Mikusiński’s Operational Calculus for Prabhakar Fractional Calculus

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Eastern Mediterranean University
June 2023
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

Mikusiński operational calculus is a powerful yet underappreciated theory, enabling differential equations to be solved using abstract algebra. We realise the fractional integrals and derivatives of Prabhakar type within Mikusiński's theory, by defining them on new function spaces and using their fundamental properties such as semigroup and series formulae.This algebraic framework allows us to construct explicit solutions for multi-term multi-order fractional differential equations, including the general linear constant-coefficient fractional differential equation posed using Prabhakar operators.

Keywords: operational calculus, fractional differential equations, Mikusinski's operational calculus, Prabhakar fractional calculus

ÖZ

Mikusinski operasyonel kalkülüs, güçlü ama yeterince takdir edilmeyen bir teori olarak, kesirli diferansiyel denklemlerin soyut cebir ile çözülmesini sağlar. Mikusinski teorisiyle, Prabhakar kesirli integral ve türevlerinin, tanımlanan yeni fonksiyon alanları ile yarı grup ve seri formülü gibi temel özellikleri kullanarak çözümünü gerçekleştiriyoruz. Bu cebirsel çercerçeve ile genel lineer sabit katsayılı diferansiyel denklemler de dahil Prabhakar operatörlerinin oluşturduğu çok terimli, çok dereceli diferansiyel denklemlerin açık çözümlerini elde ediyoruz.

Anahtar Kelimeler: operasyonel kalkülüs, kesirli diferansiyel denklemler, Mikusinski operasyonel kalkülüs, Prabhakar kesirli kalkülüs

… Dedicated to my Mother and Father, my Husband: Adil and sons: Zun--Noorain and Muhammad Zariyat Khan. To my brothers: Zafar Iqbal and Fahad Ali. I say Thank you.

ACKNOWLEDGMENTS

All gratitude, acknowledgments and approvals to Almighty Allah,the most Gracious and the Merciful. Indeed, whose grace and mercy made it possible to conduct and complete this thesis. I pay humble tribute to the Holy Prophet Hazrat Muhammad (S.A.W) Who brought the message of Allah to seek the Knowledge.

I would like to express my deepest gratitude to my supervisor Assoc. Prof. Dr. Arran Fernandez for the continuous support of PhD study and related research, for his patience, motivation and immense knowledge. His guidance and plentiful experience helped me in all the time of research and writing of this thesis.

My sincere thanks also go to all the faculty members, especially Prof. Dr. Suzan Cival Buranay and Asst. Prof. Dr. Müge Saadetoğlu who were both members of my evaluation committee in addition to Pr. Nazim Mahmudov, for their valuable guidance throughout my studies.

I am delighted to thank all my friends and classmates for their moral support and help during my course and research work.

Last but not least, I would to thank my parents and whole family members for supporting me spiritually throughout writing thesis and my life in general.

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Chapter 1

INTRODUCTION

Fractional calculus is a field of study, like many in mathematics, which arises from a type of generalisation starting from a basic concept: in this case, taking the concepts of differentiation and integration, and generalising them by allowing the orders of derivatives and integrals to take values outside of the set of integers. As well as this purely mathematical motivation, the study of such generalised operators is also useful for scientific applications. The theory of fractional calculus is introduced in textbooks such as [31, 39, 41, 47] while the applications are summarised in survey works such as [29, 52].

Differential equations are used in applied mathematics to model almost all systems and processes in the real world, and their study has also formed an important part of theoretical mathematics [13]. Both analytical and numerical methods have been widely used: either to find exact solutions of differential equations, or to estimate their solutions approximately, or to obtain qualitative results such as existence-uniqueness, regularity, etc.

An important type of analytical method, applicable to both ordinary and partial differential equations, is the family of transform methods. This includes the methods of Laplace transforms, Fourier transforms, Mellin transforms, Radon transforms, and hybrid techniques such as the Fokas method [11, 23]. Of particular interest, for many problems ranging from mathematics to engineering, is the Laplace transform [49, 56].

This is based on an integral transform involving an exponential function, which transforms a function of one variable to a different function of another variable. It is important to note that not every function has a Laplace transform: a condition such as exponential boundedness is usually imposed in order for the Laplace transform to be well-defined. Therefore, the applicability of the Laplace transform is limited to those problems where all functions concerned satisfy the appropriate conditions such as exponential boundedness.

From the pure mathematical viewpoint, then, the Laplace transform is inferior to another method, formally equivalent but more widely applicable to a broader class of functions. The latter method is Mikusiński's operational calculus, named after the Polish mathematician Jan Mikusiński who first proposed it in the 1950s [38], following earlier work on operational calculus starting from Heaviside [43], and a good survey of developments up to the 1970s can be found in [6], while Flegg [22] summarised the advantages of Mikusiński's approach over the (formally equivalent but less powerful) method of Laplace transforms, although it is less immediately grasped for non-mathematicians as its formulation requires some knowledge of algebra. There are also other types of operational calculus [4, 5, 55], but the Mikusiński formalism has drawn particular attention, having been applied to partial differential equations [26] and fractional differential equations [34], including those of Riemann–Liouville type [28], Caputo type [35], Hilfer type [30], Erdélyi–Kober type [36, 57].

There are many different operators of fractional calculus, arising from different understandings of how to interpolate between integer-order derivatives and integrals, or from different ways of combining together the basic building blocks of fractional calculus [3, 47]. Some of the definitions of fractional calculus, such as Riemann–Liouville, Caputo, Riesz, Hadamard, Weyl, and Erdélyi–Kober, have been known for decades or centuries [31, 39, 41, 47]. Other definitions, such as Hilfer, tempered, Prabhakar, and many other lesser-known models, have become popular only in the last twenty years. Some survey works [53] have attempted to list and categorise all of these, but there are so many new definitions being created every year that it is impossible to keep up with all of them. It has been proposed [3] that fractional calculus is best understood in terms of general classes of operators, classifying all the many definitions into some broad categories with shared behaviours.

General classes that have been proposed include: fractional calculus with respect to functions [2, 42]; weighted fractional calculus [1]; fractional operators with Sonine kernels [33,37]; fractional calculus with analytic kernels [17]; and some others that we will now focus on in more detail.

Prabhakar fractional calculus finds its genesis in a 1971 paper of Prabhakar [45], although the integral operator that he defined was first called a fractional integral by Kilbas et al. in 2004 [32]. The Prabhakar fractional derivatives were first defined by Garra et al. in 2014 [24], and the mathematical theory of Prabhakar fractional calculus has continued to be investigated in more recent papers [16, 27, 44] as well as discovering applications in fields including dielectrics, viscoelasticity, and options pricing [7, 25, 54]. It can be interpreted as a general class containing several other named operators of fractional calculus, both singular and nonsingular [16].

The Hilfer fractional derivative was proposed [29] as a way of interpolating between the well-established Riemann–Liouville and Caputo fractional derivatives. It has two parameters, one of which can be seen as the fractional order while the other serves as an interpolation parameter between Riemann–Liouville and Caputo. Hilfer versions of more general operators have also been defined, such as the Hilfer derivative with respect to a function [50] and the Hilfer–Prabhakar derivative [24]. Hilfer–Prabhakar derivative is a very general operator as it includes Prabhakar derivatives of both Riemann–Liouville and Caputo types, as well as all the special cases of Prabhakar listed in [16]. Thus, results achieved in the setting of Hilfer–Prabhakar fractional derivatives can be immediately applied in many useful special cases [48].

This current project is to adapt the formalism of Mikusiński's operational calculus to the setting of Prabhakar fractional calculus: defining the Prabhakar operators in such a way as to fit into a Mikusiński field, and using this notation to solve fractional differential equations within the Prabhakar model. This work will lead ultimately to an explicit solution for the general linear constant-coefficient fractional differential equation with Prabhakar operators of all three types: Riemann–Liouville type, Caputo type and the generalised Hilfer type.

Chapter 2

PRELIMINARIES

In this chapter we introduce the basic operators of fractional calculus, Riemann–Liouville, Caputo and Hilfer derivatives and integrals, and then proceed to define the main operators to be used in this project, namely the Prabhakar integrals and derivatives. We also define some function spaces required for Mikusiński method.

2.1 Fractional Calculus

We begin with key definitions and some important fundamental properties of Riemann–Liouville and Caputo fractional calculus.

Definition 2.1 ([9,31,39]): The Riemann–Liouville fractional integral operator ${}^{RL}_{0}I^{\alpha}_{x}$ is defined as follows, for $\alpha \in \mathbb{C}$ with positive real part:

$$\left({}^{RL}_{0}I^{\alpha}_{x} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \qquad x \in (0,\infty), \tag{2.1}$$

where f is any function such that this integral converges (function spaces will be considered in more detail later). The Riemann–Liouville and Caputo fractional derivative operators, ${}^{RL}_{0}D^{\alpha}_{x}$ and ${}^{C}_{0}D^{\alpha}_{x}$ respectively, are defined as follows, where $\alpha \in \mathbb{C}$ with non-negative real part and $m-1 \leq \mathrm{Re}(\alpha) < m \in \mathbb{N}$:

$$\left({}^{RL}_{0}D^{\alpha}_{x} f\right)(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left({}^{RL}_{0}I^{m-\alpha}_{x} f\right)(x),$$
$$\left({}^{C}_{0}D^{\alpha}_{x} f\right)(x) = {}^{RL}_{0}I^{m-\alpha}_{x} \left(\frac{\mathrm{d}^m}{\mathrm{d}x^m} f\right)(x),$$

where f is any function such that these expressions are well-defined (again, appropriate function spaces will be considered later).

Both of these two alternative definitions of the fractional derivative can be seen as "natural" extensions of the Riemann–Liouville integral, in two different ways. The Riemann–Liouville derivative is the unique analytic continuation of the Riemann–Liouville integral to negative orders, under the convention that

$$ {}^{RL}_{0}I^{-\alpha}_{x} = {}^{RL}_{0}D^{\alpha}_{x} \tag{2.2} $$

for all α, allowing both of these expressions to be extended to all $\alpha \in \mathbb{C}$ without restrictions. On the other hand, Riemann–Liouville fractional differential equations require fractional initial conditions, and so many applications prefer the Caputo derivative, which requires initial conditions of classical type only [9].

Lemma 2.1.1 (Composition relations [15, 31, 47])**.** Riemann–Liouville differintegrals have natural composition properties both when the inner operator is a fractional integral and when the outer operator is an ordinary repeated derivative:

$$ {}^{RL}_{0}I^{\alpha}_{x} \circ {}^{RL}_{0}I^{\beta}_{x} = {}^{RL}_{0}I^{\alpha+\beta}_{x}, \qquad \alpha,\beta \in \mathbb{C}, \quad \operatorname{Re}\beta > 0; \tag{2.3} $$

$$ \frac{\mathrm{d}^n}{\mathrm{d}x^n} \circ {}^{RL}_{0}D^{\alpha}_{x} = {}^{RL}_{0}D^{n+\alpha}_{x}, \qquad n \in \mathbb{N}, \quad \alpha \in \mathbb{C}, \tag{2.4} $$

where we follow the convention (2.2) so that the operator of order α in each case may be either a fractional integral or a fractional derivative, according to the sign of $\operatorname{Re}\alpha$.

There is also a modified composition property for fractional derivatives of fractional derivatives, which includes a finite sum of initial value terms:

$$ {}^{RL}_{0}D^{\alpha}_{x}\left({}^{RL}_{0}D^{\beta}_{x} f\right)(x) = \left({}^{RL}_{0}D^{\alpha+\beta}_{x} f\right)(x) - \sum_{k=0}^{m-1} \frac{x^{-\alpha-k-1}}{\Gamma(-\alpha-k)} \cdot \left[\left({}^{RL}_{0}D^{\beta-k-1}_{x} f\right)(x)\right]_{x=0}, \tag{2.5} $$

where $m = \lfloor \operatorname{Re}\beta \rfloor + 1$ and where we assume f is in a suitable function space (function spaces for fractional differintegrals will be discussed further below).

Definition 2.1.1 (Hilfer derivative [29, 30])**.** For any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \geq 0$ and any

$\nu \in [0,1]$, the Hilfer fractional derivative of order α and type ν is defined, acting on a suitable function $f : (0,\infty) \to \mathbb{C}$, as follows:

$$\left({}^{H}_{0}D^{\alpha,\nu}_{x} f\right)(x) = {}^{RL}_{0}I^{\nu(n-\alpha)}_{x} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left({}^{RL}_{0}I^{(1-\nu)(n-\alpha)}_{x} f\right)(x), \qquad x \in (0,\infty),$$

where again $n = \lfloor \mathrm{Re}\,\alpha \rfloor + 1 \in \mathbb{N}$. It is important to note that the special case $\nu = 0$ gives the Riemann–Liouville derivative while the case $\nu = 1$ gives the Caputo derivative; thus, the Hilfer derivative can be seen as a continuous interpolation between the Riemann–Liouville and Caputo derivatives.

Definition 2.2: [45] The Prabhakar fractional integral operator ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$ is defined as follows, for $\alpha, \beta \in \mathbb{C}$ with positive real part and $\gamma, \delta \in \mathbb{C}$ arbitrary:

$$\left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} f\right)(x) = \int_0^x (x-\xi)^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\delta(x-\xi)^{\alpha}\right) f(\xi)\mathrm{d}\xi, \qquad x > 0, \tag{2.6}$$

where f is any function such that this integral converges and $E^{\gamma}_{\alpha,\beta}$ is the 3-parameter Mittag-Leffler function defined by:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \mathrm{Re}(\alpha) > 0. \tag{2.7}$$

We will sometimes use the notation $e^{\gamma,\delta}_{\alpha,\beta}$ for the modified Mittag-Leffler function which is used here as a kernel:

$$e^{\gamma,\delta}_{\alpha,\beta}(x) = x^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\delta x^{\alpha}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \cdot \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}, \tag{2.8}$$

where $(\gamma)_n$ is the Pochhammer symbol.

Definition 2.1.2. [24, 54] For any $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}\,\beta \geq 0$, the Prabhakar fractional derivatives of Riemann–Liouville type, Caputo type, and Hilfer type are defined respectively as follows, each acting on a suitable function $f : (0,\infty) \to \mathbb{C}$:

$$
\begin{aligned}
{}^{PR}_{0}D_x^{\alpha,\beta,\gamma,\delta} f(x) &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{P}_{0}I_x^{\alpha,n-\beta,-\gamma,\delta} f\right)(x), \\
{}^{PC}_{0}D_x^{\alpha,\beta,\gamma,\delta} f(x) &= {}^{P}_{0}I_x^{\alpha,n-\beta,-\gamma,\delta}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} f\right)(x), \\
{}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu} f(x) &= {}^{P}_{0}I_x^{\alpha,\nu(n-\beta),-\gamma\nu,\delta}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{P}_{0}I_x^{\alpha,(1-\nu)(n-\beta),-\gamma(1-\nu),\delta} f\right)\right)(x),
\end{aligned}
$$

where $n = \lfloor \mathrm{Re}\,\beta \rfloor + 1 \in \mathbb{N}$ in all three cases, and $0 \leq \nu \leq 1$ in the last case.

Lemma 2.1.2 (Semigroup property [32, 45])**.** The Prabhakar integral has a semigroup property in its second and third parameters:

$$
{}^{P}_{0}I_x^{\alpha,\beta_1,\gamma_1,\delta} \circ {}^{P}_{0}I_x^{\alpha,\beta_2,\gamma_2,\delta} = {}^{P}_{0}I_x^{\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta},
$$

where $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}\,\beta_1 > 0$ and $\mathrm{Re}\,\beta_2 > 0$. As a consequence of this, we have

$$
\underbrace{{}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta} \circ {}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta} \circ \cdots \circ {}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta}}_{n \text{ times}} = {}^{P}_{0}I_x^{\alpha,n\beta,n\gamma,\delta},
$$

where $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}\,\beta > 0$ and $n \in \mathbb{N}$.

Lemma 2.1.3 (Series formulae [19])**.** The Prabhakar integral and Prabhakar derivative of Riemann–Liouville type have the following locally uniformly convergent series formulae:

$$
\left({}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta} f\right)(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k \delta^k}{k!} \left({}^{RL}_{0}I_x^{\alpha k+\beta} f\right)(x), \tag{2.9}
$$

$$
\left({}^{PR}_{0}D_x^{\alpha,\beta,\gamma,\delta} f\right)(x) = \sum_{k=0}^{\infty} \frac{(-\gamma)_k \delta^k}{k!} \left({}^{RL}_{0}I_x^{\alpha k-\beta} f\right)(x), \tag{2.10}
$$

in both cases for suitable functions $f : (0,\infty) \to \mathbb{C}$ and for any $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$ and either $\mathrm{Re}\,\beta > 0$ in the first case or $\mathrm{Re}\,\beta \geq 0$ in the second case.

2.2 Function Spaces

In the following definition, the space C_η comes from Dimovski [10], the space ${}^{R}\Omega_\eta^\alpha$ from Hadid and Luchko [28], the space ${}^{H}\Omega_\eta^\alpha$ from Hilfer, Luchko and Tomovski [30], and the space C_η^m from Luchko and Gorenflo [35].

Definition 2.3: A function $f(x), x \geq 0$, is said to be in the space C_η, for a fixed number $\eta \in \mathbb{R}$, if there exists a real number p, $p > \eta$, such that

$$f(x) = x^p f_1(x) \tag{2.11}$$

with a function $f_1 \in C[0,\infty)$. Note that any function $f \in C_\eta$ for $\eta \geq -1$ is necessarily integrable on any finite interval $[0,X]$: even if there is a singularity at $x = 0$, it must be an integrable singularity.

Clearly, C_η is a vector space, and the set of spaces C_η is ordered by inclusion according to

$$C_\eta \subseteq C_\mu \Longleftrightarrow \eta \geq \mu. \tag{2.12}$$

Another important property of the C_η spaces is how they are mapped under convolution: if $f \in C_\eta$ and $g \in C_\mu$ and $\eta, \mu \geq -1$, then $f * g \in C_{\eta+\mu+1}$. This fact can be written as follows:

$$C_\eta * C_\mu \subseteq C_{\eta+\mu+1} \subseteq C_\eta \cap C_\mu. \tag{2.13}$$

Here we use the Laplace-type convolution, defined as follows to get from any pair of functions f and g defined on the positive real axis to a new function $f * g$:

$$f * g(x) = \int_0^x f(x-t)g(t)\,\mathrm{d}t, \qquad x > 0.$$

Note that the Riemann–Liouville fractional integral ${}^{RL}_{0}I^{\mu}_{x} f(x)$, defined in (2.1), is precisely such a convolution of the function $f(x)$ with the function $\frac{x^{\mu-1}}{\Gamma(\mu)}$. Since $\mu > 0$, the latter function is in C_{-1}, so the operator ${}^{RL}_{0}I^{\mu}_{x}$ maps the space C_η into itself for any $\eta \geq -1$ [28].

Definition 2.2.1. Within the vector space C_η for a given $\eta \geq -1$, we define the following subspaces, according to the different types of differentiability conditions that may be needed to define certain fractional derivative operators:

- ${}^{R}\Omega_{\eta}^{\alpha}$, for given $\alpha > 0$, is the set of all functions $f \in C_\eta$ such that the Riemann–Liouville derivatives ${}_{0}^{RL}D_{x}^{\beta} f$ exist as functions in C_η for all β satisfying $0 \leq \beta \leq \alpha$.
- ${}^{H}\Omega_{\eta}^{\alpha}$, for given $\alpha > 0$, is the set of all functions $f \in C_\eta$ such that the Hilfer derivatives ${}_{0}^{H}D_{x}^{\beta,\nu} f$ exist as functions in C_μ for all β satisfying $0 \leq \beta \leq \alpha$ and all $\nu \in [0,1]$.
- C_{η}^{m}, for given $m \in \mathbb{N}$, is the set of all functions $f \in C_\eta$ such that the mth-order repeated derivative $f^{(m)}$ exists as a function in C_η.

All of these are vector subspaces of C_η. Again, we will be particularly interested in the case $\eta = -1$.

Proposition 2.2.1 ([35]). Let $\eta \geq -1$ and $m \geq 1$.

