Fractional Differential Equations: An Approximate – Series Form Solution

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ABSTRACT

Modern society faces many nonlinear problems, which nonlinear equations are better suited to understand and address them. Despite having access to high-performance digital computers, we still struggle to find precise and superior solutions to nonlinear problems, particularly the analytical approximation than its numerical consequence.

The Aboodh transform iterative method, based on a new iterative method and the Aboodh transform, is the methodology we suggest in this thesis work to solve fractional differential equations, and the fractional order is taken into account by the Caputo operator.

The technique combines the Aboodh transform with a fresh iterative approach to produce a series-form solution with easily calculatable components. The decomposition method is suited to visible issues, solve nonlinear problems without linearization, perturbation, or discretization methods, yet requiring less computational work than the traditional conventional methods, the numerical. The nonlinearity and linearity terms are decomposed in a series form. A few examples that are sufficiently backed up by numerical evidences have been provided to show the effectiveness of the scheme. The graphical solution showed how the fractional order of the scheme affected the results.

In general, the solution profiles have demonstrated or shown that the scheme is almost exact and strength forward, simple to apply, and computationally less expensive.

The solutions to the fractional differential equation's exact problems are completely consistent with the findings, and the proposed scheme effectively and fully captures its true behavior and fractional effects.

Keywords: Fractional calculus; Caputo Fractional Derivative; Laplace Transform Iterative Method; Space-Time Fractional Derivative.

ÖZ

Modern toplum, doğrusal olmayan denklemlerin bunları anlamak ve ele almak için daha uygun olduğu birçok doğrusal olmayan problemle karşı karşıyadır. Yüksek performanslı dijital bilgisayarlara erişimimiz olmasına rağmen, doğrusal olmayan problemlere, özellikle sayısal sonuçlarından çok analitik yaklaşımlara, kesin ve üstün çözümler bulmakta hala mücadele ediyoruz.

Yeni bir yinelemeli yönteme ve Aboodh dönüşümüne dayanan Aboodh dönüşümü yinelemeli yöntemi, bu tez çalışmasında kesirli diferansiyel denklemleri çözmek için önerdiğimiz metodolojidir ve kesir sırası Caputo operatörü tarafından dikkate alınır.

Bu teknik, kolayca hesaplanabilen bileşenlerle seri formda bir çözüm üretmek için Aboodh dönüşümünü yeni bir yinelemeli yaklaşımla birleştirir. Ayrıştırma yöntemi, görünür sorunlar için uygundur, doğrusal olmayan sorunları doğrusallaştırma, pertürbasyon veya ayrıklaştırma yöntemleri olmadan çözer, ancak geleneksel geleneksel yöntemler olan sayısal yöntemlerden daha az hesaplama çalışması gerektirir. Doğrusal olmama ve doğrusallık terimleri bir seri biçiminde ayrıştırılır. Programın etkinliğini göstermek için sayısal kanıtlarla yeterince desteklenen birkaç örnek verilmiştir. Grafik çözüm, şemanın kesirli sırasının sonuçları nasıl etkilediğini gösterdi.

Genel olarak, çözüm profilleri, şemanın neredeyse kesin ve ileriye dönük, uygulanması basit ve hesaplama açısından daha ucuz olduğunu göstermiştir veya göstermiştir. Kesirli diferansiyel denklemin kesin problemlerinin çözümleri, bulgularla tamamen tutarlıdır ve önerilen şema, gerçek davranışını ve kesirli etkilerini etkili ve tam olarak yakalar.

Anahtar Kelimeler: Kesirli Hesap; Caputo Kesirli Türev; Laplace Dönüşümü İteratif Yöntemi; Uzay-Zaman Kesirli Türev.

DEDICATION

Dedicated to Angibeku Faith

and Agivyeh Ferdinand

I Love you ALL

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To write this component of my thesis, I really lack appropriate words from my English vocabulary. This is my best, appreciate, treasure and cherish it.

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INTRODUCTION

1.1 Motivation

The advantage of using fractional order differential equations over integer-order differential equations in modeling processes and materials like fluid flow, signal processing, and dynamic systems is that the fractional order model more accurately depicts or represents the process or material action. As a more complex and dynamic field with non-local derivatives, a strong connection to the theory of fractals, and naturally fractional properties in viscoelastic materials, fractional calculus is one of the great tools for modeling these processes or materials [1, 2, 3, 4].

Different methods for solving fractional differential equations were presented by a number of academics, including the kernel discretization method [5], the Chebyshev wavelet collocation method [6], q-homotopy analysis Shehu transform method [7], fractional variational iterative approach [8], iterative Laplace transform method [9] and reduced differential transform method (RDTM) [10].

The new iterative approach presented by Daftardar-Gejii and Jafari in 2006 [11] has been used to solve many integral and fractional order differential equations [12, 13, 14], but most of these methods considered a single term time-fractional order differential equation.

In this thesis, the major *aim* is to extend the Aboodh transform iterative method to solve space-time fractional differential equations with more than one term fractional derivative.

To the best of our knowledge, no attempt has been recorded regarding the approximate analytical solution of space-time fractional differential equations using the Aboodh transform iterative method which is the novelty of this study, is our contribution to knowledge.

1.2 Review of Connected Literature

In mathematics, both the analytical and numerical method, each has a certain role to play. Whenever a problem calls for a method to be chosen for a solution, it is best to choose one that combines simplicity, speed and accuracy, [15].

In the hunt for precise and dependable ways to solve fractional differential equations, a variety of methods have been put forth in the context of science and engineering.

The Adomian decomposition method (ADM), which was created by American mathematician and aerospace engineer George Adomian (1922–1966), solves fractional differential equations [16]. The technique handles boundary value problems and the solution of both linear and nonlinear differential equations, as seen in several scientific and engineering disciplines. The initial stage in solving a nonlinear problem, the calculation of Adomian polynomials for the nonlinear variables, is necessary for its implementation [17–19].

ADM is thought of as an iterative process, and the series solution to a genuine physical problem converges quickly and with few terms [20–22]. The power series solution has an advantage over conventional traditional classical techniques in that it avoids linearization, perturbation, or discretization of the problem while still providing an effective numerical solution with high accuracy and little calculation [23–24]. While the truncated series offers an approximation, the convergent series solution is absolute and uniform. All varieties of differential and integral equations can directly benefit from ADM. Because the method comprises polynomials that need more processing, and the power series' convergence regions are limited, acceleration techniques are required to increase the size of the convergence regions. Also, the approach's possible drawbacks include the inability to use various base functions, such as the δ-expansion method, the lack of a provision for modifying the convergence zone, and the rate at which solutions are approximated (ADM) are really some of its shortcomings.

Liao Shijun established the homotopy analysis approach as an analytical technique for nonlinear issues in his doctoral dissertation [25]. The approach offers a straightforward means of regulating and adjusting the convergence zone and pace of series solutions to nonlinear problems. The method works for nonlinear problems with strong linearity, and nonlinear problem solutions can be expressed using a variety of base functions. The phrase "homotopy analysis method" merely refers to a unified or generalized version of the aforementioned techniques, which include ADM, Lyapunov's artificial small parameter method, and δ-expansion method [26]. According to [26–28], insufficient account efficiency has been taken to approximate nonlinear problem solutions; as a result, new analytical methods must be developed that will be valid for strongly nonlinear problems even without small–large parameter, offer a convenient way of adjusting the convergence region and rate of approximation of series, and create that freedom of using different base functions for nonlinear problem approximation are some of the method shortcomings.

The operational calculus approach, commonly known as the integral transform, is a fantastic tool for solving differential equations in a closed form. Heaviside Oliver (1850–1925), an English physicist, developed it in his electromagnetic theorem in London in 1899 [29].

The Fourier and Laplace transforms are the two integral transforms that are frequently utilized in many areas of mathematics and research to solve both ordinary and partial differential equations [30]. Cosine and sine transform, Mellin, Radon, and Hilbert transforms, among others, are further integral transforms [29–30].

Integral transforms with modifications, such as the Natural transform [31], Elzaki transform [32], Z.Z transform [33], Kili'cman et-al transform [34], Kashuri and Fundo transform [35], Aboodh transform [36], and others, have been created and established to solve problems. To solve fractional differential equations, some of these integral transforms are used in combination with other numerical approaches. The conclusion of the solution profile of the instances solved in [37], is application of the Mohand transformation method with ADM, established a good correlation with the precise solution.

The Fisher-Kolmogorov second order partial nonlinear differential equations published in [38] and certain significant systems of fractional-order partial differential equations were both solved using the Laplace-Adomian decomposition technique (LADM). LADM combines the advantages of the Adomian decomposition method and the Laplace transformation without incorporating the customary arbitrary constant or function of integration, Laplace transform methods enable one to quickly arrive at approximation equation solutions. Additionally, it taught us that two functions must be equal if they have the same Laplace transform [29,38].

The semi-analytical variational iteration method (VIM) can solve both linear and nonlinear ODEs and PDEs. Shou et al. [39] introduced the technique. The approximation by variational iteration method, which is flexible and has the advantage of being able to solve nonlinear equations with ease, is the necessary remedy because majority of fractional differential equations lack accurate analytic solutions. The technique minimizes the amount of the calculation and gets around the challenge of handling nonlinear terms. One of the potentials of VIM was the development of predictor corrector functional differential equations employing the general Lagrange multiplier parameter to solve nonlinear equations in quantum mechanics. Some aspects of VIM include the complete elimination of the difficulty often associated with the computation of Adomian polynomials and the use of the solution of the linearized equation as an approximation to the original problem are some of its merit [28].

He's variational iteration technique (VIM) and ADM are used to solve the fractional version of the Kdv-Burgers-Kuramoto equation. The ability and capacity to handle the solution of integral equations has been demonstrated by the HAM modified technique. The results of the HAM method and the solution profiles from the two approaches agreed with each other completely [28].

1.3 Fractional Calculus

The area of applied analysis known as fractional calculus deals with the derivatives of arbitrary orders. Outside of a collection of whole numbers, it is a generalization of differentiation and integration [30]. A branch of mathematics known as fractional calculus generalizes the classical derivatives and integrations for all orders rather than only for integer orders [40].

Fractional Calculus is a more complex and dynamic field that is researched not only by pure mathematicians but also by physicists, biologists, chemists, and engineers because of how important many of its applications are in various fields of science and engineering. As a result, fractional derivatives are non-local, have a connection to the chaotic theory's application of the fractal theory, and viscoelastic materials naturally exhibit fractional properties [30].

1.4 Brief History of Fractional Calculus

The conversations between Gottfried Wilhelm Leibniz (1646-1716) and Marquis de L' Hospital (1661-1704), which took place on September 30, 1695, are credited with the invention of fractional calculus (F.C.). see [40,41].

