Cubic Calculus

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ABSTRACT

In the study called non-Newtonian Calculus, Grossman and Katz introduced a new type of calculus that includes branches such as geometric and bi-geometric calculus. The aim of this thesis is to examine the basic features of the geometric and bi-geometric calculus.

This thesis is divided into five parts. In the first part, the literature on non-Newtonian calculus is summarized. In the second part, a sub-branch of non-Newtonian calculus called geometric arithmetic is introduced, along with the properties of geometric real numbers which is called α-arithmetic. In the third part, the applications of the α-arithmetic of non-Newtonian calculus is studied. In chapter four the arithmetic, differentiation and integration are applied to a specific example which is hyperbolic tangent. In chapter 5 the application of arithmetic to the cubic calculi is given and according to this geometric and bi-geometric differentiation and integrations are defined. Apart from this some other concepts are given such as absolute value, commutativity, associativity and distributivity properties and q-limit. In conclusion part a discussion about quadratic Calculi is given.

Keywords: Non-Newtonian Calculus, α-Arithmetic, Geometric Calculus, Bi-Geometric Calculus, Cubic Calculi and Quadratic Calculi.

ÖZ

Grossman ve Katz, Geometrik ve Bi-Geometrik kalkülüs gibi alt dalları içeren yeni bir kalkülüs türü olan Non-Newtonian Kalkülüsün tanıtımını yaptılar. Bu tezde, Geometrik ve İki-Geometrik kalkülüsün temel özelliklerini inceleme amacı taşımaktadır.

Bu tez, beş bölümden oluşur. Birinci bölümde, non-Newtonian kalkülüsün tarihçesi ve bilgileri özetlenir. İkinci bölümde, non-Newtonian kalkülüsün bir alt dalı olan Geometrik Aritmetik tanıtılır ve α-aritmetiği olarak bilinen Geometrik gerçek sayıların özellikleri verilir. Üçüncü bölümde, non-Newtonian kalkülüsün α-aritmetiğinin uygulamaları incelenir. Dördüncü bölümde aritmetik, türev alma ve tümevarım işlemleri, hiperbolik tanjant örneği üzerinde uygulanır. Beşinci bölümde aritmetik, kübik kalkülüslere uygulanır ve bu doğrultuda geometrik ve iki-geometrik türev alma ve tümevarım işlemleri tanımlanır. Ayrıca, mutlak değer, komütatiflik, örtüklük ve dağılma özellikleri ve kübik-limit gibi diğer kavramlar verilir. Sonuç bölümünde Kuadratik Kalkülüs ile ilgili tartışma yapılır.

Anahtar kelimeler: Non-Newtonian Kalkülüs, α-Aritmetik, Geometrik Kalkülüs, Bi-Geometrik Kalkülüs, Kübik Kalkülüs ve Kuadratik Kalkülüs.

DEDICATION

To My Family

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Chapter 1

INTRODUCTION

Differential and Integral Calculus, created by Isaac Newton and Gottfried Leibniz in the 17th Century, is a widely used mathematical theory for understanding and analyzing natural phenomena. The Calculus has since been applied to the study of functions by Leonard Euler and others. The development of such calculus led to the creation of advanced calculus which is also known as analysis. Nowadays, many universities include different courses on calculus or analysis in their curriculum. The usual method of teaching these courses is often based on the Newtonian Calculus, which takes the identity function into consideration. Newtonian Calculus assumes the identity function as standard and compares other functions to it to define various theorems. Differentiation and Integration are two basic concepts in analysis and calculus that involves taking infinitesimal versions of the operations such as addition and subtraction. These operations are named as arithmetic operations.

In 1972 Grossman and Katz in [11] highlighted alternative ways of presenting calculus by using different nonlinear reference functions other than identity function. This type of calculus is then named as non-Newtonian calculus. Non-Newtonian Calculus can be separated into two as geometric and bi-geometric calculus. Bi-geometric calculus is a slightly modified version of geometric Calculus which can be seen in recent research papers [7] and [10]. Also in papers, [5], [6] the geometric Calculus is taken as Multiplicative Calculus which considers the exponential function as a reference function. A real-world application of Multiplicative Calculus is economics and accounting which is explained in paper [5]. Furthermore, the development of Complex Calculus in geometric and bi-geometric forms can be found in literature [2], [3] and [4]. On the other hand the Complex Calculus can usually be applied to functions which are not one to one functions. To give an example the Quadratic Calculus.

Besides this, some other areas have been developed by using non-Newtonian Calculi such as Numerical methods. Reference [8] and [12] explains the sequence spaces for both Real and Complex spaces.

Chapter 2

α-ARITHMETIC

In this section general concept of arithmetic and alpha arithmetic is discussed. This arithmetic is used to construct the non-Newtonian calculi which is separated into two as geometric and bi-geometric calculus. Besides this, the applications of geometric and bi-geometric calculi is given in further chapter such as multiplicative derivative (geometric derivative), integrals and bi-geometric cases.

2.1 Alpha- Arithmetic

The Arithmetic is the ordered subset of real numbers R. The system of Arithmetic is defined as making algebraic operations on this ordered set.

In alpha arithmetic a generator function is considered which has a domain of real numbers R and has a range of real numbers R . There are couple of examples that can be given as the generator functions, the identity function defined as $f(x) = x$ and exponential function defined as $f(x) = e^x$. In Newtonian calculus $f(x) = x$ is taken into consideration.

In this chapter non-Newtonian calculus is explained by using a single generator function. The alpha is assumed to be a differentiable bijection from R to R_α. Besides this, the boldface letters are used to define, $\boldsymbol{x} = \alpha(x)$ throughout this thesis.

Definition 2.1.1 [11]: Let $R_\alpha = \{\alpha(x) : x\epsilon R\}$ and assume that $A \subseteq R$. To make it clear $\alpha'(x)$ is assumed to be strictly increasing function for all $x\epsilon R$. After this information the alpha arithmetic operations being $\oplus, \ominus, \otimes, \oslash$ can be defined in following way;

For all $x, y \in R_\alpha$,

α-summation: $x \oplus_\alpha y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\}$

α-subtraction: $x \ominus_\alpha y = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\}$

α-multiplication: $x \otimes_\alpha y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\}$

α-division: $x \oslash_\alpha y = \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\}$

α-ordering: $x \leq y = \alpha^{-1}(x) \leq \alpha^{-1}(y)$

From these operations it is easy to see that R_α preserves the order in R. Each bijective function generate only one arithmetic or each arithmetic is generated via single bijective function.

Definition 2.1.2 (Distributive Property): By using the operations above one can define the distributive property in a following way;

$$(x \oplus_\alpha y) \otimes_\alpha z = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} \otimes_\alpha \alpha^{-1}(z)$$

$$= \alpha\left\{\alpha^{-1}\{\alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} \times \alpha^{-1}(z)\}\right\}$$

$$= \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} \times \alpha^{-1}(z)$$

$$= \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(z)\} + \alpha\{\alpha^{-1}(y) \times \alpha^{-1}(z)\}$$

$$= (x \otimes_\alpha z) \oplus (y \otimes_\alpha z)$$

Commutativity Property 2.1.3: Commutativity can also be defined by using similar operations;

$$(x \oplus_\alpha y) \oplus_\alpha z = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} \oplus_\alpha z$$

$$= \alpha(\alpha^{-1}\left(\alpha\left(\alpha^{-1}(x) + \alpha^{-1}(y)\right) + \alpha^{-1}(z)\right)$$

$$= \alpha(\alpha^{-1}(x) + \alpha^{-1}(y) + \alpha^{-1}(z))$$

$$= \alpha(\alpha^{-1}(x) + \alpha^{-1}\left(\alpha\big(\alpha^{-1}(y) + \alpha^{-1}(z)\big)\right)$$

$$= \alpha\big(\alpha^{-1}(x) + \alpha^{-1}(y \oplus_\alpha z)\big) = x \oplus_\alpha (y \oplus_\alpha z)$$

Definition 2.1.4 ($\boldsymbol{\alpha}$-doubling): Let $x \in R_\alpha$ then,

$$x \oplus_\alpha x = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(x)\} = \alpha\{2\alpha^{-1}(x)\}$$

Since $\alpha\{\alpha^{-1}(2)\} = 2$ the following result holds,

$$\alpha\{\alpha^{-1}\alpha(2) \times \alpha^{-1}(x)\} = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(2)\} = \alpha(x) \otimes_\alpha \alpha(2)$$

This definition shows that the doubling in $R_\alpha$is different compared to R. This means that every function in basic Calculus has an analog in geometric and bi-geometric calculi.

Example 2.1.5 [3]**:** Let α be a bijection from R to R_α such that $\alpha = e^x$, by the definition of inverse functions, $\alpha: R \to R_{exp} \subseteq R^+$ and,

$$x \to \alpha(x) = e^x = y \text{ or } y \to \alpha^{-1}(x) = \ln y = x$$

According to this inverse function and the definition 1.1.1 the following algebraic operations hold, for all $x, y \in R_\alpha$,

α-summation: $\boldsymbol{x} \oplus_{exp} \boldsymbol{y} = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} = e^{(\ln(x)+\ln(y))} = xy$

α-subtraction: $\boldsymbol{x} \ominus_{exp} \boldsymbol{y} = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\} = e^{(\ln(x)+\ln(y))} = \frac{x}{y}$

α-multiplication: $\boldsymbol{x} \otimes_{exp} \boldsymbol{y} = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\} = e^{\ln(x)\ln(y)} = x^{\ln(y)} = y^{\ln(x)}$

α-division: $\boldsymbol{x} \oslash_{exp} \boldsymbol{y} = \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\} = e^{\ln(x) \div \ln(y)} = x^{\frac{1}{\ln(y)}}$

Definition 2.1.6: (alpha integer set) [11]**:** For all $k \in Z$, the Z_α set can be defined as,

$$Z_\alpha = \{\dots, \mathbf{-2}, \mathbf{-1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\} = \{\dots, \alpha(-2), \alpha(-1), \alpha(0), \dots\}$$

According to this definition of Z_α one can obtain the α-integer set,

$$Z_\alpha = \left\{\frac{\boldsymbol{k}}{\boldsymbol{k}} = \alpha(k), k \in Z\right\}$$

Using $\mathbb{Z}_\alpha$ to convert this set to $\alpha = e^x$,

$$Z_{exp} = \{\dots, e^{-2}, e^{-1}, 1, e, e^2, \dots\}$$

Note that e is taken as logarithmic number.

