with non-Hermitian potentials

As our first example, we choose a set of potentials which have a quadratic dependence on the spatial variable and, in order to deal with a complete set of exact eigenstates, we impose some restrictions over the potential parameters. In this first example we choose

$$M(x) = a_0x^2, \;\; P(x) = i(a_1x^2 + b_1), \;\; \tilde{V}(x) = a_2x^2 \tag{16}$$

In this situation, the vector potential is given by

$$V = i\left[\frac{(a_0 - a_1)x}{a_0x^2 - (a_1x^2 + b1)}\right] + a_2x^2. \tag{17}$$

Consequently the effective potential is

$$Veff = (a_0^2 - a_2^2 - a_1^2)x^4 + (2Ea_2 - 2a_1b_1)x^2 - b_1^2 - 2ia_2x. \tag{18}$$

As we asserted, we want to deal with an exactly solvable model. We restrict the potential parameters in order to eliminate the quartic term, obtaining the constraint $a_0 = \sqrt{a_1^2 + a_2^2}$. This lead us to the following effective PT-symmetric driven harmonic potential

$$Veff = \Omega^2 x^2 - 2ia_2 x - b_1^2 \tag{19}$$

where $\Omega^2 = 2Ea_2 - 2a_1 b_1$. By means of the transformation $x = y + 2ia_2/\Omega^2$ we obtain the following differential equation for $\chi(y)$

$$-\frac{d^2\chi}{dy^2} + \Omega^2 y^2 \chi = \left[E^2 + b_1^2 - \frac{a_2^2}{\Omega(E)^2}\right]\chi, \tag{20}$$

that is a quantum harmonic oscillator of unity mass. Consequently the energy E of the relativistic system is obtatined from the following equation

$$E^2 = -b_1^2 + \frac{a_2^2}{\Omega(E)^2} + (2n+1)\,\Omega(E)\,. \tag{21}$$

The above equation is considerably simplified if we set $b_1 = 0$, that is

$$E^2 = \frac{a_2}{2\,E} + (2\,n+1)\sqrt{2\,E\,a_2},$$

which can be written as

$$\left(E^3 - \frac{a_2}{2}\right) = (2\,n+1)\,E\sqrt{2\,E\,a_2},$$

and, once squared, lead us to

$$e^2 - a_2\left[1 + 2\,(2\,n+1)^2\right]\,e + \frac{a_2^2}{4} = 0,$$

where we have defined $e \equiv E^3$, leading finally to the solutions

$$E_\pm = \left(\frac{a_2}{2}\left\{1 + 2\,(2\,n+1)^2 \pm \sqrt{\left[1 + 2\,(2\,n+1)^2\right]^2 - 1}\right\}\right)^{\frac{1}{3}}.$$

One can note that it is not necessary the presence of $\tilde{V}(x)$ in order to have an effective PT-symmetric Schrödinger equation with a complex effective potential. As a matter of fact one can see that the dependence of $\tilde{V}$ on the coordinate implies in an effective potential explicitly dependent on the energy of the Dirac problem and, as a consequence one will reach invariably an intricate transcendental equation for the energy. Henceforth, we set $\tilde{V}$ a real constant.

As an example, we suggest the following configurations for the scalar and the pseudoscalar potentials

$$M(x) = i\omega_1 x + m_1,\;\; P(x) = \omega_2 x + im_2 \tag{22}$$

with ω_1, m_1, ω_2 and m_2 real constants. The vector potential, with $\widetilde{V} = 0$, is complex and given by

$$V(x) = -\frac{(\omega_1 - \omega_2)\left[(m_1 - m_2) + i(\omega_1 + \omega_2)x\right]}{2\left[(\omega_1 + \omega_2)^2 x^2 + (m_1 - m_2)^2\right]}, \tag{23}$$

and the effective potential in the Schrödinger equation obeyed by $\chi(x)$

$$V_{eff}(x) = \omega^2 x^2 + 2i(\omega_1 m_1 + \omega_2 m_2)x + (m_1^2 - m_2^2), \tag{24}$$

with $\omega^2 = \omega_2^2 - \omega_1^2$, is PT-symmetric.

