Existence of Solutions of Fractional Differential Equations with Mixed Caputo-Riemann Derivative

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Eastern Mediterranean University
September 2025
Gazimağusa, North Cyprus

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ABSTRACT

Fractional differential equation research is gaining prominence due to its extensive applications in various fields. Fixed-point theory, variational methods, and the method of lower and upper solutions are among the techniques employed to study the existence and uniqueness of solutions for fractional differential equations. The existence and uniqueness of solutions to a category of nonlinear fractional differential equation with mixed Caputo–Riemann fractional derivatives with integral initial conditions within a Banach space are examined in this thesis. The existence and uniqueness of the solutions are provided with sufficient conditions based on the parameters of the problem. These results are obtained by forming the Green's function for the initial value problem. The existence is proved through the help of Schauder's fixed-point theorem, while uniqueness is ensured by Banach's contraction mapping principle. Finally, an example is provided to illustrate the application of the results.

Keywords: Caputo and Riemann fractional derivatives, Schauder's fixed point theorem, Banach contraction mapping Principle.

ÖZ

Kesirli diferensiyel denklemler (KDD) üzerine yapılan araştırmalar, çok çeşitli alanlardaki uygulamaları sayesinde giderek daha fazla önem kazanmaktadır. Sabit nokta kuramı, varyasyonel yöntemler ve alt-üst çözümler yöntemi, kesirli diferensiyel denklemlerin çözüm varlığı ve tekliğinin incelenmesinde kullanılan başlıca teknikler arasındadır. Bu tezde, integral başlangıç koşullarına sahip karışık Caputo–Riemann kesirli türevleri içeren doğrusal olmayan bir KDD sınıfının Banach uzayında çözüm varlığı ve tekliği incelenmektedir. Çözümlerin varlığı ve tekliği, problemin parametrelerine bağlı yeterli koşullar altında sağlanmaktadır. Bu sonuçlar, başlangıç değer problemine ait Green fonksiyonu oluşturularak elde edilmiştir. Çözümün varlığı Schauder sabit nokta teoremi ile ispatlanırken, tekliği ise Banach sabit daralma ilkesiyle güvence altına alınmıştır. Son olarak, elde edilen sonuçların uygulanabilirliğini göstermek amacıyla bir örnek verilmiştir.

Anahtar Kelimeler: Caputo ve Riemann kesirli türevleri, Schauder sabit nokta teoremi, Banach daralma ilkesi.

... Dedicated to X

To loved ones...

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my supervisors, Prof. Dr. Nazım Mahmudov and Prof. Dr. Sonuç Zorlu, for their invaluable guidance, continuous support, and insightful feedback throughout the course of this research. Their expertise, patience, and encouragement were instrumental in shaping this work and in overcoming many challenges along the way.

Special thanks are extended to my colleagues and friends for their moral support, collaboration, and for creating a productive and enjoyable research atmosphere.

I am forever grateful to my family for their unconditional love, support, and encouragement throughout my academic journey. Without their unwavering belief in me, this thesis would not have been possible.

To the math itself the discipline that humbles me, frustrates me, yet endlessly fascinates me thank you for revealing your secrets slowly and beautifully.

And lastly, to the quiet nights and the sleepless mornings, the Turkish coffee and the existential crises, the fear, the fire, the sheer will thank you. You all shaped not just this thesis, but the man who wrote it.

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LIST OF SYMBOLS

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
${}_{c}^{R}I_{x}^{\alpha}$	RL operator
${}_{c}^{C}D_{x}^{\alpha}$	Caputo derivative operator
Γ	Gamma function

Chapter 1

INTRODUCTION

1.1 Introduction

The development of fractional differential equations has its conceptual roots in a question posed by Leibniz in 1695 regarding the meaning of a half-order derivative. Though initially it was a mathematical curiosity, the field matured gradually. Liouville's work was basic in the 1800s. Riemann's work during that century was foundational also. The Progress within the 20th century saw Caputo's derivative formulation. Relevant to initial value problems in physics and engineering [1, 2]. These Modern fractional calculus was built upon more advancements. fractional differential equationss, when modeling abilities are considered, have gained much traction recently. properties often found in actual systems that involve heredity and memory World excluding solutions classical differential equations inadequately address. Applications have It flourished in both biology and chemistry and mechanics with diverse disciplines that are like physics. Economics is one discipline in addition to biophysics another [3, 4]. Both are important. These equations describe phenomena with effectiveness. Complex systems do exhibit long-memory processes with viscoelasticity. Anomalous diffusion can be another characteristic that is found in such systems [5–7]. Research focus is Mixed Fractional Derivative, especially combinations within an active area. For processes with multiple memory scales, models use Caputo and Riemann-Liouville operators. Such frameworks apply to

systems exhibiting local and nonlocal. behavior. This is shown in studies with a fractional Hamiltonian approach for the Bateman–Feshbach–Tikochinsky system like Dumitru Baleanu et al. [8]. line of inquiry. Blaszczyk [9] tackled fractional oscillator equations in non-resisting media. Klimek [10] did develop symmetric fractional derivatives. Media was also a subject of their research since they derived results. Hamiltonian and Euler–Lagrange formulations correspond. These contributions Under classical mechanics, Mixed Fractional Derivative-based models converge to show they can. suitable limits. Actual problems show the relevance in practice of theoretical models. Dynamics that are both nonlocal including memory effects do govern. For instance, anomalous It is viscoelastic deformation within porous or heterogeneous materials. Gene, engineering materials, and charge transport within electrochemical systems. Fractional differential equations may all successfully depict regulation within biological networks using. Mixed Fractional Derivatives. The practical relevance of these theoretical models is evident in real-world problems governed by memory effects and nonlocal dynamics. For instance, anomalous diffusion in porous or heterogeneous materials, viscoelastic deformation in engineering materials, charge transport in electrochemical systems, and gene regulation in biological networks can all be described effectively using fractional differential equations with Mixed Fractional Derivatives. In economics, such equations capture long-memory behavior in asset volatility, and in thermal engineering, they enhance the modeling of heat conduction and control systems [11–19].

Despite their wide-ranging applicability, solving fractional differential equations

analytically remains a formidable challenge. As a result, much research has concentrated on establishing conditions under which solutions exist, are unique, and are stable. Techniques such as fixed-point theorems [20, 21], Ulam-type stability methods [18, 22], and numerical schemes [23] have all been developed to address these challenges. Furthermore, the consideration of integral initial or boundary conditions more reflective of certain physical systems has attracted growing interest [24, 25]. More recently, fractional differential equations involving mixed derivatives have become a key focus, as seen in [26, 27].

Stability theory, particularly Ulam-Hyers and Ulam-Hyers-Rassias stability, plays a central role in assessing how solutions behave under perturbations. Ibrahim [28] studied U stability for the fractional Cauchy problem in the complex unit disk, while Chen et al. [29] investigated U–H stability for Caputo-type nabla fractional differential equations on bounded intervals. Other researchers have expanded on these ideas in the context of fractional differential equations with various fractional operators [30–34].

This thesis contributes to this evolving landscape by examining the existence, uniqueness, and stability of solutions to a nonlinear fractional differential equation involving mixed Caputo–Riemann fractional derivatives under integral initial conditions, set in a Banach space such that:

$$ {}^{R}D^{b}({}^{C}D^{a}u(x) + h\,{}^{C}D^{a-1}u(x)) = g(x,u(x)), \quad x \in I = [0,1], \quad h \neq 0 \tag{1.1} $$

with the following integral initial conditions:

$$u(0) = 0, \quad u'(0) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} u(s) ds, \tag{1.2}$$

$$u''(0) = \frac{1}{\Gamma(b)} \int_0^1 (1-s)^{b-1} u(s) ds,$$

where $a \in (1,2)$, $b \in (0,1]$, $\gamma \in (0,1]$, and $f : I \times \mathbb{R} \to \mathbb{R}$. Here, ${}^{C}D^{a}$ and ${}^{R}D^{b}$ denote the Caputo and Riemann fractional derivatives, respectively. To verify the practical utility of our main findings, we provide an illustrative example.

The thesis is organized as follows: Chapter 2 provides preliminary definitions, lemmas, and notations from fractional calculus. Chapter 3 presents the formulation of the main problem, focusing on nonlinear fractional differential equations with mixed derivatives and integral conditions. In Chapter 4, we establish existence and uniqueness results via Schauder and Banach fixed-point theorems. Chapter 5 is devoted to the analysis of Ulam–Hyers and Ulam–Hyers–Rassias stability. Chapter 6 includes a worked-out example that supports the theoretical results. Chapter 7 concludes the thesis and suggests potential future research directions. Chapter 8 introduces a prospective study of a more general class of Hadamard-type fractional differential equations involving a composition of fractional operators and nonlocal integral conditions. The goal is to establish existence results under more relaxed assumptions and explore the role of advanced fixed-point methods in solving such problems.

Chapter 2

PRELIMINARIES

This section provides the essential definitions, notations, and lemmas related to fractional calculus that will be used throughout the thesis. These foundational results serve to establish the framework for the analysis presented in the subsequent chapters.

2.1 Fractional Calculus

Fractional calculus, which extends the concept of integer-order differentiation and integration to arbitrary (non-integer) orders, has become a powerful tool in modeling complex phenomena in various fields such as physics, engineering, biology, and finance. Unlike classical calculus, fractional derivatives and integrals capture memory and hereditary properties of processes, making them particularly suitable for describing anomalous diffusion, viscoelastic materials, and other systems with nonlocal behavior.

Research on fractional calculus has attracted much interest because of the capability of fractional calculus in describing memory and hereditary characteristics that are possessed by various materials and processes. Of the various forms of fractional derivatives, the Hadamard fractional operator has the peculiarity of having a logarithmic kernel, which is analytically advantageous in models that possess multiplicative structures. In this chapter, we present the definition, properties, and applications of Hadamard-type fractional derivatives and integrals. The objective is to

establish a firm theoretical background prior to using these tools for solving complicated boundary value problems in the subsequent parts of the thesis.

Among the various definitions of fractional operators, the Riemann-Liouville and Caputo formulations are the most widely used due to their well-established mathematical properties and applicability. The Riemann-Liouville fractional integral and derivative provide a foundational framework, while the Caputo fractional derivative is often preferred in initial value problems because of its more natural treatment of initial conditions.

In this section, we recall the essential definitions of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative and the Caputo fractional derivative, which form the basis of the fractional differential equation considered in this work.

Definition 2.1: [1, 2] (Riemann-Liouville Fractional Integral) Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. The Riemann-Liouville fractional integral of order α of a function $g(t)$, with respect to the variable x and lower limit c, is defined by:

$$ {}_c^R I_x^\alpha g(x) := \frac{1}{\Gamma(\alpha)} \int_c^x (x-s)^{\alpha-1} g(s)\, ds, $$

provided the integral exists for the given function g and points x and c.