F.C is as old as Calculus itself. Due to lack of accessibility to its mathematical tools, nothing was known about it for long. Leibniz's first non-integer order of differentiation was 1/2, which may be where the term *"fractional calculus"* originated from [30], when L'Hospital asked for the nth derivative of a linear function f(t)= t when n=1/2. Leibniz had a unique insight into the unknown. He stumbled onto fractional derivatives realizing that one-day great things will come from his work. What they will be, he had no idea simply replied: *"this is an apparent paradox from which, one day, useful consequences will be drawn".*

P. S. Laplace (1812), S. F. Lacroix (1819), J. B. J. Fourier (1822), N. H. Abel (1823–1826), J. Liouville (1832), Riemann (1847), A. K. Grunwald (1867), A. B. Letnikov (1868), N. Ya. Sonin (1869), H. Laurent (1884), O. Heaviside (1892-1912), A. Erdelyi

(1938-1965), H. Weyl (1917), H.T. Davis (1924–1936), J. A. Hardy and J. E. Littlewood (1917–1928), E. R. Love (1938–1996), L. Euler (1730), and Caputo (1930), among others were just a few of the notable individuals who made contributions to this field in the 19th century [41].

When B. Rose held the first conference as a specialty field where its true practical and theoretical benefits, its application drive the need for specialization, fractional calculus became a province in mathematical analysis [41]. When modeling physical occurrences, fractional derivatives produce superior results than integer order derivatives, with a higher percentage of flexibility as a potent, trustworthy tool for reality description [42]. The errors incurred for parameters overlooked in modeling real-life issues are minimized since F.C. captures phenomena and qualities that the traditional integer order could not or neglected [40]. See [40, 41] for more applications of F.C. The most widely used definition of F.C. was that of Gruwald-Letnikov, followed by that of Riemann-Liouville, Caputo, and Riesz-Feller.

In terms of history, F.C. expansion is elegantly safeguarded in the texts of Oldham and Spanier [43] and Samko, Kilbas, and Marichev's enormous 1987 volume titled *"Encyclopedia"* of Fractional Calculus [44].

1.4.1 Fractional Order Integral Operator of Riemann-Liouville

The following formula denotes the Riemann-Liouville fractional order integral operator [4]:

$$I^{\alpha}\psi(\tau) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau - r)^{\alpha-1}\psi(r)dr\,, & \alpha > 0 \\ \\ \psi(\tau) & \alpha = 0 \end{cases} \tag{1.1}$$

See [4] for some important properties of Riemann-Liouville fractional order integral operator.

1.4.2 Operator for Caputo's Fractional Order Derivatives

According to [4], the Caputo fractional order derivative operator is:

$$D_\tau^\alpha \psi(\tau) = I^{n-\alpha}\left(\frac{d^n}{d_\tau^n}\psi(\tau)\right),\ n-1<\alpha\le n, n\in\mathbb{N} \tag{1.2}$$

Also, some important properties of Caputo fractional order derivative operator can be located in [4].

1.4.3 Caputo Derivative of Order $\alpha > 0$ for Time Fractional is:

$$D_t^\alpha \psi(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\,\psi^{(n)}(x,\tau)d\tau,\ n-1<\alpha\le n \tag{1.3}$$

1.4.4 Caputo Derivative of Order $\beta > 0$ for Space Fractional is:

$$D_x^\beta \psi(x,t) = \frac{1}{\Gamma(n-\beta)}\int_0^x (x-t)^{n-\beta-1}\psi^{(n)}(x,t)dt,\quad n-1<\beta\le n \tag{1.4}$$

Remark 1.1 $D_t^\alpha \psi(x,t) = D_x^\beta \psi(x,t) = 0$, whenever $\psi(x,t)$ is a constant.

Remark 1.2 $D_t^\alpha t^b = \begin{cases} \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)} t^{b-\alpha}, & if\ n-1<\alpha\le n, b>\alpha-1 \\ 0, & if\ n-1<\alpha\le n,\ b\le\alpha-1 \end{cases}$

1.4.5 Mittag- Leffler Function

It is a special function that often occurs naturally in the solution of fraction order calculus. Is a generalization of exponential functions.

Other base functions are gamma function, beta function, error function, Mellin- Rose function, and many fractional differential equations' solutions are written in some of these special functions. Shukla and Prajapati provided a four parameter Mittag-Leffler function in 2007 [45].

$E^{p,q}_{\mathcal{B},\alpha(Z)} = \sum_{k=0}^{\infty} \frac{Z^k (p)qk}{\Gamma(k\beta+\alpha)k!}$ $, \alpha, \beta, p \in \mathbb{C}, q \in (0,1)u\mathbb{N}$,where Re$(\beta, p) > 0$ and $(p)qk = \frac{\Gamma(p+qk)}{\Gamma(p)}$

Exponential function e^z, the Mittag-Leffler function $E_\alpha(z)$, Wiman function $E_{\alpha,\beta}(z)$ and Prabhakar function $E^u_{\alpha,\beta}(z)$ are particular cases of Shukla function $E^{p,q}_{\alpha,\beta(z)}$ [45].

1.5 Summary of this Thesis

While the novel iterative approach was the brainchild of Daftardar-Gejji and Jafari, coupling of the Aboodh transform was a modification of the Laplace transform [11, 46].

We look for series approximate and analytical solutions for a variety of fractional differential equations, such as the nonlinear space-time fractional Fokker-Planck equation, the one-dimensional space-time diffusion equation, the space-time fractional Airy's partial differential equations, the nonlinear space-time fractional Fokker-Planck equation of a single term of time fractional order and three terms of space fractional order and the solution of the space-time fractional Airy's-like equation with an additional term, model.

The thesis is structured as follows: Some well-known findings and fundamental details of the Aboodh iterative approach are discussed in Chapter 2. We present fractional models of a few solutions to space-time fractional differential equations in Chapter 3. Conclusions and some suggestion(s) for additional research is/are found in Chapter 4.

Chapter 2

ABOODH TRANSFORM ITERATIVE METHOD

2.1 Section Outlook

A detail description presentation of this Chapter is devoted to Aboodh transform, Aboodh transform iterative method and for Aboodh convergence analysis see Daftardar-Gejji & Jafari, [11], Bhalekar and Daftardar-Gejji, [46], Ojo [47], Ojo and Mahmudov [12,48]

2.2 Aboodh Transform

The Aboodh transform is one of the operational calculus methods proposed by K. S Aboodh in 2013 defined in [36] as a modification of the Laplace transform whose integral transforms are defined in the time domain $t \geq 0$, and the function $\psi(t)$ is bounded with exponential rate of growth for convergence purpose. It enjoys the advantage of Laplace methods of taking one directly to the solution of the approximate problem without the introduction of conventional constant(s) or function(s) of integration [29].

Definition 2.1 (Aboodh Transform): The Aboodh transform for the function $\psi(t)$ is defined for functions of exponential order over the group of functions, see [36]. Consider the set

$$\mathcal{A} = \{\psi : |\psi(t)| < Me^{p_j|t|}, \text{ if t} \in (-1)^j x[0,\infty[, \text{ j=1,2, } (M, p_{1,} p_2 > 0)\}, \quad (2.1)$$

For any given function in set $\mathcal{A}$, $p_1 < p_2$, $p_1, p_{2.}$ may or may not be finite but M must be finite number.

The Aboodh transform of $\psi(t)$ is denoted by $\mathcal{A}[\psi(t)] = \phi(\varphi)$ (2.2)

and its integral transform version is defined by

$$\mathcal{A}[\psi(t)] = \frac{1}{\varphi}\int_0^\infty \psi(t)e^{-\varphi t}\,\mathrm{dt} = \phi(\varphi),\ t \geq 0, p_1 \leq p_2 \quad (2.3)$$

Definition 2.2 (Inverse Aboodh transform of the function $\psi(t)$): If

$\mathcal{A}[\psi(t)] = \phi(\varphi)$

then the inverse Aboodh transform of the function $\psi(t), t \in (0, \infty)$ is defined as [36]

$$\psi(t) = \mathcal{A}^{-1}[\phi(\varphi)] \quad (2.4)$$

Theorem 2.1 ([**12**]): If $\phi(\varphi)$ is the Aboodh transform of $\psi(t)$ then

$$\mathcal{A}[\psi'(t)] = \varphi\phi(\varphi) - \frac{\psi(0)}{\varphi} \quad (2.5)$$

$$\mathcal{A}[\psi''(t)] = \varphi^2\phi(\varphi) - \frac{\psi'(0)}{\varphi} - \psi(0) \quad (2.6)$$

⋮

$$\mathcal{A}[\psi^n(t)] = \varphi^n\phi(\varphi) - \sum_{\alpha=0}^{n-1}\frac{\psi^\alpha(0)}{\varphi^{2-n+\alpha}} \quad (2.7)$$

Proof: Using equation (2.3)

$$\mathcal{A}[\psi'(t)] = \frac{1}{\varphi}\int_0^\infty \psi'(t)e^{-\varphi t}dt = \phi(\varphi)$$

employing integration by parts, we have it as:

$$\frac{1}{\varphi}\int_0^\infty \psi'(t)e^{-\varphi t}dt = \varphi\phi(\varphi) - \frac{\psi(0)}{\varphi}$$

In the same vein $\mathcal{A}[\psi''(t)] = \frac{1}{\varphi}\int_0^\infty \psi''(t)e^{-\varphi t}dt$

Using integration by parts, we have

$$\frac{1}{\varphi}\int_0^\infty \psi''(t)e^{-\varphi t}dt = \varphi^2\phi(\varphi) - \frac{\psi'(0)}{\varphi} - \psi(0).$$

Therefore, for the n-th derivative, we show the proof by employing mathematical induction. So

$$\mathcal{A}[\psi^{n}(t)] = \varphi^{n}\phi(\varphi) - \sum_{\alpha=0}^{n-1}\frac{\psi^{(\alpha)}(0)}{\varphi^{2-n+\alpha}},\ n \geq 1.$$

By induction, we consider the case when n=1, and have it as:

$$\mathcal{A}[\psi'(t)] = \varphi\phi(\varphi) - \frac{\psi(0)}{\varphi}$$

It holds for n=1. Consider when n=k, we get

$$\mathcal{A}[\psi^{k}(t)] = \varphi^{k}\phi(\varphi) - \sum_{\alpha=0}^{k-1}\frac{\psi^{(\alpha)}(0)}{\varphi^{2-k+\alpha}}$$

Since it holds for n=k, we need to show that it also holds for n=k +1 that is

$$\mathcal{A}\left[\psi^{(k+1)}(t)\right] = \varphi^{k+1}\phi(\varphi) - \sum_{\alpha=0}^{k}\frac{\psi^{(\alpha)}(0)}{\varphi^{2-(k+1)+\alpha}}$$

we see that:

$$\mathcal{A}[\psi^{k+1)}(t)] = \mathcal{A}[(\psi^{(k)}(t))']$$

$$= \varphi\mathcal{A}\left[\psi^{(k)}(t)\right] - \frac{\psi^{(\alpha)}(0)}{\varphi}.$$

