

Copy Machines – Self-reproduction with 2 States on Archimedean Tilings

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ABSTRACT

Self-reproducing feature of cellular automata is one of the most important aim from the beginnings. In this thesis we show binary (2-state) infinite grid self-reproducing automata which reproduce (few copies) of the original pattern in various grids. For the three regular grids we prove their behavior based on path-counting. Similar machine also works on many semi-regular tilings. Moreover, we show that in some stage, by removing the copies and keeping only central embryo part of the pattern, it is able to reconstruct the pattern to be copied. To illustrate the self-reproduction various patterns are reproduced in various tessellations.

Keywords: Self-reproduction, Cellular automata on regular and semi-regular tessellations, Binary cellular automata, Path-counting on grids, Pattern generation, Life-like cellular automata, Path counting.

ÖZ

Hücreesel otomatların kendi kendini yeniden üretme özelliği, eskiden günümüze dayanan en önemli amaçlardan biridir. Bu tezde, orijinal şekli çeşitli ızgaralarda (birkaç kopya halinde) çoğaltan, sonsuz ızgara üzerinde ikili (2-durumlu) kendi kendini yeniden üreten otomatları gösteriyoruz. Üç düzenli ızgarada, otomatların bu davranışlarını yol sayımı kullanarak kanıtlıyoruz. Benzer makineler ayrıca birçok yarı düzenli döşemeler üzerinde de çalışır. Ayrıca, belli bir aşamada, kopyaları çıkartıp, desenin sadece merkezi embriyo kısmını tutarak, kopyalanacak deseni yeniden oluşturmanın mümkün olduğunu gösteriyoruz. Kendi kendine üremeyi göstermek için, çeşitli mozaiklerde çeşitli desenlerin kopyaları çoğaltılmıştır.

Anahtar Kelimeler: Kendi kendine üreme, Düzenli ve yarı düzenli ızgaralarda hücreesel otomatlar, İkili hücreesel otomatlar, Izzaralar üzerinde yol sayma, Model oluşturma, Gerçek gibi görünen hücreesel otomatlar, Yol sayma.

DEDICATION

For my wife Bahareh and my daughter, Artemis who are my inspiration in life.

Love you forever!

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Chapter 1

INTRODUCTION

Cellular automata are inherently parallel models of computations, mimicking living organisms in a discrete time-space environment. When the first cellular automata models were invented, in 1948 [29], one of the main aims was to give such a formal system that is able to reproduce itself, as this is one of the most important phenomena of living organisms [17], [30]. As von Neumann already gave a 29-state cellular automaton on the square grid, using only 4-neighborhood (also called von Neumann neighborhood) that was able to reproduce itself, this could be interpreted as the processes upon which life is based given in an algorithmic manner. Then, the challenge was to design cellular automata with similar feature, but with less number of states. In 1968, Codd was able to design a cellular automaton with only 8 states that had very similar properties as von Neumann's model [9]. On the other hand, cellular automata became very popular when, in 1970, Conway has introduced the Game of life, a binary model based on Moore neighborhood (8-neighborhood). In this model the states are called "alive" and "dead", and the state of cell in the next instance depends on the number of alive cells in the neighborhood: a dead cell will be alive if and only if it had exactly three alive neighbors, and a living cell stays alive if and only if it had either two or three alive neighbors [11]. The model, although its rules are very simple, produced many interesting phenomena and cellular automata became popular not only among scientists, but among many people who could write a small program to make observations. Another universal model was defined by Langton [17] with 8 states,

where looped pathway, a so-called construction arm is attached, and its idea is related to a kind of DNA self-reproduction. On the other hand, without caring the universality, relatively simple self-reproduction (replicator) cellular automata were defined by parity rules [32], [3], [10]. These automata are able to reproduce arbitrary initial patterns. In [3] it has been computed and proven how many steps are needed for the self-reproduction in one-dimensional case with any number (at least two) of states, i.e., with not necessarily binary alphabet, this depends on the size (i.e., length) of the pattern, while [32] used the square grid and binary alphabet. In [23], this result has been extended to tessellation structures of any finite dimension, i.e., to nD (hyper)cubic/rectangular grids with larger alphabet (allowing to have more than two states). In [12] the result was further expanded by allowing not only the traditional, but any types of neighborhood. However, still only the traditional nD grids were used. In [4] an alternative proof can be found for the reproduction of patterns of the same type of automata. In [5], it is shown how pattern reproducing tessellation structures can be viewed as such elements of the class of tessellation structures that have local transformations that are linear operators. A further study of these type of automata on the nD rectangular grids is in [15]. We note here that the “parity” automata are binary and based on the logical XOR operation, i.e., if the number of alive neighbor cells is even or odd, while their generalized variants with more than two states are usually referred as “additive” cellular automata.