Definition 2.2: [1, 2] (Riemann-Liouville Fractional Derivative) Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$. The Riemann-Liouville fractional derivative of order α of a function $g(x)$, taken with respect to x and starting from the point c, is given by:

$$ {}_c^R D_t^\alpha g(x) := \frac{d^m}{dt^m} \left({}_c^R I_x^{m-\alpha} g(x) \right), \quad \text{where } m = \lfloor \mathrm{Re}(\alpha) \rfloor + 1, $$

assuming the expression is well-defined for the chosen g, x, and c.

Definition 2.3: [4](Caputo Fractional Derivative).

Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$. The Caputo fractional derivative of a function $g(x)$ of order α, starting from the point c, is defined as:

$$ {}_c^C D_x^\alpha g(x) = {}_c^R I_x^{m-\alpha} \left(\frac{d^m}{dt^m} g(x) \right), \quad \text{where } m = \lfloor \mathrm{Re}(\alpha) \rfloor + 1, $$

provided the right-hand side is well-defined for the function g, and the variables x and c.

Fractional differentiation and integration generalize classical calculus to orders that are non-integer, providing greater versatility in describing dynamical systems. The Hadamard operators, in contrast to their Riemann–Liouville or Caputo analogs, are based upon logarithmic functions in their definition and are naturally adapted to problems set on unbounded or semi-bounded domains. Formal definitions of the Hadamard fractional integral and derivative are given in this section, along with examples illustrating their calculation and application in analytical contexts.

The Hadamard fractional integral allows one to accumulate the influence of a function over a logarithmically scaled interval. This is especially applicable in mathematical physics and engineering where logarithmic time or space scales are dominant. In this subsection, we give the formal definition of the operator and calculate an explicit

example for a logarithmic input function, showing how the integral enhances the memory-like behavior of the system.

Definition 2.4: [1] [35](Hadamard Fractional Integrals) Let $0 < a < b \leq \infty$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and $f \in L^1_{\text{loc}}((a,b))$. The Hadamard fractional integrals of order α are defined for $x \in (a,b)$ by

$$({}_aI^{\alpha}_H f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\, dt,$$

$$({}_HI^{\alpha}_b f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t}\, dt,$$

Lemma 2.1: [1] If $\alpha, \beta > 0$, then

$$\left(I^{\alpha}_a \left(\log \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\log \frac{x}{a}\right)^{\beta-\alpha-1}$$

Lemma 2.2: [4] Let $\alpha \in [0,\infty)$, $\beta \in [0,\infty)$, and $g(x)$ in $L_1[a,b]$.

Then $${}^R I^{\alpha}\, {}^R I^{\beta} g(x) = {}^R I^{\alpha+\beta} g(x)$$

almost everywhere on the closed interval $[a,b]$.

Furthermore, if the function $g(x)$ is continuous in $[a,b]$, then this composition property holds $\forall x \in [a,b]$.

Lemma 2.3: [36] Let $\nu > 0$, $m = \lceil \nu \rceil$ and let $g(s) \in L(a,b)$. If ${}^R_c D^{\nu-1} g$ exists and $g \in AC[a,b]$, then ${}^R_c D^{\nu-i} g = k_i$ exists for $i = 1,2,...,m$; ${}^R_c D^{\nu} g$ exists almost everywhere on $[a,b]$ is in $L(a,b)$ and

$$ {}_c^R I^\nu \quad {}_c^R D^\nu g(s) = g(s) - \sum_{i=1}^{m} \frac{k_i (s-c)^{\nu-i}}{\Gamma(\nu-i+1)}, $$

everywhere on $s \in [a,b]$

Moreover, if $g(s)$ is continuous on $(a,b]$, then the inequality is satisfied everywhere on $(a,b]$.

Lemma 2.4: [37] Let $\nu > 0$. Assuming that $u \in C(0,1) \cap L_1(0,1)$, the Caputo fractional differential equation

$$ D^\nu u(x) = 0 $$

has the solution

$$ u(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} $$

where $b_i \in \mathbb{R},\ i = 0,1,2,...,m-1$, and $m = \lfloor \nu \rfloor + 1$.

Lemma 2.5: [37] Let $u \in C(0,1) \cap L_1(0,1)$, $\nu > 0$ and ${}^C D^\nu u(x) \in C(0,1) \cap L_1(0,1)$. Then

$$ {}^R I^\nu \, {}^C D^\nu u(x) = u(x) + b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-1} x^{m-1}, $$

for $b_i \in \mathbb{R}, i = 0,1,2,...,m-1$, where $m = \lceil \nu \rceil$.

Lemma 2.6: [4, 36] Let $a,b,c \in \mathbb{R}, b > -1$. Then for all $x > c$,

$$ {}_c^R I^a \frac{(x-c)^b}{\Gamma(b+1)} = \begin{cases} \frac{(x-c)^{a+b}}{\Gamma(a+b+1)} & a+b \neq \text{negative integers} \\ 0 & a+b = \text{negative integers} \end{cases} $$

2.2 Beta and Gamma Functions and Their Properties

The Beta and Gamma functions are fundamental special functions that frequently appear in various branches of mathematics, particularly in the study of fractional calculus, differential equations, probability theory, and complex analysis. These

functions provide generalizations of the factorial function to non-integer arguments and are essential in defining other advanced mathematical constructs such as the Mittag-Leffler function.

2.2.1 Gamma Function

The Gamma function extends the concept of factorial to real as well as complex numbers. arguments save non-positive integers.

Definition 2.5: The Gamma function is defined for complex numbers with a positive real part as:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad \Re(x) > 0. \tag{2.1}$$

For positive integers n, the Gamma function satisfies:

$$\Gamma(n) = (n-1)!.$$

2.2.1.1 Properties of the Gamma Function

- **Recurrence Relation:**

$$\Gamma(x+1) = x\Gamma(x).$$

- **Value at $\frac{1}{2}$:**

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

- **Reflection Formula:**

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad x \notin \mathbb{Z}$$

- **Multiplication Formula (Gauss's Multiplication Formula):**

$$\Gamma(nx) = \frac{(2\pi)^{\frac{1-n}{2}}}{n^{nx-\frac{1}{2}}} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right).$$

The Gamma function plays a key role in fractional differential equation due to its connection to fractional-order integrals and derivatives.

2.2.2 Beta Function

The Beta function, also known as Euler's integral of the first kind, is closely related to the Gamma function and is frequently used in the evaluation of integrals involving products of powers.

Definition 2.6: The Beta function is defined for $x > 0$ and $y > 0$ by:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

2.2.2.1 Properties of the Beta Function

- **Symmetry:**

$$B(x,y) = B(y,x)$$

- **Relation to the Gamma Function:**

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

- **Reduction Formula:**

$$B(x+1,y) = \frac{x}{x+y} B(x,y)$$

- **Value when arguments equal 1:**

$$B(1,y) = \frac{1}{y}$$

2.2.2.2 Connection Between Beta and Gamma Functions

The Beta and Gamma functions are connected via the identity:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.2}$$

This connection is necessary within many fractional calculus expressions simplifying and probability theory.

2.2.2.3 Applications

In fractional calculus:

- The Gamma function is widely used in defining fractional integrals and derivatives such as the Riemann-Liouville and Caputo derivatives.
- The Beta function is essential in the evaluation of integrals and in the solution of fractional differential equations.

Both functions utilize statistical distributions as well, like beta distribution and gamma distribution.

2.2.2.4 Summary Table

Table 2.1: Summary of Beta and Gamma Functions

Function	Definition	Key Properties
Gamma Function	$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$	$\Gamma(x+1) = x\Gamma(x), \Gamma(n) = (n-1)!, \Gamma(1/2) = \sqrt{\pi}$
Beta Function	$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$	$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, B(x,y) = B(y,x)$

2.3 Green's Function

Green functions are important in the solution of fractional differential equations as they yield an explicit form of the solution to such intricate problems. Fractional differential equations differ from their classical counterparts in that they contain derivatives of non-integer order, which describe memory and hereditary effects of various processes. The Green function serves as a basic solution that contains the system's response to impulsive inputs, which facilitates the reduction of the fractional differential equation to an integral equation. In so doing, the analysis is simplified while providing a more intuitive picture of the behavior of solutions, making it amenable to investigate the implications of fractional operators in various applied disciplines like physics, engineering, and finance.

Definition 2.7: Consider a linear differential operator L acting on a function $y(t)$, such that:

$$L[y(t)] = f(t),$$

where $f(t)$ is a given continuous function.

The **Green's function** $G(t,s)$ associated with the operator L is defined as the solution to the equation:

$$L[G(t,s)] = \delta(t-s)$$

where $\delta(t-s)$ is the Dirac delta function.

The solution to the non-homogeneous fractional differential equation can then be expressed in term of the Green's function asfollows :

$$y(t) = \int_a^b G(t,s) f(s)\, ds.$$

2.3.1 Properties of Green's Function

- **Linearity:** Green's function is associated with linear differential operators.
- **Symmetry:** For self-adjoint operators, the Green's function satisfies $G(t,s) = G(s,t)$.
- **Causality:** In physical systems, the Green's function often satisfies $G(t,s) = 0$ for $t < s$, which reflects causality.
- **Dependence on Boundary Conditions:** The form of the Green's function strongly depends on the type of boundary conditions imposed on the problem (Dirichlet, Neumann, or mixed).

2.3.2 Green's Function in fractional differential equation

In fractional differential equation, the Green's function plays a key role in:

- Expressing the solution of fractional boundary value problems.
- Establishing existence and uniqueness of solutions.
- Providing an integral representation of solutions involving fractional derivatives.

The general solution of a fractional differential equation with a given fractional operator and boundary conditions can often be written in the form:

$$y(t) = \int_a^b G(t,s) f(s, y(s))\, ds, \tag{2.3}$$

where $G(t,s)$ is the Green's function corresponding to the fractional differential operator.

2.3.3 Applications

- Solving fractional boundary value problems.
- Establishing stability results using integral equations.

- Developing numerical methods based on the Green's function representation.

The Green's function is a fundamental tool in fractional calculus as it transforms complex differential problems into equivalent integral forms that are often more tractable for both analytical and numerical solutions.

2.4 Mittag-Leffler Function

The Mittag-Leffler function, introduced by Gösta Mittag-Leffler in the early 20th century, is a cornerstone special function in fractional calculus. It generalizes the classical exponential function and appears naturally as a fundamental solution to fractional differential equation.

