Blending Type Bernstein and ψ-Bernstein-Kantorovich Operators

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ABSTRACT

In this thesis, blending type Bernstein operators $L_{\zeta}^{\alpha,s}$ with parameters α and s are investigated. Despite previous research, the Lototsky matrices that produce these operators have not been discovered. By examining these matrices, insights into their role in operator generation are obtained. It is shown that these operators satisfy the Korovkin theorem, and the rate of convergence of the operators is analyzed. Additionally, it is proven that these operators preserve monotonicity and convexity properties. Finally, various functions $f(\upsilon)$ are selected, and how well the operators $L_{\zeta}^{\alpha,s}$ approximate these functions are investigated.

Moreover, blending type Bernstein-Kantorovich operators depending on ψ_{η} are introduced. Our analysis shows that the approximation results of these operators are better than the classical case for the entire closed interval $[0,1]$. Furthermore, the convergence rates of the defined operators are investigated and it is proven that they preserve the properties of monotonicity and convexity. Our findings are supported by graphical and numerical examples.

Keywords: Bernstein Operators, Bernstein-Kantorovich Operators, Korovkin Theorem, Rate of Convergence, Shape Preserving Properties

ÖZ

Bu tezde, α ve s parametrelerine sahip blending tipi Bernstein operatörleri $L_{\zeta}^{\alpha,s}$ incelenmektedir. Önceki araştırmalara rağmen, bu operatörleri üreten Lototsky matrisleri keşfedilmemiştir. Bu matrisler incelenerek, operatör üretimindeki rolleri hakkında bilgiler elde edilmektedir. Bu operatörlerin Korovkin teoremini sağladığı gösterilmekte ve operatörlerin yakınsamanın hızı analiz edilmektedir. Ayrıca, bu operatörlerin monotonluk ve konvekslik özelliklerini koruduğu kanıtlanmaktadır. Son olarak, çeşitli $f(\upsilon)$ fonksiyonları seçiliyor ve bu fonksiyonlar için $L_{\zeta}^{\alpha,s}$ operatörlerinin bu fonksiyonlara ne kadar iyi yaklaştığı inceleniyor.

Ek olarak, ψ_{η}'ya bağlı olan blending tipi Bernstein-Kantorovich operatörleri tanıtılmaktadır. Analizimiz, bu operatörlerin yaklaşım sonuçlarının klasik durumdan iyi olduğunu tüm $[0,1]$ kapalı aralığı için göstermektedir. Ayrıca tanımlanan operatörlerin yakınsama hızları araştırılıp ve monotonluk ve konvekslik özelliklerini koruduğu ispatlanmıştır. Bulgularımızı grafiksel ve sayısal örneklerle desteklenmiştir.

Anahtar Kelimeler: Bernstein Operatörleri, Bernstein-Kantorovich Operatörleri, Korovkin Teoremi, Yakınsaklık Hızı, Şekil Koruma Özellikleri.

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$\forall$	For All
$\exists$	There Exists
$\mathbb{N}$	The Set of Natural Numbers
$\mathbb{N}_0$	Union of the Set of Natural Numbers and 0
$\mathbb{R}$	The Set of Real Numbers
$\mathbb{R}^+$	The Set of Positive Real Numbers
(a,b)	An Open Interval
$[a,b]$	A Closed Interval
$C[a,b]$	The Space of All Real-Valued and Continuous Functions on the Closed Interval $[a,b]$
$C^r[a,b]$	The Space of All Real-Valued r Times Continuously Differentiable Functions on the Closed Interval $[a,b]$, Where $r \in \mathbb{N}$

Chapter 1

INTRODUCTION

In 1885, K. Weierstrass presented a theorem in his work titled "Über die Analytische Darstellbarkeit Sogenannter Willkürlicher Funktionen Einer Reelen Veranderlichen," which demonstrates the possibility of approximating continuous functions with algebraic polynomials [30]. This theorem, known as the fundamental theorem of approximation theory in mathematical analysis, is based on the existence of a sequence of polynomials that uniformly converges to any continuous function on a finite interval $[a,b]$.

While Weierstrass's theorem provides information about the existence of approximating polynomials for a given continuous function, in 1912, S. N. Bernstein [7] determined how these polynomials are related to the given function, providing a simpler and more useful proof of the Weierstrass's theorem.

The terms of the sequence of polynomials that uniformly converge to any continuous function f on the interval $[0,1]$ are defined as follows:

$$B_\zeta(f;\upsilon) = \sum_{\iota=0}^{\zeta} \binom{\zeta}{\iota} f\left(\frac{\iota}{\zeta}\right) \upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}.$$

The Bernstein polynomials, defined in the space $C[0,1]$, have been modified with the aim of achieving convergence in the space of integrable functions. This modification was first introduced by Kantorovich [16] in 1930. According to this modification, the operators $K_\zeta : L^1[0,1] \to C[0,1]$, for all $\zeta \in \mathbb{N}$ and for all $\upsilon \in [0,1]$, are defined as follows:

$$K_\zeta(f;\upsilon) = (\zeta+1)\sum_{\iota=0}^{\zeta} b_{n,\iota}(\upsilon)\int_{\frac{\iota}{\zeta+1}}^{\frac{\iota+1}{\zeta+1}} f(t)dt,$$

where $b_{\zeta,\iota}(\upsilon)$ is given by $b_{\zeta,\iota}(\upsilon) = \binom{\zeta}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-\iota}$.

These operators are called the Bernstein-Kantorovich operators.

In 1952, H. Bohman investigated the problem of approximation of a continuous function f on the closed interval $[0,1]$ using a series of linear positive operators in a summation form. Bohman's work focused on demonstrating that for any $\upsilon \in [0,1]$ and for coefficients $0 \leq \theta_{\zeta,\iota} \leq 1$, the expression for $L_\zeta(f;\upsilon)$, defined as:

$$L_\zeta(f;\upsilon) = \sum_{\iota=0}^{\zeta} f(\theta_{\zeta,\iota})p_{\zeta,\iota}(\upsilon); \quad p_{\zeta,\iota}(\upsilon) \geq 0. \tag{1.1}$$

The necessary and sufficient conditions for the uniform convergence of the sequences of linear positive operators $L_\zeta(f;\upsilon)$ on $[0,1]$ as ζ approaches infinity. These conditions were outlined as follows:

$$L_\zeta(1;\upsilon) \rightrightarrows 1 \tag{1.2}$$

$$L_\zeta(t;\upsilon) \rightrightarrows \upsilon \tag{1.3}$$

$$L_\zeta(t^2;\upsilon) \rightrightarrows \upsilon^2 \tag{1.4}$$

as $\zeta \to \infty$. The value of the operators, which is investigated by H. Bohman, is independent of the values of the function f outside on the interval $[0,1]$.

In 1953, P. P. Korovkin contributed to this field by establishing a general theorem and demonstrating that the conditions proposed by Bohman hold true under general circumstances. Korovkin's theorem, a cornerstone in approximation theory, holds significant importance.

In this thesis, two operators are introduced: "blending type Bernstein operators" and "blending type ψ-Bernstein-Kantorovich operators". The thesis comprises five chapters, each dedicated to exploring different aspects of these operators.

Chapter 2 covers the basics, like defining terms and explaining important properties. Linear and positive operators are introduced, followed by an definition of Bernstein operators and their relationship with Lototsky matrices. Additionally, α-Bernstein operators and Bernstein-Kantorovich type operators are defined and examined.

Chapter 3 focuses on blending type Bernstein operators, presenting definitions and properties, including the proof of Korovkin's Theorem specific to these operators. Convergence rates are analyzed using the first-order modulus of continuity, and Voronovskaya's Theorem is discussed. Shape-preserving properties are demonstrated. Numerical and graphical examples are illustrated for the approximation of these operators to certain functions.

In Chapter 4, blending type ψ-Bernstein-Kantorovich operators are introduced and their properties are discussed. Korovkin's Theorem is proven for this specific class of operators, and conditions for ψ functions that yield better results compared to classical Bernstein-Kantorovich operators are identified. Convergence rates are analyzed using both first and second-order modulus of continuity, and shape-preserving properties are verified. Numerical and graphical examples are provided to demonstrate the approximation of these operators to certain functions.

Chapter 5 serves as a conclusion, summarizing the findings and drawing implications from the study.

Chapter 2

PRELIMINARIES

In this chapter, we lay the foundation for our exploration into approximation theory by examining key concepts and techniques essential to our understanding. We delve into the realm of positive linear operators, convergence rates, and several specialized types of operators that form the basis of our study.

2.1 Linear Positive Operators

In this section, we focus on introducing definitions and fundamental properties that characterize linear positive operators.

Definition 2.1: [13] Let $L : M \to N$ be an operator, where M and N are vector spaces. If we have

$$L(\mu f_1 + \nu f_2) = \mu L(f_1) + \nu L(f_2)$$

for each $\mu, \nu \in \mathbb{R}$, and $f_1, f_2 \in M$, then the operator L is called linear operator.

Definition 2.2: [13] Let $L : M \to N$ be an operator, where M and N are vector spaces. If

$$L(f) \geq 0$$

when $f \geq 0 \in M$, then the operator L is called positive operator.

Proposition 2.1: [13] Linear and positive operators are monotone. In other words, if $L : M \to N$ is a positive linear operator and $f_1(\upsilon) \leq f_2(\upsilon) \in M$, then

$$L(f_1;\upsilon) \leq L(f_2;\upsilon).$$

Proof. Let $L : M \to N$ be a linear and positive operator and $f_1(\upsilon), f_2(\upsilon) \in M$ such that $f_1(\upsilon) \leq f_2(\upsilon)$ for all $\upsilon \in \mathbb{R}$. From assumption, we have

$$0 \leq f_2(\upsilon) - f_1(\upsilon).$$

Since L is a positive,

$$0 \leq L(f_2 - f_1;\upsilon).$$

Now, if we use linearity of L, we get

$$0 \leq L(f_2;\upsilon) - L(f_1;\upsilon).$$

So,

$$L(f_1;\upsilon) \leq L(f_2;\upsilon).$$

□

Proposition 2.2: [13] Consider positive linear operator $L : M \to N$, then

$$|L(f;\upsilon)| \leq L(|f|;\upsilon)$$

for each $f(\upsilon) \in M$.

Proof. For any $f(\upsilon) \in M$, we are able to write

$$-|f(\upsilon)| \leq f(\upsilon) \leq |f(\upsilon)|.$$

From Proposition 2.1, L is monotone. Therefore,

$$-L(|f|;\upsilon) \leq L(f;\upsilon) \leq L(|f|;\upsilon).$$

In other words,

$$|L(f;\upsilon)| \leq L(|f|;\upsilon).$$

□

Now, let's state the Korovkin theorem, which holds a significant place in approximation theory.

Theorem 2.1 (Bohman-Korovkin Theorem): [18] Consider the positive linear operators $\left(L_\zeta\right)_{\zeta\geq 1} : C[a,b] \to C[a,b]$ and $e_i(t) := t^i$, where $i = 0,1,2$. If the sequence $\left(L_\zeta\right)_{\zeta\geq 1}$ of positive linear operators on $[a,b]$ satisfies

$$\lim_{\zeta\to\infty} \|L_\zeta(e_i) - e_i\|_{C[a,b]} = 0,$$

where $i = 0,1,2$, then for every $f \in C[a,b]$ on the interval $[a,b]$, we get

$$\lim_{\zeta\to\infty} \|L_\zeta(f) - f\|_{C[a,b]} = 0.$$

Corollary 2.1: [19] If the sequence of operators L_ζ satisfy $L_\zeta(1;\upsilon) \rightrightarrows 1$ and $L_\zeta((t-\upsilon)^2;\upsilon) \rightrightarrows 0$ as $\zeta \to \infty$, then for each $f \in C[a,b]$, we have $L_\zeta(f;\upsilon) \rightrightarrows f(\upsilon)$ as $\zeta \to \infty$.

2.2 The Convergence Rate of Linear Positive Operators

In the field of approximation theory, one of the fundamental concerns is the convergence rate, which plays a crucial role in assessing the effectiveness of approximation methods. The convergence rate is determined by the existence of a sequence $\theta(\zeta) \to 0$ as $\zeta \to \infty$, satisfying the inequality:

$$\left|L_\zeta(f,\upsilon) - f(\upsilon)\right| \leq c\theta(\zeta)$$

where $c \in \mathbb{R}^+$ represents a positive constant.

In pursuit of understanding and improving convergence rates, various tools are employed in approximation theory. Among these tools, the modulus of continuity holds particular significance. Therefore, in this section, we will initially present the definition and properties of the modulus of continuity.

Definition 2.3: [4] Let $f \in C[a,b]$. For any $\delta > 0$, the first order modulus of continuity of f is defined by

$$\omega(f;\delta) = \sup_{|\rho-\sigma|\leq\delta} |f(\rho) - f(\sigma)|$$

where $\rho, \sigma \in [a,b]$.

Moreover, first order modulus of continuity has the following properties:

(I) $\omega(f;\delta) \geq 0$ for each $\delta \geq 0$.

(II) If $\delta_1 \leq \delta_2$, then $\omega(f;\delta_1) \leq \omega(f;\delta_2)$.

(III) $\lim_{\delta\to 0} \omega(f;\delta) = 0$.

(IV) If $p \in \mathbb{N}$, then $\omega(f;p\delta) \leq p\omega(f;\delta)$.

(V) If $\mu \in \mathbb{R}^+$, then $\omega(f;\mu\delta) \leq (1+\mu)\omega(f;\delta)$.

(VI) $|f(\rho) - f(\sigma)| \leq \omega(f;|\rho-\sigma|)$, for all $\rho, \sigma \in [a,b]$.

(VII) $|f(\rho) - f(\sigma)| \leq \left(1 + \frac{|\rho-\sigma|}{\delta}\right) \omega(f;\delta)$, for all $\rho, \sigma \in [a,b]$.

Definition 2.4 (Lipschitz Class): [19] For $R > 0$ and $0 < \mu \leq 1$, a function $f \in C[0,1]$ is considered to be in the Lipschitz class if it satisfies:

$$|f(\upsilon) - f(t)| \leq R|\upsilon - t|^{\mu}$$

for all $\upsilon, t \in [0,1]$. Here, R is referred to as the Lipschitz constant. Functions that belong to the Lipschitz class are denoted as $f \in Lip_R(\mu)$.

Definition 2.5 (Peetre-K Functional): [19] Let $f \in C[\lambda,\mu]$ and $\delta > 0$, then the Peetre-K functional is defined as:

$$K_2(f;\delta) := inf_{\tau\in\varpi^2[\lambda,\mu]} \left\{ \|f-\tau\|_{C[\lambda,\mu]} + \delta\|\tau''\|_{\varpi^2[\lambda,\mu]} \right\}.$$

Here, $\varpi^2[\lambda,\mu]$ is the second order differentiable and continous functions space. In other words,

$$\varpi^2[\lambda,\mu] = \{\tau \in C[\lambda,\mu] : \tau', \tau'' \in C[\lambda,\mu]\}.$$

Also,

$$\|\tau''\|_{\varpi^2[\lambda,\mu]} = \|\tau\|_{C[\lambda,\mu]} + \|\tau'\|_{C[\lambda,\mu]} + \|\tau''\|_{C[\lambda,\mu]}.$$

Lemma 2.1: [19] Let $\tau \in \varpi^2[\lambda,\mu]$. Then,

$$\tau(t) - \tau(\upsilon) = \tau^{'}(\upsilon)(t-\upsilon) + \int_{\upsilon}^{t} \tau^{''}(r)(t-r)dr. \tag{2.1}$$

Proof. Applying integration by parts to the integration,

$$\int_{\upsilon}^{t} \tau^{''}(r)(t-r)dr \tag{2.2}$$

where,

$$u = t - r, \quad \text{then} \quad du = -dr$$

and

$$dv = \tau^{''}(r)dr, \quad \text{then} \quad v = \tau^{'}(r),$$

we get

$$\int_{t}^{\upsilon} \tau^{''}(r)(t-r)dr = (t-r)\tau^{'}(r)\Big|_{\upsilon}^{t} + \tau(r)|_{\upsilon}^{t}$$
$$= -(t-\upsilon)\tau^{'}(\upsilon) + \tau(t) - \tau(\upsilon).$$

Therefore, we obtain

$$\tau(t) - \tau(\upsilon) = \tau^{'}(\upsilon)(t-\upsilon) + \int_{\upsilon}^{t} \tau^{''}(r)(t-r)dr.$$

□

Definition 2.6: [27] The forward difference of f is denoted by $\Delta_h f(t)$ and defined as;

$$\Delta_h f(t) = f(t+h) - f(t). \tag{2.3}$$

Moreover, the r^{th} order forward difference of f is denoted by $\Delta_h f(t)$ and defined as

$$\Delta_h^r f(t) = \sum_{i=0}^{m} \binom{r}{i} (-1)^{r-i} f(t+ih) \tag{2.4}$$

with a step size h.

2.3 Bernstein Type Operators

In this section, we give the definition of Bernstein operators and some important properties and theorems that related to these operators. In 1912, Bernstein introduced the Bernstein operators as

$$B_\zeta(f;\upsilon) = \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon) f\left(\frac{\iota}{\zeta}\right), \tag{2.5}$$

where

$$b_{\zeta,\iota}(\upsilon) = \binom{\zeta}{\iota} \upsilon^\iota (1-\upsilon)^{\zeta-\iota}, \tag{2.6}$$

$\zeta \in \mathbb{N}$ and $f(\upsilon) \in C[0,1]$ (see [7]). Here, $B_\zeta(f;\upsilon)$ is called ζ^{th}-degree Bernstein operators and $b_{\zeta,\iota}(\upsilon)$ is called ζ^{th}-degree Bernstein basis polynomials. It should be noted that, these operators are constructed based on the concept of binomial expansion. In simple terms, when we have positive numbers υ and y, and ζ is a natural number, the binomial expansion is:

$$(\upsilon + y)^\zeta = \sum_{\iota=0}^{\zeta} \binom{\zeta}{\iota} \upsilon^\iota y^{\zeta-\iota}.$$

In this expression, if we take υ to be in the range $[0,1]$, then we can set $y = 1-\upsilon$, and we get the equation:

$$(\upsilon + 1 - \upsilon)^\zeta = \sum_{\iota=0}^{\zeta} \binom{\zeta}{\iota} \upsilon^\iota (1-\upsilon)^{\zeta-\iota}. \tag{2.7}$$

Property 2.1: [27] Bernstein operators are positive operators: $\forall \upsilon \in [0,1]$, $\zeta \in \mathbb{N}$ and $\iota = 0,1,2,...\zeta$, we get

$$\binom{\zeta}{\iota} \upsilon^\iota (1-\upsilon)^{\zeta-\iota} \geq 0.$$

Therefore for each $f \geq 0$, we have

$$B_\zeta(f;\upsilon) \geq 0.$$

Property 2.2: [27] Bernstein operators are linear operators: For any $\zeta \in \mathbb{N}$, $\mu, \nu \in \mathbb{R}$ and $f_1, f_2 \in C[0,1]$, we have

$$B_\zeta(\mu f_1 + \nu f_2;\upsilon) = \mu B_\zeta(f_1;\upsilon) + \nu B_\zeta(f_2;\upsilon).$$

Property 2.3: [27] Bernstein operators have the endpoint interpolation properties: For any $\zeta \in \mathbb{N}$; we have

$$B_\zeta(f;0) = f(0)$$

and

$$B_\zeta(f;1) = f(1).$$

Lemma 2.2: [7] For any $\upsilon \in [0,1]$ and $\zeta \in \mathbb{N}$, if $f(t) = 1$, then

$$B_\zeta(1;\upsilon) = 1.$$

Proof. Consider $\zeta \in \mathbb{N}$ and $\upsilon \in [0,1]$, then

$$B_\zeta(1;\upsilon) = \sum_{\iota=0}^{\zeta} \binom{\zeta}{\iota} \upsilon^\iota (1-\upsilon)^{\zeta-\iota}.$$

Now, using the binomial expansion on (2.7), we get

$$\begin{aligned} B_\zeta(1;\upsilon) &= (\upsilon + 1 - \upsilon)^\zeta \\ &= 1. \end{aligned}$$

□

Lemma 2.3: [7] Assume that $\upsilon \in [0,1]$, $\zeta \in \mathbb{N}$ and $f(t) = t$, then

$$B_\zeta(t;\upsilon) = \upsilon.$$

Proof. Consider $\upsilon \in [0,1]$ and $\zeta \in \mathbb{N}$, then

$$\begin{aligned} B_\zeta(t;\upsilon) &= \sum_{\iota=0}^{\zeta} \binom{\zeta}{\iota} \upsilon^\iota (1-\upsilon)^{\zeta-\iota} \frac{\iota}{\zeta} \\ &= \sum_{\iota=0}^{\zeta-1} \binom{\zeta}{\iota+1} \upsilon^{\iota+1} (1-\upsilon)^{\zeta-\iota-1} \frac{\iota+1}{\zeta} \\ &= \upsilon \sum_{\iota=0}^{\zeta-1} \frac{(\zeta-1)!}{(\zeta-\iota-1)!\iota!} \upsilon^\iota (1-\upsilon)^{\zeta-\iota-1} \\ &= \upsilon(\upsilon+1-\upsilon)^{\zeta-1} \\ &= \upsilon. \end{aligned}$$

□

Using Lemma 2.2, Lemma 2.3 and linearity of Bernstein operators, we can state Corollary 2.2.

Corollary 2.2: [27] For each $\zeta \in \mathbb{N}$, and $\mu, \nu \in \mathbb{R}$, we obtain

$$B_\zeta(\mu t + \nu; \upsilon) = \mu\upsilon + \nu$$

where $\upsilon \in [0,1]$. In other words, Bernstein operators preserve linear(affine) functions.