Therefore

$$\mathcal{A}\left[\psi^{(k+1)}(t)\right] = \varphi^{k+1}\phi(\varphi) - \sum_{\alpha=0}^{k-1}\frac{\psi^{(\alpha)}(0)}{\varphi^{2-k+\alpha-1}} - \frac{\psi^{(\alpha)}(0)}{\varphi}$$

$$= \varphi^{k+1}\phi(\varphi) - \sum_{\alpha=1}^{k}\frac{\psi^{(\alpha)}(0)}{\varphi^{2-(k+1)+\alpha}}$$

When n=k+1 and n-1=k, then

$$\mathcal{A}\left[\psi^{(n)}(t)\right] = \varphi^{n}\phi(\varphi) - \sum_{\alpha=0}^{n-1}\frac{\psi^{(\alpha)}(0)}{\varphi^{2-n+\alpha}}$$

Lemma 2.1: As in [49], Caputo time-fractional derivative of order β is transformed by Aboodh as

$$\mathcal{A}\left[\left(D_t^{\beta}\psi(t)\right):\varphi\right] = \varphi^{\beta}\mathcal{A}[\psi(t)] - \sum_{r=0}^{n-1}\frac{\psi^{(r)}(0)}{\varphi^{2-\beta+r}}\ , n-1 < \beta \leq n, n \in \mathbb{N} \tag{2.8}$$

Lemma 2.2 (Linearity Property of Aboodh transform). Suppose the Aboodh transform of the functions $\psi_1(t)\ and\ \psi_2(t)$ are $\eta(\varphi) and\ \lambda(\varphi)$ in that order [50], then

$$\mathcal{A}[a\psi_1(t) \mp b\psi_2(t)] = \frac{1}{\varphi}\int_0^\infty e^{-\varphi t}\{a\psi(t) \mp b\psi_2(t)\}dt, \;\; t \geq 0, \varphi \,\epsilon (p_1, p_2)$$

$$=a\left(\frac{1}{\varphi}\int_0^\infty e^{-\varphi t}\,\psi_1(t)dt\right) \mp b\left(\frac{1}{\varphi}\int_0^\infty e^{-\varphi t}\,\psi_2(t)dt\right)$$

$$= a\mathcal{A}[\psi_1(t)] \mp b\mathcal{A}[\psi_2(t)]$$

$$= a\eta(\varphi) \mp b\lambda(\varphi) \tag{2.9}$$

where a and b are arbitrary constants.

Table 2.1: Aboodh transform table [49]

$\psi(t)$	$\mathcal{A}[\psi(t) = \phi(\varphi)$
1	$\frac{1}{\varphi^2}$
t	$\frac{1}{\varphi^3}$
t^n	$\frac{n!}{\varphi^{n+2}}$, n = 0, 1, 2, …
t^β	$\frac{\Gamma(\beta+1)}{\varphi^{\beta+2}}$

2.3 Novel Iterative Approach

Numerous problems in science, engineering, and real-world situations can be solved using integral equations.

The decomposition method, which Daftardar Gejji and Jafari first developed in 2006 [11] to solve functional equations, is the novel iterative technique. This approach has many advantages over more typical numerical methods because it does not call for discretizing the state variables, which can cause rounding off errors and reduce the precision of a problem's solution. It doesn't demand a lot of computer time or memory.

The method avoids or escapes from the rigorous manipulations and calculations of Adomian polynomials by just requiring a small number of calculations with realistic results [11].

2.3.1 The New Iterative Method's Premise

We take into account and investigate the generic functional equation

$$\psi = f + N(\psi) \tag{2.10}$$

where f is a known function and N is either linear or nonlinear operator defined from a Banach space B mapped to itself i.e B⟶B. Assume the answer to Equation (2.10) is expressed as an infinite series:

$$\psi = \sum_{i=0}^{\infty} \psi_i \tag{2.11}$$

Therefore, the N is decomposed thus:

$$N\left(\sum_{i=0}^{\infty} \psi_i\right) = N(\psi_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} \psi_j\right) - N\left(\sum_{j=0}^{i-1} \psi_j\right)\right\}, \tag{2.12}$$

Using Equations (2.12), (2.11) in (2.10), we express

$$\sum_{i=0}^{\infty} \psi_i = f + N(\psi_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} \psi_j\right) - N\left(\sum_{j=0}^{i-1} \psi_j\right)\right\}. \tag{2.13}$$

We further generate the following recurrence relation

$$\left.\begin{array}{l} \psi_0 = f \\ \psi_1 = N(\varphi_0) \\ \vdots \\ \psi_{i+1} = N(\psi_0 + \psi_1 + \cdots + \psi_i) - N(\psi_0 + \psi_1 + \cdots + \psi_{i-1}), \mathrm{i} = 1,2\,.\,.\,. \end{array}\right\} \tag{1.24}$$

2.4 Basic Concept of the Aboodh Transform Iterative Method

The space-time partial differential equation of the form below we take into consideration and analyze.

$$D_t^{\alpha}\psi(x,t) = \Phi\left(\psi(x,t), D_x^{\beta}\psi(x,t), D_x^{2\beta}\psi(x,t), D_x^{3\beta}\psi(x,t)\right), m-1 < \alpha < m,$$

$$0 < 3\beta < m-1 \tag{2.15}$$

with initial conditions

$$\psi^{(k)}(x,0) = h_k\ ,\ \ k = 0,1,2,\dots,m-1, \tag{2.16}$$

where $\psi(x,t)$ from Equation (2.15) is the unknown parameter to be determined and $\Phi\left(\psi(x,t), D_x^{\beta}\psi(x,t), D_x^{2\beta}\psi(x,t), D_x^{3\beta}\psi(x,t)\right)$ can be linear, nonlinear or both, operator of $\psi(x,t), D_x^{\beta}\psi(x,t), D_x^{2\beta}\psi(x,t), and\ D_x^{3\beta}\psi(x,t)$

We denote $\psi(x,t)\ with\ \psi,$ for ease and simplicity, and by utilizing the initial conditions on the Aboodh transform to both sides of Equation (2.15), we get the following result:

$$\mathcal{A}[\psi(x,t)] = \frac{1}{v^{\alpha}}\left(\sum_{k=0}^{m-1}\frac{\psi^{(k)}(x,0)}{v^{2-\alpha+k}} + \mathcal{A}\left[\Phi\left(\psi, D_x^{\beta}\psi, D_x^{2\beta}\psi, D_x^{3\beta}\psi\right)\right]\right),\ m\in\mathbb{N} \tag{2.17}$$

To further simplify this, we apply the Aboodh transform inverse to each side of Equation (2.17), and the result is

$$\psi(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\sum_{k=0}^{m-1}\frac{\psi^{(k)}(x,0)}{v^{2-\alpha+k}} + \mathcal{A}\left[\Phi\left(\psi, D_x^{\beta}\psi, D_x^{2\beta}\psi, D_x^{3\beta}\psi\right)\right]\right)\right] \tag{2.18}$$

The solution is provided in the form of an infinite series by the Aboodh transform iterative approach.

$$\psi(x,t) = \sum_{i=0}^{\infty}\psi_i \tag{2.19}$$

Applying $\Phi\left(\psi, D_x^{\beta}\psi, D_x^{2\beta}\psi, D_x^{3\beta}\psi\right)$ on both sides of Equation (2.19) gives

$$\Phi\left(\psi(x,t)\right) = \Phi\left(\sum_{i=0}^{\infty}\psi_i\right)$$

$$\Phi\left(\psi, D_x^{\beta}\psi, D_x^{2\beta}\psi, D_x^{3\beta}\psi\right) = \Phi\left(\varphi_{\psi}, D_x^{\beta}\psi_0, D_x^{2\beta}\psi_0, D_x^{3\beta}\psi_0\right) +$$

$$\sum_{i=1}^{\infty}\{\Phi\left(\sum_{k=0}^{i}(\psi_k, D_x^{\beta}\psi_k, D_x^{2\beta}\psi_k, D_x^{3\beta}\psi_k)\right) -$$

$$\Phi\left(\sum_{k=0}^{i-1}(\psi_k, D_x^{\beta}\psi_k, D_x^{2\beta}\psi_k, D_x^{3\beta}\psi_k)\right)\} \tag{2.20}$$

Equation (2.18) becomes the following when Equation (2.20) and Equation (2.19) are substituted:

$$\sum_{i=1}^{\infty}\psi_i(x,t) = \mathcal{A}^{-1}[\frac{1}{v^{\alpha}}(\sum_{k=0}^{m-1}\frac{\psi^{(k)}(x,0)}{v^{2-\alpha+k}} + \mathcal{A}\left[\Phi\left(\psi_0, D_x^{\beta}\psi_0, D_x^{2\beta}\psi_0, D_x^{3\beta}\psi_0\right)\right])]$$

$$+\mathcal{A}^{-1}[\frac{1}{v^{\alpha}}(\mathcal{A}[\sum_{i=1}^{\infty}\{\Phi(\sum_{k=0}^{i}(\psi_k, D_x^{\beta}\psi_k, D_x^{2\beta}\psi_k, D_x^{3\beta}\psi_k))\}])]$$

$$-\mathcal{A}^{-1}[\frac{1}{v^{\alpha}}(\mathcal{A}[\sum_{i=0}^{\infty}\{\Phi(\sum_{k=1}^{i-1}(\psi_k, D_x^{\beta}\psi_k, D_x^{2\beta}\psi_k, D_x^{3\beta}\psi_k))\}])] \text{ , m=1, 2, …} \quad (2.21)$$

From Equation (2.21), we recursively deduce the following iterations

$$\psi_0(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\sum_{k=0}^{m-1}\frac{\psi^{(k)}(x,0)}{v^{2-\alpha+k}}\right]$$

$$\psi_1(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[\Phi\left(\psi_0, D_x^{\beta}\psi_0, D_x^{2\beta}\psi_0, D_x^{3\beta}\psi_0\right)\right]\right)\right]$$

⋮

$$\psi_{m+1}(x,t) = \mathcal{A}^{-1}[\frac{1}{v^{\alpha}}(\mathcal{A}[\sum_{i=0}^{\infty}\{\Phi(\sum_{k=0}^{i}(\psi_k, D_x^{\beta}\psi_k, D_x^{2\beta}\psi_k, D_x^{3\beta}\psi_k))\}])]$$

$$-\mathcal{A}^{-1}[\frac{1}{v^{\alpha}}(\mathcal{A}[\sum_{i=0}^{\infty}\{\Phi(\sum_{k=1}^{i-1}(\psi_k, D_x^{\beta}\psi_k, D_x^{2\beta}\psi_k, D_x^{3\beta}\psi_k))\}])] \text{, m=1,2, …} \quad (2.22)$$

The m-term analytically approximate the solution of Equation (2.15) is given by

$$\psi(x,t) \approx \sum_{i=0}^{m-1}\psi_i \quad (2.23)$$

2.5 Convergence of the Iterative Aboodh Transform Method

We prove a theorem and for Aboodh transform iterative method's convergence see [11, 12, 46, 47, 48].