Definition 2.1.7[11]: Let y be in R_α. The square of y in alpha arithmetic is $y \otimes y$ and shown as y^2 as in Newtonian calculus. Let y and t be nonnegative real numbers then $\alpha\sqrt{\alpha^{-1}(y)}$ has one value which is $y = t^2$. In similar logic to the definition above, for $y \in R_\alpha$ q^{th} α-root and p^{th} α-power can be shown as $\sqrt[q]{y}$ and y^p. Using these notations and for $y \in R_\alpha$ we have,

$$y^2 = y \otimes y = \alpha\{\alpha^{-1}(y) \times \alpha^{-1}(y)\} = \alpha\{[\alpha^{-1}(y)]^2\}$$

For y^3 and y^p,

$$y^3 = y^2 \otimes y = \alpha\{[\alpha^{-1}(y)]^2\} \otimes \alpha\{\alpha^{-1}(y)\} = \alpha\{[\alpha^{-1}(y)]^3\}$$

$$y^p = y^{p-1} \otimes y = \alpha\{[\alpha^{-1}(y)]^p\}$$

Definition 2.1.8 [11]: Let $y \in R_\alpha$ the Non-Newtonian absolute value of y is represented as,

$$|y|_\alpha = \begin{cases} \boldsymbol{y}, & y > \boldsymbol{0} \\ \boldsymbol{0}, y = \boldsymbol{0} = \alpha\{|\alpha^{-1}(\boldsymbol{0})| \\ \boldsymbol{0} \ominus \boldsymbol{y}, & y < \boldsymbol{0} \end{cases}$$

Example 2.1.9: Let $y \in R_\alpha$ then one can give the example of,

$$|y|_\alpha = \begin{cases} y & if\ y \geq 1 \\ \dfrac{1}{y} & if\ y < 1 \end{cases}$$

Some applications of geometric and bi-geometric (α and biα-calculi) are given in next chapter.

2.2 $\boldsymbol{\alpha}$-Limit and $\boldsymbol{\alpha}$-Derivative

Definition 2.2.1: ($\boldsymbol{\alpha}$-limit) [8]**:** Let $X \subseteq R_\alpha,\ f: X \to R_\alpha$ and let $\boldsymbol{x_0} \in R_\alpha$ and $\boldsymbol{b} \in R_\alpha$. If for all $\boldsymbol{\varepsilon} > \boldsymbol{0}$ there exists a number $\boldsymbol{\delta} = \delta(\boldsymbol{\varepsilon}) > \boldsymbol{0}$ such that,

$$\boldsymbol{0} < |\boldsymbol{x} \ominus \boldsymbol{x_0}|_\alpha < \boldsymbol{\delta} \text{ then, } |f(x) \ominus \boldsymbol{b}|_\alpha < \boldsymbol{\varepsilon}$$

then the limit of $f(x)$ as $\boldsymbol{x}$ tends to $\boldsymbol{x_0}$ is $\boldsymbol{b}$ is,

$$\lim_{x \to x_0} {}^{\alpha} f(x) = \boldsymbol{b}$$

Since the α-limit is defined in this way one can define the α-continuity in Definition 2.2.2.

Definition 2.2.2($\boldsymbol{\alpha}$-continuity): f is continuous at $\boldsymbol{x_0}$ or,

$$\lim_{x \to x_0} {}^{\alpha} f(x) = \boldsymbol{b}$$

if for all $\boldsymbol{\varepsilon} > \boldsymbol{0}$ there exists a $\boldsymbol{\delta} > \boldsymbol{0}$ such that,

$$|\boldsymbol{x} \ominus \boldsymbol{x_0}|_\alpha < \boldsymbol{\delta}$$

implies

$$|f(x) \ominus \boldsymbol{b}|_\alpha < \boldsymbol{\varepsilon}$$

Definition 2.2.3($\boldsymbol{\alpha}$-derivative): Given that a function $f(c, d) \in R \to R_\alpha$. One can define the α-derivative $f_\alpha^*(y)$ in limit notation,

$$f_\alpha^*(y) = \lim_{x \to y} \frac{f(x) \ominus f(y)}{x \ominus y}$$

If the limit above exists then the value of this limit is the α-derivative of f at y . This limit can be represented as $f_\alpha^*(y)$.

Note that the function f is said to be alphadifferentiable if the function is positive and differentiable at y or on set A. By alpha bijection,

$$f_{\alpha}^{*}(y) = \lim_{x \to y} \frac{f(x) \ominus f(y)}{x \ominus y} = \lim_{x \to y} \frac{\alpha\{\alpha^{-1} f(x) - \alpha^{-1} f(y)\}}{x - y}$$

$$= \lim_{x \to y} \alpha \left(\frac{\{\alpha^{-1} f(x) - \alpha^{-1} f(y)\}}{\alpha^{-1}(x) - \alpha^{-1}(y)} \right)$$

Rewriting the above limit as,

$$\lim_{x \to y} \alpha \left(\frac{\{\alpha^{-1} f(x) - \alpha^{-1} f(y)\}}{x - y} \cdot \frac{x - y}{\alpha^{-1}(x) - \alpha^{-1}(y)} \right)$$

By separating the limit we obtain,

$$\lim_{x \to y} \alpha \left(\frac{\{\alpha^{-1} f(x) - \alpha^{-1} f(y)\}}{x - y} \right) . \lim_{x \to y} \alpha \left(\frac{x - y}{\alpha^{-1}(x) - \alpha^{-1}(y)} \right)$$

$$= \alpha \left(\lim_{x \to y} \alpha \left(\frac{\{\alpha^{-1} f(x) - \alpha^{-1} f(y)\}}{x - y} \right) . \lim_{x \to y} \alpha \left(\frac{x - y}{\alpha^{-1}(x) - \alpha^{-1}(y)} \right) \right)$$

By using definition 2.2.2 and the definition of Newtonian derivative, $\frac{f(x) - f(y)}{x - y} = f'(x)$

$$f_{\alpha}^{*}(y) = \alpha \left(\frac{\alpha^{-1}(f'(y))}{\alpha^{-1}(y)'} \right)$$

Example when $f(x) = x^2$, the α-derivative of function f at y is,

$$f_{\alpha}^{*}(y) = \alpha \left(\frac{\alpha^{-1}(f'(y))}{\alpha^{-1}(y)'} \right)$$

Which is equivalent to,

$$\alpha \left\{ \frac{\left(\alpha^{-1} \alpha \left\{ \left(\alpha^{-1}(x) \right)^2 \right\} \right)' \{y\}}{\alpha^{-1}(y)'} \right\}$$

Taking the derivative of above one can obtain,

$$= \alpha \left\{ \frac{2\left(\alpha^{-1}(y)\right)(\alpha^{-1})'\{y\}}{\alpha^{-1}(y)'} \right\}$$

$$= \alpha\{2\alpha^{-1}(y)\}$$

If we take $\alpha = e^x$ and $f(x) = x^2$ are taken then the above result can be written in the following way;

$$f_\alpha^*(y) = e^{2\ln(x)} = x^2$$

Note: $f_\alpha^*(y)$ differentiation is a linear operator. Suppose $l \in R_\alpha$ be a constant then,

$$f_\alpha^*(h \oplus_\alpha g) = f_\alpha^*(h) \oplus_\alpha f_\alpha^*(g)$$

Which shows the addition property of linear operators. Also,

$$f_\alpha^*(l \otimes_\alpha h) = l \otimes_\alpha f_\alpha^*(h)$$

the above result proves the multiplication with a scalar property.

Chapter 3

APPLICATIONS OF α-ARITHMETIC

In this Chapter the $\alpha = e^x$ is taken which gave rise to many theorems and definitions in non-Newtonian calculus. Also, in this section some theorems and definitions are given for Geometric derivatives, integrals, bi-geometric derivatives and integrals. Besides this, different examples are given to apply the theorems.

3.1 Multiplicative Differentiation

Definition 3.1.1(Multiplicative, geometric derivative) [5]**:** In this section geometric or in other words multiplicative derivative is defined. In this part the function f is assumed to be a positive function. One can write the limit as,

$$f^*(x) = \lim_{t\to 0}\left(\frac{f(x+t)}{f(x)}\right)^{\frac{1}{t}}$$

If $f^*(x)$ exists for all x from the open set, $A \subseteq R$ then $f^*: A(c,d) \to R$ is well defined. Besides this, by assuming that f is a positive function one can write $f^*(x)$ as,

$$f^*(x) = \lim_{t\to 0}\left(\frac{f(x+t)}{f(x)}\right)^{\frac{1}{t}} = \lim_{t\to 0}\left(\frac{f(x)}{f(x)} - \frac{f(x)}{f(x)} + \frac{f(x+t)}{f(x)}\right)^{\frac{1}{t}}$$

$$= \lim_{t\to 0}\left(\frac{f(x)}{f(x)} + \frac{f(x+t)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+t)-f(x)}\cdot\frac{1}{f(x)}\cdot\frac{f(x+t)-f(x)}{t}}$$

If we use $f'(x)$ limit definition we obtain,

$$e^{\frac{f'(x)}{f(x)}} = e^{(\ln f(x))'}$$

The higher order derivatives can be derived by using above definition. If we repeat this operation k times we have,

$$f^{*k}(x) = e^{(\ln\circ f)^{(k)}(x)} \qquad\qquad k \in \mathbb{N}$$

Corollary 3.1.2: One can write the general geometric differential rule in by using α in following way,

$$f^{\alpha}(\boldsymbol{x}) = \alpha\left(\frac{\alpha^{-1}(f(\boldsymbol{x}))'}{\alpha^{-1}(\boldsymbol{x})}\right)$$

where $\boldsymbol{x} = \alpha(x)$. This is the general geometric differentiation rule which is used in further chapters.

Theorem 3.1.3[5]**:** Let f be a positive function then we can say that function f is *differentiable at x_0 if it is positive and differentiable at x.

Proof: Taking a function f that is positive and *differentiable at the point x_0. By the $\ln\circ f$ used in the definition from 2.1.1 one can say that the $\ln\circ f$ is classically differentiable (Newtonian differentiable). On the other hand, if f is classically differentiable at x_0 then $\ln\circ f$ is also classically differentiable. Which further shows that $e^{(\ln f(x_0))'}$ exist. Therefore, one can say that f is *differentiable.

On the other hand, k^{th} Newtonian differentiation $(f^{k}(x)$ can be expressed in terms of $f^{*k}(x)$ in following way

$$(\ln\circ f)^{k}(x) = (\ln\circ f)^{*k}(x) = ((\ln\circ f)^{*n})^{k-n}(x) = ((\ln\circ f)^{n})^{k-n}$$

Using the above line one can define the first and higher order derivatives as,

$$f'(x) = f(x)(\ln\circ f^{*})(x)$$

For second order derivative product rule differentiation is used,

$$f''(x) = f(x)(\ln\circ f^{**})(x) + f'(x)(\ln\circ f^{*})(x)$$

If this procedure is followed for k times the following formula can be obtained,

$$f^{k}(x)=\sum_{n=0}^{k-1}\frac{(k-1)!}{(k-n-1)!\,k!}f^{n}(x)\big(\ln\circ f^{*(k-n)}\big)(x)$$

The *derivative of a constant function $f(x)=2a$ with a being strictly positive can be defined according to above definitions as,

$$f^{*}(x)=e^{(\ln 2a)'}=e^{0}=1, \qquad x\in(c,d)$$

Proposition 3.1.4[5]**:** If $f^{*}(x)=1$ is chosen for every $x\in(c,d)$ then,

$$f^{*}(x)=e^{(\ln\circ f)'(x)}=1$$

Proposition 3.1.5[5]**:** If function f is *differentiable at the point x_0 then the function of f is continuous at x_0.