For the Korovkin theorem conditions, we need to state Lemma 2.4.

Lemma 2.4: [7] For each $\zeta = 2,3,...$ and $f(t) = t^2$, we have

$$B_\zeta(t^2;\upsilon) = \upsilon^2 + \frac{\upsilon(1-\upsilon)}{\zeta}.$$

Proof. Let $\zeta = 2,3,...$ and $f(t) = t^2$, we get

$$
\begin{aligned}
B_\zeta(t^2;\upsilon) &= \sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-\iota}\frac{\iota^2}{\zeta^2}\\
&=\sum_{\iota=1}^{\zeta}\frac{\zeta!}{(\zeta-\iota)!\iota!}\upsilon^\iota(1-\upsilon)^{\zeta-\iota}\frac{\iota^2}{\zeta^2}\\
&=\sum_{\iota=0}^{\zeta-1}\frac{(\zeta-1)!}{(\zeta-\iota-1)!\iota!}\upsilon^{\iota+1}(1-\upsilon)^{\zeta-\iota-1}\frac{\iota+1}{\zeta}\\
&=\frac{1}{\zeta}\sum_{\iota=1}^{\zeta-1}\frac{(\zeta-1)!}{(\zeta-\iota-1)!(\iota-1)!}\upsilon^{\iota+1}(1-\upsilon)^{\zeta-\iota-1}\\
&\quad+\frac{\upsilon}{\zeta}\sum_{\iota=0}^{\zeta-1}\frac{(\zeta-1)!}{(\zeta-\iota-1)!\iota!}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota-1}\\
&=\frac{1}{\zeta}\sum_{\iota=0}^{\zeta-2}\frac{(\zeta-1)!}{(\zeta-\iota-2)!\iota!}\upsilon^{\iota+2}(1-\upsilon)^{\zeta-\iota-2}+\frac{\upsilon}{\zeta}(\upsilon+1-\upsilon)^{\zeta-1}\\
&=\frac{(\zeta-1)\upsilon^2}{\zeta}(\upsilon+1-\upsilon)^{\zeta-2}+\frac{\upsilon}{\zeta}\\
&=\upsilon^2+\frac{\upsilon(1-\upsilon)}{\zeta}.
\end{aligned}
$$

□

Theorem 2.2: [7] For each $f \in C[0,1]$ and each $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\left|B_\zeta(f;\upsilon)-f(\upsilon)\right| < \varepsilon$$

$\forall \zeta \geq N$, and $\upsilon \in [0,1]$.

Property 2.4: [27] The ζ^{th}-degree Bernstein basis polynomials can be represented as two linear combinations of $(\zeta-1)^{th}$-degree Bernstein basis polynomials such that

$$b_{\zeta,\iota}(\upsilon) = (1-\upsilon)b_{\zeta-1,\iota}(\upsilon)+\upsilon b_{\zeta-1,\iota-1}(\upsilon).$$

Property 2.5: [27] Any ζ^{th} degree Bernstein basis polynomial can be represented in terms of the power basis as:

$$b_{\zeta,\iota}(\upsilon) = \sum_{j=\iota}^{\zeta}(-1)^{j-\iota}\binom{\zeta}{j}\binom{j}{\iota}\upsilon^j. \tag{2.8}$$

Proof. Consider the definition of $b_{\zeta,\iota}(\upsilon)$, and apply binomial theorem to $(1-\upsilon)^{\zeta-\iota}$, we get

$$\begin{aligned}
b_{\zeta,\iota}(\upsilon) &= \binom{\zeta}{\iota}\upsilon^{\iota}\sum_{j=0}^{\zeta-\iota}(-1)^{j}\binom{\zeta-\iota}{j}\upsilon^{j} \\
&= \sum_{j=\iota}^{\zeta}(-1)^{j-\iota}\binom{\zeta}{\iota}\binom{\zeta-\iota}{j-\iota}\upsilon^{j} \\
&= \sum_{j=\iota}^{\zeta}(-1)^{j-\iota}\binom{\zeta}{j}\binom{j}{\iota}\upsilon^{j}.
\end{aligned}$$

□

Theorem 2.3: [27] The Bernstein operators can be expressed as

$$B_{\zeta}(f;\upsilon) = \sum_{r=0}^{\zeta}\binom{\zeta}{r}f(0)\Delta_h^r\upsilon^r, \tag{2.9}$$

where we utilize Δ_h, with $h = \frac{1}{\zeta}$.

Proof. Consider the definition of Bernstein operators,

$$B_{\zeta}(f;\upsilon) = \sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}f\left(\frac{\iota}{\zeta}\right). \tag{2.10}$$

Applying binomial expansion to $(1-\upsilon)^{\zeta-\iota}$, we get

$$\begin{aligned}
B_{\zeta}(f;\upsilon) &= \sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}f\left(\frac{\iota}{\zeta}\right)\sum_{j=0}^{\zeta-\iota}(-1)^{j}\binom{\zeta-\iota}{j}\upsilon^{j} \\
&= \sum_{\iota=0}^{\zeta}\sum_{j=0}^{\zeta-\iota}\binom{\zeta}{\iota}\binom{\zeta-\iota}{j}f\left(\frac{\iota}{\zeta}\right)(-1)^{j}\upsilon^{\iota+j}. \qquad (2.11)
\end{aligned}$$

Choosing $r = \iota + j$, we are able to write that

$$\sum_{\iota=0}^{\zeta}\sum_{j=0}^{\zeta-\iota}\cdots = \sum_{r=0}^{\zeta}\sum_{\iota=0}^{r}\cdots.$$

Also, we can write

$$\binom{\zeta}{\iota}\binom{\zeta-\iota}{j} = \binom{\zeta}{r}\binom{r}{\iota}.$$

Therefore, the equality (2.11) becomes,

$$B_{\zeta}(f;\upsilon) = \sum_{r=0}^{\zeta}\binom{\zeta}{r}\upsilon^{r}\sum_{\iota=0}^{r}\binom{r}{\iota}(-1)^{r-\iota}f\left(\frac{\iota}{\zeta}\right). \tag{2.12}$$

On the other hand, if we set $t = 0$ and $h = \frac{1}{\zeta}$ in (2.4), we get

$$\Delta_h^r f(0) = \sum_{\iota=0}^{r} \binom{r}{\iota} (-1)^{r-\iota} f\left(\frac{\iota}{\zeta}\right). \tag{2.13}$$

Therefore, combining (2.12) and (2.13), we obtain the desired result, which is

$$B_\zeta(f;\upsilon) = \sum_{r=0}^{\zeta} \binom{\zeta}{r} \upsilon^r \Delta_h^r f(0).$$

□

In recent research, Chen et al. (see [11]) have introduced a novel family of generalized Bernstein operators known as α-Bernstein operators. These operators incorporate a parameter α that allows for greater flexibility and customization in approximation theory. The α-Bernstein operators represent a significant advancement in the field, offering a versatile framework for approximating functions with enhanced precision and adaptability.

In contrast to classical Bernstein operators, which are characterized by fixed degree polynomials, α-Bernstein operators provide a broader spectrum of approximation capabilities through the inclusion of the parameter α. This parameter enables practitioners to tailor the operators to specific requirements and optimize performance across a range of applications. Now, let's introduce the α-Bernstein operators.

Definition 2.7: For each $f(\upsilon) \in C[0,1]$ and $\upsilon \in [0,1]$, the α-Bernstein operators are,

$$T_{\zeta,\alpha}(f;\upsilon) = \sum_{\iota=0}^{\zeta} p_{\zeta,\iota}^{(\alpha)}(\upsilon) f\left(\frac{\iota}{\zeta}\right), \tag{2.14}$$

where $p_{1,0}^{(\alpha)}(\upsilon) = 1-\upsilon$, $p_{1,1}^{(\alpha)}(\upsilon) = \upsilon$, and

$$\begin{aligned} p_{\zeta,\iota}^{(\alpha)}(\upsilon) = \Bigg[&(1-\alpha)\binom{\zeta-2}{\iota}\upsilon + (1-\alpha)\binom{\zeta-2}{\iota-2}(1-\upsilon) \\ &+ \alpha\binom{\zeta}{\iota}\upsilon(1-\upsilon)\Bigg] \upsilon^{\iota-1}(1-\upsilon)^{\zeta-\iota-1}, \end{aligned}$$

for $\zeta \geq 2$.

In their research, Chen et al. brought a change to the classical Bernstein operators. They achieved this by using nearby points of $f(\upsilon)$ to define new points, denoted as g_l, using a linear combination of f_l and $f_{\iota+1}$. Specifically, they expressed g_l using the following equation:

$$g_\iota = \left(1 - \frac{\iota}{\zeta - 1}\right) f_\iota + \frac{\iota}{\zeta - 1} f_{\iota+1}.$$

This method allows for the utilization of any point between consecutive points, not limited to f_l alone. This flexibility leads to a more versatile estimation of the function.

2.4 Lototsky Matrices

In the realm of approximation theory, Lototsky matrices play a pivotal role in defining a class of operators known as Lototsky-Bernstein operators. These operators, an extension of Bernstein operators, offer a versatile framework for function approximation, particularly when dealing with non-standard intervals and functions.

The foundation of Lototsky-Bernstein operators lies in the Lototsky matrix $a_{\zeta,\iota}(\upsilon)$, which is derived from the equation:

$$\prod_{\iota=1}^{\zeta}(y\upsilon + 1 - \upsilon) = \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}(\upsilon) y^\iota. \tag{2.15}$$

In essence, the Lototsky matrix $a_{\zeta,\iota}(\upsilon)$ obtained from (2.15) is equivalent to $\binom{\zeta}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-\iota}$.

However, the applicability of Lototsky-Bernstein operators extends beyond the classical Bernstein framework. For a more general case, where $h(\upsilon)$ is a function satisfying $0 \leq h(\upsilon) \leq 1$, the Lototsky matrix $a_{\zeta,\iota}(\upsilon)$ is derived from:

$$\prod_{\iota=1}^{\zeta}(yh(\upsilon) + 1 - h(\upsilon)) = \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}(\upsilon) y^\iota.$$

Utilizing this extended definition, the operators:

$$V_{\zeta}(f,\upsilon)=\sum_{\iota=0}^{\zeta}a_{\zeta,\iota}(\upsilon)f\left(\frac{\iota}{\zeta}\right),$$

are formulated, known as Lototsky-Bernstein operators. Notably, $V_{\zeta}(f,\upsilon)$ represents a generalization of $B_{\zeta}(f,\upsilon)$, providing enhanced flexibility and adaptability in function approximation.

2.5 Bernstein-Kantorovich Operators and Their Generalizations

The classical Bernstein operators are a widely-used tool for approximating functions in the continuous function space on the unit interval $[0,1]$. However, these operators have limitations when it comes to approximating functions with singularities or discontinuities. To overcome this limitation, Kantorovich [16] introduced the Bernstein-Kantorovich operators

$$K_{\zeta}(f;\upsilon)=(\zeta+1)\sum_{\iota=0}^{\zeta}b_{\zeta,\iota}(\upsilon)\int_{\frac{\iota}{\zeta+1}}^{\frac{\iota+1}{\zeta+1}}f(t)dt \quad (\upsilon\in[0,1],\zeta\in\mathbb{N}). \tag{2.16}$$

for the Lebesgue integrable function space, which includes functions that may have singularities or discontinuities. The Bernstein-Kantorovich operators are designed to approximate functions in this space by using a different basis of polynomials, called the Kantorovich basis, which is specifically designed to approximate functions with singularities or discontinuities [16]. The Bernstein-Kantorovich operators have also been subject to further research and modifications in recent years (see [1], [9], [20], [28]).

In more recent work, Özarslan and Duman have introduced a new generalization of the Bernstein-Kantorovich operators in [25] as

$$K_{\zeta,\eta}(f;\upsilon)=\sum_{\iota=0}^{\zeta}b_{\zeta,\iota}(\upsilon)\int_{0}^{1}f\left(\frac{\iota+t^{\eta}}{\zeta+1}\right)dt \tag{2.17}$$

where $b_{\zeta,\iota}(\upsilon)$ are defined in Equation (2.6). The operators $K_{\zeta,\eta}(f;\upsilon)$, utilize a different kernel than the classical Bernstein-Kantorovich operators to approximate functions in the Lebesgue integrable function space. Specifically, the kernel involves an additional parameter η, which leads to improved approximation results in some cases.

Chapter 3

BLENDING TYPE BERNSTEIN OPERATORS

In this chapter, we delve into the study of blending type Bernstein operators. Our objective is to conduct a comprehensive investigation into the properties, convergence analysis, and provide numerical and graphical examples illustrating the capabilities of blending type Bernstein operators. Through a series of illustrative examples and numerical results, we aim to demonstrate the efficacy and versatility of these operators.

3.1 Generalized Lototsky Matrices and Generalized Blending Type Bernstein Operators

In this section, we define the blending type Bernstein operators, discuss their properties, and establish their relationship with Lototsky matrices. Initially, we present an expression illustrating how the coefficients of the blending type Bernstein operators can be generated by the corresponding Lototsky matrices.

Let $\alpha \in [0,1]$ be any real number and s be a nonnegative integer, then consider the following generalization of (2.15),

$$\begin{aligned}\sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)y^{\iota} =&(1-\alpha)\left[y^s\upsilon\prod_{i=1}^{\zeta-s}(y\upsilon+1-\upsilon)+(1-\upsilon)\prod_{i=1}^{\zeta-s}(y\upsilon+1-\upsilon)\right]\\ &+\alpha\prod_{i=1}^{\zeta}(y\upsilon+1-\upsilon),\end{aligned} \tag{3.1}$$

where $\upsilon \in [0,1]$ and

$$\prod_{i=1}^{m} b = \begin{cases} b^m, & m \geq 1, \\ 1, & m = 0 \\ 0, & otherwise. \end{cases}$$

It is obvious that for $\alpha = 1$, the Equation (3.1) reduces to the Equation (2.15). Now, we proceed to introduce the blending type Bernstein operators.

Definition 3.1: Consider $f \in C[0,1]$. Then, for any $\zeta, s \in \mathbb{N}$, and $\upsilon, \alpha \in [0,1]$, the blending type Bernstein operators are

$$L_{\zeta}^{\alpha,s}(f;\upsilon) = \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon) f\left(\frac{\iota}{\zeta}\right) \tag{3.2}$$

where

$$b_{\zeta,\iota}^{\alpha,s}(\upsilon) = \begin{cases} \binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}, & if\ \zeta < s, \\ \\ a_{\zeta,\iota}^{\alpha,s}(\upsilon), & if\ \zeta \geq s. \end{cases} \tag{3.3}$$

and

$$a_{\zeta,\iota}^{\alpha,s}(\upsilon) = \left[(1-\alpha)\binom{\zeta-s}{\iota}\upsilon + (1-\alpha)\binom{\zeta-s}{\iota-s}(1-\upsilon) + \alpha\binom{\zeta}{\iota}\upsilon(1-\upsilon)\right]$$
$$\times \upsilon^{\iota-1}(1-\upsilon)^{\zeta-\iota-1}.$$

Lemma 3.1: The operators $L_{\zeta}^{\alpha,s}(f,\upsilon)$ are linear and positive operators.

Lemma 3.2: If $s = 1$, the operators $L_{\zeta}^{\alpha,1}(f,\upsilon)$, are reduced to the classical Bernstein operators, which is given in (2.5). In other words, the operators $L_{\zeta}^{\alpha,s}(f,\upsilon)$ include Bernstein operators as a special case.

Lemma 3.3: If $s = 2$, the operators $L_{\zeta}^{\alpha,2}(f,\upsilon)$, are reduced to the $B_{\zeta,\alpha}(f,\upsilon)$, which is given in (2.14). In other words, the operators $L_{\zeta}^{\alpha,s}(f,\upsilon)$ include the operators $B_{\zeta,\alpha}(f,\upsilon)$ as a special case.

Proof. It is easy to see that, for $s = 2$ we get,

$$b_{\zeta,\iota}^{\alpha,2}(\upsilon)=\begin{cases}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}, & \zeta<2,\\ \\ p_{\zeta,\iota}^{(\alpha)}(\upsilon), & \zeta\geq 2,\end{cases}$$

which completes the proof. $\square$

In the rest of the thesis, we will focus on a non-trivial generalization of the Bernstein operators, specifically emphasizing the case where $s\geq 2$.

Lemma 3.4: Consider $f\in C[0,1]$, $\alpha\in[0,1]$ and $\zeta,s\in\mathbb{N}$. The operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$ can be defined as:

$$L_{\zeta}^{\alpha,s}(f;\upsilon)=\begin{cases}B_{\zeta}(f;\upsilon), & \zeta<s,\\ \\ \sum_{\iota=0}^{\zeta}\Big[(1-\alpha)\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota} & \\ +(1-\alpha)\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} & \\ \quad+\alpha\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\Big]f\left(\frac{\iota}{\zeta}\right), & \zeta\geq s.\end{cases} \tag{3.4}$$

Proof. The case where $\zeta<s$ is straightforward. Let's assume that $\zeta\geq s$. We can derive from (3.1):

$$\begin{aligned}\sum_{\iota=0}^{\infty}a_{\zeta,\iota}^{\alpha,s}(\upsilon)y^{\iota}=&(1-\alpha)\left[y^{s}\upsilon\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}y^{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\right.\\&\left.+(1-\upsilon)\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}y^{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\right]+\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}y^{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\\=&(1-\alpha)\left[\sum_{\iota=s}^{\zeta}\binom{\zeta-s}{\iota-s}y^{\iota}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota}\right.\\&\left.+\sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota}y^{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}\right]+\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}y^{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\\=&\sum_{\iota=0}^{\zeta}\left[(1-\alpha)\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota}\right.\\&\left.+(1-\alpha)\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}+\alpha\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\right]y^{\iota}.\end{aligned}$$

$\square$

The following theorem provides an alternative representation of the operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$.

Theorem 3.1: Consider $f \in C[0,1]$, $\alpha \in [0,1]$ and $\zeta, s \in \mathbb{N}$. The operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$ can be defined as:

$$L_{\zeta}^{\alpha,s}(f;\upsilon) = \begin{cases} B_{\zeta}(f;\upsilon), & \zeta < s, \\ \\ (1-\alpha)\sum_{\iota=0}^{\zeta-s+1}\binom{\zeta-s+1}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}h_{\iota} \\ +\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}f_{\iota}, & \zeta \geq s, \end{cases}$$

where

$$h_{\iota} = \left(1 - \frac{\iota}{\zeta-s+1}\right)f_{\iota} + \frac{\iota}{\zeta-s+1}f_{\iota+s-1},$$

and

$$f_{\iota} = f\left(\frac{\iota}{\zeta}\right).$$

Proof. Consider $f \in C[0,1]$. Let $\alpha \in [0,1]$ and s be a fixed positive integer. By the definition of $b_{\zeta,\iota}^{\alpha,s}(\upsilon)$, the operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$ reduce to the operators $B_{\zeta}(f,\upsilon)$ when $\zeta < s$. Hence, the proof is straightforward. Now, assume that $\zeta \geq s$. Then, from Lemma 3.4, we have

$$\begin{aligned} L_{\zeta}^{\alpha,s}(f;\upsilon) = \sum_{\iota=0}^{\zeta}\Bigg[&(1-\alpha)\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota} \\ &+(1-\alpha)\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} \\ &+\alpha\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\Bigg]f_{\iota}. \end{aligned}$$

Let's define:

$$g_1 = \sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}f_{\iota},$$

and

$$g_2 = \sum_{\iota=0}^{\zeta} \binom{\zeta - s}{\iota - s} \upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota} f_\iota.$$

We rewrite g_1 and g_2 for clarity,

$$\begin{aligned} g_1 &= \sum_{\iota=0}^{\zeta} \binom{\zeta - s}{\iota} \upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} f_\iota \\ &= \sum_{\iota=0}^{\zeta-s+1} \binom{\zeta - s}{\iota} \upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} f_\iota \end{aligned}$$

and

$$\begin{aligned} g_2 &= \sum_{\iota=0}^{\zeta} \binom{\zeta - s}{\iota - s} \upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota} f_\iota \\ &= \sum_{\iota=0}^{\zeta-s+1} \binom{\zeta - s}{\iota - 1} \upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} f_{\iota+s-1}. \end{aligned}$$

Thus,

$$g_1 + g_2 = \sum_{\iota=0}^{\zeta-s+1} \left[\binom{\zeta - s}{\iota} f_\iota + \binom{\zeta - s}{\iota - 1} f_{\iota+s-1} \right] \upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}. \tag{3.5}$$

Substituting the expressions of $\binom{\zeta-s}{\iota}$ and $\binom{\zeta-s}{\iota-1}$, which are:

$$\binom{\zeta - s}{\iota} = \binom{\zeta - s + 1}{\iota} \left(1 - \frac{\iota}{\zeta - s + 1}\right)$$

and

$$\binom{\zeta - s}{\iota - 1} = \binom{\zeta - s + 1}{\iota} \frac{\iota}{\zeta - s + 1}$$

into (3.5), we obtain:

$$\begin{aligned} g_1 + g_2 = &\sum_{\iota=0}^{\zeta-s+1} \binom{\zeta - s + 1}{\iota} \left[\left(1 - \frac{\iota}{\zeta - s + 1}\right) f_\iota + \frac{\iota}{\zeta - s + 1} f_{\iota+s-1} \right] \\ &\times \upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}. \\ = &\sum_{\iota=0}^{\zeta-s+1} \binom{\zeta - s + 1}{\iota} h_\iota \upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}. \end{aligned}$$

Hence, for the case $\zeta \geq s$, we have

$$\begin{aligned} L_\zeta^{\alpha,s}(f;\upsilon) =&(1-\alpha) \sum_{\iota=0}^{\zeta-s+1} \binom{\zeta - s + 1}{\iota} \upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} h_\iota \\ &+ \alpha \sum_{\iota=0}^{\zeta} \binom{\zeta}{\iota} \upsilon^{\iota}(1-\upsilon)^{\zeta-\iota} f_\iota. \end{aligned}$$

$\square$

Lemma 3.5: For any $\alpha \in [0,1]$ and positive integer s, the operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$, preserve linear polynomials. In other words,

$$L_{\zeta}^{\alpha,s}(1;\upsilon) = 1 \quad \text{and} \quad L_{\zeta}^{\alpha,s}(t;\upsilon) = \upsilon.$$

Proof. When $\zeta < s$, the operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$ reduces to $B_{\zeta}(f;\upsilon)$. Therefore,

$$\begin{aligned} L_{\zeta}^{\alpha,s}(1;\upsilon) &= B_{\zeta}(1;\upsilon) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} L_{\zeta}^{\alpha,s}(t;\upsilon) &= B_{\zeta}(t;\upsilon) \\ &= \upsilon. \end{aligned}$$

So, we only need to prove both equations when $\zeta \geq s$. Let $f(t) = 1$, then $f_{\iota} = h_{\iota} = 1$. This yields:

$$\begin{aligned} L_{\zeta}^{\alpha,s}(1;\upsilon) &= (1-\alpha)\sum_{\iota=0}^{\zeta-s+1}\binom{\zeta-s+1}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} + \alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}. \\ &= (1-\alpha)B_{\zeta-s+1}(1;\upsilon) + \alpha B_{\zeta}(1;\upsilon) \\ &= 1. \end{aligned}$$

Similarly, if $f(t) = t$ then,

$$h_{\iota} = \frac{\iota}{\zeta - s + 1},$$

leading to

$$\begin{aligned}L_\zeta^{\alpha,s}(t;\upsilon) =&(1-\alpha)\sum_{\iota=0}^{\zeta-s+1}\binom{\zeta-s+1}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-s-\iota+1}\frac{\iota}{\zeta-s+1}\\&+\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-\iota}\frac{\iota}{\zeta}\\=&(1-\alpha)B_{\zeta-s+1}(t;\upsilon)+\alpha B_\zeta(t;\upsilon)\\=&\upsilon.\end{aligned}$$

□

In the following theorem, we provide an alternative expression for the operators $L_\zeta^{\alpha,s}(f;\upsilon)$ using the difference operator.

