Fractional Differential Equations with Sonine Kernels

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ABSTRACT

The method of Mikusiński operational calculus relies on an abstract algebra understanding of integral and derivative operators, which can be used for solving differential equations. This has been used for fractional operators of many types, including with Sonine kernels. Here, we use the general Sonine kernel to construct a different algebraic setting in which the general fractional derivatives and general fractional integrals can be understood in a different way. By considering algebraic properties, we construct embeddings and isomorphisms to connect the different rings and fields involved in the fractional Mikusiński calculus, and we demonstrate the advantages and limitations of the more general algebraic setting. In this work, we also study linear fractional ODEs with continuous variable coefficients and general fractional derivative (GFD) operators of Riemann–Liouville and Caputo types defined using Sonine kernels. Using the Banach fixed point theorem, we prove existence and uniqueness of continuous solution functions, and construct solutions explicitly by means of uniformly convergent infinite series involving Sonine kernels. This work is done both for the classical Luchko-type GFDs with Sonine kernels, and for the m-fold versions of these operators, akin to sequential fractional derivatives.

Keywords: Mikusiński Operational Calculus, Sonine Kernels, Fractional Calculus, Fractional Differential Equations, Fixed Point Theory.

ÖZ

Mikusiński operasyonel kalkülüs metodu, diferansiyel denklemleri çözmek için kullanılabilen integral ve türev operatörlerinin soyut cebirsel anlayışına dayanır. Bu metod, Sonine çekirdekleri de dahil olmak üzere birçok türdeki kesirli operatörler için kullanılmıştır. Burada, genel kesirli türevlerin ve genel kesirli integrallerin farklı bir şekilde anlaşılabileceği farklı bir cebirsel tanım oluşturmak için genel Sonine çekirdeğini kullanıyoruz. Cebirsel özellikleri göz önünde bulundurarak, kesirli Mikusiński kalkülüsünde yer alan farklı halkaları ve alanları birbirine bağlamak için yerleştirmeler ve izomorfizmler oluşturuyoruz ve daha genel cebirsel tanımın avantajlarını ve sınırlamalarını gösteriyoruz. Bu çalışmada, ayrıca, Sonine çekirdekleri kullanılarak tanımlanan Riemann-Liouville ve Caputo türlerinin sürekli değişken katsayılı lineer kesirli ODE'lerini ve genel kesirli türev (GFD) operatörlerini inceliyoruz. Banach sabit nokta teoremini kullanarak, sürekli çözüm fonksiyonlarının varlığını ve benzersizliğini kanıtlıyoruz ve çözümleri Sonine çekirdeklerini içeren düzgün yakınsak sonsuz seriler aracılığıyla açıkça oluşturuyoruz. Bu çalışma hem Sonine çekirdekli klasik Luchko tipi GFD'ler hem de ardışık kesirli türevlere benzer şekilde bu operatörlerin m-katlı versiyonları için yapılır.

Anahtar Kelimeler: Mikusiński Operasyonel Kalkülüs, Sonine Çekirdekleri, Kesirli Kalkülüs, Kesirli Diferansiyel Denklemler, Sabit Nokta Teorisi.

DEDICATION

To my dear family

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Chapter 1

INTRODUCTION

Fractional calculus (FC) is a generalisation of classical calculus, extending the concepts of integration and differentiation to arbitrary (non-integer) order. The concept of fractional operators emerged almost concurrently with the foundations of classical calculus. The earliest known reference appears in a correspondence between Gottfried Leibniz and Marquis de L'Hopital, wherein the notion of a derivative of order $\frac{1}{2}$ was proposed. This idea subsequently attracted the attention of many mathematicians, including Euler, Laplace, Abel, Liouville, Riemann, Grünwald, Letnikov, and others. Since the 19th century, the theory of fractional calculus has been developed significantly, serving as a theoretical basis for various applied disciplines such as fractional geometry, fractional differential equations, and fractional dynamics.

Solving fractional differential equations (FDEs) where the order of derivatives are real or complex numbers, presents significant challenges compared to classical linear constant-coefficient ordinary differential equations (ODEs), for which various well-established methods are available [1, 2]. One major reason for this difficulty is the existence of many different definitions of fractional derivatives [3]; methods that are effective for one type of fractional operator may not necessarily be applicable to others. Among the classical operators, the Riemann–Liouville and Caputo derivatives are the most well-known and widely used.

Research in generalised fractional calculus has progressed in multiple directions, including conjugated (transmuted) fractional calculus [4, 5] and fractional calculus with generalised kernel functions [6]. Among these generalised kernel functions, one particular type, the General Fractional Calculus (GFC), defined using Sonine kernel functions is gaining interest nowadays. Building on the earlier work of Anatoly Kochubei [7], Yuri Luchko introduced the general fractional integrals (GFIs) and general fractional derivatives (GFDs) with Sonine kernels [8, 9]. These operators have been further developed by Luchko and others in recent years (since 2021), leading to significant contributions to fractional calculus [10–14]. Within this new GFC with Sonine kernels, GFDs of both Riemann–Liouville and Caputo types have been defined. Here, we focus on both of these types.

Meanwhile, starting in the 1990s, Yuri Luchko and his collaborators extended the idea of Mikusiński operational calculus to the setting of fractional calculus [15–19]. Their approach provided a powerful framework for interpreting fractional operators and solving fractional differential equations. The main idea of this operational calculus is to treat operators of calculus, such as derivatives and integrals, as algebraic objects, allowing one to perform manipulations on them and solve the differential equations. The foundation of this theory was laid by Jan Mikusiński, who formalised the method and presented it in his book [20] on operational calculus. This operational perspective has since played an important role in solving fractional differential equations, especially for linear equations with constant coefficients.

There are various fractional extensions of the Mikusiński operational calculus to fractional differential equations, reflecting the diversity of fractional calculus operators. Many alternative types of fractional extension of the Mikusiński operational calculus have been proposed by Luchko and collaborators [15–19, 21, 22] and as well as by Fernandez [23].

The study of fractional differential equations (FDEs) with variable coefficients has attracted considerable attention in recent years, with a series of papers dedicated to constructing explicit solutions for such FDEs involving various types of fractional derivatives. This line of research began with the work of several authors focusing on the Riemann–Liouville and Caputo derivatives [24–26]. The research on this line then continued with two papers by Restrepo and Suragan [27, 28], who then collaborated with Fernandez [29–32]. The research in FDEs with variable coefficients is still ongoing [33, 34].

1.1 Aim and Objectives

This thesis has two main aims. Recently, Al-Kandari et al. [35] proposed an extension of the Mikusiński operational calculus by introducing a new multiplication operator that combines convolution with fractional integration. Building on this idea, one of the primary aims of this thesis is to further generalise this approach by developing a new operational calculus that combines convolution with general fractional integrals using a Sonine kernel, as presented in Chapter 3. The specific objectives related to this aim are as follows:

- to construct a new operational calculus of Mikusiński type that can be applied to general fractional derivatives and general fractional integrals with Sonine kernels;
- to examine the algebraic properties of this new operational calculus;
- to establish embeddings and isomorphisms that clarify the underlying algebraic structures;

- to demonstrate how this new operational calculus can be used in place of the classical Mikusiński operational calculus to better understand the general fractional integral and derivative operators;
- to investigate whether this new approach provides useful tools for solving fractional integro-differential equations, and to compare the obtained results with those previously reported in the literature.

The second main aim of this thesis is motivated by recent works on fractional differential equations (FDEs) with variable coefficients. Building on these ideas, we extend the study to FDEs involving general fractional derivatives (GFDs) and m-fold GFDs defined via Sonine kernels. In particular, we aim to construct explicit solutions for certain FDEs with variable coefficients involving GFD operators with Sonine kernels. We focus on solving specific initial value problems involving GFDs and m-fold GFDs with variable coefficients, considered in both the Riemann–Liouville and Caputo senses, within the space of continuous functions.

In particular, we will study the following initial value problems:

1) Chapter 4.1: Let $\mathbb{D}_{(k)}$ denote the GFD of Riemann–Liouville type with Sonine kernel k (defined in the preliminaries, Chapter 2, below). We will solve the following initial value problem for v:

$$\begin{cases} \left(\mathbb{D}_{(k)}v\right)(t) - r(t)v(t) = g(t), \qquad t \in [0,T], \\ \left(\mathbb{I}_{(k)}v\right)(0) = a_0, \end{cases} \tag{1.1}$$

where $T \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$ are fixed constants and $r \in C[0,T]$, $g \in C_{-1}[0,T]$ are fixed functions.

2) Chapter 4.2: Let ${}_*\mathbb{D}_{(k)}$ denote the GFD of Caputo type with Sonine kernel k

(defined in the preliminaries, Chapter 2, below). We will solve the following initial value problem for v:

$$\begin{cases} \left({}_*\mathbb{D}_{(k)}v\right)(t) - r(t)v(t) = g(t), \qquad t \in [0,T], \\ v(0) = a_0, \end{cases} \tag{1.2}$$

where $T \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$ are fixed constants and $r, g \in C[0,T]$ are fixed functions.

3) Chapter 4.3: Let $\mathbb{D}_{(k)}^{<m>}$ denote the m-fold GFD of Riemann–Liouville type with Sonine kernel k (defined in the preliminaries, Chapter 2, below). We will solve the following initial value problem for v:

$$\begin{cases} \mathbb{D}_{(k)}^{<m>}v(t) + \sum\limits_{i=0}^{m-1} r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}v(t) = g(t), \qquad t \in [0,T], \\ \dfrac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\mathbb{I}_{(k)}^{<m>}v_j\right)(0) = a_i, \qquad i = 0,...,m-1, \end{cases} \tag{1.3}$$

where $m \in \mathbb{N}$ and $T \in \mathbb{R}^+$ and $a_0, a_1, \cdots, a_{m-1} \in \mathbb{R}$ are fixed constants and $r_0, r_1, \cdots, r_{m-1} \in C[0,T]$, $g \in C_{-1}[0,T]$ are fixed functions.

4) Chapter 4.4: Let ${}_*\mathbb{D}_{(k)}^{<m>}$ denote the m-fold GFD of Caputo type with Sonine kernel k (defined in the preliminaries, Chapter 2, below). We will solve the following initial value problem for v:

$$\begin{cases} {}_*\mathbb{D}_{(k)}^{<m>}v(t) + \sum\limits_{i=0}^{m-1} r_i(t)\cdot{}_*\mathbb{D}_{(k)}^{<i>}v(t) = g(t), \qquad t \in [0,T], \\ \left[\dfrac{\mathrm{d}^i}{\mathrm{d}t^i}v(t)\right]_{t=0} = v^{(i)}(0) = a_i, \qquad i = 0,...,m-1, \end{cases} \tag{1.4}$$

where $m \in \mathbb{N}$ and $T \in \mathbb{R}^+$ and $a_0, a_1, \cdots, a_{m-1} \in \mathbb{R}$ are fixed constants and $r_0, r_1, \cdots, r_{m-1}, g \in C[0,T]$ are fixed functions.

Each of these problems is fully solved, with explicit general solutions constructed in terms of infinite series. We also consider certain particular cases for comparison, which have appeared in the existing literature [11, 21].

Chapter 2

PRELIMINARIES

We begin with the formal definition of the classical Riemann–Liouville integral.

Definition 2.1 (Riemann–Liouville fractional integral [36]): The fractional integral operator to an arbitrary order α of a function f, with constant of differintegration a is defined as follows:

$$ {}^{RL}_{a}I^{\alpha}_{x} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{a}^{x} (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \qquad \operatorname{Re}\alpha > 0; \tag{2.1} $$

$a \in \mathbb{R}$ and $x > a$.

Now, let us define the general fractional integral (GFI) and general fractional derivative (GFD) associated with a given pair of Sonine kernels.

Definition 2.2 (GFC with Sonine kernels [8, 37]): Let κ and k be a pair of Sonine kernels in the spaces

$$ C_{-1}(0,\infty) = \left\{ f(t) = t^p f_1(t) \,:\, p > -1, f_1 \in C[0,\infty) \right\}, $$

or for a closed interval,

$$ C_{-1}[0,T] = \left\{ f(t) = t^p f_1(t) \,:\, p > -1, f_1 \in C[0,T] \right\}, $$

i.e., functions in these spaces that satisfy $\kappa * k = 1$, where $*$ denotes the Laplace-type convolution and the result is the constant function 1. (The space $C_{-1}[0,T]$ was defined in [37] as the finite-interval version of the space defined by Dimovski in [38] much earlier.)

Then, the GFI is defined as follows:

$$\mathbb{I}_{(\kappa)} f(t) := (\kappa * f)(t) = \int_0^t \kappa(t-\tau) f(\tau) \mathrm{d}\tau,$$

while the GFDs of Riemann–Liouville type and Caputo type are defined respectively as follows:

$$\mathbb{D}_{(k)} f(t) := \frac{\mathrm{d}}{\mathrm{d}t} (k * f)(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t k(t-\tau) f(\tau) \mathrm{d}\tau,$$

$$_{*}\mathbb{D}_{(k)} f(t) := \left(k * f'\right)(t) = \int_0^t k(t-\tau) f'(\tau) \mathrm{d}\tau,$$

Definition 2.3 (m-fold GFC with Sonine kernels [8]): Let κ and k be Sonine kernels as in Definition 2.2, and let $m \in \mathbb{N}$ be any natural number. The m-fold GFI is defined as follows:

$$\mathbb{I}_{(\kappa)}^{<m>} f(t) := (\underbrace{\mathbb{I}_{(\kappa)} * \cdots * \mathbb{I}_{(\kappa)}}_{m \text{ times}} f)(t) = \left(\kappa^{<m>} * f\right)(t), \ \kappa^{<m>} := \underbrace{\kappa * \cdots * \kappa}_{m \text{ times}},$$

while the m-fold GFDs of Riemann–Liouville type and Caputo type are defined respectively as follows:

$$\mathbb{D}_{(k)}^{<m>} f(t) := \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left(k^{<m>} * f\right)(t), \ k^{<m>} := \underbrace{k * \cdots * k}_{m \text{ times}}, \tag{2.2}$$

$$_{*}\mathbb{D}_{(k)}^{<m>} f(t) := \mathbb{D}_{(k)}^{<m>} f(t) - \sum_{j=0}^{m-1} f^{(j)}(0) \frac{\mathrm{d}^{m-j-1}}{\mathrm{d}t^{m-j-1}} k^{<m>}(t), \tag{2.3}$$

Chapter 3

EXTENDING MIKUSIŃSKI OPERATIONAL CALCULUS VIA SONINE KERNELS

3.1 The Generalised Algebraic Structure

Let us briefly summarise the essential ideas of Mikusiński operational calculus used for GFDs of both Riemann–Liouville and Caputo types and GFIs with the Sonine kernels.

The fractional Mikusiński operational calculus was developed based on the space of functions that are continuous on the positive real semi-axis and can have an integrable singularity at the point zero. Using a classical theorem of Titchmarsh, it is easy to prove that this space of functions forms a commutative rng (non-unital ring) under the algebraic operations of addition and Laplace-type convolution. This rng then can be extended to a field of convolution quotients, with both GFIs and GFDs being representable by algebraic operations within this field. From this construction, it is possible to solve analytically many initial value problems for single-term and multi-term fractional differential equations involving GFDs.

More precisely, we define the function space $C_\alpha(0,\infty)$, for any $\alpha \geq -1$, with a rng structure as follows:

$$C_\alpha(0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R} \,\middle|\, f(x) = x^p f_1(x),\ p > \alpha \geq -1,\ f_1 \in C[0,\infty) \right\}, \tag{3.1}$$

$$(f+g)(x) = f(x) + g(x), \tag{3.2}$$

$$(f*g)(x) = \int_0^x f(\tau)g(x-\tau)\mathrm{d}\tau. \tag{3.3}$$

It is well known, see [38] and [18], that the space $C_\alpha(0,\infty)$ is a commutative rng, i.e. a ring without a multiplicative identity element. Further algebraic properties of this space have been investigated only very recently [39].

Throughout this chapter, let κ and k be a pair of Sonine kernels in the space S_{-1}, defined as elements of the space $C_{-1}(0,\infty)$ that satisfy the Sonine condition

$$(\kappa * k)(x) = 1. \tag{3.4}$$

(Originally, S_{-1} was defined by [8] with the requirement that κ and k should be in the space

$$C_{-1,0}(0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R} \,\middle|\, f(x) = x^p f_1(x),\, x>0,\, 0<p<1,\, f_1 \in C[0,\infty) \right\}, \tag{3.5}$$

but it has been observed [40] that this space is actually the same as $C_{-1}(0,\infty)$.)

The above constitutes a brief introduction to Luchko's theory of fractional Mikusiński operational calculus and Sonine kernels.

We now proceed to kickstart our research by defining a new general convolution and investigating its properties.

3.1.1 The Rng and Field Arising from the New Convolution

The main novelty of this work is the following definition, in which we recall that κ and k are assumed to be a Sonine pair in the space S_{-1}.

Definition 3.1: For any Sonine kernel κ in the space $S_{-1} \subseteq C_{-1}(0,\infty)$, the κ-convolution of two functions f and g is given by

$$\left(f \overset{\kappa}{*} g\right)(x) = (\kappa * f * g)(x), \qquad x > 0, \tag{3.6}$$

where $*$ denotes the usual convolution as defined in (3.3), and where f and g are functions such that this convolution is well-defined, for example arbitrary functions in $C_{-1}(0,\infty)$.

Remark 3.1: Note that the standard Laplace-type convolution $*$ is not recovered as a particular case of the new definition (3.6). This is because having $f \overset{\kappa}{*} g = f * g$ would require κ to play the role of a Dirac delta distribution, which is not in the space $C_{-1}(0,\infty)$. It is shown rigorously in Proposition 3.2 that there can be no function in $C_{-1}(0,\infty)$ playing the role of an identity under convolution.

Example 3.1: If we consider $\kappa = h_\alpha$, where in the usual notation $h_\alpha(x) = \dfrac{x^{\alpha-1}}{\Gamma(\alpha)}$ and α is a fixed constant satisfying $0 < \alpha < 1$, then the new convolution reduces to the same one considered in [35], namely:

$$\left(f \overset{\kappa}{*} g\right)(x) = (h_\alpha * f * g)(x) = \left(\left({}^{RL}_{0}I^{\alpha}_{x} f\right) * g\right)(x) = \left(f * \left({}^{RL}_{0}I^{\alpha}_{x} g\right)\right)(x),$$

where ${}^{RL}_{0}I^{\alpha}_{x}$ is the Riemann–Liouville integral operator of order α. This is the most basic example of a Sonine kernel and therefore of the general theory being presented in this chapter.

Example 3.2: If we consider $\kappa(x) = h_{1-\beta+\alpha}(x) + h_{1-\beta}(x)$, where α and β are fixed constants satisfying $0 < \alpha < \beta < 1$, then the new convolution will be as follows:

$$\begin{aligned}
\left(f \overset{\kappa}{*} g\right)(x) &= \left((h_{1-\beta+\alpha} + h_{1-\beta}) * f * g\right)(x) \\
&= \left(h_{1-\beta+\alpha} * f * g\right)(x) + \left(h_{1-\beta} * f * g\right)(x) \\
&= \left(\left({}^{RL}_{0}I^{1-\beta+\alpha}_{x} f\right) * g\right)(x) + \left(\left({}^{RL}_{0}I^{1-\beta}_{x} f\right) * g\right)(x) \\
&= \left(f * \left({}^{RL}_{0}I^{1-\beta+\alpha}_{x} g\right)\right)(x) + \left(f * \left({}^{RL}_{0}I^{1-\beta}_{x} g\right)\right)(x).
\end{aligned}$$

Note that $\kappa = h_{1-\beta+\alpha} + h_{1-\beta}$ is known to be a Sonine kernel, according to the work of Luchko [8], with its corresponding Sonine kernel function k being a modified two-parameter Mittag-Leffler function.