By defining the variable $y = x - i(\omega_1 m_1 + \omega_2 m_2)/\omega^2$ we obtain

$$-\frac{d^2\chi}{dy^2} + \omega^2\left(y^2 + \lambda^2\right)\chi = E^2\chi\ , \tag{25}$$

whose non-normalized eigenfunctions and the eigenvalues are respectively given by

$$\chi_n(y) = e^{-\omega y^2/2} H_n(\sqrt{\omega}y) \tag{26}$$

and

$$E^2 = \omega(2n+1) + \lambda^2\omega^2, \tag{27}$$

where $H_n(\sqrt{\omega}y)$ is the Hermite polynomial of degree n and $\lambda^2 = \left(\frac{\omega_1 m_2 + \omega_2 m_1}{\omega^2}\right)^2$. Consequently, the upper component of the Dirac spinor can be written as

$$\psi_{+,n}(y) = N\sqrt{\omega}(y^2+\lambda^2)^{1/4} e^{-\frac{1}{2}(\omega y^2 - i\theta(y))} H_n(\sqrt{\omega}y), \tag{28}$$

where $\theta(y) = \tan^{-1}(y/\lambda)$, and N is a normalization constant. By recalling the equation (8) and the expression for the vector potential, $V(x) = iA'_+/2A_+$, we get the following equation for the lower component of the spinor

$$\psi_-(y) = \frac{1}{\sqrt{A_+(y)}}\left(i\frac{d\chi}{dy} + E\chi\right), \tag{29}$$

which can be written in terms of Hermite polynomials as

$$\psi_{-,n}(y) = \frac{N}{\sqrt{\omega}(y^2+\lambda^2)^{1/4}} e^{-\frac{1}{2}(\omega y^2 + i\theta(y))}\left[\left(\pm|E_n| - i\omega y\right) H_n(\sqrt{\omega}y) + i2n\sqrt{\omega}H_{n-1}(\sqrt{\omega}y)\right]. \tag{30}$$

The upper (lower) sign of the factor $\pm|E_n|$ in the expression above stands for the positive (negative) energy solution and $|E_n| = \sqrt{\omega(2n+1) + \lambda^2\omega^2}$. The normalization constant N is obtained by the definition of the norm of the eigenstates

$$\int_{-\infty}^{+\infty} |\Psi_n|^2 dx = \int_{-\infty}^{+\infty} (|\psi_{+,n}(y)|^2 + |\psi_{-,n}(y)|)^2 dy = 1, \tag{31}$$

which can be written in terms of the variable $u = \sqrt{\omega}y$ as

$$2\,|N|^2 \int_0^{+\infty} du\ \frac{e^{-u^2}}{(u^2+\lambda^2\omega)^{1/2}}\left\{\left[2(u^2+\lambda^2\omega) + (2n+1)\right] H_n^2(u) - 2nH_{n-1}(u)H_{n+1}(u)\right\} = 1, \tag{32}$$

where we have used the recursion relation $2uH_n(u) = 2nH_{n-1}(u) + H_{n+1}(u)$. It is very hard to check the convergence of the above integral, for all values of n, even though each of them converges, since the weight function decays faster, for $x \to \pm\infty$, than any polynomial appearing in the numerator of the integrand. We furnish below the normalization constant for the two lowest energy levels,

$$\begin{aligned} N_0 &= \left\{e^{-\lambda^2\omega/2}(1+2\lambda^2\omega)K_0(\lambda^2\omega/2) + \sqrt{\pi}U(1/2,0,\lambda^2\omega)\right\}^{-1/2}, \\ N_1 &= \left\{e^{-\lambda^2\omega/2}\left[2(1+\lambda^4\omega^2)K_0(\lambda^2\omega/2) + 2\lambda^2\omega(1-\lambda^2\omega)K_1(\lambda^2\omega/2)\right] + \right. \\ &\quad \left. +\sqrt{\pi}(1+2\lambda^2\omega)U(1/2,0,\lambda^2\omega)\right\}^{-1/2}, \end{aligned}$$

where $K_0(z/2)$ and $K_1(z/2)$ are modified Bessel functions of the second kind and $U(a,b,z)$ is the confluent hypergeometric function.