Proof: Suppose that the function f is *differentiable at x_0.From Theorem 3.1.2 the function f is described as being positive and classically differentiable at x_0. Since, all functions which are classically differentiable are also continuous proves that function f is continuous at x_0.

Reminder: *differentiable functions are continuous functions. However, like in classical analysis the reverse statement is not always true. There are some functions which are not differentiable can be continuous. Same statement can be applied in *differentiation.

Example: Let f be a function which is continuous;

$$f(a)=\begin{cases}1+a, & \text{for} -1\le a\le 0\\ 1-a^{2}, & \text{for } 0<a<1\end{cases}$$

As it can be seen from this function the continuity condition holds,

$$\lim_{h\to 0^{-}}f(h)=\lim_{h\to 0^{+}}f(h)=f(0)=1$$

This shows that function $f(a)$ is continuous at $x_0 = 0$.

Besides this, one can calculate the *differentiation as,

$$\lim_{h\to 0^-}\left(\frac{f(0+h)}{f(0)}\right)^{\frac{1}{h}}$$

By rewriting this limit,

$$= e^{\lim\limits_{h\to 0^-}\left[\frac{(\ln\circ f)(h)-(\ln\circ f)(0)}{h}\right]}$$

As a result, one can obtain,

$$e^{\lim\limits_{h\to 0^-}\left[\frac{(\ln\circ f)(h)}{h}\right]} = e^{\lim\limits_{h\to 0^-}\left[\frac{\ln(1+h)}{h}\right]} = e$$

Now computing the right limit,

$$\lim_{h\to 0^+}\left(\frac{f(0+h)}{f(0)}\right)^{\frac{1}{h}}$$

If this limit is rewritten then,

$$= e^{\lim\limits_{h\to 0^+}\left[\frac{(\ln\circ f)(h)-(\ln\circ f)(0)}{h}\right]}$$

As above solution for lhs limit,

$$= e^{\lim\limits_{h\to 0^+}\left[\frac{(\ln\circ f)(h)}{h}\right]}$$

If the function f is substituted in this limit the result will be,

$$= e^{\lim\limits_{h\to 0^+}\left[\frac{\ln(1-h^2)}{h}\right]} = e^0 = 1$$

Since left limit is not equal to the right limit $f^*(0)$ does not exist. Hence function f is not *differentiable but continuous.

Theorem 3.1.6[5]**:** Suppose that Let f, g and h be three different functions which are *differentiable then the properties of multiplicative derivatives(*derivatives) is given below,

i) $(af)^*(x) = f^*(x),$ where $a > 0$ is a constant

ii) $(fh)^*(x) = f^*(x)h^*(x)$

iii) $\left(\frac{f}{h}\right)^*(x) = \frac{f^*(x)}{h^*(x)}$

iv) $(f^g)^*(x) = f^*(x)^{g(x)}.f(x)^{g'(x)}$

v) $(f \circ g)^*(x) = f^*\big(g(x)\big)^{g'(x)}$

Proof i) Let a be a constant and f be a *differentiable function. Therefore,

$$(af)^*(x) = e^{(\ln\circ(af))'(x)}$$

By the properties of logarithm one can write,

$$= e^{(\ln a)'(x)+(\ln\circ f)'(x)} = e^0.e^{(\ln\circ f)'(x)} = f^*(x)$$

Proof ii) Let f and h be two different functions which are *differentiable then,

$$(fh)^*(x) = e^{(\ln\circ(fh))'(x)}$$

Using the similar logic from the proof of part i;

$$= e^{(\ln\circ f)'(x)+(\ln\circ h)'(x)} = e^{(\ln\circ f)'(x)}.e^{(\ln\circ h)'(x)} = f^*(x)\cdot h^*(x)$$

Proof iii) Let f and h be two different functions which are *differentiable then,

$$\left(\frac{f}{h}\right)^*(x) = e^{\left(\ln\circ\left(\frac{f}{h}\right)\right)'(x)}$$

$$= e^{(\ln\circ f)'(x)-(\ln\circ h)'(x)} = \frac{e^{(\ln\circ f)'(x)}}{e^{(\ln\circ h)'(x)}} = \frac{f^*(x)}{h^*(x)}$$

Proof iv) Let f and g be two different functions which are *differentiable then,

$$(f^g)^*(x) = e^{(g\cdot\ln\circ(f))'(x)}$$

By using the product rule differentiation one can obtain,

$$= e^{g'(x)\cdot(\ln\circ(f)(x))+g(x)\cdot\ln\circ(f)'(x)}$$

By the properties of exponential functions,

$$= e^{g'(x)\cdot(\ln\circ(f)(x))}\cdot e^{g(x)\cdot\ln\circ(f)'(x)}$$

After doing a straightforward substitution of *differentiation the result will be obtained as,

$$= f^*(x)^{g(x)}.f(x)^{g'(x)}$$

Proof v) Let f and g be two different functions which are *differentiable then,

$$(f \circ g)^*(x) = f\big(g(x)\big)^*$$

According to the definition of multiplicative derivative one can write the above as,

$$= e^{\ln f\left(g(x)\right)'}$$

By using the appropriate chain rule one can obtain the following,

$$= e^{(\ln(f))'\left(g(x)\right)\cdot g'(x)}$$

After using the properties of logarithm and power functions one can write the above equation as,

$$= \left(e^{(\ln(f))'\left(g(x)\right)}\right)^{g'(x)}$$

By using the definition of *differentiation one can obtain the desired result,

$$= f^*\big(g(x)\big)^{g'(x)}$$

Example: Let function $h(x)$ be *differentiable and assume that $f(x) = \frac{1}{h(x)}$ then by using the theorem above,

$$f^*(x) = \frac{1}{h^*(x)}$$

To give an example assume that $f(x) = \frac{1}{x^3}$. Here the $h(x) = x^3$ by considering the above operation one can calculate the *derivative as follows.

$$f^*(x) = \frac{1}{e^{(\ln\left(\mathrm{h(x)}\right)'}}$$

By using this equation one can find the *derivative of $f(x) = \frac{1}{x^3}$ in following way,

$$f^*(x) = \frac{1}{e^{(\ln(\mathrm{x}^3)'}}$$

after taking the Newtonian derivative at the denominator the result will be,

$$= \frac{1}{e^{\frac{3x}{x^3}}}$$

If this equation is simplified the following result will be obtained,

$$= e^{-3/x^2}$$

If $\alpha = e^x$ is taken the above properties can be written in these ways, [6]

i) $\left(f \oplus_{exp} a\right)^*(x) = f^*(x),$ where $a > 0$ is a constant

ii) $\left(f \oplus_{exp} h\right)^*(x) = f^*(x) \oplus_{exp} h^*(x)$

iii) $\left(f \ominus_{exp} h\right)^*(x) = f^*(x) \ominus_{exp} h^*(x)$

iv) $\left(f \otimes_{exp} h\right)^*(x) = f^*(x) \otimes_{exp} h(x) \oplus_{exp} f(x) \otimes_{exp} h'(x)$

v) $(f \circ g)^*(x) = f^*(g(x)) \otimes_{exp} h'(x)$

3.2 Multiplicative Mean Value Theorem

In this section some theorems about Multiplicative Mean Value Theorem are given. Like in classical Newtonian Calculus Mean Value Theorem plays an important role in problem solving in Mathematical Analysis.

Proposition 3.2.1: Recalling Proposition 3.1.2 suppose that the function f is positive, $f^*(x) = 1$ if and only if $f'(x) = 0$.

Proof: If $f'(x) = 0$ then,

$$f^*(x) = e^{\frac{f'(x)}{f(x)}} = 1$$

*differentiation is used in this proof which was defined in definition 2.1.1.

Theorem 3.2.2:(Multiplicative Rolle's theorem): If a function f is *differentiable on an open interval (c, d) and continuous on the closed interval of $[c, d]$. Also, in this case $f(c) = f(d)$

$$f^*(x) = 1$$

Then there exists a constant v in the open interval (c, d).

Proof: Assume that function f is a continuous and positive function on the interval $[c, d]$. From Multiplicative Rolle's theorem it is known that function f is Newtonian differentiable and $f(c) = f(d)$. This means that there exists a constant v in the open interval (c, d) such that $f'(v) = 0$. Furthermore, by using the Proposition 2.2.1,

$$f^*(v) = 1$$

Theorem 3.2.3:(Multiplicative Mean Value Theorem) [5] : Let f be a function that is continuous on the closed interval $[c, d]$ and *differentiable on the open interval of (c, d) then there exist a constant v such that $c < v < d$. Based on these statements the mean value theorem for multiplicative calculus is given as,

$$f^*(v)^{d-c} = \frac{f(d)}{f(c)}$$

First, define an F function as below;

$$F(x) = f(c)\left[\frac{f(d)}{f(c)}\right]^{\frac{x-c}{d-c}}$$

From here $F(d) = f(d)$ and $F(c) = f(c)$.

On the other hand, suppose that there is a function H defined as, $H(x) = \frac{F(x)}{f(x)}$. Then the defined function becomes,

$$H(c) = H(d) = 1.$$

By theorem 3.2.2 we have a constant $v \in (c,d)$ such that $H^*(v) = 1$. By using the properties of *differentiation $H^*(v)$ is written as,

$$H^*(v) = \frac{F^*(v)}{f^*(v)} = 1$$

Therefore, $f^*(v) = F^*(v)$. Since this equality holds one can write,

$$f^*(v) = F^*(v) = e^{\left(\ln\left(\left[\frac{f(d)}{f(c)}\right]^{\frac{x-c}{d-c}}\right)\right)'}$$

After simplification the above equation becomes,

$$= e^{\frac{1}{d-c}\left(\ln\left(\frac{f(d)}{f(c)}\right)\right)}$$

By the properties of logarithm and simplification the Mean Value Theorem can be obtained,

$$= \left[\frac{f(d)}{f(c)}\right]^{\frac{1}{d-c}}$$

which proves the Multiplicative Mean Value Theorem

$$f^*(v)^{d-c} = \frac{f(d)}{f(c)}$$

Proposition 3.2.4 [5]: Let f be a function defined as, $f(c,d) \rightarrow R$ and suppose that function f is *differentiable. Then,

i) The function f is a decreasing function if for all $x \in (c,d)$, $f^*(x) < 1$

ii) The function f is an increasing function if for all $x \in (c,d)$, $f^*(x) > 1$.

iii) The function f is a monotonically decreasing function if for all $x \in (c,d)$, $f^*(x) \leq 1$.

iv) The function f is a monotonically increasing function if for all $x \in (c,d)$, $f^*(x) \geq 1$.