Example 3.3: If we consider $\kappa(x) = h_{\alpha,\rho}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}e^{-\rho x}$, where α and ρ are fixed constants satisfying $0 < \alpha < 1$ and $\rho \geq 0$, then the new convolution will be as follows:

$$\left(f \overset{\kappa}{*} g\right)(x) = \left(\left[\frac{x^{\alpha-1}}{\Gamma(\alpha)}e^{-\rho x}\right] * f * g\right)(x)$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^x \int_0^\tau u^{\alpha-1} e^{-\rho u} f(\tau - u) g(x - \tau) \, \mathrm{d}u \, \mathrm{d}\tau.$$

This is the Sonine kernel that can be used to construct tempered fractional calculus, see [8] again.

Example 3.4: If we consider $\kappa(x) = (\sqrt{x})^{\alpha-1} J_{\alpha-1}(2\sqrt{x})$, where α is a fixed constant satisfying $0 < \alpha < 1$ and J is the Bessel function, then the new convolution becomes as follows:

$$\left(f \overset{\kappa}{*} g\right)(x) = \left(\left((\sqrt{x})^{\alpha-1} J_{\alpha-1}(2\sqrt{x})\right) * f * g\right)(x)$$
$$= \int_0^x \int_0^\tau (\sqrt{u})^{\alpha-1} J_{\alpha-1}(2\sqrt{u}) f(\tau - u) g(x - \tau) \, \mathrm{d}u \, \mathrm{d}\tau.$$

Note that the function κ used here is known to be a Sonine kernel [41], with its corresponding Sonine kernel function k being similarly related to the modified Bessel function.

Having discussed some examples to verify that the general theory we are constructing is not trivial, we move on to proving the theorems which establish proper algebraic structures (rng and field) under the action of the newly defined binary operation $\overset{\kappa}{*}$.

Theorem 3.1: Let $\alpha \geq -1$ be a fixed real number such that the Sonine kernel κ is contained in $C_\alpha(0,\infty) \subset C_{-1}(0,\infty)$. Let R_α denote the commutative rng $(C_\alpha(0,\infty), +, *)$. The function space $C_\alpha(0,\infty)$, equipped with the operations of the usual addition of functions and the new modified convolution $\overset{\kappa}{*}$ as multiplication, becomes a commutative rng $R_{\alpha,\kappa} = \left(C_\alpha(0,\infty), +, \overset{\kappa}{*}\right)$ without divisors of zero and without any multiplicative unity.

Proof. It is a general fact in abstract algebra that, given any rng $(R,+,\times)$ and element $a \in R$, we can create a new rng $(R,+,\overset{a}{\times})$ with a new multiplication given by $r_1 \overset{a}{\times} r_2 = a \times r_1 \times r_2$ for any $r_1, r_2 \in R$. The rng axioms are straightforward to check for the new rng, so it remains to check that $R_{\alpha,\kappa}$ has no zero divisors and no multiplicative unity.

The well-known theorem of Titchmarsh, on the function space $C[0,\infty)$ having no zero divisors under convolution, implies the same fact for the rng R_α. If the new rng $R_{\alpha,\kappa}$ had any zero divisors, say $f \overset{\kappa}{*} g = 0$, then we have $\kappa * f * g = 0$, and we know $\kappa \neq 0$, so either $f = 0$ or $g = 0$, giving a contradiction.

Let's assume that there exists a unity $e \in R_{\alpha,\kappa}$. Then, $e \overset{\kappa}{*} f = f$ for all $f \in C_\alpha(0,\infty)$, which is the same as saying $\kappa * e * f = f$ for all $f \in C_\alpha(0,\infty)$. But then the element $\kappa * e$ would be an identity in the original rng R_α, which is a contradiction.

Hence we have proved all the required facts for the new rng $R_{\alpha,\kappa}$. □

It is another general fact in abstract algebra that every ring or rng without zero divisors can be extended to its field of fractions, which here we denote $F_{\alpha,\kappa}$ and call the field of convolution quotients. For the benefit of the reader, we recall briefly the procedure for defining this field: as a set, it is

$$F_{\alpha,\kappa} = C_\alpha(0,\infty) \times \left(C_\alpha(0,\infty) \setminus 0\right) / \overset{\kappa}{\sim},$$

where we are quotienting the Cartesian product by the equivalence relation $\overset{\kappa}{\sim}$ defined by

$$(f_1, g_1) \overset{\kappa}{\sim} (f_2, g_2) \Longleftrightarrow (f_1 \overset{\kappa}{*} g_2)(x) = (g_1 \overset{\kappa}{*} f_2)(x).$$

In the field, we denote the equivalence class of the pair (f,g) by $\frac{f}{g}^{\kappa}_{*}$, and the two algebraic field operations are defined by

$$\left({}^{\kappa}_{*}\frac{f_1}{g_1}\right)+\left({}^{\kappa}_{*}\frac{f_2}{g_2}\right):={}^{\kappa}_{*}\frac{f_1\overset{\kappa}{*}g_2+g_1\overset{\kappa}{*}f_2}{g_1\overset{\kappa}{*}g_2}, \tag{3.7}$$

$$\left({}^{\kappa}_{*}\frac{f_1}{g_1}\right)\overset{\kappa}{*}\left({}^{\kappa}_{*}\frac{f_2}{g_2}\right):={}^{\kappa}_{*}\frac{f_1\overset{\kappa}{*}f_2}{g_1\overset{\kappa}{*}g_2}. \tag{3.8}$$

With these definitions, we have the following theorem, whose proof we omit as it is a standard fact of abstract algebra given Theorem 3.1 above.

Theorem 3.2: Equipped with the two binary operations defined by (3.7) and (3.8) respectively, the set $F_{\alpha,\kappa}$ becomes a field, called the field of convolution quotients.

The ring $R_{\alpha,\kappa}$ can be embedded into the field $F_{\alpha,\kappa}$ by the map

$$f\to{}^{\kappa}_{*}\frac{\kappa\overset{\kappa}{*}f}{\kappa}. \tag{3.9}$$

Also, the set $C_\alpha(0,\infty)$ is a vector space in such a way that the space $R_{\alpha,\kappa}$ is a non-unital algebra, and therefore the set $F_{\alpha,\kappa}$ is also a vector space and an algebra.

3.1.2 The Corresponding Integral Transform

The relation between our new convolution operation and the Laplace transform is clear. By the usual convolution theorem for Laplace transforms, we have:

$$\mathscr{L}\left[f\overset{\kappa}{*}g\right](s)=K(s)F(s)G(s), \tag{3.10}$$

where $\mathscr{L}$ denotes the Laplace transform and the functions K and F and G are respectively the Laplace transforms of κ and f and g, and where κ, f, g are functions in the subspace $C_{\alpha;L}(0,\infty)$ of functions in $C_\alpha(0,\infty)$ which are exponentially bounded and therefore have well-defined Laplace transforms.

In the subspace $C_{\alpha;L}(0,\infty)$, many algebraic properties of the new convolution operation can therefore be deduced by passing to the Laplace domain and back again. In the

Laplace domain, the new convolution of f and g and the classical convolution of f and g correspond to elements that differ only by some fixed factor (namely a factor of K, the Laplace transform of κ). However, as noted by many authors who work with Mikusiński operational calculus [42], the advantage of such abstract algebraic approaches over the formally equivalent Laplace transform approach is that the latter requires us to assume exponential boundedness. Any algebraic results derived using Laplace transforms will only be valid in the subspace $C_{\alpha;L}(0,\infty)$, whereas the ring-theory approach of the previous subsection enabled us to achieve results on the whole space $C_{\alpha}(0,\infty)$, which is much larger.

On the other hand, any given convolution operation often has a corresponding integral transform [43], where the correspondence is given by a convolution theorem: the desired transform should convert the given convolution to a simple product. As such, we may ask the question: which integral transform corresponds to our new convolution? Given (3.10), it is unsurprising that the answer to this question is closely related to the classical Laplace transform. We define it as follows.

Definition 3.2: Assuming that $\kappa \in C_{\alpha;L}(0,\infty)$ (i.e. that the function κ is exponentially bounded), we define the κ-modified Laplace transform of any function $f \in C_{\alpha;L}(0,\infty)$ to be the following function:

$$\mathscr{L}_{\kappa}\left[f\right](s) = K(s) \cdot \mathscr{L}\left[f\right](s) = \mathscr{L}\left[\kappa * f\right](s),$$

where again K denotes the classical Laplace transform of κ.

Proposition 3.1 (Convolution theorem for the new convolution and transform): For any $f, g \in C_{\alpha;L}(0,\infty)$, we have the following relation:

$$\mathscr{L}_{\kappa}\left[f \overset{\kappa}{*} g\right] = \mathscr{L}_{\kappa}[f] \cdot \mathscr{L}_{\kappa}[g].$$

Proof. This follows directly from the definition of $\mathscr{L}_{\kappa}$ together with (3.10). We have:

$$
\begin{aligned}
\mathscr{L}_\kappa\left[f \overset{\kappa}{*} g\right](s) &= K(s) \cdot \mathscr{L}\left[f \overset{\kappa}{*} g\right](s) \\
&= K(s) \cdot K(s) \cdot \mathscr{L}[f](s) \cdot \mathscr{L}[g](s) \\
&= \Big(K(s) \cdot \mathscr{L}[f](s)\Big) \cdot \Big(K(s) \cdot \mathscr{L}[g](s)\Big) \\
&= \mathscr{L}_\kappa[f] \cdot \mathscr{L}_\kappa[g],
\end{aligned}
$$

as required. $\square$

Of course, the new transform defined here is a trivial modification of the Laplace transform, and all of its properties follow immediately from those of the classical Laplace transform, which are well known. There is no need to develop its theory further, but we leave this subsection here to indicate that there is, as expected, an integral transform corresponding to the newly defined convolution operator.

3.1.3 Embeddings and Isomorphisms

In the usual fractional Mikusiński operational calculus, the rng R_α is extended to a field F_α, in which there is a "division" which is the inverse of the "multiplication" given by the classical convolution: for given $f \in C_\alpha(0,\infty)$ and $g \in C_\alpha(0,\infty)\backslash\{0\}$, we define $f \div g$ to be the unique element of the field F_α such that $(f \div g) * g = f$.

Using the new multiplication given in (3.6) by κ-convolution, we have correspondingly a new division in the field $F_{\alpha,\kappa}$, which we denote as follows:

$$
f \overset{\kappa}{\div} g = \overset{\kappa}{*}\frac{f}{g} \tag{3.11}
$$

Note that this field element may or may not be a function in the rng $R_{\alpha,\kappa} \subset F_{\alpha,\kappa}$, so it does not make sense to write $\left(f \overset{\kappa}{\div} g\right)(x)$ as the algebraic element we are dealing with might not be a function of x at all.

For example, both fields have identity elements given by setting both f and g in the quotient to be the function κ. We have:

$$I = \kappa \div \kappa \in F_\alpha,$$

$$I_\kappa = \kappa \overset{\kappa}{\div} \kappa = \frac{\overset{\kappa}{*}\kappa}{\kappa} \in F_{\alpha,\kappa}.$$

Proposition 3.2: Neither I nor I_κ is a function in the space $C_\alpha(0,\infty)$. (This makes sense because $C_\alpha(0,\infty)$ is a space of functions, and a convolutional identity would have properties analogous to the Dirac delta distribution, which is not a function.)

Proof. If I were a function in $C_\alpha(0,\infty)$, then it would have the property that its standard Laplace-type convolution with any function $f \in C_\alpha(0,\infty)$ is the function f itself. By the definition of $C_\alpha(0,\infty)$, we must have $I \in C_\beta(0,\infty)$ for some $\beta > \alpha$ (given p from (3.1), take any β satisfying $-1 \leq \alpha < \beta < p$). Then, for any $f \in C_\alpha(0,\infty) \backslash C_\beta(0,\infty)$, the result of [39, Lemma 1] gives $I * f \in C_{\alpha+\beta+1}(0,\infty) \subsetneq C_\beta(0,\infty)$. Contradiction, so I cannot be a function.

If I_κ were a function, then its standard Laplace-type convolution with $\kappa \in C_\alpha(0,\infty)$ would be also a function, but this is I which we already showed is not a function. Contradiction, so I_κ cannot be a function. □

Clearly, I and I_κ are algebraically useful as field elements, although they cannot be understood as functions. But can they be understood as being both elements of the same field? Although both rngs R_α and $R_{\alpha,\kappa}$ are based on the same set $C_\alpha(0,\infty)$, the constructed fields are both strictly larger than this set and cannot necessarily be understood as intersecting each other except in the function space $C_\alpha(0,\infty)$ that we started with.

A theorem at the end of [35] showed that there are homomorphisms between some of the modified fields of convolution quotients. But this is not a particularly strong statement, since many rings with totally different structures have homomorphisms between them: for example, every ring has a unique homomorphism mapping the ring of integers into it and a unique homomorphism mapping it into the trivial ring (these are, respectively, initial and terminal objects in the category of rings). Here, we aim to do better and find embeddings and isomorphisms, which will be stronger results on the algebraic structure of the spaces involved.

Theorem 3.3: For any $\kappa \in C_{-1}(0,\infty)$ and any $\alpha \geq -1$, there is an injective rng homomorphism from $R_{\alpha,\kappa}$ into R_α, or more generally from $R_{\alpha,\kappa*\kappa_1}$ into R_{α,κ_1} for any $\kappa_1 \in C_{-1}(0,\infty)$.

Proof. We consider the following map between sets:

$$\phi : C_\alpha(0,\infty) \longrightarrow C_\alpha(0,\infty)$$

$$f \longmapsto f * \kappa.$$

It is clear that ϕ maps $C_\alpha(0,\infty)$ into $C_\alpha(0,\infty)$, and that it is a linear map between vector spaces. It is also a rng homomorphism from $R_{\alpha,\kappa}$ to R_α, with the given multiplication on both of these rngs, because

$$\phi(f) * \phi(g) = f * \kappa * g * \kappa = \left(f \overset{\kappa}{*} g\right) * \kappa = \phi\left(f \overset{\kappa}{*} g\right).$$

More generally, the same map ϕ is a rng homomorphism from $R_{\alpha,\kappa*\kappa_1}$ to R_{α,κ_1}, with the given multiplication on both of these rngs, because

$$\phi(f) \overset{\kappa_1}{*} \phi(g) = f * \kappa * \kappa_1 * g * \kappa = \left(f \overset{\kappa*\kappa_1}{*} g\right) * \kappa = \phi\left(f \overset{\kappa*\kappa_1}{*} g\right).$$

Finally, the map ϕ is injective because

$$f * \kappa = g * \kappa \Rightarrow (f-g) * \kappa = 0 \Rightarrow f - g = 0 \Rightarrow f = g,$$

as we know that there are no zero divisors in the rng R_α by the theorem of Titchmarsh.

□

Now we have proved the following rng embeddings for any $\kappa \in C_{-1}(0,\infty)$:

$$R_{\alpha,\kappa} \lesssim R_{\alpha},$$

and for any $\kappa_1 \in C_{-1}(0,\infty)$, also

$$R_{\alpha,\kappa*\kappa_1} \lesssim R_{\alpha,\kappa_1}.$$

Every injective homomorphism between rngs extends to an injective homomorphism between their fields of fractions, so the map ϕ defined above can be extended to an injective field homomorphism

$$\begin{aligned} \widetilde{\phi} : F_{\alpha,\kappa} &\longrightarrow F_{\alpha} \\ \frac{f}{g}\overset{\kappa}{*} &\longmapsto \frac{f*\kappa}{g*\kappa} = \frac{f}{g}. \end{aligned}$$

This is, in fact, a bijection, because for every $f \in C_{\alpha}(0,\infty)$ and $g \in C_{\alpha}(0,\infty)\backslash\{0\}$, the general element $\frac{f}{g} \in F_{\alpha}$ is the image under $\widetilde{\phi}$ of a corresponding element $\overset{\kappa}{*}\frac{f}{g} \in F_{\alpha,\kappa}$. Thus, $\widetilde{\phi}$ is actually a field isomorphism, and we have proved the following result.

Theorem 3.4: For each given $\alpha \geq -1$, all convolution quotient fields generated from $C_{\alpha}(0,\infty)$ are isomorphic: we have

$$F_{\alpha} \simeq F_{\alpha,\kappa} \quad \text{as fields,}$$

for every $\kappa \in C_{\alpha}(0,\infty)$.

It should be noted that $F_{\alpha,\kappa}$ denotes the field generated from $R_{\alpha,\kappa}$, i.e. the field of fractions of the rng generated from R_{α} by combining κ with convolution. It is also possible to apply the same two processes in the opposite order: to consider the field generated from F_{α} (the field of fractions of R_{α}) by combining κ with convolution. Let us denote this field F_{α}^{κ}.

In the same way as Theorem 3.3, we can construct a field homomorphism into F_α from the κ-modified version F_α^κ:

$$\begin{aligned}\psi : F_\alpha^\kappa &\longrightarrow F_\alpha \\ \frac{f}{g} &\longmapsto \frac{f}{g} * \kappa = \frac{f*\kappa}{g}.\end{aligned}$$

This is an isomorphism because κ is invertible in F_α, so there is an inverse homomorphism given by

$$\begin{aligned}\psi^{-1} : F_\alpha &\longrightarrow F_\alpha^\kappa \\ \frac{f}{g} &\longmapsto \frac{f}{g} * \frac{I}{\kappa} = \frac{f}{g*\kappa}.\end{aligned}$$

(Note that the concept of inversion here is unambiguous because everything is taking place in F_α and not $F_{\alpha,\kappa}$, so we do not need to think about the other identity I_κ or the κ-division from (3.11) above.)

By combining ψ with $\widetilde{\phi}^{-1}$, we obtain an isomorphism from F_α^κ to $F_{\alpha,\kappa}$, giving us a way of identifying these two fields that takes into account the different natures of I and I_κ rather than automatically identifying them with each other. We have:

$$\begin{aligned}F_\alpha^\kappa \xrightarrow{\psi} \; &F_\alpha \xrightarrow{\widetilde{\phi}^{-1}} F_{\alpha,\kappa} \\ \frac{f}{g} \longmapsto \; &\frac{f*\kappa}{g} \longmapsto {}^{\kappa}_{*}\frac{f*\kappa}{g}.\end{aligned}$$

In the remainder of this chapter, we use this isomorphism to identify the sets F_α^κ and $F_{\alpha,\kappa}$ with their corresponding field structures. This means that we have

$$\frac{I}{\kappa} = \frac{f}{f*\kappa} \longmapsto \frac{f*\kappa}{f*\kappa} \longmapsto {}^{\kappa}_{*}\frac{f*\kappa}{f*\kappa} = I_\kappa \tag{3.12}$$

and

$$I = \frac{f}{f} \longmapsto \frac{f*\kappa}{f} \longmapsto {}^{\kappa}_{*}\frac{f*\kappa}{f}, \tag{3.13}$$

thus we identify I_κ, the identity element of $F_{\alpha,\kappa}$, with the element $\frac{I}{\kappa} \in F_\alpha$, and we can also consider I as an element of $F_{\alpha,\kappa}$. This will be required in the work below, in which we will need to be able to consider κ-convolution with the element I in order to interpret GFIs in the field $F_{\alpha,\kappa}$.

3.2 Fractional Calculus Understood Using the New Convolution

3.2.1 Interpretation of the GFI and Fundamental Theorems of Calculus

Let us recall from Luchko [8] the definition of the general fractional integral (GFI) and general fractional derivative (GFD) associated with a pair of Sonine kernels. Given Sonine kernels κ and k satisfying (3.4), the GFI is defined by

$$\mathbb{I}_{(\kappa)}f(x) = (\kappa * f)(x) = \int_0^x \kappa(x-\tau)f(\tau)\mathrm{d}\tau, \qquad x > 0,$$

and the GFD of Riemann–Liouville type is defined by

$$\mathbb{D}_{(k)}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}(k * f)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x k(x-\tau)f(\tau)\mathrm{d}\tau, \qquad x > 0,$$

while the GFD of Caputo type, also called the regularised GFD, is defined by

$${}_*\mathbb{D}_{(k)}f(x) = \left(k * f'\right)(x) = \int_0^x k(x-\tau)f'(\tau)\mathrm{d}\tau, \qquad x > 0.$$

These operators have already been much studied in the literature, but now we can express them in a new way by using the new multiplication and division operations defined above, and re-investigate their properties using this new notation.