Definition 2.8: [4] The definition of the Mittag-Leffler function with two parameters is given by,

$$E_{a,b}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(ma+b)}, \quad a,b,x \in \mathbb{C}, \quad Re(a) > 0, \text{ and } Re(b) > 0.$$

This function reduces to the classical exponential function when $a = b = 1$, i.e.,

$$E_{1,1}(x) = e^x$$

The two-parameter Mittag-Leffler function provides flexibility to model various fractional processes due to its additional parameter b, which influences the function's behavior and convergence properties. Beyond this, there are further generalizations of the Mittag-Leffler function that involve three or more parameters. One prominent example is the Prabhakar function (also known as the three-parameter Mittag-Leffler function), defined as

$$E_{a,b}^{c}(x) = \sum_{m=0}^{\infty} \frac{(c)_m x^m}{m!\Gamma(ma+b)},$$

where $(c)_m$ is the Pochhammer symbol (rising factorial) given by $(c)_m = c(c+1)\cdots(c+m-1)$.

These multi-parameter generalizations of the Mittag-Leffler function have found applications in more advanced fractional models, including distributed order fractional differential equation and fractional integral equations with nonlocal kernels.

Theorem 2.1: [38] (Ascoli-Arzela Theorem). A subset $\mathscr{F}$ of $C(K)$ is totaly bounded iff it is point-wise bounded and equicontinuous.

Fixed point theorems are essential instruments in the investigation of fractional differential equations since they yield a strict means of ensuring the existence and uniqueness of solutions. Among them, the Banach fixed point theorem, or the contraction mapping principle, is very crucial as it guarantees the existence of a unique fixed point under contraction conditions, providing a constructive approach to approximating solutions via iterative schemes. Conversely, the Schauder fixed point theorem proves vital in the case of nonlinear or non-contractive mappings as it assures the existence of at least one solution under more relaxed compactness and continuity assumptions. Both theorems play important roles in surmounting the obstacles presented by the complexity of fractional derivatives, enabling the mathematical handling of such equations and allowing researchers to ascertain that well-defined solutions exist, which is vital for both theoretical exploration and practical uses.

Theorem 2.2: [38] (Schauder Fixed Point Theorem). Let M be a convex, closed, and bounded subset of a Banach space A. If $U : M \to M$ is a continuous operator where UM is a relatively compact subset of A, then U has at least one fixed point in M.

A fundamental tool employed in this study is the Schauder Fixed Point Theorem, which serves as a cornerstone in the analysis of nonlinear problems where contractive conditions may not hold. Unlike Banach Fixed Poin theorem, which demands the existence of astrict contraction, Schauder Fixed Poin theorem operates under more general assumptions and is particularly well-suited for applications in infinite-dimensional spaces. Specifically, the theorem states that any continuous mapping from a closed, convex, and bounded subset of a Banach space into itself that maps bounded sets into relatively compact sets must possess at least one fixed point.

The power of Schauder Fixed Poin theorem lies in its reliance on topological compactness rather than metric contraction. This distinction is crucial when dealing with nonlinear fractional differential equations, especially those involving Hadamard-type derivatives or integral operators, where the mappings constructed from the equation are not necessarily contractions in the normed sense. In many cases, such operators are continuous and compact mapping bounded subsets into subsets with compact closure but do not reduce distances between functions. Schauder's framework accommodates this by allowing fixed points to emerge from structural properties of the operator rather than its numerical behavior.

In the context of the problem considered in this thesis, the solution operator associated with the integral formulation of the differential equation is constructed within an appropriate Banach space, such as $C([0,T],\mathbb{R})$, and shown to satisfy the conditions required by the Schauder theorem. This involves verifying that the operator is well-defined, continuous, and compact, and that it maps a closed, bounded, convex subset of the space into itself. Once these properties are confirmed, the theorem guarantees the existence of a fixed point, which corresponds to a solution of the original equation.

It is worth noting that Schauder's theorem, while not providing uniqueness, ensures the presence of at least one solution under minimal assumptions. This aspect is particularly valuable when handling equations involving memory terms, nonlocal conditions, or nonlinearities dependent on fractional derivatives. Such generality makes the Schauder fixed point theorem an indispensable tool in modern fractional calculus and justifies its central role in the analytical strategy adopted in this research.

Theorem 2.3: [38] (Banach contraction mapping principle). Let (A,d) be a complete metric space, and $U : M \to M$ a contraction mapping. If for all $a,b \in M$, and $c \in (0,1)$

$$d(Ua,Ub) \leq cd(a,b),$$

then, the operator U has a unique fixed point x in M, i.e., $UM = x$.

2.5 Banach spaces

In the framework of functional analysis, Banach spaces serve as a foundational structure for the investigation of a wide range of problems (FATIMA) involving differential and integral equations. A Banach space is formally defined as a vector space over the field of real or complex numbers that is equipped with a norm and is complete with respect to that norm. Completeness here implies that every sequence which is Cauchy (i.e., a sequence whose elements become arbitrarily close to each other as the sequence progresses) has a limit that also lies within the same space. This property is not merely technical; it plays a critical role in ensuring that limit-based arguments, such as those used in fixed-point theory or iterative schemes, remain within the bounds of the space under consideration.

In this study, we consider spaces of continuous real-valued functions defined on closed intervals, specifically the space $C([0,T],\mathbb{R})$, which consists of all continuous functions mapping the interval $[0,T]$ into $\mathbb{R}$. This space becomes a Banach space when endowed with the supremum norm, given by

$$\|z\|_\infty = \sup_{t\in[0,T]} |z(t)|.$$

The choice of this space is not arbitrary. It reflects the nature of the solutions we are seeking functions that are continuous on a compact domain and can be analyzed using tools of nonlinear analysis.

The importance of Banach spaces in the context of fractional differential equations stems from their compatibility with fixed-point theorems, which are powerful techniques for proving the existence (and sometimes uniqueness) of solutions. In

particular, Banach's Fixed Point Theorem (also known as the Contraction Mapping Principle) requires a complete metric space as a setting. It asserts that if a self-mapping on a complete metric space is contractive meaning it brings points closer together by a uniform factor then there exists a unique point in the space that remains invariant under the mapping. Similarly, Schauder's Fixed Point Theorem, which applies to compact and continuous self-maps on closed, convex, and bounded subsets of a Banach space, is another essential tool in establishing existence results when contractiveness is not guaranteed.

Furthermore, the structure of Banach spaces allows us to define operators such as fractional integrals and Hadamard-type fractional derivatives as bounded linear (or nonlinear) mappings between appropriately defined spaces. This becomes crucial when dealing with integral equations or differential equations reformulated as operator equations. The availability of a Banach space structure ensures that convergence of sequences of functions (e.g., approximations or successive iterations) leads to well-defined and meaningful limits that preserve the necessary continuity or integrability properties.

In summary, the adoption of Banach spaces in this thesis is not merely for mathematical formality but stems from a deep necessity to provide a rigorous and stable environment in which analytical techniques particularly those related to existence, uniqueness, and stability can be applied with confidence. This setting supports the application of various abstract theorems and ensures that all constructed arguments are grounded in a robust topological and algebraic framework.

2.6 Ulam-Hyers and Ulam-Hyers-Rassias Stability

In the study of differential and fractional differential equation, the stability of solutions is a key concept that ensures the reliability of approximate or perturbed solutions. Two important stability notions in this context are the Ulam-Hyers stability and the Ulam-Hyers and Ulam-Hyers-Rassias Stability.

The Ulam-Hyers stability was first introduced in response to a problem posed by Ulam and later solved by Hyers. It provides a guarantee that if a function approximately satisfies a given differential equation within a fixed bound, then there exists a true solution nearby. This is crucial in applications where small errors or perturbations naturally occur in modeling or numerical simulations.

Formally, the equation under consideration is said to be Ulam-Hyers stable if there exists a positive constant such that for every small perturbation, any approximate solution that satisfies the differential equation within a constant bound will have a corresponding exact solution close to it. The distance between the approximate and the exact solution is directly proportional to the size of the perturbation.

Specifically, the Ulam-Hyers-Rassias stability guarantees that if an approximate solution satisfies the differential equation within a bound that depends on a given function, there still exists an exact solution nearby, and the distance between them is controlled by this function. This generalization is particularly useful in problems where errors are not constant but vary with respect to the independent variable.

Definition 2.9: [7] The equation (1.1) is Ulam-Hyers stable if there is $m_g \in \mathbb{R}^+$ such that for all positive ε and solution $y \in C^1(I,\mathbb{R})$ of the inequality

$$|{}^R D^b({}^C D^a y(x) + h\,{}^C D^{a-1} y(x)) - g(x,y(x))| \leq \varepsilon \quad x \in I,$$

there is a solution $v \in C^1(I,\mathbb{R})$, of equation (1) with

$$|y(x) - v(x)| \leq m_g \cdot \varepsilon \quad , x \in I.$$

Definition 2.10: [25] The equation (1.1) is Ulam-Hyers-Rassia stable with respect to nondecreasing $\phi \in C(I,\mathbb{R}_+)$, if there exists a real number $m_g > 0$ such that for all positive ε and solution $y \in C^1(I,\mathbb{R})$ of the inequality

$$|{}^R D^\beta({}^C D^\alpha y(x) + h\,{}^C D^{\alpha-1} y(x)) - g(x,y(x))| \leq \varepsilon\phi(x), \quad x \in I,$$

there is a solution $v \in C^1(I,\mathbb{R})$, of equation (1) with

$$|y(x) - v(x)| \leq m_g \varepsilon \phi(x), \quad x \in I.$$

Chapter 3

THE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION

In this chapter, we derived the general solution of the fractional differential equation subject to the given integral initial conditions. By applying these conditions, the constants c_0, c_1, and c_2 were determined explicitly. Subsequently, the obtained solution was expressed in terms of the Green's function, providing a more concise representation that is well-suited for analytical purposes. This Green's function formulation will serve as the basis for proving the existence and uniqueness of the solution in the subsequent chapter. The resulting solution is given by the following lemma:

Lemma 3.1: Let $g \in C[0,1], u \in C^1[0,1]$, then the initial value problem (1.1)-(1.2) has a solution

$$u(x) = \int_0^1 F(x,z)g(z,u(z))dz,$$

where $F(x,z)$ is the Green function described by

$$F(x,z)=\begin{cases} e^{-h(x-z)}\frac{1}{\Gamma(a+b-1)}\int_0^z(z-s)^{a+b-2}ds+\frac{1}{\varpi_1(E_{1,2b+a}(-h)+h)+\varpi_2(1-E_{1,2\gamma+a}(-h)}\\ \int_\tau^1\int_r^z\left[\frac{\varpi_2}{\Gamma(\gamma)}(1-z)^{\gamma-1}-\frac{\varpi_1}{\Gamma(b)}(1-z)^{b-1}\right]\frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)}(\tau-r)^{a+b-2}g(r,u(r))d\tau dz\\ x^{a+b-1}E_{1,a+b}(-hx)+\frac{1}{\Gamma(\gamma)h\varpi_1(E_{1,2b+a}(-h)+h)+\varpi_2(1-E_{1,2\gamma+a}(-h)}\\ \int_\tau^1\int_r^z\left[\frac{(E_{1,2b+a}(-h)-1)(1-z^{b-1})}{\gamma(b)}-\frac{(E_{1,2\gamma+a}(-h)+h)(1-z)^{\gamma-1}}{\gamma(\gamma)}\right]\frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)}(\tau-r)^{a+b-2}\\ g(r,u(r))d\tau dz[1-e^{-hx}]\quad \text{if}\quad 0\le z\le x,\\ \\ \frac{1}{\varpi_1(E_{1,2b+a}(-h)+h)+\varpi_2(1-E_{1,2\gamma+a}(-h)}\\ \int_\tau^1\int_r^z\left[\frac{\varpi_2}{\Gamma(\gamma)}(1-z)^{\gamma-1}-\frac{\varpi_1}{\Gamma(b)}(1-z)^{b-1}\right]\frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)}(\tau-r)^{a+b-2}g(r,u(r))d\tau dz\\ x^{a+b-1}E_{1,a+b}(-hx)+\frac{1}{\Gamma(\gamma)h\varpi_1(E_{1,2b+a}(-h)+h)+\varpi_2(1-E_{1,2\gamma+a}(-h)}\\ \int_\tau^1\int_r^z\left[\frac{(E_{1,2b+a}(-h)-1)(1-z^{b-1})}{\gamma(b)}-\frac{(E_{1,2\gamma+a}(-h)+h)(1-z)^{\gamma-1}}{\gamma(\gamma)}\right]\frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)}(\tau-r)^{a+b-2}\\ g(r,u(r))d\tau dz[1-e^{-hx}]\qquad \text{if}\quad x\le z\le 1, \end{cases} \tag{3.1}$$

here,

$$\Psi_1=\frac{1}{\Gamma(b)}\int_0^1(1-z)^{b-1}\int_0^z\frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)}\int_0^\tau(\tau-r)^{a+b-2}g(r,u(r))drd\tau dz$$

$$\Psi_2=\frac{1}{\Gamma(b)}\int_0^1(1-z)^{b-1}\int_0^z\frac{e^{-h(z-\tau)}\tau^{a+b-2}}{\Gamma(a+b-1)}d\tau dz$$

$$\Psi_3=\frac{1}{\Gamma(\gamma)}\int_0^1(1-z)^{\gamma-1}\int_0^z\frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)}\int_0^\tau(\tau-r)^{a+b-2}g(r,u(r))\,drd\tau dz$$

$$\Psi_4=\frac{1}{\Gamma(\gamma)}\int_0^1(1-z)^{\gamma-1}\int_0^z\frac{e^{-h(z-\tau)}\tau^{a+b-2}}{\Gamma(a+b-1)}d\tau dz$$

$$\varpi_1=\frac{1}{h}[\frac{1}{\Gamma(\gamma+1)}-\frac{1}{\Gamma(\gamma)}\int_0^1(1-z)^{\gamma-1}e^{-hz}dz=\frac{1}{h}(\frac{1}{\Gamma(\gamma+1)}-E_{1,\gamma+1}(-h))$$

$\varpi_2 = \frac{1}{h}[\frac{1}{\Gamma(b+1)} - \frac{1}{\Gamma(b)}\int_0^1 (1-z)^{b-1}e^{-hz}dz = \frac{1}{h}(\frac{1}{\Gamma(b+1)} - E_{1,b+1}(-h))$

By simplifying ψ_2 as follows:

$$\frac{1}{\Gamma(b)}\int_0^1 (1-z)^{b-1}\int_0^z \frac{e^{-h(z-\tau)}\tau^{a+b-2}}{\Gamma(a+b-1)}d\tau dz$$

$$= \frac{1}{\Gamma(b)\Gamma(a+b-1)}\int_0^1 (1-z)^{b-1}\int_0^z \sum_{n=0}^{\infty}\frac{(-h(z-\tau))^n}{n!}\tau^{a+b-2}\,d\tau\,dz$$

$$= \frac{1}{\Gamma(b)\Gamma(a+b-1)}\int_0^1 (1-z)^{b-1}\sum_{n=0}^{\infty}\frac{(-h)^n}{n!}\int_0^z (z-\tau)^n\tau^{a+b-2}\,d\tau\,dz$$

$$= \frac{1}{\Gamma(b)\Gamma(a+b-1)}\int_0^1 (1-z)^{b-1}\sum_{n=0}^{\infty}\frac{(-h)^n z^n}{n!}\int_0^z \left(1-\frac{\tau}{z}\right)^n\tau^{a+b-2}\,d\tau\,dz$$

$$= \frac{1}{\Gamma(b)\Gamma(a+b-1)}\int_0^1 (1-z)^{b-1}\sum_{n=0}^{\infty}\frac{(-h)^n z^n}{n!}\int_0^1 (1-u)^n(zu)^{a+b-2}\,du\,dz$$

$$= \frac{1}{\Gamma(b)\Gamma(a+b-1)}\int_0^1 (1-z)^{b-1}\sum_{n=0}^{\infty}\frac{(-h)^n z^{n+a+b-1}}{n!}\int_0^1 (1-u)^n u^{a+b-2}\,du\,dz$$

$$= \frac{1}{\Gamma(b)\Gamma(a+b-1)}\int_0^1 (1-z)^{b-1}\sum_{n=0}^{\infty}\frac{(-h)^n z^{n+a+b-1}}{n!}\mathrm{B}(n+1,a+b-1)\,dz$$

$$= \frac{1}{\Gamma(b)\Gamma(a+b-1)} \int_0^1 (1-z)^{b-1} \sum_{n=0}^{\infty} \frac{(-h)^n z^{n+a+b-1}}{n!} \frac{\Gamma(n+1)\Gamma(a+b-1)}{\Gamma(n+a+b)}\, dz$$

$$= \frac{1}{\Gamma(b)} \int_0^1 (1-z)^{b-1} \sum_{n=0}^{\infty} \frac{(-h)^n z^{n+a+b-1}}{\Gamma(n+a+b)}\, dz$$

$$= \frac{1}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(-h)^n}{\Gamma(n+a+b)} \int_0^1 (1-z)^{b-1} z^{n+a+b-1}\, dz$$

$$= \frac{1}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(-h)^n}{\Gamma(n+a+b)} \mathrm{B}(b, n+a+b)$$

$$= \frac{1}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(-h)^n}{\Gamma(n+a+b)} \cdot \frac{\Gamma(b)\Gamma(n+a+b)}{\Gamma(b+n+a+b)}$$

$$= \sum_{n=0}^{\infty} \frac{(-h)^n}{\Gamma(2b+n+a)} = E_{(1,2b+a)}(-h)$$

By employing similar computations, one can readily derive $\psi_4 = E_{1,2\gamma+a}(-h)$

Proof. Given $${}^{R}D^{b}({}^{C}D^{\alpha}u(x) + h\,{}^{C}D^{a-1}u(x)) = g(x,u(x)),$$

we apply the operator ${}^{R}I^{b}$ on (1.1) to set

$${}^{C}D^{a}u(x) + h{}^{C}D^{a-1}u(x) - \sum_{i=1}^{1} \frac{k_i x^{b-i}}{\Gamma(b-i+1)} = {}^{R}I^{b} g(x,u(x))$$

Now applying ${}^{R}I^{a}$on both sides:

$${}^{R}I^{a}({}^{C}D^{a}u(x) + {}^{C}D^{a-1}(hu(x)) - \frac{k_1 {}^{R}I^{a} x^{b-1}}{\Gamma(b)} = {}^{R}I^{a+b} g(x,u(x)).$$

Using lemmas (2.3) and (2.5),

$$u(x)+h\int_0^x u(z)dz+c_0+c_1x-k_1\frac{\Gamma(b)x^{a+b-1}}{\Gamma(a+b-1+1)\Gamma(b)} =^R I^{a+b}g(x,u(x)).$$

That is

$$u(x)+h\int_0^x u(z)dz =^R I^{a+b}g(x,u(x))+\frac{k_1x^{a+b-1}}{\Gamma(a+b)}-c_0-c_1x.$$

Differentiating both sides:

$$Du(x)+hu(x) =^R I^{a+b-1}g(x,u(x))+\frac{k_1(a+b-1)x^{a+b-2}}{\Gamma(a+b)}-c_1$$

$$=^R I^{a+b-1}g(x,u(x))+\frac{k_1x^{a+b-2}}{\Gamma(a+b-1)}-c_1$$

$$(D+h)u(x) =^R I^{a+b-1}g(x,u(x))+\frac{k_1x^{a+b-2}}{\Gamma(a+b-1)}-c_1$$

$$D(e^{hx}u(x))=[^RI^{a+b-1}g(x,u(x))+k_1\frac{x^{a+b-2}}{\Gamma(a+b-1)}+c_1]e^{hx}. \tag{3.2}$$

Integrating both sides of equation (3.2) from 0 to x yields:

$$e^{hx}u(x) - u(0) = \int_0^x \left[{}^{R}I^{a+b-1} g(z,u(z)) + k_1 \frac{z^{a+b-2}}{\Gamma(a+b-1)} + c_1 \right] e^{hz}\, dz$$

$$u(x) = u(0)e^{-hx} + \int_0^x e^{-h(x-z)} {}^{R}I^{a+b-1} g(z,u(z))\, dz$$

$$+ k_1 \int_0^x \frac{e^{-h(x-z)} z^{a+b-2}}{\Gamma(a+b-1)}\, dz + c_1 \int_0^x e^{-h(x-z)}\, dz$$

$$\Rightarrow \quad u(x) = \int_0^x e^{-h(x-z)} {}^{R}I^{a+b-1} g(z,u(z))\, dz$$

$$+ k_1 \int_0^x \frac{e^{-h(x-z)} z^{a+b-2}}{\Gamma(a+b-1)}\, dz + \frac{c_1}{h}\left(1 - e^{-hx}\right)$$