Proof i) Let f be continuous on the interval (x, y) and assume that f is *differentiable on closed interval $[x, y]$. Also, suppose that the inequality $c < x < y < d$. By the use of Mean Value Theorem for some $v \in (x, y)$,

$$f^*(v)^{y-x} = \frac{f(y)}{f(x)}$$

Since, $f^*(v) > 1$ the above equation becomes,

$$1 > \left(\frac{f(y)}{f(x)}\right)^{\frac{1}{y-x}}$$

If the above equation is simplified then,

$$1 > \frac{f(y)}{f(x)}$$

Which leads into,

$$f(x) > f(y)$$

Proof ii) Let f be continuous on the interval (x, y) and assume that f is *differentiable on closed interval $[x, y]$. Also, suppose that the inequality $c < x < y < d$. By the use of Mean Value Theorem for some $v \in (x, y)$,

$$f^*(v)^{y-x} = \frac{f(y)}{f(x)}$$

As in the proof of i since $f^*(v) > 1$ the result will be,

$$1 < \left(\frac{f(y)}{f(x)}\right)^{\frac{1}{y-x}}$$

If continued in this way similar to part I proof one can obtain,

$$1 < \frac{f(y)}{f(x)}$$

Which leads into,

$$f(x) < f(y)$$

which completes the proof.

Proof iii) The proof of iii is very similar to the proof of i. Let f be continuous on the interval (x, y) and assume that f is *differentiable on closed interval $[x, y]$. Also, suppose that the inequality $c < x < y < d$.

$$1 \geq \frac{f(y)}{f(x)}$$

Which leads into,

$$f(x) \geq f(y)$$

Which completes the proof. The proof of iv is almost identical. If the inequality signs are changed the final result will be,

$$f(x) \geq f(y)$$

3.3 Multiplicative Integral

In this section, the multiplicative antiderivative definition is given. On the other hand, some important properties of multiplicative integral are going to be discussed. Besides this, the definite multiplicative integral definition and the relationship between the Newtonian integration is given.

Definition 3.3.1: Let B be an open set representing (r, s). If $h^{*}(x) = g(x)$ for each $x \in (r, s)$ then $h: B \to R$ is said to be the *antiderivative of $g: B \to R$. In this section, the *antiderivative is going to be represented as,

$$* \int g(x)^{dx} = h(x)$$

As it can be seen from above, unlike the classical integration the dx is written as a power of $g(x)$. According to the *differentiation definition one can define the *antiderivative as,

$$g(x) = e^{\left(\ln\left(h(x)\right)\right)'}$$

As a result if the relevant substitution is done one can obtain the *integration

$$* \int g(x)^{dx} = e^{\int \ln\left(h(x)\right)dx}$$

Based on this equation the antiderivative u of $\ln \circ\, h$ function between $\int \ln\bigl(h(x)\bigr)\, dx$ and $\int g(x)^{dx}$ and the *antiderivative of h can be defined in this way,

$$h(x) = e^{u(x)}, r \leq x \leq s$$

Theorem 3.3.2:(Definite Multiplicative integration) [5]: Let h be a function defined in the closed interval $[r, s]$. Then by taking the partition of the closed interval $[r, s]$, $r = x_0 < x_1 < x_2 \ldots < x_n$. Then one can take the inequality of $x_i < a_i < x_{i+1}$ for any $i = 0,1, \ldots k - 1$ and define the integral summation by using Riemann Integration

$$T_k = \sum_{i=0}^{k-1} h(a_i)[x_{i+1} - x_i]$$

To define *integration of function h on $[r, s]$ the summation notation should change so that the following is obtained,

$$C_k = \prod_{i=0}^{k-1} h(a_i)^{x_{i+1}-x_i}$$

If C_k has a limit that exists and if this limit is independent on the selection of numbers which are associated with partition defined above, also if $\underset{i=0,1,\ldots k-1}{\mathrm{A} = \max}(x_{i+1} - x_i)$ tends to zero, the *definite integral can be defined as,

$$* \int_r^s g(x)^{dx} = \lim_{A \to 0} \prod_{i=0}^{k-1} h(a_i)^{x_{i+1}-x_i}$$

Proposition 3.3.3 [5]: If function $g(x)$ is positive and continuous on the closed interval $[r, s]$ then function $g(x)$ is said to be *integrable on the open interval (r, s). Considering this statement one can write,

$$*\int_r^s g(x)^{dx} = e^{\left(\int_r^s \ln(g(x))dx\right)}$$

Proof: Assume that the function $g(x)$ is positive and continuous on the closed interval $[r, s]$ then,

$$*\int_r^s g(x)^{dx} = \lim_{A\to 0} \prod_{i=0}^{k-1} h(a_i)^{x_{i+1}-x_i}$$

By converting the right hand side of the above equation one can obtain,

$$= \lim_{A\to 0} e^{\sum_{i=0}^{k-1} \ln g(x)[x_{i+1}-x_i]}$$

By using Proposition 3.3.3 the above limit will become,

$$= e^{\left(\int_r^s \ln(g(x))dx\right)}$$

Corollary 3.3.4: One can define the general version of this integration by considering α in following way,

$$\int_r^s \boldsymbol{g}(\boldsymbol{x})^{d^\alpha \boldsymbol{x}} = \alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{g}(\boldsymbol{x}))d\boldsymbol{x}\right)$$

Reminder: In this case boldface letters represent, $\alpha(g(x)) = \boldsymbol{g}(\boldsymbol{x})$ to prevent confusion.

Theorem 3.3.5:(Properties of *integration) [5]: Suppose that h and g are *integrable on the open interval (r, s) and let both of these functions be positive and continuous on this interval then $g \cdot h$, g^n,g/h are differentiable and the following properties hold;

i) $* \int_r^s \left(g(x)h(x)\right)^{dx} = * \int_r^s \left(g(x)\right)^{dx} \cdot * \int_r^s \left(h(x)\right)^{dx}$

ii) $* \int_r^s (g(x)^n)^{dx} = \left(* \int_r^s \left(g(x)\right)^{dx}\right)^n$

iii) $* \int_r^s \left(\frac{g(x)}{h(x)}\right)^{dx} = \frac{*\int_r^s \left(g(x)\right)^{dx}}{*\int_r^s \left(h(x)\right)^{dx}}$

Proof i) Assume that the functions h and g are *integrable on the closed interval of $[r, s]$. By using the proposition 2.3.3 the *integral can be written as,

$$*\int_r^s \big(g(x)h(x)\big)^{dx} = e^{\int_r^s \ln(g(x)\cdot h(x))dx}$$

By using the properties of logarithm, the above equation becomes,

$$= e^{\int_r^s \ln(g(x)\cdot h(x))dx} = e^{\int_r^s \ln(g(x)+h(x))dx}$$

One can separate inside of the integral as,

$$= e^{(\int_r^s \ln(g(x))dx)+(\int_r^s \ln(h(x))dx)}$$

After applying the *integration the result will be obtained as,

$$*\int_r^s \big(g(x)h(x)\big)^{dx} = *\int_r^s \big(g(x)\big)^{dx} \cdot *\int_r^s \big(h(x)\big)^{dx}$$

Proof ii) Let g be a function that is *integrable on the closed interval $[r, s]$. Since, it is assumed that g is positive and continuous on this closed interval, for all $n \in R$,

$$*\int_r^s (g(x)^n)^{dx} = e^{\int_r^s \ln(g(x))^n dx}$$

in this case Proposition 2.3.3 is used. Continuing with the rules of the logarithm one can write the above equation in the following way,

$$= e^{\int_r^s \mathrm{n}\ln(\mathrm{g(x)})dx} = e^{n\int_r^s \ln(\mathrm{g(x)})dx}$$

after simplifying this equation,

$$= \left(e^{\int_r^s \ln(\mathrm{g(x)})dx}\right)^n = \left(*\int_r^s \big(g(x)\big)^{dx}\right)^n$$

Which proves part ii.

Proof iii) Similar to the proof of part i let h and g be functions which are *integrable on the closed interval of $[r, s]$. Then,

$$*\int_r^s \left(\frac{g(x)}{h(x)}\right)^{dx} = e^{\int_r^s \ln\left(\frac{g(x)}{h(x)}\right)dx}$$

By the properties of logarithm one can convert the above equation into,

$$= e^{\int_r^s \ln(g(x)-h(x))dx} = e^{\int_r^s \ln(g(x))dx-\int_r^s \ln(h(x))dx}$$

If the definition of *integration is applied then,

$$*\int_r^s \left(\frac{g(x)}{h(x)}\right)^{dx} = \frac{*\int_r^s (g(x))^{dx}}{*\int_r^s (h(x))^{dx}}$$

which proves iii.

Theorem 3.3.6 (Fundamental Theorem of Multiplicative Calculus): Suppose that function g is positive and continuous on the closed interval $[r, s]$. Then one can define the function G as,

$$G(x) = \int_r^x g(x)^{dx}$$

where $r \leq x \leq s$ is assumed. If function G is defined in this way, then G becomes one of the *antiderivatives of the function g. Furthermore, if $F(x)$ is also a *antiderivative of g inside of the interval $[r, s]$ then the following equation holds,

$$\int_r^s g(x)^{dx} = \frac{F(s)}{F(r)}$$

Proof: To begin with one can use *differentiation to obtain the following limit,

$$F'(x) = \lim_{t\to 0}\left[\frac{G(x+t)}{G(x)}\right]^{\frac{1}{t}}$$

if the integral definition of $F(x)$ is used then this limit becomes,

$$= \lim_{t\to 0}\left[\frac{e^{\left(\int_r^{x+h} \ln(g(y))dy\right)}}{e^{\left(\int_r^{x} \ln(g(y))dy\right)}}\right]^{\frac{1}{t}}$$

Using the properties of exponential function,

$$\lim_{t\to 0}\left[e^{\left(\int_r^{x+h} \ln(g(y))dy\right)-\left(\int_r^{x} \ln(g(y))dy\right)}\right]^{\frac{1}{t}}$$

By simplifying the limit above,

$$e^{\lim_{t\to 0}\left[e^{\left(\int_r^{x+h}\ln(g(y))dy\right)-\left(\int_r^{x}\ln(g(y))dy\right)}\right]^{\frac{1}{t}}}$$

This limit looks identical to the *differentiation and classical differentiation so,

$e^{\frac{d}{dz}(\int_r^z \ln(g(x)dx)}$ where $z = x$

$$= e^{\ln(g(x))} = g(x)$$

If $F'(x) = g(x)$ then $G(x) = CF(x)$. In this case constant $C \in [r, s]$. Thered=fore the following is true,

$$*\int_r^x g(x)^{dx} = G(x) = CF(x)$$

Choosing $x = b$ the $CF(x) = 1$ and if the inverse is taken on both sides,

$$\left(F(b)\right)^{-1} = C$$

holds. Furthermore, if $x = a$ is taken then the above * integration will be,

$$*\int_r^a g(x)^{dx} = G(x) = CF(a) = \frac{F(a)}{F(b)}$$

The result follows if $a = s$ and $b = r$.

3.4 Bi-geometric Differentiation and Integration

In this section the general definitions of bi-geometric differentiation and integration are given. On the other hand, $\alpha = e^x$ example is given. Bi-geometric differentiation and Integration have some differences compared to multiplicative one.