Definition 3.3: The GFI can now be expressed using κ-convolution as follows:

$$\mathbb{I}_{(\kappa)}f = I \overset{\kappa}{*} f = \kappa * I * f = \kappa * f, \tag{3.14}$$

and the GFD of Riemann–Liouville type can be expressed as

$$\mathbb{D}_{(k)}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left((k \overset{\kappa}{\div} I) \overset{\kappa}{*} f\right)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa * (k \div (\kappa * I)) * f\right)(x) = \frac{\mathrm{d}}{\mathrm{d}x}(k * f)(x), \tag{3.15}$$

while the GFD of Caputo type can be expressed as

$${}_*\mathbb{D}_{(k)}f = \left(k \overset{\kappa}{\div} I\right) \overset{\kappa}{*} f' = k * f'. \tag{3.16}$$

Note that here we are using the fact that I, the identity element from the field F_α, can be understood as an element of $F_{\alpha,\kappa}$, and we can use both convolutions $*$ and $\overset{\kappa}{*}$ keeping in mind their relationship with each other. These facts were justified in the previous section by algebraic arguments. Specifically, equation (3.13) justifies treating I as an element of $F_{\alpha,\kappa}$ via the identification of the fields F_α^κ and $F_{\alpha,\kappa}$ with each other that was discussed there.

It is known [8] that the GFDs of Riemann–Liouville and Caputo types are related by a simple subtraction formula as follows:

$$ {}_*\mathbb{D}_{(k)} f(x) = \mathbb{D}_{(k)} f(x) - f(0)k(x), \qquad x > 0. \tag{3.17}$$

Now, using the new formulations of the GFI and GFDs in terms of κ-convolution, we can prove the first and second fundamental theorems of calculus for these operators. These results were already proved by Luchko [8], but it is interesting to see how they can also be proved using a κ-convolution approach.

Proposition 3.3 (First fundamental theorem of calculus [8, Theorem 3]): Let κ and k be Sonine kernels as above. Then we have

$$\left(\mathbb{D}_{(k)} \mathbb{I}_{(\kappa)} f\right)(x) = f(x), \qquad x > 0, \tag{3.18}$$

for all $f \in C_{-1}(0,\infty)$, and

$$\left({}_*\mathbb{D}_{(k)} \mathbb{I}_{(\kappa)} f\right)(x) = f(x), \qquad x > 0, \tag{3.19}$$

for all f in the space $C_{-1,k}(0,\infty)$ characterised as follows:

$$C_{-1,k}(0,\infty) := \left\{ f \,:\, f = (\mathbb{I}_{(k)}\phi),\ \ \phi \in C_{-1}(0,\infty) \right\}.$$

Proof. By using (3.14) and (3.15), we have directly

$$\left(\mathbb{D}_{(k)}\mathbb{I}_{(\kappa)}f\right)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left((k \overset{\kappa}{\div} I) \overset{\kappa}{*} (I \overset{\kappa}{*} f)\right)(x)$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(k \overset{\kappa}{*} f\right)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\{1\} * f\Big)(x)$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x f(\tau)\,\mathrm{d}\tau\right) = f(x).$$

To prove the formula (3.19), we use the following chain of implications:

$$f = \mathbb{I}_{(k)}\phi = (k \overset{\kappa}{\div} I) \overset{\kappa}{*} \phi$$
$$\Rightarrow \mathbb{I}_{(\kappa)}f = I \overset{\kappa}{*} \left((k \overset{\kappa}{\div} I) \overset{\kappa}{*} \phi\right) = k \overset{\kappa}{*} \phi$$
$$\Rightarrow (\mathbb{I}_{(\kappa)}f)' = \phi \in C_{-1}(0,\infty)$$
$$\Rightarrow {}_*\mathbb{D}_{(k)}\mathbb{I}_{(\kappa)}f = (k \overset{\kappa}{\div} I) \overset{\kappa}{*} \left(\mathbb{I}_{(\kappa)}f\right)'$$
$$= (k \overset{\kappa}{\div} I) \overset{\kappa}{*} \phi = \mathbb{I}_{(k)}\phi = f.$$

Thus, the result is proved. □

Proposition 3.4 (Second fundamental theorem of calculus [8, Theorem 4]): Let κ and k be Sonine kernels as above. Then we have

$$\left(\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}f\right)(x) = f(x) - f(0), \qquad x > 0, \tag{3.20}$$

and

$$\left(\mathbb{I}_{(\kappa)}\mathbb{D}_{(k)}f\right)(x) = f(x), \qquad x > 0, \tag{3.21}$$

for all f in the space $C^1_{-1}(0,\infty)$ of functions whose 1st derivatives are in $C_{-1}(0,\infty)$.

Proof. By using (3.14) and (3.16),

$$\left(\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}f\right)(x) = \left(I \overset{\kappa}{*} (k \overset{\kappa}{\div} I) \overset{\kappa}{*} f'\right)(x)$$
$$= \left(k \overset{\kappa}{*} f'\right)(x) = \Big(\{1\} * f'\Big)(x)$$
$$= \int_0^x f'(\tau)\,\mathrm{d}\tau = f(x) - f(0).$$

To prove the formula (3.21), we use the formula (3.17) and deduce it from (3.20) in the same way as in [8, Theorem 4]. □

In the same way as above, the fundamental theorems of calculus for n-fold GFIs and GFDs [8, Theorems 6–7] can also be proved again using the new κ-convolution. We omit the proofs here, as the results are already known from Luchko's work, but we include the definitions of the n-fold GFI and GFD, as follows.

Definition 3.4 ([8]): For any $\kappa \in S_{-1}$ and $n \in \mathbb{N}$, the n-fold GFI is defined as the composition of n GFIs each with kernel κ:

$$\mathbb{I}^{n}_{(\kappa)} f := \underbrace{\mathbb{I}_{(\kappa)} \circ \cdots \circ \mathbb{I}_{(\kappa)}}_{n \text{ times}} f = \kappa^{n-1} \overset{\kappa}{*} f = I \overset{\kappa^n}{*} f = \kappa^n * f, \qquad x > 0, \tag{3.22}$$

where κ^n denotes the convolution power of κ according to the original convolution $*$ (note that this is a function in the space $C_{-1}(0,\infty)$, but it is not always a Sonine kernel).

Definition 3.5: Let κ and k be a Sonine pair in S_{-1} as above. The n-fold sequential GFDs in the Riemann–Liouville and Caputo senses, respectively, are defined as follows:

$$\mathbb{D}^{<n>}_{(k)} f := \underbrace{\mathbb{D}_{(k)} \circ \cdots \circ \mathbb{D}_{(k)}}_{n \text{ times}} f, \tag{3.23}$$

$$_*\mathbb{D}^{<n>}_{(k)} f := \underbrace{_*\mathbb{D}_{(k)} \circ \cdots \circ {_*\mathbb{D}_{(k)}}}_{n \text{ times}} f. \tag{3.24}$$

3.2.2 Generalized Convolution Taylor Series

In this subsection, we shall closely follow Luchko's work in [44] in order to construct, in the setting of our new convolution, the concept of convolutional powers and Taylor series.

Definition 3.6: Let κ^n denote the convolution power of κ according to the original convolution $*$, in the same way as in [8, Definition 5]:

$$\kappa^n = \underbrace{\kappa * \kappa * \cdots * \kappa}_{n \text{ times}}.$$

Let $\kappa^{<n>}$ denote the convolution power of κ according to the new convolution $\overset{\kappa}{*}$, defined inductively by:

$$\kappa^{<n>}(x) := \begin{cases} \kappa(x), & n=1 \\ \left(\kappa \overset{\kappa}{*} \kappa^{<n-1>}\right)(x), & n=2,3,\cdots \end{cases}$$

Thus, we have:

$$\kappa^{<1>}(x) = \kappa(x),$$

$$\kappa^{<2>}(x) = \kappa \overset{\kappa}{*} \kappa^{<1>}(x) = \kappa^3(x),$$

$$\kappa^{<3>}(x) = \kappa \overset{\kappa}{*} \kappa^{<2>}(x) = \kappa^5(x),$$

$$\vdots$$

$$\kappa^{<n>}(x) = \kappa \overset{\kappa}{*} \kappa^{<n-1>}(x) = \kappa^{2n-1}(x),$$

and, again by induction, we have shown that

$$\kappa^{<n>}(x) = \kappa^{2n-1}(x), \qquad n \in \mathbb{N}.$$

Theorem 3.5: Let $\kappa \in C_{-1}(0,\infty)$, which by definition we can write as

$$\kappa(x) = x^p \kappa_1(x), \qquad x > 0, \tag{3.25}$$

for some $p > -1$ and $\kappa_1 \in C[0,\infty)$. Let A be an analytic function on some disc, given by a power series

$$A(z) = \sum_{j=0}^{\infty} a_j z^j, \;\; a_j \in \mathbb{C}, \;\; z \in \mathbb{C} \;\; be \;\; non-zero, \tag{3.26}$$

with non-zero radius of convergence, where $a_j \in \mathbb{C}$ for all j. Then the convolution series

$$A_\kappa(x) = \sum_{j=0}^{\infty} a_j \kappa^{<j+1>}(x) \tag{3.27}$$

is convergent for all $x > 0$ and defines a function in the space $C_{-1}(0,\infty)$. Moreover, for $\alpha = \min\{p, 1\}$, the series

$$x^{-\alpha} A_\kappa(x) = \sum_{j=0}^{\infty} a_j x^{-\alpha} \kappa^{<j+1>}(x)$$

is locally uniformly convergent on $[0,\infty)$.

Proof. For the same reason that $C_{-1,0}(0,\infty)$ is the same function space as $C_{-1}(0,\infty)$ as mentioned above, we can assume without loss of generality that $p \in (-1,0]$. Let us fix a compact interval $[0,X]$ with $X > 0$, and work in this interval.

The function $\kappa_1(x) = x^{-p}\kappa(x)$ from (3.25) is continuous on $[0,\infty)$, so we can fix $C_X > 0$ such that $|\kappa_1(x)| \le \frac{C_X}{\Gamma(p+1)}$ for all $x \in [0,X]$. Now let $P(n)$ be the statement that

$$|\kappa^{<n>}(x)| \le \frac{C_X^{2n-1} x^{(2n-1)(p+1)-1}}{\Gamma((2n-1)(p+1))}, \qquad x \in (0,X],$$

which we will prove by induction on n. It is clearly true for $n = 1$. For $n = 2$, we have:

$$\begin{aligned}
\left|\kappa^{<2>}(x)\right| &= \left|(\kappa \overset{\kappa}{*} \kappa)(x)\right| \\
&= \left|\int_0^x \int_0^\tau \kappa(u)\kappa(\tau-u)\kappa(x-\tau)\,\mathrm{d}u\,\mathrm{d}\tau\right| \\
&\le \int_0^x \int_0^\tau u^p|\kappa_1(u)|(\tau-u)^p|\kappa_1(\tau-u)|(x-\tau)^p|\kappa_1(x-\tau)|\,\mathrm{d}u\,\mathrm{d}\tau \\
&\le \left(\frac{C_X}{\Gamma(p+1)}\right)^3 \int_0^x \int_0^\tau u^p(\tau-u)^p(x-\tau)^p\,\mathrm{d}u\,\mathrm{d}\tau \\
&= C_X^3 \cdot \frac{x^{3p+2}}{\Gamma(3p+3)}, \qquad 0 < x \le X,
\end{aligned}$$

where in the last step we have used properties of the beta function and the fact that $p+1 > 0$.

For the inductive step, we assume $P(j)$ and $P(j+1)$ are true and try to prove $P(j+1)$. We have:

$$
\begin{aligned}
\left|\kappa^{<j+1>}(x)\right| &= \left|\kappa \overset{\kappa}{*} \kappa^{<j>}(x)\right| = \left|\kappa * \kappa * \kappa^{<j>}(x)\right| \\
&= \left|\kappa * \kappa * \kappa * \kappa * \kappa^{<j-1>}(x)\right| \\
&= \left|\kappa * \kappa^{<2>} * \kappa^{<j-1>}(x)\right| \\
&= \left|\int_0^x \int_0^\tau \kappa(u)\kappa^{<2>}(\tau-u)\kappa^{<j-1>}(x-\tau)\,\mathrm{d}u\,\mathrm{d}\tau\right| \\
&\leq \frac{C_X}{\Gamma(p+1)} \cdot \frac{C_X^3}{\Gamma(3p+3)} \cdot \frac{C_X^{2n-3}}{\Gamma((2n-3)(p+1))} \\
&\qquad\qquad \times \int_0^x \int_0^\tau u^p(\tau-u)^{3p+2}(x-\tau)^{(2n-3)(p+1)-1}\,\mathrm{d}u\,\mathrm{d}\tau \\
&= \frac{C_X^{2n+1}}{\Gamma((2n+1)(p+1))} \cdot x^{(2n+1)(p+1)-1},
\end{aligned}
$$

where in the last step we have used properties of the beta function and the fact that all multiples of $p+1$ are positive.

Thus, by induction we have proved the statement $P(n)$ for all $n \in \mathbb{N}$, or equivalently $P(j+1)$ for all $j \in \mathbb{N}_0$. The latter can also be written as follows:

$$
\left|x^{-p}\kappa^{<j+1>}(x)\right| \leq \frac{C_X^{2j+1}x^{2j(p+1)}}{\Gamma(2j(p+1)+p+1)}, \qquad 0 < x \leq X, \quad j \in \mathbb{N}_0. \tag{3.28}
$$

Let $z_0 \in \mathbb{C}$ be any point in the disc of convergence of the power series (3.26). Because the series converges at z_0, we can let $C = \sup_{j\in\mathbb{N}_0}\left|a_j z_0^j\right|$ so that

$$
|a_j| \leq \frac{C}{|z_0|^j} \ \ \forall j \in \mathbb{N}_0. \tag{3.29}
$$

Combining the inequalities (3.28) and (3.29), we find

$$
\begin{aligned}
\left|x^{-p}a_j\kappa^{<j+1>}(x)\right| &\leq \frac{C}{|z_0|^j} \cdot \frac{C_X^{2j+1}x^{2j(p+1)}}{\Gamma(2j(p+1)+p+1)} &(3.30)\\
&\leq \frac{CC_X}{\Gamma(2j(p+1)+p+1)}\left(\frac{C_X^2 x^{2p+2}}{|z_0|}\right)^j, \ \ j \in \mathbb{N}_0, \ \ x \in [0,X]. &(3.31)
\end{aligned}
$$

Now, by the Weierstrass M-test and comparison with the power series for the Mittag-Leffler function, we can see that the series $x^{-p}\sum_{j=0}^{\infty} a_j\kappa^{<j+1>}(x)$ is absolutely and uniformly convergent on the interval $[0,X]$ by Weierstrass M-test. Because the functions $x^{-p}a_j\kappa^{<j+1>}(x)$ are continuous on $[0,X]$ for all $j \in \mathbb{N}_0$, we find that the above series defines a function in $C[0,X]$. Because X can be any large positive real number, we have now proved that the convolution series (3.27) is convergent for all $x > 0$, locally uniformly on $[0,\infty)$, and it defines a function in the space $C_{-1}(0,\infty)$. □

Remark 3.2: The above proof is inspired by that of [44, Theorem 3], but with some extra difficulties arising from the double convolution of three functions that is needed to define the κ-convolution in our setting.

Theorem 3.6: For any function $f \in C_{-1}(0,\infty)$ that can be expressed in the form of a convolution series (3.27) using the Sonine kernel $\kappa \in S_{-1}$, the coefficients of that series can be found as follows:

$$f(x) = \sum_{j=0}^{\infty} a_j\kappa^{<j+1>}(x), \qquad a_j = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{I}_{(k^2)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \left(\mathbb{I}_{(k)}f\right)(0), \qquad j = 0,1,2,\cdots. \tag{3.32}$$

Proof. By the local uniform convergence result proved in Theorem 3.5, we can apply the GFI (3.14) with the kernel k (the other element of the Sonine pair with κ) to the series in order to cancel some of the κ factors:

$$\begin{aligned}(\mathbb{I}_{(k)}f)(x) &= \left((k \overset{\kappa}{\div} I) \overset{\kappa}{*} f\right)(x) = (k*f)(x) = \left(k*\sum_{j=0}^{\infty} a_j\kappa^{<j+1>}\right)(x) \qquad (3.33)\\
&= a_0\left(k*\kappa^{<1>}\right)(x) + a_1\left(k*\kappa^{<2>}\right)(x) + a_2\left(k*\kappa^{<3>}\right)(x) + \cdots\\
&= a_0(k*\kappa)(x) + a_1(k*\kappa^3)(x) + a_2(k*\kappa^5)(x) + \cdots\\
&= a_0(k*\kappa)(x) + a_1\left(k*\kappa\overset{\kappa}{*}\kappa^{<1>}\right)(x) + a_2\left(k*\kappa\overset{\kappa}{*}\kappa^{<2>}\right)(x) + \cdots\\
&= a_0\{1\} + a_1\left(\{1\}\overset{\kappa}{*}\kappa^{<1>}\right)(x) + a_2\left(\{1\}\overset{\kappa}{*}\kappa^{<2>}\right)(x) + \cdots,\end{aligned}$$

where $\{1\}$ denotes the constant function identically equal to 1, so that ordinary $*$-convolution with $\{1\}$ is equivalent to ordinary integration. So we have

$$(\mathbb{I}_{(k)}f)(x) = a_0 + \left(\{1\} \overset{\kappa}{*} \sum_{j=0}^{\infty} a_{j+1}\kappa^{<j+1>}\right)(x) = a_0 + \left(\{1\} \overset{\kappa}{*} f_1\right)(x), \qquad (3.34)$$

where f_1 is a function in $C_{-1}(0,\infty)$ and therefore $\{1\} \overset{\kappa}{*} f_1$ is a continuous function on the whole interval $[0,\infty)$ which takes the value zero at $x = 0$. Thus, we can determine the first coefficient a_0 of the convolution series by substituting $x = 0$ in the equation (3.34):

$$a_0 = (\mathbb{I}_{(k)}f)(0). \qquad (3.35)$$

To find the next coefficient, we differentiate (3.34) and get the following formula:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathbb{I}_{(k)}f\right)(x) = (\kappa * f_1)(x) &= \sum_{j=0}^{\infty} a_{j+1}\kappa^{2(j+1)}(x) \\ &= a_1\kappa^2(x) + a_2\kappa^4(x) + a_3\kappa^6(x) + \cdots\end{aligned}$$

To find the coefficient a_1, we must apply a similar argument, but taking a convolution with k twice and then differentiating, in order to cancel the double factor of κ^2:

$$\mathbb{I}_{(k^2)}\left(\frac{\mathrm{d}}{\mathrm{d}x}(\mathbb{I}_{(k)}f)\right)(x) = a_1 x + a_2\left(\{1\}^2 * \kappa^2\right)(x) + a_3\left(\{1\}^2 * \kappa^4\right)(x) + \cdots,$$

therefore

$$a_1 = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{I}_{(k^2)}\frac{\mathrm{d}}{\mathrm{d}x}(\mathbb{I}_{(k)}f)\right)(0).$$

Note that this is not the same as

$$(\mathbb{D}_{(k)}^{<2>}\mathbb{D}_{(k)}f)(0) = (\mathbb{D}_{(k)}^{<3>}f)(0),$$

because the operators $\frac{\mathrm{d}}{\mathrm{d}x}$ and $\mathbb{I}_{(k)}$ do not commute with each other – in the same way that sequential fractional derivatives are not the same as ordinary fractional derivatives [36].

For the next step, we firstly differentiate again to get a simple series of κ powers:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathbb{I}_{(k^2)}\frac{\mathrm{d}}{\mathrm{d}x}(\mathbb{I}_{(k)}f)\right)(x) = a_2\kappa^2(x) + a_3\kappa^4(x) + a_4\kappa^6(x) + \cdots,$$

and then again we take a convolution with k twice and differentiate to find the coefficient a_2.