Theorem 2.2 ([54]): If $\mathcal{A}[\psi(t)], and\ \mathcal{A}[\|\psi(t)\|]$ exist, where $\mathcal{A}[\psi(t)]$ is the Aboodh transform for any function ψ in the Banach space B, then

$$\|\mathcal{A}[\psi(t)]\| \leq \mathcal{A}[\|\psi(t)\|],$$

Proof:

From the Aboodh transform definition of a function $\psi(t)$ given by

$$\mathcal{A}[\psi(t)] = \frac{1}{v}\int_0^\infty \psi(t)e^{-vt}\,dt,$$

then

$$\|\mathcal{A}[\psi(t)]\| = \left\|\frac{1}{v}\int_0^\infty \psi(t)e^{-vt}\,dt\right\|$$

$$\leq \frac{1}{v}\int_0^\infty \|\psi(t)e^{-vt}\|\;dt$$

$$= \frac{1}{v}\int_0^\infty \|\psi(t)\|\,e^{-vt}dt$$

$$= \mathcal{A}[\|\psi(t)\|]$$

$$\Rightarrow \|\mathcal{A}[\psi(t)]\| \leq \mathcal{A}[\|\psi(t)\|].$$

Chapter 3

SPACE-TIME FRACTIONAL DIFFERENTIAL EQUATIONS SOLUTION USING ABOODH ITERATIVE TECHNIQUE

3.1 Chapter Summary

The major goal of this chapter is to use the iterative Aboodh transform method to solve space-time fractional differential equations using a model with fractional orders in both space and time as described in [52]. The numerous graphs generated agreed with the fractional order solution's convergence study. We use the following notations:

$\psi_e(x,t)$: The exact solution of the components of infinite series solution when $\alpha = \beta = 1$

$\gamma^{(n)}(x,t), n = 3$: The series approximate solution for various values of $\alpha = \beta$ at the 3rd approximate.

3.2 Problem Statement

We take into account and investigate the space-time partial differential equation of the type

$$D_t^{\alpha}\psi(x,t) = \Phi(\psi(x,t), D_x^{\beta}\psi(x,t), D_x^{2\beta}\psi(x,t), D_x^{3\beta}\psi(x,t)),\ 0 < \alpha, \beta \leq 1 \quad (3.1)$$

with initial conditions

$$\varphi^{(k)}(x,0) = h_k,\ \text{k=0, 1, 2, …m-1} \quad (3.2)$$

$\psi(x,t)$ is unknown function to be determined (3.3)

$\Phi(\psi(x,t), D_x^{\beta}\psi(x,t), D_x^{2\beta}\psi(x,t), D_x^{3\beta}\psi(x,t))$ can be both linear and nonlinear or one of them. (3.4)

$D_x^{2\beta}, D_x^{3\beta}$ where 2, 3 are the constant coefficients of $\beta,$ of space order (3.5)

D_t^{α} denotes the time-fractional differential operator in Caputo sense (3.6)

D_x^{β} denotes the Caputo sense space-fractional differential operator (3.7)

$\psi(x,t) = \sum_{i=0}^{\infty} \psi_i$ is an infinite series of iterative Aboodh transform of Equation (3.1) (3,8)

3.3 Explanatory Applications

This section's focus is on demonstrating the viability and effectiveness of the Aboodh iterative approach in the context of five different space-time fractional differential equations with different initial conditions.

3.3.1 Application 1

Consider the fractional Air's-like equation with an additional term [10]

$$D_t^{\alpha}\psi(x,t) = D_x^{\beta}\psi(x,t) + \psi(x,t), 0 < \alpha, \beta \leq 1, \tag{3.9}$$

with the initial condition

$$\psi(x,0) = x^3. \tag{3.10}$$

Applying the Aboodh transform iterative procedure described in Chapter 2,

$$\psi_0(x,t) = \mathcal{A}^{-1}[\frac{1}{v^{\alpha}}(\sum_{k\ =0}^{m-1} \frac{\psi^{(k)}Cx,0)}{v^{2-\alpha+k}}]$$

$$= \mathcal{A}^{-1}\left[\frac{\psi(x,0)}{v^2}\right]$$

$$= x^3, \tag{3.11}$$

$$\psi_1(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{\beta}\psi_0(x,t) + \psi_0(x,t)\right]\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{\Gamma(4)x^{3-\beta}}{v^{2+\alpha}\Gamma(4-\beta)} + \frac{x^3}{v^{2+\alpha}}\right]$$

$$= \frac{\Gamma(4)x^{3-\beta}t^{\alpha}}{\Gamma(\alpha+1)\Gamma(4-\beta)} + \frac{x^3t^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.12}$$

$$\psi_2(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{\beta}\big(\psi_0(x,t)+\psi_1(x,t)\big) + \big(\psi_0(x,t)+\psi_1(x,t)\big)\right]\right)\right] -$$

$$\mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{\beta}\psi_0(x,t) + \psi_0(x,t)\right]\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{\Gamma(4)x^{3-\beta}}{v^{2+\alpha}\Gamma C4{=}\beta)} + \frac{\Gamma(4)x^{3-2\beta}}{v^{2+2\beta}\Gamma(4-2\beta)} + \frac{2\Gamma(4)x^{3-\beta}}{v^{2+2\alpha}\Gamma(4-\beta)} + \frac{x^3}{v^{2+\alpha}} + \frac{x^3}{v^{2+2\alpha}}\right] -$$

$$\mathcal{A}^{-1}\left[\frac{\Gamma(4)x^{3-\beta}}{v^{2+\alpha}\Gamma(4-\beta)} + \frac{x^3}{v^{2+\alpha}}\right]$$

$$= \frac{\Gamma(4)x^{3-2\beta}t^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(4-2\beta)} + \frac{2\Gamma(4)x^{3-\beta}t^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(4-\beta)} + \frac{x^3t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{3.13}$$

Other terms can be obtained by analogy. The series solution is obtained as

$$\psi(x,t) = \psi_0 + \psi_1 + \psi_2 + \cdots$$

$$= x^3 + \frac{\Gamma(4)x^{3-\beta}t^{\alpha}}{\Gamma(\alpha+1)\Gamma(4-\beta)} + \frac{x^3t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(4)x^{3-2\beta}t^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(4-2\beta)} + \frac{2\Gamma(4)x^{3-\beta}t^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(4-\beta)}$$

$$+ \frac{x^3t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \tag{3.14}$$

Setting $\beta = 1$ in Equation (3.14), we obtained

$$\psi(x,t) = x^3\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right) + 3x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} +..\right)$$

$$+ \frac{6xt^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \tag{3.15}$$

$$= x^3\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} + 3x^2\sum_{i=1}^{\infty}\frac{2^{i-1}t^{i\alpha}}{\Gamma(i\alpha+1)} + 3x\sum_{i=2}^{\infty}\frac{2^{i-1}t^{i\alpha}}{\Gamma(i\alpha+1)} \tag{3.16}$$

Solution to Equation (3.16) leads to exact answer in closed form as $i \longrightarrow \infty$.

$$\psi(x.t) = x^3\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} + 3x^2\sum_{i=1}^{\infty}\frac{2^{i-1}t^{i\alpha}}{\Gamma(i\alpha+1)} + 3x\ \sum_{i=2}^{\infty}\frac{2^{i-1}t^{i\alpha}}{\Gamma(i\alpha+1)}$$

$= x^3 E_\alpha(t^\alpha) + 3x^2(0) + 3x(0)$

$$= x^3 E_\alpha(t^\alpha) \tag{3.17}$$

If $\alpha = 1$ in Equation (3.17), then $\psi(x,t) = x^3 e^t$ (3.18)

which is the same solution obtained in [10]. We compare the absolute difference

$E_{dif} = \left|\psi(0.25,t) - \gamma^{(3)}(0.25,t)\right|$ between the series approximate solution and the close form solution for various $\alpha = \beta$ and obtain the absolute error

$E_{abs} = \left|\psi_e(0.25,t) - \gamma^{(3)}(0.25,t)\right|$ at $\alpha = \beta = 1$, and $t = 0.25, 0{,}50, 0.75, 1.00$

when $x = 0.25$ in Table 3.1.

Figures 3.1(a) and 3.1(b) represent the Application 1 solution charts when

$\alpha = \beta =$ 0.02,0.04, 0.06, 0.08, 0.2, 0.4, 0.6. 0.8 for x=1 and t=1, respectively.

Figures 3.2(a) and 3.2(b) is the exact and approximate surface plots for $\alpha = \beta = 1$ and $\alpha = 0.75, \beta = 0.25$ respectively for Application 1.

Figures 3.3(a) and 3.3(b) is the exact and approximate surface plots for $\alpha = \beta = 1$ and $\alpha = 0.25, \beta = 0.85$ respectively for Application 1.

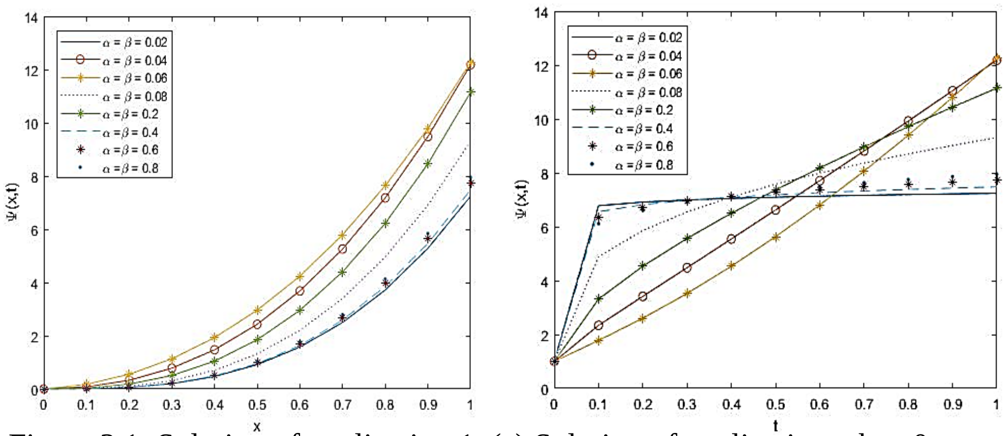

Figure 3.1: Solution of application 1. (a) Solution of application when 0 ≤ x ≤ 1 (b) Solution of application 1 when 0 ≤ t ≤ 1