Definition 3.4.1:(General definition of Bi-geometric differentiation)[10],[7]**:** Let α be a bijection from $\alpha: R \to R_\alpha$, besides this assume that $h' = f$ is the Newtonian derivative of h then the analog $\boldsymbol{f}$ of f is the biα-differentiation of h then one can define this biα-differentiation as,

$$\boldsymbol{h}^{bi\alpha}(\boldsymbol{y}) = \alpha\left(\frac{\alpha^{-1}\big(\boldsymbol{h}(\boldsymbol{y})\big)'}{\alpha^{-1}(\boldsymbol{y})'}\right)$$

where $\alpha\big(h(y)\big) = \boldsymbol{h}(\boldsymbol{y})$.

Proof: By using the limit definition of the $bi\alpha$-differentiation,

$$\boldsymbol{h}^{bi\alpha}(\boldsymbol{y}) = \lim_{\boldsymbol{x}\to\boldsymbol{y}}\big(\boldsymbol{h}(\boldsymbol{x}) \ominus \boldsymbol{h}(\boldsymbol{y})\big) \oslash (\boldsymbol{x} - \boldsymbol{y})$$

By using a Similar calculation as the one in Chapter 2. The above equation can be rewritten as,

$$=\lim_{\boldsymbol{x}\to\boldsymbol{y}}\left(\alpha\left(h\big(\alpha^{-1}(\boldsymbol{x})\big)\right) \ominus \alpha\left(h\big(\alpha^{-1}(\boldsymbol{y})\big)\right)\right) \oslash (\boldsymbol{x} - \boldsymbol{y})$$

By using the fact that $\alpha\big(h(y)\big) = \boldsymbol{h}(\boldsymbol{y})$ one can convert the boldface $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$= \lim_{x\to y}\Big(\alpha\big(h(x)\big) \ominus \alpha\big(h(y)\big)\Big) \oslash (\alpha(x) - \alpha(y))$$

Simplifying this limit,

$$= \lim_{x\to y} \alpha\left(\frac{\left(\alpha^{-1}\Big(\big(\alpha(h(x)\big) \ominus \alpha\big(h(y)\big)\Big)\right)}{\alpha^{-1}\Big(\big(\alpha(x) - \alpha(y)\big)\Big)}\right)$$

This limit will become,

$$= \lim_{x\to y} \alpha\left(\frac{h(x) - h(y)}{x - y}\right)$$

As it can be seen from here the fraction inside of the bracket is Newtonian differentiation of $h(x)$. Hence,

$$= \alpha\big(h'(y)\big) = \boldsymbol{f}(\boldsymbol{x})$$

To complete the proof the rearranged version of $\alpha\big(h'(y)\big)$ is used, if $\alpha\big(h'(y)\big) = \boldsymbol{h}'(\boldsymbol{y})$ then,

$$\frac{\alpha^{-1}\big(\boldsymbol{h}(\boldsymbol{y})\big)'}{\alpha^{-1}(\boldsymbol{y})'} = h'\big(\alpha^{-1}(\boldsymbol{y})\big)$$

If α is taken on both sides of this equation the result follows,

$$= \alpha\left(\frac{\alpha^{-1}\big(\boldsymbol{h}(\boldsymbol{y})\big)'}{\alpha^{-1}(\boldsymbol{y})'}\right) = \alpha(h'\big(\alpha^{-1}(y)\big) = \boldsymbol{h}^{bi\alpha}(\boldsymbol{y})$$

This general rule can be applied to $\alpha(y) = e^{y}$ which is given in next example,

Example [3]**:** If $\alpha(y) = e^{y}$ is taken the bigeometric differentiation can be given as,

$$h^{biexp}(y) = e^{\left(\frac{\ln\left(h(y)\right)'}{(\ln(y))'}\right)}$$

Proof: By using the definition that has been used to prove the general case,

$$\boldsymbol{h}^{bi\alpha}(\boldsymbol{y}) = \lim_{\boldsymbol{x}\to\boldsymbol{y}}\big(\boldsymbol{h}(\boldsymbol{x}) \ominus \boldsymbol{h}(\boldsymbol{y})\big) \oslash (\boldsymbol{x}-\boldsymbol{y})$$

Here one can take inside of the limit to write,

$$=(\boldsymbol{h}(\boldsymbol{x}) \ominus \boldsymbol{h}(\boldsymbol{y})) \oslash (\boldsymbol{x}-\boldsymbol{y}) = \left(\frac{h(x)}{h(y)}\right)^{\frac{1}{\ln\left(\frac{x}{y}\right)}}$$

After doing some manipulations to the above equation,

$$= \left(\frac{h(x)}{h(y)}\right)^{\frac{1}{x-y}\cdot\frac{x-y}{\ln(x)-\ln(y)}}$$

This resulting equation implies,

$$= e^{\left(\frac{\ln\left(h(x)\right)'}{(\ln(x))'}\right)}$$

Which proves the result.

Definition 3.4.2: (Bi$\boldsymbol{\alpha}$-integration) [10], [7]: If function $h(y)$ is positive and continuous on the closed interval $[r,s]$ then function $h(y)$ is said to be Biα-integration integrable on the open interval (r,s). Considering this statement one can write,

$$\int_r^s h(y)^{d^{bi\alpha}y} = \alpha\left(\int_r^s h(y)dy\right) = \alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{y})'\alpha^{-1}\big(\boldsymbol{h}(\boldsymbol{y})\big)^{dy}\right)$$

Proof: Assume that the function $h(y)$ is positive and continuous on the closed interval $[r,s]$ then,

$$\int_r^s h(y)^{d^{bi\alpha}y} = \lim_{A\to 0}\prod_{i=0}^{k-1} h(a_i)^{\alpha^{-1}(y_{i+1}/y_i)}$$

By converting the right hand side of the above equation one can obtain,

$$= \lim_{A\to 0}\alpha\left(\sum_{i=0}^{k-1}\alpha^{-1}(\boldsymbol{h}(\boldsymbol{a_i}))(\alpha^{-1}(\boldsymbol{y_{i+1}}) - \alpha^{-1}(\boldsymbol{y_i})\right)$$

By simplifying the limit above,

$$= \lim_{A\to 0}\alpha\left(\sum_{i=0}^{k-1} h(a_i)\big((y_{i+1}) - (y_i)\big)\right)$$

As a result the above series and limit can be written as,

$$= \alpha\left(\int_r^s h\,(y)dy\right)$$

One can use this result and $\alpha^{-1}\big(\boldsymbol{h}(\boldsymbol{y})\big) = h(\alpha^{-1}(\boldsymbol{y}))$ to reach desired integral,

$$= \int_r^s \alpha^{-1}(\boldsymbol{y})'\alpha^{-1}\big(\boldsymbol{h}(\boldsymbol{y})\big)^{dy} = \int_r^s \alpha^{-1}(\boldsymbol{y})'h\big(\alpha^{-1}(\boldsymbol{y})\big)^{dy} = \int_r^s h\,(y)dy$$

In this case the substitution of $y = \alpha^{-1}(\boldsymbol{y})$ is used. The result follows after taking α on both sides of the above equation,

$$\int_r^s h(y)^{d^{bi\alpha}y} = \alpha\left(\int_r^s h(y)dy\right) = \alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{y})'\alpha^{-1}\big(\boldsymbol{h}(\boldsymbol{y})\big)^{dy}\right)$$

Example[3]: If $\alpha(y) = e^y$ is taken and substituted in above definition one can obtain bi-geometric integral for exponential function,

$$\int_r^s h(y)^{d^{biexp}y} = e^{\int_r^s \frac{\ln(h(y))}{y}dy}$$

Chapter 4

HYPERBOLIC TANGENT CALCULI

In this section the application of geometric and bi-geometric calculi is given for a different alpha function. This alpha function is taken as the hyperbolic tangent, which is represented by, $\alpha = \tanh(y)$. Also, in this Chapter the interval $\mathbb{L} = (-1,1)$. Since hyperbolic function of tangent is defined in this open interval.

4.1 The Hyperbolic Tangent Arithmetic

Theorem 4.1.1:(The Hyperbolic Tangent Arithmetic): Let $L = (-1,1)$ then one can define $\alpha(y)$ as,

$$\alpha(y) = \tanh(y)$$

This hyperbolic tangent function can be expanded by using the exponential functions as,

$$= \tanh(y) = \frac{e^{2y} - 1}{e^{2y} + 1}$$

here $y \in R$. Considering the above function the $\alpha^{-1}(y)$ can be computed as,

$$\alpha^{-1}(y) = \tanh^{-1} y = \frac{1}{2} \ln\left(\frac{1+y}{1-y}\right)$$

From now on the bold letters are used to represent elements of $\mathbb{L}$ so for example let $\boldsymbol{y}= \alpha(y)$. In this case, $\boldsymbol{y} \in L$ and $y \in R$. This is done to reduce the confusion between the elements of L and R. Taking this note and the above theorem into consideration the following algebraic operations can be defined.

Definition 4.1.2:(Operations of tanh function): Let $\boldsymbol{d} \in L$ and $\boldsymbol{c} \in L$. Then,

i) $\alpha\left(\alpha^{-1}(d) \oplus_{\boldsymbol{tanh}} \alpha^{-1}(c)\right) = \boldsymbol{d} \oplus_{\boldsymbol{tanh}} \boldsymbol{c} = \frac{\boldsymbol{d}+\boldsymbol{c}}{1+\boldsymbol{dc}}$

ii) $\alpha\left(\alpha^{-1}(d) \ominus_{\boldsymbol{tanh}} \alpha^{-1}(c)\right) = \boldsymbol{d} \ominus_{\boldsymbol{tanh}} \boldsymbol{c} = \frac{\boldsymbol{d}-\boldsymbol{c}}{1-\boldsymbol{dc}}$

iii) $\alpha\left(\alpha^{-1}(d) \otimes_{\boldsymbol{tanh}} \alpha^{-1}(c)\right) = \boldsymbol{d} \otimes_{\boldsymbol{tanh}} \boldsymbol{c} = \frac{e^{\frac{1}{2}\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)}-1}{e^{\frac{1}{2}\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)}+1}$

iv) $\alpha\left(\alpha^{-1}(d) \oslash_{\boldsymbol{tanh}} \alpha^{-1}(c)\right) = \boldsymbol{d} \oslash_{\boldsymbol{tanh}} \boldsymbol{c} = \frac{e^{2\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)/\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)}-1}{e^{2\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)/\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)}+1}$

Proof: For part i one can use the theorem 3.1.1 and definition 1.1.1 to get,

$$\boldsymbol{d} \oplus_{\boldsymbol{tanh}} \boldsymbol{c} = \alpha\{\alpha^{-1}(\boldsymbol{d}) + \alpha^{-1}(\boldsymbol{c})\} = \frac{e^{2\left(\frac{1}{2}\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)+\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)\right)}-1}{e^{2\left(\frac{1}{2}\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)+\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)\right)}+1}$$