Repeating the same process as in the derivation for a_1, we get in general the formula

$$a_j = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{I}_{(k^2)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \left(\mathbb{I}_{(k)}f\right)(0), \tag{3.36}$$

which establishes the stated result. $\square$

Remark 3.3: Note that the formulae for the coefficients in Theorem 3.6 are not simply sequential GFDs, as they were in [44, Theorem 4]. This is one of the key differences showing the different structure of our results using κ-convolution rather than classical convolution.

Example 3.5: For $0 < \alpha < 1$, we can consider the Sonine kernel $\kappa = h_\alpha$, forming a pair with the associated kernel $k = h_{1-\alpha}$. Applying the above results, we have

$$\kappa^{<j+1>}(x) = \kappa^{2(j+1)-1}(x) = \kappa^{2j+1}(x) = h_\alpha^{2j+1}(x) = h_{\alpha(2j+1)}(x) = \frac{x^{\alpha(2j+1)-1}}{\Gamma(\alpha(2j+1))},$$

for all $x > 0$ and $j \in \mathbb{N}_0$. Thus, the generalised convolution series (3.32) takes the following form:

$$f(x) = x^{\alpha-1}\sum_{j=0}^{\infty} a_j \frac{x^{2\alpha j}}{\Gamma(2\alpha j+\alpha)},$$

where the coefficients a_j are given by

$$\begin{aligned} a_j &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\,{}^{RL}_{0}I^{2-2\alpha}_{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \left({}^{RL}_{0}I^{1-\alpha}_{x}f\right)(0) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\,{}^{RL}_{0}I^{2-2\alpha}_{x}\left({}^{RL}_{0}D^{2\alpha}_{x}\right)^{j-1}\left({}^{RL}_{0}D^{\alpha}_{x}f\right)(0). \end{aligned}$$

Example 3.6: If we set $\alpha = 1$ in the previous example, then since $h_1 = \{1\}$ is not a Sonine kernel, we cannot directly apply the theorem given above. However, taking the limit as $\alpha \to 1^-$ in the previous example, we can derive the following classical

Taylor series for an even function:

$$f(x) = \sum_{j=0}^{\infty} a_j \frac{x^{2j}}{(2j)!}, \tag{3.37}$$

where the coefficients

$$a_j = \left(\frac{\mathrm{d}}{\mathrm{d}x} \circ \frac{\mathrm{d}}{\mathrm{d}x}\right)^j \left({}^{RL}_{0}I^{0}_{x} f\right)(0) = f^{(2j)}(0)$$

are consistent with the classical results for the coefficients of a Taylor series.

In the previous work of Luchko [21, 44], it has been seen that some particularly important convolution series arise from the geometric series

$$A(z) = \sum_{j=0}^{\infty} \lambda^j z^j, \qquad z \in \mathbb{C}, \tag{3.38}$$

where $\lambda \in \mathbb{C}\backslash\{0\}$, which has radius of convergence $r = \frac{1}{|\lambda|}$. By Theorem 3.5, the convolution series

$$\ell_{\kappa,\lambda}(x) = \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>}(x), \tag{3.39}$$

is convergent for all $x > 0$ and $\lambda \in \mathbb{C}$, defining a function from the space $C_{-1}(0,\infty)$. Let us consider some particularly interesting examples of (3.39) arising from different choices of the Sonine kernel κ.

Example 3.7: For $0 < \alpha < 1$, considering the kernel function $\kappa = h_\alpha$, we know from above that $\kappa^{<j+1>} = h_{(2j+1)\alpha}$, and so the convolution series (3.39) takes the form

$$\ell_{\kappa,\lambda}(x) = \sum_{j=0}^{\infty} \lambda^j h_{(2j+1)\alpha}(x) = x^{\alpha-1} \sum_{j=0}^{\infty} \frac{\lambda^j x^{2j\alpha}}{\Gamma(2j\alpha+\alpha)} = x^{\alpha-1} E_{2\alpha,\alpha}(\lambda x^{2\alpha}), \tag{3.40}$$

in terms of the well-known two-parameter Mittag-Leffler function [45].

Example 3.8: If we set $\alpha = 1$ in the previous example, then the kernel function $\kappa = h_1 = \{1\}$ is not a Sonine kernel, but we can still write $\kappa^{<j+1>} = \kappa^{2j+1} = h_1^{2j+1} = h_{2j+1}$, and the convolution series (3.39) would be

$$\ell_{\kappa,\lambda}(x) = \sum_{j=0}^{\infty} \lambda^j h_{2j+1}(x) = \sum_{j=0}^{\infty} \frac{\lambda^j x^{2j}}{(2j)!} = \cosh(\sqrt{\lambda} x), \tag{3.41}$$

which is of course a well-defined power series and is the result of letting $\alpha \to 1^-$ in the previous example.

Example 3.9: For $0 < \alpha < 1$ and $\rho \in \mathbb{C}$, considering the kernel function $\kappa(x) = h_{\alpha,\rho}(x) = h_\alpha(x) \cdot e^{-\rho x}$, we find the convolution power as

$$\kappa^{<j+1>}(x) = \kappa^{2j+1}(x) = h_{(2j+1)\alpha,\rho}(x) = h_\alpha^{2j+1}(x) e^{-\rho x} = h_{(2j+1)\alpha}(x) e^{-\rho x}.$$

Therefore, the convolution series takes the form:

$$\ell_{\kappa,\lambda}(x) = e^{-\rho x} x^{\alpha-1} \sum_{j=0}^{\infty} \frac{\lambda^j x^{2j\alpha}}{\Gamma(2j\alpha + \alpha)} = e^{-\rho x} x^{\alpha-1} E_{2\alpha,\alpha}(\lambda x^{2\alpha}), \tag{3.42}$$

again in terms of the two-parameter Mittag-Leffler function $E_{2\alpha,\alpha}$.

Example 3.10: For $0 < \alpha < \beta < 1$, considering the kernel function $\kappa = h_{1-\beta+\alpha} + h_{1-\beta}$, we need to use the binomial formula in the rng $R_{-1,\kappa}$ to find the convolution power $\kappa^{<j+1>}(x)$ as follows:

$$\begin{aligned}
\kappa^{<j+1>}(x) &= \left(h_{1-\beta+\alpha}(x) + h_{1-\beta}\right)^{2j+1}(x) \\
&= \sum_{i=0}^{2j+1} \binom{2j+1}{i} \left(h_{1-\beta+\alpha}^i * h_{1-\beta}^{2j+1-i}\right)(x) \\
&= \sum_{i=0}^{2j+1} \binom{2j+1}{i} h_{i\alpha+(2j+1)(1-\beta)}(x).
\end{aligned}$$

Thus the corresponding convolution series takes the form:

$$\begin{aligned}
\ell_{\kappa,\lambda}(x) &= \sum_{j=0}^{\infty} \lambda^j \sum_{i=0}^{2j+1} \binom{2j+1}{i} h_{i\alpha+2j(1-\beta)+(1-\beta)}(x) \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{2j+1} \binom{2j+1}{i} \lambda^j \frac{x^{i\alpha+2j(1-\beta)+(1-\beta)-1}}{\Gamma(i\alpha + 2j(1-\beta) + (1-\beta))}.
\end{aligned}$$

It seems that this function should be understandable in terms of bivariate Mittag-Leffler functions [46], but we were unable to construct a direct connection, due to $2j+1$ being forced to be odd. If $2j+1$ were replaced by simply j, allowing odd or even values, then we would have

$$\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}\lambda^{(j-1)/2}\frac{x^{i\alpha+j(1-\beta)-1}}{\Gamma(i\alpha+j(1-\beta))}$$
$$=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(i+k)!}{i!k!}\lambda^{(i+k-1)/2}\frac{x^{i(\alpha+1-\beta)+k(1-\beta)-1}}{\Gamma(i(\alpha+1-\beta)+k(1-\beta))}$$
$$=\frac{1}{x\sqrt{\lambda}}E^{1}_{\alpha+1-\beta,1-\beta,0}\left(\sqrt{\lambda}x^{\alpha+1-\beta},\sqrt{\lambda}x^{1-\beta}\right),$$

where E denotes the bivariate Mittag-Leffler function defined in [46] that appears naturally in many fractional differential equations and systems.

However, the function above is not the same as our $\ell_{\kappa,\lambda}(x)$. Even splitting the sum over j into odd and even parts did not yield any way of extracting the odd-only part as a difference of two bivariate Mittag-Leffler functions, as is possible with the classical Mittag-Leffler functions defined by single sums [45].

3.2.3 Interpretation of the GFD and Fractional Differential Equations

Convolution series, similar to those constructed above, were used by Luchko [21] for fractional differential equations with Sonine kernels. Here we show how this method can be extended to the new algebraic structures $R_{\alpha,\kappa}$ and $F_{\alpha,\kappa}$ given by the new κ-convolution.

In the field $F_{\alpha,\kappa}$, we know that every element has a multiplicative inverse. Since the GFI (3.14) is precisely multiplication in $F_{\alpha,\kappa}$ with the element I (which was originally defined in F_α but can be identified as an element of $F_{\alpha,\kappa}$ via (3.13) above), it is natural to consider the multiplicative inverse of the element I in the field $F_{\alpha,\kappa}$.

Definition 3.7: Let T_κ denote the multiplicative inverse of κ in the field $F_{\alpha,\kappa}$, and let S_κ denote the multiplicative inverse of I in the field $F_{\alpha,\kappa}$. By definition of the κ-convolution, we know that

$$\kappa=\overset{\kappa}{*}\frac{\kappa^{<2>}}{\kappa}\Longrightarrow T_\kappa=\overset{\kappa}{*}\frac{\kappa}{\kappa^{<2>}}=\overset{\kappa}{*}\frac{\kappa}{\kappa^{3}},$$

and similarly, from (3.13) we have

$$I = \overset{\kappa}{\ast}\!\frac{\kappa * \kappa}{\kappa} = \overset{\kappa}{\ast}\!\frac{\kappa^2}{\kappa} \Longrightarrow S_\kappa = \overset{\kappa}{\ast}\!\frac{\kappa}{\kappa^2}. \tag{3.43}$$

Note that both S_κ and T_κ are not functions, only a useful algebraic abstraction: they are not in the rng $R_{\alpha,\kappa}$ that is also a function space. Interestingly, in I and S_κ we have discovered two field elements that are inverse to each other and neither of which are functions.

Theorem 3.7: Let $k \in S_{-1}$ be the Sonine kernel associate to the kernel κ. The algebraic GFD of Caputo type has the following algebraic representation in $F_{-1,\kappa}$, for any $f \in C^1_{-1}(0,\infty)$:

$$_*\mathbb{D}_{(k)} f = S_\kappa \overset{\kappa}{\ast} f - f(0)k. \tag{3.44}$$

Proof. This follows from the second fundamental theorem of calculus, Proposition 3.4. From there we have

$$\left(\mathbb{I}_{(\kappa)}\, {}_*\mathbb{D}_{(k)} f\right)(x) = f(x) - f(0), \qquad x > 0,$$

which from (3.14) can be written as

$$I \overset{\kappa}{\ast} \left({}_*\mathbb{D}_{(k)} f\right) = f - f(0)\{1\},$$

and therefore the algebraic representation of the Caputo-type GFD is

$$_*\mathbb{D}_{(k)} f = \overset{\kappa}{\ast}\!\frac{f}{I} - f(0)\overset{\kappa}{\ast}\!\frac{\{1\}}{I} = S_\kappa \overset{\kappa}{\ast} f - f(0)\left(S_\kappa \overset{\kappa}{\ast} \{1\}\right),$$

which is almost the same as the claimed result. We just need to prove that $S_\kappa \overset{\kappa}{\ast} \{1\} = k$, which follows from (3.43) and the definitions of κ-convolution and the associated division. We have

$$S_\kappa \overset{\kappa}{\ast} \{1\} = \left(\overset{\kappa}{\ast}\!\frac{\kappa}{\kappa^2}\right) \overset{\kappa}{\ast} (\kappa * k) = \overset{\kappa}{\ast}\!\frac{\kappa \overset{\kappa}{\ast} (\kappa * k)}{\kappa^2} = \overset{\kappa}{\ast}\!\frac{\kappa * \kappa * \kappa * k}{\kappa * \kappa},$$

which equals k because the κ-convolution of k with the denominator equals the numerator. $\square$

Theorem 3.8: Let $k \in S_{-1}$ be the Sonine kernel associate to the kernel κ. The 2-fold sequential GFD of Caputo type has the following algebraic representation in $F_{-1,\kappa}$, for any $f \in C^2_{-1}(0,\infty)$:

$$ {}_*\mathbb{D}^{<2>}_{(k)} f = T_\kappa \overset{\kappa}{*} f - f'(0)k^2 - f(0) T_\kappa \overset{\kappa}{*} \{1\}. \tag{3.45} $$

Proof. This follows from the second fundamental theorem of calculus for 2-fold GFDs [21, Theorem 3.5]. From there we have

$$ \left(\mathbb{I}^2_{(\kappa)}\, {}_*\mathbb{D}^{<2>}_{(k)} f \right)(x) = f(x) - f(0) - f'(0)x, \qquad x > 0, $$

or in other words,

$$ \kappa \overset{\kappa}{*} \left({}_*\mathbb{D}^{<2>}_{(k)} f \right) = f - f(0)\{1\} - f'(0)\{1\}^2, $$

and therefore the algebraic representation of the 2-fold sequential Caputo-type GFD is

$$ {}_*\mathbb{D}^{<2>}_{(k)} f = \overset{\kappa}{*}\frac{f}{\kappa} - f(0) \overset{\kappa}{*}\frac{\{1\}}{\kappa} - f'(0) \overset{\kappa}{*}\frac{\{1\}^2}{\kappa}. $$

Recalling the definition of T_κ, this becomes

$$ {}_*\mathbb{D}^{<2>}_{(k)} f = T_\kappa \overset{\kappa}{*} f - f(0) \left(T_\kappa \overset{\kappa}{*} \{1\} \right) - f'(0)k^2, $$

where the last term has been simplified using the fact that

$$ \kappa \overset{\kappa}{*} k^2 = \{1\}^2, $$

making use of the Sonine condition between k and κ. $\square$

Theorem 3.9: For any $\lambda \in \mathbb{C}$, the function $\ell_{\kappa,\lambda} \in R_{\alpha,\kappa} \subset F_{\alpha,\kappa}$ defined by the convolution series (3.39) is inverse to the element $T_\kappa - \lambda I_\kappa \in F_{\alpha,\kappa}$, i.e. we have:

$$ \overset{\kappa}{*}\frac{I_\kappa}{T_\kappa - \lambda I_\kappa} = \ell_{\kappa,\lambda}. \tag{3.46} $$

Proof. If $\lambda = 0$, the convolution series (3.39) takes the form $\ell_{\kappa,\lambda} = \kappa$ and the relation (3.46) takes the form $\overset{\kappa}{*}\frac{I_\kappa}{T_\kappa} = \kappa$, which is true by definition of T_κ. For the remaining cases $\lambda \neq 0$, we first note that

$$\kappa \overset{\kappa}{*} (T_\kappa - \lambda I_\kappa) = \kappa \overset{\kappa}{*} T_\kappa - \lambda \kappa = I_\kappa - \lambda \kappa. \tag{3.47}$$

Then, we have

$$\begin{aligned}
(I_\kappa - \lambda \kappa) \overset{\kappa}{*} \ell_{\kappa,\lambda} &= \ell_{\kappa,\lambda} - \lambda \kappa \overset{\kappa}{*} \ell_{\kappa,\lambda} \\
&= \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>} - \sum_{j=0}^{\infty} \lambda^{j+1} \kappa \overset{\kappa}{*} \kappa^{<j+1>} \\
&= \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>} - \sum_{j=0}^{\infty} \lambda^{j+1} \kappa^{<j+2>} \\
&= \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>} - \sum_{j=1}^{\infty} \lambda^j \kappa^{<j+1>} = \kappa.
\end{aligned}$$

Hence,

$$\kappa \overset{\kappa}{*} (T_\kappa - \lambda I_\kappa) \overset{\kappa}{*} \ell_{\kappa,\lambda} = (I_\kappa - \lambda \kappa) \overset{\kappa}{*} \ell_{\kappa,\lambda} = \kappa,$$

which leads to the identity

$$(T_\kappa - \lambda I_\kappa) \overset{\kappa}{*} \ell_{\kappa,\lambda} = I_\kappa,$$

which proves the desired identity (3.46). □

Theorem 3.10: For any $\lambda \in \mathbb{C}$, the inverse in $F_{\alpha,\kappa}$ of the element $S_\kappa - \lambda I_\kappa \in F_{\alpha,\kappa}$ is given by a classical convolution series (not κ-convolution as defined in the previous subsection) which is not a function in $R_{\alpha,\kappa}$:

$$\overset{\kappa}{*}\frac{I_\kappa}{S_\kappa - \lambda I_\kappa} = \sum_{j=0}^{\infty} \lambda^j \kappa^j = I + \lambda \sum_{j=0}^{\infty} \lambda^j \kappa^{j+1}, \tag{3.48}$$

where the last sum here (without the I term) is actually a function in $R_{\alpha,\kappa} = C_\alpha(0,\infty)$.

Proof. From the expression (3.43) for S_κ, we have

$$\overset{\kappa}{*}\frac{I_\kappa}{S_\kappa - \lambda I_\kappa} = \overset{\kappa}{*}\frac{I_\kappa}{\overset{\kappa}{*}\frac{\kappa}{\kappa^2} - \lambda I_\kappa} = \overset{\kappa}{*}\frac{\kappa^2}{\kappa - \lambda \kappa^2},$$

and now it simply remains to take the κ-convolution of $\kappa - \lambda \kappa^2$ with the stated convolution series and check that the answer is κ^2. We have:

$$\left(\kappa-\lambda\kappa^2\right)\overset{\kappa}{*}\left(\sum_{j=0}^{\infty}\lambda^j\kappa^j\right)=\sum_{j=0}^{\infty}\lambda^j\kappa^{j+2}-\sum_{j=0}^{\infty}\lambda^{j+1}\kappa^{j+3}$$
$$=\sum_{j=0}^{\infty}\lambda^j\kappa^{j+2}-\sum_{j=1}^{\infty}\lambda^j\kappa^{j+2}=\kappa^2,$$

and the result is proved. □

We can now examine some examples to see how the new Mikusiński operational calculus applies to some initial value problems for fractional differential equations involving GFDs with Sonine kernels.

Consider the following initial value problem:

$$\begin{cases}\left({}_*\mathbb{D}_{(k)}y\right)(x)-\lambda y(x)=f(x), \qquad x>0;\\ y(0)=y_0,\end{cases} \tag{3.49}$$

where $\lambda, y_0\in\mathbb{R}$ are fixed constants and f is fixed in the space $C_{-1}(0,\infty)$ and the unknown function y is assumed to be in the space $C_{-1}^1(0,\infty)$.