Theorem 3.2: Consider $f\in C[0,1]$. The operators $L_\zeta^{\alpha,s}(f;\upsilon)$ can be defined as:

$$\begin{aligned}L_\zeta^{\alpha,s}(f;\upsilon) =&(1-\alpha)\sum_{r=0}^{\zeta-s+1}\binom{\zeta-s+1}{r}\left(\triangle^r f_0+\frac{r}{\zeta-s+1}(\triangle^{r-1}f_s-\triangle^{r-1}f_1)\right)\upsilon^r\\&+\alpha\sum_{r=0}^{\zeta}\binom{\zeta}{r}\upsilon^r\triangle^r f_0\end{aligned}\tag{3.6}$$

where $\alpha\in[0,1]$ and $\zeta,s\in\mathbb{N}$ such that $\zeta\geq s\geq 2$.

Proof. From Theorem 3.1,

$$L_\zeta^{\alpha,s}(f;\upsilon)=(1-\alpha)\sum_{\iota=0}^{\zeta-s+1}\binom{\zeta-s+1}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-s-\iota+1}h_\iota+\alpha B_\zeta(f;\upsilon),$$

where $\zeta\geq s\geq 2$. Now let's denote the first term of the above equation as,

$$L_\zeta^{\star,s}(f;\upsilon):=\sum_{\iota=0}^{\zeta-s+1}\binom{\zeta-s+1}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-s-\iota+1}h_\iota\tag{3.7}$$

then using the following equation,

$$(1-\upsilon)^{\zeta-s-\iota+1}=\sum_{j=0}^{\zeta-s-\iota+1}\binom{\zeta-s-\iota+1}{j}(-1)^j\upsilon^j$$

in (3.7), we have,

$$\begin{aligned}L_\zeta^{\star,s}(f;\upsilon)&=\sum_{\iota=0}^{\zeta-s+1}h_\iota\binom{\zeta-s+1}{\iota}\upsilon^\iota\sum_{j=0}^{\zeta-s-\iota+1}\binom{\zeta-s-\iota+1}{j}(-1)^j\upsilon^j\\&=\sum_{\iota=0}^{\zeta-s+1}\sum_{j=0}^{\zeta-s-\iota+1}h_\iota\binom{\zeta-s+1}{\iota}\binom{\zeta-s-\iota+1}{j}(-1)^j\upsilon^{\iota+j}.\end{aligned}$$

However,

$$\binom{\zeta-s+1}{\iota}\binom{\zeta-s-\iota+1}{j}=\binom{r}{\iota}\binom{\zeta-s+1}{r}$$

where $r=\iota+j$. Therefore,

$$\begin{aligned}L_{\zeta}^{\star,s}(f;\upsilon)&=\sum_{r=0}^{\zeta-s+1}\binom{\zeta-s+1}{r}\upsilon^{r}\sum_{\iota=0}^{r}\binom{r}{\iota}(-1)^{r-\iota}h_{\iota}\\&=\sum_{r=0}^{\zeta-s+1}\binom{\zeta-s+1}{r}\upsilon^{r}\triangle^{r}h_{0}.\end{aligned}\tag{3.8}$$

On the other hand, it is known that from (2.9),

$$B_{\zeta}(f,\upsilon)=\sum_{r=0}^{\zeta}\binom{\zeta}{r}\upsilon^{r}\triangle^{r}f_{0}.\tag{3.9}$$

Using (3.8) and (3.9) in the definition of $L_{\zeta}^{\alpha,s}(f;\upsilon)$, we have,

$$L_{\zeta}^{\alpha,s}(f;\upsilon)=(1-\alpha)\sum_{r=0}^{\zeta-s+1}\binom{\zeta-s+1}{r}\upsilon^{r}\triangle^{r}h_{0}+\alpha\sum_{r=0}^{\zeta}\binom{\zeta}{r}\upsilon^{r}\triangle^{r}f_{0}.$$

Finally, by using mathematical induction and the Leibniz rule

$$\triangle^{\zeta}(g_{i}f_{i})=\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}(\triangle^{\iota}f_{i})(\triangle^{\zeta-\iota}g_{i+\iota})$$

for $\zeta=1$, we can establish that,

$$\begin{aligned}\triangle^{r}h_{i}=&\left(1-\frac{i}{\zeta-s+1}\right)\triangle^{r}f_{i}+\frac{i}{\zeta-s+1}\triangle^{r}f_{i+s-1}\\&+\frac{r}{\zeta-s+1}\left(\triangle^{r-1}f_{i+s}-\triangle^{r-1}f_{i+1}\right)\end{aligned}$$

which implies

$$\triangle^{r}h_{0}=\triangle^{r}f_{0}+\frac{r}{\zeta-s+1}\left(\triangle^{r-1}f_{s}-\triangle^{r-1}f_{1}\right)$$

for $i=0$. This concludes the proof. □

Recall that, we have the following relation between derivatives and differences;

$$\triangle^{\iota}f_{i}=\frac{1}{\zeta^{\iota}}f^{(\iota)}(\xi_{i}),\ \text{ for some }\xi_{i}\in\left(\frac{i}{\zeta},\frac{i+\iota}{\zeta}\right).$$

where $\iota\in\mathbb{N}$.

Also, if $f(t)=t^m$ then,

$$\triangle^l f_0 = 0 \text{ for } l > m,$$

and

$$\triangle^m f_i = \frac{1}{\zeta^m} f^{(m)}(\xi_i) = \frac{m!}{\zeta^m}.$$

Moreover,

$$\triangle^l f_1 = 0 \text{ for } l > m.$$

Lemma 3.6: For any $\alpha \in [0,1]$ and $\zeta, s \in \mathbb{N}$ such that $\zeta \geq s \geq 2$, we have

$$L_{\zeta}^{\alpha,s}(t^2;\upsilon) = \upsilon^2 + \frac{\upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)\right]}{\zeta^2} \tag{3.10}$$

$$L_{\zeta}^{\alpha,s}(t^3;\upsilon) = \upsilon^3 + \upsilon^2(1-\upsilon)\left[\frac{3\zeta-2}{\zeta^2}+(1-\alpha)\frac{s(s-1)(3\zeta-2s-2)}{\zeta^3}\right] + \upsilon(1-\upsilon)\left[\frac{1}{\zeta^2}+(1-\alpha)\frac{s(s-1)(s+1)}{\zeta^3}\right] \tag{3.11}$$

$$\begin{aligned} L_{\zeta}^{\alpha,s}(t^4;\upsilon) = {}&\upsilon^4 + \upsilon^3(1-\upsilon)\left[\frac{6\zeta^2-11\zeta+6}{\zeta^3}\right. \\ &\left.+\,(1-\alpha)\frac{s(1-s)\left[3(s+1)(s+2)+2\zeta(3\zeta-4s-7)\right]}{\zeta^4}\right] \\ &+\upsilon^2(1-\upsilon)\left[\frac{7(\zeta-1)}{\zeta^3}+(1-\alpha)\frac{s(s-1)\left[(\zeta-s)(4s+10)-7\right]}{\zeta^4}\right] \\ &+\upsilon(1-\upsilon^2)\left[\frac{1}{\zeta^3}+(1-\alpha)\frac{s(s-1)(s^2+s+1)}{\zeta^4}\right]. \end{aligned} \tag{3.12}$$

Proof. From equation (3.6) we have,

$$\begin{aligned} L_{\zeta}^{\alpha,s}(t^2;\upsilon) = {}&(1-\alpha)\left[(\zeta-s+1)\left(\triangle f_0+\frac{1}{\zeta-s+1}(f_s-f_1)\right)\upsilon\right. \\ &\left.+\frac{(\zeta-s+1)(\zeta-s)}{2}\left(\triangle^2 f_0+\frac{2}{\zeta-s+1}(\triangle f_s-\triangle f_1\right)\upsilon^2\right] \\ &+\alpha\left[\zeta\triangle f_0\upsilon+\frac{\zeta(\zeta-1)}{2}\triangle^2 f_0\upsilon^2\right] \\ ={}&(1-\alpha)\left[\left(\frac{s^2-s+\zeta}{\zeta^2}\right)\upsilon+\upsilon^2+\left(\frac{-s^2+s-\zeta}{\zeta^2}\right)\upsilon^2\right]+\alpha\left[\frac{\upsilon}{\zeta}+\upsilon^2-\frac{\upsilon^2}{\zeta}\right] \\ ={}&\upsilon^2+\frac{\upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)\right]}{\zeta^2} \end{aligned}$$

which proves (3.10). Equations (3.11) and (3.12), can be proved in a parallel way. $\square$

3.2 Approximation Properties of $L_{\zeta}^{\alpha,s}(f;\upsilon)$

In this section, we state the Korovkin theorem, analyze convergence rates through the first-order modulus of continuity, investigate Voronovskaya's theorem, and demonstrate approximation results for the Lipschitz class for the operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$.

Theorem 3.3: Let $f(\upsilon)\in C[0,1]$. Then, for any $\zeta,s\in\mathbb{N}$, and $\alpha\in[0,1]$, the operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$ uniformly convergent to f on $[0,1]$.

Proof. As a consequence of the Korovkin Theorem (see [4, 19]), it suffices to demonstrate the conditions in (1.2), (1.3), and (1.4) for $L_{\zeta}^{\alpha,s}(f;\upsilon)$. By Lemmas 2.2, 2.3, and 3.5, we establish that $L_{\zeta}^{\alpha,s}(1;\upsilon)=1$ and $L_{\zeta}^{\alpha,s}(t;\upsilon)=\upsilon$ for any $s\in\mathbb{N}$ and $\alpha\in[0,1]$. On the other hand, according to Lemma 2.4 and the Equation (3.10), we find that:

$$L_{\zeta}^{\alpha,s}(t^2;\upsilon)=\begin{cases}\upsilon^2+\frac{\upsilon(1-\upsilon)}{\zeta}, & \zeta<s,\\ \\ \upsilon^2+\frac{\upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)\right]}{\zeta^2}, & \zeta\geq s.\end{cases}$$

Clearly, in both cases the operators $L_{\zeta}^{\alpha,s}(t^2;\upsilon)$ uniformly convergent to $f(\upsilon)=\upsilon^2$. So, this completes the proof. $\square$

Now, our aim is to demonstrate Voronovskaya's theorem. To accomplish this, we give the next two lemmas.

Lemma 3.7: Consider

$$S_m=\sum_{\iota=0}^{\zeta}(l-\zeta\upsilon)^m a_{\zeta,\iota}^{\alpha,s}(\upsilon).$$

Therefore, for $\zeta\geq s$ we get

$$
\begin{aligned}
S_0(\upsilon) =& 1 \\
S_1(\upsilon) =& 0 \\
S_2(\upsilon) =& \upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)\right] \\
S_3(\upsilon) =& \upsilon(1-\upsilon)(1-2\upsilon)\left[\zeta+(1-\alpha)s(s-1)(s+1)\right] \\
S_4(\upsilon) =& 3\upsilon^2(1-\upsilon)^2\left[\zeta(\zeta-2)+(1-\alpha)s(s-1)[\zeta-(s+1)(s+2)]\right] \\
& +\upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)(s^2+s+1)\right].
\end{aligned}
$$

Proof. It is obvious that $S_0(\upsilon)=1$ and $S_1(\upsilon)=0$.

$$
\begin{aligned}
S_2(\upsilon) &= \sum_{\iota=0}^{\zeta}(\iota-\zeta\upsilon)^2 a_{\zeta,\iota}^{\alpha,s}(\upsilon) = \sum_{\iota=0}^{\zeta}(\iota^2-2\zeta\iota\upsilon+\upsilon^2)a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&= \zeta^2\sum_{\iota=0}^{\zeta}\left(\frac{\iota^2}{\zeta^2}-2\upsilon\frac{\iota}{\zeta}+\upsilon^2\right)a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&= \zeta^2\sum_{\iota=0}^{\zeta}\frac{\iota^2}{\zeta^2}a_{\zeta,\iota}^{\alpha,s}(\upsilon)-2\upsilon\zeta^2\sum_{\iota=0}^{\zeta}\frac{\iota}{\zeta}a_{\zeta,\iota}^{\alpha,s}(\upsilon)+\upsilon^2\zeta^2\sum_{\iota=0}^{\zeta}a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&= \zeta^2\upsilon^2+\Big(\upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)\right]\Big)-2\zeta^2\upsilon^2+\zeta^2\upsilon^2 \\
&= \upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)\right].
\end{aligned}
$$

Equations for $S_3(\upsilon)$ and $S_4(\upsilon)$ can be obtained in a parallel way. $\square$

Lemma 3.8: For any $\alpha\in[0,1]$, and positive integer s, there exists a constant C independent of ζ such that,

$$
\sum_{|\frac{\iota}{\zeta}-\upsilon|\geq\zeta^{-\frac{1}{8}}} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\leq\frac{C}{\zeta^{\frac{3}{2}}}, \tag{3.13}
$$

for all $\upsilon\in[0,1]$.

Proof. Given a positive integer s. For the case $\zeta<s$, $b_{\zeta,\iota}^{\alpha,s}(\upsilon)=\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}$ and (3.13) holds. Assume that $\zeta\geq s$. Since $S_4(\upsilon)$ is a second order polynomial in ζ. There exists C such that $S_4(\upsilon)\leq C\zeta^2$ for each $\upsilon\in[0,1]$. On the other hand, $|\frac{\iota}{\zeta}-\upsilon|\geq\zeta^{-\frac{1}{8}}$ implies that $\frac{(\iota-\zeta\upsilon)^4}{\zeta^{\frac{7}{2}}}\geq 1$. Thus,

$$
\begin{aligned}
\sum_{|\frac{\iota}{\zeta}-\upsilon|\geq\zeta^{-\frac{1}{8}}} a_{\zeta,\iota}^{\alpha,s}(\upsilon) &\leq \frac{1}{\zeta^{\frac{7}{2}}}\sum_{\iota=0}^{\zeta}(\iota-\zeta\upsilon)^4 a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&\leq \frac{1}{\zeta^{\frac{7}{2}}}S_4(\upsilon) \\
&\leq \frac{1}{\zeta^{\frac{7}{2}}}C\zeta^2 \\
&\leq \frac{C}{\zeta^{\frac{3}{2}}}.
\end{aligned}
$$

□

Now we can prove the following Voronovskaya's type theorem [29] for $L_{\zeta}^{\alpha,s}(f;\upsilon)$.

Theorem 3.4: Let $f(\upsilon)$ be a bounded function on $[0,1]$, if $f''(\upsilon)$ exists at a point $\upsilon \in [0,1]$, then

$$
\lim_{\zeta\to\infty}\zeta\left[L_{\zeta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right]=\frac{1}{2}\upsilon(1-\upsilon)f''(\upsilon) \tag{3.14}
$$

where $\alpha \in [0,1]$ and $s \in \mathbb{N}$.

Proof. Given a positive integer s. If $\zeta < s$, $L_{\zeta}^{\alpha,s}(f;\upsilon) = B_{\zeta}(f;\upsilon)$ and (3.14) holds. Let $\zeta \geq s$, by the Taylor's formula we have

$$
f(t)=f(\upsilon)+(t-\upsilon)f'(\upsilon)+\frac{1}{2}(t-\upsilon)^2f''(\upsilon)+p(t)(t-\upsilon)^2,
$$

where $\lim_{t\to x}p(t)=0$. Taking $t=\frac{\iota}{\zeta}$ where $l \leq \zeta$ gives,

$$
\begin{aligned}
f\left(\frac{\iota}{\zeta}\right)=&f(\upsilon)+\left(\frac{\iota}{\zeta}-\upsilon\right)f'(\upsilon)+\frac{1}{2}\left(\frac{\iota}{\zeta}-\upsilon\right)^2f''(\upsilon) \\
&+p\left(\frac{\iota}{\zeta}\right)\left(\frac{\iota}{\zeta}-\upsilon\right)^2.
\end{aligned} \tag{3.15}
$$

Equation (3.15) gives that,

$$
\begin{aligned}
\zeta[L_{\zeta}^{\alpha,s}(f;\upsilon)-f(\upsilon)] &= \frac{1}{2\zeta}S_2(\upsilon)f''(\upsilon)+\zeta\sum_{\iota=0}^{\zeta}p\left(\frac{\iota}{\zeta}\right)\left(\frac{\iota}{\zeta}-\upsilon\right)^2 a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&= \upsilon(1-\upsilon)\left[\frac{1}{2}+\frac{(1-\alpha)s(s-1)}{2\zeta}\right]f''(\upsilon)+\zeta P_{\zeta}^{\alpha,s}(\upsilon),
\end{aligned}
$$

for any $0 \leq \alpha \leq 1$ and s, where

$$P_\zeta^{\alpha,s}(\upsilon) = \sum_{\iota=0}^{\zeta} p\left(\frac{\iota}{\zeta}\right)\left(\frac{\iota}{\zeta}-\upsilon\right)^2 a_{\zeta,\iota}^{\alpha,s}(\upsilon).$$

On the other hand,

$$\begin{aligned}\left|P_\zeta^{\alpha,s}(\upsilon)\right| \le & \sum_{\left|\frac{\iota}{\zeta}-\upsilon\right|<\zeta^{-\frac{1}{8}}} \left|p\left(\frac{\iota}{\zeta}\right)\right|\left(\frac{\iota}{\zeta}-\upsilon\right)^2 a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\ & + \sum_{\left|\frac{\iota}{\zeta}-\upsilon\right|\ge\zeta^{-\frac{1}{8}}} \left|p\left(\frac{\iota}{\zeta}\right)\right|\left(\frac{\iota}{\zeta}-\upsilon\right)^2 a_{\zeta,\iota}^{\alpha,s}(\upsilon).\end{aligned}$$

Given $\varepsilon > 0$, we can find ζ, large enough such that

$$\left|\frac{\iota}{\zeta}-\upsilon\right| < \zeta^{-\frac{1}{8}}$$

implies

$$p\left(\frac{\iota}{\zeta}\right) < \varepsilon.$$

Therefore,

$$\begin{aligned}\left|P_\zeta^{\alpha,s}(\upsilon)\right| &\le \frac{\varepsilon}{\zeta^2} S_2(\upsilon) + K \sum_{\left|\frac{\iota}{\zeta}-\upsilon\right|\ge\zeta^{-\frac{1}{8}}} \left|a_{\zeta,\iota}^{\alpha,s}(\upsilon)\right| \\ &\le \varepsilon\upsilon(1-\upsilon)\left[\frac{1}{2\zeta}+\frac{(1-\alpha)s(s-1)}{2\zeta^2}\right]+\frac{KC}{\zeta^{\frac{3}{2}}},\end{aligned}$$

where $L = \sup\limits_{0\le t\le 1} p(t)(t-\upsilon)^2$. In other words,

$$\zeta\left|P_\zeta^{\alpha,s}(\upsilon)\right| \le \varepsilon\upsilon(1-\upsilon)\left[\frac{1}{2}+\frac{(1-\alpha)s(s-1)}{2\zeta}\right]+\frac{KC}{\zeta^{\frac{1}{2}}}.$$