If the properties of logarithm is used one can write this operation as,

$$= \frac{e^{\left(\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)+\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)\right)}-1}{e^{\left(\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)+\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)\right)}+1} = \frac{\frac{(1+\boldsymbol{d})(1+\boldsymbol{c})}{(1-\boldsymbol{d})(1-\boldsymbol{c})}-1}{\frac{(1+\boldsymbol{d})(1+\boldsymbol{c})}{(1-\boldsymbol{d})(1-\boldsymbol{c})}+1}$$

Simplifying the above equation,

$$= \frac{\frac{(1+\boldsymbol{d})(1+\boldsymbol{c})}{(1-\boldsymbol{d})(1-\boldsymbol{c})}-\frac{(1-\boldsymbol{d})(1-\boldsymbol{c})}{(1-\boldsymbol{d})(1-\boldsymbol{c})}}{\frac{(1+\boldsymbol{d})(1+\boldsymbol{c})}{(1-\boldsymbol{d})(1-\boldsymbol{c})}+\frac{(1-\boldsymbol{d})(1-\boldsymbol{c})}{(1-\boldsymbol{d})(1-\boldsymbol{c})}} = \frac{2\boldsymbol{d}+2\boldsymbol{c}}{2+2\boldsymbol{dc}}$$

If the above equation is simplified further the result will follow,

$$= \frac{\boldsymbol{d}+\boldsymbol{c}}{1+\boldsymbol{dc}}$$

Proof of ii) is almost identical,

$$= \alpha\{\alpha^{-1}(\boldsymbol{d}) - \alpha^{-1}(\boldsymbol{c})\} = \boldsymbol{d} \ominus_{\boldsymbol{tanh}} \boldsymbol{c} = \frac{e^{2\left(\frac{1}{2}\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)-\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)\right)}-1}{e^{2\left(\frac{1}{2}\ln\left(\frac{1+\boldsymbol{d}}{1-\boldsymbol{d}}\right)-\ln\left(\frac{1+\boldsymbol{c}}{1-\boldsymbol{c}}\right)\right)}+1}$$

After simplifying and using the properties of logarithm,

$$= \frac{(1+\boldsymbol{d})(1-\boldsymbol{c})-(1-\boldsymbol{d})(1+\boldsymbol{c})}{(1+\boldsymbol{d})(1-\boldsymbol{c})+(1-\boldsymbol{d})(1+\boldsymbol{c})} = \frac{2\boldsymbol{d}-2\boldsymbol{c}}{2-2\boldsymbol{dc}}$$

As a result,

$$\boldsymbol{d} \ominus_{\boldsymbol{tanh}} \boldsymbol{c} = \frac{\boldsymbol{d}-\boldsymbol{c}}{\boldsymbol{1}-\boldsymbol{dc}}$$

is obtained.

The proof of iii) and iv) follows from the theorem 3.1.1 immediately.

Theorem 4.1.3:(Geometric differentiation of $\boldsymbol{tanh}$): Let $\alpha = \tanh(y)$. From the definition of multiplicative derivative one can write g^* as,

$$g^*(\boldsymbol{y}) = \alpha\left(\frac{\alpha^{-1}(\boldsymbol{g}(\boldsymbol{y}))'}{\alpha^{-1}(\boldsymbol{y})}\right)$$

Since $\alpha^{-1}(\boldsymbol{g}(\boldsymbol{y})) = \tanh^{-1}\boldsymbol{g}(\boldsymbol{y})$, the derivative of $\alpha^{-1}(y)$ can be written as,

$$\alpha^{-1}(\boldsymbol{g}(\boldsymbol{y}))' = (\tanh^{-1}\boldsymbol{g}(\boldsymbol{y}))'$$

After substitution,

$$= \left(\frac{1}{2}\ln\left(\frac{1+\boldsymbol{g}(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})}\right)\right)' = \frac{\boldsymbol{g}'(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})^2}$$

By using this definition one can obtain,

$$g^*(\boldsymbol{y}) = \frac{e^{\left(\frac{2\boldsymbol{g}'(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})^2}\right)}-1}{e^{\left(\frac{2\boldsymbol{g}'(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})^2}\right)}+1}$$

Theorem 4.1.4:(Geometric integration of hyperbolic tangent): By using the similar idea from *integration the integration of hyperbolic tangent can be expressed as,

$$\int_r^s \boldsymbol{g}(\boldsymbol{y})^{dy} = \frac{e^{2\left(\int_r^s \ln\frac{1+g(y)}{1-g(y)}dy\right)}-1}{e^{2\left(\int_r^s \ln\frac{1+g(y)}{1-g(y)}dy\right)}+1}$$

Proof: By using the geometric integration,

$$\int_r^s \boldsymbol{g}(\boldsymbol{y})^{dy} = \alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{g}(\boldsymbol{y}))d\boldsymbol{y}\right)$$

Since $\alpha^{-1}(\boldsymbol{g}(\boldsymbol{y})) = \tanh^{-1} \boldsymbol{g}(\boldsymbol{y}) = \frac{1}{2}\ln\left(\frac{1+\boldsymbol{g}(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})}\right)$ one can write,

$$\alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{g}(\boldsymbol{y}))d\boldsymbol{y}\right) = \alpha\left(\int_r^s \frac{1}{2}\ln\left(\frac{1+\boldsymbol{g}(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})}\right)d\boldsymbol{y}\right)$$

If this integral is substituted into the definition of hyperbolic tangent the result will follow,

$$= \frac{e^{2\left(\int_r^s \ln\frac{1+g(y)}{1-g(y)}dy\right)} - 1}{e^{2\left(\int_r^s \ln\frac{1+g(y)}{1-g(y)}dy\right)} + 1}$$

4.2 Bi-Geometric Calculus of Hyperbolic Tangent

Theorem 4.2.1:(Bi-geometric integration for the hyperbolic tangent): By using the definition of Bi-geometric integral in general,

$$\int_r^s \boldsymbol{g}(\boldsymbol{y})^{dbi*\boldsymbol{y}} = \frac{e^{2\left(\int_r^s \frac{1}{1-y^2}\ln\frac{1+g(y)}{1-g(y)}dy\right)} - 1}{e^{2\left(\int_r^s \frac{1}{1-y^2}\ln\frac{1+g(y)}{1-g(y)}dy\right)} + 1}$$

Proof: By using the general definition of bigeometric integration,

$$\int_r^s \boldsymbol{g}(\boldsymbol{y})^{d\boldsymbol{y}} = \alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{y})'\alpha^{-1}(\boldsymbol{g}(\boldsymbol{y}))d\boldsymbol{y}\right)$$

Since $\alpha^{-1}(\boldsymbol{g}(\boldsymbol{y})) = \tanh^{-1} \boldsymbol{g}(\boldsymbol{y}) = \frac{1}{2}\ln\left(\frac{1+\boldsymbol{g}(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})}\right)$ one can write,

$$\alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{y})'\alpha^{-1}(\boldsymbol{g}(\boldsymbol{y}))d\boldsymbol{y}\right) = \alpha\left(\int_r^s \alpha^{-1}(\boldsymbol{y})'\frac{1}{2}\ln\left(\frac{1+\boldsymbol{g}(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})}\right)d\boldsymbol{y}\right)$$

Since $\alpha^{-1}(\boldsymbol{y})' = \frac{1}{1+\boldsymbol{y}^2}$ the above integral becomes,

$$\alpha\left(\int_r^s \left(\frac{1}{1+\boldsymbol{y}^2}\right)\frac{1}{2}\ln\left(\frac{1+\boldsymbol{g}(\boldsymbol{y})}{1-\boldsymbol{g}(\boldsymbol{y})}\right)d\boldsymbol{y}\right)$$

If this integral is substituted into the definition of hyperbolic tangent the result will follow,

$$\int_r^s \boldsymbol{g}(\boldsymbol{y})^{dbi*\boldsymbol{y}} = \frac{e^{2\left(\int_r^s \frac{1}{1-y^2}\ln\frac{1+g(y)}{1-g(y)}dy\right)} - 1}{e^{2\left(\int_r^s \frac{1}{1-y^2}\ln\frac{1+g(y)}{1-g(y)}dy\right)} + 1}$$

Theorem 4.2.2:(Bi-geometric differentiation of hyperbolic tangent): By using the definition of bi-geometric derivatives,

$$g^{bi*}(\boldsymbol{y}) = \alpha\left(\frac{\alpha^{-1}\big(\boldsymbol{g}(\boldsymbol{y})\big)'}{\alpha^{-1}(\boldsymbol{y})'}\right)$$

Considering this and theorem 3.1.3 the bigeometric derivative for hyperbolic tangent is,

$$g^{bi*}(\boldsymbol{y}) = \frac{e^{\left(\frac{2\boldsymbol{g}'(\boldsymbol{y})(1-\boldsymbol{y}^2)}{1-\boldsymbol{g}(\boldsymbol{y})^2}\right)} - 1}{e^{\left(\frac{2\boldsymbol{g}'(\boldsymbol{y})(1-\boldsymbol{y}^2)}{1-\boldsymbol{g}(\boldsymbol{y})^2}\right)} + 1}$$

Chapter 5

CUBIC CALCULUS

In this section, one of the subcategories of α-arithmetic is explained which has a name of cubic calculi with γ-arithmetic. As the name of the arithmetic suggests, γ bijection is chosen in cubic calculi. The definitions and applications of γ-arithmetic is given throughout this chapter and some rules of differentiation and integration are given. Besides this, in final parts of this chapter some information about quadratic calculi is given.

5.1 Cubic Arithmetic and Cubic Real Numbers

Definition 5.1.1 [11] : In this definition the general case is given by taking $p \in N$, for all $y \in R$ the $\gamma_p: R \rightarrow R_\gamma \subseteq R$ function is,

$$\gamma_p(y) = \begin{cases} y^{\frac{1}{p}}, & y > 0 \\ 0, & y = 0 \\ -(-y)^{\frac{1}{p}}, & y < 0 \end{cases}$$

and the inverse of this function can be defined as,

$$\gamma_p^{-1}(y) = \begin{cases} y^p, & y > 0 \\ 0, & y = 0 \\ -(-y)^p, & y < 0 \end{cases}$$

Considering this definition p-root and p-power bijection can be created. Furthermore, by the help of these bijections γ_p-arithmetic can be generated $\alpha(y) = \gamma_p(y)$. This is the subclass of α-arithmetic defined in Chapter 2.

If $p = 3$ is taken then γ-arithmetic can be defined in the following way, For all $y \in R$ and $p = 3$,

$$\gamma(y) = \begin{cases} y^{\frac{1}{3}}, & y > 0 \\ 0, & y = 0 \\ -(-y)^{\frac{1}{3}}, & y < 0 \end{cases}$$

and inverse can be defined as,

$$\gamma^{-1}(y) = \begin{cases} y^3, & y > 0 \\ 0, & y = 0 \\ -(-y)^3, & y < 0 \end{cases}$$

using the function and inverse above one can define the algebraic operations in general which is given in the next definition.