Using the idea of Mikusiński operational calculus from (3.44), we can transform the initial value problem (3.49) into a single algebraic equation in the field $F_{\alpha,\kappa}$:

$$S_\kappa\overset{\kappa}{*}y-y_0k-\lambda y=f, \tag{3.50}$$

where the functions $y,f\in R_{\alpha,\kappa}$ are interpreted as elements of $F_{\alpha,\kappa}$. The solution of (3.50) in $F_{\alpha,\kappa}$ is therefore

$$y=f\overset{\kappa}{*}\overset{\kappa}{*}\frac{I_\kappa}{S_\kappa-\lambda I_\kappa}+y_0\left(k\overset{\kappa}{*}\overset{\kappa}{*}\frac{I_\kappa}{S_\kappa-\lambda I_\kappa}\right). \tag{3.51}$$

As an element of the field $F_{\alpha,\kappa}$, this is not necessarily a function. However, using the result of Theorem 3.10, we know that $\overset{\kappa}{*}\frac{I_\kappa}{S_\kappa-\lambda I_\kappa}$ is a function plus I, so taking its convolution with the functions f and k is definitely going to result in a function in $R_{\alpha,\kappa}$. More explicitly, let us write

$$y(x) = y_f(x) + y_0 \cdot y_{iv}(x), \quad y_f := f \overset{\kappa}{*} \overset{\kappa}{*} \frac{I_\kappa}{S_\kappa - \lambda I_\kappa}, \quad y_{iv} := k \overset{\kappa}{*} \overset{\kappa}{*} \frac{I_\kappa}{S_\kappa - \lambda I_\kappa}, \tag{3.52}$$

and from (3.48) we have

$$y_f = f \overset{\kappa}{*} \overset{\kappa}{*} \frac{I_\kappa}{S_\kappa - \lambda I_\kappa} = f \overset{\kappa}{*} \left(\sum_{j=0}^{\infty} \lambda^j \kappa^j \right) = \sum_{j=0}^{\infty} \lambda^j f * \kappa^{j+1},$$
$$y_{iv} = k \overset{\kappa}{*} \overset{\kappa}{*} \frac{I_\kappa}{S_\kappa - \lambda I_\kappa} = k \overset{\kappa}{*} \left(\sum_{j=0}^{\infty} \lambda^j \kappa^j \right) = \sum_{j=0}^{\infty} \lambda^j k * \kappa^{j+1}.$$

Since $k * \kappa = \{1\}$, the last of these expressions becomes

$$y_{iv}(x) = 1 + \sum_{j=0}^{\infty} \lambda^{j+1} k * \kappa^{j+2} = 1 + \{1\} * \sum_{j=0}^{\infty} \lambda^{j+1} \kappa^{j+1}.$$

Hence, we have found the unique solution $y \in C^1_{-1}(0,\infty)$ to the initial value problem (3.49) for the fractional differential equation with a GFD of Caputo type. It can be written either algebraically as

$$y = (f + y_0 k) \overset{\kappa}{*} \overset{\kappa}{*} \frac{I_\kappa}{S_\kappa - \lambda I_\kappa},$$

or analytically as

$$y(x) = y_0 + (f(x) + y_0 \lambda) * \left(\sum_{j=0}^{\infty} \lambda^j \kappa^{j+1}(x) \right).$$

This is consistent with the result found by Luchko [21, §5], but here we have expressed it in terms of the new κ-convolution in the new field $F_{\alpha,\kappa}$.

In Luchko's paper [21, §5], he proceeded from problems of the form (3.49) to multi-term fractional differential equations. Can we do the same here? Let us consider firstly the 2-term case:

$$\begin{cases} a_2 \left({}_*\mathbb{D}^{<2>}_{(k)} y \right)(x) + a_1 \left({}_*\mathbb{D}_{(k)} y \right)(x) + a_0 y(x) = f(x), \qquad x > 0; \\ y(0) = y_0, \qquad y'(0) = y_1, \end{cases} \tag{3.53}$$

where $a_0, a_1, a_2, y_0, y_1 \in \mathbb{R}$ are fixed constants and f is fixed in the space $C_{-1}(0,\infty)$ and the unknown function y is assumed to be in the space $C^2_{-1}(0,\infty)$.

Using the results from above, (3.44) and (3.45), we can transform the initial value problem (3.53) into a single algebraic equation in the field $F_{\alpha,\kappa}$:

$$a_2\left(T_\kappa \overset{\kappa}{*} y - y_1 k^2 - y_0 T_\kappa \overset{\kappa}{*} \{1\}\right) + a_1\left(S_\kappa \overset{\kappa}{*} y - y_0 k\right) + a_0 y = f,$$

which rearranges to the following algebraic formula for the solution y:

$$y = \overset{\kappa}{*}\frac{f + a_2 y_1 k^2 + a_2 y_0 T_\kappa \overset{\kappa}{*} \{1\} + a_1 y_0 k}{a_2 T_\kappa + a_1 S_\kappa + a_0 I_\kappa}. \tag{3.54}$$

Unfortunately, the denominator of this expression involves both S_κ and T_κ in separate terms, and so we cannot easily expand (3.54) into convolution Taylor series as we did above with (3.52). This is in contrast to the solution of Luchko [21, §5], where the corresponding algebraic formula has a denominator $P_n(S_\kappa)$ which is a polynomial in S_κ, and therefore it can be expanded in partial fractions which can be understood as series by the analogues of our Theorem 3.9 and Theorem 3.10.

Therefore, it seems that solving multi-term fractional differential equations involving n-fold GFDs, even in the simplest case $n = 2$, will be more complicated using our new convolution than it was using the classical Laplace-type convolution. And of course, even when we put in the work to simplify (3.54), finally the same solution function will be reached, since the uniqueness of solutions has already been proved.

We were not able to find any other fractional differential equations which are more easily solved using the new convolution compared with the classical one. The theorems in this subsection are new, but when compared with the corresponding results in Luchko's previous work [21, 44, 47], they express the same functions and operators using different algebraic language, and therefore the same differential equations will be solved in essentially the same way, via a different rng structure but ultimately deriving the same solution functions.

Chapter 4

FRACTIONAL DIFFERENTIAL EQUATIONS WITH CONTINUOUS VARIABLE COEFFICIENTS AND SONINE KERNELS

4.1 FDEs with GFDs in Riemann–Liouville Sense

We start by considering the case of an FDE involving a single GFD of Riemann–Liouville type, namely the initial value problem given by (1.1). This section is divided into three parts: firstly, we solve the differential equation from (1.1) under a zero initial condition; secondly, we solve the corresponding homogeneous differential equation under the general initial condition from (1.1); thirdly, we use the superposition principle to solve (1.1) in general.

4.1.1 Inhomogeneous FDE under Zero Initial Condition

Theorem 4.1: Let $T \in \mathbb{R}^+$ and $r \in C[0,T]$ and $g \in C_{-1}[0,T]$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair. The following initial value problem:

$$\begin{cases} \left(\mathbb{D}_{(k)}v\right)(t) - r(t)v(t) = g(t), \qquad t \in [0,T], \\ \left(\mathbb{I}_{(k)}v\right)(0) = 0, \end{cases} \tag{4.1}$$

has a unique solution in the function space

$$\left\{ v \in C_{-1}[0,T] \,:\, \mathbb{I}_{(k)}v \in C^1_{-1}[0,T] \right\}, \tag{4.2}$$

and this solution is:

$$v(t) = \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)} \Big(r(t) \cdot \mathbb{I}_{(\kappa)} \Big)^j g(t), \tag{4.3}$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. Following the structure from [31], the proof consists of four main parts.

Part 1: transforming the initial value problem (4.1) **into an equivalent integral equation.** To prove equivalence, we need to go both ways.

Firstly, let $v \in C_{-1}[0,T]$ with $w = \mathbb{I}_{(k)}v \in C^1_{-1}[0,T]$. Define $u = w' = \mathbb{D}_{(k)}v$, and notice that the given condition implies $u \in C_{-1}[0,T]$. Now, by the equivalent definitions [8, Eq. (47) and (50)] of the GFD of Caputo type for $w \in C^1_{-1}[0,T]$, we have

$$\begin{aligned}\mathbb{I}_{(\kappa)}u(t) &= \mathbb{I}_{(\kappa)}w'(t) = {}_*\mathbb{D}_{(\kappa)}w(t) = \mathbb{D}_{(\kappa)}w(t) - w(0)\kappa(t)\\ &= \mathbb{D}_{(\kappa)}\mathbb{I}_{(k)}v(t) - \mathbb{I}_{(k)}v(0)\cdot\kappa(t) = v(t) - \mathbb{I}_{(k)}v(0)\cdot\kappa(t).\end{aligned}$$

Therefore, the initial value problem (4.1) for v gives precisely the following integral equation for the function $u \in C_{-1}[0,T]$:

$$u(t) - r(t)\cdot\mathbb{I}_{(\kappa)}u(t) = g(t). \tag{4.4}$$

Now one direction of the equivalence is proved.

Secondly, let $u \in C_{-1}[0,T]$ and define $v = \mathbb{I}_{(\kappa)}u$. It was shown by Luchko [8, Eq. (53)] that, if v is expressible as $\mathbb{I}_{(\kappa)}u$ for some $u \in C_{-1}[0,T]$, then $\mathbb{I}_{(k)}v \in C^1_{-1}[0,T]$ and $\mathbb{I}_{(k)}v(0) = 0$. (In fact, Luchko's work was on the half-line $[0,\infty)$, but it is obvious that the same proof works on the finite interval $[0,T]$.) Also, $\mathbb{D}_{(k)}v = u$ by the first fundamental theorem of fractional calculus [8, Theorem 3]. Therefore, if u satisfies the integral equation (4.4), then v satisfies both parts of the initial value problem (4.1), and also $v \in C_{-1}[0,T]$ with $\mathbb{I}_{(k)}v \in C^1_{-1}[0,T]$.

Now we have constructed a bijection between solutions $v \in C_{-1}[0,T]$ with $\mathbb{I}_{(k)}v \in C^1_{-1}[0,T]$ of the initial value problem (4.1) and solutions $u \in C_{-1}[0,T]$ of the integral equation (4.4). If necessary, we can also restrict this bijection to one between smaller function spaces: for example, between solutions $v \in C[0,T]$ with $\mathbb{I}_{(k)}v \in C^1[0,T]$ of the initial value problem (4.1) and solutions $u \in C[0,T]$ of the integral equation (4.4).

Part 2: constructing a contraction mapping. In order to use the Banach fixed point theorem to prove that the integral equation (4.4) has a unique solution $u \in C_{-1}[0,T]$, we need to find some contraction mapping whose fixed points are exactly solutions of (4.4).

First of all, since $\kappa \in C_{-1}[0,T]$, we can write

$$\kappa(t) = t^p \kappa_1(t)$$

for some $p > -1$ and $\kappa_1 \in C[0,T]$. Without loss of generality, we can assume that p is as small as desired, by multiplying κ_1 by some positive power if necessary. Thus, we assume at least $-1 < p < 0$.

Next, we notice that $C_{-1}[0,T]$ is not naturally a Banach space, so let us fix $\alpha \in (-1,0)$ and consider the space $t^\alpha \cdot C[0,T]$ of functions $v(t)$ which are expressible as t^α times a continuous function of $t \in [0,T]$. This space is a Banach space under the following norm:

$$\|v\|_{\alpha,\lambda} = \sup_{t\in(0,T]} \left| t^{-\alpha} e^{-\lambda t} v(t) \right|,$$

where $\lambda \gg 0$ is a fixed real number to be determined later. For reasons that will become clear later, we also require that $\alpha < -p-1$.

Next, we need an operator in order for the Banach fixed point theorem to be used. Let us define the operator $\mathscr{O}$ as follows:

$$(\mathscr{O}u)(t) = g(t) + r(t) \cdot \mathbb{I}_{(\kappa)} u(t).$$

Then, clearly the integral equation (4.4) is equivalent to $\mathscr{O}u = u$. It remains to check the function space mapping and boundedness properties of this operator.

It is clear that $\mathscr{O}$ maps the space $C_{-1}[0,T]$ into itself. More strongly, by [39, Lemma 1], we know that $\mathscr{O}$ maps the space $t^{\alpha} \cdot C[0,T]$ into itself. What about boundedness?

Let v_1 and v_2 be in the space $t^{\alpha} \cdot C[0,T]$. For a given $t \in (0,T]$, we have the following calculations:

$$\begin{aligned}
&\left| t^{-\alpha} e^{-\lambda t} \left(\mathscr{O}v_1(t) - \mathscr{O}v_2(t) \right) \right| \qquad (4.5) \\
&\quad = \left| t^{-\alpha} e^{-\lambda t} r(t) \cdot \mathbb{I}_{(\kappa)} (v_1 - v_2)(t) \right| \\
&\quad \leq t^{-\alpha} e^{-\lambda t} \|r\|_{\infty} \left| \int_0^t \kappa(t-\tau) \tau^{\alpha} e^{\lambda \tau} \left[\tau^{-\alpha} e^{-\lambda \tau} (v_1 - v_2)(\tau) \right] \mathrm{d}\tau \right| \\
&\quad \leq t^{-\alpha} e^{-\lambda t} \|r\|_{\infty} \|v_1 - v_2\|_{\alpha,\lambda} \left| \int_0^t (t-\tau)^p \kappa_1(t-\tau) \tau^{\alpha} e^{\lambda \tau} \, \mathrm{d}\tau \right| \\
&\quad \leq t^{-\alpha} \|r\|_{\infty} \|\kappa_1\|_{\infty} \|v_1 - v_2\|_{\alpha,\lambda} \left| \int_0^t (t-\tau)^p \tau^{\alpha} e^{-\lambda(t-\tau)} \, \mathrm{d}\tau \right|, \qquad (4.6)
\end{aligned}$$

where $\|\cdot\|_{\infty}$ denotes the usual supremum norm on $C[0,T]$. It remains to find a suitable bound for the integral expression, namely for

$$\int_0^t (t-\tau)^p \tau^{\alpha} e^{-\lambda(t-\tau)} \, \mathrm{d}\tau = \lambda^{-p-\alpha-1} \int_0^{\lambda t} (\lambda t - x)^{\alpha} x^p e^{-x} \, \mathrm{d}x.$$

When evaluated exactly, this integral is a confluent hypergeometric function; however, the known asymptotic formulae for such functions (at least those that we were able to find) were not enough to get a bound that would suffice for our purposes.

Noting that both p and α are between -1 and 0, so the integrand has singularities at both ends of the interval $x \in (0, \lambda t)$, we realised that the best way to bound the integral at both ends is to split it. It turned out that finding a reasonable bound for the lower part of the interval (starting from $x = 0$) was much easier than finding one for the upper part (going up to $x = \lambda t$), so we let the splitting point be at $x = \lambda t - \lambda^c t$ for some $c \in (0,1)$ to be determined.

On the lower part of the interval, we construct a bound as follows:

$$\begin{aligned}\int_0^{\lambda t - \lambda^c t} (\lambda t - x)^\alpha x^p e^{-x}\,\mathrm{d}x &\leq \left(\lambda^c t\right)^\alpha \int_0^{\lambda t - \lambda^c t} x^p e^{-x}\,\mathrm{d}x \\ &\leq \left(\lambda^c t\right)^\alpha \int_0^{\infty} x^p e^{-x}\,\mathrm{d}x = \left(\lambda^c t\right)^\alpha \cdot \Gamma(p+1).\end{aligned}$$

On the upper part of the interval, we construct a bound as follows:

$$\begin{aligned}\int_{\lambda t - \lambda^c t}^{\lambda t} (\lambda t - x)^\alpha x^p e^{-x}\,\mathrm{d}x &= \int_0^{\lambda^c t} (\lambda t - x)^p x^\alpha e^{x - \lambda t}\,\mathrm{d}x \\ &\leq (\lambda t - \lambda^c t)^p\, e^{\lambda^c t - \lambda t} \int_0^{\lambda^c t} x^\alpha\,\mathrm{d}x \\ &= t^p\,(\lambda - \lambda^c)^p\, e^{\lambda^c t - \lambda t} \cdot \frac{(\lambda^c t)^{\alpha+1}}{\alpha+1} \\ &\leq t^{\alpha+p+1} \lambda^{p + c\alpha + c} \cdot \frac{2^{-p}}{\alpha+1},\end{aligned}$$

where in the last step we used the fact that, since $\lambda \gg 0$ is large and p is negative, $\lambda^c \leq \lambda/2$ and therefore $(\lambda - \lambda^c)^p \leq \lambda^p 2^{-p}$.

Combining the above two estimates and inserting them back into (4.6), we find:

$$\begin{aligned}&\left| t^{-\alpha} e^{\lambda t} \left(\mathscr{O} v_1(t) - \mathscr{O} v_2(t) \right) \right| \\ &\qquad \leq t^{-\alpha} \lambda^{-p-\alpha-1} \|r\|_\infty \|\kappa_1\|_\infty \|v_1 - v_2\|_{\alpha,\lambda} \\ &\qquad\qquad \times \left(\lambda^{c\alpha} t^\alpha \cdot \Gamma(p+1) + t^{\alpha+p+1} \lambda^{p+c\alpha+c} \cdot \frac{2^{-p}}{\alpha+1} \right).\end{aligned}$$

Simplifying the right-hand side turns it into the following expression:

$$\|r\|_\infty \|\kappa_1\|_\infty \|v_1 - v_2\|_{\alpha,\lambda} \left(\frac{\Gamma(p+1)}{\lambda^{\alpha(1-c)+p+1}} + \frac{t^{p+1}}{\lambda^{(1-c)(\alpha+1)}} \cdot \frac{2^{-p}}{\alpha+1} \right).$$

Finally, taking the supremum over $t \in [0,T]$ on both sides of the inequality, and using $p+1>0$, we have:

$$\left\|\mathscr{O}v_1 - \mathscr{O}v_2\right\|_{\alpha,\lambda} \leq \|r\|_\infty \|\kappa_1\|_\infty \left(\frac{\Gamma(p+1)}{\lambda^{\alpha(1-c)+p+1}} + \frac{T^{p+1}}{\lambda^{(1-c)(\alpha+1)}} \cdot \frac{2^{-p}}{\alpha+1} \right) \|v_1 - v_2\|_{\alpha,\lambda} \cdot$$

Clearly, $(1-c)(\alpha+1)>0$, so we can guarantee being able to choose a large enough $\lambda \gg 0$ in order to have

$$\|r\|_\infty \|\kappa_1\|_\infty \left(\frac{\Gamma(p+1)}{\lambda^{\alpha(1-c)+p+1}} + \frac{T^{p+1}}{\lambda^{(1-c)(\alpha+1)}} \cdot \frac{2^{-p}}{\alpha+1} \right) < 1,$$

so that $\mathscr{O}$ is a contraction mapping on the space $t^\alpha \cdot C[0,T]$ under the norm $\|\cdot\|_{\alpha,\lambda}$, as long as we have $\alpha(1-c)+p+1>0$. The required condition can be rewritten as follows:

$$\alpha(1-c)+p+1>0 \Longleftrightarrow 1-c < \frac{p+1}{-\alpha},$$

and it is always possible to pick $c \in (0,1)$ satisfying this, since we assumed $\alpha < -p-1$ which implies $p+1<-\alpha$, meaning that the required upper bound for $1-c$ is in the interval $(0,1)$.

Part 3: existence, uniqueness, and solution as a limit. So far, we have shown that solutions of (4.4) in the space $C_{-1}[0,T]$ are precisely fixed points of the mapping $\mathscr{O}$: $C_{-1}[0,T] \to C_{-1}[0,T]$, and we have also shown that $\mathscr{O}$ is a contraction mapping on any space $t^\alpha \cdot C[0,T]$ as long as $-1<\alpha<-p-1$.

By the Banach fixed point theorem, this means that the integral equation (4.4) has a unique solution in the space $t^\alpha \cdot C[0,T]$ for any $\alpha \in (-1,-p-1)$. Since the union of these spaces over all such α is the space $C_{-1}[0,T]$, we have thus proved that the integral equation (4.4) has a unique solution $u \in C_{-1}[0,T]$. By the equivalence constructed in

Part 1 of this proof, we now know that the initial value problem (4.1) has a unique solution v in the function space (4.2).