- If $f \in C_{\eta}^{m}$, then the limit $f^{(k)}(0^+)$ is finite for all $k = 0, 1, \cdots, m-1$, and the function $\tilde{f}$ given by appending the value $\tilde{f}(0) = f(0^+)$ to the function $\tilde{f}(x) = f(x)$, $x > 0$, is in $C^{m-1}[0,\infty)$.
- If $f \in C_{\eta}^{m}$, then $f \in C^{m}(0,\infty) \cap C^{m-1}[0,\infty)$.
- A function f is in C_{η}^{m} iff it can be written as

$$f(x) = \left({}_{0}I_{x}^{m} g\right)(x) + \sum_{k=0}^{m-1} c_k \frac{x^k}{k!}, \qquad x \geq 0, \tag{2.14}$$

  for some $g \in C_\eta$ and some constants $c_0, c_1, \cdots, c_{m-1}$.

Proposition 2.2.2 ([35]). Let $m \geq 0$ in $\mathbb{Z}$. If $f \in C_{-1}^{m}$ and $g \in C_{-1}^{1}$ and $f(0) = \cdots = f^{(m-1)}(0) = 0$, then their convolution $h = f * g$ is in C_{-1}^{m+1} and $h(0) = \cdots = h^{(m)}(0) = 0$.

Remark 2.1: It would also be possible to define spaces like ${}^{R}\Omega_{\eta}^{\alpha}$ and ${}^{H}\Omega_{\eta}^{\alpha}$ for complex values of α, so we take this opportunity to explain why we are not doing so.

In fact, some of our other definitions and results in this chapter have assumed complex parameters even where previous sources have assumed real parameters. But, in all of these cases, things work in essentially the same way under complex assumptions as under real assumptions, so the extension is basically trivial.

Here, taking α to be complex would potentially lead to a stronger assumption, as requiring fractional derivatives to be in C_η for all complex β satisfying $0 \leq \operatorname{Re}\beta \leq \operatorname{Re}\alpha$ would also mean a stronger assumption even in the case where α is real. This is undesirable as it would change the definitions of the ${}^R\Omega_\eta^\alpha$ and ${}^H\Omega_\eta^\alpha$ spaces for real α too. Our results in section 2.2 will prove some fairly strong restrictions on the ${}^R\Omega_\eta^\alpha$ and ${}^H\Omega_\eta^\alpha$ spaces, and it is important to be aware that these restrictions apply even according to the classical definitions of the spaces.

So, we keep the classical definitions of ${}^R\Omega_\eta^\alpha$ and ${}^H\Omega_\eta^\alpha$, with both η and α assumed to be real parameters.

For the sake of simplicity, we will consider in our further investigation the case of the space C_{-1}, the largest C_η space where we can usefully utilise convolution. The following theorem was established by the work of Luchko and others on Mikusińksi's operational calculus for fractional operators [28,34,35]; however, those papers referred to it as a ring rather than a rng, so we also cite [14] which acknowledged the lack of multiplicative identity.

Theorem 2.1 ([14,28]): The space C_{-1} with the operations of the Laplace convolution $*$ and ordinary addition becomes a commutative rng $\left(C_{-1}, *, +\right)$ without divisors of zero (and also without a multiplicative identity, hence rng rather than ring).

From this commutative rng without zero divisors, we follow the classical reasoning of Mikusiński to construct the field of fractions M_{-1}, defined as the quotient of the product set $C_{-1} \times (C_{-1} \backslash \{0\})$ by the equivalence relation

$$(f,g) \sim (f_1,g_1) \Longleftrightarrow (f*g_1)(x) = (g*f_1)(x), \tag{2.15}$$

with the operations in M_{-1} defined as usual by

$$\begin{aligned} \frac{f}{g} + \frac{f_1}{g_1} &= \frac{f*g_1 + g*f_1}{g*g_1}, \\ \frac{f}{g} \cdot \frac{f_1}{g_1} &= \frac{f*f_1}{g*g_1}. \end{aligned}$$

This gives rise to a field M_{-1}, with all the field axioms easily checkable based on the rng axioms for C_{-1}. Note that, even though C_{-1} does not have a multiplicative identity, M_{-1} does, namely the field element given by $I = \frac{f}{f}$ for any $f \in C_{-1} \backslash \{0\}$. Both the ring C_{-1} and the field $\mathbb{C}$ of complex numbers can be naturally embedded in M_{-1}.

For the current work, we wish to consider in particular the following function, an element of the ring C_{-1} and therefore the field M_{-1}:

$$P_{\alpha,\beta,\gamma,\delta} = P_{\beta,\gamma} := x^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\delta x^{\alpha}\right). \tag{2.16}$$

(This function depends on four parameters, but we will usually suppress the α-dependence and δ-dependence in the notation. The dependence on β and γ is more important because of the semigroup property (3.4) in these parameters.)

The Prabhakar integral operator ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$, acting on the function space C_{-1}, is precisely multiplication by the element $P_{\alpha,\beta,\gamma,\delta}$ in the algebraic rng.

Chapter 3

PRABHAKAR FRACTIONAL CALCULUS OF RIEMANN–LIOUVILLE TYPE

In this chapter, we establish how the Prabhakar fractional integral and derivative, as introduced in chapter 2, act with respect to some of the function spaces defined in section 2.2. We interpret the Prabhakar operator of Riemann–Liouville type from an algebraic viewpoint, using Mikusiński operational calculus, and utilise this algebraic formalism to solve some fractional differential equations.

3.1 Action of Operators on Spaces

Theorem 3.1: Let $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$. The Prabhakar fractional integral operator ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$ is a linear map of the space C_η into itself for any given $\eta > -1$, because

$$ {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} : C_\eta \to C_{\eta+\beta} \subseteq C_\eta . \tag{3.1} $$

Proof. The Prabhakar fractional integral is clearly a linear operator. For the C_η mapping, we write $f(x) = x^p f_1(x)$, $p > \eta$, $f_1 \in C[0,\infty)$, and set $t = x\tau$ in Chapter 2 Definition (2.2):

$$ \begin{aligned} {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} f(x) &= \int_0^1 (x - x\tau)^{\beta-1} E^{\gamma}_{\alpha,\beta}\big(\delta(x - x\tau)^{\alpha}\big)(x\tau)^p f_1(x\tau) x \,\mathrm{d}\tau \\ &= x^{p+\beta} \int_0^1 \tau^p (1-\tau)^{\beta-1} E^{\gamma}_{\alpha,\beta}\big(\delta x^{\alpha}(1-\tau)^{\alpha}\big) f_1(x\tau) \mathrm{d}t, \\ &= x^{p+\beta} f_2(x) \end{aligned} $$

where $f_2 \in C[0,\infty)$ because the last integral is uniformly convergent with respect to x in every closed interval $[0,X]$, $X>0$, since $p>\eta\geq-1$ and $\beta-1\geq-1$ so both potential singularities are integrable. Therefore, since $p+\beta\geq\eta+\beta$, we have ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}f\in C_{\eta+\beta}$, which is contained in C_{η} by (2.12). $\square$

It is important to note that the operator ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}f$ has the following convolution representation in the function space C_{η}, $\eta\geq-1$:

$$ {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}f(x)=(P_{\alpha,\beta,\gamma,\delta}*f)(x),\qquad P_{\alpha,\beta,\gamma,\delta}:=x^{\beta-1}E^{\gamma}_{\alpha,\beta}\big(\delta x^{\alpha}\big),f\in C_{\eta}, \tag{3.2} $$

This convolution representation would have enabled a quick proof of Theorem 3.1, simply by using (2.13), because the Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}$ is continuous and therefore the modified version (2.7) is in C_{-1}.

It is known [32, 45] that this function has the following convolution relationship (i.e. Riemann–Liouville fractional integral relationship):

$$ h_{\mu}*P_{\beta,\gamma}=P_{\beta+\mu,\gamma},\qquad \beta,\mu>0,\gamma\in\mathbb{R}, \tag{3.3} $$

where h_{μ} denotes (following the notation of [28, 35]) the power function $h_{\mu}(x)=\frac{x^{\mu-1}}{\Gamma(\mu)}\in C_{-1}$ for any $\mu>0$.

It is also known [32, Theorem 2] that the modified Mittag-Leffler function (2.7) has a semigroup property in the parameters β and γ, namely:

$$ \left(x^{\beta_1-1}E^{\gamma_1}_{\alpha,\beta_1}(\delta x^{\alpha})\right)*\left(x^{\beta_2-1}E^{\gamma_2}_{\alpha,\beta_2}(\delta x^{\alpha})\right)=x^{\beta_1+\beta_2-1}E^{\gamma}_{\alpha,\beta_1+\beta_2}(\delta x^{\alpha}). \tag{3.4} $$

This leads directly to the semigroup property in the parameters β and γ of the Prabhakar fractional integral operator, which was shown in [32, Theorem 8] for L^1 functions but which we now state in the C_{η} function space:

$$ \left({}^{P}_{0}I^{\alpha,\beta_1,\gamma_1,\delta}_{x}\circ{}^{P}_{0}I^{\alpha,\beta_2,\gamma_2,\delta}_{x}f\right)(x)={}^{P}_{0}I^{\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta}_{x}f(x),\qquad f\in C_{\eta},\ \eta\geq-1. \tag{3.5} $$

In particular, taking the n times composition of a single Prabhakar integral operator with itself is equivalent to multiplying the β and γ parameters by n:

$$\left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\circ\cdots\circ{}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x)={}^{P}_{0}I^{\alpha,n\beta,n\gamma,\delta}_{x}f(x),\qquad f\in C_\eta,\ \eta\geq-1,\ n\in\mathbb{N}.$$

Thus, we can think about defining "fractional powers" of the Prabhakar integral operator, by multiplying the β and γ coefficients by an arbitrary real number, similar to the technique used in [20]. We shall make this concept rigorous later when we have an algebraic structure for these operators.

Theorem 3.2: Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ with $\alpha>0$ and $\beta>0$. The Prabhakar fractional derivative operator ${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ is a left inverse of the Prabhakar fractional integral ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$ on the function space C_η, $\eta\geq-1$.

Proof. Let $f\in C_\eta$, $\eta\geq-1$. Making use of the semigroup property, and the fact that the Prabhakar fractional integral reduces to the Riemann–Liouville fractional integral of order β when $\gamma=0$, we get

$$\begin{aligned}\left({}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}\,{}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x)&=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left({}^{P}_{0}I^{\alpha,n-\beta,-\gamma,\delta}_{x}\,{}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x)\\&=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left({}^{RL}_{0}I^{n}_{x}f\right)(x)\\&=f(x),\end{aligned}$$

where $n=\lfloor\beta\rfloor+1$ is a positive integer, and in the last step we used the fundamental theorem of calculus. □

Theorem 3.3: Let $\alpha,\beta,\gamma,\delta,\eta\in\mathbb{R}$ with $\alpha>0$ and $\beta>0$ and $\eta>-1$. The Prabhakar fractional derivative operator ${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ maps the function space ${}^{R}\Omega^{\beta}_{\eta}$ into C_η:

$$\left({}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x)\in C_\eta. \tag{3.6}$$

Proof. We consider the series formula (2.10) and split it into two sub-series according to the sign of $n\alpha-\beta$. Let $N:=\lfloor\beta/\alpha\rfloor$ denote the cutoff point, and we have:

$$ {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) = \sum_{n=0}^{\lfloor \beta/\alpha \rfloor} \frac{(-\gamma)_n \delta^n}{n!} \, {}^{RL}_{0}D^{\beta-n\alpha}_{x} f(x) + \sum_{n=\lfloor \beta/\alpha \rfloor+1}^{\infty} \frac{(-\gamma)_n \delta^n}{n!} \, {}^{RL}_{0}I^{n\alpha-\beta}_{x} f(x). $$

Since $f \in C_\eta$, all fractional integrals of f are also in C_η, so the second sub-series is a locally uniformly convergent sum of functions in C_η and therefore in C_η itself. Since $f \in {}^{R}\Omega^{\beta}_{\eta}$, all fractional derivatives of f up to order at most β are in C_η, so the first sub-series is a finite sum of functions in C_η and therefore in C_η itself. Since C_η is a vector space, the proof is complete. □

Remark 3.1: Let $g \in C_\eta$ and

$$ f(x) = {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} g(x). $$

Making use of Theorem 3.2, we get the inclusion $f \in \Omega^{\mu}_{\eta}$ and the formula

$$ \begin{aligned} \left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} \circ {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f \right)(x) &= \left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} \circ {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} \circ {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} g \right)(x) \\ &= {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} g(x) = f(x). \end{aligned} $$

This means that the Prabhakar fractional integral operator ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$ is a left inverse, therefore a two-sided inverse, of the Prabhakar fractional derivative operator ${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ on some subspace of ${}^{R}\Omega^{\beta}_{\eta}$ which contains in particular the functions $f = {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} g$ that can be written as Prabhakar derivatives of functions in C_η.

For a more general understanding of what happens when we apply a Prabhakar integral to a Prabhakar derivative, not just in a subspace but for any function in the space ${}^{R}\Omega^{\beta}_{\eta}$, we have the following theorem.

Theorem 3.4: Let $f \in {}^{R}\Omega^{\beta}_{\eta}$, $0 < \beta \leq 1$, $\eta \geq -1$. Then, for any $\alpha, \gamma, \delta \in \mathbb{R}$ with $\alpha > 0$,

$$ (Ff)(x) := \left(f - {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} \, {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f \right)(x) = x^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\delta x^{\alpha}\right) \left({}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x} f \right)(0), \tag{3.7} $$

where $F := E - {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\, {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ (E being the identity operator on the space ${}^{R}\Omega^{\beta}_{\eta}$) is called the projection of the Prabhakar fractional operator, following the terminology of [28], and where $\left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} f\right)(0)$ means $\lim\limits_{x\to 0}\left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} f\right)(x)$.

Proof. Let

$$\phi := {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\, {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f. \tag{3.8}$$

By Theorem 3.3, this is well-defined for $f \in \Omega^{\beta}_{\eta}$, and by Remark 3.1, we have

$$\phi \in C_{\eta+\beta} \cap \Omega^{\beta}_{\eta}. \tag{3.9}$$

Applying the Prabhakar fractional derivative ${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ to the function $\phi(x)$ and using Theorem 3.2, we readily obtain

$${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}\phi(x) = {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}\, {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\, {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) = {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) \tag{3.10}$$

which implies that

$$f - \phi \in \ker\left({}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}\right). \tag{3.11}$$

Writing $g = f - \phi$ and using the definition of the Prabhakar fractional derivative, we have

$${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} g(x) = 0 \Longleftrightarrow \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{P}_{0}I^{\alpha,n-\beta,-\gamma,\delta}_{x} g(x)\right) = 0.$$

By standard properties of classical derivatives, this means ${}^{P}_{0}I^{\alpha,n-\beta,-\gamma,\delta}_{x} g(x)$ must be a polynomial of degree less than n:

$${}^{P}_{0}I^{\alpha,n-\beta,-\gamma,\delta}_{x} g(x) = \sum_{k=0}^{n-1} a_k x^k,$$

where the a_k are constants. This means

$$g(x) = \sum_{k=0}^{n-1} a_k \cdot {}^{P}_{0}D^{\alpha,n-\beta,-\gamma,\delta}_{x}(x^k).$$

Since we assumed $0 < \beta \leq 1$, this means $n = 1$ and the kernel is generated by the function

$$
\begin{aligned}
{}^{PR}_{\;\;0}D_x^{\alpha,\beta,\gamma,\delta}(1) &= \sum_{m=0}^{\infty} \frac{\Gamma(\delta+m)}{\Gamma(\delta)m!}\, {}^{RL}_{\;\;0}I^{\alpha m+\beta-1}(1) \\
&= \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+m)}{\Gamma(\delta)m!} \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} \\
&= x^{\beta-1} E_{\alpha,\beta}^{\gamma}(\delta x^{\alpha})
\end{aligned}
$$

By the definition of $g = f - \phi$, we get

$$
f(x) = \phi(x) + a x^{\beta-1} E_{\alpha,\beta}^{\gamma}(\delta x^{\alpha}), \tag{3.12}
$$

for some constant a. It follows from this last formula that

$$
{}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} f(x) = {}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} \phi(x) + a\, {}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} \left(x^{\beta-1} E_{\alpha,\beta}^{\gamma}(\delta x^{\alpha})\right). \tag{3.13}
$$

Now, the last term simplifies as follows:

$$
\begin{aligned}
{}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} \left(x^{\beta-1} E_{\alpha,\beta}^{\gamma}(\delta x^{\alpha})\right) &= \sum_{n=0}^{\infty} \frac{(-\gamma)_n \delta^n}{n!}\, {}^{RL}_{\;\;0}I_x^{\alpha n+1-\beta} \left(x^{\beta-1} E_{\alpha,\beta}^{\gamma}(\delta x^{\alpha})\right) \\
&= \sum_{n=0}^{\infty} \frac{(-\gamma)_n \delta^n}{n!}\, {}^{RL}_{\;\;0}I_x^{\alpha n+1-\beta} \left(x^{\beta-1} \sum_{m=0}^{\infty} \frac{(\gamma)_m \delta^m x^{\alpha m}}{m!\Gamma(\alpha m+\beta)} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-\gamma)_n \delta^n}{n!} \sum_{m=0}^{\infty} \frac{(-\gamma)_m \delta^m}{m!} \frac{x^{m\alpha+n\alpha}}{\Gamma((m+n)\alpha+1)} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\gamma)_n (\gamma)_m \delta^{n+m} x^{n\alpha+m\alpha}}{n!m!\Gamma((n+m)\alpha+1)} \\
&= \sum_{k=0}^{\infty} \left[\sum_{n+m=0}^{\infty} \frac{(-\gamma)_n (\gamma)_m}{n!m!} \right] \frac{\delta^k x^{2k}}{\Gamma(k\alpha+1)} \\
&= \sum_{k=0}^{\infty} \left[\frac{(-\gamma+\gamma)_k}{k!} \right] \frac{\delta^k x^{2k}}{\Gamma(k\alpha+1)} \\
&= 1,
\end{aligned}
$$

where in the last step we used the Chu–Vandermonde identity to simplify the finite sum of double Pochhammer symbols. Using this in equation (3.13), we get

$$
\left({}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} f\right)(x) = \left({}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} \phi\right)(x) + a. \tag{3.14}
$$

The inclusion (3.9), Theorem 3.1, and the condition $\eta \geq -1$ together lead to the fact that the function ${}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} \phi$ is in C_0 and therefore continuous on the interval

$[0,\infty)$. Then, due to relation 3.14, the function ${}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x} f$ is also continuous on the interval $[0,\infty)$. Using the definition of ϕ and some well-known properties of Prabhakar operators, we have

$$
\begin{aligned}
{}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x}\phi(x) &= {}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x}\,{}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\,{}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f(x)\\
&= {}^{P}_{0}I^{\alpha,1-\beta+\beta,-\gamma+\gamma,\delta}_{x}\,{}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f(x)\\
&= {}^{P}_{0}I^{\alpha,1,0,\delta}_{x}\,{}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f(x)\\
&= {}^{RL}_{0}I^{1}_{x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\,{}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x}f(x)\right)\\
&= \left({}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x}f\right)(x)-\left({}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x}f\right)(0),
\end{aligned}
\tag{3.15}
$$

where the last step comes simply from the fundamental theorem of calculus. Comparing (3.15) with (3.14), we obtain the value of a as:

$$a = \left({}^{P}_{0}I^{\alpha,1-\beta,-\gamma,\delta}_{x}f\right)(0). \tag{3.16}$$

Using equation (3.16) in equation (3.14) leads us to the relation (3.7). □

3.2 Algebraic Structure

Definition 3.1: The algebraic inverse of the Prabhakar fractional integral operator ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$ is said to be the element $S_{\alpha,\beta,\gamma,\delta} = S_{\beta,\gamma}$ of the field M_{-1} which is reciprocal to the element $P_{\beta,\gamma} \in C_{-1} \hookrightarrow M_{-1}$. That is,

$$S_{\beta,\gamma} = \frac{I}{P_{\beta,\gamma}} \equiv \frac{P_{\beta,\gamma}}{P_{\beta,\gamma} * P_{\beta,\gamma}} \equiv \frac{P_{\beta,\gamma}}{P_{2\beta,2\gamma}}, \tag{3.17}$$

where $I = \frac{P_{\beta,\gamma}}{P_{\beta,\gamma}}$ denotes the multiplicative identity element of the field M_{-1}, and where we have used the semigroup property (3.4) of the Mittag-Leffler function.