Since ε is arbitrary (3.14) follows. □

Theorem 3.5: Let f be a bounded function on $[0,1]$ and let $0 \le \alpha \le 1$ be any real number. For the case $\zeta \ge s$, we have

$$\left|L_\zeta^{\alpha,s}(f;\upsilon) - f(\upsilon)\right| \le \frac{3}{2}\omega\left(f, \frac{\sqrt{\zeta+(1-\alpha)s(s-1)}}{\zeta}\right).$$

Proof. For any $\alpha \in [0,1]$, and $\zeta \ge s$ we have,

$$
\begin{aligned}
\left|L_{\zeta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| &= \left|\sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon) f\left(\frac{\iota}{\zeta}\right)-f(\upsilon)\right| \\
&\leq \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\left|f\left(\frac{\iota}{\zeta}\right)-f(\upsilon)\right| \\
&\leq \sum_{\iota=0}^{\zeta} \omega\left(f;\left|\upsilon-\frac{\iota}{\zeta}\right|\right) a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&\leq \sum_{\iota=0}^{\zeta}\left(1+\frac{\zeta}{\sqrt{\zeta+(1-\alpha)s(s-1)}}\left|\upsilon-\frac{\iota}{\zeta}\right|\right) \\
&\quad \times \omega\left(f;\frac{\sqrt{\zeta+(1-\alpha)s(s-1)}}{\zeta}\right) a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&\leq \omega\left(f;\frac{\sqrt{\zeta+(1-\alpha)s(s-1)}}{\zeta}\right) \\
&\quad \times\left(1+\frac{\zeta}{\sqrt{\zeta+(1-\alpha)s(s-1)}}\sum_{\iota=0}^{\zeta}\left|\upsilon-\frac{\iota}{\zeta}\right| a_{\zeta,\iota}^{\alpha,s}(\upsilon)\right). \qquad (3.16)
\end{aligned}
$$

On the other hand as a consequence of Schwarz's inequality we have,

$$
\begin{aligned}
\sum_{\iota=0}^{\zeta}\left|\upsilon-\frac{\iota}{\zeta}\right| a_{\zeta,\iota}^{\alpha,s}(\upsilon) &= \sum_{\iota=0}^{\zeta}\left|\upsilon-\frac{\iota}{\zeta}\right| \sqrt{a_{\zeta,\iota}^{\alpha,s}(\upsilon)}\sqrt{a_{\zeta,\iota}^{\alpha,s}(\upsilon)} \\
&\leq \left[\sum_{\iota=0}^{\zeta}\left(\upsilon-\frac{\iota}{\zeta}\right)^2 a_{\zeta,\iota}^{\alpha,s}(\upsilon)\right]^{\frac{1}{2}}\left[\sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\right]^{\frac{1}{2}} \\
&\leq \left[\sum_{\iota=0}^{\zeta}\left(\upsilon-\frac{\iota}{\zeta}\right)^2 a_{\zeta,\iota}^{\alpha,s}(\upsilon)\right]^{\frac{1}{2}}.
\end{aligned}
$$

For $\zeta \geq s \geq 2$ we get,

$$
\begin{aligned}
\sum_{\iota=0}^{\zeta}\left(\upsilon-\frac{\iota}{\zeta}\right)^2 a_{\zeta,\iota}^{\alpha,s} &= \frac{\upsilon(1-\upsilon)\left[\zeta+(1-\alpha)s(s-1)\right]}{\zeta^2} \\
&\leq \frac{\zeta+(1-\alpha)s(s-1)}{4\zeta^2}.
\end{aligned}
$$

Therefore,

$$
\left|L_{\zeta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| \leq \frac{3}{2}\omega\left(f,\frac{\sqrt{\zeta+(1-\alpha)s(s-1)}}{\zeta}\right).
$$

□

3.3 Shape Preserving Properties of $L_{\zeta}^{\alpha,s}(f;\upsilon)$

In this section we examine the shape-preserving properties of $L_{\zeta}^{\alpha,s}(f;\upsilon)$. Particularly, we aim to demonstrate that $L_{\zeta}^{\alpha,s}(f;\upsilon)$ preserve monotonicity and convexity properties.

Theorem 3.6: Let $f \in C[0,1]$. If f is monotonically increasing (or monotonically decreasing) on $[0,1]$, then for any positive integer s, and for any $0 \leq \alpha \leq 1$, the operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$ are also monotonically increasing (or monotonically decreasing) on $[0,1]$.

Proof. If $s=1$ or $\zeta < s$, then $L_{\zeta}^{\alpha,s}(f;\upsilon) = B_{\zeta}(f,\upsilon)$, which preserve monotonicity property. Now, assuming $\zeta \geq s \geq 2$, according to Lemma 3.4, we have:

$$\begin{aligned}L_{\zeta}^{\alpha,s}(f;\upsilon) =&(1-\alpha)\sum_{\iota=0}^{\zeta}\left[\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota}\right.\\ &\left.+\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}\right]f_{\iota}+\alpha B_{\zeta}(f,\upsilon).\end{aligned}$$

Say

$$L_{\zeta}^{\star,s}(f;\upsilon) := \sum_{\iota=0}^{\zeta}\left[\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota}+\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}\right]f_{\iota}.$$

Then, we have

$$\begin{aligned}(L_{\zeta}^{\star,s})'(f;\upsilon) =&\sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota-s}(\iota-s+1)\upsilon^{\iota-s}(1-\upsilon)^{\zeta-\iota}f_{\iota}\\ &-\sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota-s}(\zeta-\iota)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}f_{\iota}\\ &+\sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota}\iota\upsilon^{\iota-1}(1-\upsilon)^{\zeta-s-\iota+1}f_{\iota}\\ &-\sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota}(\zeta-\iota-s+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}f_{\iota}.\end{aligned}$$

The first and third summations are zero for $\iota = 0$ and second and fourth summations are zero for $\iota = \zeta$. So if we modify the summations and then if we replace ι by $\iota+1$ in first and third summations we get,

$$
\begin{aligned}
(L_{\zeta}^{\star,s})'(f;\upsilon) = &\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}(\iota-s+2)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}f_{\iota+1}\\
&-\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s}(\zeta-\iota)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota+1}(\iota+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}f_{\iota+1}\\
&-\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota}(\zeta-\iota-s+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}f_{\iota}. \qquad (3.17)
\end{aligned}
$$

Using the equations

$$\binom{\zeta-s}{\iota-s}(\zeta-\iota)=\binom{\zeta-s}{\iota-s+1}(\iota-s+1)$$

and

$$\binom{\zeta-s}{\iota+1}(\iota+1)=\binom{\zeta-s}{\iota}(\zeta-s-\iota)$$

in (3.17) we have,

$$
\begin{aligned}
(L_{\zeta}^{\star,s})'(f;\upsilon) = &\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota}(\zeta-s-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}f_{\iota+1}\\
&-\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}f_{\iota}.
\end{aligned}
$$

Since $\binom{\zeta-s}{\iota-s+1}=0$ when $\iota<s-1$, and $\binom{\zeta-s}{\iota}=0$ when $\iota>\zeta-s$, we get

$$
\begin{aligned}
(L_{\zeta}^{\star,s})'(f;\upsilon) = &\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota}(\zeta-s-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\triangle f_{\iota}\\
&+\sum_{\iota=s-1}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}f_{\iota+1}\\
&-\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}f_{\iota}.
\end{aligned}
$$

Replace ι by $\iota+s-1$ in the third summation we get,

$$
\begin{aligned}
(L_{\zeta}^{\star,s})'(f;\upsilon) =& \sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota}(\zeta-s-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}f_{\iota+s}-\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}f_{\iota},
\end{aligned}
$$

or

$$
\begin{aligned}
(L_{\zeta}^{\star,s})'(f;\upsilon) =& \sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota}(\zeta-s-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}(f_{\iota+s}-f_{\iota}).
\end{aligned}
\tag{3.18}
$$

Therefore, using (3.18) and the fact that

$$
B'_{\zeta}(f,\upsilon)=\sum_{\iota=0}^{\zeta-1}\binom{\zeta}{\iota}(\zeta-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota-1}\triangle f_{\iota}
$$

we get,

$$
\begin{aligned}
(L_{\zeta}^{\alpha,s})'(f;\upsilon) =&(1-\alpha)\Bigg[\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota-s+1}(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-1}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-1}\binom{\zeta-s}{\iota}(\zeta-s-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\triangle f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}(f_{\iota+s}-f_{\iota})\Bigg]\\
&+\alpha\sum_{\iota=0}^{\zeta-1}\binom{\zeta}{\iota}(\zeta-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota-1}\triangle f_{\iota}.
\end{aligned}
\tag{3.19}
$$

Now, if $f(\upsilon)$ is monotonically increasing, then $\triangle f_{\iota}\geq 0$, and $f_{\iota+s}-f_{\iota}\geq 0$ which implies

$$
(L_{\zeta}^{\alpha,s})'(f;\upsilon)\geq 0.
$$

Similarly, if $f(\upsilon)$ is monotonically decreasing, then $\triangle f_{\iota}\leq 0$ and $f_{\iota+s}-f_{\iota}\leq 0$ which implies

$$
(L_{\zeta}^{\alpha,s})'(f;\upsilon)\leq 0.
$$

$\square$

Theorem 3.7: Let $f(\upsilon)$ be a continuous function on the interval $[0,1]$. If $f(\upsilon)$ is convex (or concave) on $[0,1]$, then for any positive integer s, and for any $0 \leq \alpha \leq 1$, $L_{\zeta}^{\alpha,s}(f;\upsilon)$ are also convex (or concave) on $[0,1]$.

Proof. If $s = 1$ or $\zeta < s$, then $L_{\zeta}^{\alpha,s}(f;\upsilon) = B_{\zeta}(f,\upsilon)$, which preserve convexity property. Now we can assume that $\zeta \geq s \geq 2$. The derivative of (3.18) gives that,

$$\begin{aligned}
(L_{\zeta}^{\star,s})''(f;\upsilon) =& \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota-s+1}(\iota-s+1)(\iota-s+2)\upsilon^{\iota-s}(1-\upsilon)^{\zeta-\iota-1}f_{\iota+1} \\
&- \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota-s+1}(\iota-s+2)(\zeta-\iota-1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota+1} \\
&- \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota-s}(\zeta-\iota)(\iota-s+1)\upsilon^{\iota-s}(1-\upsilon)^{\zeta-\iota-1}f_{\iota} \\
&+ \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota-s}(\zeta-\iota)(\zeta-\iota-1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota} \\
&+ \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota+1}\iota(\iota+1)\upsilon^{\iota-1}(1-\upsilon)^{\zeta-s-\iota}f_{\iota+1} \\
&- \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota+1}(\zeta-s-\iota)(\iota+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota+1} \\
&- \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota}(\zeta-s-\iota+1)\iota\upsilon^{\iota-1}(1-\upsilon)^{\zeta-s-\iota}f_{\iota} \\
&+ \sum_{\iota=0}^{\zeta-1} \binom{\zeta-s}{\iota}(\zeta-s-\iota+1)(\zeta-s-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota}.
\end{aligned}$$

First, third, fifth and seventh summations are zero for $\iota = 0$ and other summations are zero for $\iota = \zeta - 1$. Modify end-points of each summation accordingly and replace ι by $\iota + 1$ in first, third, fifth and seventh summations we have,

$$
\begin{aligned}
(L_{\zeta}^{\star,s})''(f;\upsilon) &= \sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s+2}(\iota-s+2)(\iota-s+3)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota+2}\\
&\quad -2\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\iota-s+2)(\zeta-\iota-1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota+1}\\
&\quad +\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s}(\zeta-\iota)(\zeta-\iota-1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota}\\
&\quad +\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota+2}(\iota+1)(\iota+2)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota+2}\\
&\quad -2\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota+1}(\zeta-s-\iota)(\iota+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota+1}\\
&\quad +\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota}(\zeta-s-\iota+1)(\zeta-s-\iota)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota}. \qquad (3.20)
\end{aligned}
$$

Using following equations

$$\binom{\zeta-s}{\iota-s+2}(\iota-s+2)=\binom{\zeta-s}{\iota-s+1}(\zeta-\iota-1),$$

$$\binom{\zeta-s}{\iota-s}(\zeta-\iota)=\binom{\zeta-s}{\iota-s+1}(\iota-s+1),$$

$$\binom{\zeta-s}{\iota+2}(\iota+2)=\binom{\zeta-s}{\iota+1}(\zeta-s-\iota-1),$$

and

$$\binom{\zeta-s}{\iota}(\zeta-s-\iota)=\binom{\zeta-s}{\iota+1}(\iota+1)$$

in (3.20) we get,

$$
\begin{aligned}
(L_{\zeta}^{\star,s})''(f;\upsilon) = &\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\zeta-\iota-1)(\iota-s+3)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota+2}\\
&-2\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\iota-s+2)(\zeta-\iota-1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota+1}\\
&+\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\zeta-\iota-1)(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota+1}(\iota+1)(\zeta-s-\iota-1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota+2}\\
&-2\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota+1}(\zeta-s-\iota)(\iota+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota+1}\\
&+\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota+1}(\iota+1)(\zeta-s-\iota+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}f_{\iota}
\end{aligned}
$$

So,

$$
\begin{aligned}
(L_{\zeta}^{\star,s})''(f;\upsilon) = &\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\zeta-\iota-1)(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}\bigtriangleup^2 f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\zeta-\iota-1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}(2f_{\iota+2}-2f_{\iota+1})\\
&+\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota+1}(\iota+1)(\zeta-s-\iota-1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}\bigtriangleup^2 f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-2}\binom{\zeta-s}{\iota+1}(\iota+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}(-2f_{\iota+1}+2f_{\iota}).
\end{aligned}
$$

The first two summations are zero for $\iota < s-1$ and last two summations are zero for $\iota > \zeta-s-1$, so we get

$$
\begin{aligned}
(L_{\zeta}^{\star,s})''(f;\upsilon) = &\sum_{\iota=s-1}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\zeta-\iota-1)(\iota-s+1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}\bigtriangleup^2 f_{\iota}\\
&+\sum_{\iota=s-1}^{\zeta-2}\binom{\zeta-s}{\iota-s+1}(\zeta-\iota-1)\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota-2}(2f_{k+2}-2f_{\iota+1})\\
&+\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota+1}(\iota+1)(\zeta-s-\iota-1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}\bigtriangleup^2 f_{\iota}\\
&+\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota+1}(\iota+1)\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota-1}(-2f_{\iota+1}+2f_{\iota}).
\end{aligned}
$$

$$
\begin{aligned}
(L_\zeta^{\star,s})''(f;\upsilon) = &\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota}\iota(\zeta-\iota-s)\upsilon^\iota(1-\upsilon)^{\zeta-\iota-s-1}\triangle^2 f_{\iota+s-1}\\
&+\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota+1}(\iota+1)(\zeta-s-\iota-1)\upsilon^\iota(1-\upsilon)^{\zeta-s-\iota-1}\triangle^2 f_\iota\\
&+2\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota}(\zeta-\iota-s)\upsilon^\iota(1-\upsilon)^{\zeta-\iota-s-1}\\
&\times(f_{\iota+s+1}-f_{\iota+s}-f_{\iota+1}+f_\iota).
\end{aligned}
\tag{3.21}
$$

By using mathematical induction, one can prove that

$$
f_{\iota+s+1}-f_{\iota+s}-f_{\iota+1}+f_\iota=\sum_{m=0}^{s-1}\triangle^2 f_{\iota+m}. \tag{3.22}
$$

Therefore, using (3.22) in (3.21) we get,

$$
\begin{aligned}
(L_\zeta^{\star,s})''(f;\upsilon) = &\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota}(\zeta-\iota-s)\upsilon^\iota(1-\upsilon)^{\zeta-\iota-s-1}(\iota)\triangle^2 f_{\iota+s-1}\\
&+\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota+1}(\iota+1)\upsilon^\iota(1-\upsilon)^{\zeta-s-\iota-1}(\zeta-s-\iota-1)\triangle^2 f_\iota\\
&+2\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s}{\iota}(\zeta-\iota-s)\upsilon^\iota(1-\upsilon)^{\zeta-\iota-s-1}\sum_{m=0}^{s-1}\triangle^2 f_{\iota+m}.
\end{aligned}
$$

If f is convex then

$$
\triangle^2 f_\iota \geq 0,
$$

which means $(L_\zeta^{\star,s})''(f;\upsilon)\geq 0$ and $B''_\zeta(f;\upsilon)\geq 0$ or equivalently,

$$
(L_\zeta^{\alpha,s})''(f;\upsilon)=(1-\alpha)(L_\zeta^{\star,s})''(f;\upsilon)+\alpha B''_\zeta(f;\upsilon)\geq 0,
$$

which completes the proof. □

3.4 Graphical Analysis and Error Estimation of $L_\zeta^{\alpha,s}(f;\upsilon)$

In this section, we explore how well the operators $L_\zeta^{\alpha,s}(f;\upsilon)$ approximate different functions f, using various values for ζ, s, and α by graphically and numerically.

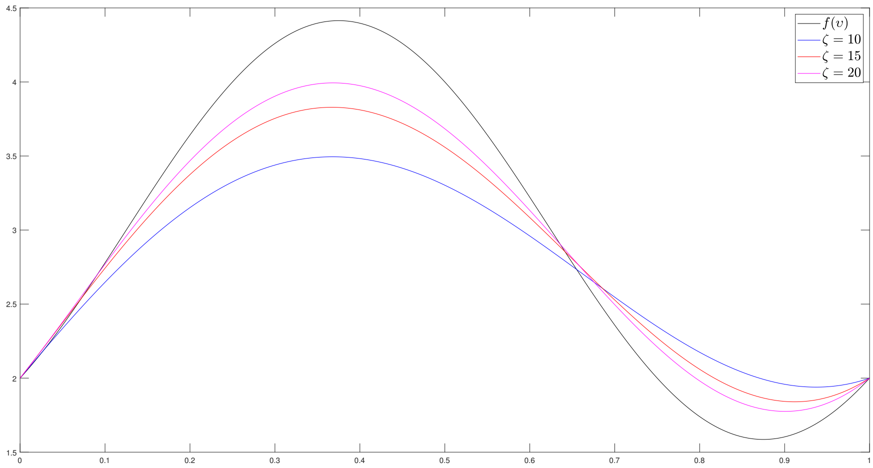

Figure 3.1: Approximation of $L_{\zeta}^{\alpha,s}(f;\upsilon)$ to $f(\upsilon) = 3 - \cos(2\pi\upsilon) + \sin(2\pi\upsilon)$ for $s = 5$, $\alpha = \frac{1}{2}$ and $\zeta = 10, 15, 20$.

Table 3.1: Estimation of error.

υ	$L_{10}^{\frac{1}{2},5}(f;\upsilon) - f(\upsilon)$	$L_{15}^{\frac{1}{2},5}(f;\upsilon) - f(\upsilon)$	$L_{20}^{\frac{1}{2},5}(f;\upsilon) - f(\upsilon)$
0.0	0.0000	0.0000	0.0000
0.1	0.1312	0.0364	0.0091
0.2	0.4905	0.2673	0.1741
0.3	0.8213	0.5060	0.3562
0.4	0.9142	0.5853	0.4218
0.5	0.6973	0.4406	0.3161
0.6	0.2613	0.1371	0.0887
0.7	0.1887	0.1697	0.1382
0.8	0.4346	0.3195	0.2411
0.9	0.3552	0.2395	0.1722
1.0	0.0000	0.0000	0.0000

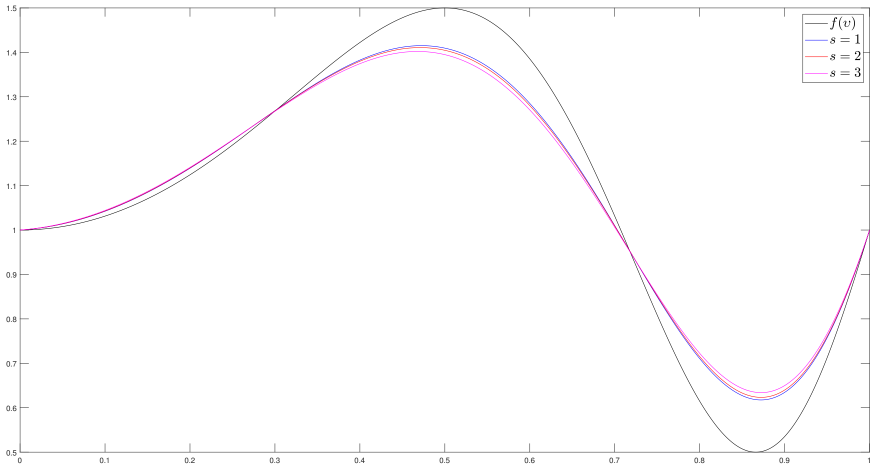

Figure 3.2: Approximation of $L_{\zeta}^{\alpha,s}(f;\upsilon)$ to $f(\upsilon)=\sin(\pi\upsilon^2)\cos(\pi\upsilon^2)+1$ for $\zeta=25$, $\alpha=\frac{1}{4}$ and $s=1,2,3$.

Table 3.2: Estimation of error.