The Banach fixed point theorem also tells us that u is the limit, under the norm $\|\cdot\|_{\alpha,\lambda}$ on the space $t^{\alpha}\cdot C[0,T]$ for each $\alpha\in(-1,-p-1)$, of the following sequence of functions:

$$\begin{cases} u_0(t)=g(t), \\ u_n(t)=\mathscr{O}u_{n-1}(t)=g(t)+r(t)\cdot\mathbb{I}_{(\kappa)}u_{n-1}(t), \qquad n\geq 1. \end{cases} \tag{4.7}$$

The sequence $\left(t^{-\alpha}e^{-\lambda t}u_n(t)\right)$ converges uniformly on $[0,T]$, so the sequence (u_n) itself converges locally uniformly on $(0,T]$. By Young's convolution inequality, the GFI with Sonine kernel κ is a bounded operator on the space of continuous functions under the supremum norm, and Part 2 of this proof shows that it is also a bounded operator on the space $t^{\alpha}\cdot C[0,T]$ as above. So the sequence of functions $v_n=\mathbb{I}_{(\kappa)}u_n$ is also locally uniformly convergent on $(0,T]$ with limit $v=\mathbb{I}_{(\kappa)}u$. Namely, the following sequence:

$$\begin{cases} v_0(t)=\mathbb{I}_{(\kappa)}u_0(t)=\mathbb{I}_{(\kappa)}g(t), \\ v_n(t)=\mathbb{I}_{(\kappa)}g(t)+\mathbb{I}_{(\kappa)}\left(r(t)\cdot\mathbb{I}_{(\kappa)}u_{n-1}(t)\right)=v_0(t)+\mathbb{I}_{(\kappa)}\big(r(t)\cdot v_{n-1}(t)\big), \\ \qquad n\geq 1. \end{cases} \tag{4.8}$$

converges locally uniformly to the function $v=\mathbb{I}_{(\kappa)}u$, which is the unique solution of the initial value problem (4.1) in the function space (4.2).

Part 4: explicit solution as a series. Let us write explicitly the first few functions in the sequence (v_n):

$$
\begin{aligned}
v_0(t) &= \mathbb{I}_{(\kappa)} g(t);\\
v_1(t) &= \mathbb{I}_{(\kappa)} g(t) + \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)} g(t)\Big)\\
&= \sum_{j=0}^{1} \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)}\Big)^j g(t);\\
v_2(t) &= \mathbb{I}_{(\kappa)} g(t) + \mathbb{I}_{(\kappa)}\big(r(t)\cdot v_1(t)\big)\\
&= \mathbb{I}_{(\kappa)} g(t) + \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \Big[\mathbb{I}_{(\kappa)} g(t) + \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)} g(t)\Big)\Big]\Big)\\
&= \mathbb{I}_{(\kappa)} g(t) + \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)} g(t)\Big) + \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)} g(t)\Big)\Big)\\
&= \sum_{j=0}^{2} \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)}\Big)^j g(t);\\
&\dots
\end{aligned}
$$

By induction, it is straightforward to prove that for all $n \in \mathbb{N}$, we have

$$
v_n(t) = \sum_{j=0}^{n} \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)}\Big)^j g(t),
$$

and thus, taking the limit,

$$
v(t) = \lim_{n\to\infty} v_n(t) = \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)}\Big)^j g(t),
$$

where the limit, and therefore the infinite summation, are locally uniformly convergent on $(0,T]$, since the integral of a uniformly convergent sum is also uniformly convergent. □

4.1.2 Homogeneous FDE under General Initial Condition

Theorem 4.2: Let $T \in \mathbb{R}^+$ and $r \in C[0,T]$ and $a_0 \in \mathbb{R}$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair. The following initial value problem:

$$
\begin{cases}
\left(\mathbb{D}_{(k)} v\right)(t) - r(t)v(t) = 0, \qquad t \in [0,T],\\
\left(\mathbb{I}_{(k)} v\right)(0) = a_0,
\end{cases}
\tag{4.9}
$$

has a unique solution v in the function space (4.2), namely:

$$
v(t) = a_0\kappa(t) + a_0 \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)}\Big(r(t)\cdot \mathbb{I}_{(\kappa)}\Big)^j \Big(r(t)\cdot \kappa(t)\Big),
\tag{4.10}
$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. The aim here is to reduce the FDE (4.9) to an inhomogeneous FDE under zero initial condition, so that we can directly use Theorem 4.1. Since κ and k form a Sonine pair, we have $\mathbb{I}_{(k)}\kappa = k*\kappa = 1$ and $\mathbb{D}_{(k)}\kappa = \frac{\mathrm{d}}{\mathrm{d}t}\left(k*\kappa\right) = \frac{\mathrm{d}}{\mathrm{d}t}(1) = 0$, so κ is in the function space (4.2).

Defining $w = v - a_0\kappa$, we find that the initial value problem (4.9) is equivalent to the following initial value problem for w:

$$\begin{cases} \left(\mathbb{D}_{(k)}w\right)(t) - r(t)w(t) = a_0 r(t)\kappa(t), \qquad t\in[0,T], \\ \left(\mathbb{I}_{(k)}w\right)(0) = 0, \end{cases} \tag{4.11}$$

and also that v is in the function space (4.2) if and only if w is.

Now, (4.11) is an initial value problem exactly in the form of (4.1), because $a_0 r\cdot\kappa$ is in the space $C_{-1}[0,T]$. Using Theorem 4.1 with $g(t) = a_0 r(t)\kappa(t)$, we find

$$w(t) = \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)}\Big(r(t)\cdot\mathbb{I}_{(\kappa)}\Big)^j\Big(a_0 r(t)\cdot\kappa(t)\Big),$$

which gives the stated result (4.10) for v, where we know from Theorem 4.1 that the sum is locally uniformly convergent on $(0,T]$. $\square$

4.1.3 The General Case: Inhomogeneous FDE under General Initial Condition

Theorem 4.3: Let $T\in\mathbb{R}^+$, $r\in C[0,T]$, $g\in C_{-1}[0,T]$, and $a_0\in\mathbb{R}$, and let $\kappa, k\in C_{-1}[0,T]$ be a Sonine pair. Then the general initial value problem (1.1) has a unique solution v in the space (4.2), namely:

$$v(t) = a_0\kappa(t) + \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)}\Big(r(t)\cdot\mathbb{I}_{(\kappa)}\Big)^j\Big(g(t) + a_0 r(t)\cdot\kappa(t)\Big),$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. By applying the superposition principle to the results of Theorem 4.1 and

Theorem 4.2. $\square$

To verify the consistency of the above result with previous work in the literature, we consider a particular case in the following example.

Example 4.1: Let us consider the case where $r(t) = \lambda$ is a constant function. Then, we have the following initial value problem:

$$\begin{cases} \left(\mathbb{D}_{(k)}v\right)(t) - \lambda v(t) = g(t), \qquad t \geq 0; \\ \left(\mathbb{I}_{(k)}v\right)(0) = a_0. \end{cases}$$

In Luchko's paper [11, Theorem 5.1], the solution of this equation is found to be

$$v(t) = \left(g * l_{\kappa,\lambda}\right)(t) + a_0 l_{\kappa,\lambda}(t),$$

where $l_{\kappa,\lambda}$ is the convolution series [48] defined as follows:

$$l_{\kappa,\lambda}(t) = \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>}(t). \tag{4.12}$$

From our Theorem 4.3 above, with $r(t) = \lambda$, we find the following general solution:

$$\begin{aligned} v(t) &= a_0\kappa(t) + \sum_{j=0}^{\infty} \lambda^j \mathbb{I}_{(\kappa)}^{<j+1>}\Big(g(t) + a_0\lambda\kappa(t)\Big) \\ &= a_0\kappa(t) + \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>} * \Big(g(t) + a_0\lambda\kappa(t)\Big) \\ &= \left(g * l_{\kappa,\lambda}\right)(t) + a_0\left(\kappa(t) + \sum_{j=0}^{\infty} \lambda^{j+1}\kappa^{<j+2>}(t)\right) \\ &= \left(g * l_{\kappa,\lambda}\right)(t) + a_0 l_{\kappa,\lambda}(t), \end{aligned}$$

which is exactly the same solution as Luchko found. Thus, our result on variable-coefficient FDEs is consistent with previous work in the literature for the particular case of constant coefficients.

4.2 FDEs with GFDs in Caputo Sense

In this section, we will consider an FDE involving a single GFD of Caputo type, namely the initial value problem given by (1.2), which is the same as the

Riemann–Liouville problem of the previous section except with a different GFD operator and a different initial condition. In the previous section, it was non-trivial to understand what type of initial condition should be used to make a well-posed initial value problem; but for Caputo operators, it is well known [49] that classical-type initial conditions are usually needed.

Again, this section is divided into three parts: for the differential equation from (1.2) under a zero initial condition, the corresponding homogeneous differential equation under the general initial condition from (1.2), and finally solving (1.2) in general. However, we can speed up the work compared with the previous section, by noting some similarities and reducing the new problem to one already solved wherever possible.

4.2.1 Inhomogeneous FDE under Zero Initial Condition

Theorem 4.4: Let $T \in \mathbb{R}^+$ and $r \in C[0,T]$ and $g \in C[0,T]$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair. The following initial value problem:

$$\begin{cases} \left({}_*\mathbb{D}_{(k)}v\right)(t) - r(t)v(t) = g(t), \qquad t \in [0,T], \\ v(0) = 0, \end{cases} \tag{4.13}$$

has a unique solution in the function space

$$\left\{ v \in C_{-1}[0,T] \, : \, \mathbb{I}_{(k)}v \in C^1_{-1}[0,T] \right\} \cap C[0,T], \tag{4.14}$$

and this solution is precisely (4.3).

Proof. By Theorem 4.1 and its proof, it will be sufficient to construct a bijection between solutions v of the initial value problem (4.13) and solutions u of the integral equation (4.4), in some suitable function spaces.

Firstly, if v in the function space (4.14) satisfies the initial value problem (4.13), then we know from the proof of Theorem 4.1 that $\mathbb{D}_{(k)}v$ exists in $C_{-1}[0,T]$, and the GFD in Caputo sense can be written as

$$\left({}_*\mathbb{D}_{(k)}v\right)(t) = \left(\mathbb{D}_{(k)}v\right)(t) - v(0)k(t) = \left(\mathbb{D}_{(k)}v\right)(t).$$

So, defining $u = {}_*\mathbb{D}_{(k)}v \in C_{-1}[0,T]$, we have $\mathbb{I}_{(\kappa)}u = v - \mathbb{I}_{(k)}v(0)\cdot\kappa$ as in the proof of Theorem 4.1. By [39, Lemma 1], the assumption $v \in C[0,T]$ implies that $\mathbb{I}_{(k)}v(t)$ equals a positive power of t times a function in $C[0,T]$, so $\mathbb{I}_{(k)}v(0) = 0$. This means $\mathbb{I}_{(\kappa)}u = v$, and then (4.13) becomes

$$u(t) - r(t)\cdot\mathbb{I}_{(\kappa)}u(t) = g(t),$$

which is exactly the integral equation (4.4) as required, with $u \in C_{-1}[0,T]$ such that $\mathbb{I}_{(\kappa)}u \in C[0,T]$.

Conversely, if $u \in C_{-1}[0,T]$ satisfies (4.4) and $\mathbb{I}_{(\kappa)}u \in C[0,T]$, then we know from the proof of Theorem 4.1 that $v = \mathbb{I}_{(\kappa)}u$ is in the space (4.2) and satisfies the Riemann–Liouville initial value problem (4.1). We also know $v \in C[0,T]$, so v must be in the space (4.14). We also have g and r and $\mathbb{I}_{(\kappa)}u$ all in $C[0,T]$, so the integral equation itself implies that $u \in C[0,T]$, which means $v(0) = 0$ and ${}_*\mathbb{D}_{(k)}v = \mathbb{D}_{(k)}v$, so that v satisfies the Caputo initial value problem (4.13).

Now we have established a bijection between solutions v of the initial value problem (4.13) in the space (4.14) and solutions u of the integral equation (4.4) in the space

$$\left\{u \in C_{-1}[0,T] \,:\, \mathbb{I}_{(\kappa)}u \in C[0,T]\right\}. \tag{4.15}$$

Thus, the result is proved as long as we can be sure that the unique solution $u \in C_{-1}[0,T]$ of the integral equation (4.4) is in the space (4.15): in other words, as long as the function defined by (4.3) is in $C[0,T]$. And this is true, since $g \in C[0,T]$

implies that we remain in $C[0,T]$ no matter how many times we apply the GFI or multiply by r, so the entire sequence (v_n) is in $C[0,T]$. $\square$

4.2.2 Homogeneous FDE under General Initial Condition

Theorem 4.5: Let $T \in \mathbb{R}^+$ and $r \in C[0,T]$ and $a_0 \in \mathbb{R}$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair. The following initial value problem:

$$\begin{cases} \left({}_*\mathbb{D}_{(k)}v\right)(t) - r(t)v(t) = 0, & t \in [0,T], \\ v(0) = a_0, \end{cases} \tag{4.16}$$

has a unique solution v in the function space (4.14), namely:

$$v(t) = a_0 + a_0 \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)} \Big(r(t) \cdot \mathbb{I}_{(\kappa)}\Big)^j \big(r(t)\big), \tag{4.17}$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. We can again reduce (4.16) to an inhomogeneous FDE under zero initial conditions, in order to use Theorem 4.4.

Let $w(t) = v(t) - a_0$. Since the Caputo-type GFD of a constant is zero, we find that the initial value problem (4.16) is equivalent to the following initial value problem for w:

$$\begin{cases} \left({}_*\mathbb{D}_{(k)}w\right)(t) - r(t)w(t) = a_0 r(t), & t \in [0,T], \\ w(0) = 0, \end{cases} \tag{4.18}$$

and also that v is in the function space (4.14) if and only if w is.

Now, (4.18) is an initial value problem exactly in the form of (4.13), because $a_0 r$ is in the space $C[0,T]$. Using Theorem 4.4 with $g(t) = a_0 r(t)$, we find

$$w(t) = \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)} \Big(r(t) \cdot \mathbb{I}_{(\kappa)}\Big)^j \big(a_0 r(t)\big),$$

which gives the stated result (4.17) for v. Local uniform convergence of the series is guaranteed since we are using Theorem 4.4, where the solution is given by (4.3) which we already showed to be locally uniformly convergent on $(0,T]$ in the proof of Theorem 4.1. $\square$

4.2.3 The General Case: Inhomogeneous FDE under General Initial Condition

Theorem 4.6: Let $T \in \mathbb{R}^+$ and $r,g \in C[0,T]$ and $a_0 \in \mathbb{R}$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair. Then the general initial value problem (1.2) has a unique solution v in the space (4.14), namely:

$$v(t) = a_0 + \sum_{j=0}^{\infty} \mathbb{I}_{(\kappa)} \Big(r(t) \cdot \mathbb{I}_{(\kappa)} \Big)^j \Big(g(t) + a_0 r(t) \Big),$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. By applying the superposition principle to the results of Theorem 4.4 and Theorem 4.5. $\square$

To verify the consistency of the above result with previous work in the literature, we consider a particular case in the following example.

Example 4.2: Let us consider the case where $r(t) = \lambda$ is a constant function. Then, we have the following initial value problem:

$$\begin{cases} \big({}_*\mathbb{D}_{(k)} v\big)(t) - \lambda v(t) = g(t), & t \geq 0; \\ v(0) = a_0. \end{cases}$$

In Luchko's paper [21, Theorem 5.1], the solution of this equation is found to be

$$v(t) = (g * l_{\kappa,\lambda})(t) + a_0 L_{\kappa,\lambda}(t),$$

where $l_{\kappa,\lambda}$ and $L_{\kappa,\lambda}$ are the convolution series [48] defined as follows:

$$l_{\kappa,\lambda}(t) = \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>}(t), \qquad L_{\kappa,\lambda}(t) = \big(k * l_{\kappa,\lambda}\big)(t) = 1 + \{1\} * \sum_{j=1}^{\infty} \lambda^j \kappa^{<j>}(t). \tag{4.19}$$

From our Theorem 4.6 above, with $r(t) = \lambda$, we find the following general solution:

$$
\begin{aligned}
v(t) &= a_0 + \sum_{j=0}^{\infty} \lambda^j \mathbb{I}_{(\kappa)}^{<j+1>} \Big(g(t) + a_0 \lambda \{1\} \Big) \\
&= a_0 + \sum_{j=0}^{\infty} \lambda^j \kappa^{<j+1>} * \Big(g(t) + a_0 \lambda \{1\} \Big) \\
&= \big(g * l_{\kappa,\lambda} \big)(t) + a_0 \left(1 + \sum_{j=0}^{\infty} \lambda^{j+1} \kappa^{<j+1>} * \{1\} \right) \\
&= \big(g * l_{\kappa,\lambda} \big)(t) + a_0 L_{\kappa,\lambda}(t),
\end{aligned}
$$

which is exactly the same solution as Luchko found. Thus, our result on variable-coefficient FDEs is consistent with previous work in the literature for the particular case of constant coefficients.

4.3 FDEs with M-fold GFDs in Riemann–Liouville Sense

In the above sections, we have considered only FDEs involving a single GFD. These were good model problems to start developing the theory of variable-coefficient FDEs with Sonine kernels, but now we are ready to move on to multi-term FDEs. The goal of this section is to solve the initial value problem (1.3), and again we split the work into three cases: the general equation from (1.3) under zero initial conditions, the homogeneous version of the FDE under the general initial conditions from (1.3), and finally the full problem (1.3).

4.3.1 Inhomogeneous FDE under Zero Initial Conditions

Theorem 4.7: Let $T \in \mathbb{R}^+$ and $m \in \mathbb{N}$ and $r_0, r_1, \cdots, r_{m-1} \in C[0,T]$ and $g \in C_{-1}[0,T]$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair. The following initial value problem:

$$
\begin{cases}
\mathbb{D}_{(k)}^{<m>} v(t) + \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} v(t) = g(t), \qquad t \in [0,T], \\
\frac{\mathrm{d}^i}{\mathrm{d}t^i} \Big(\mathbb{I}_{(k)}^{<m>} v \Big)(0) = 0, \qquad i = 0, \ldots, m-1,
\end{cases}
\tag{4.20}
$$

has a unique solution in the function space

$$
\Big\{ v \in C_{-1}[0,T] \, : \, \mathbb{I}_{(k)}^{<m>} v \in C_{-1}^{m}[0,T] \Big\},
\tag{4.21}
$$

and this solution is:

$$v(t)=\sum_{p=0}^{\infty}(-1)^p\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\right)^p\mathbb{I}_{(\kappa)}^{<m>}g(t), \tag{4.22}$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. As before, the proof consists of four main parts: constructing an equivalent integral equation, a contraction mapping, the solution as a limit, and the solution as a convergent infinite series.

Part 1: transforming the initial value problem (4.20) **into an equivalent integral equation.** We can already guess how this is going to go, similar to the proof of Theorem 4.1.

Let $v\in C_{-1}[0,T]$ with $w=\mathbb{I}_{(k)}^{<m>}v\in C_{-1}^m[0,T]$, and define $u=w^{(m)}=\mathbb{D}_{(k)}^{<m>}v\in C_{-1}[0,T]$. By the equivalent definitions [8, Eq. (67) and (69)] of the m-fold GFD of Caputo type for $w\in C_{-1}^m[0,T]$, we have

$$\begin{aligned}\mathbb{I}_{(\kappa)}^{<m>}u(t)&=\mathbb{I}_{(\kappa)}^{<m>}w^{(m)}(t)={}_*\mathbb{D}_{(\kappa)}^{<m>}w(t)\\&=\mathbb{D}_{(\kappa)}^{<m>}w(t)-\sum_{j=0}^{m-1}w^{(j)}(0)\cdot\frac{\mathrm{d}^{m-j-1}}{\mathrm{d}t^{m-j-1}}\left(\kappa^{<m>}(t)\right)\\&=\mathbb{D}_{(\kappa)}^{<m>}\mathbb{I}_{(k)}^{<m>}v(t)-\sum_{j=0}^{m-1}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\mathbb{I}_{(k)}^{<m>}v\right)(0)\cdot\frac{\mathrm{d}^{m-j-1}}{\mathrm{d}t^{m-j-1}}\left(\kappa^{<m>}(t)\right).\end{aligned} \tag{4.23}$$

By the first fundamental theorem of fractional calculus for the m-fold GFD, we have $v=\mathbb{I}_{(\kappa)}^{<m>}u$ provided that the initial condition in (4.20) is satisfied. Furthermore, for every $i=0,1,...,m-1$,

$$\mathbb{D}_{(k)}^{<i>}v=\mathbb{D}_{(k)}^{<i>}\mathbb{I}_{(\kappa)}^{<m>}u=\mathbb{D}_{(k)}^{<i>}\mathbb{I}_{(\kappa)}^{<i>}\mathbb{I}_{(\kappa)}^{<m-i>}u=\mathbb{I}_{(\kappa)}^{<m-i>}u,$$

and thus the FDE from (4.20) for v becomes the following integral equation for u:

$$u(t)+\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{I}_{(\kappa)}^{<m-i>}u(t)=g(t),\qquad t\in[0,T]. \tag{4.24}$$

Now it is proved that if v in the function space (4.21) satisfies the initial value problem (4.20), then $u = \mathbb{D}_{(k)}^{<m>} v \in C_{-1}[0,T]$ satisfies the integral equation (4.24).