From the semigroup property (3.4), we can easily define positive integer powers of the function $P_{\beta,\gamma}$ in the ring C_{-1}:

$$P^{n}_{\beta,\gamma}(x) := (P_{\beta,\gamma} * \cdots * P_{\beta,\gamma}) = P_{n\beta,n\gamma}(x). \tag{3.18}$$

It is natural to extend this relation to arbitrary positive real powers of $P_{\beta,\gamma}$, using the multiplication by the exponent as in [20]:

$$P^{\lambda}_{\beta,\gamma} = P_{\lambda\beta,\lambda\gamma}(x), \qquad \lambda > 0. \tag{3.19}$$

Making use of the algebraic inverse as well, we obtain fractional powers to any real order of the element $S_{\beta,\gamma}$:

$$S^{\lambda}_{\beta,\gamma} = \begin{cases} P^{-\lambda}_{\beta,\gamma}(x), & \lambda < 0. \\ I, & \lambda = 0. \\ \frac{I}{P^{\lambda}_{\beta,\gamma}}, & \lambda > 0. \end{cases} \tag{3.20}$$

These satisfy the following semigroup property:

$$S^{\lambda}_{\beta,\gamma} \cdot S^{\nu}_{\beta,\gamma} = S^{\lambda+\nu}_{\beta,\gamma}, \qquad \lambda, \nu \in C. \tag{3.21}$$

Therefore, the set of all $P_{\beta,\gamma}$ and $S_{\beta,\gamma}$, for fixed parameters $\alpha > 0$ and $\delta \in \mathbb{R}$, forms a multiplicative group, isomorphic to $(\mathbb{R}^2, +)$, within the field M_{-1}.

We know that the Prabhakar fractional integral can be identified with the element $P_{\beta,\gamma}$ of the field M_{-1}, but what about the Prabhakar fractional derivative? Intuitively, the derivative is expected to be the inverse of an integral, so we expect to find a close relationship between the Prabhakar derivative ${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ and the algebraic inverse $S_{\beta,\gamma}$ of the Prabhakar integral. This will be important for applications, if we can express Prabhakar derivatives in terms of algebraic operations.

Theorem 3.5: The Prabhakar fractional differential operator ${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ may be represented in the field M_{-1} in the form

$${}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f = S_{\beta,\gamma} \cdot f - S_{\beta,\gamma} \cdot Ff, \qquad f \in \Omega^{\beta}_{-1}, \tag{3.22}$$

where $F := E - {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\, {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ is the projection of the Prabhakar integral, which according to Theorem 3.4 is given by a constant (depending on f) times $P_{\beta,\gamma}$.

Proof. Let $f \in {}^{R}\Omega^{\beta}_{-1}$, and write it in the following form using the definition of F:

$$f(x) = \big(Ff\big)(x) + \big({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\, {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f\big)(x),$$

which can be rewritten algebraically (in the ring C_{-1}) as

$$f = Ff + P_{\beta,\gamma} * \big({}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f\big)$$

Upon multiplying both sides by $S_{\beta,\gamma}$, and using the algebraic inverse relation, we get the required result (3.22). □

Another important step in application of an operational calculus is to represent solutions of some equations in the field M_{-1} by elements of the initial ring of functions C_{-1}. This is required because weak solutions using the algebraic abstraction of the field (whose elements in general are not necessarily realisable as functions) are less useful than explicit solutions as functions. We can obtain a useful class of such functions for our operational calculus using the following result.

Theorem 3.6: Consider a multiple power series as a function of several complex variables $z_1, \cdots, z_m$:

$$\sum_{k_1,\dots,k_m=0}^{\infty} c_{k_1,\dots,k_m} z_1^{k_1} \times \dots \times z_m^{k_m}, \qquad c_{k_1,\dots,k_m} \in \mathbb{C} \text{ fixed coefficient,} \tag{3.23}$$

and assume it is convergent at a given point $z_0 = (z_{10}, \dots, z_{m0}) \in \mathbb{C}^m$.

Then, for any given $\kappa, \lambda_1, \cdots, \lambda_m > 0$ and $\gamma, \delta \in \mathbb{R}$, $\alpha, \beta > 0$, the field element

$$S^{-\kappa}_{\beta,\gamma} \sum_{k_1,\dots,k_m=0}^{\infty} c_{k_1,\dots,k_m} \big(S^{-\lambda_1}_{\beta,\gamma}\big)^{k_1} \times \dots \times \big(S^{-\lambda_m}_{\beta,\gamma}\big)^{k_m} \tag{3.24}$$

represents a function in the ring C_{-1}.

Proof. Given the semigroup property of the $S_{\beta,\gamma}$ elements and the fact that negative powers of $S_{\beta,\gamma}$ are Mittag-Leffler functions in the ring, we have

$$S_{\beta,\gamma}^{-\kappa}\sum_{k_1,\ldots,k_m=0}^{\infty}c_{k_1,\ldots,k_m}\left(S_{\beta,\gamma}^{-\lambda_1}\right)^{k_1}\times\cdots\times\left(S_{\beta,\gamma}^{-\lambda_m}\right)^{k_m}$$
$$=\sum_{k_1,\ldots,k_m=0}^{\infty}c_{k_1,\ldots,k_m}P_{(\lambda_1k_1+\ldots+\lambda_mk_m+\kappa)(\beta,\gamma)}(x).$$

Expanding the Mittag-Leffler function summand as an infinite series itself, we can write this expression as a multiple series of power functions:

$$x^{\kappa\beta-1}\sum_{k_1,\ldots,k_m,k=0}^{\infty}c_{k_1,\ldots,k_m}\left(x^{\beta\lambda_1}\right)^{k_1}\cdots\left(x^{\beta\lambda_m}\right)^{k_m}\left(x^{\alpha}\right)^{k}$$
$$\times\frac{\delta^k\Gamma(\gamma\lambda_1k_1+\cdots+\gamma\lambda_mk_m+k+\kappa\gamma)}{\Gamma(\gamma\lambda_1k_1+\cdots+\gamma\lambda_mk_m+\kappa\gamma)\Gamma(k+1)\Gamma(\beta\lambda_1k_1+\cdots+\beta\lambda_mk_m+\alpha k+\kappa\beta)}.$$

Since $\kappa\beta-1>-1$, it will be sufficient to show that the multiple sum here is convergent for $x\in[0,\infty)$.

In the case $\gamma>0$, we use the fact that

$$\frac{\Gamma(\gamma\lambda_1k_1+\cdots+\gamma\lambda_mk_m+k+\kappa\gamma)}{\Gamma(\gamma\lambda_1k_1+\cdots+\gamma\lambda_mk_m+\kappa\gamma)\Gamma(k+1)}\leq 1$$

for all but finitely many values of the indices $k_1,\cdots,k_m,k$, and then proceed as in the proof of [28, Theorem 6] to bound the other parts of the series summand and obtain convergence as desired.

In the case $\gamma<0$, we use the well-known identity

$$\frac{\Gamma(\rho+k)}{\Gamma(\rho)}=(-1)^k\frac{\Gamma(1-\rho)}{\Gamma(1-\rho-k)},\qquad k\in\mathbb{N},\rho\in\mathbb{C},$$

to replace the quotient of two gamma functions of negative arguments by a quotient of two gamma functions of positive arguments, and repeat the same procedure as above.

In the case $\gamma=0$, the Mittag-Leffler function reduces to a power function, and our result reduces to exactly [28, Theorem 6]. □

Applying Theorem 3.6 enables us to consider various functions of $S_{\beta,\gamma}$ in the field M_{-1} and express them as functions in the ring C_{-1}. We consider some examples as follows.

Example 3.1: For any constants $a \in \mathbb{R}$ and $m \in \mathbb{N}$, as well as $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha, \beta > 0$, we consider the field element $\frac{I}{(S_{\beta,\gamma}-aI)^m}$ and rewrite it as a sum of negative powers of $S_{\beta,\gamma}$ in order to express it as a function in C_{-1}:

$$\begin{aligned}
\frac{I}{(S_{\beta,\gamma}-aI)^m} &= S_{\beta,\gamma}^{-m}\frac{I}{(I-aS_{\beta,\gamma})^m} = S_{\beta,\gamma}^{-m}\sum_{i=0}^{\infty}\frac{(m)_i}{i!}a^i S_{\beta,\gamma}^{-i} \\
&= \sum_{i=0}^{\infty}\frac{(m)_i}{i!}a^i P_{(m+i)(\beta,\gamma)}(x) = \sum_{i=0}^{\infty}\frac{(m)_i}{i!}a^i x^{(m+i)\beta-1}E_{\alpha,(m+i)\beta}^{(m+i)\gamma}(\delta x^{\alpha}) \\
&= \sum_{i=0}^{\infty}a^i\frac{(m)_i}{i!}x^{(m+i)\beta-1}\sum_{j=0}^{\infty}\frac{((m+i)\gamma)_j}{j!}\cdot\frac{\delta^j x^{\alpha j}}{\Gamma(\alpha j+(m+i)\beta)} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a^i\frac{(m)_i}{i!}\cdot\frac{((m+i)\gamma)_j}{j!}\cdot\frac{\delta^j x^{\alpha j+(m+i)\beta-1}}{\Gamma(\alpha j+(m+i)\beta)}.
\end{aligned} \tag{3.25}$$

This double series is of the general category investigated by Srivastava and Daoust [51] and has a similar form to some recently defined bivariate Mittag-Leffler functions [21].

Example 3.2: Let $0 < \beta_1 < \beta_2 < ... < \beta_m$ and $\alpha > 0, \gamma_1, \cdots, \gamma_m, \delta$ be real numbers. We consider the general element $\dfrac{I}{\sum_{i=1}^{m}\lambda_i S_{\beta_i,\gamma_i}}$ in the field M_{-1} and manipulate it as follows:

$$\begin{aligned}
\frac{I}{\sum\limits_{i=1}^{m}\lambda_i S_{\beta_i,\gamma_i}} &= \frac{\lambda_m^{-1}S_{\beta_m,\gamma_m}^{-1}}{I-\sum\limits_{i=1}^{m-1}\frac{-\lambda_i}{\lambda_m}S_{\beta_i,\gamma_i}S_{\beta_m,\gamma_m}^{-1}} \\
&= \frac{\lambda_m^{-1}S_{-\beta_m,-\gamma_m}}{I-\sum\limits_{i=1}^{m-1}\frac{-\lambda_i}{\lambda_m}S_{\beta_i-\beta_m,\gamma_i-\gamma_m}} \\
&= \frac{S_{-\beta_m,-\gamma_m}}{\lambda_m}\sum_{k=0}^{\infty}\left(\sum_{i=1}^{m-1}\frac{-\lambda_i}{\lambda_m}S_{\beta_i-\beta_m,\gamma_i-\gamma_m}\right)^k.
\end{aligned} \tag{3.26}$$

We then use the multinomial identity for powers of a finite sum:

$$\left(\sum_{i=1}^{m-1}\frac{-\lambda_i}{\lambda_m}S_{\beta_i-\beta_m,\gamma_i-\gamma_m}\right)^k$$
$$=\sum_{n_1+\ldots+n_{m-1}=k}\frac{k!}{n_1!\ldots n_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1}\ldots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}}$$
$$\times S^{n_1}_{\beta_i-\beta_m,\gamma_i-\gamma_m}\ldots S^{n_{m-1}}_{\beta_i-\beta_m,\gamma_i-\gamma_m}. \quad (3.27)$$

Using equation (3.27) in (3.26), we get:

$$\frac{I}{\sum_{i=1}^{m}\lambda_i S_{\beta_i,\gamma_i}} \quad (3.28)$$
$$=\frac{S_{-\beta_m,-\gamma_m}}{\lambda_m}\sum_{k=0}^{\infty}\left(\sum_{n_1+\cdots+n_{m-1}=k}\frac{k!}{n_1!\cdots n_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1}\cdots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}}\right.$$
$$\left.\times S^{n_1}_{\beta_i-\beta_m,\gamma_i-\gamma_m}\cdots S^{n_{m-1}}_{\beta_i-\beta_m,\gamma_i-\gamma_m}\right)$$
$$=\frac{S_{-\beta_m,-\gamma_m}}{\lambda_m}\sum_{n_1,\cdots,n_{m-1}=0}^{\infty}\frac{(n_1+\cdots+n_{m-1})!}{n_1!\cdots n_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1}\cdots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}}$$
$$\times S_{n_1(\beta_1-\beta_m)+\cdots+n_{m-1}(\beta_{m-1}-\beta_m),n_1(\gamma_1-\gamma_m)+\cdots+n_{m-1}(\gamma_{m-1}-\gamma_m)}$$
$$=\sum_{n_1,\cdots,n_{m-1}=0}^{\infty}\frac{(n_1+\cdots+n_{m-1})!}{n_1!\cdots n_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1}\cdots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}} \quad (3.29)$$
$$\times S_{n_1(\beta_1-\beta_m)+\cdots+n_{m-1}(\beta_{m-1}-\beta_m)-\beta_m,n_1(\gamma_1-\gamma_m)+\cdots+n_{m-1}(\gamma_{m-1}-\gamma_m)-\gamma_m}$$

Now, using the fact that the field element $S_{\beta,\gamma}$ is a modified Mittag-Leffler function for any negative value of β, we find that equation (3.29) becomes

$$\frac{I}{\sum_{i=1}^{m}\lambda_i S_{\beta_i,\gamma_i}}=\sum_{n_1,\cdots,n_{m-1}=0}^{\infty}\frac{(n_1+\cdots+n_{m-1})!}{n_1!\cdots n_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1}\cdots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}}$$
$$\times x^{n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m-1}E^{n_1(\gamma_m-\gamma_1)+\cdots+n_{m-1}(\gamma_m-\gamma_{m-1})+\gamma_m}_{\alpha,n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m}(\delta x^\alpha). \quad (3.30)$$

3.3 Fractional Differential Equations

In this section, we demonstrate with a simple model example to begin with, followed by a much more general Prabhakar differential equation, linear and with constant coefficients, but with an arbitrary number of terms.

Example 3.3: We consider the following initial value problem:

$$ {}^{PR}_{\ 0}D_x^{\alpha,\beta,\gamma,\delta} y(x) - \lambda y(x) = f(x), \qquad x > 0; $$
$$ \left({}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta}\right) y(0) = a, $$

where $\alpha, \beta, \gamma, \delta, a$ are fixed constants with $0 < \beta \leq 1$.

Making use of Theorem 3.5 and Theorem 3.4, we can rewrite this problem in the form of an algebraic equation in the field M_{-1}:

$$ S_{\beta,\gamma} \cdot y - \lambda y = f + S_{\beta,\gamma} \cdot y_0, $$

where

$$ y_0 := x^{\beta-1} E_{\alpha,\beta}^{\gamma}(\delta x^{\alpha}) \times {}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} f(0) = a P_{\beta,\gamma}. $$

Clearly, this algebraic equation has the following unique solution in the field M_{-1}:

$$ \begin{aligned} y &= \frac{I}{S_{\beta,\gamma} - \lambda} \cdot f + \frac{S_{\beta,\gamma}}{S_{\beta,\gamma} - \lambda} \cdot y_0 \\ &= \frac{I}{S_{\beta,\gamma} - \lambda} \cdot f + \frac{S_{\beta,\gamma}}{S_{\beta,\gamma} - \lambda} \cdot a P_{\beta,\gamma} \\ &= \frac{I}{S_{\beta,\gamma} - \lambda} \cdot f + \frac{aI}{S_{\beta,\gamma} - \lambda}. \end{aligned} \tag{3.31} $$

Using the above-proved relation (3.25) in the case $m = 1$, we find

$$ \frac{I}{S_{\beta,\gamma} - \lambda} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \lambda^i \frac{((1+i)\gamma)_j}{j!} \cdot \frac{\delta^j x^{\alpha j + (i+1)\beta - 1}}{\Gamma(\alpha j + (1+i)\beta)}, $$

which means that (3.31) becomes

$$y(x) = \int_0^x \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lambda^i \frac{((1+i)\gamma)_j}{j!} \cdot \frac{\delta^j (x-t)^{\alpha j+(i+1)\beta-1}}{\Gamma(\alpha j+(1+i)\beta)} f(t)\,\mathrm{d}t$$
$$+ a \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lambda^i \frac{((1+i)\gamma)_j}{j!} \cdot \frac{\delta^j x^{\alpha j+(i+1)\beta-1}}{\Gamma(\alpha j+(1+i)\beta)},$$

which is the solution function for the given Prabhakar differential equation.

Remark 3.2: The above argument improves slightly on the method of [28, Example 1] for solving a similar problem, by our more efficient way of dealing with the initial value term, noting that $S_{\beta,\gamma}$ precisely cancels the $P_{\beta,\gamma}$ which is part of the formula for y_0. Following the methodology of [28, Example 1] would lead us instead to the following:

$$y_1 = \frac{I}{S_{\beta,\gamma}-\lambda} \cdot f = \int_0^x \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lambda^i \frac{((1+i)\gamma)_j}{j!} \cdot \frac{\delta^j (x-t)^{\alpha j+(i+1)\beta-1}}{\Gamma(\alpha j+(1+i)\beta)} f(t)\,\mathrm{d}t;$$

$$\begin{aligned}
y_2 &= \frac{S_{\beta,\gamma}}{S_{\beta,\gamma}-\lambda} \cdot y_0 = y_0 + \frac{\lambda}{S_{\beta,\gamma}-\lambda} \cdot y_0 \\
&= x^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha})\, {}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} f(0) + \frac{\lambda}{S_{\beta,\gamma}-\lambda} \cdot x^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha})\, {}^{P}_{0}I_x^{\alpha,1-\beta,-\gamma,\delta} f(0) \\
&= a x^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha}) \qquad (3.32)\\
&\quad + \lambda \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lambda^i \frac{((1+i)\gamma)_j}{j!} \cdot \frac{\delta^j x^{\alpha j+(i+1)\beta-1}}{\Gamma(\alpha j+(1+i)\beta)} \right) * \left(a x^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha}) \right). \qquad (3.33)
\end{aligned}$$

Since convolution of power function works by

$$\frac{x^{\sigma-1}}{\Gamma(\sigma)} * \frac{x^{\rho-1}}{\Gamma(\rho)} = \frac{x^{\sigma+\rho-1}}{\Gamma(\sigma+\rho)},$$

we can simplify (3.33) to:

$$
\begin{aligned}
y_2 &= ax^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha}) + a\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda^{i+1}\frac{((1+i)\gamma)_j(\gamma)_k}{j!k!}\frac{\delta^{j+k}x^{\alpha(j+k)+(i+2)\beta-1}}{\Gamma(\alpha(j+k)+(i+2)\beta)} \\
&= ax^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha}) + a\sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty}\left[\sum_{j+k=\ell}\frac{((1+i)\gamma)_j(\gamma)_k}{j!k!}\right]\lambda^{i+1}\frac{\delta^{\ell}x^{\alpha\ell+(i+2)\beta-1}}{\Gamma(\alpha\ell+(i+2)\beta)} \\
&= ax^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha}) + a\sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{((2+i)\gamma)_{\ell}}{\ell!}\cdot\lambda^{i+1}\frac{\delta^{\ell}x^{\alpha\ell+(i+2)\beta-1}}{\Gamma(\alpha\ell+(i+2)\beta)}.
\end{aligned}
$$

Replacing $i+2$ by $i+1$ to get a double sum over $1 \le i < \infty$ and $0 \le \ell < \infty$, and then folding in the first term as an extra sum over $0 \le \ell < \infty$ in the case $i = 0$, we obtain the same result as above.

Example 3.4: Let us now consider the following more general initial value problem:

$$
\left.
\begin{aligned}
&\sum_{i=1}^{m}\lambda_i\left({}^{P}_{0}D_x^{\alpha,\beta_i,\gamma_i,\delta}y\right)(x) = f(x), \qquad x > 0, \\
&\left({}^{P}_{0}I_x^{\alpha,1-\beta_i,-\gamma_i,\delta}y\right)(0) = a_i, \qquad 1 \le i \le m,
\end{aligned}
\right\}
\tag{3.34}
$$

where $0 \le \beta_1 < \cdots < \beta_m \le 1$ and $\alpha > 0$ and $\gamma_i, \lambda_i, \delta, a_i \in \mathbb{R}$ are constants ($1 \le i \le m$) and $f \in C_{-1}$, the problem to be solved for y in the space $\Omega^{\beta_m}_{-1}$.