υ	$L_{25}^{\frac{1}{4},1}(f;\upsilon)-f(\upsilon)$	$L_{25}^{\frac{1}{4},2}(f;\upsilon)-f(\upsilon)$	$L_{25}^{\frac{1}{4},3}(f;\upsilon)-f(\upsilon)$
0.0	0.0000	0.0000	0.0000
0.1	0.0110	0.0116	0.0128
0.2	0.0148	0.0155	0.0167
0.3	0.0007	0.0004	0.0006
0.4	0.0396	0.0422	0.0477
0.5	0.0905	0.0954	0.1052
0.6	0.1002	0.1051	0.1149
0.7	0.0214	0.0224	0.0252
0.8	0.0959	0.1004	0.1078
0.9	0.0994	0.1047	0.1147
1.0	0.0000	0.0000	0.0000

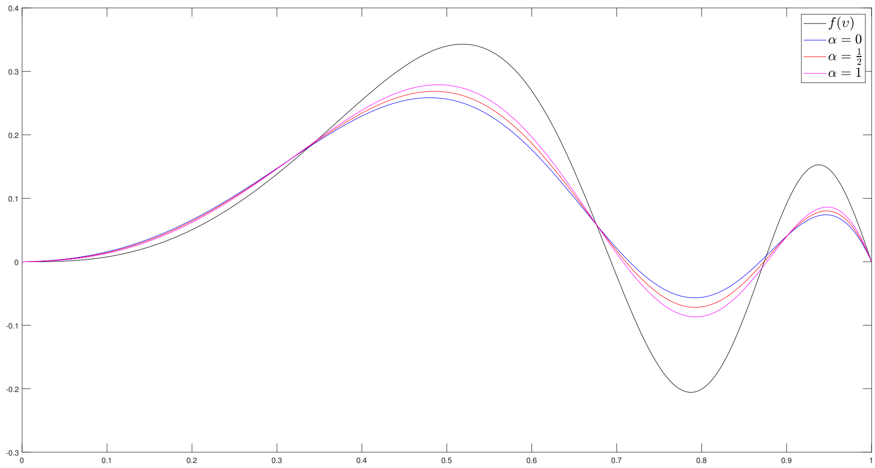

Figure 3.3: Approximation of $L_{\zeta}^{\alpha,s}(f;\upsilon)$ to $f(\upsilon) = e^{-2\upsilon}\sin(3\pi\upsilon^3)$ for $\zeta = 30$, $s = 4$ and $\alpha = 0, \frac{1}{2}, 1$.

Table 3.3: Estimation of error.

υ	$\left\|L_{30}^{0,4}(f;\upsilon) - f(\upsilon)\right\|$	$\left\|L_{30}^{\frac{1}{2},4}(f;\upsilon) - f(\upsilon)\right\|$	$\left\|L_{30}^{1,4}(f;\upsilon) - f(\upsilon)\right\|$
0.0	0.0000	0.0000	0.0000
0.1	0.0080	0.0068	0.0056
0.2	0.0155	0.0135	0.0116
0.3	0.0092	0.0087	0.0082
0.4	0.0252	0.0207	0.0161
0.5	0.0836	0.0726	0.0615
0.6	0.0929	0.0827	0.0726
0.7	0.0436	0.0396	0.0355
0.8	0.1446	0.1295	0.1145
0.9	0.0512	0.0515	0.0518
1.0	0.0000	0.0000	0.0000

Chapter 4

BLENDING TYPE ψ-BERNSTEIN-KANTOROVICH OPERATORS

In this section, we introduce the blending type ψ-Bernstein-Kantorovich operators as follows:

$$K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)=\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \tag{4.1}$$

for all $\upsilon\in[0,1]$, where $b_{\zeta,\iota}^{\alpha,s}(\upsilon)$ is defined in (3.3), and $\psi_\eta(t)$ is any integrable function on $[0,1]$ such that

$$\begin{aligned} 0 &\leq \psi_\eta(t)\leq 1,\\ \psi_\eta(0)&=0,\ \psi_\eta(1)=1, \text{ and}\\ M_{p,\psi_\eta} &:= \int_0^1 \psi_\eta^p(t)dt \end{aligned} \tag{4.2}$$

exits where p is any non-negative integer. Obviously, $0\leq M_{p,\psi_\eta}\leq 1$ for all p.

It should be noted that if we choose $\psi(t)=t$, then the operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ reduce to the classical blending type Bernstein-Kantorovich operators $K_{\zeta}^{\alpha,s}(f;\upsilon)$.

Remark 4.1: The operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ have the following special cases:

(I) If $\psi_\eta(t)=t$, and $s=1$ then $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)=K_\zeta(f;\upsilon)$, which is given in (2.16)

(II) If $\psi_\eta(t)=t^\eta$ and $s=1$, then $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)=K_{\zeta,\eta}(f;\upsilon)$, which is given in (2.17).

(III) If $\psi_\eta(t)=t$ and $s=2$, then $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)=K_{\zeta}^{\alpha,2}(f;\upsilon)$, which is given in [22].

(IV) If $s=1$, then $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)=K_{\zeta,\psi_\eta}(f;\upsilon)$, which is introduced in [3].

4.1 Basic Results and Properties of $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$

In this section, we provide some results and properties of the operator $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$, that will be utilized in the following sections. Owing to the next lemma, we will be able to find moments of the operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ by using the moments of the blending type Bernstein operators $L_{\zeta}^{\alpha,s}(f;\upsilon)$.

Lemma 4.1: For any $\zeta \in \mathbb{N}$ and for any function $\psi_\eta(t)$ satisfying (4.2) we have;

- $K_{\zeta,\psi_\eta}^{\alpha,s}(1;\upsilon) = 1.$
- $K_{\zeta,\psi_\eta}^{\alpha,s}(t;\upsilon) = \frac{\zeta\upsilon}{\zeta+1} + \frac{M_{1,\psi_\eta}}{\zeta+1}.$
- $K_{\zeta,\psi_\eta}^{\alpha,s}(t^2;\upsilon) = \begin{cases} \frac{\zeta^2\upsilon^2}{(\zeta+1)^2} + \frac{\upsilon(1-\upsilon)\zeta}{(\zeta+1)^2} + \frac{2\zeta\upsilon M_{1,\psi_\eta}}{(\zeta+1)^2} + \frac{M_{2,\psi_\eta}}{(\zeta+1)^2}, & \zeta < s, \\ \\ \frac{\zeta^2\upsilon^2+\upsilon(1-\upsilon)[\zeta+(1-\alpha)s(s-1)]+2\zeta\upsilon M_{1,\psi_\eta}+M_{2,\psi_\eta}}{(\zeta+1)^2}, & \zeta \geq s. \end{cases}$

Proof. For the case $\zeta < s$, we have $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon) = K_{\zeta,\psi_\eta}(f;\upsilon)$. Therefore,

$$\begin{aligned} K_{\zeta,\psi_\eta}^{\alpha,s}(1;\upsilon) &= \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\int_0^1 1dt \\ &= \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon) \\ &= 1. \end{aligned}$$

$$\begin{aligned} K_{\zeta,\psi_\eta}^{\alpha,s}(t;\upsilon) &= \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\int_0^1 \frac{\iota+\psi_\eta(t)}{\zeta+1}dt \\ &= \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\int_0^1 \frac{\iota}{\zeta+1}dt + \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\int_0^1 \frac{\psi_\eta(t)}{\zeta+1}dt \\ &= \frac{\zeta}{\zeta+1}\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\frac{\iota}{\zeta} + \frac{M_{1,\psi_\eta}}{\zeta+1}\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon) \\ &= \frac{\zeta\upsilon}{\zeta+1} + \frac{M_{1,\psi_\eta}}{\zeta+1}. \end{aligned}$$

$$
\begin{aligned}
K^{\alpha,s}_{\zeta,\psi_\eta}(t^2;\upsilon) =& \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\int_0^1 \left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)^2 dt\\
=& \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\int_0^1 \frac{\iota^2+2\iota\psi_\eta(t)+\psi_\eta^2(t)}{(\zeta+1)^2}dt\\
=& \frac{\zeta^2}{(\zeta+1)^2}\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\frac{\iota^2}{\zeta^2}+\frac{2\zeta M_{1,\psi_\eta}}{(\zeta+1)^2}\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\frac{\iota}{\zeta}\\
&+\frac{M_{2,\psi_\eta}}{(\zeta+1)^2}\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}(\upsilon)\\
=& \frac{\zeta^2\upsilon^2+\upsilon(1-\upsilon)\zeta+2\zeta M_{1,\psi_\eta}\upsilon+M_{2,\psi_\eta}}{(\zeta+1)^2}.
\end{aligned}
$$

For the case $\zeta \geq s$,

$$
\begin{aligned}
K^{\alpha,s}_{\zeta,\psi_\eta}(1;\upsilon) &= \sum_{\iota=0}^{\zeta} a^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1 1dt\\
&= \sum_{\iota=0}^{\zeta} a^{\alpha,s}_{\zeta,\iota}(\upsilon)\\
&= 1.
\end{aligned}
$$

$$
\begin{aligned}
K^{\alpha,s}_{\zeta,\psi_\eta}(t;\upsilon) &= \sum_{\iota=0}^{\zeta} a^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1 \frac{\iota+\psi_\eta(t)}{\zeta+1}dt\\
&= \sum_{\iota=0}^{\zeta} a^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1 \frac{\iota}{\zeta+1}dt+\sum_{\iota=0}^{\zeta} a^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1 \frac{\psi_\eta(t)}{\zeta+1}dt\\
&= \frac{\zeta}{\zeta+1}\sum_{\iota=0}^{\zeta} a^{\alpha,s}_{\zeta,\iota}(\upsilon)\frac{\iota}{\zeta}+\frac{M_{1,\psi_\eta}}{\zeta+1}\sum_{\iota=0}^{\zeta} a^{\alpha,s}_{\zeta,\iota}(\upsilon)\\
&= \frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}.
\end{aligned}
$$

$$
\begin{aligned}
K_{\zeta,\psi_\eta}^{\alpha,s}(t^2;\upsilon) &= \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)^2 dt \\
&= \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \frac{\iota^2+2\iota\psi_\eta(t)+\psi^2(t)}{(\zeta+1)^2}dt \\
&= \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \frac{\iota^2}{(\zeta+1)^2}dt + \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \frac{2\iota\psi_\eta(t)}{(\zeta+1)^2}dt \\
&\quad + \sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \frac{\psi^2(t)}{(\zeta+1)^2}dt \\
&= \frac{\zeta^2}{(\zeta+1)^2}\sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\frac{\iota^2}{\zeta^2} + \frac{2\zeta M_{1,\psi_\eta}}{(\zeta+1)^2}\sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon)\frac{\iota}{\zeta} \\
&\quad + \frac{M_{2,\psi_\eta}}{(\zeta+1)^2}\sum_{\iota=0}^{\zeta} a_{\zeta,\iota}^{\alpha,s}(\upsilon) \\
&= \frac{\zeta^2\upsilon^2}{(\zeta+1)^2} + \frac{\upsilon(1-\upsilon)[\zeta+(1-\alpha)s(s-1)]}{(\zeta+1)^2} + \frac{2\zeta M_{1,\psi_\eta}\upsilon}{(\zeta+1)^2} + \frac{M_{2,\psi_\eta}}{(\zeta+1)^2}.
\end{aligned}
$$

$\square$

Corollary 4.1: We have the following special cases of the above lemma.

i) Let $\psi_\eta(t) = t$, then we have $K_{\zeta,\psi_\eta}^{\alpha,s}(t^i;\upsilon) = K_{\zeta}^{\alpha,s}(t^i;\upsilon)$ with $M_{1,\psi_\eta} = \frac{1}{2}$ and $M_{2,\psi_\eta} = \frac{1}{3}$, for $i = 0,1,2$.

ii) Let $\psi_\eta(t) = t^\eta$, then we have $K_{\zeta,\psi_\eta}^{\alpha,s}(t^i;\upsilon) = K_{\zeta,\eta}^{\alpha,s}(t^i;\upsilon)$ with $M_{1,\psi_\eta} = \frac{1}{\eta+1}$ and $M_{2,\psi_\eta} = \frac{1}{2\eta+1}$ for $i = 0,1,2$.

Lemma 4.2: For any $\zeta \in \mathbb{N}$ and for each $\psi_\eta(t)$, given in (4.2), we have the following relation;

$$
K_{\zeta,\psi_\eta}^{\alpha,s}(t^m;\upsilon) = \frac{1}{(\zeta+1)^m}\sum_{i=0}^{m}\binom{m}{i}\zeta^i M_{m-i,\psi_\eta} L_{\zeta}^{\alpha,s}(t^i;\upsilon)
$$

for each $m \in \mathbb{N}$.

Proof. For any $m \in \mathbb{N}$,

$$\begin{aligned}K^{\alpha,s}_{\zeta,\psi_\eta}(t^m;\upsilon) &= \sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)^m dt\\ &= \frac{1}{(\zeta+1)^m}\sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1 (\iota+\psi_\eta(t))^m dt\\ &= \frac{1}{(\zeta+1)^m}\sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\sum_{i=0}^{m}\binom{m}{i}\iota^i M_{m-i,\psi_\eta}\\ &= \frac{1}{(\zeta+1)^m}\sum_{i=0}^{m}\binom{m}{i}\zeta^i M_{m-i,\psi_\eta}\sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\frac{\iota^i}{\zeta^i}\\ &= \frac{1}{(\zeta+1)^m}\sum_{i=0}^{m}\binom{m}{i}\zeta^i M_{m-i,\psi_\eta}L^{\alpha,s}_{\zeta}(t^i;\upsilon).\end{aligned}$$

where M_{m-i,ψ_η} is given in (4.2). □

Following lemma gives the relation between central moments of the operators $K^{\alpha,s}_{\zeta,\psi_\eta}(f,\upsilon)$ and $L^{\alpha,s}_{\zeta}(f;\upsilon)$.

Lemma 4.3: For all $\zeta \in \mathbb{N}$,

$$K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^m;\upsilon) = \sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^{m-r}\upsilon^{m-r}}{(\zeta+1)^r}\sum_{i=0}^{r}\binom{r}{i}\zeta^i M_{r-i,\psi_\eta}L^{\alpha,s}_{\zeta}(t^i;\upsilon),$$

where $m \in \mathbb{N}$ and $L^{\alpha,s}_{\zeta}(f;\upsilon)$ are defined in (3.2).

Proof. Let $m \in \mathbb{N}$. Since $K^{\alpha,s}_{\zeta,\psi_\eta}$ are linear operators, we obtain

$$\begin{aligned}K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^m;\upsilon) &= K^{\alpha,s}_{\zeta,\psi_\eta}\left(\sum_{r=0}^{m}\binom{m}{r}t^r(-\upsilon)^{m-r};\upsilon\right)\\ &= \sum_{r=0}^{m}\binom{m}{r}(-1)^{m-r}\upsilon^{m-r}K^{\alpha,s}_{\zeta,\psi_\eta}(t^r;\upsilon).\end{aligned}$$

From Lemma 4.2, we get

$$K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^m;\upsilon) = \sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^{m-r}\upsilon^{m-r}}{(\zeta+1)^r}\sum_{i=0}^{r}\binom{r}{i}\zeta^i M_{r-i,\psi_\eta}L^{\alpha,s}_{\zeta}(t^i;\upsilon)$$

which completes the proof. □

Corollary 4.2: For any $\zeta \in \mathbb{N}$ and function $\psi_\eta(t)$ given in (4.2) we have

- $K^{\alpha,s}_{\zeta,\psi_\eta}(t-\upsilon;\upsilon) := \varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon) = \frac{-\upsilon}{\zeta+1} + \frac{M_{1,\psi\eta}}{\zeta+1}$

•

$$K_{\zeta,\psi_\eta}^{\alpha,s}\left((t-\upsilon)^2;\upsilon\right) := \phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon) = \begin{cases} \frac{\upsilon^2}{(\zeta+1)^2}+\frac{\upsilon(1-\upsilon)\zeta}{(\zeta+1)^2} \\ -\frac{M_{1,\psi_\eta}\upsilon}{(\zeta+1)^2}+\frac{M_{2,\psi_\eta}}{(\zeta+1)^2}, & if\ \zeta<s, \\ \\ \frac{\upsilon^2}{(\zeta+1)^2} \\ +\frac{\upsilon(1-\upsilon)[\zeta+(1-\alpha)s(s-1)]}{(\zeta+1)^2} \\ -\frac{M_{1,\psi_\eta}\upsilon}{(\zeta+1)^2}+\frac{M_{2,\psi_\eta}}{(\zeta+1)^2}, & if\ \zeta\geq s. \end{cases}$$

Using Remark 4.1, the following central moments for $K_\zeta(f;\upsilon)$ and $K_{\zeta,\eta}(f;\upsilon)$ can be obtain from Corollary 4.2.

Corollary 4.3: Taking $M_{1,\psi_\eta}=\frac{1}{2}$ and $M_{2,\psi_\eta}=\frac{1}{3}$ in Lemma 4.3 (see [16]), we get

i) $K_\zeta^{\alpha,s}(t-\upsilon;\upsilon)=\frac{-\upsilon}{\zeta+1}+\frac{1}{2(\zeta+1)}$.

ii) $$K_\zeta^{\alpha,s}\left((t-\upsilon)^2;\upsilon\right) = \begin{cases} \frac{\upsilon^2}{(\zeta+1)^2}+\frac{\upsilon(1-\upsilon)\zeta}{(\zeta+1)^2} \\ -\frac{\upsilon}{2(\zeta+1)^2}+\frac{1}{3(\zeta+1)^2}, & if\ \zeta<s, \\ \\ \frac{\upsilon^2}{(\zeta+1)^2} \\ +\frac{\upsilon(1-\upsilon)[\zeta+(1-\alpha)s(s-1)]}{(\zeta+1)^2} \\ -\frac{\upsilon}{2(\zeta+1)^2}+\frac{1}{3(\zeta+1)^2}, & if\ \zeta\geq s. \end{cases}$$

Corollary 4.4: For, $\psi_\eta(t)=t^\eta$ we have, $M_{1,\psi_\eta}=\frac{1}{\eta+1}$, $M_{2,\psi_\eta}=\frac{1}{2\eta+1}$ and from Lemma 4.3 (see [6]), we obtain

i) $K_{\zeta,\eta}^{\alpha,s}(t-\upsilon;\upsilon)=\frac{-\upsilon}{\zeta+1}+\frac{1}{(\eta+1)(\zeta+1)}$.

$$\text{ii) } K_{\zeta,\eta}^{\alpha,s}\left((t-\upsilon)^2;\upsilon\right)=\begin{cases}\frac{\upsilon^2}{(\zeta+1)^2}+\frac{\upsilon(1-\upsilon)\zeta}{(\zeta+1)^2} \\ -\frac{\upsilon}{(\eta+1)(\zeta+1)^2}+\frac{1}{(2\eta+1)(\zeta+1)^2}, & if\ \zeta<s, \\ \\ \frac{\upsilon^2}{(\zeta+1)^2} \\ +\frac{\upsilon(1-\upsilon)[\zeta+(1-\alpha)s(s-1)]}{(\zeta+1)^2} \\ -\frac{\upsilon}{(\eta+1)(\zeta+1)^2}+\frac{1}{(2\eta+1)(\zeta+1)^2}, & if\ \zeta\geq s.\end{cases}$$

4.2 Direct and Local Approximation Properties of $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$

In this section, we present the direct and local approximation properties of the operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f,\upsilon)$. Initially, we introduce the Bohman-Korovkin Theorem, (see [19]).

Theorem 4.1: For any $\zeta\in\mathbb{N}$, $0\leq\alpha\leq 1$, nonnegative integer s, and $\psi_\eta(t)$ which satisfy the conditions in (4.2) and for each $f\in C[0,1]$, the operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ are uniformly convergent to f on $[0,1]$.

Proof. Let $f(\upsilon)\in C[0,1]$. Since M_{1,ψ_η} and M_{2,ψ_η} are positive real numbers, from Lemma 4.1, we obtain that

$$\lim_{\zeta\to\infty} K_{\zeta,\psi_\eta}^{\alpha,s}(1;\upsilon)=1,$$

$$\lim_{\zeta\to\infty} K_{\zeta,\psi_\eta}^{\alpha,s}(t;\upsilon)=\upsilon,$$

and

$$\lim_{\zeta\to\infty} K_{\zeta,\psi_\eta}^{\alpha,s}\left(t^2;\upsilon\right)=\upsilon^2.$$

Therefore, the operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ satisfies the Bohman-Korovkin conditions. □

Note that the positive linear operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ on $C[0,1]$, have the property that $K_{\zeta,\psi_\eta}^{\alpha,s}(1;\upsilon)=1$ for all ζ. Recall that operators satisfying the property $K_{\zeta,\psi_\eta}^{\alpha,s}(1;\upsilon)=1$, also satisfy:

$$\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right|\leq\varepsilon+\frac{2\|f\|}{\delta^2}K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon), \tag{4.3}$$

where δ is determined by the uniform continuity of the function f which is being approximated. In other words, equation (4.3) shows that the order of approximation to a function f by $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ is more influenced by the term $K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon)$. Therefore, we can evaluate how well the blending type ψ-Bernstein Kantorovich operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ and classical blending type Bernstein-Kantorovich operators $K_{\zeta}^{\alpha,s}(f;\upsilon)$ approximate the function $f(\upsilon)$ by using second central moments. In the lights of the above information, the inequality

$$K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon)<K_{\zeta}^{\alpha,s}((t-\upsilon)^2;\upsilon), \tag{4.4}$$

will be used to compare the order of the approximation to a function $f(\upsilon)$ by operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ and $K_{\zeta}^{\alpha,s}(f;\upsilon)$.

The advantage of our operators is that, $K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon)$ terms are given depending on the numbers M_{1,ψ_η} and M_{2,ψ_η}; therefore, different $K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon)$ terms will be obtained for different functions $\psi_\eta(t)$ satisfying (4.2). This allows us to investigate different values of M_{1,ψ_η} and M_{2,ψ_η} to obtain a better approximation. In other words, using our operators, the problem of obtaining a better approximation is reduced to the following two problems: for what values of M_{1,ψ_η} and M_{2,ψ_η} does Equation (4.4) hold, and is there any function $\psi_\eta(t)$ with these values of M_{1,ψ_η} and M_{2,ψ_η}? In this thesis, we showed that both problems have solutions.