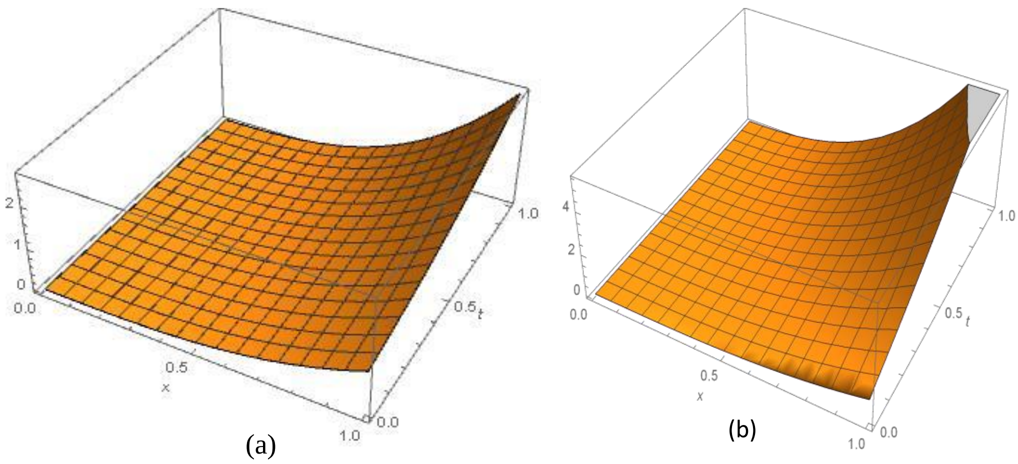

Figure 3.2: Exact and approximate surface plots. (a) Exact plot at α=β=1 and (b) Approximate plot at α=0.75, β=0.25 of application 1

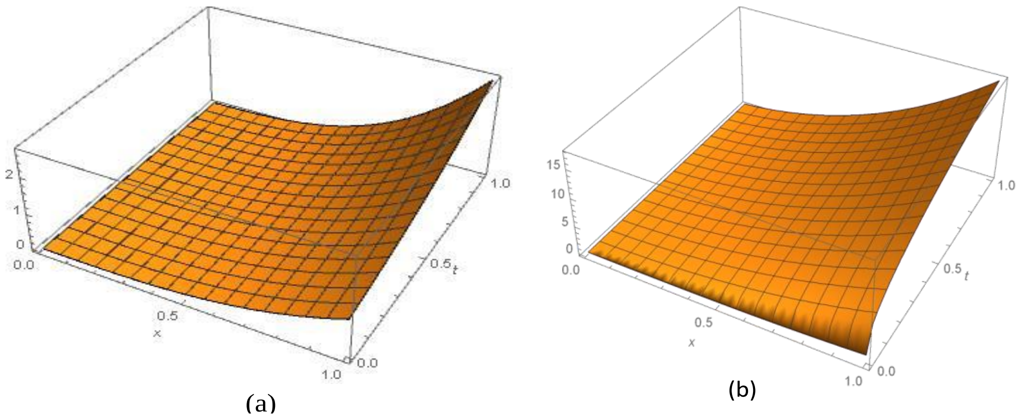

Figure 3.3: Exact and approximate surface plots. (a) Exact plot at α=β=1 and (b) Approximate plot at α=0.25, β=0.85 of application 1.

Table 3.1: Comparison of error difference for diverse values of α=β

t	**E_{abs} where α = β = 1.00**
0.25	6.939×10^{-18}
0.50	3.469×10^{-18}
0.75	6.939×10^{-18}
1.00	6.938×10^{-18}

3.3.2 Application 2

Consider the nonlinear equation space-time fractional Fokker-Planck Equation [53]

$$D_t^{\alpha}\psi(x,t) = D_x^{\beta}\left(\frac{x\psi}{3}\right) - \left(\frac{4\psi^2}{x}\right)_x + (\psi^2)_{xx}, \quad 0 < \alpha, \beta \leq 1, \tag{3.19}$$

with initial condition

$$\psi(x,0) = x^2. \tag{3.20}$$

Using the iterative Aboodh transform process given in Chapter 2, the first three terms of the Equation (3.19)

$$\psi_0(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^\alpha}\left(\sum_{k=0}^{m-1}\frac{\psi^{(k)}}{v^{2-\alpha+k}}\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{\psi(x,0)}{v^2}\right]$$

$$= x^2, \tag{3.21}$$

$$\psi_1(x,t) = \mathcal{A}^{-1}[\frac{1}{v^\alpha}\{\mathcal{A}\left[D_x^\beta\left(\frac{x\psi_0}{3}\right) - \left(\frac{4\psi_0^2}{x}\right)_x + (\psi_0^2)_{xx}]\ \right)\}\right]$$

$$= \mathcal{A}^{-1}\left[\frac{2x^{3-\beta}}{\Gamma(4-\beta)v^{2+\alpha}}\right]$$

$$= \frac{2x^{3-\beta}t^\alpha}{\Gamma(\alpha+1)\Gamma(4-\beta)}, \tag{3.22}$$

$$\psi_2(x,t) = \mathcal{A}^{-1}[\frac{1}{v^\alpha}(\mathcal{A}[D_x^\beta\left(\frac{x}{3}(\psi_0+\psi_1)\right) - (\frac{4}{x}(\psi_0+\psi_1)^2)_x + (\psi_0+\psi_1)^2_{xx]})]$$

$$-\mathcal{A}^{-1}\left[\frac{(5-2\beta)16x^{4-2\beta}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2v^{2+3\beta}} + \frac{4(5-\beta)(4-\beta)x^{3-\beta}}{\Gamma(4-\beta)x^{2+2\alpha}}\right] + \mathcal{A}^{-1}[\frac{4(6-2\beta)(5-2\beta)\Gamma(2\alpha+1)x^{4-2\beta}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2v^{2+3\alpha}} - \frac{2x^{3-\beta}}{\Gamma(4-\beta)v^{2+\alpha}}]$$

$$= \frac{2\Gamma(5-\beta)x^{4-2\beta}t^{2\alpha}}{3\Gamma(4-\beta)\Gamma(5-2\beta)\Gamma(2\alpha+1)} + \frac{(4-4\beta)(4-\beta)x^{3-\beta}t^{2\alpha}}{\Gamma(4-\beta)\Gamma(2\alpha+1)} + \frac{(8-8\beta)(5-2\beta)\Gamma(2\alpha+1)x^{4-2\beta}t^{3\alpha}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)}. \tag{3.23}$$

Other terms can be obtained by analogy. The series solution is obtained by

$$\psi(x,t) = \psi_0 + \psi_1 + \psi_2 + \cdots$$

$$= x^2 + \frac{2x^{3-\beta}t^\alpha}{\Gamma(4-\beta)\Gamma(\alpha+1)} + \frac{2\Gamma(5-\beta)x^{4-2\beta}t^{2\alpha}}{3\Gamma(4-\beta)\Gamma(5-2\beta)\Gamma(2\alpha+1)} + \frac{(4-4\beta)(4-\beta)x^{3-\beta}t^{2\alpha}}{\Gamma(4-\beta)\Gamma(2+1)} +$$

$$\frac{(8-8\beta)(5-2\beta)\Gamma(2\alpha+1)x^{4-2\beta}t^{3\alpha}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} + \cdots \tag{3.24}$$

Setting $\beta = 1$ in Equation (3.24), we obtained

$$\psi(x,t) = x^2(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \quad = x^2 \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \tag{3.25}$$

The answer from Equation (3.25) converges to the precise answer in a closed form as $i \longrightarrow \infty$ $i.e,$

$$\psi(x,t) = x^2 \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \quad = x^2 E_\alpha(t^\alpha) \tag{3.26}$$

Also letting $\alpha = 1$, we obtained $\psi(x,t) = x^2 e^t$ (3.27)

which is the same solution obtained in [53].

Since grids system method was not used, we compare the absolute difference $E_{dif} = \left|\psi(0.25,t) - \gamma^{(3)}(0.25,t)\right|$ between the series approximate solution and the closed form solution for various $\alpha = \beta$ and obtain the absolute error $E_{abs} = \left|\psi_e(0.25,t) - \gamma^{(3)}(0.25,t)\right|$ at $\alpha = \beta = 1$, and t = 0.25, 0.50, 0.75, 1.00 when x=0.25 in the Table 3.2.

Figures 3.4(a) and 3.4(b) display the Equation (3.19) solution charts when $\alpha = \beta = 0.02, 0.04, 0.06, 0.08, 0.2, 0.4, 0.6, \; 0.8, \; for\ x = 1\ and\ t = 1$ in that order.

Figures 3.5(a) and 3.5(b) is the exact and approximate surface plots for $\alpha = \beta = 1$ and $\alpha = 0.75, \beta = 0.25$ respectively for Application 2.

Figures 3.6(a) and 3.6(b) is the exact and approximate surface plots for $\alpha = \beta = 1$ and $\alpha = 0.25, \beta = 0.85$ respectively for Application 2.

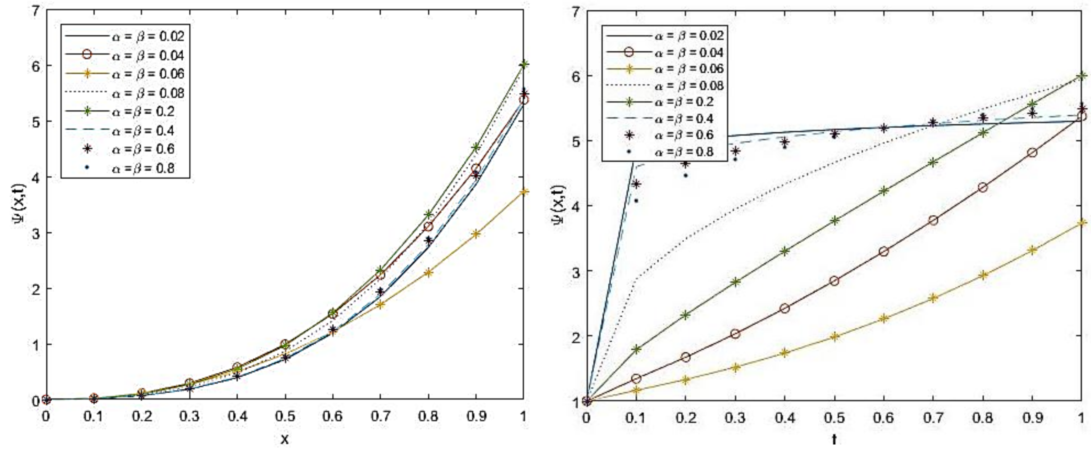

Figure 3.4: Solution of application 2. (a) Solution of application when $0 \leq x \leq 1$ (b) Solution of application 2 when $0 \leq t \leq 1$

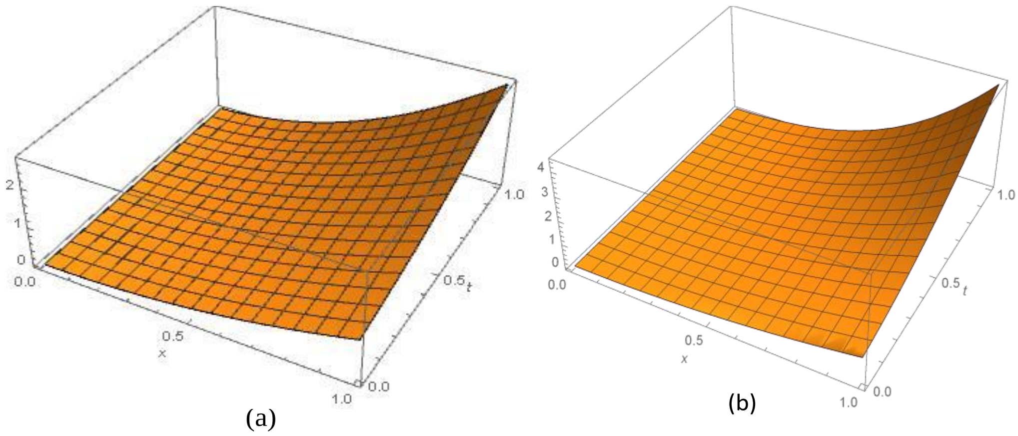

Figure 3.5: Exact and approximate surface plots. (a) Exact plot at $\alpha=\beta=1$ and (b) Approximate plot at $\alpha=0.75$, $\beta=0.25$ for application 2.