Since $y \in {}^{R}\Omega^{\beta_m}_{-1}$ and therefore $y \in {}^{R}\Omega^{\beta_m}_{-1}$ for all i, the Cauchy problem (3.34) can be reduced, by Theorem 3.5 and Theorem 3.4, to the following algebraic equation in the field M_{-1}:

$$
\sum_{i=1}^{m}\lambda_i\left(S_{\beta_i,\gamma_i}\cdot y - S_{\beta_i,\gamma_i}\cdot y_{i0}\right) = f(x),
\tag{3.35}
$$

where for each $i = 1, \cdots, m$

$$
y_{i0} = x^{\beta_i-1}E^{\gamma_i}_{\alpha,\beta_i}(\delta x^{\alpha})\left({}^{P}_{0}I_x^{\alpha,1-\beta_i,-\gamma_i,\delta}y\right)(0) = \left({}^{P}_{0}I_x^{\alpha,1-\beta_i,-\gamma_i,\delta}y\right)(0)\cdot P_{\beta_i,\gamma_i}.
$$

By the initial conditions given in (3.34), we know that $y_{i0} = a_i P_{\beta_i,\gamma_i}$ for $1 \le i \le m$. Therefore, the algebraic equation (3.35) simplifies to:

$$
\sum_{i=1}^{m}\lambda_i S_{\beta_i,\gamma_i}\cdot y = f + \sum_{i=1}^{m}\lambda_i S_{\beta_i,\gamma_i}\cdot y_{i0} = f + \left(\sum_{i=1}^{m}a_i\lambda_i\right)I,
\tag{3.36}
$$

using the definition of S_{β_i,γ_i} as the algebraic inverse of P_{β_i,γ_i}. Now it is clear that equation (3.36) has a unique solution in the field M_{-1} , which is given by:

$$y = \frac{I}{\sum_{i=1}^{m} \lambda_i S_{\beta_i,\gamma_i}} \cdot f + \frac{\left(\sum_{i=1}^{m} a_i \lambda_i\right) I}{\sum_{i=1}^{m} \lambda_i S_{\beta_i,\gamma_i}}. \tag{3.37}$$

By using the representation (3.30) of the field element $\frac{I}{\sum_{i=1}^{m} \lambda_i S_{\beta_i,\gamma_i}}$ as a function, we get:

$$\begin{aligned} y(x) = &\int_0^x \sum_{n_1,\cdots,n_{m-1}=0}^{\infty} \frac{(n_1+\cdots+n_{m-1})!}{n_1!\cdots n_{m-1}!} \left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1} \cdots \left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}} \\ &\times t^{n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m-1} E_{\alpha,n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m}^{n_1(\gamma_m-\gamma_1)+\cdots+n_{m-1}(\gamma_m-\gamma_{m-1})+\gamma_m}(\delta t^\alpha) f(x-t)\,\mathrm{d}t \\ &+ \left(\sum_{i=1}^{m} a_i\lambda_i\right) \sum_{n_1,\cdots,n_{m-1}=0}^{\infty} \frac{(n_1+\cdots+n_{m-1})!}{n_1!\cdots n_{m-1}!} \left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1} \cdots \left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}} \\ &\times x^{n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m-1} E_{\alpha,n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m}^{n_1(\gamma_m-\gamma_1)+\cdots+n_{m-1}(\gamma_m-\gamma_{m-1})+\gamma_m}(\delta x^\alpha), \end{aligned} \tag{3.38}$$

and this function is the final solution in ${}^R\Omega_{-1}^{\beta}$ to the initial value problem (3.34).

Chapter 4

PRABHAKAR FRACTIONAL CALCULUS OF CAPUTO TYPE

In this chapter, we shall construct the theory of the Caputo-type Prabhakar derivative operator ${}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$: defining function spaces for this operator, checking its relationships with the corresponding integral operator and Riemann–Liouville-type derivative operator, and finally embedding it as an element of the abstract algebraic field M_{-1}.

4.1 Action of Operators on Spaces

Theorem 4.1.1. For fixed $m \in \mathbb{N}$, the Caputo-type Prabhakar fractional derivative ${}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f$ is well-defined for any $f \in C^{m}_{-1}$ and $\alpha \in \mathbb{R}^{+}$, $\beta \in [0,m)$, $\gamma,\delta \in \mathbb{R}$, and we have the following inclusion:

$${}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f \in C_{-1}. \tag{4.1}$$

Proof. Let $f \in C^{m}_{-1}$; then by Definition (2.3) in Chapter 2 we have $f^{(m)} \in C_{-1}$ and therefore $f^{(k)} \in C_{-1}$ for all $k = 0,1,\cdots,m$. By Theorem 3.1 in Chapter 3 and the definition of the Prabhakar–Caputo derivative, we have that

$${}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f = {}^{P}_{0}I^{\alpha,k-\beta,-\gamma,\delta}_{x} f^{(k)} \in C_{-1},$$

where $k = \lfloor \beta \rfloor + 1 \leq m$. □

Theorem 4.1.2. Let $\alpha,\beta \in \mathbb{R}^{+}$ and $\gamma,\delta \in \mathbb{R}$ and $m = \lfloor \beta \rfloor + 1 \in \mathbb{N}$. For any $f \in C^{m}_{\eta}$ (where $\eta \geq -1$), its Prabhakar fractional derivatives of Riemann–Liouville and Caputo type are connected by the following relation:

$$ {}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) = {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) - \sum_{k=0}^{m-1} x^{k-\beta} E^{\gamma}_{\alpha,k-\beta}(\delta(x)^{\alpha}) f^{(k)}(0). \tag{4.2} $$

Proof. We use the definitions of the Prabhakar–Caputo derivative and the Prabhakar integral, the semigroup property for the Riemann–Liouville differintegral of an integral, and then the classical formula for the mth integral of the mth derivative of a function:

$$
\begin{aligned}
{}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) &= {}^{P}_{0}I^{\alpha,m-\beta,-\gamma,\delta}_{x} f^{(m)}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \, {}^{RL}_{0}I^{n\alpha+m-\beta}_{x} f^{(m)}(x) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \, {}^{RL}_{0}I^{n\alpha-\beta}_{x} \left[{}_{0}I^{m}_{x} f^{(m)}(x) \right] \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \, {}^{RL}_{0}I^{n\alpha-\beta}_{x} \left[f(x) - \sum_{k=0}^{m-1} \frac{x^k}{\Gamma(k+1)} f^{(k)}(0) \right] \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \, {}^{RL}_{0}I^{n\alpha-\beta}_{x} f(x) \\
&\qquad - \sum_{n=0}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \sum_{k=0}^{m-1} \frac{x^{n\alpha-\beta+k}}{\Gamma(n\alpha-\beta+k+1)} f^{(k)}(0) \\
&= {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \sum_{n=0}^{\infty} \frac{(\gamma)_n \delta^n}{n!} \frac{x^{n\alpha-\beta+k}}{\Gamma(n\alpha-\beta+k+1)} \\
&= {}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f(x) - \sum_{k=0}^{m-1} x^{k-\beta} E^{\gamma}_{\alpha,k-\beta}(\delta(x)^{\alpha}) \cdot f^{(k)}(0).
\end{aligned}
$$

The swapping of summations is justified since all of the infinite sums involved are locally uniformly convergent: the series formula (analogous to (2.9)) for the Prabhakar fractional derivative of Riemann–Liouville type, and the power series for the three-parameter Mittag-Leffler function. We have also used standard results on the Riemann–Liouville differintegrals of a power function. □

Theorem 4.1.3. Let $\alpha,\beta \in \mathbb{R}^+$ and $\gamma,\delta \in \mathbb{R}$ and $m = \lfloor \beta \rfloor + 1 \in \mathbb{N}$. For any $f \in C^m_\eta$ (where $\eta \geq -1$), its Prabhakar fractional derivative of Caputo type has the following inversion relation with the corresponding Prabhakar fractional integral:

$$ \left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x} \, {}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x} f \right)(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \qquad x \geq 0. $$

Proof. By using the definition of the Prabhakar–Caputo derivative and the semigroup property (3.5) of Prabhakar integrals, we get:

$$\begin{aligned}\left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\,{}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x) &= {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\left[{}^{P}_{0}I^{\alpha,m-\beta,-\gamma,\delta}_{x}f^{(m)}(x)\right]\\ &= {}^{P}_{0}I^{\alpha,m,0,\delta}_{x}f^{(m)}(x)\\ &= {}^{R}_{0}I^{m}_{x}f^{(m)}(x)\\ &= f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^{+})\frac{x^{k}}{k!}. \qquad x\geq 0,\end{aligned}$$

where we have used the fact (clear from the series formula (2.1.3)) that the Prabhakar integral ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$ in the case $\gamma=0$ is just the Riemann–Liouville integral ${}^{RL}_{0}I^{\beta}_{x}$. □

4.2 Algebraic Structure

Theorem 4.2.1. Let $\alpha,\beta\in\mathbb{R}^{+}$ and $\gamma,\delta\in\mathbb{R}$ and $m=\lfloor\beta\rfloor+1\in\mathbb{N}$. The Prabhakar fractional derivative operator ${}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}$ may be represented in the field M_{-1} in the form

$$ {}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f = S_{\beta,\gamma}\cdot f - S_{\beta,\gamma}\cdot Ff, \qquad f\in C^{m}_{-1}, \tag{4.3}$$

where the element Ff is given by initial values of f as

$$Ff=\sum_{k=0}^{m-1}f^{(k)}(0^{+})\cdot\frac{x^{k}}{k!}.$$

Proof. Let $f\in C^{m}_{-1}$, and use the result of Theorem 4.1.3 with the notation of Ff:

$$f(x)=\big(Ff\big)(x)+\left({}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\,{}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x).$$

Algebraically, in the ring C_{-1}, this is the same as

$$f=Ff+P_{\beta,\gamma}*\left({}^{PC}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f\right).$$

Multiplying both sides by the algebraic inverse $S_{\beta,\gamma}$ in the field M_{-1}, we find exactly the required result (4.3). □

We mention the following two examples of expressions in M_{-1} which are defined in terms of the abstract field elements $S_{\beta,\gamma}$ but which can be written as functions in the ring C_{-1}. Both of these are taken from Chapter 3 where they are consequences of Theorem 3.6.

- If $\alpha, \beta \in \mathbb{R}^+$ and $\gamma, \delta \in \mathbb{R}$ and $a \in \mathbb{R}$ and $m \in \mathbb{N}$, then the field element $\frac{I}{(S_{\beta,\gamma}-aI)^m}$ is expressed as the following function in C_{-1}:

$$\frac{I}{(S_{\beta,\gamma}-aI)^m} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a^i \frac{(m)_i}{i!} \cdot \frac{((m+i)\gamma)_j}{j!} \cdot \frac{\delta^j x^{\alpha j+(m+i)\beta-1}}{\Gamma(\alpha j+(m+i)\beta)}. \tag{4.4}$$

- If $\alpha \in \mathbb{R}^+$ and $\beta_m > \beta_{m-1} > \cdots > \beta_2 > \beta_1 > 0$ and $\gamma_1, \cdots, \gamma_m, \delta \in \mathbb{R}$, then the field element $\frac{I}{\sum_{i=1}^m \lambda_i S_{\beta_i,\gamma_i}}$ is expressed as the following function in C_{-1}:

$$\frac{I}{\sum\limits_{i=1}^{m} \lambda_i S_{\beta_i,\gamma_i}} = \sum_{n_1,\cdots,n_{m-1}=0}^{\infty} \frac{(n_1+\cdots+n_{m-1})!}{n_1!\cdots n_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{n_1}\cdots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{n_{m-1}}$$
$$\times x^{n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m-1} E^{n_1(\gamma_m-\gamma_1)+\cdots+n_{m-1}(\gamma_m-\gamma_{m-1})+\gamma_m}_{\alpha,n_1(\beta_m-\beta_1)+\cdots+n_{m-1}(\beta_m-\beta_{m-1})+\beta_m}(\delta x^{\alpha}). \tag{4.5}$$

4.3 Fractional Differential Equations

We now reach the main purpose of this chapter, to solve fractional differential equations with Prabhakar derivatives of Caputo type. Some such equations have already been studied in the literature [12, 46], but, by making use of the Mikusiński's operational calculus framework, we will be able to solve a wider class of equations, including a general linear constant-coefficient (incommensurate, multi-term) fractional differential equation of Prabhakar–Caputo type.

Example 4.1: As a model problem to begin with, let us consider the following fractional initial value problem:

$$
\begin{cases}
{}^{PC}_{\;0}D_x^{\alpha,\beta,\gamma,\delta} y(x) - \lambda y(x) = f(x), & x > 0; \\
y^{(k)}(0) = a_k, & k = 0,1,2,\cdots,m-1,
\end{cases}
\tag{4.6}
$$

where $\alpha, \beta \in \mathbb{R}^+$ and $\gamma, \delta, a_0, a_1, \cdots, a_{m-1} \in \mathbb{R}$ are fixed constants with $m = \lfloor \beta \rfloor + 1$ so that $m-1 \leq \beta < m$.

Using the results of Theorem (4.2.1), the initial value problem (4.6) can be written as the following algebraic equation in M_{-1}:

$$
S_{\beta,\gamma} \cdot y - \lambda y = f + S_{\beta,\gamma} \cdot Fy,
$$

where
$$
Fy = \sum_{k=0}^{m-1} y^{(k)}(0^+) \frac{x^k}{k!} = \sum_{k=0}^{m-1} y^{(k)}(0^+) h_{k+1},
$$

using the same notation h_k as at (3.3). Now it is clear that the unique solution in the field M_{-1} has the form

$$
y = \frac{I}{S_{\beta,\gamma} - \lambda} \cdot f + \frac{S_{\beta,\gamma}}{S_{\beta,\gamma} - \lambda} \cdot Fy. \tag{4.7}
$$

Using (4.4) with $m = 1$, we have

$$
\frac{I}{S_{\beta,\gamma} - \lambda} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{((1+i)\gamma)_j}{j!} \cdot \frac{\lambda^i \delta^j x^{\alpha j + (i+1)\beta - 1}}{\Gamma(\alpha j + (1+i)\beta)},
$$

and therefore y is given as a function in C_{-1} by the sum of the following functions y_1 and y_2 coming from the two terms in (4.7):

$$
\begin{aligned}
y_1 &= \frac{I}{S_{\beta,\gamma}-\lambda}\cdot f = \int_0^x \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{((1+i)\gamma)_j}{j!}\cdot\frac{\lambda^i\delta^j(x-t)^{\alpha j+(i+1)\beta-1}}{\Gamma(\alpha j+(1+i)\beta)}f(t)\,\mathrm{d}t \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda^i\delta^j\frac{((1+i)\gamma)_j}{j!}\cdot {}^{RL}_{\ 0}I_x^{\alpha j+(i+1)\beta}f(x); \\
y_2 &= \frac{S_{\beta,\gamma}}{S_{\beta,\gamma}-\lambda}\cdot Fy = Fy+\frac{\lambda}{S_{\beta,\gamma}-\lambda}\cdot Fy \\
&= \sum_{k=0}^{m-1}y^{(k)}(0^+)\frac{x^k}{k!}+\lambda\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda^i\delta^j\frac{((1+i)\gamma)_j}{j!}\cdot h_{\alpha j+(i+1)\beta}\right)*\left(\sum_{k=0}^{m-1}y^{(k)}(0^+)h_{k+1}\right) \\
&= \sum_{k=0}^{m-1}y^{(k)}(0^+)\frac{x^k}{k!}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m-1}\lambda^{i+1}\delta^j\frac{((1+i)\gamma)_j}{j!}y^{(k)}(0^+)\cdot h_{\alpha j+(i+1)\beta+k+1} \\
&= \sum_{k=0}^{m-1}y^{(k)}(0^+)\frac{x^k}{k!}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m-1}\frac{((1+i)\gamma)_j}{j!}y^{(k)}(0^+)\cdot\frac{\lambda^{i+1}\delta^j x^{\alpha j+(i+1)\beta+k}}{\Gamma(\alpha j+(i+1)\beta+k+1)}.
\end{aligned}
$$

Thus, we obtain the following explicit formula for the unique solution to the initial value problem

$$
\begin{aligned}
y = \sum_{k=0}^{m-1}\frac{a_k x^k}{k!}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m-1}\frac{((1+i)\gamma)_j}{j!}\cdot\frac{a_k\lambda^{i+1}\delta^j x^{\alpha j+(i+1)\beta+k}}{\Gamma(\alpha j+(i+1)\beta+k+1)} \\
+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda^i\delta^j\frac{((1+i)\gamma)_j}{j!}\cdot {}^{RL}_{\ 0}I_x^{\alpha j+(i+1)\beta}f(x),
\end{aligned}
\tag{4.8}
$$

or more explicitly (without using Riemann–Liouville integrals in the solution)

$$
\begin{aligned}
y = \sum_{k=0}^{m-1}\frac{a_k x^k}{k!}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{m-1}\frac{((1+i)\gamma)_j}{j!}\cdot\frac{a_k\lambda^{i+1}\delta^j x^{\alpha j+(i+1)\beta+k}}{\Gamma(\alpha j+(i+1)\beta+k+1)} \\
+\int_0^x\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{((1+i)\gamma)_j}{j!}\cdot\frac{\lambda^i\delta^j(x-t)^{\alpha j+(i+1)\beta-1}}{\Gamma(\alpha j+(1+i)\beta)}f(t)\,\mathrm{d}t.
\end{aligned}
\tag{4.9}
$$

Example 4.2: We now study a more general initial value problem, a multi-term fractional differential equation using Prabhakar–Caputo derivatives, as follows:

$$
\begin{cases}
\displaystyle\sum_{i=1}^{n}\lambda_i\left({}^{P}_{0}D_x^{\alpha,\beta_i,\gamma_i,\delta}y\right)(x) = f(x), & x>0, \\
\\
y^{(k)}(0) = c_k, & k=0,\cdots,m-1,
\end{cases}
\tag{4.10}
$$

where $\alpha \in \mathbb{R}^+$ and $\beta_1 > \cdots > \beta_m > 0$ and $\gamma_i, \delta, \lambda_i, a_i \in \mathbb{R}$ are constants for $1 \le i \le m$ and $f \in C_{-1}$ is a fixed forcing function.

By Theorem 4.2.1, the fractional differential equation from (4.10) becomes an algebraic equation in M_{-1} as follows:

$$\sum_{i=1}^{n} \lambda_i \big(S_{\beta_i,\gamma_i} \cdot y - S_{\beta_i,\gamma_i} \cdot F_i y \big) = f, \tag{4.11}$$

where for $i = 1, \cdots, n$ we write $m_i = \lfloor \beta_i \rfloor + 1$, so that $m_i - 1 \le \beta < m_i$, and define the expression $F_i y$ by

$$F_i y = \sum_{k=0}^{m_i-1} y^{(k)}(0^+) \frac{x^k}{k!} = \sum_{k=0}^{m_i-1} c_k h_{k+1}.$$

The algebraic equation (4.11) can be rearranged to solve for y as follows:

$$\sum_{i=1}^{n} \lambda_i S_{\beta_i,\gamma_i} \cdot y = f + \sum_{i=1}^{n} \lambda_i S_{\beta_i,\gamma_i} \cdot F_i y = f + \sum_{i=1}^{n} \lambda_i S_{\beta_i,\gamma_i} \cdot \sum_{k=0}^{m_i-1} c_k h_{k+1}, \tag{4.12}$$

then

$$y = \frac{I}{\sum_{i=1}^{n} \lambda_i S_{\beta_i,\gamma_i}} \cdot f + \frac{\sum_{i=1}^{n} \lambda_i S_{\beta_i,\gamma_i} \cdot \sum_{k=0}^{m_i-1} c_k h_{k+1}}{\sum_{i=1}^{n} \lambda_i S_{\beta_i,\gamma_i}}. \tag{4.13}$$

The first term of (4.13) can be immediately written as a function in C_{-1} by using (4.5) from above. The second term requires some further manipulation, transforming the S_{β_i,γ_i} terms in the numerator in order to obtain overall a function in C_{-1}:

$$
\begin{aligned}
\frac{\sum_{i=1}^{n}\lambda_i S_{\beta_i,\gamma_i}\cdot\sum_{k=0}^{m_i-1}c_k h_{k+1}}{\sum_{i=1}^{n}\lambda_i S_{\beta_i,\gamma_i}} &= \frac{\sum_{i=1}^{n}\lambda_i S_{\beta_i-\beta_1,\gamma_i-\gamma_1}\cdot\sum_{k=0}^{m_i-1}c_k h_{k+1}}{\sum_{i=1}^{n}\lambda_i S_{\beta_i-\beta_1,\gamma_i-\gamma_1}} \\
&= \frac{\sum_{i=1}^{n}\lambda_i P_{\beta_1-\beta_i,\gamma_1-\gamma_i}\cdot\sum_{k=0}^{m_i-1}c_k h_{k+1}}{\lambda_1+\sum_{i=2}^{n}\lambda_i P_{\beta_1-\beta_i,\gamma_1-\gamma_i}} \\
&= \frac{\sum_{i=1}^{n}\sum_{k=0}^{m_i-1}c_k\lambda_i P_{\beta_1-\beta_i+k+1,\gamma_1-\gamma_i}}{\lambda_1+\sum_{i=2}^{n}\lambda_i P_{\beta_1-\beta_i,\gamma_1-\gamma_i}} \\
&= \frac{\sum_{k=0}^{m_1-1}\sum_{i=0}^{\ell_k}c_k\lambda_i P_{\beta_1-\beta_i+k+1,\gamma_1-\gamma_i}}{\lambda_1+\sum_{i=2}^{n}\lambda_i P_{\beta_1-\beta_i,\gamma_1-\gamma_i}} \qquad (4.14)
\end{aligned}
$$

where the index ℓ_k for each $k = 0, 1, \cdots, m_1 - 1$ is defined so that $k \le m_i - 1 \Leftrightarrow i \le \ell_k$. In other words, since $k \le m_i - 1 \Rightarrow k \le m_j - 1$ whenever $j \le i$, we define

$$\ell_k = \max\{i \le n : k \le m_i - 1\}$$

for all $k = 0, 1, \cdots, m_1 - 1$.