Lemma 4.4: Let ψ_η be a function satisfying (4.2), then Equation (4.4) holds;

i) for all $\upsilon<\frac{\frac{1}{3}-M_{2,\psi\eta}}{1-2M_{1,\psi_\eta}}$, if $M_{1,\psi_\eta}<\frac{1}{2}$ and $M_{2,\psi_\eta}<\frac{1}{3}$.

ii) for all $\frac{\frac{1}{3}-M_{2,\psi\eta}}{1-2M_{1,\psi_\eta}}<\upsilon$, if $M_{1,\psi_\eta}>\frac{1}{2}$ and $M_{2,\psi_\eta}>\frac{1}{3}$.

iii) for all $\upsilon \in [0,1]$, if $M_{1,\psi_\eta} > \frac{1}{2}$ and $M_{2,\psi_\eta} < \frac{1}{3}$.

Proof. Utilizing Corollary 4.2 and the inequality (4.4), we obtain for any fixed s,

$$\begin{aligned}
K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon) &< K^{\alpha,s}_{\zeta}((t-\upsilon)^2;\upsilon) \\
-\frac{2M_{1,\psi_\eta}\upsilon}{(\zeta+1)^2}+\frac{M_{2,\psi_\eta}}{(\zeta+1)^2} &< -\frac{\upsilon}{(\zeta+1)^2}+\frac{1}{3(\zeta+1)^2} \\
-2M_{1,\psi_\eta}\upsilon+M_{2,\psi_\eta} &< -\upsilon+\frac{1}{3}, \\
(1-2M_{1,\psi_\eta})\upsilon+M_{2,\psi_\eta}-\frac{1}{3} &< 0. \qquad (4.5)
\end{aligned}$$

The solution of the inequality given in (4.5) completes the proof. □

The following theorem shows under which conditions and for what values of υ, the order of approximation to a function $f(\upsilon)$ by our operators is as good as the classical blending type Bernstein-Kantorovich operators $K^{\alpha,s}_{\zeta}(f,\upsilon)$.

Theorem 4.2: Let ψ_η be a function satisfying (4.2), then the order of approximation to $f(\upsilon)$ by $K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)$ is at least as good as the order of approximation by $K^{\alpha,s}_{\zeta}(f;\upsilon)$ for the following cases.

i. If, $M_{1,\psi_\eta} < \frac{1}{2}$ and $M_{2,\psi_\eta} < \frac{1}{3}$, then for all $\upsilon \in \left[0, \frac{\frac{1}{3}-M_{2,\psi_\eta}}{1-2M_{1,\psi_\eta}}\right) \cap [0,1]$.

ii. If, $M_{1,\psi_\eta} > \frac{1}{2}$ and $M_{2,\psi_\eta} > \frac{1}{3}$, then for all $\upsilon \in \left(\frac{\frac{1}{3}-M_{2,\psi_\eta}}{1-2M_{1,\psi_\eta}}, \infty\right) \cap [0,1]$.

iii. If, $M_{1,\psi_\eta} > \frac{1}{2}$ and $M_{2,\psi_\eta} < \frac{1}{3}$, then for all $\upsilon \in [0,1]$.

Proof. Since the domain of the given operators are [0,1], the proof is an obvious consequence of Lemma 4.4. □

In Theorem 4.2, the solutions of the first problem were obtained. Now we will observe that the second problem which is, there is at least one function $\psi_\eta(t)$ with values of M_{1,ψ_η} and M_{2,ψ_η} satisfying Theorem 4.2 exists. Now consider the the following function,

$$\psi_\eta^*(t) := \begin{cases} a^{\eta-\frac{1}{\eta}} t^{\frac{1}{\eta}} & 0 \le t \le a \\ \\ t^\eta & a \le t \le 1, \end{cases} \tag{4.6}$$

where $a \in [0,1]$ and $\eta > 0$. Obviously, $\psi_\eta^*(t)$ is continuous for all $a \in [0,1]$, $\eta > 0$, and satisfies (4.2), with

$$M_{1,\psi_\eta^*} = \Big(\frac{\eta-1}{\eta+1}\Big)a^{\eta+1} + \frac{1}{\eta+1} \text{ and } M_{2,\psi_\eta^*} = \frac{2(\eta-1)(\eta+1)}{(\eta+2)(2\eta+1)}a^{2\eta+1} + \frac{1}{2\eta+1}.$$

Now, we define 3 examples that show that the operators have better approximation than classical case. Let $\psi_\eta^{*,1}(t)$ where $\eta = 1.10$ and $a = 0.75$, $\psi_\eta^{*,2}(t)$ where $\eta = 1.09$ and $a = 0.78$ and $\psi_\eta^{*,3}(t)$ where $\eta = 1.12$ and $a = 0.74$, then it is easy to obtain that

$$M_{1,\psi_\eta^{*,1}} = 0.5022 \text{ and } M_{2,\psi_\eta^{*,1}} = 0.3294$$

$$M_{1,\psi_\eta^{*,2}} = 0.5041 \text{ and } M_{2,\psi_\eta^{*,2}} = 0.3318$$

$$M_{1,\psi_\eta^{*,3}} = 0.5016 \text{ and } M_{2,\psi_\eta^{*,3}} = 0.3276.$$

Therefore,

$$K^{\alpha,s}_{\zeta,\psi_\eta^{*,i}}((t-\upsilon)^2 : \upsilon) \le K^{\alpha,s}_{\zeta}((t-\upsilon)^2 : \upsilon)$$

for all $i = 1,2,3$. From Theorem 4.2-iii. $K^{\alpha,s}_{\zeta,\psi_\eta^{*,i}}(f : \upsilon)$, have better approximation properties than the classical blending type Bernstein-Kantorovich operators for all $i = 1,2,3$ on $[0,1]$.

For the case $s=1$, the operators $K^{\alpha,s}_{\zeta,\psi_\eta}$ reduce to K_{ζ,ψ_η}. Morover choosing $\psi(t)=t$, we get the classical Bernstein-Kantorovich operators. Figure 4.1 shows the comparison between classical Bernstein-Kantorovich operators and ψ-Bernstein-Kantorovich operators for different functions ψ_η.

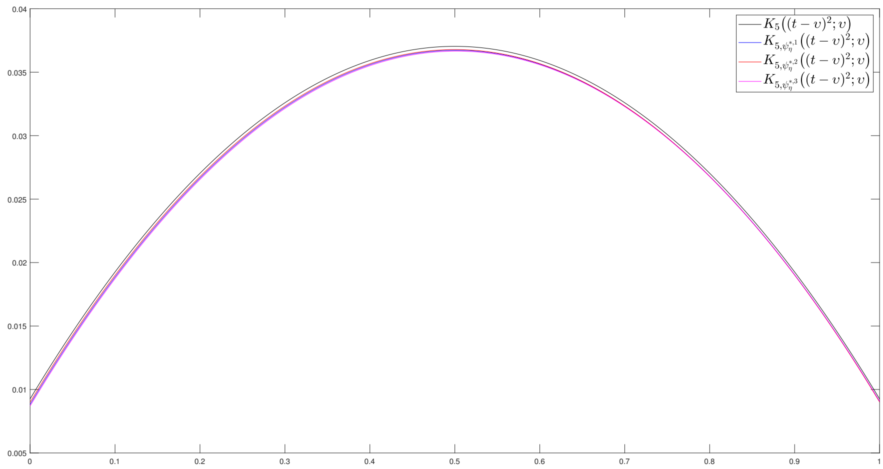

Figure 4.1: Graphical representations of 2^{nd} central moments of K_5, $K_{5,\psi_\eta^{*,1}}$, $K_{5,\psi_\eta^{*,2}}$, and $K_{5,\psi_\eta^{*,3}}$.

Table 4.1: Numerical representations of 2^{nd} central moments of K_5, $K_{5,\psi_\eta^{*,1}}$, $K_{5,\psi_\eta^{*,2}}$, and $K_{5,\psi_\eta^{*,3}}$.

υ	K_5	$K_{5,\psi_\eta^{*,1}}$	$K_{5,\psi_\eta^{*,2}}$	$K_{5,\psi_\eta^{*,3}}$
0.0	0.0093	0.0088	0.0090	0.0087
0.1	0.0193	0.0189	0.0190	0.0187
0.2	0.0270	0.0267	0.0268	0.0265
0.3	0.0326	0.0322	0.0323	0.0321
0.4	0.0359	0.0356	0.0357	0.0355
0.5	0.0370	0.0367	0.0368	0.0367
0.6	0.0359	0.0356	0.0357	0.0356
0.7	0.0326	0.0323	0.0323	0.0323
0.8	0.0270	0.0268	0.0268	0.0268
0.9	0.0193	0.0190	0.0190	0.0190
1.0	0.0093	0.0091	0.0090	0.0091

For the case $s = 2$, the operators $K_{\zeta,\psi_\eta}^{\alpha,s}$ reduce to $K_{\zeta,\psi_\eta}^{\alpha,2}$. Morover choosing $\psi(t) = t$, we get the operators which are introduced in [22]. Figure 4.2 shows the comparison between the operators in [22] and $K_{\zeta,\psi_\eta}^{\alpha,2}$ for different functions ψ_η.

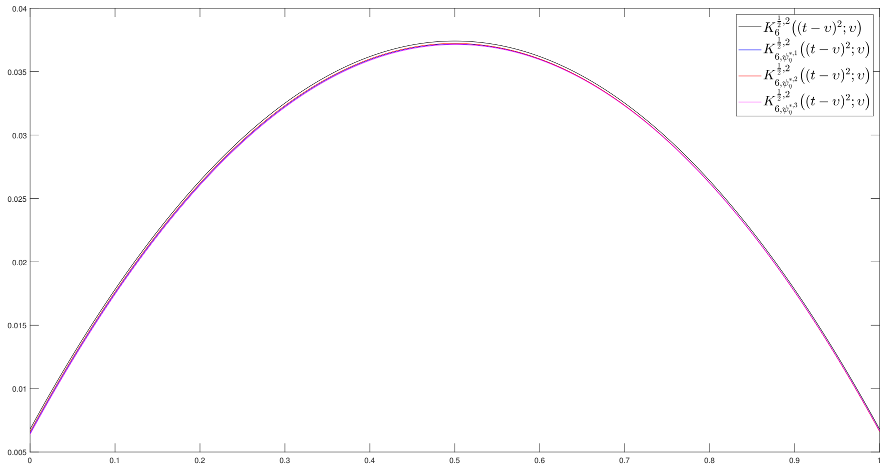

Figure 4.2: Graphical representations of 2^{nd} central moments of $K_6^{\frac{1}{2},2}$, $K_{6,\psi_\eta^{*,1}}^{\frac{1}{2},2}$,$K_{6,\psi_\eta^{*,1}}^{\frac{1}{2},2}$ and $K_{6,\psi_\eta^{*,1}}^{\frac{1}{2},2}$.

Table 4.2: Numerical representations of 2^{nd} central moments of $K_6^{\frac{1}{2},2}$, $K_{6,\psi_\eta^{*,1}}^{\frac{1}{2},2}$,$K_{6,\psi_\eta^{*,1}}^{\frac{1}{2},2}$ and $K_{6,\psi_\eta^{*,1}}^{\frac{1}{2},2}$.

υ	$K_6^{\frac{1}{2},2}$	$K_{6,\psi_\eta^{*,1}}^{\frac{1}{2},2}$	$K_{6,\psi_\eta^{*,2}}^{\frac{1}{2},2}$	$K_{6,\psi_\eta^{*,3}}^{\frac{1}{2},2}$
0.0	0.0068	0.0065	0.0066	0.0064
0.1	0.0178	0.0175	0.0176	0.0174
0.2	0.0264	0.0261	0.0262	0.0260
0.3	0.0325	0.0323	0.0323	0.0322
0.4	0.0362	0.0359	0.0360	0.0359
0.5	0.0374	0.0372	0.0372	0.0371
0.6	0.0362	0.0360	0.0360	0.0359
0.7	0.0325	0.0323	0.0323	0.0323
0.8	0.0264	0.0262	0.0262	0.0262
0.9	0.0178	0.0177	0.0176	0.0177
1.0	0.0068	0.0067	0.0066	0.0067

For $s > 2$, choosing $\psi(t) = t$, we get classical blending type Bernstein-Kantorovich operators. Figure 4.3 shows the comparison between classical case and blending type ψ-Bernstein-Kantorovich operators for different functions ψ_η when $s = 5$.

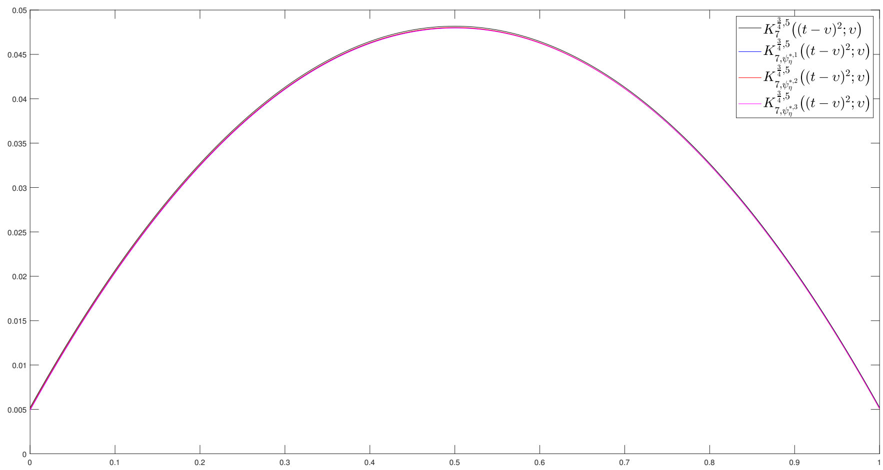

Figure 4.3: Graphical representations of 2^{nd} central moments of $K_7^{\frac{3}{4},5}$, $K_{7,\psi_\eta^{*,1}}^{\frac{3}{4},5}$,$K_{7,\psi_\eta^{*,1}}^{\frac{3}{4},5}$ and $K_{7,\psi_\eta^{*,1}}^{\frac{3}{4},5}$.

Table 4.3: Numerical representations of 2^{nd} central moments of $K_7^{\frac{3}{4},5}$, $K_{7,\psi_\eta^{*,1}}^{\frac{3}{4},5}$,$K_{7,\psi_\eta^{*,1}}^{\frac{3}{4},5}$ and $K_{7,\psi_\eta^{*,1}}^{\frac{3}{4},5}$.

υ	$K_7^{\frac{3}{4},5}$	$K_{7,\psi_\eta^{*,1}}^{\frac{3}{4},5}$	$K_{7,\psi_\eta^{*,2}}^{\frac{3}{4},5}$	$K_{7,\psi_\eta^{*,3}}^{\frac{3}{4},5}$
0.0	0.0052	0.0050	0.0051	0.0049
0.1	0.0207	0.0204	0.0205	0.0204
0.2	0.0327	0.0325	0.0326	0.0324
0.3	0.0413	0.0411	0.0412	0.0410
0.4	0.0465	0.0463	0.0463	0.0462
0.5	0.0482	0.0480	0.0480	0.0480
0.6	0.0465	0.0463	0.0463	0.0463
0.7	0.0413	0.0412	0.0412	0.0411
0.8	0.0327	0.0326	0.0326	0.0326
0.9	0.0207	0.0206	0.0205	0.0205
1.0	0.0052	0.0051	0.0051	0.0051

The figures and tables above indicate that, for fixed values of s, the ψ-Bernstein-Kantorovich operators in the blending type demonstrate better estimation than their classical counterparts. Hence, regardless of the values of α and s, blending type ψ-Bernstein-Kantorovich operators consistently yield superior results. It should be noted that these examples are based on fixed ψ_η functions, and there exist other examples that may outperform even these functions.

Remark 4.2: Intervals, where $K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)$ have better error estimation than classical blending type Bernstein-Kantorovich operators $K^{\alpha,s}_{\zeta}(f;\upsilon)$, can also be found from parts i. and ii. of Theorem 4.2. For example for $\psi^*_\eta(t)$ given in (4.6) we have the following cases,

i. if $\eta = 1.1$ and $a = 0.7$ then, $M_{1,\psi^*_\eta} = 0.4987$ and $M_{2,\psi^*_\eta} = 0.3260$ therefore from Theorem 4.2, i. the order of approximation to $f(\upsilon)$ by $K_{\zeta,\psi^*}(f:\upsilon)$ is as good as the classical blending type Bernstein-Kantorovich operators on the interval $[0,0.921)$.

ii. if $\eta = 1.1$ and $a = 0.9$ then $M_{1,\psi^*_\eta} = 0.5144$ and $M_{2,\psi^*_\eta} = 0.3427$ therefore from Theorem 4.2-ii. the order of approximation to $f(\upsilon)$ by $K_{\zeta,\psi^*_\eta}(f:\upsilon)$ is as good as the classical blending type Bernstein-Kantorovich operators on the interval $(0.3269,1]$.

Remark 4.3: If $\psi_\eta(t) = t^\eta$ and $s = 1$, then $K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon) = K_{\zeta,\eta}(f;\upsilon)$ with $M_{1,\psi_\eta} = \frac{1}{\eta+1}$ and $M_{2,\psi_\eta} = \frac{1}{(2\eta+1)}$. Therefore, Theorem 4.2, gives the following results which are obtained in [25].

i) For any fixed $0 < \eta < 1$, we have $M_{1,\psi_\eta} > \frac{1}{2}$ and $M_{2,\psi_\eta} > \frac{1}{3}$. Therefore, $K_{\zeta,\eta}(f;\upsilon)$ have better approximation results than Bernstein Kantorovich operators $K_\zeta(f;\upsilon)$ for all $\upsilon \in \left(\frac{2\eta+2}{6\eta+3}, 1\right]$.

ii) For any fixed $\eta > 1$, we have $M_{1,\psi_\eta} < \frac{1}{2}$ and $M_{2,\psi_\eta} < \frac{1}{3}$. Therefore, $K_{\zeta,\eta}(f;\upsilon)$ have better approximation results than Bernstein Kantorovich operators $K_\zeta(f;\upsilon)$ for all $\upsilon \in \left[0, \frac{2\eta+2}{6\eta+3}\right)$.

Please note that, the operators defined in [25] and [6] are depending on a varible $\eta > 0$, but there is no $\eta > 0$ which makes $M_{1,\psi_\eta} = \frac{1}{\eta+1} > \frac{1}{2}$ and $M_{2,\psi_\eta} = \frac{1}{2\eta+1} < \frac{1}{3}$, therefore the operators $K_{\zeta,\eta}(f;\upsilon)$ and $K_{\zeta,\eta}^{\alpha,s}(f;\upsilon)$ can not be better than their classical cases for the all closed interval $[0,1]$.

On the other hand, the function $\psi_\eta^*(t)$ given in 4.6 is not unique, different function can be found which gives better approximation properties.

Theorem 4.3: Let $f \in C[0,1]$ and $\psi_\eta(t)$ be a function satisfying (4.2), then we have,

$$\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon) - f(\upsilon)\right| \leq 2\omega\left(f, \sqrt{\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon)}\right)$$

for each $\upsilon, \alpha \in [0,1]$ and $\zeta \in \mathbb{N}$ and nonnegative integer s where $\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon)$ is defined in Corollary 4.2.

Proof. Since $K_{\zeta,\psi_\eta}^{\alpha,s}(1;\upsilon) = 1$ and $b_{\zeta,\iota}^{\alpha,s}(\upsilon) \geq 0$ on $[0,1]$, we are able to write,

$$\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon) - f(\upsilon)\right| \leq \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon) \int_0^1 \left| f\left(\frac{\iota + \psi_\eta(t)}{\zeta+1}\right) - f(\upsilon)\right| dt. \tag{4.7}$$

Also, using modulus of continuity properties, we obtain

$$\begin{aligned} \left| f\left(\frac{\iota + \psi_\eta(t)}{\zeta+1}\right) - f(\upsilon)\right| &\leq \omega\left(f; \frac{1}{\delta}\left|\frac{\iota + \psi_\eta(t)}{\zeta+1} - \upsilon\right| \delta\right) \\ &\leq \left(1 + \frac{1}{\delta}\left|\frac{\iota + \psi_\eta(t)}{\zeta+1} - \upsilon\right|\right)\omega(f;\delta). \end{aligned} \tag{4.8}$$

If we use (4.8) in (4.7), we get

$$\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| \leq \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \left(1+\frac{1}{\delta}\left|\frac{\iota+\psi_\eta}{\zeta+1}-\upsilon\right|\right)\omega(f;\delta)dt$$

$$= \left(1+\frac{1}{\delta}\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \left|\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right|dt\right)\omega(f;\delta). \tag{4.9}$$

Now, using Cauchy-Schwarz inequality,

$$\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \left|\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right|dt \leq \left(\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 \left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right)^2 dt\right)^{\frac{1}{2}}$$

$$= \sqrt{\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon)}. \tag{4.10}$$

Therefore, we combine (4.9) and (4.10) we get,

$$\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| \leq \left(1+\frac{\sqrt{\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon)}}{\delta}\right)\omega(f;\delta).$$

Finally, choosing $\delta = \sqrt{\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon)}$, we get the result. □

Now, we need to state the following lemma to prove the local approximation properties of the operators $K_{\zeta,\psi_\eta}^{\alpha,s}$ depending on $\omega_2(f;\delta)$.

Lemma 4.5: For any $f \in C[0,1]$, and $\psi_\eta(t)$ be a function satisfying (4.2), we have

$$\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)\right| \leq \|f\| \tag{4.11}$$

where $\|.\|$ is the uniform norm of $C[0,1]$.