We should mention that cellular automata on some non-traditional grids were already defined: on the triangular grid, see, e.g., [7] for cellular automata with totalistic rules, [2] where binary totalistic cellular automata with memory was used or [14], [26] and [25] where cellular automata generating various patterns and performing simple mathematical morphological operations were presented for image manipulation.

Reversibility of cellular automata were studied in [28]. Other grids were also involved [6]. In [24] cellular automata performing various image processing methods, e.g., resizing, skeletonization, edge detection on the square grid are presented, but some work on the hexagonal and triangular grids are also shown. In [1] cellular automata are used in production of art works with rich morphological behavior, showing the interconnection of the fields of mathematics, computer science, physics and biology.

The very sophisticated nature of the 1D binary cellular automata was studied in detail by Wolfram [33],[34]. We should recall that the 1D case of the replicator is already shown in [31] and it is actually, the cellular automata with Wolfram number 90, however, in [34] the replicator property of these automata was not studied. In [16], 1D reversible cellular automata with six states were analyzed where the forward orbit of every (non-uniform) finite configuration contains copies of all finite patterns over the state alphabet.

2D cellular automata are based on a tessellation of the plane, we have already recalled models working on the regular grids. However, there are many other tessellations and we consider here also the eight semi-regular tessellations. Those with the regular grids are also frequently referred as Archimedean tilings. They have an old history as ancient Greeks knew them, and Kepler classified them in 1619. They play important role in discrete and digital geometry [8], [27], [20], in chemistry [22] and other fields.

Here, we have expanded and investigated these parity cellular automata on various other tessellations. As we can see, they can reproduce any pattern on all regular tessellations. We have modeled them mathematically and proven how and when the copies of the original pattern appear. Then, we study similar automata on the semi-

regular tessellations, and we show that some of them have the self-reproducing feature. In this way, on the one hand, we still work with only binary cellular automata (for simplicity), but on the other hand, we shift the domain to a more general family of grids which includes several grids that are not closed under translation by every grid vector and the neighborhood structure is not uniform in the sense that various cells may have neighbors not exactly in the same respective directions.

The structure of this thesis is as follows. In the next section we define the cellular automata, we use in general named “copy-machine”, while in chapters 3 and 4 we specify it for the three regular and for the eight semi-regular grids, respectively. As one will see the “copy machine” works nicely on the regular grids and also it has self-reproducing feature on some of the semi-regular grids. However, it does not work on every grid, we investigate also that issue.

Chapter 2

COPY MACHINE CELLULAR AUTOMATON

A “Cellular Automaton” (shortly a CA) consists of a tessellation with finite number of dimensions. Each cell in the tessellation has a finite number of states. For cell α there is a set of cells called its neighbors represented by $N(\alpha)$. An initial state ($n = 0$) is where each cell has one specified state (e.g., “1” or “-1”) and shown by $S_0(\alpha)$. A new generation ($n + 1$) is created according to some fixed rules that specify the new state of each cell based on its current state and its neighbors’ state. The rules are the same for each cell and do not change. They are simultaneously applied to the whole grid.

2.1 CMCA General Definition

Definition 1. Copy Machine Cellular Automaton or shortly CMCA is a cellular automaton on periodic Tessellation T such that $S_0(\alpha) \in \{-1, 1\}$, $N(\alpha) = \{\beta | \alpha \text{ and } \beta \text{ share exactly one side.}\}$ And $S_n(\alpha) = \prod_{\beta \in N(\alpha)} S_{n-1}(\beta)$ where, $\alpha, \beta \in T$, $n \in \mathbb{N}$ and $S_n(\alpha)$ is the state of the cell α at the time step n .