Using an argument similar to that used in Chapter 3 to express $\frac{I}{\sum_{i=1}^{n}\lambda_i S_{\beta_i,\gamma_i}}$ as a function in C_{-1} via (4.5), we get:

$$
\begin{aligned}
\frac{I}{\lambda_1+\sum\limits_{i=2}^{n}\lambda_i P_{\beta_1-\beta_i,\gamma_1-\gamma_i}} &= \lambda_1^{-1}\sum_{r=0}^{\infty}\left(\sum_{i=2}^{n}-\frac{\lambda_i}{\lambda_1}P_{\beta_1-\beta_i,\gamma_1-\gamma_i}\right)^r \\
&= \lambda_1^{-1}\sum_{r=0}^{\infty}\sum_{r_2+\cdots+r_n=r}\frac{r!}{r_2!\cdots r_n!}\left(-\frac{\lambda_2}{\lambda_1}\right)^{r_2}\cdots\left(-\frac{\lambda_n}{\lambda_1}\right)^{r_n} \\
&\qquad\times P^{r_2}_{\beta_1-\beta_2,\gamma_1-\gamma_2}P^{r_n}_{\beta_1-\beta_n,\gamma_1-\gamma_n} \\
&= \lambda_1^{-1}\sum_{r_2,\cdots,r_n=0}^{\infty}\frac{(r_2+\cdots+r_n)!}{r_2!\cdots r_n!}\left(-\frac{\lambda_2}{\lambda_1}\right)^{r_2}\cdots\left(-\frac{\lambda_n}{\lambda_1}\right)^{r_n} \\
&\qquad\times P_{r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n),r_2(\gamma_1-\gamma_2)+\cdots+r_n(\gamma_1-\gamma_n)} \\
&= \lambda_1^{-1}\sum_{r_2,\cdots,r_n=0}^{\infty}\frac{(r_2+\cdots+r_n)!}{r_2!\cdots r_n!}\cdot\frac{\lambda_2^{r_2}\cdots\lambda_n^{r_n}}{(-\lambda_1)^{r_2+\cdots+r_n}} \\
&\qquad\times x^{r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)-1}E^{r_2(\gamma_1-\gamma_2)+\cdots+r_n(\gamma_1-\gamma_n)}_{\alpha,r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)}\left(\delta x^{\alpha}\right),
\end{aligned}
$$

and so the field element (4.14) becomes a function as follows:

$$
\begin{aligned}
\frac{\sum\limits_{i=1}^{n}\lambda_i S_{\beta_i,\gamma_i}\cdot\sum\limits_{k=0}^{m_i-1}c_k h_{k+1}}{\sum\limits_{i=1}^{n}\lambda_i S_{\beta_i,\gamma_i}} &= \lambda_1^{-1}\left(\sum_{k=0}^{m_1-1}\sum_{i=0}^{\ell_k}c_k\lambda_i P_{\beta_1-\beta_i+k+1,\gamma_1-\gamma_i}\right)* \\
&\quad *\Bigg(\sum_{r_2,\cdots,r_n=0}^{\infty}\frac{(r_2+\cdots+r_n)!}{r_2!\cdots r_n!}\cdot\frac{\lambda_2^{r_2}\cdots\lambda_n^{r_n}}{(-\lambda_1)^{r_2+\cdots+r_n}} \\
&\qquad\times P_{r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n),r_2(\gamma_1-\gamma_2)+\cdots+r_n(\gamma_1-\gamma_n)}\Bigg) \\
&= \sum_{r_2,\cdots,r_n=0}^{\infty}\sum_{k=0}^{m_1-1}\sum_{i=0}^{\ell_k}\frac{(r_2+\cdots+r_n)!}{r_2!\cdots r_n!}\cdot\frac{\lambda_2^{r_2}\cdots\lambda_n^{r_n}}{(-\lambda_1)^{r_2+\cdots+r_n}}c_k \\
&\qquad\times P_{r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)+(\beta_1-\beta_i),r_2(\gamma_1-\gamma_2)+\cdots+r_n(\gamma_1-\gamma_n)+(\gamma_1-\gamma_i)} \\
&= \sum_{r_2,\cdots,r_n=0}^{\infty}\sum_{k=0}^{m_1-1}\sum_{i=0}^{\ell_k}\frac{(r_2+\cdots+r_n)!}{r_2!\cdots r_n!}\cdot\frac{\lambda_2^{r_2}\cdots\lambda_n^{r_n}}{(-\lambda_1)^{r_2+\cdots+r_n}}c_k \\
&\qquad\times x^{r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)+(\beta_1-\beta_i)-1} \\
&\qquad\times E^{r_2(\gamma_1-\gamma_2)+\cdots+r_n(\gamma_1-\gamma_n)+(\gamma_1-\gamma_i)}_{\alpha,r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)+(\beta_1-\beta_i)}\left(\delta x^{\alpha}\right).
\end{aligned}
$$

Combining this with (4.5) and using them in (4.13), we get the following explicit solution function for the initial value problem (4.10):

$$y(x) = \int_0^x \sum_{r_2,\cdots,r_n=0}^{\infty} \frac{(r_2+\cdots+r_n)!}{r_2!\cdots r_n!} \cdot \frac{\lambda_2^{r_2}\cdots\lambda_n^{r_n}}{(-\lambda_1)^{r_2+\cdots+r_n}}$$

$$\times (x-t)^{r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)+\beta_1-1} E_{\alpha,r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)+\beta_1}^{r_2(\gamma_1-\gamma_2)+\cdots+r_n(\gamma_1-\gamma_n)+\gamma_1} f(x-t)\,\mathrm{d}t$$

$$+ \sum_{r_2,\cdots,r_n=0}^{\infty} \sum_{k=0}^{m_1-1} \sum_{i=0}^{\ell_k} \frac{(r_2+\cdots+r_n)!}{r_2!\cdots r_n!} \cdot \frac{\lambda_2^{r_2}\cdots\lambda_n^{r_n}}{(-\lambda_1)^{r_2+\cdots+r_n}} c_k$$

$$\times x^{r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)+(\beta_1-\beta_i)-1} E_{\alpha,r_2(\beta_1-\beta_2)+\cdots+r_n(\beta_1-\beta_n)+(\beta_1-\beta_i)}^{r_2(\gamma_1-\gamma_2)+\cdots+r_n(\gamma_1-\gamma_n)+(\gamma_1-\gamma_i)} \left(\delta x^{\alpha}\right). \quad (4.15)$$

Chapter 5

PRABHAKAR FRACTIONAL CALCULUS OF HILFER TYPE

In this chapter, we investigate the general Hilfer–Prabhakar fractional derivative operator using the algebraic machinery of Mikusiński's operational calculus. We also provide some important characterisations in terms of initial conditions for the function spaces used in the fractional Mikusiński method. Armed with an algebraic interpretation of the Hilfer–Prabhakar derivative, we solve a variety of fractional differential equations involving this operator, including multi-order and multi-term ones.

5.1 Characterising Function Spaces

We begin with some important results concerning the ${}^{R}\Omega_{\mu}^{\alpha}$ and ${}^{H}\Omega_{\mu}^{\alpha}$ spaces, which help us to understand the nature of these spaces more clearly than what can be seen in the existing literature.

Theorem 5.1.1. Let $\alpha > 0$, $\mu \geq -1$ be fixed and let $n = \lfloor \alpha \rfloor + 1$. Then we have

$$ {}^{R}\Omega_{\mu}^{n} \subset {}^{H}\Omega_{\mu}^{\alpha} \subset {}^{R}\Omega_{\mu}^{\alpha}. $$

Proof. The second inclusion is clear from the definition: if all Hilfer derivatives exist in C_{μ}, then so do the corresponding Riemann–Liouville derivatives (by putting $\nu = 0$). It remains to prove the first inclusion, i.e. to find a sufficient condition, in terms of Riemann–Liouville derivatives, for a function to be in ${}^{H}\Omega_{\mu}^{\alpha}$.

Let us fix ν, β with $0 \leq \nu \leq 1$ and $0 \leq \beta \leq \alpha$, and define $m = \lfloor \beta \rfloor + 1$ so that $m - 1 \leq \beta < m$. Then, by definition of the Hilfer derivative,

$$\left({}^{H}_{0}D^{\beta,\nu}_{x} f\right)(x) = {}^{RL}_{0}I^{\nu(m-\beta)}_{x} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left({}^{RL}_{0}I^{(1-\nu)(m-\beta)}_{x} f\right)(x),$$

so

$${}^{H}_{0}D^{\beta,\nu}_{x} f \in C_\mu \Longleftarrow \frac{\mathrm{d}^m}{\mathrm{d}x^m} {}^{RL}_{0}I^{(1-\nu)(m-\beta)}_{x} f \in C_\mu, \tag{5.1}$$

where the latter function is a fractional differintegral of f, which can be rewritten as follows using (2.4):

$$\begin{aligned}\frac{\mathrm{d}^m}{\mathrm{d}x^m} \left({}^{RL}_{0}I^{(1-\nu)(n-\beta)}_{x} f\right)(x) &= \left({}^{RL}_{0}D^{m-(1-\nu)(m-\beta)}_{x} f\right)(x) \\ &= \left({}^{RL}_{0}D^{\beta+(m-\beta)\nu}_{x} f\right)(x) = \frac{\mathrm{d}^N}{\mathrm{d}x^N} \left({}^{RL}_{0}I^{N-m+(1-\nu)(n-\beta)}_{x} f\right)(x),\end{aligned}$$

where $N := \lfloor \beta + (m-\beta)\nu \rfloor + 1$.

Because $0 \leq (m-\beta)\nu \leq 1$, there are only two possible values for N: either $N = m$ or $N = m+1$. The only case when N can equal $m+1$ is if we have $\beta + (m-\beta)\nu \geq m$, which is equivalent to $m - \beta \leq (m-\beta)\nu$, which can only happen if $\nu = 1$. So there are two cases:

- either $\nu = 1$ and ${}^{H}_{0}D^{\beta,\nu}_{x} f = {}^{C}_{0}D^{\beta}_{x} f$,
- or $\nu < 1$ and $N = m$, meaning that the right-hand function in (5.1) is just an Riemann–Liouville fractional derivative of f, and we have ${}^{H}_{0}D^{\beta,\nu}_{x} f = {}^{RL}_{0}I^{\nu(m-\beta)}_{x} \left({}^{RL}_{0}D^{\beta+(m-\beta)\nu}_{x} f\right)$, where the inner fractional order satisfies $\beta + (m-\beta)\nu < \beta + (m-\beta) = m \leq n$.

Now we can find sufficient conditions for a function to be in the space ${}^{H}\Omega^{\alpha}_{\mu}$, as follows:

$$
\begin{aligned}
f \in {}^H\Omega^\alpha_\mu &\Longleftarrow \left({}^C_0D^\beta_x f \in C_\mu \quad \text{and} \quad {}^{RL}_0D^\gamma_x f \in C_\mu \quad \text{for all } \beta \in [0,\alpha] \text{ and } \gamma \in [0,n) \right) \\
&\Longleftarrow \left({}^{RL}_0D^\gamma_x f \in C_\mu \quad \text{for all } \gamma \in [0,n] \right) \\
&\Longleftrightarrow f \in {}^R\Omega^n_\mu.
\end{aligned}
$$

This completes the proof. □

Theorem 5.1.2. Let $\alpha > 1$ be a fixed non-integer with $n = \lfloor \alpha \rfloor + 1 \in \mathbb{N}\backslash\{1\}$. If a function f is in ${}^H\Omega^\alpha_{-1}$, then it has zero initial conditions $\left({}^{RL}_0D^\mu_x f\right)(0) = 0$ for all μ with $0 \leq \mu \leq n-2$.

Proof. We recall from [35, Corollary 2.1] that, for a function $f \in C^n_{-1}$ (the natural space to ensure that all Caputo derivatives of f up to order n are in C_{-1}), its Riemann–Liouville derivative ${}^{RL}_0D^\alpha_x f$ is generally not in C_{-1}, except in some special cases. These special cases are enumerated as follows: either when $\alpha \in \mathbb{N}$, or when $0 < \alpha < 1$ (here we have excluded both of these cases), or when $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. To understand why the latter case is needed here, we recall [9] the following relation between Riemann–Liouville and Caputo derivatives:

$$
\left({}^C_0D^\alpha_x f\right)(x) = \left({}^{RL}_0D^\alpha_x f\right)(x) - \sum_{k=0}^{n-1} \frac{x^{k-\alpha} f^{(k)}(0)}{\Gamma(k-\alpha+1)}.
$$

Therefore, if both ${}^C_0D^\alpha_x f$ and ${}^{RL}_0D^\alpha_x f$ are in C_{-1}, then (since C_{-1} is a vector space) we must have $\frac{x^{k-\alpha} f^{(k)}(0)}{\Gamma(k-\alpha+1)} \in C_{-1}$ for $k = 0,1,2,\cdots,n-1$. Since the exponent $k-\alpha$ is not greater than -1 for $k = 0,1,2,\cdots,n-2$, and the gamma function reciprocal is not zero as $\alpha \notin \mathbb{N}$, this means $f(0) = f'(0) = \cdots = f^{(n-2)}(0) = 0$.

So far, we have only used the Riemann–Liouville and Caputo derivatives, i.e. the cases $\nu = 0$ and $\nu = 1$. Let us now consider the general Hilfer derivative, with $0 \leq \nu \leq 1$ and $0 \leq \beta \leq \alpha$, and with $\ell = \lfloor \beta \rfloor + 1 \leq n$. Using the composition relations (2.4) and (2.5), we have:

$$
\begin{aligned}
\left({}^{H}_{0}D^{\beta,\nu}_{x} f\right)(x) &= {}^{RL}_{0}D^{-\nu(\ell-\beta)}_{x} \left({}^{RL}_{0}D^{\ell-(1-\nu)(\ell-\beta)}_{x} f\right)(x) \\
&= {}^{R}_{0}D^{-\nu(\ell-\beta)}_{x} \left({}^{RL}_{0}D^{\beta+\nu(\ell-\beta)}_{x} f\right)(x) \\
&= \left({}^{RL}_{0}D^{\alpha}_{x} f\right)(x) - \sum_{k=0}^{m-1} \frac{x^{\nu(\ell-\beta)-k-1}}{\Gamma(\nu(\ell-\beta)-k)} \cdot \left({}^{RL}_{0}D^{\beta+\nu(\ell-\beta)-k-1}_{x} f\right)(0),
\end{aligned}
$$

where $m = \lfloor \beta + \nu(\ell-\beta) \rfloor + 1$.

This means, for every $\nu \in [0,1]$ and $\beta \in [0,\alpha]$ and for every k such that $\nu(\ell-\beta)-k$ is a negative non-integer, we must have $\left({}^{RL}_{0}D^{(1-\nu)\beta+\ell\nu-k-1}_{x} f\right)(0) = 0$. Ignoring the case of integer-order derivatives which has already been covered above, requiring $\nu(\ell-\beta)-k$ to be a negative non-integer means we are interested in every $k = 1, 2, \cdots, m-1$, since the number $\nu(\ell-\beta)$ is in $[0,1]$. It remains to prove that the numbers $(1-\nu)\beta + \ell\nu - k - 1$ cover all non-integer values between 0 and $n-2$, for which we use the following elementary lemma. □

Lemma 5.1.1. Let $\alpha > 1$ be a fixed non-integer with $n = \lfloor \alpha \rfloor + 1 \in \mathbb{N}\backslash\{1\}$. For any non-integer $\mu \in [0, n-2]$, there exist $\beta \in (n-1, \alpha]$ and $\nu \in [0,1]$ and $k \in \mathbb{N}$ such that $k \leq \lfloor (1-\nu)\beta + n\nu \rfloor + 1$ and $\beta + (n-\beta)\nu - k - 1 = \mu$.

Proof. We split into two cases.

If the fractional part of μ is less than or equal to the fractional part of α, then there exists $\beta \in (n-1, \alpha]$ such that the fractional parts of β and μ are equal, and we can choose $\nu = 0$ to get $\mu = \beta - k - 1$ for some $k \leq \lfloor \beta \rfloor + 1$. Note that k must be positive since $\beta > n-1$ and $\mu < n-2$.

If the fractional part of μ is greater than the fractional part of α, then we let $\beta = \alpha$ and note that $\beta + (n-\beta)\nu$ varies between α and n, so we can find $\nu \in [0,1]$ such that the fractional parts of μ and $\beta+(n-\beta)\nu$ are equal. Then k is chosen in the same way as above. $\square$

Corollary 5.1.1. If $\alpha \geq 2$ and $f \in {}^{R}\Omega_{-1}^{\alpha}$, then $\left({}_{0}^{R}D_{x}^{\mu} f\right)(0)=0$ for all μ with $0 \leq \mu \leq \lfloor\alpha\rfloor-2$.

Proof. By Theorem 5.1.1, we have

$$
{}^{R}\Omega_{-1}^{\alpha} \subset {}^{R}\Omega_{-1}^{\lfloor\alpha\rfloor} \subset {}^{H}\Omega_{-1}^{\lfloor\alpha\rfloor-1/2},
$$

so $f \in {}^{H}\Omega_{-1}^{\lfloor\alpha\rfloor-1/2}$. By Theorem 5.1.2 with $n=\lfloor\alpha\rfloor$, the result follows. $\square$

We have now demonstrated that, while C_{-1} is a very large function space compared to many that are used for strong solutions of differential equations, the Ω spaces (of both Riemann–Liouville and Caputo types) imply rather strong assumptions on the initial values, both classical and fractional, of any function contained in them. These facts may have significant repercussions in the study of fractional differential equations set in these function spaces.

5.2 Action of Operators on Spaces

Theorem 5.2.1. Let $\alpha>0$, $\beta \geq 0$ with $n=\lfloor\beta\rfloor+1 \in \mathbb{N}$, and $\gamma, \delta \in \mathbb{C}$, with $\nu \in[0,1]$. The Hilfer–Prabhakar fractional derivative maps the function space ${}^{H}\Omega_{-1}^{\beta}$ into C_{-1}:

$$
{}_{0}^{PH}D_{x}^{\alpha,\beta,\gamma,\delta;\nu} : {}^{H}\Omega_{-1}^{\beta} \longrightarrow C_{-1}. \tag{5.2}
$$

Moreover, it can be related to the Prabhakar derivative of Riemann–Liouville type by the following relation:

$$
\begin{aligned}
&\left({}_{0}^{PH}D_{x}^{\alpha,\beta,\gamma,\delta;\nu} f\right)(x)=\left({}_{0}^{PR}D_{x}^{\alpha,\beta,\gamma,\delta} f\right)(x) \\
&-\sum_{j=0}^{N} \sum_{k=0}^{M_j-1} \frac{(-\gamma(1-\nu))_j}{j!} \delta^j \cdot\left({}_{0}^{RL}D_{x}^{-\alpha j+(1-\nu)\beta+n\nu-k-1} f\right)(0) \cdot e_{\alpha,\nu(n-\beta)-k}^{-\gamma\nu,\delta}(x),
\end{aligned} \tag{5.3}
$$

for any function $f \in {}^{H}\Omega_{-1}^{\alpha}$, where $N=\left\lfloor\frac{(1-\nu)\beta+n\nu}{\alpha}\right\rfloor$ and $M_j=\lfloor-\alpha j+(1-\nu)\beta+n\nu\rfloor+1 \in \mathbb{Z}^{+}$ for each $j=0,1,\cdots,N$, and we recall the notation $e_{\alpha,\beta}^{\gamma,\delta}(x)$ from (2.8).