Now by applying the initial conditions given in 1.2, we get :

$$y'(0) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} \int_0^z \frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)} \int_0^\tau (\tau - r)^{a-b-2} g(r, u(r))\, dr\, d\tau\, dz$$

$$+ c_1 \frac{1}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} \int_0^z \frac{e^{-h(z-\tau)} \tau^{a+b-2}}{\Gamma(a+b-1)}\, d\tau\, dz$$

$$+ \frac{k_1}{h} \left[\frac{1}{\Gamma(\gamma+1)} - \frac{1}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} e^{-hz}\, dz \right]$$

$$\frac{1}{\Gamma(b)} \int_0^1 (1-z)^{b-1} \int_0^z \frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)} \int_0^\tau (\tau - r)^{a+b-2} g(r, u(r))\, dr\, d\tau\, dz$$

$$+ c_1 \frac{1}{\Gamma(b)} \int_0^1 (1-z)^{b-1} \int_0^z \frac{e^{-h(z-\tau)} \tau^{a+b-2}}{\Gamma(a+b-1)}\, d\tau\, dz$$

$$+ \frac{k_1}{h} \left[\frac{1}{\Gamma(b+1)} - \frac{1}{\Gamma(b)} \int_0^1 (1-z)^{b-1} e^{-hz}\, dz \right]$$

$$\frac{1}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} \int_0^z \frac{e^{-h(z-\tau)}}{\Gamma(a+b-1)} \int_0^\tau (\tau - r)^{a+b-2} g(r, u(r))\, dr\, d\tau\, dz$$

$$+ c_1 \frac{1}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} \int_0^z \frac{e^{-h(z-\tau)} \tau^{a+b-2}}{\Gamma(a+b-1)}\, d\tau\, dz$$

$$+ \frac{k_1}{h} \left[\frac{1}{\Gamma(\gamma+1)} - \frac{1}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} e^{-hz}\, dz \right]$$

To find c_1 and k_1

$$\psi_3 + c_1 \psi_4 + k_1 \varpi_1 = c_1 \tag{3.3}$$

$$\psi_1 + c_1\psi_2 + k_1\varpi_2 = -hc_1 \tag{3.4}$$

$$k_1 = \frac{\psi_1(\psi_4 - 1) - \psi_3(\psi_2 + h)}{\varpi_1(\psi_2 + h) + \varpi_2(1 - \psi_4)}.$$

$$= \frac{\psi_1(E_{1,2\gamma+a}(-h) - 1) - \psi_3(E_{1,2b+a}(-h) + h)}{\varpi_1(E_{1,2b+a}(-h) + h) + \varpi_2(1 - E_{1,2\gamma+a}(-h))}.$$

$$c_1 = \frac{\psi_3\varpi_2 - \psi_1\varpi_1}{\varpi_1(\psi_2 + h) + \varpi_2(1 - \psi_4)}$$

$$= \frac{\psi_3\varpi_2 - \psi_1\varpi_1}{\varpi_1(E_{1,2b+a}(-h) + h) + \varpi_2(1 - E_{1,2\gamma+a}(-h))}$$

Substituting k_1 and c_1, we reach

$$u(x) = \int_0^x e^{-h(x-z)R} I^{a+b-1} g(z,u(z))dz$$

$$+\left(\frac{\psi_3\varpi_2 - \psi_1\varpi_1}{\varpi_1(E_{1,2b+a}(-h)+h) + \varpi_2(1-E_{1,2\gamma+a}(-h))}\right)$$

$$\times \int_0^x \frac{e^{-h(x-z)} z^{a+b-2}}{\Gamma(a+b-1)} dz$$

$$+\left(\frac{\psi_1(E_{1,2\gamma+a}(-h)-1) - \psi_3(E_{1,2b+a}(-h)+h)}{h\varpi_1(E_{1,2b+a}(-h)+h) + h\varpi_2(1-E_{1,2\gamma+a}(-h))}\right)[1-e^{-hx}]$$

$$= \int_0^x e^{-h(x-z)R} I^{a+b-1} g(z,u(z))dz$$

$$+\left(\frac{\psi_3\varpi_2 - \psi_1\varpi_1}{\varpi_1(E_{1,2b+a}(-h)+h) + \varpi_2(1-E_{1,2\gamma+a}(-h))}\right) x^{a+b-1} E_{1,a+b(-hx)}$$

$$+\left(\frac{\psi_1(E_{1,2\gamma+a}(-h)-1) - \psi_3(E_{1,2b+a}(-h)+h)}{h\varpi_1(E_{1,2b+a}(-h)+h) + h\varpi_2(1-E_{1,2\gamma+a}(-h))}\right)[1-e^{-hx}],$$

showing that

$$u(x) = \int_0^1 F(x,z)g(z,u(z))dz.$$

□

Chapter 4

EXISTENCE AND UNIQUENESS

Existence and uniqueness of solutions is a basic concern in the investigation of fractional differential equations, as it guarantees that mathematical models formulated in terms of such equations are well-posed and meaningful. This chapter is committed to deriving rigorous existence and uniqueness of solutions results for a variety of classes of fractional differential equations based on two cornerstone tools from functional analysis: the Banach and Schauder fixed point theorems. These theorems provide effective means of dealing with linear as well as nonlinear problems. The Banach fixed point theorem, through its use of contraction mappings, yields a simple mechanism for proving uniqueness, whereas the Schauder fixed point theorem provides an effective tool for proving existence results when compactness and continuity are available but contraction is not. Applying these theorems in suitable function spaces, this chapter provides a firm theoretical basis for investigating fractional models, which are being increasingly utilized to model complicated phenomena in physics, biology, and engineering.

Let $C([0,1],\mathbb{R})$ denotes the set of all continuous function in the Banach space of C from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|u\| = \sup\{|u(x)|; x \in I\}.$$

The following assumptions are needed to show the main results :

(**A1**) There exists a positive constant γ_1, γ_2 such that

$|g(x,u(x))| \leq \gamma_1 + \gamma_2|u(x)|, \forall x \in I$ and $u \in \mathbb{R}$

(**A2**) There exists $k \in \mathbb{R}^+$ such that $|g(x,t(x)) - g(x,u(x))| \leq k|t(x) - u(x)|, \forall x \in I$ and $t, u \in \mathbb{R}$.

(**A3**) There exists an increasing function $\varphi \in C(I, \mathbb{R}_+)$ and $\zeta_1 > 0$ such that for any $x \in I$

$$\int_0^x |F(x,z)dz| \leq \zeta_1 \varphi(x)$$

For simplicity, we introduce the following notations:

$$\eta_1 = \left| \frac{E_{1,a+2\gamma+1}(-h)\varpi_2 - E_{1,a+2b+1}(-h)\varpi_1}{\varpi_1(E_{1,2b+a}(-h)+h) + \varpi_2(1 - E_{1,2\gamma+a}(-h))} \right|$$

$$\eta_2 = \left| \frac{E_{1,2b+1}(-h)(E_{1,2\gamma+a}(-h)-1) - E_{1,a+2\gamma+1}(-h)(E_{1,2b+a}(-h)+h)}{(h\varpi_1(E_{1,2b+a}(-h)+h) + h\varpi_2(1 - E_{1,2\gamma+a}(-h))} \right|$$

$$\zeta_1 = |E_{1,a+b}(-hx)| + \eta_1 x^{a+b-1}|E_{1,a+b}(-hx)| + \eta_2[1 + e^{-hx}].$$

Applying the Schauder fixed point theorem allows us to obtain the result of existence.

Theorem 4.1: Let $g : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy the condition (**A1**). Then the problem (1.1)-(1.2) has a solution.

Proof. Consider the operator $\mathscr{T}$ defined on $C(I)$ by

$$(\mathscr{T}u)(x) = \int_0^1 F(x,z)g(z,u(z))dz.$$

since $F(x,z)$ and $g(x,u(x))$ are continuous functions, we have $\mathscr{T}u \in C(I)\ \forall u \in C(I)$. Define the set $B_r = \{u(x) \in C(I,\mathbb{R}) : \|u\| \leq r\}$ by choosing $r \geq \frac{\gamma_1\zeta_1}{(1-\gamma_2\zeta_1)}$.

First we need to show that $\mathscr{T}B_r \subseteq B_r$, for $u \in B_r$. To do this consider

$$\|(\mathscr{T}u)(x)\| = \sup_{x\in I}\int_0^1 |F(x,z)g(z,u(z))|dz.$$

Applying (**A1**)
$$\|(\mathscr{T}u)(x)\| \leq (\gamma_1+\gamma_2 r)\sup_{x\in I}\int_0^1 |F(x,z)|dz.$$

Now we can obtain the following inequality by the definition of the Green function

$$\|(\mathscr{T}u)(x)\| \leq (\gamma_1+\gamma_2 r)(|E_{1,a+b+1}(-h)|+$$

$$\left|\frac{E_{1,a+2\gamma+1}(-h)\varpi_2 - E_{1,a+2b+1}(-h)\varpi_2}{\varpi_1(E_{1,2b+a}(-h)+h)+\varpi_2(1-E_{1,2\gamma+a}(-h))}\right| + |E_{1,a+b}(-h)$$

$$\left|\frac{E_{1,a+2b+1}(-h)(E_{1,2\gamma+a}(-h)-1) - E_{1,a+2\gamma+1}(-h)(E_{1,2b+a}(-h)+h)}{h\varpi_1(E_{1,2b+a}(-h)+h)+h\varpi_2(1-E_{1,2\gamma+a}(-h))}\right|$$

$$\times [1+e^{-hx}])$$

which implies that

$$\|(\mathscr{T}u)(x)\| \leq (\gamma_1+\gamma_2 r)\zeta_1 \leq r.$$

Therefore, $\mathscr{T}B_r \subseteq B_r$.

By fixing $Q = \sup_{x\in I}|g(x,u(x))|$, where $u \in B_r$, and $x, \chi \in I$ with $x < \chi$. we can prove that the operator $\mathscr{T}$ is completely continuous.

$$\|\mathscr{T}(u)(x)-\mathscr{T}(u)(\chi)\| \leq Q \sup_{x\in J}\int_0^1 |F(x,z)-F(\chi,z)|dz.$$

Therefore,

$$\|\mathscr{T}(u)(x)-\mathscr{T}(u)(\chi)\| \leq Q(|x^{a+b}E_{1,a+b+1}(-hx)-\chi^{a+b}E_{1,a+b+1}(-h\chi)|$$

$$+|x^{a+b-1}E_{1,a+b}(-hx)-\chi^{a+b-1}E_{1,a+b}(-h\chi)|\eta_1+\eta_2|e^{-hx}-e^{-h\chi}|).$$

We note that as x tends to χ, the right-hand side of the above inequality approaches zero. Consequently, $\mathscr{T}$ is uniformly bounded and equicontinuous. By the Ascoli-Arzelà theorem, $\mathscr{T}$ is completely continuous. Therefore, the problem (1.1)-(1.2) has a solution in $C(J,\mathbb{R})$ by applying Schauder's fixed point theorem. □