Binary systems usually use values 0 and 1, however, to make our description easier, here, we use -1 and +1. Moreover, the value -1 of a cell is referred as an alive cell and shown by black color in our figures, while 1 represents dead cells and shown by white color.

Lemma 1: In CMCA,

$$S_n(\alpha) = \prod_{d[\alpha, \beta] \leq n} (S_0(\beta))^{C_n(\beta)}$$

where $C_n(\beta)$ is the number of paths from the cell α to the cell β with length n .

Proof: In each step every cell can affect only its direct neighbors so in step n , the number of times cell β can affect cell α equals to the number of paths from the cell α to the cell β with length n . □

In the next sections we are specifying and analyzing CMCA on the Archimedean tessellations. Whenever we are specializing CMCA on a specific grid, the description of the grid plays importance, e.g., the coordinate system and how it reflects the neighbor relation among tiles.

Chapter 3

CMCA ON REGULAR TESSALATIONS

A regular tessellation is a highly symmetric, edge-to-edge tiling made up of regular polygons, all the same shape. There are only three regular tessellations: those made up of equilateral triangles, squares, or regular hexagons. In the next subsections we investigate CMCA on those grids. Note that in fact, on the square grid our automaton is similar to [3] and [10], we shall show this case mostly for the completeness and to show our technique for the analogy for other tessellations. Nevertheless, we give a new alternative proof of the self-reproduction also in this case.

3.1 Square Tiling

A square grid or square tiling is a grid formed by tiling the plane regularly with squares.

3.1.1 Coordinate System

To specify the CMCA on the square tiling, we formally give the coordinate system and the neighborhood used. We use the Cartesian coordinate system, as usual for the square grid, i.e., $\mathbb{Z}^2$ is addressing the cells. Each cell has 2 coordinates i and j where $i, j \in \mathbb{Z}$. Closest neighbors, also called side-neighbors are used throughout this thesis. Two cells are neighbors if exactly one of their coordinates differs by ± 1 . Consequently, each cell has 4 neighbors, the cells with which it shares a side.