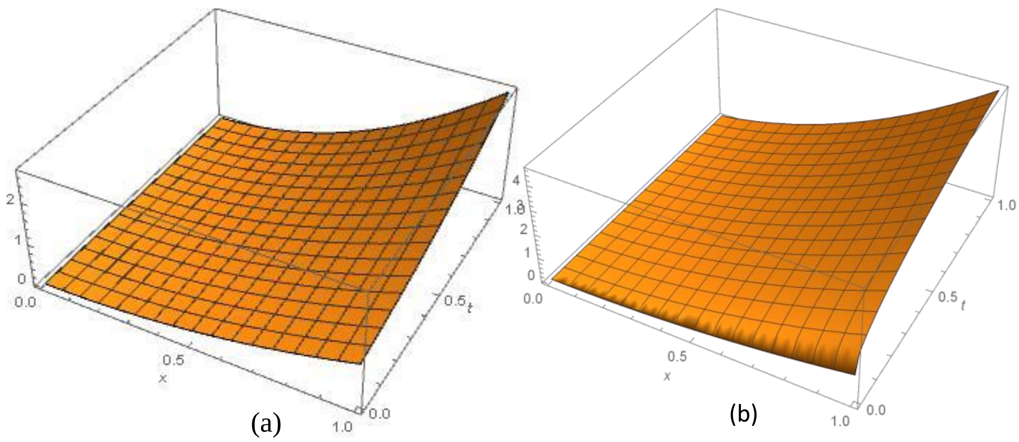

Figure 3.6: Exact and approximate surface plots. (a) Exact plot at $\alpha=\beta=1$ and (b) approximate plot at $\alpha=0.25$, $\beta=0.85$ for application 2.

Table 3.2: Comparison of error difference for diverse values of α=β

T	**E_{abs} where α = β = 1.00**
0.25	2.776x10^{-17}
0.50	1.388x10^{-17}
0.75	2.776x10^{-17}
1.00	2.776x10^{-17}

3.3.3 Application 3

Consider the one-dimensional space-time diffusion equation [54]

$$D_t^{\alpha}\psi(x,t) = D_x^{2\beta}(\psi(x,t)) + D_x^{\beta}(x\psi(x,t)) \qquad 0 < \alpha, \beta \le 1, \tag{3.28}$$

with initial condition

$$\psi(x,0) = 1. \tag{3.29}$$

We calculate the following using the first three terms of iterative Aboodh transform method described in Chapter 2,

$$\psi_0(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\sum_{k=0}^{\infty}\frac{\psi^{(k)}(x,0)}{v^{2-\alpha+k}}\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{\psi(x,0)}{v^2}\right] = 1, \tag{3.30}$$

$$\psi_1(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{2\beta}\psi_0 + D_x^{\beta}(x\psi_0)\right]\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{x^{1-\beta}}{v^{2+\alpha}\Gamma(2-\beta)}\right]$$

$$= \frac{x^{1-\beta}t^{\alpha}}{\Gamma(\alpha+1)\Gamma(2-\beta)}, \tag{3.31}$$

$$\psi_2(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{2\beta}(\psi_0+\psi_1) + D_x^{\beta}\left(x(\psi_0+\psi_1)\right)\right]\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{x^{1-\beta}}{\Gamma(2-\beta)v^{2+\alpha}} + \frac{\Gamma(3-\beta)x^{2-2\beta}}{\Gamma(3-2\beta)\Gamma(2-\beta)v^{3+2}}\right] - \mathcal{A}^{-1}\left[\frac{x^{1-\beta}}{\Gamma(2-\beta)v^{2+\alpha}}\right]$$

$$= \frac{(2-\beta)x^{2-2\beta}t^{2\alpha}}{\Gamma(3-2\beta)\Gamma(2\alpha+1)}. \tag{3.32}$$

Remaining terms can be given by analogy. The series solution is then obtained as

$$\psi(x,t) = \psi_0 + \psi_1 + \psi_2 + \dots$$

$$= 1 + \frac{x^{1-\beta}t^{\alpha}}{\Gamma(2-\beta)\Gamma(\alpha+1)} + \frac{(2-\beta)x^{2-2\beta}t^{2\alpha}}{\Gamma(3-2\beta)\Gamma(2\alpha+1)} + \cdots \quad (3.33)$$

Putting $\beta = 1$ in Equation (3.33), we obtained

$$\psi(x,t) = 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \quad = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \quad (3.34)$$

The series in Equation (3.34) converges to the exact solution in closed form

as $i \longrightarrow \infty$ i.e,

$$\psi(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} = E_{\alpha}(t^{\alpha}) \quad (3.35)$$

Fixing $\alpha = \frac{1}{2}$ into Equation (3.35), we get $\psi(x,t) = E_{\frac{1}{2}}\left(t^{\frac{1}{2}}\right)$ (3.36)

Utilizing the natural transform method, this solution is same as in [54].

In Table 3.3, we present E_{dif} and E_{abs}, with same values as in Application 2, of α, β and t for Application 3. The solution charts for Application 3 for $\alpha = \beta = 0.02, 0.04, 0.06, 0.08, 0.2, 0.4, 0.6, 0.8$ $for\ x = 1\ and\ t = 1$ are shown in Figures 3.7(a) and 3.7(b) respectively.

Figures 3.8(a) and 3.8(b) is the comparison of exact and approximate surface plots for $\alpha = \beta = 1$, and $\alpha = 0.75, \beta = 0.25$ in that order for Application 3.

Figures 3.9(a) and 3.9(b) is the exact and approximate surface plots comparison for $\alpha = \beta = 1$, and $\alpha = 0.25, \beta = 0.85$ respectively for Application 3.

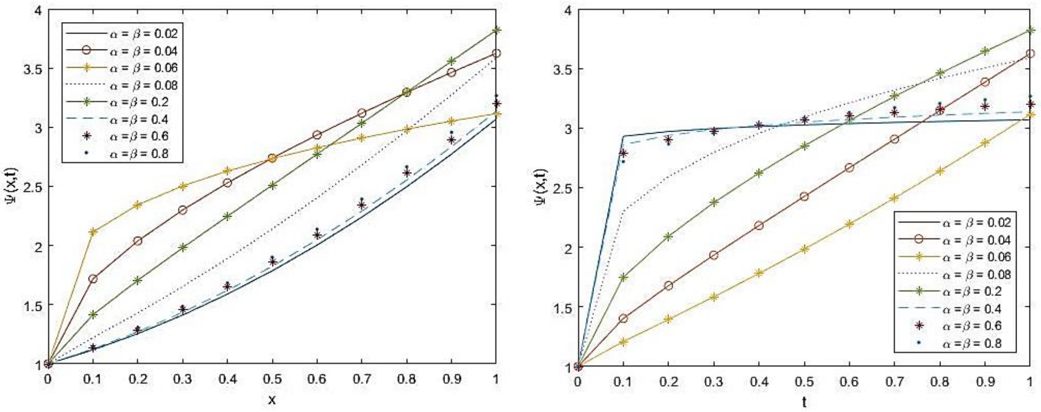

Figure 3.7: Solution of application 3. (a) Solution of application when $0 \leq x \leq 1$. (b) Solution of application 3 when $0 \leq t \leq 1$.

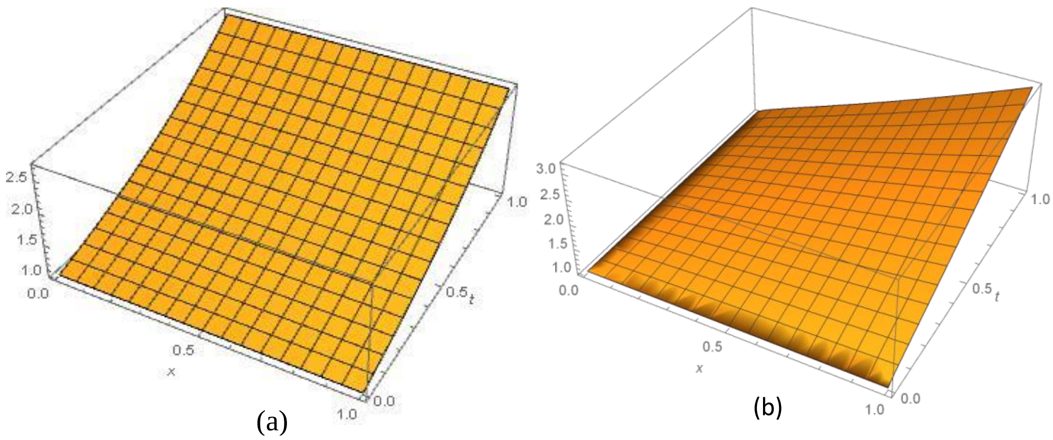

Figure 3.8: Exact and approximate surface plots. (a) Exact plot at α=β=1 and (b) Approximate plot at α=0.75, β=0.25 for application 3.

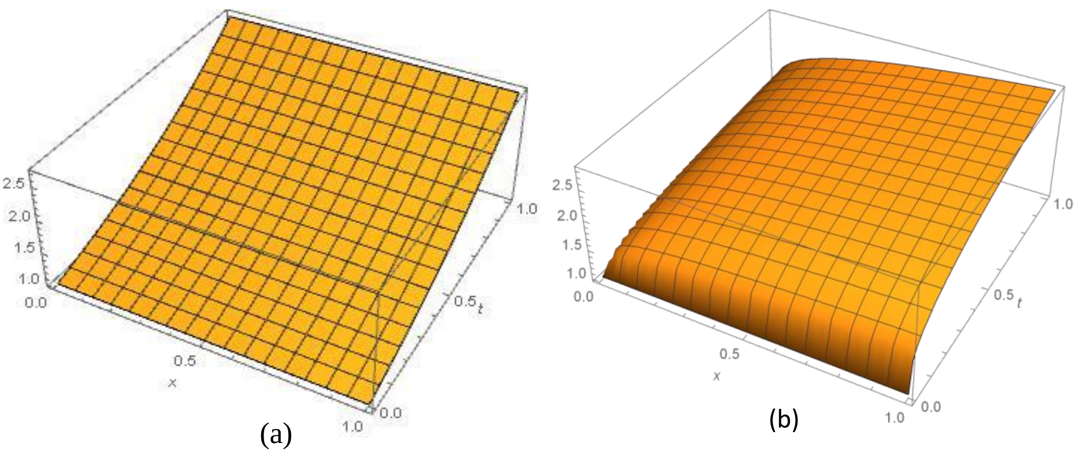

Figure 3.9: Exact and approximate surface plots. (a) Exact plot at α=β=1 and (b) Approximate plot at α=0.25, β=0.85 for application 3.