Proof. Let $f \in {}^{H}\Omega^{\alpha}_{-1}$, therefore $f \in {}^{R}\Omega^{\alpha}_{-1}$ by Theorem 5.1.1. Using the series formula (Lemma 2.1.3) and the convention (2.2), we have:

$$\begin{aligned}
&\left({}^{PH}_{\ 0}D^{\alpha,\beta,\gamma,\delta;\nu}_{x} f\right)(x) = {}^{P}_{0}I^{\alpha,\nu(n-\beta),-\gamma\nu,\delta}_{x} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{P}_{0}I^{\alpha,(1-\nu)(n-\beta),-\gamma(1-\nu),\delta}_{x} f\right)\right)(x) \\
&= \sum_{i=0}^{\infty} \frac{(-\gamma\nu)_i}{i!}\delta^i\, {}^{RL}_{\ 0}I^{\alpha i+\nu(n-\beta)}_{x} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sum_{j=0}^{\infty}\frac{(-\gamma(1-\nu))_j}{j!}\delta^j \cdot \left({}^{RL}_{\ 0}I^{\alpha j+(1-\nu)(n-\beta)}_{x} f\right)(x)\right) \\
&= \sum_{i=0}^{\infty} \frac{(-\gamma\nu)_i}{i!}\delta^i\, {}^{RL}_{\ 0}I^{\alpha i+\nu(n-\beta)}_{x} \left(\sum_{j=0}^{\infty}\frac{(-\gamma(1-\nu))_j}{j!}\delta^j \cdot \left({}^{RL}_{\ 0}I^{\alpha j+(1-\nu)(n-\beta)-n}_{x} f\right)(x)\right) \\
&= \sum_{i=0}^{\infty} \frac{(-\gamma\nu)_i}{i!}\delta^i\, {}^{RL}_{\ 0}D^{-\alpha i-\nu(n-\beta)}_{x} \left(\sum_{j=0}^{\infty}\frac{(-\gamma(1-\nu))_j}{j!}\delta^j \cdot \left({}^{RL}_{\ 0}D^{-\alpha j+(1-\nu)\beta+n\nu}_{x} f\right)(x)\right)
\end{aligned}$$

where the outer operator ${}^{RL}_{\ 0}D^{-\alpha i-\nu(n-\beta)}_{x}$ is a fractional integral for all values of i and the inner operator ${}^{RL}_{\ 0}D^{-\alpha j+(1-\nu)\beta+n\nu}$ is a fractional derivative if j is small enough that $-\alpha j+(1-\nu)\beta+n\nu \geq 0$ or a fractional integral for all j large enough that $-\alpha j+(1-\nu)\beta+n\nu < 0$. This distinction is important because of the composition relations (2.3)–(2.5): we have

$$ {}^{RL}_{\ 0}D^{-\alpha i-\nu(n-\beta)}_{x}\left({}^{RL}_{\ 0}D^{-\alpha j+(1-\nu)\beta+n\nu}_{x} f\right)(x) = \left({}^{RL}_{\ 0}D^{-\alpha i-\alpha j+\beta}_{x} f\right)(x)$$

if $-\alpha j+(1-\nu)\beta+n\nu < 0$, but the same left-hand side equals

$$\begin{aligned}
\left({}^{RL}_{\ 0}D^{-\alpha i-\alpha j+\beta}_{x} f\right)(x) - \sum_{k=0}^{M_j-1} & \frac{x^{-(-\alpha i-\nu(n-\beta))-k-1}}{\Gamma(-(-\alpha i-\nu(n-\beta))-k)} \\
& \cdot \left({}^{RL}_{\ 0}D^{-\alpha j+(1-\nu)\beta+n\nu-k-1}_{x} f\right)(0)
\end{aligned}$$

if $-\alpha j+(1-\nu)\beta+n\nu \geq 0$, where $M_j = \lfloor -\alpha j+(1-\nu)\beta+n\nu \rfloor + 1 \in \mathbb{Z}^+$.

Splitting the inner series in the above double sum according to the sign of $-\alpha j+(1-\nu)\beta+n\nu$, and letting N denote the cutoff point, we have:

$$\begin{aligned}\left({}^{PH}_{\ 0}D_x^{\alpha,\beta,\gamma,\delta;\nu}f\right)(x) &= \sum_{i=0}^{\infty}\sum_{j=0}^{N}\frac{(-\gamma\nu)_i}{i!}\cdot\frac{(-\gamma(1-\nu))_j}{j!}\,\delta^{i+j}\Bigg[\left({}^{RL}_{\ 0}D_x^{-\alpha i-\alpha j+\beta}f\right)(x) \\ &\quad -\sum_{k=0}^{M_j-1}\frac{x^{\alpha i+\nu(n-\beta)-k-1}}{\Gamma(\alpha i+\nu(n-\beta)-k)}\left({}^{RL}_{\ 0}D_x^{-\alpha j+(1-\nu)\beta+n\nu-k-1}f\right)(0)\Bigg] \\ &\quad +\sum_{i=0}^{\infty}\sum_{j=N+1}^{\infty}\frac{(-\gamma\nu)_i}{i!}\cdot\frac{(-\gamma(1-\nu))_j}{j!}\,\delta^{i+j}\cdot\left({}^{RL}_{\ 0}D_x^{-\alpha i-\alpha j+\beta}f\right)(x) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\gamma\nu)_i}{i!}\cdot\frac{(-\gamma(1-\nu))_j}{j!}\,\delta^{i+j}\cdot\left({}^{RL}_{\ 0}I_x^{\alpha i+\alpha j-\beta}f\right)(x) \\ &\quad -\sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M_j-1}\frac{(-\gamma\nu)_i}{i!}\cdot\frac{(-\gamma(1-\nu))_j}{j!}\,\delta^{i+j} \\ &\quad \times\frac{x^{\alpha i+\nu(n-\beta)-k-1}}{\Gamma(\alpha i+\nu(n-\beta)-k)}\cdot\left({}^{RL}_{\ 0}D_x^{-\alpha j+(1-\nu)\beta+n\nu-k-1}f\right)(0).\end{aligned}$$

The first (double) sum here can be rewritten as follows, using the Chu–Vandermonde identity:

$$\begin{aligned}S_1 &:= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\gamma\nu)_i}{i!}\cdot\frac{(-\gamma(1-\nu))_j}{j!}\,\delta^{i+j}\cdot\left({}^{RL}_{\ 0}I_x^{\alpha i+\alpha j-\beta}f\right)(x) \\ &= \sum_{r=0}^{\infty}\left(\sum_{i+j=r}^{\infty}\frac{(-\gamma\nu)_i(-\gamma(1-\nu))_j}{i!j!}\right)\delta^r\cdot\left({}^{RL}_{\ 0}I_x^{\alpha r-\beta}f\right)(x) \\ &= \sum_{r=0}^{\infty}\frac{(-\gamma)_r}{r!}\delta^r\cdot\left({}^{RL}_{\ 0}I_x^{\alpha r-\beta}f\right)(x) \\ &= \left({}^{PR}_{\ 0}D_x^{\alpha,\beta,\gamma,\delta}f\right)(x).\end{aligned}$$

Since $f\in{}^{H}\Omega_{-1}^{\alpha}\subset{}^{R}\Omega_{-1}^{\alpha}$, this Prabhakar derivative is in the space C_{-1}. Meanwhile, the second (triple) sum can be rewritten by moving the infinite summation over i to the inside and noting that it is simply a Mittag-Leffler power series:

$$
\begin{aligned}
S_2 &:= \sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M_j-1} \frac{(-\gamma\nu)_i}{i!}\cdot\frac{(-\gamma(1-\nu))_j}{j!}\,\delta^{i+j} \\
&\qquad\qquad \times \frac{x^{\alpha i+\nu(n-\beta)-k-1}}{\Gamma(\alpha i+\nu(n-\beta)-k)}\cdot\left({}^{RL}_{0}D_x^{-\alpha j+(1-\nu)\beta+n\nu-k-1} f\right)(0) \\
&= \sum_{j=0}^{N}\sum_{k=0}^{M_j-1} \frac{(-\gamma(1-\nu))_j}{j!}\,\delta^j x^{\nu(n-\beta)-k-1}\cdot\left({}^{RL}_{0}D_x^{-\alpha j+(1-\nu)\beta+n\nu-k-1} f\right)(0) \\
&\qquad\qquad \times \sum_{i=0}^{\infty}\frac{(-\gamma\nu)_i}{i!}\,\delta^i\frac{x^{\alpha i}}{\Gamma(\alpha i+\nu(n-\beta)-k)} \\
&= \sum_{j=0}^{N}\sum_{k=0}^{M_j-1} x^{\nu(n-\beta)-k-1}E^{-\gamma\nu}_{\alpha,\nu(n-\beta)-k}(\delta x^{\alpha})\frac{(-\gamma(1-\nu))_j}{j!}\delta^j \\
&\cdot\left({}^{RL}_{0}D_x^{-\alpha j+(1-\nu)\beta+n\nu-k-1} f\right)(0).
\end{aligned}
$$

To show that this function is in the space C_{-1}, we need that, for every value of k such that $\nu(n-\beta)-k\leq 0$, the initial condition $\left({}^{RL}_{0}D_x^{-\alpha j+(1-\nu)\beta+n\nu-k-1} f\right)(0)$ should be zero.

The condition $j\leq N$ guarantees that $-\alpha j+(1-\nu)\beta+n\nu\geq 0$. The condition $k\leq M_j-1$ guarantees that $-\alpha j+(1-\nu)\beta+n\nu-k-1\geq 0$. Ignoring the trivial case $\nu=0$ (in which case Hilfer is Riemann–Liouville and the result of this theorem is trivial), we have $\nu(n-\beta)\in(0,1]$, and therefore $\nu(n-\beta)-k\leq 0$ is equivalent to $k\geq 1$. Given also that $j\geq 0$, the maximum possible value (as j and k vary) of the fractional order $-\alpha j+(1-\nu)\beta+n\nu-k-1$ is $(1-\nu)\beta+n\nu-2=\beta-2+\nu(n-\beta)$, whose maximum possible value (as ν varies) is $n-2$. Thus, by Theorem 5.1.2, we have the required inclusion in C_{-1}. □

Theorem 5.2.2. Let $\alpha>0$, $\beta\geq 0$ with $n=\lfloor\beta\rfloor+1\in\mathbb{N}$, and $\gamma,\delta\in\mathbb{C}$, with $\nu\in[0,1]$. We introduce the notation $\beta'=\beta+\nu(n-\beta)$. For any $f\in{}^{H}\Omega^{\beta}_{-1}$, the Hilfer–Prabhakar fractional derivative has the following inversion relation:

$$
{}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta}\left({}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu}f\right)(x)=f(x)-\sum_{k=0}^{M-1}e_{\alpha,\beta'-k}^{\gamma(1-\nu),\delta}(x)\sum_{j=0}^{N_k}\frac{(-\gamma(1-\nu))_j\delta^j}{j!}
\cdot\left({}^{RL}_{0}D_x^{-\alpha j+\beta'-k-1}f\right)(0),
$$

where the upper bounds M and N_k of the two finite sums are defined as follows:

$$
M=\lfloor\beta+\nu(n-\beta)\rfloor+1=\begin{cases}n & \text{if } \nu<1,\\ n+1 & \text{if } \nu=1,\end{cases}
$$

$$
N_k=\left\lfloor\frac{\beta-k+\nu(n-\beta)}{\alpha}\right\rfloor, \tag{5.4}
$$

and we recall the notation $e_{\alpha,\beta}^{\gamma,\delta}(x)$ from (2.8).

Proof. First of all, we use the definitions of the Prabhakar integral and Hilfer–Prabhakar derivative, and then various composition relations between Prabhakar operators, namely the semigroup property of Lemma 2.1.2 and the differentiation property [24, Equation (17)] that follows immediately from Lemma 2.1.3.

$$
\begin{aligned}
&{}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta}\left({}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu}f\right)(x)\\
&\quad={}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta}\left({}^{P}_{0}I_x^{\alpha,\nu(n-\beta),-\gamma\nu,\delta}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{P}_{0}I_x^{\alpha,(1-\nu)(n-\beta),-\gamma(1-\nu),\delta}f\right)\right)\right)(x)\\
&\quad={}^{P}_{0}I_x^{\alpha,\beta+\nu(n-\beta),\gamma-\gamma\nu,\delta}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{P}_{0}I_x^{\alpha,(1-\nu)(n-\beta),-\gamma(1-\nu),\delta}f\right)\right)(x)\\
&\quad={}^{P}_{0}P_x^{\alpha,\beta+\nu(n-\beta),\gamma-\gamma\nu,\delta}\left({}^{PR}_{0}D_x^{\alpha,n-(1-\nu)(n-\beta),\gamma(1-\nu),\delta}f\right)(x)\\
&\quad={}^{P}_{0}I_x^{\alpha,\beta+\nu(n-\beta),\gamma(1-\nu),\delta}\left({}^{PR}_{0}D_x^{\alpha,\beta+\nu(n-\beta),\gamma(1-\nu),\delta}f\right)(x).
\end{aligned}\tag{5.5}
$$

Now it only remains to prove an inversion result for the Prabhakar integral of an Riemann–Liouville type Prabhakar derivative, a generalisation of the result in Chapter 3, Theorem 3.4. We do this using the series formulae of Lemma 2.1.3, as follows:

$$ {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\left({}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x)=\sum_{i=0}^{\infty}\frac{(\gamma)_i\delta^i}{i!}\cdot{}^{RL}_{0}I^{\alpha i+\beta}\left(\sum_{j=0}^{\infty}\frac{(-\gamma)_j\delta^j}{j!}\cdot\left({}^{RL}_{0}I^{\alpha j-\beta}f\right)(x)\right) $$

$$ =\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\gamma)_i(-\gamma)_j\delta^{i+j}}{i!j!}\cdot{}^{RL}_{0}D^{-\alpha i-\beta}\left({}^{RL}_{0}D^{-\alpha j+\beta}f\right)(x). $$

Now we split the inner sum into two parts according to whether $-\alpha j+\beta$ is non-negative or negative, with $j=N$ being the cutoff point, and use the composition relations (2.3)–(2.5):

$$ =\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\gamma)_i(-\gamma)_j\delta^{i+j}}{i!j!}\cdot\left({}^{RL}_{0}D^{-\alpha i-\alpha j}f\right)(x) $$

$$ -\sum_{i=0}^{\infty}\sum_{j=0}^{N}\frac{(\gamma)_i(-\gamma)_j\delta^{i+j}}{i!j!}\sum_{k=0}^{M_j-1}\frac{x^{\alpha i+\beta-k-1}}{\Gamma(\alpha i+\beta-k)}\cdot\left({}^{RL}_{0}D^{-\alpha j+\beta-k-1}f\right)(0), $$

where $M_j=\lfloor-\alpha j+\beta\rfloor+1$. The first (double) sum simplifies by the Chu–Vandermonde identity to $f(x)$, so we have:

$$ =f(x)-\sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M_j-1}\frac{(\gamma)_i(-\gamma)_j\delta^{i+j}}{i!j!}\cdot\frac{x^{\alpha i+\beta-k-1}}{\Gamma(\alpha i+\beta-k)}\cdot\left({}^{RL}_{0}D^{-\alpha j+\beta-k-1}f\right)(0) $$

$$ =f(x)-\sum_{j=0}^{N}\sum_{k=0}^{M_j-1}\frac{(-\gamma)_j\delta^j}{j!}\cdot\left({}^{RL}_{0}D^{-\alpha j+\beta-k-1}f\right)(0)\left[\sum_{i=0}^{\infty}\frac{(\gamma)_i\delta^i}{i!}\cdot\frac{x^{\alpha i+\beta-k-1}}{\Gamma(\alpha i+\beta-k)}\right] $$

$$ =f(x)-\sum_{j=0}^{N}\sum_{k=0}^{M_j-1}\frac{(-\gamma)_j\delta^j}{j!}\cdot\left({}^{RL}_{0}D^{-\alpha j+\beta-k-1}f\right)(0)\cdot x^{\beta-k-1}E^{\gamma}_{\alpha,\beta-k}\big(\delta x^{\alpha}\big). $$

To rearrange this double sum, we note that

$$ k\le M_j-1\Longleftrightarrow k\le-\alpha j+\beta\Longleftrightarrow j\le\frac{-k+\beta}{\alpha}\Longleftrightarrow j\le\left\lfloor\frac{\beta-k}{\alpha}\right\rfloor, $$

so the result becomes:

$$ {}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}\left({}^{PR}_{0}D^{\alpha,\beta,\gamma,\delta}_{x}f\right)(x)=f(x)-\sum_{k=0}^{\lfloor\beta\rfloor}x^{\beta-k-1}E^{\gamma}_{\alpha,\beta-k}\big(\delta x^{\alpha}\big) $$

$$ \left[\sum_{j=0}^{\lfloor\frac{\beta-k}{\alpha}\rfloor}\frac{(-\gamma)_j\delta^j}{j!}\cdot\left({}^{RL}_{0}D^{-\alpha j+\beta-k-1}f\right)(0)\right],\quad(5.6) $$

where the expression in square brackets is a constant.

Substituting (5.6) back into (5.5), we obtain the desired result of the theorem. $\square$

Remark 5.1: Let us verify that the $n = 1$ case of our result (5.6) above is consistent with that found in Chapter 3, Theorem 3.4 for $0 < \beta < 1$. Setting $n = 1$ reduces the summation over k to simply $k = 0$:

$$
\begin{aligned}
{}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta}\left({}^{PR}_{0}D_x^{\alpha,\beta,\gamma,\delta}f\right)(x) &= f(x) - x^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\delta x^{\alpha}\right) \\
&\qquad \times \left[\sum_{j=0}^{\lfloor \beta/\alpha \rfloor} \frac{(-\gamma)_j \delta^j}{j!} \cdot \left({}^{RL}_{0}D_x^{-\alpha j+\beta-1}f\right)(0)\right].
\end{aligned}
$$

This is now the same as the result in Chapter 3, Theorem 3.4 except for the value of the constant: for full consistency, the summation over j should also reduce to simply $j = 0$. This happens for the following reason.

Since $-\alpha j + \beta - 1 < 0$ for all values of j, all of these differintegral operators are actually fractional integrals rather than fractional derivatives. If a fractional integral to order κ is bounded at the initial value 0, then the higher-order fractional integrals (order greater than κ) must be zero at the initial value 0. So, given that $\left({}^{RL}_{0}I_x^{\alpha j-\beta+1}f\right)(0)$ is defined when $j = 0$, it must be zero for all higher values of j. Thus, the summation over j reduces to just $j = 0$, and our result yields Chapter 3, Theorem 3.4 as a special case.

For the same reason, when $0 < \beta < 1$ we have

$$
\begin{aligned}
{}^{P}_{0}I_x^{\alpha,\beta,\gamma,\delta}\left({}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu}f\right)(x) &= f(x) - x^{\beta'-1}E_{\alpha,\beta'}^{\gamma(1-\nu)}\left(\delta x^{\alpha}\right) \\
&\qquad \times \sum_{j=0}^{\lfloor \beta'/\alpha \rfloor} \frac{(-\gamma(1-\nu))_j \delta^j}{j!} \cdot \left({}^{RL}_{0}D_x^{-\alpha j+\beta'-1}f\right)(0) \\
&= f(x) - x^{\beta'-1}E_{\alpha,\beta'}^{\gamma(1-\nu)}\left(\delta x^{\alpha}\right) \cdot \left({}^{R}_{0}I_x^{1-\beta'}f\right)(0), \qquad (5.7)
\end{aligned}
$$

as the $n=1$ case of our Theorem 5.2.2.