Definition 5.1.2a: From definition 5.1.1 the γ_p and the inverse is used to define the following algebraic operations:

For all $\boldsymbol{x}, \boldsymbol{y} \in R_\gamma$, where $R_\gamma \subseteq R$ and $R_\gamma = \{\boldsymbol{y} = \gamma(y) \colon y \in R\}$

γ-summation: $\boldsymbol{x} \oplus_\gamma \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) + \gamma^{-1}(\boldsymbol{y})\}$

γ-subtraction: $\boldsymbol{x} \ominus_\gamma \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) - \gamma^{-1}(\boldsymbol{y})\}$

γ-multiplication: $\boldsymbol{x} \otimes_\gamma \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) \times \gamma^{-1}(\boldsymbol{y})\}$

γ-division: $\boldsymbol{x} \oslash_\gamma \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) \div \gamma^{-1}(\boldsymbol{y})\}$

γ-ordering: $\boldsymbol{x} \leq \boldsymbol{y} = \gamma^{-1}(\boldsymbol{x}) \leq \gamma^{-1}(\boldsymbol{y})$

Definition 5.1.2b: From definition 4.1.1 or in other words if $p = 3$ is taken, then there will be two sets which are,

$R_\gamma^+ = \left\{x^{\frac{1}{3}} \in R_\gamma : x > 0\right\}$ and $R_\gamma^- = \left\{-(-x)^{\frac{1}{3}} : x < 0\right\}$

For all $x, y \in R_\gamma^+$

γ-summation: $\boldsymbol{x} \oplus \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) + \gamma^{-1}(\boldsymbol{y})\} = (x + y)^{1/3}$

γ-subtraction: $\boldsymbol{x} \ominus \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) - \gamma^{-1}(\boldsymbol{y})\} = (x - y)^{1/3}$

γ-multiplication: $\boldsymbol{x} \otimes \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) \times \gamma^{-1}(\boldsymbol{y})\} = (xy)^{1/3}$

γ-division: $\boldsymbol{x} \oslash \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) \div \gamma^{-1}(\boldsymbol{y})\} = \left(\frac{x}{y}\right)^{1/3}$

γ-ordering: $\boldsymbol{x} \leq \boldsymbol{y} = \gamma^{-1}(x) \leq \gamma^{-1}(y)$

The proof of γ-summation can be obtained by rewriting the first operation in the following way,

$$\gamma\{\gamma^{-1}(\boldsymbol{x}) + \gamma^{-1}(\boldsymbol{y})\} = \gamma\left\{\left(x^{\frac{1}{3}}\right)^3 + \left(y^{\frac{1}{3}}\right)^3\right\}$$

If this equation is simplified further,

$$= (x + y)^{1/3}$$

can be found.

for γ-subtraction one can follow a similar way as above,

$$\gamma\{\gamma^{-1}(\boldsymbol{x}) - \gamma^{-1}(\boldsymbol{y})\} = \gamma\left\{\left(x^{\frac{1}{3}}\right)^3 - \left(y^{\frac{1}{3}}\right)^3\right\} = (x - y)^{1/3}$$

for γ-multiplication,

$$\gamma\{\gamma^{-1}(\boldsymbol{x}) \times \gamma^{-1}(\boldsymbol{y})\} = \gamma\left\{\left(x^{\frac{1}{3}}\right)^3 \times \left(y^{\frac{1}{3}}\right)^3\right\} = (xy)^{1/3}$$

finally for γ-division,

$$\gamma\{\gamma^{-1}(\boldsymbol{x}) \div \gamma^{-1}(\boldsymbol{y})\} = \left(\frac{x}{y}\right)^{1/3}$$

one can define these operations in negative set,

For all $x, y \in R_\gamma^-$

γ-summation: $\boldsymbol{x} \oplus \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) + \gamma^{-1}(\boldsymbol{y})\} = (x + y)^{1/3}$

γ-subtraction: $\boldsymbol{x} \ominus \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) - \gamma^{-1}(\boldsymbol{y})\} = (x - y)^{1/3}$

γ-multiplication: $\boldsymbol{x} \otimes \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) \times \gamma^{-1}(\boldsymbol{y})\} = (xy)^{1/3}$

γ-division: $\boldsymbol{x} \oslash \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) \div \gamma^{-1}(\boldsymbol{y})\} = \left(\frac{x}{y}\right)^{1/3}$

γ-ordering: $\boldsymbol{x} \leq \boldsymbol{y} = \gamma^{-1}(x) \leq \gamma^{-1}(y)$

The proofs are omitted since the results are same for operations.

Definition 5.1.3:(Cubic Integer Set): Let Z_γ represent the Cubic integer set. Then,

$$Z_\gamma = \{\gamma(y) : y \in R\}$$

for all $y \in Z^+$, $\boldsymbol{y} = y^{1/3}$ and for all $y \in Z^-$, $\boldsymbol{y} = -(-y)^{1/3}$. If integer set is defined in this way the Cubic integer set can be written as,

$$Z_\gamma = \{\dots, -(-2)^{\frac{1}{3}}, -(-1)^{\frac{1}{3}}, 0, (1)^{\frac{1}{3}}, (2)^{\frac{1}{3}}, \dots\}$$

Theorem 5.1.4:$(R_\gamma, \oplus, \otimes)$ is a field. Then one can define commutativity, associativity and distributivity.

Proof: First, assume that for any $\boldsymbol{x}, \boldsymbol{y} \in R_\gamma^+$ are given which are cubic real numbers then,

$$\boldsymbol{x} \oplus \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{x}) + \gamma^{-1}(\boldsymbol{y})\} = \gamma\left\{\left(x^{\frac{1}{3}}\right)^3 + \left(y^{\frac{1}{3}}\right)^3\right\}$$

the above equation becomes,

$$= (x + y)^{1/3} \in R_\gamma^+$$

this means that,

$$\boldsymbol{x} \oplus \boldsymbol{y} \in R_\gamma^+$$

Since the result is same for $\boldsymbol{x}, \boldsymbol{y} \in R_\gamma^-$ proof is omitted.

By using the result above one can prove the commutativity rule.

Assume that for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in R_\gamma^+$ which are cubic numbers then,

$$(\boldsymbol{x} \oplus \boldsymbol{y}) \oplus \boldsymbol{z} = \gamma\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right) + \gamma^{-1}\left(y^{\frac{1}{3}}\right)\right\} \oplus z^{\frac{1}{3}}$$

If the above equation is rewritten then,

$$=\gamma\left\{\gamma^{-1}\left[\gamma\left(\gamma^{-1}\left(x^{\frac{1}{3}}\right)+\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right)+\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right]\right\}$$

after further simplification,

$$=\gamma\left[\left(\gamma^{-1}\left(x^{\frac{1}{3}}\right)+\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right)+\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right]$$

the above result gives,

$$=\boldsymbol{x}\oplus(\boldsymbol{y}\oplus\boldsymbol{z})$$

since the result is same for $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in R_{\gamma}^{-}$ proof is omitted.

For distributive property Assume that for any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in R_{\gamma}^{+}$ which are cubic numbers,

$$(\boldsymbol{x}\oplus\boldsymbol{y})\otimes\boldsymbol{z}=\gamma\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right)+\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right\}\otimes\gamma^{-1}\left(z^{\frac{1}{3}}\right)$$

$$=\gamma\left\{\gamma^{-1}\left\{\gamma\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right)+\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right\}\times\gamma^{-1}\left(z^{\frac{1}{3}}\right)\right\}\right\}$$

$$=\gamma\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right)+\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right\}\times\gamma^{-1}\left(z^{\frac{1}{3}}\right)$$

$$=\gamma\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right)\times\gamma^{-1}\left(z^{\frac{1}{3}}\right)\right\}+\gamma\left\{\gamma^{-1}\left(y^{\frac{1}{3}}\right)\times\gamma^{-1}\left(z^{\frac{1}{3}}\right)\right\}$$

$$=(\boldsymbol{x}\otimes\boldsymbol{z})\oplus(\boldsymbol{y}\otimes\boldsymbol{z})$$

For associativity, for any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in R_{\gamma}^{+}$ are cubic numbers then,

$$(\boldsymbol{x}\otimes\boldsymbol{y})\otimes\boldsymbol{z}=\gamma\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right)\times\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right\}\otimes\gamma^{-1}\left(z^{\frac{1}{3}}\right)$$

$$=\gamma\left\{\gamma^{-1}\left\{\gamma\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right)\times\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right\}\times\gamma^{-1}\left(z^{\frac{1}{3}}\right)\right\}\right\}$$

$$=\gamma\left[\left\{\gamma^{-1}\left(x^{\frac{1}{3}}\right)\times\gamma^{-1}\left(y^{\frac{1}{3}}\right)\right\}\times\gamma^{-1}\left(z^{\frac{1}{3}}\right)\right]$$

one can write the top operation as,

$$=\gamma\left[\gamma^{-1}\left(x^{\frac{1}{3}}\right)\times\left\{\gamma^{-1}\left(y^{\frac{1}{3}}\right)\times\gamma^{-1}\left(z^{\frac{1}{3}}\right)\right\}\right]$$

which by the operations the result for associativity will be shown,

$$=\boldsymbol{x}\otimes(\boldsymbol{y}\otimes\boldsymbol{z})$$

This result is easy to show for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in R_\gamma^-$.

Definition 5.1.5: (cubic power): The cubic square of number $\boldsymbol{y}$ in $A \in R_\gamma$ is written as,

$$\boldsymbol{y} \otimes \boldsymbol{y} = \boldsymbol{y}^{2_\gamma}$$

Proof: Let $\boldsymbol{y} \in R_\gamma^+$,

$$\boldsymbol{y}^{2_\gamma} = \boldsymbol{y} \otimes \boldsymbol{y} = \gamma\left(\gamma^{-1}(\boldsymbol{y}) \cdot \gamma^{-1}(\boldsymbol{y})\right)$$

From the definition of $\gamma^{-1}(\boldsymbol{y})$ and $\gamma(\boldsymbol{y})$ one can simplify this operation to find,

$$= \left(\left(y^{\frac{1}{3}}\right)^3 \left(y^{\frac{1}{3}}\right)^3\right)^{1/3} = y^{2/3}$$

Similar result can be found if $\boldsymbol{y} \in R_\gamma^-$

$$\left(\left(-(-y)^{\frac{1}{3}}\right)^3 \left(-(-y)^{\frac{1}{3}}\right)^3\right)^{1/3} = (y \cdot y)^{1/3} = y^{2/3}$$

If $\boldsymbol{y}^{2_\gamma}$ is defined in this way then one can define the higher powers by choosing p In definition 5.1.5, $p = 2$ is taken. To generalize one has,

For $\boldsymbol{y} \in R_\gamma$,

$$\boldsymbol{y}^{2_\gamma} = \boldsymbol{y} \otimes \boldsymbol{y} = \gamma\{\gamma^{-1}(\boldsymbol{y}) \cdot \gamma^{-1}(\boldsymbol{y})\} = \gamma\{[\gamma^{-1}(\boldsymbol{y})]^2\}$$

$$\boldsymbol{y}^{3_\gamma} = \boldsymbol{y}^{2_\gamma} \otimes \boldsymbol{y} = \gamma\{\gamma^{-1}\{\gamma[\gamma^{-1}(\boldsymbol{y}) \cdot \gamma^{-1}(\boldsymbol{y})]\} \cdot \gamma^{-1}(\boldsymbol{y})\} = \gamma\{[\gamma^{-1}(\boldsymbol{y})]^3\}$$

Continuing with this logic,

$$\boldsymbol{y}^{p_\gamma} = \boldsymbol{y}^{(p-1)_\gamma} \otimes \boldsymbol{y} = \gamma\{[\gamma^{-1}(\boldsymbol{y})]^p\}$$

Can be defined generally.