Theorem 4.2: the condition (**A2**) holds. if

$$|E_{1,a+b+1}(-h)|+\left|\frac{E_{1,a+2\gamma+1}(-h)\varpi_2-E_{1,a+2b+1}(-h)\varpi_2}{\varpi_1(E_{1,2b+a}(-h)+h)+\varpi_2(1-E_{1,2\gamma+a}(-h))}\right|$$

$$+|E_{1,a+b}(-h)|\left|\frac{E_{1,a+2b+1}(-h)(E_{1,2\gamma+a}(-h)-1)-E_{1,a+2\gamma+1}(-h)(E_{1,2b+a}(-h)+h)}{h\varpi_1(E_{1,2b+a}(-h)+h)+h\varpi_2(1-E_{1,2\gamma+a}(-h))}\right|$$

$$\times[1+e^{-hx}]<1. \tag{4.1}$$

Proof. Assume $t,v\in C(I,\mathbb{R})$. Then

$$\|\mathscr{T}(t)(x)-\mathscr{T}(v)(x)\| \leq \sup_{x\in I}\int_0^1 |F(x,z)||g(z,t(z))-g(z,v(z))|dz.$$

Now using **(A2)**

$$\|\mathscr{T}(t)(x)-\mathscr{T}(v)(x)\| \leq k\sup_{x\in I}\int_0^1 |F(x,z)||t(z)-v(z)|dz,$$

which can also be written as

$$\|\mathscr{T}(t)(x)-\mathscr{T}(v)(x)\| \leq k\|t(x)-v(x)\||E_{1,a+b+1}(-h)|+$$

$$\left|\frac{E_{1,a+2\gamma+1}(-h)\varpi_2 - E_{1,a+2b+1}(-h)\varpi_2}{\varpi_1(E_{1,2b+a}(-h)+h)+\varpi_2(1-E_{1,2\gamma+a}(-h))}\right| + |E_{1,a+b}(-h)$$

$$\left|\frac{E_{1,a+2b+1}(-h)(E_{1,2\gamma+a}(-h)-1) - E_{1,a+2\gamma+1}(-h)(E_{1,2b+a}(-h)+h)}{h\varpi_1(E_{1,2b+a}(-h)+h)+h\varpi_2(1-E_{1,2\gamma+a}(-h))}\right|$$

$$\times [1+e^{-hx}])$$

Hence

$$\|\mathscr{T}(t)(x)-\mathscr{T}(v)(x)\| \leq k\zeta_1\|t(x)-v(x)\|.$$

Utilizing condition (4.1), we establish that $\mathscr{T}$ satisfies the properties of a contraction mapping. Consequently, according to the Banach contraction mapping, it is ensured that $\mathscr{T}$ possesses a fixed point, which corresponds to the exclusive solution of problem (1.1)-(1.2). This concludes the proof. □

Chapter 5

STABILITY THEOREMS

This chapter is devoted to the stability concept in the framework of fractional differential equations, specifically Ulam–Hyers and Ulam–Hyers–Rassias stability. These functional analysis-based concepts address how small initial condition changes or system perturbations influence solution behavior. Ulam–Hyers stability is concerned with whether near-solutions stay close to exact solutions and thus gives information on the system's robustness to small disturbances. Ulam–Hyers–Rassias stability is a generalization that makes the stability dependent on the type and magnitude of the perturbation by using functional inequalities. These stability types are particularly important in the fractional context, where the non-locality of the fractional derivatives causes memory effects and adds complexity. Definitions and theoretical outcomes are given in this chapter, to show the relevance of these stability types in guaranteeing the reliability of mathematical models described by fractional differential equations.

Theorem 5.1: If $g : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and (**A2**) is satisfied with $k\zeta_1 < 1$. Then the problem (1.1)-(1.2) is Ulam-Hyers stable.

Proof. Let $y(x) \in C(I, \mathbb{R})$ be a solution of the inequality in definition (2.9) and there is a solution $v \in C(I, \mathbb{R})$ of equation (1.1) - (1.2).

Then, we get

$$u(x) = \int_0^1 F(x,z)g(z,u(z))dz.$$

$$\max_{0\leq x\leq 1} \int_0^1 F(x,z)dz = \zeta_1$$

From Definition (2.9)

$$\begin{aligned} |y(x) - \int_0^1 F(x,z)g(z,y(z))dz| &\leq \int_0^1 |F(x,z)g(z,u(z))\psi(z)|dz \\ &\leq \int_0^1 |F(x,z)g(z,u(z))||\psi(z)|dz \\ &\leq \varepsilon \int_0^1 |F(x,z)g(z,u(z))|dz \\ &\leq \varepsilon\zeta_1 \end{aligned} \tag{5.1}$$

Using **(A2)**, we have

$$|y(x) - u(x)| \leq |y(x) - \int_0^1 F(x,z)g(z,y(z))dz| + k\int_0^1 F(x,z)|y(z) - u(z)|dz, \forall x \in I.$$

From equation (5.1)

$$|y(x) - u(x)| \leq \frac{\varepsilon\zeta_1}{1-k\zeta_1}, \quad 1 - k\zeta_1 \neq 0,$$

letting $C_g = \frac{\zeta_1}{1-k\zeta_1}$, we have

$$|y(x) - u(x)| \leq C_g\varepsilon, \quad x \in I,$$

Thus the problem (1.1) -(1.2) is Ulam-Hyers stable. □

Theorem 5.2: Suppose that $g : I \times R \to R$ is a continuous function and (**A2**)-(**A3**) hold with $k\zeta_1 < 1$. Then the integral initial value problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable.

Proof. Let $y(x) \in C(I,\mathbb{R})$ be a solution of the inequality in definition (2.10), and there exixts a solution $v \in C(I,\mathbb{R})$ of equation (1.1-1.2). Now from the inequality in definition (2.10) and for all $x \in I$, we have

$$|y(x) - \int_0^1 F(x,z)g(z,y(z))dz| \leq \varepsilon \int_0^1 |F(x,z)g(z,y(z))|\varphi(z)dz \tag{5.2}$$
$$\leq \varepsilon\zeta_1\varphi(x).$$

By (**A2**), for all $x \in I$, we have

$$|y(x) - u(x)| \leq |y(x) - \int_0^1 F(x,z)g(z,y(z))dz| + k\int_0^1 F(x,z)|y(z) - u(z)|dz.$$

Using equation (5.2) we get

$$|y(x) - u(x)| \leq \frac{\varepsilon\zeta_1\varphi(x)}{1-k\zeta_1}, \quad 1 - k\zeta_1 \neq 0$$

Letting $C_g = \frac{\zeta_1}{1-k\zeta_1}$, then $|y(x) - u(x)| \leq C_g\varepsilon\varphi(x), \quad x \in I,$

which prooves that the integral initial value problem (1.1-1.2) is Ulam-Hyers-Rassias stable. □

Chapter 6

APPLICATION OF THEORETICAL RESULTS

This chapter offers a practical demonstration of the theoretical outcomes presented in the previous Chapters by giving a concrete numerical example of a fractional differential equation with both Caputo and Riemann–Liouville derivatives. The example is specially designed to prove the existence and uniqueness of the solution via fixed point theorems, namely the Banach and Schauder methods. Moreover, the stability of the solution is examined in the Ulam–Hyers and Ulam–Hyers–Rassias sense, showing how the solution performs when it is subject to small perturbations or approximations. By concretely applying the concepts to a problem, this chapter fills the gap between theory and application, providing an insight into how abstract mathematical techniques are effectively employed to guarantee the reliability and stability of solutions in systems modeled by mixed-type fractional derivatives. This application not only strengthens the relevance of the analytical techniques but also highlights their usefulness in solving practical problems described by complicated fractional dynamics.

Consider the following problem

$$\begin{cases} {}^{R}D^{0.4}({}^{C}D^{1.2}u(x) + {}^{C}D^{0.2}u(x)) = \frac{x}{5}\cos u(x), \quad x,u \in I, \\ u(0) = 0, \quad u'(0) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-z)^{\gamma-1}u(z)dz, \\ u''(0) = \frac{1}{\Gamma(b)}\int_0^1 (1-z)^{b-1}u(z)dz \end{cases} \tag{6.1}$$

Taking $a = 1.2, b = 0.4, \gamma = 0.4$ and $h = 1$ and also by the Lipschitz condition we have $k = 0.2$. Using Theorem 2.3, we get $k\zeta_1 = 0.351601312 < 1$ which proves that the problem (6.1) has a unique solution.

Now by Theorem (5.1),

$$|y(x) - u(x)| \leq \frac{\varepsilon\zeta_1}{1-k\zeta_1}, \text{ where } \frac{\zeta_1}{1-k\zeta_1} = 2.711304931 > 0,$$

which proves the problem (6.1) is Ulam-Hyers stable.

Now, in order to examine the actions of the operator $\mathscr{T}$, it becomes apparent that $|g(x,u(x)| \leq 0.1080604612$ and $|\mathscr{T}u(x)| \leq 0.1080604612\zeta_1$

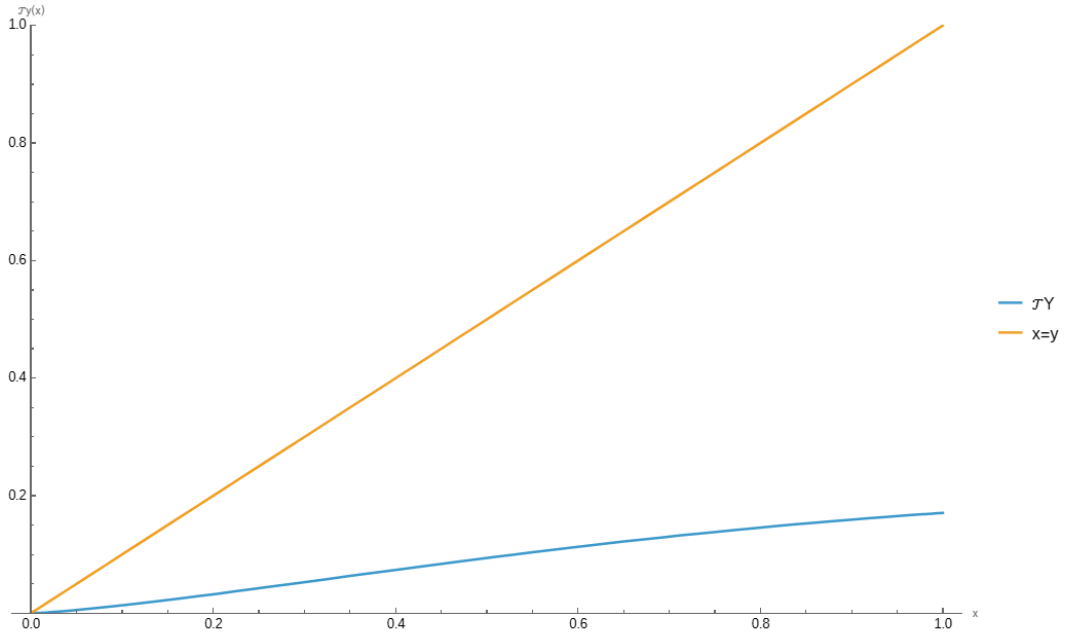

Figure 6.1: The response of the operator $\mathscr{T}$ for values of x in I.