5.3 Algebraic Interpretation

Now we are ready to discover the algebraic interpretation, in the field of fractions M_{-1} generated from the function space C_{-1}, of the Hilfer–Prabhakar derivative. Recall the notation $S_{\alpha,\beta,\gamma,\delta} = S_{\beta,\gamma}$ for the algebraic inverse in M_{-1} of the element $P_{\alpha,\beta,\gamma,\delta} = P_{\beta,\gamma} = x^{\beta-1}E^{\gamma}_{\alpha,\beta}(\delta x^{\alpha})$ which can be associated with the Prabhakar integral operator ${}^{P}_{0}I^{\alpha,\beta,\gamma,\delta}_{x}$.

Theorem 5.3.1. Let $\alpha>0$, $\beta\geq 0$ with $n=\lfloor\beta\rfloor+1\in\mathbb{N}$, and $\gamma,\delta\in\mathbb{C}$, with $\nu\in[0,1]$. The Hilfer–Prabhakar fractional differential operator ${}^{PH}_{0}D^{\alpha,\beta,\gamma,\delta;\nu}_{x}$ may be represented in the field M_{-1} in the following form, for $f\in{}^{H}\Omega^{\beta}_{-1}$:

$$ {}^{PH}_{0}D^{\alpha,\beta,\gamma,\delta;\nu}_{x} f = S_{\beta,\gamma} * f - \sum_{k=0}^{M-1} c_{f,k} S_{\beta,\gamma} * P_{\beta+\nu(1-\beta)-k,\gamma-\gamma\nu}, \tag{5.8} $$

where the constant $c_{f,k}$ is defined by

$$ c_{f,k} = \sum_{j=0}^{N_k} \frac{(-\gamma(1-\nu))_j \delta^j}{j!} \cdot \left({}^{RL}_{0}D^{-\alpha j+\beta+\nu(n-\beta)-k-1}_{x} f \right)(0), \tag{5.9} $$

or equivalently by

$$ c_{f,k} = \left({}^{PR}_{0}D^{\alpha,\beta+\nu(n-\beta)-k-1,\gamma(1-\nu),\delta}_{x} f \right)(0), \tag{5.10} $$

for each $k=0,1,\cdots,M-1$ with M and N_k defined as in Theorem 5.2.2. For $\nu\neq 1$ (ignoring the Caputo case which is already covered in Chapter 4), we have $M=n$ and so the formula (5.8) can be rewritten as follows:

$$ {}^{PH}_{0}D^{\alpha,\beta,\gamma,\delta;\nu}_{x} f = S_{\beta,\gamma} * f - c_{f,0} P_{\nu(1-\beta),-\gamma\nu} - \sum_{k=1}^{n-1} c_{f,k} S_{k-\nu(1-\beta),\gamma\nu}. \tag{5.11} $$

Proof. Let $f\in{}^{H}\Omega^{\beta}_{-1}$. By Theorem 5.2.2, we have the following algebraic relation in the field M_{-1}:

$$P_{\beta,\gamma} * \left({}^{PH}_{\ 0}D^{\alpha,\beta,\gamma,\delta;\nu}_x f\right) = f - \sum_{k=0}^{M-1} \left[\sum_{j=0}^{N_k} \frac{(-\gamma(1-\nu))_j \delta^j}{j!} \cdot \left({}^{RL}_{\ 0}D^{-\alpha j+\beta'-k-1}_x f\right)(0)\right] P_{\beta'-k,\gamma(1-\nu)}, \quad (5.12)$$

where we have again used the notation $\beta' = \beta + \nu(n-\beta)$. Upon multiplying both sides by the algebraic inverse $S_{\beta,\gamma}$ of the function $P_{\beta,\gamma}$, we get:

$${}^{PH}_{\ 0}D^{\alpha,\beta,\gamma,\delta;\nu}_x f = S_{\beta,\gamma} * f - \sum_{k=0}^{M-1} \left[\sum_{j=0}^{N_k} \frac{(-\gamma(1-\nu))_j \delta^j}{j!} \cdot \left({}^{RL}_{\ 0}D^{-\alpha j+\beta'-k-1}_x f\right)(0)\right] S_{\beta,\gamma} * P_{\beta+\nu(1-\beta)-k,\gamma-\gamma\nu}. \quad (5.13)$$

This is already in the form (5.8), and it yields (5.11) after using the semigroup relation of the P and S elements of M_{-1}, noting that the first parameter (β) in $P_{\beta,\gamma}$ and $S_{\beta,\gamma}$ should be positive for clarity of the notation.

It remains to prove the equivalence of the two expressions for $c_{f,k}$, namely to show that each given finite sum of Riemann–Liouville initial values is identical to a single Prabhakar initial value. From the series formula in Lemma 2.1.3, we have

$$\left({}^{PR}_{\ 0}D^{\alpha,\beta+\nu(n-\beta)-k-1,\gamma(1-\nu),\delta}_x f\right)(0) = \sum_{j=0}^{\infty} \frac{(-\gamma(1-\nu))_j \delta^j}{j!} \left({}^{RL}_{\ 0}D^{-\alpha j+\beta+\nu(n-\beta)-k-1}_x f\right)(0), \quad (5.14)$$

so it will suffice to show that

$$j \geq N_k \Longrightarrow \left({}^{RL}_{\ 0}D^{-\alpha j+\beta+\nu(n-\beta)-k-1}_x f\right)(0) = 0.$$

This can be done by using the definition of N_k and the convolution properties of the C_μ spaces. Since

$$j \geq \left\lfloor \frac{\beta - k + \nu(n-\beta)}{\alpha} \right\rfloor + 1$$
$$\implies j > \frac{\beta - k + \nu(n-\beta)}{\alpha} \implies -\alpha j + \beta + \nu(n-\beta) - k < 0,$$

we have in this case that the operator ${}^{RL}_{0}D_x^{-\alpha j+\beta+\nu(n-\beta)-k-1}$ is an Riemann–Liouville integral of order greater than 1. Since it is known [28] that an Riemann–Liouville integral of order greater than μ is equivalent to convolution with an element of C_μ, and $f \in C_{-1}$ by assumption, we have here that the function ${}^{RL}_{0}D_x^{-\alpha j+\beta+\nu(n-\beta)-k-1} f$ is a convolution between an element of C_μ for some positive μ and an element of C_{-1}. Another fundamental fact about the C_μ spaces [10, 28] is that $C_\mu * C_\nu \subset C_{\mu+\nu+1}$, so now we have the inclusion ${}^{RL}_{0}D_x^{-\alpha j+\beta+\nu(n-\beta)-k-1} f \in C_\mu$ for some $\mu > 0$, which means $\left({}^{RL}_{0}D_x^{-\alpha j+\beta+\nu(n-\beta)-k-1} f\right)(0) = 0$ as required. □

Corollary 5.3.1. Let $\alpha > 0$, $0 \leq \beta < 1$, and $\gamma, \delta \in \mathbb{C}$, with $\nu \in [0,1]$. The Hilfer–Prabhakar fractional differential operator ${}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu}$ may be represented in the field M_{-1} in the following form, for $f \in {}^{H}\Omega^{\beta}_{-1}$:

$$ {}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu} f = S_{\beta,\gamma} f - P_{\nu(n-\beta),-\gamma\nu} \cdot \left({}^{RL}_{0}I_x^{1-\beta+\nu(n-\beta)} f\right)(0). \tag{5.15}$$

Proof. Similar to the previous proof, but starting from (5.7), whose algebraic form is:

$$P_{\beta,\gamma} * \left({}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu} f\right) = f - P_{\beta+\nu(n-\beta),\gamma(1-\nu)} \cdot \left({}^{RL}_{0}I_x^{1-\beta+\nu(n-\beta)} f\right)(0),$$

we obtain after multiplication

$${}^{PH}_{0}D_x^{\alpha,\beta,\gamma,\delta;\nu} f = S_{\beta,\gamma} * \left(f - P_{\beta+\nu(n-\beta),\gamma-\gamma\nu} \cdot \left({}^{RL}_{0}I_x^{1-\beta+\nu(n-\beta)} f\right)(0)\right),$$

which is the desired result. □

5.4 Fractional Differential Equations

In this section, we shall illustrate some applications of the results proved above. This will be done by solving some fractional differential equations and integral equations involving Hilfer–Prabhakar derivatives, by using the algebraic Mikusiński method.

Example 5.1: Let us consider the following simple initial value problem involving a single Hilfer–Prabhakar derivative:

$$\left({}^{PH}_{\ 0}D_x^{\alpha,\beta,\gamma,\delta;\nu}y\right)(x)-\lambda y(x)=f(x),\qquad x>0; \tag{5.16}$$

$$\left({}^{RL}_{\ 0}D_x^{-\alpha j+\beta+\nu(n-\beta)-k-1}y\right)(0)=a_{j,k},\qquad j,k\in\mathbb{Z}_0^+ \text{ such that } \alpha j+k\le\beta \tag{5.17}$$

where $\alpha,\beta,\gamma,\delta,\nu$, and all the $a_{j,k}$ are fixed constants with $0\le\nu<1$.

We firstly rewrite the problem in an algebraic form, using (5.11) from Theorem 5.3.1:

$$S_{\beta,\gamma}*y-c_0P_{\nu(1-\beta),-\gamma\nu}-\sum_{k=1}^{n-1}c_kS_{k-\nu(1-\beta),\gamma\nu}-\lambda y=f,$$

where each constant $c_k=c_{y,k}$ is given in terms of the known initial values $a_{j,k}$. Rearranging the algebraic equation gives

$$\left(S_{\beta,\gamma}-\lambda I\right)*y=f+c_0P_{\nu(1-\beta),-\gamma\nu}+\sum_{k=1}^{n-1}c_kS_{k-\nu(1-\beta),\gamma\nu},$$

where I denotes the multiplicative identity in the field M_{-1}. Thus, we have

$$\begin{aligned}
y&=\frac{f+c_0P_{\nu(1-\beta),-\gamma\nu}}{S_{\beta,\gamma}-\lambda I}+\sum_{k=1}^{n-1}\frac{c_kS_{k-\nu(1-\beta),\gamma\nu}}{S_{\beta,\gamma}-\lambda I}\\
&=\frac{f*P_{\beta,\gamma}+c_0P_{\nu(1-\beta),-\gamma\nu}*P_{\beta,\gamma}}{I-\lambda P_{\beta,\gamma}}+\sum_{k=1}^{n-1}\frac{c_kS_{k-\nu(1-\beta),\gamma\nu}*P_{\beta,\gamma}}{I-\lambda P_{\beta,\gamma}}\\
&=\frac{f*P_{\beta,\gamma}}{I-\lambda P_{\beta,\gamma}}+\frac{c_0P_{\beta+\nu(1-\beta),\gamma-\gamma\nu}}{I-\lambda P_{\beta,\gamma}}+\sum_{k=1}^{n-1}\frac{c_kP_{\beta-k+\nu(1-\beta),\gamma-\gamma\nu}}{I-\lambda P_{\beta,\gamma}}\\
&=f*P_{\beta,\gamma}*\left(I-\lambda P_{\beta,\gamma}\right)^{-1}+\sum_{k=0}^{n-1}c_kP_{\beta-k+\nu(1-\beta),\gamma-\gamma\nu}*\left(I-\lambda P_{\beta,\gamma}\right)^{-1}.
\end{aligned}$$

Now everything can be written in terms of functions in C_{-1}, since we know as in Chapter 3 Example 3.3 that

$$\left(I-\lambda P_{\beta,\gamma}\right)^{-1}=\sum_{i=0}^{\infty}\lambda^iP_{i\beta,i\gamma},$$

so

$$y=\sum_{i=0}^{\infty}\lambda^if*P_{(i+1)\beta,(i+1)\gamma}+\sum_{i=0}^{\infty}\sum_{k=0}^{n-1}\lambda^ic_kP_{\beta-k+\nu(1-\beta)+i\beta,\gamma-\gamma\nu+i\gamma}.$$

This can be written explicitly as a function in the following way:

$$y(x)=\sum_{i=0}^{\infty}\lambda^i\int_0^x(x-\xi)^{(i+1)\beta-1}E_{\alpha,(i+1)\beta}^{(i+1)\gamma}\left(\delta(x-\xi)^{\alpha}\right)f(\xi)\,\mathrm{d}\xi$$
$$+\sum_{i=0}^{\infty}\sum_{k=0}^{n-1}\lambda^i c_k x^{\beta-k+\nu(1-\beta)+i\beta-1}E_{\alpha,\beta-k+\nu(1-\beta)+i\beta}^{\gamma-\gamma\nu+i\gamma}\left(\delta x^{\alpha}\right),$$

or in terms of the Prabhakar integral operator and the modified Mittag-Leffler function $e_{\alpha,\beta}^{\gamma,\delta}$ as follows:

$$y(x)=\sum_{i=0}^{\infty}\lambda^i\left({}_0^{P}I_x^{\alpha,(i+1)\beta,(i+1)\gamma,\delta}f\right)(x)+\sum_{i=0}^{\infty}\sum_{k=0}^{n-1}\lambda^i c_k\cdot e_{\alpha,\beta-k+\nu(1-\beta)+i\beta}^{\gamma-\gamma\nu+i\gamma,\delta}(x).$$

Now we have found a strong solution, unique in the space ${}^{H}\Omega_{-1}^{\beta}$, to the initial value problem that we started with. Note that the constant c_k is defined in this case by

$$c_k=\sum_{j=0}^{N_k}\frac{(-\gamma(1-\nu))_j\delta^j a_{j,k}}{j!},\tag{5.18}$$

in terms of the given initial values $a_{j,k}$, following the formula (5.9) from above and recalling from (5.4) the definition of N_k.

Remark 5.2: The initial conditions (5.17) may seem strange to the reader. They are a finite collection of initial values, as would be expected for such a problem, but they are enumerated by two separate indices j and k.

In fact, this collection of initial conditions on the Riemann–Liouville derivatives of y can be replaced by a smaller collection of initial conditions on the Riemann–Liouville-type Prabhakar derivatives of y. From the equivalence between (5.9) and (5.10) that we showed above, it follows that the constants c_k used in Example 5.1 could also be expressed from (5.18) as

$$
\begin{aligned}
c_k &= \sum_{j=0}^{N_k} \frac{(-\gamma(1-\nu))_j \delta^j}{j!} \cdot \left({}^{RL}_{\ 0}D_x^{-\alpha j+\beta+\nu(n-\beta)-k-1} y \right)(0) \\
&= \left({}^{PR}_{\ 0}D_x^{\alpha,\beta+\nu(n-\beta)-k-1,\gamma(1-\nu),\delta} f \right)(0),
\end{aligned}
$$

for $k = 0, 1, \cdots, n-1$.

Thus, we could replace the initial conditions (5.17) by the following alternative (Prabhakar-type) initial conditions:

$$
\left({}^{PR}_{\ 0}D_x^{\alpha,\beta+\nu(n-\beta)-k-1,\gamma(1-\nu),\delta} f \right)(0) = c_k, \qquad k = 0, 1, \cdots, n-1, \tag{5.19}
$$

and the solution to the initial value problem given by (5.16) and (5.19), unique in ${}^{H}\Omega^{\beta}_{-1}$, would be exactly the same as before, with the constants c_k defined directly from the initial conditions (5.19) rather than indirectly from the initial conditions (5.17) via the relation (5.18).

Example 5.2: Let us consider a Prabhakar version of the free electron laser integro-differential equation [8], namely the following fractional integro-differential equation involving both a Prabhakar integral and a single Hilfer–Prabhakar derivative:

$$
\left({}^{PH}_{\ 0}D_x^{\alpha,\beta,\gamma,\delta;\nu} y \right)(x) - \lambda \left({}^{p}_{0}I_x^{\alpha,\beta',\gamma',\delta} y \right)(x) = f(x), \qquad x > 0; \tag{5.20}
$$

$$
\left({}^{RL}_{\ 0}D_x^{-\alpha j+\beta+\nu(n-\beta)-k-1} y \right)(0) = a_{j,k}, \qquad j,k \in \mathbb{Z}_0^+ \text{ such that } \alpha j + k \leq \beta \tag{5.21}
$$

where $\alpha, \beta, \beta', \gamma, \gamma', \delta, \nu$, and all the $a_{j,k}$ are fixed constants with $0 \leq \nu < 1$.

We firstly rewrite the problem in an algebraic form, using (5.11) from Theorem 5.3.1 with the same notation $c_k = c_{y,k}$ as in Example 5.1:

$$S_{\beta,\gamma} * y - c_0 P_{\nu(1-\beta),-\gamma\nu} - \sum_{k=1}^{n-1} c_k S_{k-\nu(1-\beta),\gamma\nu} - \lambda P_{\beta',\gamma'} * y = f,$$

where each constant c_k is given in terms of the known initial values $a_{j,k}$. Rearranging the algebraic equation gives

$$y = \frac{f + c_0 P_{\nu(1-\beta),-\gamma\nu} + \sum_{k=1}^{n-1} c_k S_{k-\nu(1-\beta),\gamma\nu}}{S_{\beta,\gamma} - \lambda P_{\beta',\gamma'}},$$

which we rearrange in terms of P elements to give a function as follows:

$$\begin{aligned} y &= \frac{f * P_{\beta,\gamma} + c_0 P_{\nu(1-\beta),-\gamma\nu} * P_{\beta,\gamma} + \sum_{k=1}^{n-1} c_k S_{k-\nu(1-\beta),\gamma\nu} * P_{\beta,\gamma}}{I - \lambda P_{\beta',\gamma'} * P_{\beta,\gamma}} \\ &= \frac{f * P_{\beta,\gamma} + c_0 P_{\beta+\nu(1-\beta),\gamma-\gamma\nu} + \sum_{k=1}^{n-1} c_k P_{\beta-k+\nu(1-\beta),\gamma-\gamma\nu}}{I - \lambda P_{\beta+\beta',\gamma+\gamma'}} \\ &= f * P_{\beta,\gamma} * \left(I - \lambda P_{\beta+\beta',\gamma+\gamma'}\right)^{-1} + \sum_{k=0}^{n-1} c_k P_{\beta-k+\nu(1-\beta),\gamma-\gamma\nu} * \left(I - \lambda P_{\beta+\beta',\gamma+\gamma'}\right)^{-1}. \end{aligned}$$

Using the same power series approach to $\left(I - \lambda P_{\beta+\beta',\gamma+\gamma'}\right)^{-1}$ as in the previous Example above, we have

$$y = \sum_{i=0}^{\infty} \lambda^i f * P_{(i+1)\beta+i\beta',(i+1)\gamma+i\gamma'} + \sum_{i=0}^{\infty}\sum_{k=0}^{n-1} \lambda^i c_k P_{(i+1)\beta+\nu(1-\beta)+i\beta'-k,(i+1-\nu)\gamma+i\gamma'}.$$

This can be written explicitly as a function in the following way:

$$\begin{aligned} y(x) = \sum_{i=0}^{\infty} \lambda^i \int_0^x (x-\xi)^{(i+1)\beta+i\beta'-1} E_{\alpha,(i+1)\beta+i\beta'}^{(i+1)\gamma+i\gamma'}\left(\delta(x-\xi)^{\alpha}\right) f(\xi)\,\mathrm{d}\xi \\ + \sum_{i=0}^{\infty}\sum_{k=0}^{n-1} \lambda^i c_k x^{(i+1)\beta+\nu(1-\beta)+i\beta'-k-1} E_{\alpha,(i+1)\beta+\nu(1-\beta)+i\beta'-k}^{(i+1-\nu)\gamma+i\gamma'}\left(\delta x^{\alpha}\right), \end{aligned}$$

or in terms of the Prabhakar integral operator and the modified Mittag-Leffler function $e_{\alpha,\beta}^{\gamma,\delta}$ as follows:

$$y(x) = \sum_{i=0}^{\infty} \lambda^i \left({}^{P}_{0}I_x^{\alpha,(i+1)\beta+i\beta',(i+1)\gamma+i\gamma',\delta} f\right)(x) + \sum_{i=0}^{\infty}\sum_{k=0}^{n-1} \lambda^i c_k \cdot e_{\alpha,(i+1)\beta+\nu(1-\beta)+i\beta'-k}^{(i+1-\nu)\gamma+i\gamma',\delta}(x).$$

Now we have found a strong solution, unique in the space ${}^{H}\Omega_{-1}^{\beta}$, to the initial value problem that we started with. Note that the constant c_k is defined in terms of the given initial values $a_{j,k}$ in exactly the same way (5.18) as in Example 5.1.