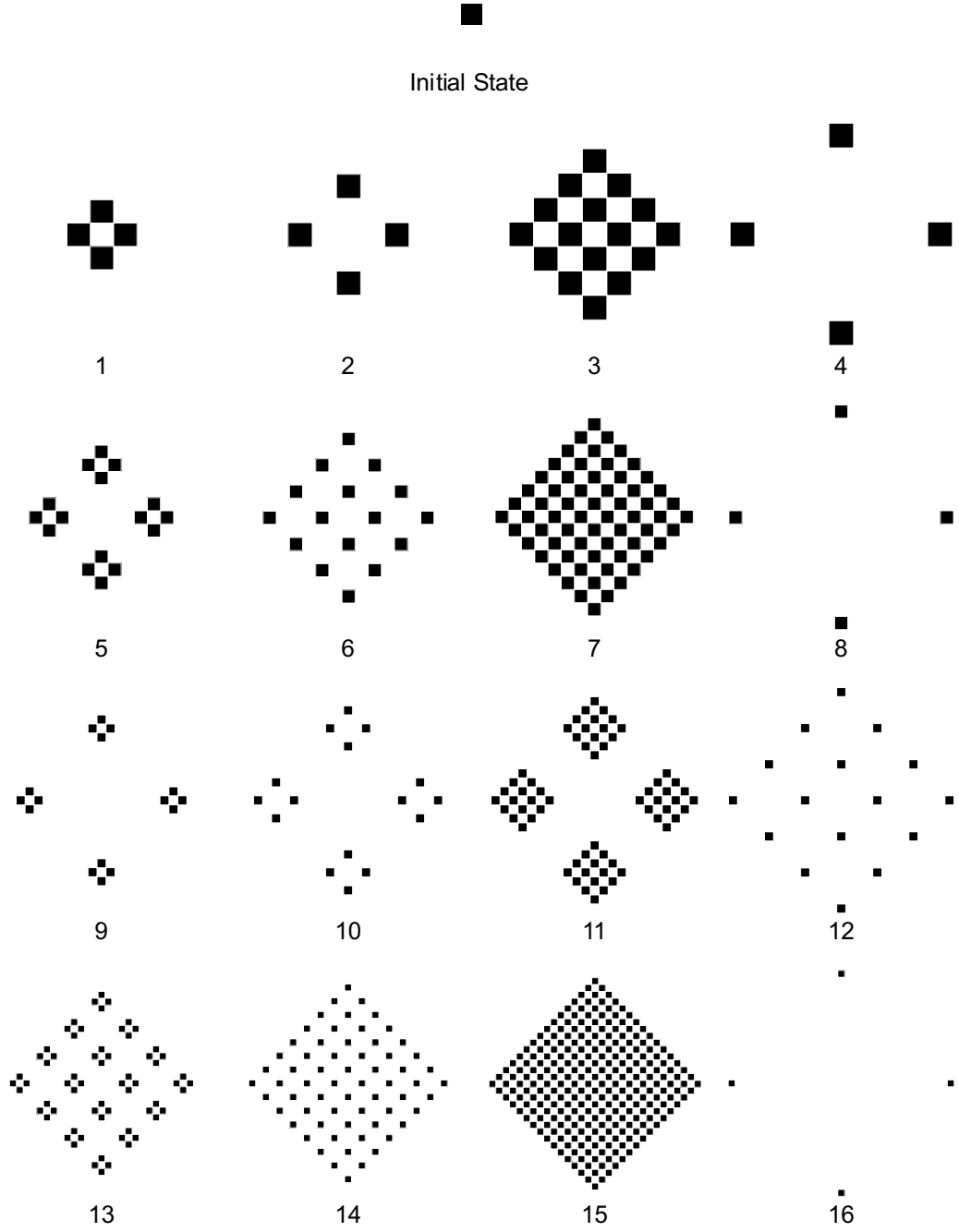


Figure 1: Square Grid, Sample 1

3.1.2 Automaton Study

Definition 2. CMCA on Square grid is a cellular automaton on square grid such that:

$$S_0(i, j) \in \{-1, 1\},$$

$$N(i, j) = \{(i + 1, j), (i - 1, j), (i, j + 1), (i, j - 1)\} \text{ and}$$

$$S_n(i, j) = \prod_{(i', j') \in N(i, j)} S_{n-1}(i', j') \text{ where } i, j \in \mathbb{Z}, n \in \mathbb{N}.$$

This automaton replicates the initial pattern at steps $n = 2^k, k \in \mathbb{N}$. Figure 1 and Figure 2 show two sample initial patterns and the first 16 generations of the automaton.