Table 3.3: Comparison of error difference for different values of α=β

t	**E_{abs} where α = β = 1.00**
0.25	4.448x10^{-16}
0.50	4.441x10^{-16}
0.75	8.882x10^{-16}
1.00	9.548x10^{-13}

3.3.4 Application 4

Consider the space-time fractional Air's partial differential equations [10]

$$D_t^{\alpha}\psi(x,t) = D_x^{3\beta}\psi(x,t), 0 < \alpha, \beta \leq 1, \tag{3.37}$$

with the initial condition

$$\psi(x,0) = \frac{1}{6}x^3. \tag{3.38}$$

We obtained the following using the iterative Aboodh transform process described in Chapter 2,

$$\psi_0(x,t) = \mathcal{A}^{-1}[\frac{1}{v^{\alpha}}(\sum_{k=0}^{m-1}\frac{\psi^{(k)}(x,0)}{v^{2-\alpha+k}})]$$

$$= \mathcal{A}^{-1}\left[\frac{\psi(x,0)}{v^2}\right]$$

$$= \frac{1}{6}x^3, \tag{3.39}$$

$$\psi_1(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{3\beta}\psi_0\right]\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{x^{3-3\beta}}{\Gamma(4-3\beta)v^{2+\alpha}}\right]$$

$$= \frac{x^{3-3\beta}t^{\alpha}}{\Gamma 4-3\beta)\Gamma(\alpha+1)}, \tag{3.40}$$

$$\psi_2(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{3\beta}(\psi_1+\psi_0)\right]\right)\right] - \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{3\beta}\psi_0\right]\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{\Gamma(4)x^{3-3\beta}}{6\Gamma(4-3\beta)v^{2+\alpha}}\right] - \mathcal{A}^{-1}\left[\frac{x^{3-3\beta}}{\Gamma(4-3\beta)v^{2+\alpha}}\right]$$

$$= 0. \tag{3.41}$$

By analogy, the remaining terms can be computed and verified to be 0.

This is how the series solution is reached:

$$\psi(x,t) = \psi_0 + \psi_1 + \psi_2 + \cdots$$

$$= \frac{1}{6}x^3 + \frac{x^{3-3\beta}t^{\alpha}}{\Gamma(4-3\beta)\Gamma(\alpha+1)} + 0 + \cdots \tag{3.42}$$

for all $i > 1, \psi_i(x,t) = 0$ from the above and substituting $\beta = 1$ in Application 4, we have it as

$$\psi(x,t) = \frac{1}{6}x^3 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 0 + 0 \ldots. \quad = \frac{1}{6}x^3 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{3.43}$$

From Equation (3.43), replacing $\alpha = 1$, we discovered the exact answer as follows

$$\psi(x,t) = \frac{1}{6}x^3 + t \tag{3.44}$$

which is the same solution in [10].

Since we did not use grids method system, we compare the absolute difference

$E_{dif} = \left|\psi(0.25,t) - \gamma^{(3)}\ (0.25,t)\right|$ between the series approximate solution and the close form solution for various $\alpha = \beta$ and obtain the absolute error

$E_{abs} = \left|\psi_e(0.25,t) - \gamma^{(3)}(0.25,t)\right|$ at $\alpha = \beta = 1$, and $t = 0.25, 0{,}50, 0.75, 1.00$ when $x = 0.25$ in Table 3.4.

The solution charts for Application 4 are shown in Figures 3.10(a) and 3.10(b)

respectively, when $\alpha = \beta = 0.02{,}0.04{,}0.06{,}0.08{,}0.2, 0.4, 0.8\ for\ x = 1\ and\ t = 1$. Figures 3.11(a) and 3.11(b) is the exact and the approximate surface plots comparison for $\alpha = \beta = 1$, and $\alpha = 0.75, \beta = 0.25$ in that order for Application 4.

Figures 3.12(a) and 3.12(b) is the exact and approximate surface plots comparison for $\alpha = \beta = 1$, and $\alpha = 0.25, \beta = 0.85$ respectively for Application 4.

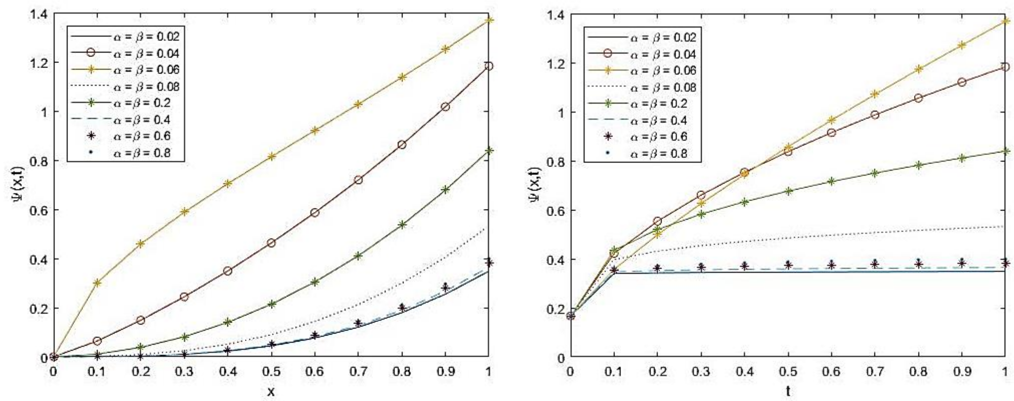

Figure 3.10: Solution of Application 4. (a) Solution of application when $0 \leq x \leq 1$ (b) Solution of application when $0 \leq t \leq 1$

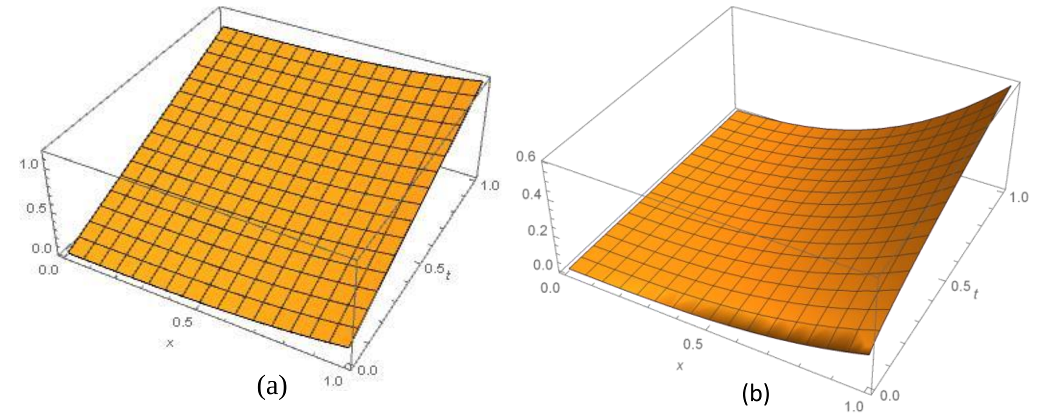

Figure 3.11: Exact and approximate surface plots. (a) Exact plot at $\alpha=\beta=1$ and (b) approximate plot at $\alpha=0.75$, $\beta=0.25$ for application 4

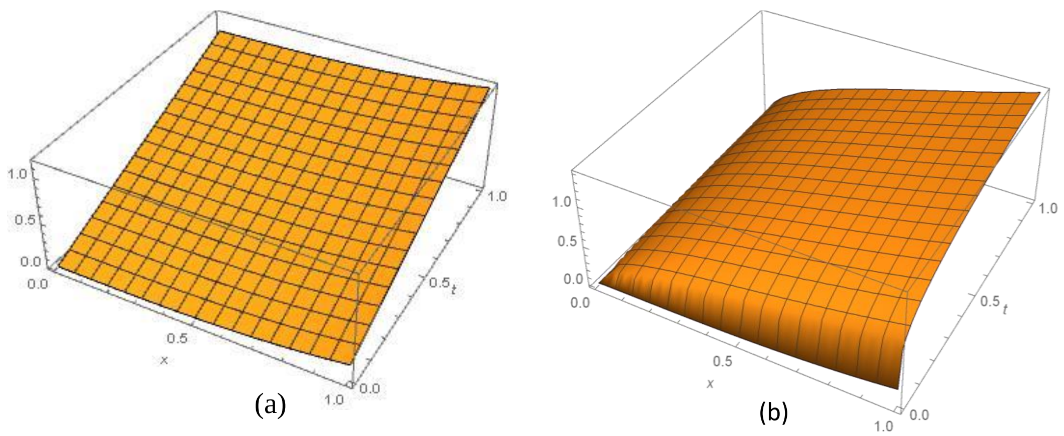

Figure 3.12: Exact and approximate surface plots. (a) Exact plot at $\alpha=\beta=1$ and (b) approximate plot at $\alpha=0.25$, $\beta=0.85$ for application 4

Table 3.4: Comparison of error difference for diverse values of α=β

t	**E_{abs} where α = β = 1.00**
0.25	1.110x10^{-16}
0.50	1.110x10^{-16}
0.75	1.110x10^{-16}
1.00	1.110x10^{-16}

3.3.5 Application 5

Take into account the nonlinear space-time fractional Fokker-Planck equation, which has three terms of space fractional order and one term of time fractional order [53].