Remark 5.3: A problem almost the same as in Example 5.2, but specifically with $n = 1$ and $\beta = \beta'$, was considered in [24, Section 5.2]. The initial conditions used there were of Prabhakar form, like (5.19) which we saw above is equivalent to (5.21).

Our solution obtained here using Mikusiński's method is identical, after setting $n = 1$ and $\beta = \beta'$ and modifying notation appropriately, to the solution found in [24, Theorem 5.3] using Laplace transforms. This verifies the correctness of our result.

Remark 5.4: The relationship between the $a_{j,k}$ and c_k given by (5.18), i.e. between initial conditions of Riemann–Liouville type and of Prabhakar type, is worth exploring in a little more detail, as the form of the sum in (5.18) may inspire thoughts of combinatorial sum identities. We note a few interesting cases:

1. If $a_{j,k} = 0$ for all j,k, then $c_k = 0$ for all k. This is the case of homogeneous initial conditions, where the solution to a problem like (5.16) will be simply a convolution of f with an appropriate kernel function.
2. If $a_{j,k} = 1$ for all j,k and $\delta = 1$, then using the notations $\theta = -\gamma(1-\nu) \in \mathbb{C}$ and $p = \left\lfloor \frac{\beta-k+\nu(n-\beta)}{\alpha} \right\rfloor \in \mathbb{Z}_0^+$, we find
$$c_k = \binom{\theta+p}{\theta}.$$
This can be proved using the following combinatorial identity [40, Chapter 3, Example 3.52] in (5.18):
$$\sum_{j=0}^{p} \binom{\theta+j-1}{j} = \binom{\theta+p}{\theta}.$$
3. If $a_{j,k} = (-1)^j \binom{p}{j}$ for all j,k, using the same notations p and θ as above, and

$\delta = 1$, then we find

$$c_k = (-1)^p \binom{\theta - 1}{p}.$$

This can be proved using the following combinatorial identity [40, Chapter 6, Example 3.5] in (5.18):

$$\sum_{j=0}^{p} (-1)^j \binom{\theta + j - 1}{\theta - 1} \binom{p}{j} = (-1)^p \binom{\theta - 1}{\theta - 1 - p}.$$

Example 5.3: The above two Examples have been devoted to solving integro-differential equations involving only one Hilfer Prabhakar derivative each. Such equations have also been solved by other methods in the literature [24, 55], so we now consider a more general multi-term fractional differential equation involving multiple Hilfer Prabhakar derivatives with independent parameters. Namely, the following problem:

$$\sum_{i=1}^{m} \lambda_i \left({}^{PH}_{\ 0}D_x^{\alpha,\beta_i,\gamma_i,\delta;\nu_i} y \right)(x) = f(x), \qquad x > 0; \tag{5.22}$$

$$\left({}^{PR}_{\ 0}D_x^{\alpha,\beta_i+\nu_i(n_i-\beta_i)-k-1,\gamma_i(1-\nu_i),\delta} y \right)(0) = c_k^{(i)}, \tag{5.23}$$

$$k = 0, 1, \cdots, n_i - 1, \quad i = 1, 2, \cdots, m,$$

where $\alpha, \beta_i, \gamma_i, \delta, \nu_i$, and all the $c_k^{(i)}$ are fixed constants with $0 \leq \nu_i < 1$ and $n_i := \lfloor \beta_i \rfloor$ for $i = 1, 2, \cdots, m$, and where we assume without loss of generality that β_m is the largest of all the β_i.

We firstly rewrite the problem in an algebraic form, using (5.11) from Theorem 5.3.1:

$$\sum_{i=1}^{m} \lambda_i \left[S_{\beta_i,\gamma_i} * y - c_0^{(i)} P_{\nu_i(1-\beta_i),-\gamma_i\nu_i} - \sum_{k=1}^{n_i-1} c_k^{(i)} S_{k-\nu_i(1-\beta_i),\gamma_i\nu_i} \right] = f,$$

where each constant $c_k^{(i)}$ plays a role according to (5.10). Rearranging the algebraic equation gives

$$y=\frac{f+\sum_{i=1}^{m}\lambda_i c_0^{(i)} P_{\nu_i(1-\beta_i),-\gamma_i\nu_i}+\sum_{i=1}^{m}\sum_{k=1}^{n_i-1}\lambda_i c_k^{(i)} S_{k-\nu_i(1-\beta_i),\gamma_i\nu_i}}{\lambda_1 S_{\beta_1,\gamma_1}+\cdots+\lambda_m S_{\beta_m,\gamma_m}}. \tag{5.24}$$

We will simplify this by using the following relation from Chapter3, Example 3.1:

$$\begin{aligned}
&\frac{I}{\lambda_1 S_{\beta_1,\gamma_1}+\cdots+\lambda_m S_{\beta_m,\gamma_m}} \\
&\qquad =\sum_{j_1=0}^{\infty}\cdots\sum_{j_{m-1}=0}^{\infty}\frac{(j_1+\cdots+j_{m-1})!}{j_1!\cdots j_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{j_1}\cdots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{j_{m-1}} \\
&\qquad\qquad \times P_{\beta_m+j_1(\beta_m-\beta_1)+\cdots+j_{m-1}(\beta_m-\beta_{m-1}),\gamma_m+j_1(\gamma_m-\gamma_1)+\cdot+j_{m-1}(\gamma_m-\gamma_{m-1})},
\end{aligned} \tag{5.25}$$

which can be written as an explicit function in two ways: either, following Chapter 3, Example 3.2 , as a multiple ($m-1$ times) sum of univariate Mittag-Leffler functions of Prabhakar type,

$$\begin{aligned}
&\sum_{j_1=0}^{\infty}\cdots\sum_{j_{m-1}=0}^{\infty}\frac{(j_1+\cdots+j_{m-1})!}{j_1!\cdots j_{m-1}!}\left(\frac{-\lambda_1}{\lambda_m}\right)^{j_1}\cdots\left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{j_{m-1}} \\
&\qquad \times x^{\beta_m+j_1(\beta_m-\beta_1)+\cdots+j_{m-1}(\beta_m-\beta_{m-1}-1} E^{\gamma_m+j_1(\gamma_m-\gamma_1)+\cdot+j_{m-1}(\gamma_m-\gamma_{m-1})}_{\alpha,\beta_m+j_1(\beta_m-\beta_1)+\cdots+j_{m-1}(\beta_m-\beta_{m-1})}\left(\delta x^{\alpha}\right),
\end{aligned}$$

or, following [18], as a single sum of multivariate Mittag-Leffler functions of Gorenflo–Luchko type [35],

$$\begin{aligned}
&\sum_{r=0}^{\infty}\frac{\left(\gamma_m+j_1(\gamma_m-\gamma_1)+\cdots+j_{m-1}(\gamma_m-\gamma_{m-1})\right)_r\delta^r}{r!} \\
&\qquad \times x^{r\alpha+\beta_m-1}E_{\beta_m-\beta_1,\cdots,\beta_m-\beta_{m-1};r\alpha+\beta_m}\left(\frac{-\lambda_1}{\lambda_m}x^{\beta_m-\beta_1},\cdots,\frac{-\lambda_{m-1}}{\lambda_m}x^{\beta_m-\beta_{m-1}}\right).
\end{aligned}$$

(Convergence of the multiple series is guaranteed by [28, Theorem 6].)

Substituting (5.25) into (5.24), we find

$$
\begin{aligned}
y = \sum_{j_1=0}^{\infty} \cdots \sum_{j_{m-1}=0}^{\infty} & \frac{(j_1+\cdots+j_{m-1})!}{j_1!\cdots j_{m-1}!} \left(\frac{-\lambda_1}{\lambda_m}\right)^{j_1} \cdots \left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{j_{m-1}} \\
& \times P_{\beta_m+j_1(\beta_m-\beta_1)+\cdots+j_{m-1}(\beta_m-\beta_{m-1}),\gamma_m+j_1(\gamma_m-\gamma_1)+\cdot\cdot+j_{m-1}(\gamma_m-\gamma_{m-1})} * f \\
& + \sum_{i=1}^{m} \sum_{k=0}^{n_i-1} \lambda_i c_k^{(i)} \sum_{j_1=0}^{\infty} \cdots \sum_{j_{m-1}=0}^{\infty} \frac{(j_1+\cdots+j_{m-1})!}{j_1!\cdots j_{m-1}!} \left(\frac{-\lambda_1}{\lambda_m}\right)^{j_1} \cdots \left(\frac{-\lambda_{m-1}}{\lambda_m}\right)^{j_{m-1}} \\
& \times P_{\beta_m-k+\nu_i(1-\beta_i)+j_1(\beta_m-\beta_1)+\cdots+j_{m-1}(\beta_m-\beta_{m-1}),\gamma_m-\gamma_i\nu_i+j_1(\gamma_m-\gamma_1)+\cdot\cdot+j_{m-1}(\gamma_m-\gamma_{m-1})}.
\end{aligned}
$$

This is the solution to the fractional differential equation that we started with. The two possible formulations of (5.25) as a function, seen above, lead to two possible ways of writing this solution function $y(x)$ explicitly as a function. We choose here the second way, in order to use only single sums rather than multiple sums:

$$
\begin{aligned}
y(x) = \sum_{r=0}^{\infty} & \frac{\left(\gamma_m + j_1(\gamma_m-\gamma_1)+\cdots+j_{m-1}(\gamma_m-\gamma_{m-1})\right)_r \delta^r}{r!} \Big[\int_0^x \xi^{r\alpha+\beta_m-1} \\
& \times E_{\beta_m-\beta_1,\cdots,\beta_m-\beta_{m-1};r\alpha+\beta_m}\left(\frac{-\lambda_1}{\lambda_m}\xi^{\beta_m-\beta_1},\ldots,\frac{-\lambda_{m-1}}{\lambda_m}\xi^{\beta_m-\beta_{m-1}}\right) f(x-\xi)\,\mathrm{d}\xi\Big] \\
& + \sum_{i=1}^{m}\sum_{k=0}^{n_i-1}\sum_{r=0}^{\infty} \frac{\lambda_i c_k^{(i)}\left(\gamma_m-\gamma_i\nu_i+j_1(\gamma_m-\gamma_1)+\cdots+j_{m-1}(\gamma_m-\gamma_{m-1})\right)_r \delta^r}{r!} \\
& \qquad \times x^{r\alpha+\beta_m-k+\nu_i(1-\beta_i)-1} \\
& \times E_{\beta_m-\beta_1,\cdots,\beta_m-\beta_{m-1};r\alpha+\beta_m-k+\nu_i(1-\beta_i)}\left(\frac{-\lambda_1}{\lambda_m}x^{\beta_m-\beta_1},\ldots,\frac{-\lambda_{m-1}}{\lambda_m}x^{\beta_m-\beta_{m-1}}\right).
\end{aligned}
$$

This is a strong solution, unique in the space ${}^{H}\Omega_{-1}^{\beta_m}$, to the initial value problem (5.22)–(5.23) that we started with.

REFERENCES

[1] Agrawal, O. P. (2012). “Some generalized fractional calculus operators and their applications in integral equations”, *Fractional Calculus and Applied Analysis* 15(4), 700–711.

[2] Almeida, R. (2017). “A Caputo fractional derivative of a function with respect to another function”, *Communications in Nonlinear Science and Numerical Simulation* 44, 460–481.

[3] Baleanu, D., Fernandez, A. (2019). *On fractional operators and their classifications*, Math., 7(9), 830.

[4] Bengochea, G. (2014). “Operational solution of fractional differential equations”, *Applied Mathematics Letters* 32, 48–52.

[5] Bengochea, G., Verde-Star, L. (2011). “Linear algebraic foundations of the operational calculi”, *Advances in Applied Mathematics* 47(2), 330–351.

[6] Brychkov, Y. A., Prudnikov, A. P., Shishov, V. S. (1981). “Operational Calculus”, *Journal of Soviet Mathematics* 15(6), 733–765.

[7] Colombaro, I., Giusti, A., Vitali, S. (2018). “Storage and Dissipation of Energy in Prabhakar Viscoelasticity”, *Mathematics* 6(2), 15.

[8] Dattoli, G., Giannessi, L., Mezi, L., Torre, A., Caloi, R. (1991). "FEL time-evolution operator", *Nuclear Instruments and Methods in Physics Research Section A* 304(1–3), 541–544.

[9] Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer, Heidelberg, 2010.

[10] Dimovski, I. (1966). "Operational calculus for a class of differentional operators", *Comptes Rendus de l'Academie Bulgare des Sciences* 19(12), 1111–1114.

[11] Duffy, D. G. (2004). *Transform methods for solving partial differential equations*, Chapman and Hall.

[12] Eshaghi, S., Ghaziani, R. K., Ansari, A. (2020). "Stability and dynamics of neutral and integro-differential regularized Prabhakar fractional differential systems", *Computational and Applied Mathematics* 39, 250.

[13] Evans, L. C. (1998). *Partial Differential Equations: Graduate Studies in Mathematics, Volume 19*, American Mathematical Society.

[14] Fahad, H. M., Fernandez, A. (2021). *Operational calculus for Riemann–Liouville fractional calculus with respect to functions and the associated fractional differential equations*, Frac. Calc. Appl. Anal., 24(2), 518–540.

[15] Fernandez, A. "Tables of composition properties of fractional integrals and derivatives", preprint.

[16] Fernandez, A., Baleanu, D. (2011). "Classes of Operators in Fractional Calculus: A Case Study", *Mathematical Methods in the Applied Sciences* 44(11), 9143–9162.

[17] Fernandez, A., Özarslan, M. A., Baleanu, D. (2019). "On fractional calculus with general analytic kernels", *Applied Mathematics and Computation* 354, 248–265.

[18] Fernandez, A., Restrepo, J. E., Suragan, D. "Prabhakar-type linear differential equations with variable coefficients", *Differential and Integral Equations* 35 (2022), pp. 581–610.

[19] Fernandez, A., Baleanu, D., Srivastava, H. M. (2019)."Series representations for fractional-calculus operators involving generalised Mittag-Leffler functions", *Communications in Nonlinear Science and Numerical Simulation* 67, 517–527.

[20] Fernandez, A., Baleanu, D. (2019). *On a new definition of fractional differintegrals with Mittag-Leffler kernel*, Filomat, 33(1), 245–254.

[21] Fernandez, A., Kürt, C., Özarslan, M. A. (2020). *A naturally emerging bivariate Mittag-Leffler function and associated fractional-calculus operators*, Comp. Appl. Math. 39, 200.

[22] Flegg, H. G. (1974). *Mikusinski's Operational Calculus*, Int. J. Math. Educ. Sci. Tech., 5(2), 131–137.

[23] Fokas, A. S. (2008). *A Unified Approach to Boundary Value Problems*, SIAM, Philadelphia.

[24] Garra, R., Gorenflo, R., Polito, F., Tomovski, Ž. (2014). "Hilfer–Prabhakar derivatives and some applications", *Applied Mathematics and Computation* 242, 576–589.

[25] Garrappa, R., Maione, G. (2017). Fractional Prabhakar Derivative and Applications in Anomalous Dielectrics: A Numerical Approach. In: A. Babiarz, A. Czornik, J. Klamka, M. Niezabitowski, eds., Theory and Applications of Non-integer Order Systems, Springer, Cham.

[26] Gutterman, M. (1969). *An operational method in partial differential equations*, SIAM J. Appl. Math., 17(2), 468–493.

[27] Giusti, A., Colombaro, I., Garra, R., Garrappa, R., Polito, F., Popolizio, M., Mainardi, F. (2020). "A practical guide to Prabhakar fractional calculus", *Fractional Calculus and Applied Analysis* 23(1), 9–54.

[28] Hadid, S. B., Luchko, Y. F. (1996). "An Operational Method for Solving Fractional Differential Equations of an Arbitrary Real Order", *Panamerican Mathematical Journal* 6, 57–73.

[29] Hilfer, R., ed. (2000). *Applications of Fractional Calculus in Physics*, World Scientific, Singapore.

[30] Hilfer, R., Luchko, Y. F., Tomovski, Ž. (2009). “Operational method for the solution of fractional differential equations with generalized Riemann–Liouville fractional derivatives”, *Fractional Calculus and Applied Analysis* 12(3), 299–318.

[31] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam.

[32] Kilbas, A. A., Saigo, M., Saxena, R. K. (2004). “Generalized Mittag-Leffler function and generalized fractional calculus operators”, *Integral Transforms and Special Functions* 15(1), 31–49.

[33] Kochubei, A. N. (2011). “General Fractional Calculus, Evolution Equations, and Renewal Processes”, *Integral Equations and Operator Theory* 71 , 583–600.

[34] Luchko, Y. (1999). “Operational method in fractional calculus”, *Fractional Calculus and Applied Analysis* 2(4), 463–488.

[35] Luchko, Y., Gorenflo, R. (1999). “An operational method for solving fractional differential equations”, *Acta Mathematica Vietnamica* 24, 207–234.

[36] Luchko, Y., Yakubovich, S. (1994). “An operational method for solving some classes of integro-differential equations", *Differentsial’nye Uravneniya* 30(2),

269–280.

[37] Luchko, Y. (2021). “General fractional integrals and derivatives of arbitrary order”, *Symmetry* 13, 755.

[38] Mikusiński, J. (1959). *Operational Calculus*, Pergamon Press, Oxford.

[39] Miller, K. S., Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley.

[40] Mitrinović, D. S., Kečkić, J. D. (1990). *Metodi Izračunavanja Konačnih Zbirova*, Naučna Knjiga, Beograd.

[41] Oldham, K. B., Spanier, J. (1974). *The Fractional Calculus*, Academic Press.

[42] Osler, T. J. (1970). “Leibniz rule for fractional derivatives generalized and an application to infinite series”, *SIAM Journal of Applied Mathematics* 18(3) , 658–674.

[43] Pipes, L. A. (1939). “The Operational Calculus I”, *Journal of Applied Physics* 10, 172.

[44] Polito, F., Tomovski, Ž. (2016). “Some properties of Prabhakar-type fractional calculus operators”, *Fractional Differential Calculus* 6(1), 73–94.

[45] Prabhakar, T. R. (1971). “A singular integral equation with a generalized Mittag-Leffler function in the kernel”, *Yokohama Mathematical Journal* 19, 7–15.

[46] Restrepo, J. E., Suragan, D. (2021). “Oscillatory solutions of fractional integro-differential equations II”, *Mathematical Methods in the Applied Sciences* 44(8), 7262–7274.

[47] Samko, S. G., Kilbas, A. A., Marichev, O. I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon.

[48] Sandev, T., Tomovski, Ž. (2019). *Fractional Equations and Models: Theory and Applications*, Springer Nature.

[49] Schiff, J. L. (1999). *The Laplace transform: theory and applications*, Springer-Verlag.

[50] Sousa, J. V. d. C., De Oliveira, E. C. (2018). “On the Ψ-Hilfer fractional derivative”, *Communications in Nonlinear Science and Numerical Simulation* 60 72–91.

[51] Srivastava, H. M., Daoust, M. C. (1972). *A note on the convergence of Kampé de Fériet’s Double Hypergeometric series*, Math. Nachrichten, 53, 151–159.

[52] Sun, H. G., Zhang, Y., Baleanu, D., Chen, W., Chen, Y. Q. (2018). “A new collection of real world applications of fractional calculus in science and

engineering", *Communications in Nonlinear Science and Numerical Simulation* 64 , 213–231.

[53] Teodoro, G. S., Machado, J. T., De Oliveira, E. C. (2019). "A review of definitions of fractional derivatives and other operators", *Journal of Computational Physics* 388, 195–208.

[54] Tomovski, Ž, Dubbeldam, J. L. A., Korbel. J. (2020). Applications of Hilfer–Prabhakar operator to option pricing financial model. Fract. Calc. Appl. Anal. 23(4), 996–1012.

[55] Tomovski, Ž., Garra, R. (2014). "Analytic solutions of fractional integro-differential equations of Volterra type with variable coefficients", *Fractional Calculus and Applied Analysis* 17(1), 38–60.

[56] Widder, D. V. (1941). *The Laplace Transform*, Princeton University Press.

[57] Yakubovich, S., Luchko, Y. (1994). *The Hypergeometric Approach to Integral Transforms and Convolutions: Mathematics and Applications, Volume 287*, Kluwer Academic Publishers, Amsterdam, 1994.