Conversely, let $u \in C_{-1}[0,T]$ be a solution of (4.24), and define $v = \mathbb{I}_{(\kappa)}^{<m>} u$. By the first fundamental theorem of fractional calculus for m-fold GFDs, we have $\mathbb{D}_{(k)}^{<m>} v = u$ and $\mathbb{D}_{(k)}^{<i>} v = \mathbb{I}_{(\kappa)}^{<m-i>} u$ for every $i = 0, 1, ..., m-1$, so the integral equation (4.24) gives the following FDE for v:

$$\mathbb{D}_{(k)}^{<m>} v(t) + \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} v(t) = g(t).$$

Since $\mathbb{I}_{(k)}^{<m>} v = \{1\}^{<m>} * u$ is the mth-order integral of u, it follows immediately that v is in the function space (4.21) and the initial conditions from (4.20) are valid, meaning that v satisfies the initial value problem (4.20), as required.

Thus, we have established a bijection between solutions v in the function space (4.21) of the initial value problem (4.20) and solutions $u \in C_{-1}[0,T]$ of the integral equation (4.24).

Part 2: constructing a contraction mapping. As before, we need to find a contraction mapping whose fixed points are exactly solutions of (4.24).

Let us define the operator $\mathscr{O}_m$ as follows:

$$\big(\mathscr{O}_m u\big)(t) = g(t) - \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{I}_{(\kappa)}^{<m-i>} u(t).$$

Clearly, $\mathscr{O}_m$ maps the space $C_{-1}[0,T]$ into itself, and the integral equation (4.24) is equivalent to $\mathscr{O}_m u = u$. We just need to show that $\mathscr{O}_m$ has a unique fixed point in the space $C_{-1}[0,T]$.

For any $t \in [0,T]$ and $u_1, u_2 \in C_{-1}[0,T]$, we have:

$$\mathscr{O}_m u_1(t) - \mathscr{O}_m u_2(t) = \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{I}_{(\kappa)}^{<m-i>} (u_2 - u_1)(t).$$

In Part 2 of the proof of Theorem 4.1, we showed that, for any $\kappa \in C_{-1}[0,T]$ and $r \in C[0,T]$ and for any sufficiently small $\alpha \in (-1,0)$ and sufficiently large $\lambda \gg 0$, the operator $r(t) \cdot \mathbb{I}_{(\kappa)}$ is bounded on the space $t^{\alpha} \cdot C[0,T]$ under the norm $\| \cdot \|_{\alpha,\lambda}$, with an operator norm that can be made as small as desired by choosing λ sufficiently large. Here, $\mathbb{I}_{(\kappa)}^{<m-i>}$ is also convolution with a function in $C_{-1}[0,T]$, namely the function $\kappa^{<m-i>}$, so every one of the terms $r_i(t) \cdot \mathbb{I}_{(\kappa)}^{<m-i>}$ can be bounded with an arbitrary small operator norm. Thus, by choosing $\lambda \gg 0$ sufficiently large for a fixed $\alpha \in (-1,0)$ sufficiently close to -1, we can ensure that $\mathscr{O}_m$ is a contraction mapping on the space $t^{\alpha} \cdot C[0,T]$ under the norm $\| \cdot \|_{\alpha,\lambda}$.

Part 3: existence, uniqueness, and solution as a limit. By the Banach fixed point theorem, the operator $\mathscr{O}_m$ has a unique fixed point in the space $t^{\alpha} \cdot C[0,T]$ for any sufficiently small $\alpha \in (-1,0)$, which means that the integral equation (4.24) has a unique solution in every such space. Since the union of these spaces over all such α is the space $C_{-1}[0,T]$, we have now proved that (4.24) has a unique solution $u \in C_{-1}[0,T]$. By the equivalence constructed in Part 1 of this proof, the initial value problem (4.20) has a unique solution v in the function space (4.21).

The Banach fixed point theorem also tells us that the solution function u is the limit, under the norm $\| \cdot \|_{\alpha,\lambda}$ on the space $t^{\alpha} \cdot C[0,T]$ for each sufficiently small $\alpha \in (-1,0)$, of the following sequence:

$$\begin{cases} u_0(t) = g(t) \\ u_n(t) = g(t) - \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{I}_{(\kappa)}^{<m-i>} u_{n-1}(t), \qquad n \geq 1. \end{cases} \tag{4.25}$$

The sequence $\left(t^{-\alpha} e^{-\lambda t} u_n(t)\right)$ converges uniformly on $[0,T]$, so the sequence (u_n) itself converges locally uniformly on $(0,T]$. By Part 2 of the proof, the m-fold GFI with Sonine kernel κ is a bounded operator on the space $t^{\alpha} \cdot C[0,T]$, so the sequence of functions $v_n = \mathbb{I}_{(\kappa)}^{<m>} u_n$ is also locally uniformly convergent on $(0,T]$ with limit $v = \mathbb{I}_{(\kappa)}^{<m>} u$. In other words, the following sequence:

$$\begin{cases} v_0(t) &= \mathbb{I}_{(\kappa)}^{<m>} g(t) \\ v_n(t) &= \mathbb{I}_{(\kappa)}^{<m>} g(t) - \mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{I}_{(\kappa)}^{<m-i>} u_{n-1}(t) \\ &= v_0(t) - \mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} v_{n-1}(t). \end{cases} \tag{4.26}$$

converges locally uniformly on $(0,T]$ to the unique solution function v in the space (4.21) of the initial value problem (4.20).

Part 4: explicit solution as a series. Let us write explicitly the first few functions in the sequence (v_n):

$$
\begin{aligned}
v_0(t) &= \mathbb{I}_{(\kappa)}^{<m>} g(t); \\
v_1(t) &= \mathbb{I}_{(\kappa)}^{<m>} g(t) - \mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \mathbb{I}_{(\kappa)}^{<m>} g(t) \\
&= \sum_{p=0}^{1} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \right)^p \mathbb{I}_{(\kappa)}^{<m>} g(t); \\
v_2(t) &= \mathbb{I}_{(\kappa)}^{<m>} g(t) - \mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \\
&\quad \times \left(\mathbb{I}_{(\kappa)}^{<m>} g(t) - \mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \mathbb{I}_{(\kappa)}^{<m>} g(t) \right) \\
&= \mathbb{I}_{(\kappa)}^{<m>} g(t) - \mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \mathbb{I}_{(\kappa)}^{<m>} g(t) \\
&\quad + \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \right)^2 \mathbb{I}_{(\kappa)}^{<m>} g(t) \\
&= \sum_{p=0}^{2} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \right)^p \mathbb{I}_{(\kappa)}^{<m>} g(t); \\
&\cdots
\end{aligned}
$$

By induction, it is straightforward to prove that for all $n \in \mathbb{N}$, we have

$$
v_n(t) = \sum_{p=0}^{n} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \right)^p \mathbb{I}_{(\kappa)}^{<m>} g(t),
$$

and thus, taking the limit,

$$
v(t) = \lim_{n \to \infty} v_n(t) = \sum_{p=0}^{\infty} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \right)^p \mathbb{I}_{(\kappa)}^{<m>} g(t).
$$

where the limit, and therefore the infinite summation, are locally uniformly convergent on $(0,T]$, since the integral of a uniformly convergent sum is also uniformly convergent. □

4.3.2 Homogeneous FDE under General Initial Conditions

In this subsection and the next one, an extra assumption will be needed on the Sonine kernel κ, in order for the dependence on initial conditions to work out as desired. This assumption is stated as follows, and in the proof of Theorem 4.8 we will see why it is needed.

Assumption 4.3.1: The Sonine kernel functions $\kappa, k \in C_{-1}[0,T]$ are assumed to be such that
$$\left(\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\kappa^{<m>}\right)\right)(0)=0, \qquad i=0,1,\cdots,m-1.$$

Theorem 4.8: Let $T\in\mathbb{R}^+$ and $m\in\mathbb{N}$ and $r_0,r_1,\cdots,r_{m-1}\in C[0,T]$, and let $\kappa,k\in C_{-1}[0,T]$ be a Sonine pair satisfying Assumption 4.3.1. For each $j=0,1,\cdots,m-1$, the following homogeneous FDE:
$$\mathbb{D}_{(k)}^{<m>}v_j(t)+\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}v_j(t)=0, \qquad t\in[0,T], \tag{4.27}$$
under the initial conditions
$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\mathbb{I}_{(k)}^{<m>}v_j\right)(0)=\delta_{ij}, \qquad i=0,...,m-1,$$
where δ_{ij} denotes the Kronecker delta taking the value 1 if $i=j$ and 0 otherwise, has a unique solution v_j in the function space (4.21), namely:
$$v_j(t)=\mathbb{D}_{(\kappa)}^{<m>}\left(\frac{t^j}{j!}\right)-\sum_{p=0}^{\infty}(-1)^p\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\right)^{p+1}\left(\mathbb{D}_{(\kappa)}^{<m>}\left(\frac{t^j}{j!}\right)\right), \tag{4.28}$$
where this series is locally uniformly convergent on $(0,T]$.

Proof. Our first task is to find a family of functions $\phi_0,\phi_1,\cdots,\phi_{m-1}$ satisfying
$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\mathbb{I}_{(k)}^{<m>}\phi_j\right)(0)=\delta_{ij}, \qquad i,j=0,...,m-1. \tag{4.29}$$
For this, it would suffice to have $\mathbb{I}_{(k)}^{<m>}\phi_j(t)=\frac{t^j}{j!}$, which implies $\phi_j(t)=\mathbb{D}_{(\kappa)}^{<m>}\left(\frac{t^j}{j!}\right)$, since it is well known that
$$\left[\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\frac{t^j}{j!}\right)\right]_{t=0}=\delta_{ij}, \qquad i,j=0,...,m-1. \tag{4.30}$$
Thus, we define $\phi_j(t)=\mathbb{D}_{(\kappa)}^{<m>}\left(\frac{t^j}{j!}\right)$ and check that (4.29) is satisfied. Writing $\frac{t^j}{j!}=\{1\}^{<j+1>}$ in the notation of convolutional algebra, and using the second fundamental theorem of fractional calculus in the form (4.23), we have:

$$
\begin{aligned}
&\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\mathbb{I}_{(k)}^{<m>}\phi_j\right)(0)\\
&\quad=\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\mathbb{I}_{(k)}^{<m>}\mathbb{D}_{(\kappa)}^{<m>}\left(\{1\}^{<j+1>}\right)\right)(0)\\
&\quad=\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\{1\}^{<j+1>}-\sum_{r=0}^{m-1}\frac{\mathrm{d}^r}{\mathrm{d}t^r}\left(\mathbb{I}_{(\kappa)}^{<m>}\left(\{1\}^{<j+1>}\right)\right)(0)\right.\\
&\qquad\qquad\qquad\qquad\left.\times\frac{\mathrm{d}^{m-r-1}}{\mathrm{d}t^{m-r-1}}\left(k^{<m>}(t)\right)\right)(0)\\
&\quad=\delta_{ij}-\sum_{r=0}^{m-1}\frac{\mathrm{d}^r}{\mathrm{d}t^r}\left(\kappa^{<m>}*\{1\}^{<j+1>}\right)(0)\cdot\frac{\mathrm{d}^{m+i-r-1}}{\mathrm{d}t^{m+i-r-1}}\left(k^{<m>}\right)(0)\\
&\quad=\delta_{ij}-\sum_{r=j+1}^{m-1}\frac{\mathrm{d}^{r-j-1}}{\mathrm{d}t^{r-j-1}}\left(\kappa^{<m>}\right)(0)\cdot\frac{\mathrm{d}^{m+i-r-1}}{\mathrm{d}t^{m+i-r-1}}\left(k^{<m>}\right)(0),
\end{aligned}
$$

which equals simply δ_{ij} when we use Assumption 4.3.1, because

$$
\frac{\mathrm{d}^{r-j-1}}{\mathrm{d}t^{r-j-1}}\left(\kappa^{<m>}\right)(0)=0 \qquad \text{whenever } 0\le j<r\le m-1.
$$

Now we have our family ϕ_j as required. Defining $w_j=v_j-\phi_j$, we see that the given initial value problem for v_j is equivalent to the following initial value problem for w_j:

$$
\begin{cases}
\mathbb{D}_{(k)}^{<m>}w_j(t)+\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}w_j(t)=-\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\phi_j(t),\\
\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\mathbb{I}_{(k)}^{<m>}w_j\right)(0)=0, \qquad i=0,...,m-1,
\end{cases}
\tag{4.31}
$$

where we used the fact that $\mathbb{D}_{(k)}^{<m>}\phi_j=0$ because $\mathbb{I}_{(k)}^{<m>}\phi_j=\frac{t^j}{j!}$ and $j\le m-1$.

Now, the initial value problem (4.31) is exactly of the form discussed in Theorem 4.7, with $g(t)$ being the function on the right-hand side. Hence, by using the result of Theorem 4.7, we have

$$
\begin{aligned}
w_j(t)&=\sum_{p=0}^{\infty}(-1)^p\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\right)^p\mathbb{I}_{(\kappa)}^{<m>}\left(-\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\phi_j(t)\right)\\
&=\sum_{p=0}^{\infty}(-1)^{p+1}\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\mathbb{D}_{(k)}^{<i>}\right)^{p+1}\phi_j(t),
\end{aligned}
$$

which gives the formula for v_j as stated, where we know from Theorem 4.7 that the sum is locally uniformly convergent on $(0,T]$. □

Corollary 4.1: Let $T \in \mathbb{R}^+$ and $m \in \mathbb{N}$ and $r_0, r_1, \cdots, r_{m-1} \in C[0,T]$ and $a_0, a_1, \cdots, a_{m-1} \in \mathbb{R}$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair satisfying Assumption 4.3.1. The following general homogeneous initial value problem:

$$\begin{cases} \mathbb{D}_{(k)}^{<m>} v(t) + \sum_{i=0}^{m-1} r_i(t) \mathbb{D}_{(k)}^{<i>} v(t) = 0, & t \in [0,T], \\ \dfrac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\mathbb{I}_{(k)}^{<m>} v_j \right)(0) = a_i, & i = 0, ..., m-1, \end{cases} \tag{4.32}$$

has a unique solution v in the space (4.21), namely:

$$v(t) = \mathbb{D}_{(\kappa)}^{<m>} \left(\sum_{j=0}^{m-1} a_j \frac{t^j}{j!} \right) - \sum_{p=0}^{\infty} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \mathbb{D}_{(k)}^{<i>} \right)^{p+1} \mathbb{D}_{(\kappa)}^{<m>} \left(\sum_{j=0}^{m-1} a_j \frac{t^j}{j!} \right).$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. The functions $v_0, v_1, \cdots, v_{m-1}$ found in Theorem 4.8 form a fundamental set of solutions for the homogeneous ODE (4.27). By the superposition principle, the unique solution of (4.32) is precisely $v = a_0 v_0 + a_1 v_1 + \cdots + a_{m-1} v_{m-1}$. From (4.28), we get the stated result. □

4.3.3 The General Case: Inhomogeneous FDE under General Initial Conditions

Theorem 4.9: Let $T \in \mathbb{R}^+$ and $m \in \mathbb{N}$ and $g \in C_{-1}[0,T]$ and $r_0, r_1, \cdots, r_{m-1} \in C[0,T]$ and $a_0, a_1, \cdots, a_{m-1} \in \mathbb{R}$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair satisfying Assumption 4.3.1. Then the general initial value problem (1.3) has a unique solution v in the space (4.21), namely:

$$v(t) = \sum_{p=0}^{\infty} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \right)^p \left(\mathbb{I}_{(\kappa)}^{<m>} g(t) + \mathbb{D}_{(\kappa)}^{<m>} \left(\sum_{j=0}^{m-1} a_j \frac{t^j}{j!} \right) \right),$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. By applying the superposition principle to the results of Theorem 4.7 and Corollary 4.1, we have

$$v(t)=\sum_{p=0}^{\infty}(-1)^p\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\right)^p\mathbb{I}_{(\kappa)}^{<m>}g(t)$$

$$+\mathbb{D}_{(\kappa)}^{<m>}\left(\sum_{j=0}^{m-1}a_j\frac{t^j}{j!}\right)-\sum_{p=0}^{\infty}(-1)^p\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\mathbb{D}_{(k)}^{<i>}\right)^{p+1}\mathbb{D}_{(\kappa)}^{<m>}\left(\sum_{j=0}^{m-1}a_j\frac{t^j}{j!}\right),$$

which gives the stated result after some manipulation of sums. □

As far as we know, multi-term fractional differential equations involving such m-fold GFDs of Riemann–Liouville type have not been solved before in the literature. Some of Luchko's papers deal with multi-term fractional equations involving m-fold GFDs, but these are either of Caputo type [21] or of sequential type [22,47], neither of which is the case that we are dealing with here. Thus, verifying consistency of the above result with previous work in the literature is only possible in the case $m=1$, which was already done in a previous section.

4.4 FDEs with *M*-fold GFDs in Caputo Sense

In this section, we will consider a multi-term FDE involving GFDs of Caputo type, namely the initial value problem given by (1.4), which is the same as the Riemann–Liouville problem of the previous section except with a different type of GFD operator and with different initial conditions. As in section 4.2, the initial conditions will now be of classical type and the part of the solution process that deals with them will be much easier.

Again, this section is divided into three parts: for the differential equation from (1.4) under a zero initial condition, the corresponding homogeneous differential equation under the general initial condition from (1.4), and finally solving (1.4) in general. We will speed up the work by noting similarities with the Riemann–Liouville problem of section 4.3 and reducing the new problem to one already solved wherever possible.

4.4.1 Inhomogeneous FDE under Zero Initial Conditions

Theorem 4.10: Let $T \in \mathbb{R}^+$ and $m \in \mathbb{N}$ and $r_0, r_1, \cdots, r_{m-1} \in C[0,T]$ and $g \in C[0,T]$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair satisfying Assumption 4.3.1. The following initial value problem:

$$\begin{cases} {}_*\mathbb{D}_{(k)}^{<m>} v(t) + \sum_{i=0}^{m-1} r_i(t)\, {}_*\mathbb{D}_{(k)}^{<i>} v(t) = g(t), \qquad t \in [0,T], \\ \left[\dfrac{\mathrm{d}^i}{\mathrm{d}t^i} v(t)\right]_{t=0} = v^{(i)}(0) = 0, \qquad i = 0,1,...,m-1, \end{cases} \tag{4.33}$$

has a unique solution in the function space

$$\left\{ v \in C_{-1}^m[0,T] \,:\, \mathbb{I}_{(k)}^{<m>} v \in C_{-1}^m[0,T] \right\}, \tag{4.34}$$

and this solution is precisely (4.22).

Proof. By Theorem 4.7 and its proof, it will be sufficient to construct a bijection between solutions v of the initial value problem (4.33) and solutions u of the integral equation (4.24), in some suitable function spaces.

Firstly, if v in the function space (4.34) satisfies the initial value problem (4.33), then we know from the proof of Theorem 4.7 that $\mathbb{D}_{(k)}^{<m>} v$ exists in $C_{-1}[0,T]$. By the definition (2.3) of the m-fold GFD of Caputo type, we have ${}_*\mathbb{D}_{(k)}^{<i>} v = \mathbb{D}_{(k)}^{<i>} v$ for each $i = 0,1,\cdots,m$. Defining then $u = {}_*\mathbb{D}_{(k)}^{<m>} v \in C_{-1}[0,T]$, the second fundamental theorem of fractional calculus [8, Theorem 7] gives $\mathbb{I}_{(\kappa)}^{<m>} u = v$, and then the first fundamental theorem of fractional calculus gives ${}_*\mathbb{D}_{(k)}^{<i>} v = \mathbb{D}_{(k)}^{<i>} v = \mathbb{I}_{(\kappa)}^{<m-i>} u$ for each $i = 0,1,\cdots,m-1$, so we have the integral equation (4.24) for $u \in C_{-1}[0,T]$ with $\mathbb{I}_{(\kappa)}^{<m>} u \in C_{-1}^m[0,T]$.