$$D_t^{\alpha}\psi(x,t) = D_x^{\beta}\left(\frac{x\psi}{3}\right) - D_x^{\beta}\left(\frac{4\psi^2}{x}\right) + D_x^{2\beta}(\psi^2)\,, 0 < \alpha,\beta \leq 1, \tag{3.45}$$

subject to initial condition

$$\psi(x,0) = x^2. \tag{3.46}$$

We deduce the following using the Aboodh transform iterative technique as described in Chapter 2,

$$\psi_0(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\sum_{k=0}^{m-1}\frac{\psi^{(k)}(x,0)}{v^{2-\alpha+k}}\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{\psi(x,0)}{v^2}\right]$$

$$= x^2, \tag{3.47}$$

$$\psi_1(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{\beta}\left(\frac{x\psi_0}{3}\right) - D_x^{\beta}\left(\frac{4\psi_0^2}{x}\right) + D_x^{2\beta}(\psi_0^2)\right]\right)\right]$$

$$= \mathcal{A}^{-1}\left[\frac{24x^{4-2\beta}}{\Gamma(5-2\beta)v^{2+\alpha}} - \frac{22x^{3-\beta}}{\Gamma(4-\Sigma\beta)v^{2+\alpha}}\right]$$

$$= \frac{24x^{4-2\beta}t^{\alpha}}{\Gamma(5-2\beta)\Gamma(\alpha+1)} - \frac{22x^{3-\beta}t^{\alpha}}{\Gamma(4-\beta)\Gamma(\alpha+1)}, \tag{3.48}$$

$$\psi_2(x,t) = \mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{\beta}\left(\frac{x(\psi_0+\psi_1)}{3}\right) - D_x^{\beta}\left(\frac{4(\psi_0+\psi_1)^2}{x}\right) + D_x^{2\beta}(\psi_0+\psi_1)^2\right]\right)\right] -$$

$$\mathcal{A}^{-1}\left[\frac{1}{v^{\alpha}}\left(\mathcal{A}\left[D_x^{\beta}\left(\frac{x\psi_0}{3}\right)-D_x^{\beta}\left(\frac{4\psi_0^2}{x}\right)+D_x^{2\beta}(\psi_0^2)\right]\right)\right]$$

$$=\mathcal{A}^{-1}\left[\frac{8\Gamma(6-2\beta)x^{5-3\beta}}{\Gamma(5-2\beta)\Gamma(6-3)v^{2+2\alpha}}-\frac{22\Gamma(5-\beta)x^{4-2\beta}}{\Gamma(4-\beta)\Gamma(5-2\beta)v^{22+2\alpha}}+\frac{\Gamma(4)x^{3-\beta}}{3\Gamma(4-\beta)v^{2+\alpha}}\right]$$

$$+\mathcal{A}^{-1}\left[\frac{-1152\Gamma(8-4\beta)\Gamma(2\alpha+1)v^{7-5\beta}}{[\Gamma(5-2\beta)\Gamma(\alpha+1)]^2\Gamma(8-5\beta)v^{2+3\alpha}}-\frac{192\Gamma(6-2\beta)x^{5-3\beta}}{\Gamma(5-2\beta)\Gamma(6-3\beta)v^{2+2\alpha}}\right]$$

$$+\mathcal{A}^{-1}\left[\frac{-1936\Gamma(6-2\beta)\Gamma(2\alpha+1)x^{5-3\beta}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2\Gamma(6-3\beta)v^{2+3\alpha}}+\frac{4224\Gamma(7-3\beta)\Gamma(2\alpha+1)x^{6-4\beta}}{\Gamma(5-2\beta)\Gamma(\alpha+1)^2\Gamma(4-\beta)\Gamma(7-4\beta)v^{2+3\alpha}}\right]+$$

$$\mathcal{A}^{-1}\left[\frac{176\Gamma(5-\beta)x^{4-2\beta}}{\Gamma(4-\beta)\Gamma(5-2\beta)v^{2+2\alpha}}-\frac{4\Gamma(4)x^{3-\beta}}{\Gamma(4-\beta)v^{2+\alpha}}\right.$$

$$\left.+\frac{576\Gamma(9-4\beta)\Gamma(2\alpha+1)x^{8-6\beta}}{[\Gamma(5-2\beta)\Gamma(\alpha+1)]^2\Gamma(9-6\beta)v^{2+3\alpha}}\right]$$

$$+\mathcal{A}^{-1}\left[\frac{-1056\Gamma(8-3\beta)\Gamma(2\alpha+1)x^{7-5\beta}}{[\Gamma(5-2\beta)\Gamma(\alpha+1)]^2\Gamma(4-\beta)\Gamma(8-5\beta)v^{2+3\alpha}}\right.$$

$$\left.+\frac{48\Gamma(7-2\beta)x^{6-4\beta}}{\Gamma(5-2\beta)\Gamma(7-4\beta)v^{2+2\alpha}}\right]$$

$$+\mathcal{A}^{-1}\left[\frac{484\Gamma(7-2\beta)\Gamma(2\alpha+1)x^{6-4\beta}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2\Gamma(7-4\beta)v^{2+3\alpha}}+\frac{44\Gamma(6-\beta)x^{5-3\beta}}{\Gamma(4-\beta)\Gamma(6-3\beta)v^{2+2\alpha}}\right.$$

$$\left.+\frac{24x^{4-2\beta}}{\Gamma(5-2\beta)v^{2+\alpha}}\right]-\mathcal{A}^{-1}\left[\frac{24x^{4-2\beta}}{\Gamma(5-2\beta)v^{2+\alpha}}-\frac{22x^{3-\beta}}{\Gamma(4-\beta)v^{2+\alpha}}\right]$$

$$=\frac{8\Gamma(6-2\beta)x^{5-3\beta}t^{2\alpha}}{\Gamma(5-2\beta)\Gamma(6-3\beta)\Gamma(2\alpha+1)}-\frac{1152\Gamma(8-4\beta)\Gamma(2\alpha+1)x^{7-5\beta}t^{3\alpha}}{[\Gamma(5-2\beta)\Gamma(\alpha+1)]^2\Gamma(8-5\beta)\Gamma(3\alpha+1)}$$

$$-\frac{22\Gamma(5-\beta)x^{4-2\beta}t^{2\alpha}}{3\Gamma(4-\beta)\Gamma(5-2\beta)\Gamma(2\alpha+1)}$$

$$+\frac{4224\Gamma(7-3\beta)\Gamma(2\alpha+1)x^{6-4\beta}t^{3\alpha}}{\Gamma(5-2\beta)\Gamma(\alpha+1)^2\Gamma(4-\beta)\Gamma(7-4\beta)\Gamma(3\alpha+1)}$$

$$-\frac{1436\Gamma(6-2\beta)\Gamma(2\alpha+1)x^{5-3\beta}t^{3\alpha}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2\Gamma(6-3\beta)\Gamma(3\alpha+1)}-\frac{192\Gamma(6-2\beta)x^{5-3\beta}t^{2\alpha}}{\Gamma(5-2\beta)\Gamma(6-3\beta)\Gamma2\alpha+1)}$$

$$+\frac{176\Gamma(5-\beta)x^{4-2\beta}t^{2\alpha}}{\Gamma(4-\beta)\Gamma(5-2\beta)\Gamma(2\alpha+1)}+\frac{576\Gamma(9-4\beta)\Gamma(2\alpha+1)x^{8-6\beta}t^{3\alpha}}{[\Gamma(5-2\beta)\Gamma(\alpha+1)]^2\Gamma(9-6\beta)\Gamma(3\alpha+1)}$$

$$-\frac{1056\Gamma(8-3\beta)\Gamma(2\alpha+1)x^{7-5\beta}t^{3\alpha}}{[\Gamma(5-2\beta)\Gamma(\alpha+1)]^2\Gamma(4-\beta)\Gamma(8-5\beta)\Gamma(3\alpha+1)}$$

$$+\frac{48\Gamma(7-2\beta)x^{6-4\beta}t^{2\alpha}}{\Gamma(5-2\beta)\Gamma(7-4\beta))\Gamma(2\alpha+1)}$$

$$+\frac{484\Gamma(7-2\beta)\Gamma(2\alpha+1)x^{6-4\beta}t^{3\alpha}}{[\Gamma(4-\beta)\Gamma(\alpha+1)]^2\Gamma(7-4\beta)\Gamma(3\alpha+1)}+\frac{44\Gamma(6-\beta)x^{5-3\beta}t^{2\alpha}}{\Gamma(4-\beta)\Gamma(6-3\beta)\Gamma(2\alpha+1)}. \quad (3.49)$$

By analogy, the remaining terms are computed same way. The series' solution is

$$\psi(x,t)=\psi_0+\psi_1+\psi_2+\cdots$$

Setting $\alpha=\beta=1$, we get

$$\psi(x,t)=x^2+x^2t+\frac{x^2t^2}{2}+\cdots$$

$$=x^2(1+t+\frac{t^2}{2}+\cdots)$$

Therefore

$$\psi(x,t)=x^2\sum_{i=0}^{\infty}\frac{t^i}{i!}=x^2e^t \quad (3.50)$$

which agreed with the exact solution in [53] and is similar to the answer found in Application 2. The difference between Application 2 and Application 5 can be explained by the fact that in Application 2, only one term of the space fractional derivative was taken into account while three terms of space fractional derivatives were considered in Application 5. Therefore, increasing the number of terms of space-fractional derivatives do not change the solution of the same problem.

Chapter 4

CONCLUSION AND SUGGESTED RESEARCH QUESTION(S)

We have successfully in this thesis work *provided, presented and applied* the proposed Aboodh transform iterative technique for the solution of space-time fractional differential equation with fractional order derivative in more than one term to actualize accurately the series approximate analytical solution of space-time fractional Air's-like equation, nonlinear space-time fractional Fokker-Planck equation, one dimensional space-time diffusion equation, space-time fractional Air's partial differential equation and nonlinear space-time fractional Fokker-Planck equation with single term of time fractional order and three terms of space fractional order.

The new iterative approach decomposes both the linear and nonlinear term. Some examples considered and when the value at $\alpha=\beta=1$ is taken into account, the approach makes it feasible to recover the classical differential equations from the fractional differential equations.

It is apparent that the Caputo operator takes into account both the space and time fractional derivatives since, given the right forcing term or function and initial values, the solution's uniqueness is guaranteed. To put it another way, we chose the Caputo derivative because it naturally acquires the initial and boundary conditions that are typically used in the real physical formulation of the physical or real-life situations [55]. A global operator with precise and consistent outputs is the Caputo operator.

Aboodh transform iterative method produces closed form solutions in this thesis and exact solutions in some cases.

Aboodh transform is a refinement of the Laplace transform, which benefits all those advantages that Laplace transform methods have, because they can take one directly with just small effort of calculation to the solution of the approximate problem without the usual introduction of arbitrary constant(s) or integration function(s).

The new iterative method, decomposes both the linear and nonlinearity terms in a series form in order to solve the nonlinear problems without linearizing, perturbating or discretizing the state variables, all of which frequently result in rounding off errors that have a significant impact on the accuracy of a problem solution. In other words, this strategy uses less computer time and memory and avoids rounding-off errors.

The graphically shown solution demonstrated the impact of the scheme's fractional orders $\alpha \; and \; \beta$ as displayed in figures 1 to 4, which the readers in different fields of study are at liberty to study and transcribe for different applications.

Generally, the solution profiles have shown or revealed that the scheme is precise and strength forwardness, easy to implement, computational cost is less expensive. The results are in complete agreement towards the exact solutions and the scheme actually and adequately capture the real behavior and fractional effect of the fractional differential equation.

In future, we investigated extending the Aboodh transform iterative method to handle boundary value problems with consideration for other fractional order differential equations which till date have not been solved either analytically or numerically.

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