As a third example we take the scalar and pseudoscalar potentials given respectively by

$$M = iM_1 \tanh(\mu x) + M_0, \;\; P = P_1 \tanh(\mu x) + iP_0, \tag{33}$$

with M_1, P_1, M_0, P_0 reals and constants. Thus, the vector potential is also complex and, with $\tilde{V} = V_0$, is given by,

$$V = -\frac{\mu}{2} \frac{(P_1 + M_1)\; [(M_0 - P_0) - i(M_1 + P_1)\tanh(\mu x)]\,\mathrm{sech}(\mu x)}{(M_0 - P_0)^2 + (M_1 + P_1)^2 \tanh^2(\mu x)} + V_0, \tag{34}$$

and the effective Schrödinger equation (14) can be written as

$$-\chi'' - V_{eff}\chi = \varepsilon\chi \;, \tag{35}$$

where

$$V_{eff}(x) = -(P_1^2 - M_1^2)\mathrm{sech}^2(\mu x) + 2i(M_0 M_1 + P_0 P_1)\tanh(\mu x) \tag{36}$$

and

$$\varepsilon = (E - V_0)^2 - (P_1^2 - M_1^2) + (P_0^2 - M_0^2). \tag{37}$$

This effective potential is a generalization of the real Rosen-Morse II (hyperbolic) [38, 39] and was already considered in the literature concerning PT symmetry in non-Hermitian Schrödinger problems [5],[6]. The eigenvalues of the effective energy can be conveniently written as

$$\varepsilon_n = -\mu^2(a_n^2 - b_n^2), \tag{38}$$

where

$$a_n = (1/2)\left[\sqrt{1 + (2U_1/\mu)^2} - (2n+1)\right] \;\; \text{and} \;\; b_n = \Omega/\mu^2 a_n, \tag{39}$$

with $U_1^2 = P_1^2 - M_1^2$, $\Omega = M_0 M_1 + P_0 P_1$ and $n = n = 0, 1, 2, ... < s = (1/2)\left[\sqrt{1 + (2U_1/\mu)^2} - 1\right]$. Thus, the energy eigenvalues of the Dirac problem are given by the following relation

$$E_n = V_0 \pm \sqrt{U_1^2 - U_0^2 - \mu^2(a_n^2 - b_n^2)}, \tag{40}$$

where $U_0^2 = P_0^2 - M_0^2$. In order to have a real spectra, the relation $U_1^2 - U_0^2 \geq \mu^2(a_n^2 - b_n^2)$ among the parameters of the potential must be imposed. It worths mentioning that this kind of restriction is not a consequence of a complex potential in the effective Schrödinger equation. It appears, instead, due to the kind of coupling and to the configuration of the potentials we have considered in the relativistic problem.

If the potential supports $n_{\max} + 1$ bound states, then we can say that $s = n_{\max} + \delta$, with $0 < \delta < 1$. We have observed that, for $\Omega^2 = \mu^4\delta^4$, the factor $\mu^2(a_n^2 - b_n^2)$ decreases in the range $[0, n_{\max}]$. The maximum value achieved by $\mu^2(a_n^2 - b_n^2)$ is $\mu^2(s^2 - \delta^4/s^2)$. Therefore, the relations

$$\Omega^2 = \mu^4\delta^4, \; U_1^2 - U_0^2 > \mu^2\left[(n_{\max} + \delta)^2 - \frac{\delta^4}{(n_{\max} + \delta)^2}\right] \tag{41}$$

are necessary to have a real spectrum. In the case we have only one bound state for example, corresponding to $n_{\max} = 0$, we obtain $M_0 M_1 + P_0 P_1 = \mu^2\delta^2$, $P_0^2 - M_0^2 < P_1^2 - M_1^2 = \mu^2\delta(\delta + 1)$.