Definition 5.1.6: (Cubic Absolute value): Let $A \in R_\gamma$.The absolute value of the number $\boldsymbol{x}$ is defined as,

$$|\boldsymbol{x}|_{\gamma} = \begin{cases} x^{\frac{1}{3}}, & x > 0 \\ 0, & x = 0 \\ -(-x)^{\frac{1}{3}}, & x < 0 \end{cases}$$

Since for all number $\boldsymbol{x}$ in the set $A \in R_{\gamma}$,

$$\left(\sqrt[3]{\boldsymbol{x}^{3_{\gamma}}}\right) = \gamma\left[\sqrt[3]{\gamma^{-1}(\boldsymbol{x}^{3_{\gamma}})}\right] = \gamma\left[\sqrt[3]{\gamma^{-1}\left(\gamma\left(\gamma^{-1}(\boldsymbol{x})\right)^{3}\right)}\right]$$

If the above equation is simplified one can write,

$$\gamma\left[\sqrt[3]{\gamma^{-1}\left(\gamma\left(\gamma^{-1}(\boldsymbol{x})\right)^{3}\right)}\right] = \gamma\left[\sqrt[3]{\left(\gamma^{-1}(\boldsymbol{x})\right)^{3}}\right] = |\boldsymbol{x}|_{\boldsymbol{\gamma}}$$

for all $x \in R$,

$$|\boldsymbol{x}|_{\boldsymbol{\gamma}} = \left|x^{1/3}\right|_{\gamma} = \gamma\left(\left|\gamma^{-1}\left(x^{1/3}\right)\right|\right) = \gamma\left(\left|\left(x^{\frac{1}{3}}\right)^{3}\right|\right) = |x|^{1/3}$$

Definition 5.1.7: ($\boldsymbol{\gamma}$-limit): Let $P \subseteq R_{\gamma},\ f\colon P \to R_{\alpha}$ and let $\boldsymbol{x_0} \in P_{\gamma}$ and $\boldsymbol{b} \in R_{\gamma}$. Then for all $\boldsymbol{\varepsilon} > \boldsymbol{0}$ there exists a number $\boldsymbol{\delta} = \delta(\boldsymbol{\varepsilon}) > \boldsymbol{0}$ such that,

$$\boldsymbol{0} < |\boldsymbol{x} \ominus \boldsymbol{x_0}|_{\gamma} < \boldsymbol{\delta} \text{ then, } |f(x) \ominus \boldsymbol{b}|_{\gamma} < \boldsymbol{\varepsilon}$$

then the limit of $f(x)$ as $\boldsymbol{x}$ tends to $\boldsymbol{x_0}$ is $\boldsymbol{b}$ and it can be written as,

$$\lim_{x \to x_0} {}^{\alpha} f(x) = \boldsymbol{b}$$

5.2 Cubic Differentiation and Integration

Definition 5.2.1: (Cubic derivative): Let $\boldsymbol{y} \in R_{\gamma}^{+}$ then the geometric derivative of $\gamma_p\colon R \to R_{\gamma} \subseteq R$ can be written as,

$$g^{*}(\boldsymbol{y}) = \gamma\left(\frac{\gamma^{-1}\left(\boldsymbol{g}(\boldsymbol{y})\right)'}{\gamma^{-1}(\boldsymbol{y})}\right)$$

If $\gamma^{-1}(\boldsymbol{y}) = (\boldsymbol{y})^{3}, \gamma(\boldsymbol{y}) = \boldsymbol{y}^{1/3}$ and $\gamma^{-1}\left(\boldsymbol{g}(\boldsymbol{y})\right) = \boldsymbol{g}(\boldsymbol{y})^{3}, \gamma(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{g}(\boldsymbol{y})^{1/3}$ for $\boldsymbol{y} > 0$ then one can obtain the cubic differentiation,

$$g^{*cubic}(\boldsymbol{y}) = \left(\frac{3(\boldsymbol{g}(\boldsymbol{y}))^2(\boldsymbol{g}'(\boldsymbol{y}))}{\boldsymbol{y}^3}\right)^{1/3}$$

Definition 5.2.2: (cubic integral): From

$$\int_r^s \boldsymbol{g}(\boldsymbol{y})^{dy} = \gamma\left(\int_r^s \gamma^{-1}(\boldsymbol{g}(\boldsymbol{y}))d\boldsymbol{y}\right)$$

If $\gamma^{-1}(\boldsymbol{y}) = (\boldsymbol{y})^3, \gamma(\boldsymbol{y}) = \boldsymbol{y}^{1/3}$ and $\gamma^{-1}(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{g}(\boldsymbol{y})^3, \gamma(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{g}(\boldsymbol{y})^{1/3}$ for $\boldsymbol{y} > 0$ then one can obtain the cubic integration for cubic functions,

$$\int_r^s \boldsymbol{g}(\boldsymbol{y})^{dy} = \left(\int_r^s (\boldsymbol{g}(\boldsymbol{y}))^3 d\boldsymbol{y}\right)^{\frac{1}{3}}$$

Definition 5.2.3:(Bi-cubic derivative): From the general definition of bi-geometric derivatives,

$$\boldsymbol{g}^{bi\alpha}(\boldsymbol{y}) = \gamma\left(\frac{\gamma^{-1}(\boldsymbol{g}(\boldsymbol{y}))'}{\gamma^{-1}(\boldsymbol{y})'}\right)$$

If $\gamma^{-1}(\boldsymbol{y}) = (\boldsymbol{y})^3, \gamma(\boldsymbol{y}) = \boldsymbol{y}^{1/3}$ and $\gamma^{-1}(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{g}(\boldsymbol{y})^3, \gamma(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{g}(\boldsymbol{y})^{1/3}$ for $\boldsymbol{y} > 0$ then one can obtain the bi-cubic differentiation,

$$\boldsymbol{g}^{bicubic}(\boldsymbol{y}) = \left(\frac{3(\boldsymbol{g}(\boldsymbol{y}))^2(\boldsymbol{g}'(\boldsymbol{y}))}{3\boldsymbol{y}^2}\right)^{1/3} = \left(\frac{(\boldsymbol{g}(\boldsymbol{y}))^2(\boldsymbol{g}'(\boldsymbol{y}))}{\boldsymbol{y}^2}\right)^{1/3}$$

Definition 5.2.4: (Bi-cubic integral): From the general definition of bi-geometric integration,

$$\left(\gamma\int_r^s \gamma^{-1}(\boldsymbol{y})'\gamma^{-1}(\boldsymbol{g}(\boldsymbol{y}))d\boldsymbol{y}\right)$$

If $\gamma^{-1}(\boldsymbol{y}) = (\boldsymbol{y})^3, \gamma(\boldsymbol{y}) = \boldsymbol{y}^{1/3}$ and $\gamma^{-1}(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{g}(\boldsymbol{y})^3, \gamma(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{g}(\boldsymbol{y})^{1/3}$ for

$\boldsymbol{y} > 0$ then one can obtain the bi-cubic integral for cubic functions,

$$= \left(\int_r^s (3(\boldsymbol{y}))^2 \big(\boldsymbol{g}(\boldsymbol{y})\big)^3 d\boldsymbol{y} \right)^{1/3}$$

From definition 5.2.1 to 5.2.4 the boldface letters $\boldsymbol{y}$ and $\boldsymbol{g}(\boldsymbol{y})$ **is** represented by,

$\boldsymbol{y} = \gamma(y), \boldsymbol{g}(\boldsymbol{y}) = \gamma\big(g(y)\big)$.

Chapter 6

CONCLUSION AND DISCUSSION

In this study, the relationship between various calculi and classical arithmetic was investigated such as exponential, hyperbolic tangent and cubic calculi. On the other hand, the equivalents of some fundamental definitions and theorems are given related to these calculi similar to Newtonian Calculi such as differentiation, integration, mean value theorem and fundamental theorem of calculus.

In final chapter, the general theorems explained in chapter 2 and 3 are applied to Cubic Arithmetic, geometric cubic calculus and bi-cubic calculus by taking $p = 3$. The theorems and definitions are similar to the ones which is explained in the final chapter in Quadratic calculus however, since quadratic functions are not differentiable bijections defining the differentiation and integration might be problematic R. To overcome this, one can carry the real number space to complex space by using a field in [3]. Besides this, the definitions and theorems in [2] can be used in this area. Hopefully, this thesis helps on bringing new ideas for further research in this area of Quadratic Calculi and Cubic Calculi.

REFERENCES

[1] Bashirov, A. E. (2013). On line and double multiplicative integrals. *TWMS Journal of Applied and Engineering Mathematics*, 103-107.

[2] Bashirov, A. E., & Norozpour, S. (2017). On complex multiplicative integration. *TWMS Journal of Applied and Engineering Mathematics*, 82-93.

[3] Bashirov, A. E., & Norozpour, S. (2018). On an alternative view to complex calculus. *Mathematical Methods in the Applied Sciences*, 7313-7324.

[4] Bashirov, A. E., & Rıza, M. (2011). On complex multiplicative differentiation. *TWMS Journal of applied and engineering mathematics*, 75-85.

[5] Bashirov, A. E., Kurpınar, E. M., & Özyapıcı, A. (2008). Multiplicative calculus and its applications. *Journal of Mathematical Analysis and Applications*, 36-48.

[6] Bashirov, A. E., Mısırlı, E., Tandoğdu, Y., & Özyapıcı, A. (2012). On modeling with multiplicative differential equations. *Applied Mathematics-A Journal of Chinese Universities*, 425-438.

[7] Boruah, K., & Hazarika, B. (2016). *Bigeometric Calculus and its applications.* arXiv preprint arXiv:1608.08088.

[8] Çakmak, A. F., & Başar, F. (2012). Some new results on sequence spaces with respect to non-Newtonian calculus. *Journal of Inequalities and Applications*, 1-17.

[9] Grossman, J. (2006). *Meta-Calculus: Differential and Integral. Non-Newtonian Calculus.* Massachusetts: University of Lowell and Archimedes Foundation.

[10] Grossman, M. (1983). *BIGEOMETRIC CALCULUS : A System with a Scale-Free Derivative.* Massachusetts: Archimedes Foundation: Rockport.

[11] Grossman, M., & Katz, R. (1972). *Non-Newtonian Calculi.* Massachusetts: Lee Press: Pigeon Cove.

[12] Tekin, S., & Başar, F. (2013). Certain sequence spaces over the non-Newtonian complex field. *Abstract and Applied Analysis*, 1-11.