Proof. Using the definition of $K_{\zeta,\eta}^{(\alpha,s)}$,

$$\begin{aligned}\left|K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)\right| &= \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{(\alpha,s)}(\upsilon)\left|\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt\right| \\ &\leq \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{(\alpha,s)}(\upsilon)\int_0^1 \left|f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right|dt \\ &\leq \|f\|\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{(\alpha,s)}(\upsilon)\int_0^1 1dt \\ &= \|f\|.\end{aligned}$$

□

Theorem 4.4: Let $f \in C[0,1]$, and $\psi_\eta(t)$ be a function satisfying (4.2), then there exists $R \in \mathbb{R}^+$ such that

$$\begin{aligned}\left|K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)-f(\upsilon)\right| \leq& R\omega_2\left(f;\frac{1}{2}\sqrt{\phi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)+\left(\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right)^2}\right)\\ &+\omega\left(f;\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right)\end{aligned}$$

where $\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)$ and $\phi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)$ are defined in Corollary 4.2.

Proof. Firstly, we define the auxiliary operator as

$$\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(f;\upsilon) := K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)+f(\upsilon)-f\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}\right). \tag{4.12}$$

From Lemma 4.1, we have,

$$K^{*,\alpha,s}_{\zeta,\psi_\eta}(1;\upsilon)=1,$$

and

$$K^{*,\alpha,s}_{\zeta,\psi_\eta}(t-\upsilon;\upsilon)=0,$$

where $\upsilon \in [0,1]$ and $\zeta \in \mathbb{N}$.

Now, suppose that $g \in W^2[0,1]$. We use Taylor's expansion to get,

$$g(t)=g(\upsilon)+(t-\upsilon)g'(\upsilon)+\int_\upsilon^t (t-u)g''(u)du. \tag{4.13}$$

If we apply $\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}$ to (4.13), we obtain

$$\begin{aligned}\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(g;\upsilon) =& g(\upsilon)+\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}\left(\int_\upsilon^t (t-u)g''(u)du;\upsilon\right)\\ =& g(\upsilon)+K^{\alpha,s}_{\zeta,\psi_\eta}\left(\int_\upsilon^t (t-u)g''(u)du;\upsilon\right)\\ &-\int_\upsilon^{\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}}\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}-u\right)g''(u)du.\end{aligned}$$

Then,

$$\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(g;\upsilon)-g(\upsilon)=K^{\alpha,s}_{\zeta,\psi_\eta}\left(\int_\upsilon^t (t-u)g''(u)du;\upsilon\right)$$
$$-\int_\upsilon^{\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi\eta}}{\zeta+1}}\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}-u\right)g''(u)du. \tag{4.14}$$

Here, if we apply absolute value to both sides of (4.14) and using triangular inequality we obtain,

$$\begin{aligned}
\left|\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(g;\upsilon)-g(\upsilon)\right| \leq & \left|K^{\alpha,s}_{\zeta,\psi_\eta}\left(\int_\upsilon^t (t-u)g''(u)du;\upsilon\right)\right| \\
& +\left|\int_\upsilon^{\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi\eta}}{\zeta+1}}\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}-u\right)g''(u)du\right| \\
\leq & K^{\alpha,s}_{\zeta,\psi_\eta}\left(\left|\int_\upsilon^t (t-u)g''(u)du\right|;\upsilon\right) \\
& +\int_\upsilon^{\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi\eta}}{\zeta+1}}\left|\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}-u\right|\left|g''(u)\right|du \\
\leq & K^{\alpha,s}_{\zeta,\psi_\eta}\left(\left|\int_\upsilon^t |(t-u)|du\right|;\upsilon\right)\|g''\| \\
& +\int_\upsilon^{\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi\eta}}{\zeta+1}}\left|\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}-u\right|du\|g''\| \\
\leq & K^{\alpha,s}_{\zeta,\psi_\eta}\left((t-\upsilon)^2;\upsilon\right)\|g''\|+\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}-\upsilon\right)^2\|g''\| \\
\leq & \left[\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)+\left(\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right)^2\right]\|g''\|.
\end{aligned} \tag{4.15}$$

So,

$$\left|K^{\alpha,s}_{\zeta,\psi_\eta}(g;\upsilon)-g(\upsilon)\right|\leq\left[\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)+\left(\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right)^2\right]\|g''\|.$$

On the other hand, we combine (4.11) and (4.12) to get

$$\begin{aligned}
\left|\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(f;\upsilon)\right| & \leq\left|K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)\right|+|f(\upsilon)|+\left|f\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}\right)\right| \\
& \leq\|f\|\left|K^{\alpha,s}_{\zeta,\psi_\eta}(1;\upsilon)\right|+2\|f\| \\
& \leq 3\|f\|
\end{aligned} \tag{4.16}$$

for all $f\in C[0,1]$ and $\upsilon\in[0,1]$.

Finally, for any $f \in C[0,1]$, $g \in W^2[0,1]$ and using (4.15) and (4.16), we obtain

$$\begin{aligned}
\left|K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)-f(\upsilon)\right| =& \left|\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(f;\upsilon)-f(\upsilon)+f\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}\right)-f(\upsilon)\right| \\
=& \left|\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(f;\upsilon)-\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(g;\upsilon)+\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(g;\upsilon)-g(\upsilon)\right. \\
&\left. +g(\upsilon)-f(\upsilon)+f\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}\right)-f(\upsilon)\right| \\
\leq& \left|\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(f;\upsilon)-\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(g;\upsilon)\right|+\left|\widetilde{K^{\alpha,s}_{\zeta,\psi_\eta}}(g;\upsilon)-g(\upsilon)\right| \\
&+|g(\upsilon)-f(\upsilon)|+\left|f\left(\frac{\zeta\upsilon}{\zeta+1}+\frac{M_{1,\psi_\eta}}{\zeta+1}\right)-f(\upsilon)\right| \\
\leq& 4\|f-g\|+\left[\phi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)+\left(\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right)^2\right]\|g''\| \\
&+\omega\left(f;\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right).
\end{aligned}$$

Thus, by taking infimum on the right-hand side over all $g \in W^2[0,1]$, we get

$$\begin{aligned}
\left|K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)-f(\upsilon)\right| \leq& 4K_2\left(f;\frac{\phi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)+\left(\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right)^2}{4}\right) \\
&+\omega\left(f;\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right),
\end{aligned}$$

Which complete the proof. □

Theorem 4.5: Let ψ_η be a function satisfying (4.2), $f \in Lip_R(\mu)$, where $R > 0$ and $0 < \mu \leq 1$, then we have,

$$\left|K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)-f(\upsilon)\right| \leq R\sqrt{\left(K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon)^{\mu}\right)},$$

where $\upsilon \in [0,1]$.

Proof. Let $f \in Lip_R(\mu)$, we use the linearity and positivity properties of $K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)$ to obtain

$$
\begin{aligned}
\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| &\leq K_{\zeta,\psi_\eta}^{\alpha,s}\left(|f(t)-f(\upsilon)|;\upsilon\right)\\
&=\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left|f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)-f(\upsilon)\right|dt\\
&\leq R\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left|\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right|^{\mu}dt.
\end{aligned}
$$

Now, we apply the Hölder's inequality to get

$$
\begin{aligned}
\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| \leq & R\left[\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right)^2 dt\right]^{\frac{\mu}{2}}\\
&\times\left[\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1 dt\right]^{\frac{2-\mu}{2}}\\
=&R\left[K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon)\right]^{\frac{\mu}{2}}.
\end{aligned}
$$

which gives the desired result. □

Now, we invastigate the rate of convergence of the operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ for a function space $C^1[0,1]$.

Theorem 4.6: Let $f\in C^1[0,1]$ and $\psi_\eta(t)$ be a function satisfying (4.2), then we have

$$
\begin{aligned}
\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| \leq &\left|\varphi_{\zeta,M_{1,\psi_\eta}}^{(\alpha,s)}(\upsilon)\right|\left|f'(\upsilon)\right|+2\sqrt{\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon)}\\
&\times\omega\left(f';\sqrt{\phi_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}^{(\alpha,s)}(\upsilon)}\right)
\end{aligned}
$$

Proof. Let $f\in C^1[0,1]$, then by the mean value theorem of differential calculus, we have

$$
f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)-f(\upsilon)=\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right)f'(\upsilon)+\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right)\left[f'(c)-f'(\upsilon)\right]
$$

where $c=c_{\zeta,\iota,t}(\upsilon)\in\left(\upsilon,\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)$.

If we multiply both sides of the above equality by $b_{\zeta,\iota}^{\alpha,s}(\upsilon)$ and sum from 0 to ζ, we get

$$
\begin{aligned}
\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\left(f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)-f(\upsilon)\right) &= \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right)f^{'}(\upsilon) \\
&+ \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right) \\
&\times \left(f^{'}(c)-f^{'}(\upsilon)\right).
\end{aligned}
$$

Now, if we take integral from 0 to 1 then we have,

$$
\begin{aligned}
\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left(f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)-f(\upsilon)\right)dt &= \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right) \\
&\times f^{'}(\upsilon)dt \\
&+ \sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right) \\
&\times \left(f^{'}(c)-f^{'}(\upsilon)\right)dt.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon) =& K_{\zeta,\psi_\eta}^{\alpha,s}(t-\upsilon;\upsilon)f^{'}(\upsilon) \\
&+\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right)\left(f^{'}(c)-f^{'}(\upsilon)\right)dt.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\left|K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)-f(\upsilon)\right| \leq& \left|K_{\zeta,\psi_\eta}^{\alpha,s}(t-\upsilon;\upsilon)\right|\left|f^{'}(\upsilon)\right| \\
&+\sum_{\iota=0}^{\zeta} b_{\zeta,\iota}^{\alpha,s}(\upsilon)\int_0^1\left|\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right|\left|f^{'}(c)-f^{'}(\upsilon)\right|dt. \quad (4.17)
\end{aligned}
$$

On the other hand, note that

$$
\begin{aligned}
\left|f^{'}(c)-f^{'}(\upsilon)\right| &= \left(1+\frac{1}{\delta}|c-\upsilon|\right)\omega\left(f^{'};\delta\right) \\
&\leq \left(1+\frac{1}{\delta}\left|\frac{\iota+\psi_\eta(t)}{\zeta+1}-\upsilon\right|\right)\omega\left(f^{'};\delta\right) \qquad (4.18)
\end{aligned}
$$

where δ is any positive number which does not depend on ι and ω is the usual modulus.

Consequently, combining (4.17) and (4.18), we obtain

$$
\begin{aligned}
\left|K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)-f(\upsilon)\right| \leq & \left|K^{\alpha,s}_{\zeta,\psi_\eta}(t-\upsilon;\upsilon)\right|\left|f^{'}(\upsilon)\right| \\
& +\sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1\left|\frac{\iota+\psi\eta(t)}{\zeta+1}-\upsilon\right| dt\ \omega\left(f^{'};\delta\right) \\
& +\frac{1}{\delta}\sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1\left(\frac{\iota+\psi\eta(t)}{\zeta+1}-\upsilon\right)^2 dt\ \omega\left(f^{'};\delta\right) \\
\leq & \left|K^{\alpha,s}_{\zeta,\psi_\eta}(t-\upsilon;\upsilon)\right|\left|f^{'}(\upsilon)\right| \\
& +\sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1\left|\frac{\iota+\psi\eta(t)}{\zeta+1}-\upsilon\right| dt\ \omega(f^{'};\delta) \\
& +\frac{1}{\delta}K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon)\omega\left(f^{'};\delta\right).
\end{aligned}
$$

We apply Cauchy-Schwarz inequality to get,

$$
\begin{aligned}
\left|K^{\alpha,s}_{\zeta,\psi_\eta}(f;\upsilon)-f(\upsilon)\right| \leq & \left|K^{\alpha,s}_{\zeta,\psi_\eta}(t-\upsilon;\upsilon)\right|\left|f^{'}(\upsilon)\right| \\
& +\left[\sum_{\iota=0}^{\zeta} b^{\alpha,s}_{\zeta,\iota}(\upsilon)\int_0^1\left(\frac{\iota+\psi\eta(t)}{\zeta+1}-\upsilon\right)^2 dt\right]^{\frac{1}{2}}\omega\left(f^{'};\delta\right) \\
& +\frac{1}{\delta}K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon)\omega\left(f^{'};\delta\right) \\
= & \left|K^{\alpha,s}_{\zeta,\psi_\eta}(t-\upsilon;\upsilon)\right|\left|f^{'}(\upsilon)\right|+\left[K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon)\right]^{\frac{1}{2}}\omega\left(f^{'};\delta\right) \\
& +\frac{1}{\delta}K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon)\omega\left(f^{'};\delta\right) \\
= & \left|K^{\alpha,s}_{\zeta,\psi_\eta}(t-\upsilon;\upsilon)\right|\left|f^{'}(\upsilon)\right| \\
& +\left(1+\frac{\left[K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon)\right]^{\frac{1}{2}}}{\delta}\right)\left[K^{\alpha,s}_{\zeta,\psi_\eta}((t-\upsilon)^2;\upsilon)\right]^{\frac{1}{2}}\omega\left(f^{'};\delta\right) \\
= & \left|\varphi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta}}(\upsilon)\right|\left|f^{'}(\upsilon)\right| \\
& +\left(1+\frac{\sqrt{\phi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)}}{\delta}\right)\sqrt{\phi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)}\omega\left(f^{'};\delta\right)
\end{aligned}
$$

choosing $\delta=\sqrt{\phi^{(\alpha,s)}_{\zeta,M_{1,\psi_\eta},M_{2,\psi_\eta}}(\upsilon)}$, completes the proof. □

4.3 Shape Preserving Properties of $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$

In this section, we give the two most important shape pereserving properties, which are called monotonicity and covexity, of the operators $K_{\zeta,\eta}^{(\alpha,s)}$. But first, in order to make some calculations easier we use the following representation of these operators.

Lemma 4.6: For each $\alpha\in[0,1]$, $\psi_\eta(t)$ which satisfy the conditions in (4.2), $\zeta\in\mathbb{N}$ and nonnegative integer s, we have

$$K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)=\begin{cases} K_{\zeta,\psi_\eta}(f;\upsilon) & \text{if } \ \zeta<s \\ (1-\alpha)K_{\zeta,\psi_\eta}^{(\star,s)}(f;\upsilon)+\alpha K_{\zeta,\psi_\eta}(f;\upsilon) & \text{if } \ \zeta\geq s \end{cases}$$

where

$$K_{\zeta,\psi_\eta}(f;\upsilon)=\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt$$

and

$$\begin{aligned} K_{\zeta,\psi_\eta}^{(\star,s)}(f;\upsilon)=&\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota} \\ &\times\int_0^1\left(\upsilon f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)+(1-\upsilon)f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt. \end{aligned}$$

Proof. For $\zeta<s$, using the definition of $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$, it is obvious that

$$K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)=\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt.$$

For $\zeta\geq s$, we have

$$
\begin{aligned}
K^{(\alpha,s)}_{\zeta,\psi_\eta}(f;\upsilon) =& \sum_{\iota=0}^{\zeta}\Bigg((1-\alpha)\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota} \\
&+(1-\alpha)\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1} \\
&+\alpha\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\Bigg)\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
=&(1-\alpha)\sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
&+(1-\alpha)\sum_{\iota=0}^{\zeta}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
&+\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
=&(1-\alpha)\sum_{\iota=s}^{\zeta}\binom{\zeta-s}{\iota-s}\upsilon^{\iota-s+1}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
&+(1-\alpha)\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
&+\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
=&(1-\alpha)\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota+1}(1-\upsilon)^{\zeta-s-\iota}\int_0^1 f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)dt \\
&+(1-\alpha)\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota+1}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
&+\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
=&(1-\alpha)\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota} \\
&\times\int_0^1\left(\upsilon f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)+(1-\upsilon)f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt \\
&+\alpha\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt \\
=&(1-\alpha)K^{(\star,s)}_{\zeta,\psi_\eta}(f;\upsilon)+\alpha K_{\zeta,\psi_\eta}(f;\upsilon)
\end{aligned}
$$

where

$$
K_{\zeta,\psi_\eta}(f;\upsilon)=\sum_{\iota=0}^{\zeta}\binom{\zeta}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-\iota}\int_0^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt
$$

and

$$K_{\zeta,\psi_\eta}^{(\star,s)}(f;\upsilon)=\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}$$
$$\times\int_0^1\left(\upsilon f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)+(1-\upsilon)f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt.$$

□

Using mathematical by induction, we give r^{th} derivative of the operators in the following lemma, where $r\in\mathbb{N}$.

Lemma 4.7: For any $\alpha\in[0,1]$, $\psi_\eta(t)$ which satisfy the conditions in (4.2), $\zeta,r\in\mathbb{N}$ such that $\zeta\geq r$, and nonnegative integer s, we get the r^{th} derivative of the operators $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$ as:

$$K_{\zeta,\psi_\eta}^{(\alpha,s)^{(r)}}(f;\upsilon)=\begin{cases}K_{\zeta,\psi_\eta}{}^{(r)}(f;\upsilon) & if\quad \zeta<s\\ (1-\alpha)K_{\zeta,\psi_\eta}^{(\star,s)^{(r)}}(f;\upsilon)+\alpha K_{\zeta,\psi_\eta}{}^{(r)}(f;\upsilon) & if\quad \zeta\geq s\end{cases}$$

where

$$K_{\zeta,\psi_\eta}^{(\star,s)^{(r)}}(f;\upsilon)=\frac{(\zeta-s)!}{(\zeta-s-r)!}\sum_{\iota=0}^{\zeta-s-r}\binom{\zeta-s-r}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-r-\iota}$$
$$\times\int_0^1\left(\upsilon\Delta_h^r f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)+(1-\upsilon)\Delta_h^r f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt$$
$$+\frac{r(\zeta-s)!}{(\zeta-s-r+1)!}\sum_{\iota=0}^{\zeta-s-r+1}\binom{\zeta-s-r+1}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-r+1-\iota}$$
$$\times\int_0^1\left(\Delta_h^{r-1}f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)-\Delta_h^{r-1}f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt$$

and,

$$K_{\zeta,\psi_\eta}{}^{(r)}(f;\upsilon)=\frac{\zeta!}{(\zeta-r)!}\sum_{\iota=0}^{\zeta-r}\binom{\zeta-r}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-r-\iota}\int_0^1\Delta_h^r f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt.$$

Also,

$$\Delta_h^r f(t)=\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}f(t+ih)$$

where $t\in[0,1]$ and the step size $h=\frac{1}{\zeta+1}$.

Now, we are able to prove the shape preserving properties of $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$. Firstly, we start with monotonicity property of $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$

Theorem 4.7: Let $f \in C[0,1]$, and suppose that f is monotonically increasing (or decreasing) on the interval $[0,1]$. Then, for any $0 \leq \alpha \leq 1$, $\psi_\eta(t)$ which satisfy the conditions in (4.2), and positive integer s, the operators $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$ are also monotonically increasing (or decreasing) on $[0,1]$.

Proof. For the case $\zeta < s$, Let f be a monotonically increasing function on the interval $[0,1]$ then

$$K_{\zeta,\psi_\eta}{}^{(1)}(f;\upsilon) = \zeta \sum_{\iota=0}^{\zeta-1} b_{\zeta-1,\iota}(\upsilon) \int_0^1 \left(f\left(\frac{\iota+1+\psi_\eta(t)}{\zeta+1}\right) - f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right) dt$$

Obviously, if $\psi_\eta(t)$ is a function satisfying (4.2) then for all $n \in \mathbb{N}$, $\iota = 0,1,\cdots\zeta-1$,

$$f\left(\frac{\iota+1+\psi_\eta(t)}{\zeta+1}\right) - f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right) \geq 0$$

which implies that $\forall \upsilon \in [0,1]$,

$$K_{\zeta,\psi_\eta}{}^{(1)}(f;\upsilon) \geq 0.$$

The same proof, can easily be modified for monotonically decreasing functions.

Now, consider the case $\zeta \geq s$. From Lemma 4.7, the first derivative of the operators $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$ can be written as

$$K_{\zeta,\psi_\eta}^{(\alpha,s)\,(1)}(f;\upsilon) = (1-\alpha) K_{\zeta,\psi_\eta}^{(\star,s)\,(1)}(f;\upsilon) + \alpha K_{\zeta,\psi_\eta}{}^{(1)}(f;\upsilon)$$

where,

$$\begin{aligned}K_{\zeta,\psi_\eta}^{(\star,s)\,(1)}(f;\upsilon) =&(\zeta-s)\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s-1}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-1-\iota}\\&\times\int_0^1\left(\upsilon\Delta_h^1 f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)+(1-\upsilon)\Delta_h^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt\\&+\sum_{\iota=0}^{\zeta-s}\binom{\zeta-s}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-s-\iota}\\&\times\int_0^1\left(f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)-f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt\end{aligned}$$

and,

$$K_{\zeta,\psi_\eta}{}^{(1)}(f;\upsilon)=\zeta\sum_{\iota=0}^{\zeta-1}\binom{\zeta-1}{\iota}\upsilon^{\iota}(1-\upsilon)^{\zeta-1-\iota}\int_0^1\Delta_h^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt.$$

Since f is a monotonically increasing function, then we get for all $0\le\alpha\le1$, positive integer s, $\zeta\in\mathbb{N}$ such that $\zeta\ge s$, and $\iota=0,1,2,...,\zeta-s-1$

$$\Delta_h^1 f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)\ge0,$$

and,

$$f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)-f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\ge0.$$

Also, for any $\iota=0,1,2,...\zeta-1$

$$\Delta_h^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\ge0.$$

This proves that, $K_{\zeta,\psi_\eta}^{(\star,s)\,(1)}(f;\upsilon)\ge0$ and $K_{\zeta,\psi_\eta}{}^{(1)}(f;\upsilon)\ge0$. Therefore,

$$K_{\zeta,\psi_\eta}^{(\alpha,s)\,(1)}(f;\upsilon)\ge0$$

for all $\upsilon\in[0,1]$, which implies that $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$ are monotonically increasing on $[0,1]$. Using the same technique, one can also prove this result for monotonically decreasing functions on $[0,1]$. $\square$

Now, we prove the convexity property of the operators $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$.