In general the upper component of the spinor is given by

$$\psi_{+,n}(z) = N_+|A_+(z)|^{1/2}e^{i\alpha(z)/2}(1-z^2)^{a_n/2}\left(\frac{1-z}{1+z}\right)^{b_n} P_n^{(a_n+b_n,a_n-b_n)}(z), \qquad (42)$$

where, $z = \tanh(\mu x)$, $P_n^{(\alpha,\beta)}(z)$ are the Jacobi polynomials of degree n [42]. On its turn the lower component can be written as

$$\begin{aligned}\psi_{-,n}(z) &= N_-|A_+(z)|^{-1/2}e^{-i\alpha(z)/2}\mu(1-z^2)^{a_n/2}\left(\frac{1-z}{1+z}\right)^{b_n}\times \\ &\times\left[(E_n - a_n z - 2b_n - V_0)P_n^{(a_n+b_n,a_n-b_n)} + (1-z^2)\frac{d}{dz}P_n^{(a_n+b_n,a_n-b_n)}\right]. \qquad (43)\end{aligned}$$

In the expressions (42) and (43) we have $|A_+(z)| = \sqrt{(M_0 - P_1 z)^2 + (M_1 z + P_0)^2}$ and $\alpha(z) = \tan^{-1}((M_1 z + P_0)/(M_0 - P_1 z))$.

The normalization constants are found by the definition of the norm, namely

$$\int_{-1}^{+1} dz\ (1-z^2)^{-1}(|\psi_{+,n}|^2 + |\psi_{-,n}|^2) = 1.$$

5 Final considerations

We have extended the problem of fermions in 1+1 dimensions interacting with complex potentials by considering the most general coupling of fermions which is possible in the time-independent Dirac equation in two dimensions. The potentials are considered to be of scalar, pseudoscalar and vector natures. We have shown that one can map the Dirac equation with all those complex potentials into a PT-symmetric effective Schrödinger equation, which can be exactly solved and present real spectra. Indeed, the PT-symmetry is also observed already in the Dirac equation when the imaginary part of each one of the potentials transforms, under parity transformation, in the reversed manner that the real part of the potential. In this way, the PT-symmetry seems to play a fundamental role for the reality of the spectrum. There still remains the open question about pseudo-hermiticity of the Dirac operator with the mixing of potentials we have considered.

Another remarkable point of those examples, besides being exactly solvable, is that they bind fermions despite the presence of the vector potential. Inhomogeneous vector potentials give rise to electric fields which are responsible for pair production. Based on what happens when we have a convenient mixing of scalar and vector potential [35], we think that the presence of the scalar and pseudoscalar potentials, although complex, are responsible for giving an effective mass to the fermion and, as a consequence, the threshold for pair production becomes higher than the energy that the electric field can supplies to the fermion. Confinement of fermions with a vector potential, as in the second example treated here, is a peculiar case. Indeed, one can check that the relativistic spectra we have found is very similar to that of the "Dirac oscillator" in 1+1 dimensions [43]. In that reference a massive fermion is coupled to a linear potential. That coupling, being of a pseudoscalar type, is the two-dimensional version of the four-dimensional coupling of an anomalous magnetic moment with an electric field which emerges in the "Dirac oscillator" [44]-[47], and that can be seen as a mechanism for confinement of fermions in 3+1 dimensions.

We have noted that in the absence of the scalar and pseudoscalar couplings, the effective equation for the upper component of the spinor is given by

$$-\psi_+'' + \left[2\ i\ V' - (E-V)^2)\right]\psi_+ = 0,$$

that is, we have a PT-symmetric effective Schrödinger equation with a vector potential which is real and even under parity transformation. This is a situation where the PT-symmetry becomes quite important, because it opens the possibility of having real spectra in a model which, otherwise, would be discarded due to the fact that the effective Schrödinger equation corresponds to a non-Hermitian operator. This opens the promising perspective of finding exactly solvable real potentials where this feature could be accomplished.

Acknowledgments:

This work was supported in part by means of funds provided by CAPES, CNPq and FAPESP. This work was partially done during a visit (ASD) within the Associate Scheme of the Abdus Salam ICTP.

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