Chapter 7

PROSPECTIVE STUDY ON NONLINEAR HADAMARD-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL CONDITIONS

Building upon the results obtained in this thesis, future research will focus on the analysis of a more general class of fractional differential equations involving Hadamard-type derivatives and integral boundary conditions. In particular, we aim to study the following nonlinear fractional differential equation:

$$ {}^{H}D^{r}({}^{H}D^{s}+\lambda)z(t) = f(t,z(t),{}^{H}D^{\rho}z(t)), \quad t\in[0,T], \quad \lambda\neq 0, \tag{7.1} $$

subject to the integral initial conditions:

$$ z(1)=0, \quad z(T)=\alpha\int_{0}^{T} z(u)\,du, \tag{7.2} $$

$$ z(\zeta)=\beta\int_{0}^{\zeta} z(u)\,du, $$

where $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $1<r<2$, $0<\rho\leq s\leq 1$, and ${}^{H}D^{\gamma}$ denotes the Hadamard fractional derivative of order γ. The presence of the composition ${}^{H}D^{r}({}^{H}D^{s}+\lambda)$ and the mixed-order derivative ${}^{H}D^{\rho}$ introduces analytical challenges not present in standard fractional models.

This investigation will aim to establish conditions ensuring the existence and uniqueness of solutions to this problem using tools such as the fixed point theorems, measure of noncompactness, and fractional integral inequalities. Special attention will also be given to the qualitative behavior of solutions, including continuity, dependence on parameters, and stability under perturbations.

The motivation behind this direction stems from the increasing relevance of such models in engineering, viscoelasticity, and control theory, where memory and nonlocal effects play a significant role. Extending the theoretical framework to such general equations may open new possibilities for modeling complex systems governed by Hadamard-type fractional dynamics.

Let $f \in C([0,T],\mathbb{R})$.then the unique root of the problem given in (7.1) and (7.2) can be expressed as

$$z(t) = I^{s+r} f(t) - \lambda I^{s} z(t)$$

$$+\Omega_1 \left(\frac{\psi_\beta}{(\log \zeta)^{r+s-2}} - \frac{\psi_\alpha}{(\log T)^{r+s-2}} \right) \tag{7.3}$$

$$+\Omega_2 \left(\frac{\psi_\beta}{(\log \zeta)^{r+s-1}} - \frac{\psi_\alpha}{(\log T)^{r+s-1}} \right)$$

Proof. $(D^s+\lambda)z(t) = c_1(\log t)^{r-1} + c_2(\log t)^{r-2} + I^r f(t),$

$$z(t) = I^{r+s}f(t) - \lambda I^s z(t) + c_0(\log t)^{s-1} + \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log t)^{r+s-1}, \qquad (7.4)$$

$$+ \frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log t)^{r+s-2},$$

After applying the first condition, we observed that $c_0 = 0$,

$$z(t) = I^{r+s}f(t) - \lambda I^s z(t) + \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log t)^{r+s-1},$$
$$+ \frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log t)^{r+s-2},$$

Applying the second condition we get

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log T)^{r+s-1} + \frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log T)^{r+s-2} = \alpha I z(t) - I^{r+s}f(t) + \lambda I^s z(t),$$

Now applying the third condition,

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log \zeta)^{r+s-1} + \frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log \zeta)^{r+s-2} = \beta I z(t) - I^{r+s}f(t) + \lambda I^s z(t).$$

By taking

$\alpha I z(t) - I^{r+s}f(t) + \lambda I^s z(t) = \Psi_\alpha,$

$\beta I z(t) - I^{r+s}f(t) + \lambda I^s z(t) = \Psi_\beta,$

we get

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log T)^{r+s-1} + \frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log T)^{r+s-2} = \Psi_\alpha, \qquad (7.5)$$

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log\zeta)^{r+s-1}+\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log\zeta)^{r+s-2}=\Psi_\beta, \tag{7.6}$$

From equation (7.5),

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}=\frac{\Psi_\alpha}{(\log T)^{r+s-1}}-\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log T)^{-1}, \tag{7.7}$$

And from (7.6),

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}=\frac{\Psi_\beta}{(\log\zeta)^{r+s-1}}-\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log\zeta)^{-1}, \tag{7.8}$$

By (7.7) and (7.8),

$$\frac{\Psi_\alpha}{(\log T)^{r+s-1}}-\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log T)^{-1}=\frac{\Psi_\beta}{(\log\zeta)^{r+s-1}}-\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log\zeta)^{-1},$$

$$\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log\zeta)^{-1}-\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log T)^{-1}=\frac{\Psi_\beta}{(\log\zeta)^{r+s-1}}-\frac{\Psi_\alpha}{(\log T)^{r+s-1}},$$

$$\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}[(\log\zeta)^{-1}-(\log T)^{-1}]=\frac{\Psi_\beta}{(\log\zeta)^{r+s-1}}-\frac{\Psi_\alpha}{(\log T)^{r+s-1}},$$

$$\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}=\frac{\Psi_\beta}{[(\log\zeta)^{-1}-(\log T)^{-1}](\log\zeta)^{r+s-1}}-\frac{\Psi_\alpha}{[(\log\zeta)^{-1}-(\log T)^{-1}](\log T)^{r+s-1}}.$$

For c_1 from (7.5),

$$\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log T)^{r+s-2}=\Psi_\alpha-\frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log T)^{r+s-1},$$

$$\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)} = \frac{\Psi_\alpha}{(\log T)^{r+s-2}} - \frac{c_1\Gamma(r)}{(\log T)^{r+s-2}\Gamma(r-s)}(\log T)^{r+s-1},$$

$$= \frac{\Psi_\alpha}{(\log T)^{r+s-2}} - \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log T),$$

Now from (7.6),

$$\frac{c_2\Gamma(r-1)}{\Gamma(r-s-1)}(\log \zeta)^{r+s-2} = \Psi_\beta - \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log \zeta)^{r+s-1},$$

$$= \frac{\Psi_\beta}{(\log \zeta)^{r+s-2}} - \frac{c_1\Gamma(r)}{(\log \zeta)^{r+s-2}\Gamma(r-s)}(\log \zeta)^{r+s-1},$$

$$= \frac{\Psi_\beta}{(\log \zeta)^{r+s-2}} - \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log \zeta),$$

$$\frac{\Psi_\beta}{(\log \zeta)^{r+s-2}} - \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log \zeta) = \frac{\Psi_\alpha}{(\log T)^{r+s-2}} - \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log T),$$

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log T) - \frac{c_1\Gamma(r)}{\Gamma(r-s)}(\log \zeta) = \frac{\Psi_\alpha}{(\log T)^{r+s-2}} - \frac{\Psi_\beta}{(\log \zeta)^{r+s-2}},$$

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)}[(\log T) - (\log \zeta)] = \frac{\Psi_\alpha}{(\log T)^{r+s-2}} - \frac{\Psi_\beta}{(\log \zeta)^{r+s-2}},$$

$$\frac{c_1\Gamma(r)}{\Gamma(r-s)} = \frac{\Psi_\alpha}{[(\log T) - (\log \zeta)](\log T)^{r+s-2}} - \frac{\Psi_\beta}{[(\log T) - (\log \zeta)](\log \zeta)^{r+s-2}},$$

Plugging c_1 and c_2 into (7.4),

$$z(t) = I^{r+s} f(t) - \lambda I^s z(t) + \left[\frac{\Psi_\alpha}{[(\log T) - (\log \zeta)](\log T)^{r+s-2}} \right.$$

$$\left. - \frac{\Psi_\beta}{[(\log T) - (\log \zeta)](\log \zeta)^{r+s-2}} \right] (\log t)^{r+s-1}$$

$$+ \left[\frac{\Psi_\beta}{[(\log \zeta)^{-1} - (\log T)^{-1}](\log \zeta)^{r+s-1}} \right.$$

$$\left. - \frac{\Psi_\alpha}{[(\log \zeta)^{-1} - (\log T)^{-1}](\log T)^{r+s-1}} \right] (\log t)^{r+s-2},$$

$$\Omega_1 = \frac{(\log t)^{r+s-1}}{(\log T) - (\log \zeta)},$$

and

$$\Omega_2 = \frac{(\log t)^{r+s-2}}{(\log \zeta)^{-1} - (\log T)^{-1}},$$

$$z(t) = I^{r+s} f(t) - \lambda I^s z(t) + \left[\frac{\Psi_\alpha}{(\log T)^{r+s-2}} - \frac{\Psi_\beta}{(\log \zeta)^{r+s-2}} \right] \Omega_1$$

$$+ \left[\frac{\Psi_\beta}{(\log \zeta)^{r+s-1}} - \frac{\Psi_\alpha}{(\log T)^{r+s-1}} \right] \Omega_2.$$

□

The outline of the proof for the existence of solutions to the nonlinear Hadamard-type fractional differential equation with integral conditions is given below. This outline uses fixed point theory and standard functional analysis tools, tailored to the structure of the problem.

Considering the nonlinear fractional differential equation,

$$^{H}D^{r}(^{H}D^{s}+\lambda)z(t)=f(t,z(t),^{H}D^{\rho}z(t)),\quad t\in[0,T],\quad \lambda\neq 0,$$

subject to the integral initial conditions:

$$z(1)=0,\quad z(T)=\alpha\int_{0}^{T}z(u)\,du,$$

$$z(\zeta)=\beta\int_{0}^{\zeta}z(u)\,du,$$

Here the aim is to prove existence of a solution using Schauder's Fixed Point Theorem.

The Step-by-Step Outline of the Proof for the Existence:

Step 1: convert the problem to an integral equation using the properties of the Hadamard fractional integral operator.

We aim to rewrite the differential equation as an equivalent Volterra-type integral equation.

Step 2: Define a Suitable Function Space.

Step 3: Define an Operator

our goal is to show that the defined operator has a fixed point, which will be a solution to the original problem.

Step 4: Show that this operator maps a convex closed subset into itself.

We want to show that the operator maps bounded functions to bounded functions. **Step**

5: Show the operator is completely continuous.

continuity will follow from the continuity of the function and the integral kernel ,and compactness will be shown by the Arzela–Ascoli theorem.

Step 6: Apply Schauder's Fixed Point Theorem

Schauder's theorem says that if a completely continuous operator maps a convex, closed, bounded subset of a Banach space into itself, then it has at least one fixed point. Using Steps 4 and 5, one can easily prove that the operator has a fixed point.

The Step-by-Step Proof Outline for Uniqueness

The outline of the proof for the uniqueness of the solution to the nonlinear Hadamard-type fractional differential equation with integral conditions, using Banach's Fixed Point Theorem.