Theorem 4.8: If $f(\upsilon)$ is a continuous and convex (concave) function on the interval $[0,1]$, then for any $0\le\alpha\le1$, $\psi_\eta(t)$ which satisfy the conditions in (4.2), and positive integer s, the operators $K^{(\alpha,s)}_{\zeta,\psi_\eta}(f;\upsilon)$ are also convex (concave) on $[0,1]$.

Proof. If $\zeta < s$, then the operators $K^{(\alpha,s)}_{\zeta,\psi_\eta}(f;\upsilon)$ reduce to the operators $K_{\zeta,\psi_\eta}(f;\upsilon)$.

Let f be a convex function on the interval $[0,1]$ then from (2.4) we have,

$$K_{\zeta,\psi_\eta}{}^{(2)}(f;\upsilon) = \zeta(\zeta-1)\sum_{\iota=0}^{\zeta-2} b_{\zeta-2,\iota}(\upsilon)\int_0^1 \Delta_h^2 f\left(\frac{\iota+\psi_\eta}{\zeta+1}\right)dt$$

where $h = \frac{1}{\zeta+1}$. It is well-known that (see [12]), for a convex function $f(\upsilon)$, $\Delta_h^2 f(\frac{\iota+\psi_\eta}{\zeta+1}) > 0$ which completes the proof. Similar proof can be written for concave functions.

Now, consider the case $\zeta \geq s$. From Lemma 4.7, the second derivative of the operators $K^{(\alpha,s)}_{\zeta,\psi_\eta}(f;\upsilon)$ can be written as

$$K^{(\alpha,s)(2)}_{\zeta,\psi_\eta}(f;\upsilon) = (1-\alpha)K^{(\star,s)}_{\zeta,\psi_\eta}{}^{(2)}(f;\upsilon) + \alpha K_{\zeta,\psi_\eta}{}^{(2)}(f;\upsilon)$$

where,

$$\begin{aligned} K^{(\star,s)}_{\zeta,\psi_\eta}{}^{(2)}(f;\upsilon) =& (\zeta-s)(\zeta-s-1)\sum_{\iota=0}^{\zeta-s-2}\binom{\zeta-s-2}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-s-2-\iota} && (4.19)\\ &\times\int_0^1\left(\upsilon\Delta_h^2 f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)+(1-\upsilon)\Delta_h^2 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt \\ &+2(\zeta-s)\sum_{\iota=0}^{\zeta-s-1}\binom{\zeta-s-1}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-s-1-\iota} && (4.20)\\ &\times\int_0^1\left(\Delta_h^1 f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right)-\Delta_h^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)\right)dt && (4.21)\end{aligned}$$

and,

$$\begin{aligned} K_{\zeta,\psi_\eta}{}^{(2)}(f;\upsilon) =& \zeta(\zeta-1)\sum_{\iota=0}^{\zeta-2}\binom{\zeta-2}{\iota}\upsilon^\iota(1-\upsilon)^{\zeta-2-\iota}\\ &\times\int_0^1\Delta_h^2 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right)dt. && (4.22)\end{aligned}$$

Since f is convex function on $[0,1]$, it follows from Theorem 3.2.2 in [12] we get for any $0 \leq \alpha \leq 1$, positive integer s, $\zeta \in \mathbb{N}$ such that $\zeta \geq s$, and $\iota = 0,1,2,...,\zeta-s-2$

$$\Delta_h^2 f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right) \geq 0, \tag{4.23}$$

and, for any $\iota = 0,1,2,...\zeta-2$

$$\Delta_h^2 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right) \geq 0. \tag{4.24}$$

On the other hand, by using mathematical induction one can prove that

$$\Delta_h^1 f\left(\frac{\iota+s+\psi_\eta(t)}{\zeta+1}\right) - \Delta_h^1 f\left(\frac{\iota+\psi_\eta(t)}{\zeta+1}\right) = \sum_{m=0}^{s-1} \Delta_h^2 f\left(\frac{\iota+m+\psi_\eta(t)}{\zeta+1}\right). \tag{4.25}$$

This implies that, ${K_{\zeta,\psi_\eta}^{(\star,s)}}^{(2)}(f;\upsilon) \geq 0$ and ${K_{\zeta,\psi_\eta}}^{(2)}(f;\upsilon) \geq 0$. Thus,

$${K_{\zeta,\psi_\eta}^{(\alpha,s)}}^{(2)}(f;\upsilon) \geq 0$$

for all $\upsilon \in [0,1]$, which proves that $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$ are convex on $[0,1]$. Similarly, one can prove that for concave functions on $[0,1]$, the operators $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$ are also concave.

□

4.4 Graphical Analysis and Error Estimation of $K_{\zeta,\psi_\eta}^{(\alpha,s)}(f;\upsilon)$

In this section, we give some graphics to illustrate approximation of $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ to certain function $f(\upsilon)$, and shape preserving properties of $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$. For the following figures, $\psi_\eta(t)$ is the function given in (4.6).

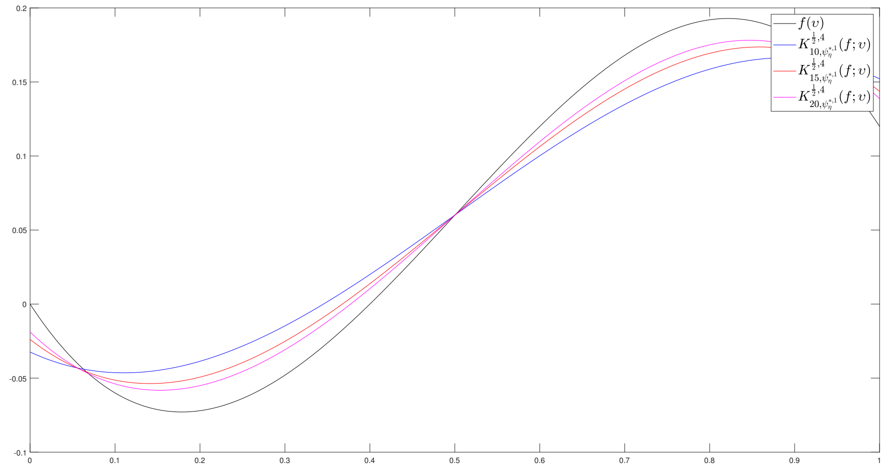

Figure 4.4: Approximation of $K^{\alpha,s}_{\zeta,\psi^{*,1}_{\eta}}(f;\upsilon)$ to $f(\upsilon) = -2\upsilon^3+3\upsilon^2-\frac{22}{25}\upsilon$ when $s=4$, $\alpha=\frac{1}{2}$ and $\zeta = 10,15,20$.

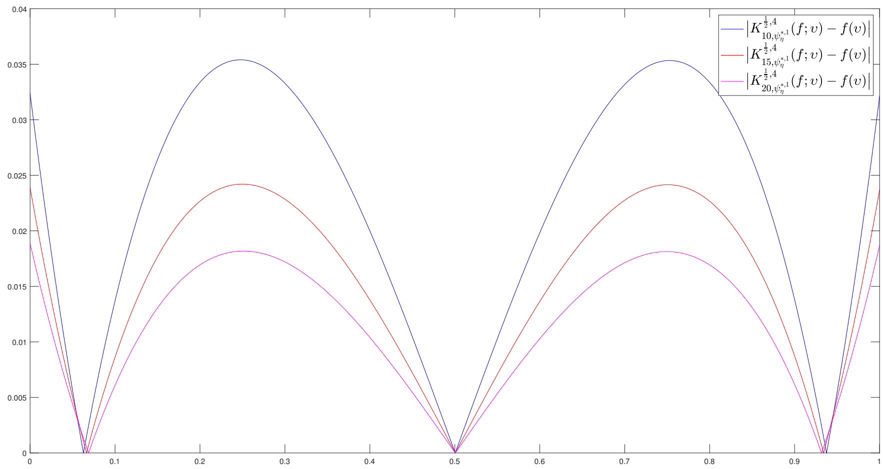

Figure 4.5: Error estimation of $K^{\alpha,s}_{\zeta,\psi^{*,1}_{\eta}}(f;\upsilon)$ to $f(\upsilon) = -2\upsilon^3+3\upsilon^2-\frac{22}{25}\upsilon$ when $s=4$, $\alpha=\frac{1}{2}$ and $\zeta = 10,15,20$.

Table 4.4: Estimation of error.

υ	$\left\lvert K^{\frac{1}{2},4}_{10,\psi^{*,1}_{\eta}}(f;\upsilon)-f(\upsilon)\right\rvert$	$\left\lvert K^{\frac{1}{2},4}_{15,\psi^{*,1}_{\eta}}(f;\upsilon)-f(\upsilon)\right\rvert$	$\left\lvert K^{\frac{1}{2},4}_{20,\psi^{*,1}_{\eta}}(f;\upsilon)-f(\upsilon)\right\rvert$
0.0	0.0324	0.0239	0.0189
0.1	0.0137	0.0086	0.0061
0.2	0.0334	0.0227	0.0169
0.3	0.0333	0.0228	0.0172
0.4	0.0200	0.0138	0.0104
0.5	0.0001	0.0001	0.0001
0.6	0.0198	0.0137	0.0103
0.7	0.0332	0.0228	0.0171
0.8	0.0334	0.0227	0.0169
0.9	0.0138	0.0087	0.0062
1.0	0.0321	0.0237	0.0187

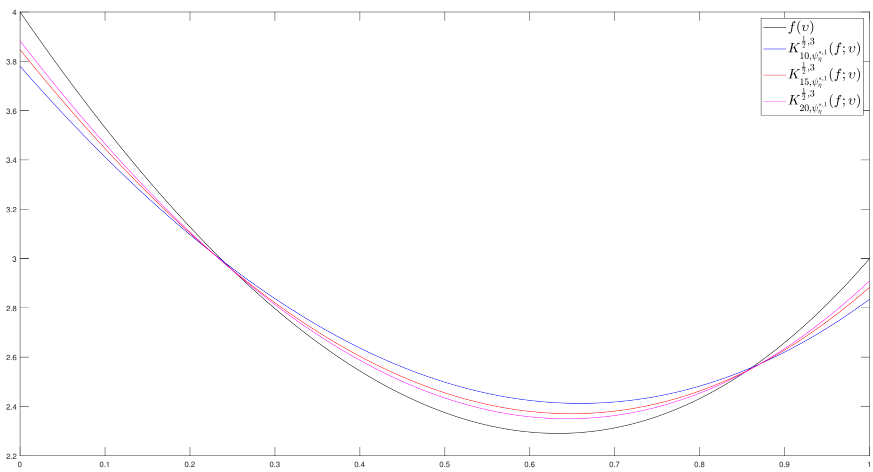

Figure 4.6: $K^{\alpha,s}_{\zeta,\psi^{*,1}_{\eta}}(f;\upsilon)$, preserves the concavity of $f(\upsilon)=\upsilon^3+3\upsilon^2-5\upsilon+4$ when $s=3$, $\alpha=\frac{1}{2}$ and $\zeta=10,15,20$.

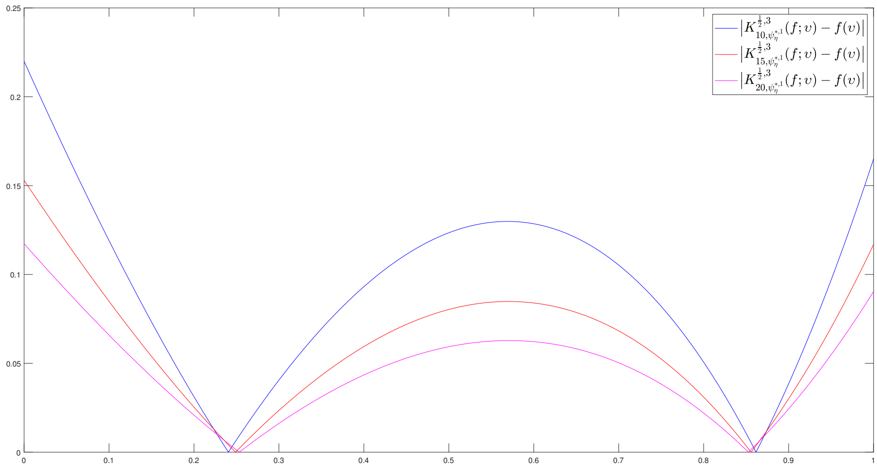

Figure 4.7: Error estimation of $K^{\alpha,s}_{\zeta,\psi^{*,1}_{\eta}}(f;\upsilon)$ to $f(\upsilon)=-2\upsilon^3+3\upsilon^2-\frac{22}{25}\upsilon$ when $s=3$, $\alpha=\frac{1}{2}$ and $\zeta=10,15,20$.

Table 4.5: Estimation of error.

υ	$\left\|K^{\frac{1}{2},3}_{10,\psi^{*,1}_{\eta}}(f;\upsilon)-f(\upsilon)\right\|$	$\left\|K^{\frac{1}{2},3}_{15,\psi^{*,1}_{\eta}}(f;\upsilon)-f(\upsilon)\right\|$	$\left\|K^{\frac{1}{2},3}_{20,\psi^{*,1}_{\eta}}(f;\upsilon)-f(\upsilon)\right\|$
0.0	0.2200	0.1530	0.1173
0.1	0.1191	0.0849	0.0659
0.2	0.0312	0.0253	0.0208
0.3	0.0406	0.0235	0.0162
0.4	0.0931	0.0594	0.0434
0.5	0.1235	0.0803	0.0593
0.6	0.1285	0.0839	0.0622
0.7	0.1053	0.0682	0.0503
0.8	0.0506	0.0310	0.0220
0.9	0.0385	0.0300	0.0243
1.0	0.1651	0.1169	0.0904

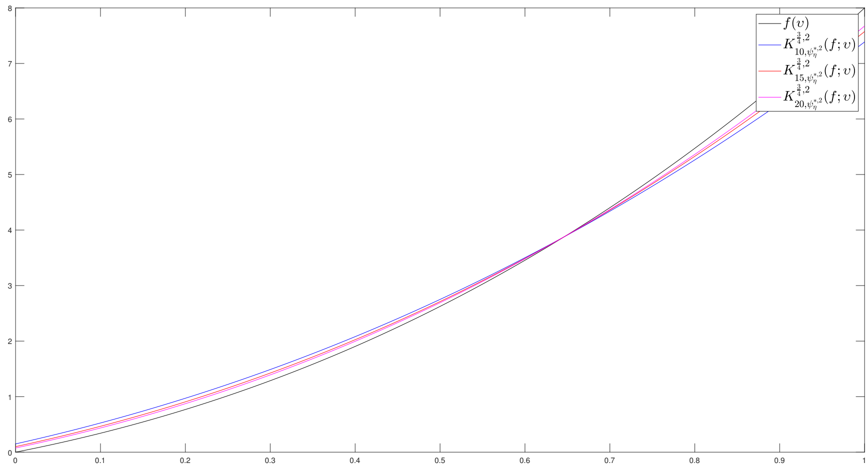

Figure 4.8: $K_{\zeta,\psi^{*,2}_{\eta}}(f;\upsilon)$ is increasing for an increasing function $f(\upsilon) = \upsilon^3 + 4\upsilon^2 + 3\upsilon$ when $s = 2$, $\alpha = \frac{3}{4}$ and $\zeta = 10, 15, 20$.

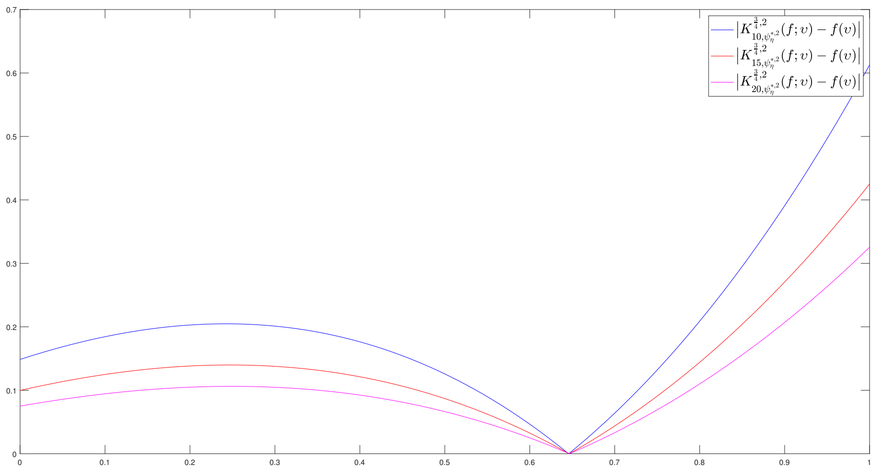

Figure 4.9: Error estimation of $K_{\zeta,\psi^{*,2}_{\eta}}(f;\upsilon)$ to $f(\upsilon) = \upsilon^3 + 4\upsilon^2 + 3\upsilon$ when $s = 2$, $\alpha = \frac{3}{4}$ and $\zeta = 10, 15, 20$.

Table 4.6: Estimation of error.

υ	$\left\|K^{\frac{3}{4},2}_{10,\psi^{*,2}_{\eta}}(f;\upsilon)-f(\upsilon)\right\|$	$\left\|K^{\frac{3}{4},2}_{15,\psi^{*,2}_{\eta}}(f;\upsilon)-f(\upsilon)\right\|$	$\left\|K^{\frac{3}{4},2}_{20,\psi^{*,2}_{\eta}}(f;\upsilon)-f(\upsilon)\right\|$
0.0	0.1486	0.0998	0.0750
0.1	0.1844	0.1250	0.0945
0.2	0.2029	0.1383	0.1049
0.3	0.2011	0.1378	0.1047
0.4	0.1763	0.1213	0.0924
0.5	0.1258	0.0869	0.0664
0.6	0.0466	0.0326	0.0250
0.7	0.0641	0.0438	0.0333
0.8	0.2089	0.1442	0.1101
0.9	0.3908	0.2706	0.2069
1.0	0.6126	0.4251	0.3255

Chapter 5

CONCLUSION

In this thesis, we introduced the blending type Bernstein operators $L_{\zeta}^{\alpha,s}$, which depends on two parameters α and s. Despite extensive prior research, the Lototsky matrices underlying these operators remained unexplored. By introducing and investigating these matrices, that generates the blending type Bernstein operators, we uncovered crucial insights into their role in operator generation. It should be noted that Lototsky matrix that generates classical Bernstein operators (2.5) and α-Bernstein operators (2.14) can be obtained from (3.1) for $s=1$ and $s=2$, respectively.

The generalized blending type Bernstein operators has the following properties:

(i) For $\alpha=1$ or $s=1$, $L_{\zeta}^{\alpha,s}(f;\upsilon)$ reduces to Bernstein operators.

(ii) For $s=2$, $L_{\zeta}^{\alpha,s}(f;\upsilon)$ reduces to α-Bernstein operators which is given by Chen et al. [11].

(iii) For any $0\leq\alpha\leq 1$ and positive integer s, $L_{\zeta}^{\alpha,s}(f;\upsilon)$ reproduce the linear functions.

(iv) If $f(\upsilon)$ is a continuous function on $[0,1]$, then for any real number $\alpha\in[0,1]$ and for any s, $L_{\zeta}^{\alpha,s}(f;\upsilon)$ converges uniformly to $f(\upsilon)$.

(v) For any $0\leq\alpha\leq 1$ and positive integer s, $L_{\zeta}^{\alpha,s}(f;\upsilon)$ have monotonicity and convexity properties.

(vi) The Lototsky matrices, introduced in (3.1), generates $L_{\zeta}^{\alpha,s}(f;\upsilon)$. The particular case of these Lototsky matrices for $s=2$ generates α-Bernstein operators given by Chen et al. [11] .

(vii) We establish an upper bound for the approximation error of these operators in terms of the modulus of continuity.

In Chapter 4, we introduced a family of blending type Bernstein-Kantorovich operators which depend on a function ψ_η satisfying the conditions given in (4.2). Different from other papers in the same field, we obtained first three moments and first two central moments of the new operators in terms of two variables M_{1,ψ_η} and M_{2,ψ_η}, integrals of ψ_η and ψ_η^2 on $[0,1]$ respectively. Following this new approach and considering the fact that the order of approximation to a function f by any operators $K_{\zeta,\psi_\eta}^{\alpha,s}(f;\upsilon)$ is more controlled by the term $K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon)$, two problems have considered. Are there M_{1,ψ_η} and M_{2,ψ_η} values so that

$$K_{\zeta,\psi_\eta}^{\alpha,s}((t-\upsilon)^2;\upsilon) < K_{\zeta}^{\alpha,s}((t-\upsilon)^2;\upsilon)$$

and is there a function ψ_η which have these M_{1,ψ_η} and M_{2,ψ_η} values? In this thesis, we proved that both problems have solutions. Moreover, our analysis showcased that the new operators outperform classical blending type Bernstein-Kantorovich operators in terms of approximation properties on $[0,1]$. We provided insights into their uniform convergence and rate of convergence, leveraging the first and second-order modulus of continuity. Notably, our operators exhibit shape-preserving properties, as confirmed by numerical examples.

In summary, this thesis contributes to a deeper understanding of generalized blending type Bernstein and ψ-Bernstein-Kantorovich operators, offering new insights, analytical results, and practical implications for mathematical approximation theory.

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