Conversely, if $u \in C_{-1}[0,T]$ satisfies (4.24) and $\mathbb{I}_{(\kappa)}^{<m>} u \in C_{-1}^m[0,T]$, then we know from the proof of Theorem 4.7 that $v = \mathbb{I}_{(\kappa)}^{<m>} u$ is in the space (4.34) and satisfies the Riemann–Liouville initial value problem (4.20). It remains only to prove the zero initial conditions from (4.33), because then the Riemann–Liouville type and Caputo type GFDs of v will be the same, and the Riemann–Liouville type FDE from (4.20) will become exactly the Caputo type FDE from (4.33). To prove the zero initial conditions for v, we need to use $v = \mathbb{I}_{(\kappa)}^{<m>} u = \kappa^{<m>} * u$ together with Assumption 4.3.1. We have

$$v'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \kappa^{<m>}(t-\tau) u(\tau)\,\mathrm{d}\tau = \kappa^{<m>}(0) u(t) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \kappa^{<m>}(t-\tau) u(\tau)\,\mathrm{d}\tau$$
$$= \left(\frac{\mathrm{d}}{\mathrm{d}t} \kappa^{<m>} \right) * u,$$

and similarly,

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i} v = \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i} \kappa^{<m>} \right) * u, \qquad i = 0, 1, \cdots, m-1.$$

Then, because $\kappa^{<m>} \in C_{-1}^m[0,T]$ with all of its derivatives up to order $m-1$ being zero at 0, the same is true for v, which gives the desired initial conditions.

Now we have established a bijection between solutions v of the initial value problem (4.33) in the space (4.34) and solutions u of the integral equation (4.24) in the space

$$\left\{ u \in C_{-1}[0,T] \,:\, \mathbb{I}_{(\kappa)} u \in C_{-1}^m[0,T] \right\}. \tag{4.35}$$

Thus, the result is proved as long as we can be sure that the unique solution $u \in C_{-1}[0,T]$ of the integral equation (4.24) is in the space (4.35): in other words, as long as the function defined by (4.22) is in $C_{-1}^m[0,T]$. And this is true, since $g \in C[0,T]$ together with Assumption 4.3.1 implies that $\mathbb{I}_{(\kappa)}^{<m>} g \in C_{-1}^m[0,T]$ by a similar argument to the one used above for the initial values of v, then we remain in this space no matter how many times we apply GFIs and multiply by r, so the entire sequence (v_n) is in $C_{-1}^m[0,T]$. $\square$

4.4.2 Homogeneous FDE under General Initial Conditions

Theorem 4.11: Let $T \in \mathbb{R}^+$ and $m \in \mathbb{N}$ and $r_0, r_1, \cdots, r_{m-1} \in C[0,T]$ and $a_0, a_1, \cdots, a_{m-1} \in \mathbb{R}$, and let $\kappa, k \in C_{-1}[0,T]$ be a Sonine pair satisfying Assumption 4.3.1. For each $j = 0, 1, \cdots, m-1$, the following homogeneous initial value problem:

$$\begin{cases} {}_{*}\mathbb{D}_{(k)}^{<m>} v(t) + \sum_{i=0}^{m-1} r_i(t) \cdot {}_{*}\mathbb{D}_{(k)}^{<i>} v(t) = 0, \qquad t \in [0,T], \\ \left[\frac{\mathrm{d}^i}{\mathrm{d}t^i} v(t) \right]_{t=0} = v^{(i)}(0) = a_i, \qquad i = 0, ..., m-1 \end{cases} \tag{4.36}$$

has a unique solution v in the space (4.34), namely:

$$v(t) = \sum_{j=0}^{m-1} \frac{a_j t^j}{j!} - \sum_{p=0}^{\infty} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<m>} \sum_{i=0}^{m-1} r_i(t) \cdot \mathbb{D}_{(k)}^{<i>} \right)^p \times \mathbb{I}_{(\kappa)}^{<m>} \left(\sum_{i,j\,:\,i \le j} a_j r_i(t) \cdot {}_{*}\mathbb{D}_{(k)}^{<i>} \left(\frac{t^j}{j!} \right) \right),$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. We will reduce (4.36) to an inhomogeneous FDE under zero initial conditions, in order to use Theorem 4.10. Let us define a function w by

$$w(t) = v(t) - \sum_{j=0}^{m-1} \frac{a_j t^j}{j!},$$

so that v is in the space (4.34) if and only if w is. By the relation (4.30) for derivatives of $t^j/j!$, we have $w^{(i)}(0) = v^{(i)}(0) - \sum_j a_j \delta_{ij} = a_i - a_i = 0$ for all $j = 0, 1, \cdots, m-1$. Also, ${}_{*}\mathbb{D}_{(k)}^{<i>} \left(t^j/j! \right) = 0$ whenever $i > j$, so the initial value problem (4.36) for v is equivalent to the following initial value problem for w:

$$\begin{cases} {}_{*}\mathbb{D}_{(k)}^{<m>} w(t) + \sum_{i=0}^{m-1} r_i(t) \cdot {}_{*}\mathbb{D}_{(k)}^{<i>} w(t) = - \sum_{i,j\,:\,i \le j} a_j r_i(t) \, {}_{*}\mathbb{D}_{(k)}^{<i>} \left(\frac{t^j}{j!} \right), \quad t \in [0,T], \\ \left[\frac{\mathrm{d}^i}{\mathrm{d}t^i} w(t) \right]_{t=0} = w^{(i)}(0) = 0, \qquad i = 0, ..., m-1, \end{cases}$$

where we note that the inhomogeneous term on the right-hand side is in $C[0,T]$ by [39, Lemma 1], because it is the convolution of k with a polynomial function in $C[0,T]$. Therefore, we can use Theorem 4.10, to find the following unique solution of the initial value problem for w:

$$w(t)=\sum_{p=0}^{\infty}(-1)^p\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\right)^p\mathbb{I}_{(\kappa)}^{<m>}\left(-\sum_{i,j\,:\,i\leq j}a_jr_i(t)\cdot{}_*\mathbb{D}_{(k)}^{<i>}\left(\frac{t^j}{j!}\right)\right).$$

This gives the stated result for v. Local uniform convergence of the series is guaranteed since we are using Theorem 4.10, where the solution is given by (4.22) which we already showed to be locally uniformly convergent on $(0,T]$ in the proof of Theorem 4.7. $\square$

4.4.3 The General Case: Inhomogeneous FDE under General Initial Conditions

Theorem 4.12: Let $T\in\mathbb{R}^+$ and $m\in\mathbb{N}$ and $r_0,r_1,\cdots,r_{m-1},g\in C[0,T]$ and $a_0,a_1,\cdots,a_{m-1}\in\mathbb{R}$, and let $\kappa,k\in C_{-1}[0,T]$ be a Sonine pair satisfying Assumption 4.3.1. Then the general initial value problem (1.4) has a unique solution v in the space (4.34), namely:

$$\begin{aligned}v(t)=\sum_{j=0}^{m-1}\frac{a_jt^j}{j!}+\sum_{p=0}^{\infty}(-1)^p&\left(\mathbb{I}_{(\kappa)}^{<m>}\sum_{i=0}^{m-1}r_i(t)\cdot\mathbb{D}_{(k)}^{<i>}\right)^p\\&\times\mathbb{I}_{(\kappa)}^{<m>}\left(g(t)-\sum_{i,j\,:\,i\leq j}a_jr_i(t)\cdot{}_*\mathbb{D}_{(k)}^{<i>}\left(\frac{t^j}{j!}\right)\right),\end{aligned}$$

where this series is locally uniformly convergent on $(0,T]$.

Proof. By applying the superposition principle to the results of Theorem 4.10 and Theorem 4.11. $\square$

To verify the consistency of the above result with previous work in the literature, we consider a particular case in the following example.

Example 4.3: Let us consider the case where $m=2$ and the coefficients $r_0(t)=2$, $r_1(t)=-3$ are constant. Then, we have the following initial value problem:

$$\begin{cases}{}_*\mathbb{D}_{(k)}^{<2>}v(t)-3\,{}_*\mathbb{D}_{(k)}v(t)+2v(t)=g(t), & t\in[0,T],\\[2ex] v(0)=a_0, \qquad v'(0)=a_1.\end{cases}\tag{4.37}$$

Our general solution from Theorem 4.12 becomes in this case:

$$v(t) = a_0 + a_1 t + \sum_{p=0}^{\infty} (-1)^p \left(\mathbb{I}_{(\kappa)}^{<2>}(-3\mathbb{D}_{(k)}^{<1>} + 2)\right)^p \mathbb{I}_{(\kappa)}^{<2>} \Big(g(t) - (a_0 r_0(t)$$

$$+ a_1 r_0(t) + a_1 r_1(t) \cdot {}_* \mathbb{D}_{(k)}(t)\Big)$$

$$= a_0 + a_1 t + \sum_{p=0}^{\infty} \left(3\mathbb{I}_{(\kappa)} - 2\mathbb{I}_{(\kappa)}^{<2>}\right)^p \mathbb{I}_{(\kappa)}^{<2>} \Big(g(t) - 2a_0 - 2a_1 t + 3a_1 k * \{1\}\Big).$$

To simplify the series here, we note that

$$\begin{aligned}
\sum_{p=0}^{\infty} \left(3\mathbb{I}_{(\kappa)} - 2\mathbb{I}_{(\kappa)}^{<2>}\right)^p h(t) &= \sum_{p=0}^{\infty} \left(3\kappa - 2\kappa^{<2>}\right)^{<p>} \\
&= \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} (3)^{p-q} \kappa^{<p-q>} * (-2)^q \kappa^{<2q>} \\
&= \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{q+r}{q} (-2)^q (3)^r \kappa^{<2q+r>} \\
&= \sum_{j=0}^{\infty} \left[\sum_{\substack{r,q \geq 0 \\ 2q+r=j}} \binom{q+r}{q} (-2)^q (3)^r \right] \kappa^{<j>},
\end{aligned}$$

and it is possible to observe that the sum in square brackets equals $2^{j+1} - 1$ for every value of j, so our solution reduces to

$$v(t) = a_0 + a_1 t + \sum_{j=1}^{\infty} (2^j - 1) \kappa^{<j+1>} * \Big(g(t) - 2a_0 - 2a_1 t + 3a_1 k * \{1\}\Big). \tag{4.38}$$

The solution found by Luchko [21, §5] is constructed in a more indirect way, relying on the partial fractions expansions of rational functions in an abstract field whose denominators are the polynomial corresponding to the left-hand side of the fractional differential equation. Here, the polynomial is $S^2 - 3S + 2$ and the partial fractions expansion for its inverse is

$$\frac{I}{S^2 - 3S + 2} = \frac{I}{S-2} - \frac{I}{S-1},$$

so, in the notation of [21, Eq. (5.25)], we have $l = 2$ and $p_1 = p_2 = 1$ and $\lambda_1 = 2, \lambda_2 = 1$ and $c_{11} = 1, c_{21} = -1$. Then, the function $G_{\kappa,n}$ from [21, Eq. (5.26)] is

$$G_{\kappa,n}(t) = l_{\kappa,2}(t) - l_{\kappa,1}(t),$$

where the function $l_{\kappa,\lambda}$ is the convolution series defined by (4.12). Continuing with [21, Eq. (5.30)], the expressions $P_i(S)$ are $P_0(S) = -3+S$ and $P_1(S) = 1$, giving the partial fractions expansions

$$\frac{S-3}{S^2-3S+2} = \frac{2I}{S-1} - \frac{I}{S-2} \qquad \text{and} \qquad \frac{I}{S^2-3S+2} = \frac{I}{S-2} - \frac{I}{S-1},$$

so, in the notation of [21, Eq. (5.31)], we have $d_{01} = -1$ and $d_{02} = 2$ and $d_{11} = 1, d_{12} = -1$. Finally, the solution function $v(t)$ found by Luchko [21, Eq. (5.34)] can now be written as:

$$\begin{aligned}
v(t) &= \big(g * G_{\kappa,n}\big)(t) + a_0\widetilde{y}_0(t) + a_1\widetilde{y}_1(t) \\
&= \big(g * l_{\kappa,2}\big)(t) - \big(g * l_{\kappa,1}\big)(t) + a_0\big(-L_{\kappa,2}(t) + 2L_{\kappa,1}(t)\big) + a_1 k * \big(L_{\kappa,2} - L_{\kappa,1}\big)(t),
\end{aligned}$$

where the function $L_{\kappa,\lambda}$ is the convolution series defined by (4.19). Using the definitions of $l_{\kappa,\lambda}$ and $L_{\kappa,\lambda}$, we have:

$$\begin{aligned}
v(t) = g * \sum_{j=0}^{\infty}(2^j-1)\kappa^{<j+1>} - a_0 + 2a_0 + a_0\{1\} * \sum_{j=1}^{\infty}(-2^j+2)\kappa^{<j>} \\
+ a_1 k * \{1\} * \sum_{j=1}^{\infty}(2^j-1)\kappa^{<j>} \\
= g * \sum_{j=1}^{\infty}(2^j-1)\kappa^{<j+1>} + a_0 + a_0\sum_{j=2}^{\infty}(2-2^j)\kappa^{<j>} * \{1\} + a_1 t \\
+ a_1 k * \{1\} * \sum_{j=2}^{\infty}(2^j-1)\kappa^{<j>} \\
= a_0 + a_1 t + \sum_{j=1}^{\infty}(2^j-1)\kappa^{<j+1>} * g - 2a_0\sum_{j=1}^{\infty}(2^j-1)\kappa^{<j+1>} * \{1\} \\
+ a_1\sum_{j=1}^{\infty}(2^{j+1}-1)\kappa^{<j>} * t,
\end{aligned}$$

where we have used the Sonine condition to know that $\kappa * k * \{1\} = t$. Comparing this with the solution (4.38) found by our approach, it remains only to prove that

$$\sum_{j=1}^{\infty}(2^j-1)\kappa^{<j+1>} * \Big(-2t + 3k * \{1\}\Big) = \sum_{j=1}^{\infty}(2^{j+1}-1)\kappa^{<j>} * t.$$

We can do this by using again the fact that $\kappa * k * \{1\} = t$ to simplify the left-hand side

as follows:
$$
\begin{aligned}
LHS &= -2\sum_{j=1}^{\infty}(2^j-1)\kappa^{<j+1>}*t+3\sum_{j=1}^{\infty}(2^j-1)\kappa^{<j>}*t\\
&= \sum_{j=2}^{\infty}(2-2^j)\kappa^{<j>}*t+\sum_{j=1}^{\infty}\big(3(2^j)-3\big)\kappa^{<j>}*t\\
&= 3\kappa*t+\sum_{j=2}^{\infty}\big(2(2^j)-1\big)\kappa^{<j>}*t\\
&= \sum_{j=1}^{\infty}\big(2^{j+1}-1\big)\kappa^{<j>}*t = RHS.
\end{aligned}
$$

Thus, finally, we have demonstrated the consistency between our result and that of Luchko [21], for the particular initial value problem (4.37).

Remark 4.1: Given the complex and lengthy calculations required for even the simple case considered above, with $m=2$ and $r_0(t)=2$ and $r_1(t)=-3$ for an easily factorisable polynomial, the reader may understand why we did not try to extend this verification of consistency to any further cases. The form given by Luchko [21] for the solution function is not fully explicit in terms of the coefficients, depending rather on some partial fractions expansions whose forms are not written explicitly in general, so it would be extremely difficult to check consistency of our results with his for the general m-fold multi-term FDE with constant coefficients. However, verifying consistency in a single non-trivial case is enough to give some confirmation of the correctness of our results.

Chapter 5

CONCLUSION

5.1 Conclusion

This thesis has the results of two papers which are given in Chapter 3 and 4. In the first work (Chapter 3), we have constructed a new convolution operator by combining generalised fractional integrals with Laplace-type convolution. This extends the work of Al-Kandari et al [35], who did the same thing in the particular case of Riemann–Liouville integration, and our more general convolution is valid on any of the C_α spaces that were defined by Dimovski [38] and used for fractional Mikusiński operational calculus.

The ideas of Sonine kernels and Mikusiński operational calculus have previously been combined in a different way: namely, by using the standard Mikusiński algebraic structure to solve fractional differential equations involving operators with Sonine kernels [21, 47]. Our work is different, because we used Sonine kernels to construct a new type of Mikusiński algebraic structure. There are at least two viewpoints from which to understand this work: looking at the purely algebraic structure of the new convolution rng, and investigating how it can be used to solve problems from analysis such as fractional differential equations.

From the algebraic viewpoint, we have examined the structure of the set C_α under the different convolution operations, and of the associated fields of fractions. We constructed embeddings of rngs and isomorphisms of fields which allow the recently discovered [39] algebraic properties of C_α to be extended to the new setting, essentially showing that the new algebraic structures are similar to the classical ones.

From the fractional calculus viewpoint, we have considered how the GFI and GFD can be interpreted in terms of the new convolution, and how new types of convolution series can be constructed using it. These results are new, but our investigations showed that they are not particularly useful in solving differential equations: we verified that solutions obtained using our new convolution are consistent with those previously obtained by Luchko [21], but using the new convolution did not enable us to solve any new problems in the area of fractional differential equations.

The advantage of this work has been to showcase the fusion of abstract algebra and fractional calculus together, following in the same line as previous work combining these two areas. Our work has generalised the previously published work of Al-Kandari et al [35], and constructed new embeddings and isomorphisms which demonstrate the limitations of this approach. We constructed an integral transform corresponding to the new convolution, but it is essentially a trivial variation of the Laplace transform, and therefore also of limited usefulness.

In the second work (Chapter 4), we have solved several types of linear ordinary differential equations with continuous variable coefficients, where the derivative operators are general fractional derivatives with Sonine kernels. This represents a significant step forwards in the theory of fractional calculus with Sonine kernels and the associated differential equations, following in the footsteps of Luchko and others who have worked on these operators and equations.

This work also represents a step forward in the theory of fractional differential equations with variable coefficients. Most of the previous works in this area have proved uniqueness of solutions only in spaces of continuous functions, but here we have proved the stronger result of uniqueness in a larger function space, by allowing integrable power-function-type singularities at the initial point of the domain. This was one of the major challenges of this work, since establishing a contraction mapping (in order to use the Banach fixed point theorem) was significantly harder than in previous works on this topic, due to the necessity of bounding an integral where the integrand is unbounded at both ends of the interval. This work, found in Part 2 of our Theorem (4.1) above, will be a useful contribution to the study of fractional differential equations with variable coefficients, more generally, representing a new technique that can be used in other papers in this area.

We have verified our results by comparison with previous work in the literature on constant-coefficient differential equations with Sonine kernel derivative operators [11, 21]. This ensures that our work is consistent with previous results of Luchko, at least in this particular case. As well as the standard GFDs using Sonine kernels k and κ, our work in Chapter 4 also covers the m-fold GFDs using convolutional powers of the same Sonine kernels k and κ.

A future direction of investigation would be to extend our results to the so-called Sonine kernels of arbitrary order, satisfying a modified (generalised) version of the Sonine condition [9], but this will be more challenging to manage for multi-term FDEs due to the lack of a semigroup property for such operators, unless we work under a restricting assumption of some relation between all the different Sonine kernels involved (i.e. assuming a sort of commensurate FDE). Another direction would be to consider other families of general kernels, of which there are many, unrelated to the particular family named after Sonine, as surveyed for example in [6]. Thus, it seems that research on fractional differential equations with variable coefficients will continue into the future, with yet more general fractional derivative operators being involved.

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