The aim is to show that the problem,

$$^{H}D^{r}(^{H}D^{s}+\lambda)z(t)=f(t,z(t),^{H}D^{\rho}z(t)),\quad t\in[0,T],\quad \lambda\neq 0,$$

subject to the integral initial conditions:

$$z(1)=0,\quad z(T)=\alpha\int_{0}^{T}z(u)\,du,$$

$$z(\zeta)=\beta\int_{0}^{\zeta}z(u)\,du,$$

has a unique solution.

Conclusion

The existence of at least one solution to the nonlinear Hadamard-type fractional differential equation with integral conditions will be established by reformulating the problem as an integral equation, showing the associated operator is completely continuous and finally applying Schauder's fixed point theorem, we want to show that the problem has a unique solution under a Lipschitz condition on the defined function.

Step 1: Reformulate the Problem.

Convert the differential equation to an equivalent Volterra-type integral equation using Hadamard integral identities.

Step 2: Assume a Lipschitz Condition.

Assume that the nonlinear term satisfies the Lipschitz condition. The key condition is the Lipschitz continuity of function with sufficiently small Lipschitz constants.

Step 3: Define a Metric Space.

Step 4: Show that the operator is a contraction and ensure contraction condition.

Step 5: Apply Banach Fixed Point Theorem.

Since the function space is a complete metric space, and the operator is a contraction on this space, Banach's fixed point theorem guarantees the unique fixed point which will conclude that the problem has a unique solution.

Note that, the key condition is the Lipschitz continuity of the function with sufficiently small Lipschitz constants. The Hadamard derivative is handled by its integral form and boundedness properties on compact domains. The combination of the Lipschitz condition and Volterra integral formulation uniqueness.

Chapter 8

CONCLUSIONS

This research focused on the analytical study of a class of nonlinear fractional differential equations involving mixed Caputo–Riemann derivatives and integral initial conditions. These types of equations are particularly suited for modeling complex physical and engineering systems characterized by memory effects and nonlocal dynamics.

By applying the Schauder fixed-point theorem and the contraction mapping principle, we established sufficient conditions for the existence and uniqueness of solutions to the nonlinear problem. These foundational results ensure that the model is mathematically well-posed and open the door for both theoretical extensions and practical applications.

Furthermore, the Ulam-Hyers and Ulam-Hyers-Rassias stability of the equations (1.1)-(1.2) were thoroughly examined. These stability analyses provide insight into the robustness of the system under small perturbations, a critical aspect in real-world scenarios where exact modeling is rarely possible. Our findings contribute to the growing literature on the qualitative behavior of fractional systems and support their reliability for modeling dynamic processes in various scientific fields.

To reinforce the theoretical developments, we presented a concrete illustrative

example that verifies the applicability of the obtained results. This example not only validates the assumptions under which the theorems hold but also serves as a guideline for further implementation in applied contexts.

In addition to the classical fractional operators, this thesis also incorporated the Hadamard fractional derivative, which is particularly effective in capturing logarithmic-type memory kernels. Its inclusion broadens the analytical framework and demonstrates the flexibility of fractional calculus in modeling systems with non-uniform scaling properties. The analysis involving the Hadamard operator enriches the theoretical depth of the study and underscores its relevance in handling more intricate dynamical systems governed by nonlocal behaviors.

Overall, this thesis provides a solid analytical foundation for the study of fractional differential equations with mixed derivatives and paves the way for future work in numerical approximation, optimal control, and the extension to more general boundary value problems and higher-dimensional systems.

REFERENCES

[1] A. A. Kilbas, O. I. Marichev, and S. G. Samko, "Fractional integrals and derivatives (theory and applications)," 1993.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.

[3] R. Herrmann, *Fractional Calculus An Introduction for Physicists*. Singapor:World Scientific Publication Company, February 2011.

[4] I. Podlubny, "Fractional differential equations, mathematics in science and engineering, academic press, san diego," 1999.

[5] T. Blaszczyk and M. Ciesielski, "Numerical solution of euler-lagrange equation with caputo derivatives," *Advances in Applied Mathematics and Mechanics*, vol. 9, pp. 173–185, 2017.

[6] G. Chatzarakis, M. Deepa, N. Nagajothi, and V. Sadhasivam, "Oscillatory properties of a certain class of mixed fractional differential equations," *Appl. Math. Inf. Sci*, vol. 14, no. 1, pp. 123–131, 2020.

[7] S. I. Muslih and D. Baleanu, "Fractional euler—lagrange equations of motion in fractional space," *Journal of Vibration and Control*, vol. 13, no. 9-10, pp. 1209–

1216, 2007.

[8] D. Baleanu, J. H. Asad, and I. Petras, "Fractional bateman—feshbach tikochinsky oscillator," *Communications in Theoretical Physics*, vol. 61, no. 2, p. 221, 2014.

[9] T. Blaszczyk, "A numerical solution of a fractional oscillator equation in a non-resisting medium with natural boundary conditions," *Rom. Rep. Phys*, vol. 67, no. 2, pp. 350–358, 2015.

[10] M. Klimek, "Fractional sequential mechanics—models with symmetric fractional derivative," *Czechoslovak Journal of Physics*, vol. 51, no. 12, pp. 1348–1354, 2001.

[11] O. Agrawal, "Fractional variational calculus and the transversality conditions," *Journal of Physics A: Mathematical and General*, vol. 39, no. 33, p. 10375, 2006.

[12] T. Blaszczyk, "A numerical solution of a fractional oscillator equation in a non-resisting medium with natural boundary conditions," *Rom. Rep. Phys*, vol. 67, no. 2, pp. 350–358, 2015.

[13] R. Khaldi and A. Guezane-Lakoud, "Higher order fractional boundary value problems for mixed type derivatives," *J. Nonlinear Funct. Anal*, vol. 30, no. 9, 2017.

[14] A. G. Lakoud, R. Khaldi, and A. Kılıçman, "Existence of solutions for a mixed fractional boundary value problem," *Advances in difference equations*, vol. 2017, pp. 1–9, 2017.

[15] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, "Fractional order differential systems involving right caputo and left riemann–liouville fractional derivatives with nonlocal coupled conditions," *Boundary value problems*, vol. 2019, pp. 1–12, 2019.

[16] S. Song and Y. Cui, "Existence of solutions for integral boundary value problems of mixed fractional differential equations under resonance," *Boundary Value Problems*, vol. 2020, no. 1, p. 23, 2020.

[17] Y. Liu, C. Yan, and W. Jiang, "Existence of the unique nontrivial solution for mixed fractional differential equations," *Journal of Function Spaces*, vol. 2021, no. 1, p. 5568492, 2021.

[18] S. A. Murad and Z. A. Ameen, "Existence and ulam stability for fractional differential equations of mixed caputo-riemann derivatives," *AIMS Mathematics*, vol. 7, no. 4, pp. 6404–6419, 2022.

[19] S. K. Ntouyas, A. Alsaedi, and B. Ahmad, "Existence theorems for mixed riemann–liouville and caputo fractional differential equations and inclusions

with nonlocal fractional integro-differential boundary conditions," *Fractal and Fractional*, vol. 3, no. 2, p. 21, 2019.

[20] D. N. ABDULQADER, S. A. MURAD *et al.*, "Existence and uniqueness results for certain fractional boundary value problems," *Journal of Duhok University*, vol. 22, no. 2, pp. 76–88, 2019.

[21] S. A. Murad, R. W. Ibrahim, and S. B. Hadid, "Existence and uniqueness for solution of differential equation with mixture of integer and fractional derivative," *Pak. Acad. Sci*, vol. 49, pp. 33–37, 2012.

[22] S. A. Jose, W. Yukunthorn, J. E. N. Valdes, and H. Leiva, "Some existence, uniqueness and stability results of nonlocal random impulsive integro-differential equations," *Appl. Math. E-Notes*, vol. 20, pp. 481–492, 2020.

[23] H. A. Jalab, R. W. Ibrahim, S. A. Murad, and S. B. Hadid, "Exact and numerical solution for fractional differential equation based on neural network," *Proc. Pakistan Aca. Sci*, vol. 49, pp. 199–208, 2012.

[24] M. Hu and L. Wang, "Existence of solutions for a nonlinear fractional differential equation with integral boundary condition," *International Journal of Mathematical and Computational Sciences*, vol. 5, no. 1, pp. 55–58, 2011.

[25] M. Benchohra, S. Bouriah, and J. J. Nieto, "Existence and ulam stability

for nonlinear implicit differential equations with riemann-liouville fractional derivative," *Demonstratio Mathematica*, vol. 52, no. 1, pp. 437–450, 2019.

[26] B. Ahmad, J. J. Nieto *et al.*, "Boundary value problems for a class of sequential integrodifferential equations of fractional order," *Journal of Function Spaces*, vol. 2013, 2013.

[27] B. Ahmad and J. J. Nieto, "Sequential fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3046–3052, 2012.

[28] R. W. Ibrahim, "Ulam stability of boundary value problem," *Kragujevac Journal of Mathematics*, vol. 37, no. 2, pp. 287–297, 2013.

[29] C. Chen, M. Bohner, and B. Jia, "Ulam-hyers stability of caputo fractional difference equations," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 18, pp. 7461–7470, 2019.

[30] N. I. Mahmudov, "Analytical solution of the fractional linear time-delay systems and their ulam-hyers stability," *Journal of Applied Mathematics*, vol. 2022, 2022.

[31] N. I. Mahmudov and A. M. Almatarneh, "Stability of ulam–hyers and existence of solutions for impulsive time-delay semi-linear systems with non-permutable matrices," *Mathematics*, vol. 8, no. 9, p. 1493, 2020.

[32] R. I. Butt, T. Abdeljawad *et al.*, "Stability analysis by fixed point theorems for a class of non-linear caputo nabla fractional difference equation," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–11, 2020.

[33] K. Liu, M. Fečkan, and J. Wang, "Hyers–ulam stability and existence of solutions to the generalized liouville–caputo fractional differential equations," *Symmetry*, vol. 12, no. 6, p. 955, 2020.

[34] K. Liu, J. Wang, Y. Zhou, and D. O'Regan, "Hyers–ulam stability and existence of solutions for fractional differential equations with mittag–leffler kernel," *Chaos, Solitons & Fractals*, vol. 132, p. 109534, 2020.

[35] J. Hadamard, "Essai sur l'étude des fonctions données par leur développement de taylor," *Journal de Mathématiques Pures et Appliquées*, vol. 8, no. 4, pp. 101–186, 1892.

[36] J. H. Barrett, "Differential equations of non-integer order," *Canadian Journal of Mathematics*, vol. 6, pp. 529–541, 1954.

[37] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. elsevier, 2006, vol. 204.

[38] Y. Zhou, J. Wang, and L. Zhang, *Basic theory of fractional differential equations*. World scientific, 2016.