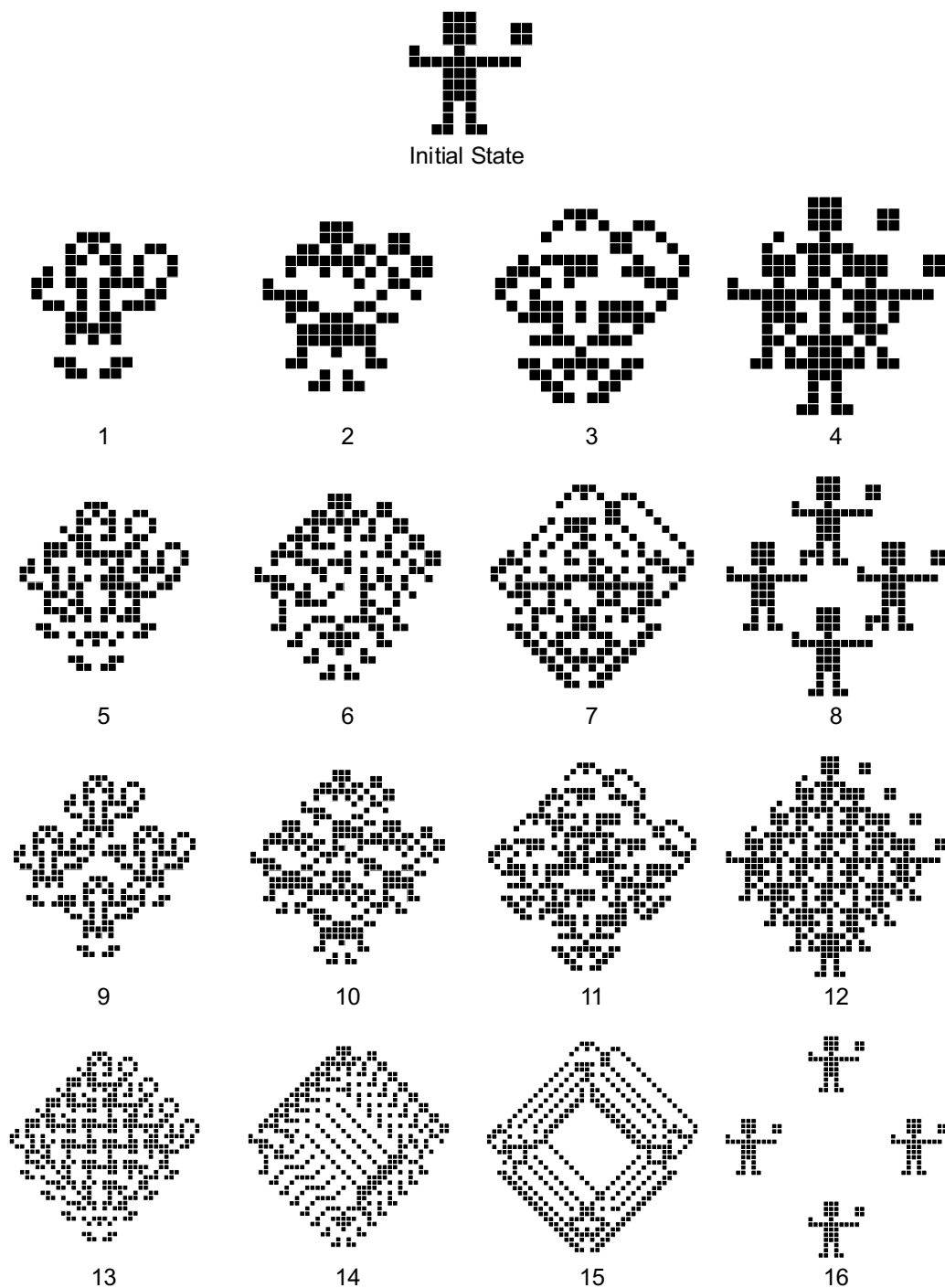


Figure 2: Square Grid, Sample 2

3.1.3 Mathematical Proof of Reproduction

To prove that the Copy Machine CA reproduces the initial pattern first we need to solve recursive equation of the automaton.

Theorem 1. If n is a power of 2, then binomial coefficient $\binom{n}{k}$ is odd if and only if $k = n$ or $k = 0$.

Proof. In general, we may apply the well-known theorem of Lucas [18]: a binomial coefficient $\binom{n}{k}$ is divisible by a prime p if and only if at least one of the base p digits of k is greater than the corresponding digit of n .

Here, $p = 2$ and since n is a power of 2, its base 2 representation is as below format:

$$1000 \dots 0$$

Since all the digits of n except the first is zero, for every k , $\binom{n}{k}$ is divisible by 2 unless if $k = 0$ or $k = n$. □

Our analysis is based on path-counting, as the next part of the section shows.

Theorem 2. Number of paths in the square tiling from cell $(0,0)$ to cell (i,j) with length n is shown by $C_n(i,j)$ where:

$$C_n(i,j) = \begin{cases} 0 & , \quad i+j+n \text{ is odd} \\ \sum_{a=|i|}^{\frac{n+|i|-|j|}{2}} \binom{n}{a} \times \binom{n-a}{a-|i|} \times \binom{\frac{n-2a+|i|}{2}}{\frac{n+|i|+|j|}{2}-a} & , \quad i+j+n \text{ is even} \end{cases}$$

Proof. Without loss of generality, we assume $i, j \geq 0$.

Case 1: $i + j + n$ is odd. If the distance between the cell (i, j) and the starting cell $(0, 0)$ is an odd number, then it's only possible to reach it by odd number of steps. On the other hand, if the distance between the cell (i, j) and the starting cell $(0, 0)$ is an even number then it's only possible to reach it by even number of steps. From these 2 possible cases we can conclude that if $i + j + n$ is odd then no path with length n is possible from the starting cell $(0, 0)$ to the cell (i, j) .

Case 2: $i + j + n$ is even. We use '+' when we increase the coordinate by 1 and '-' when we decrease it by 1. This leads to 4 different movements:

$$i^+, i^-, j^+, j^-$$

Let us denote the number of each of them by a , b , c and d respectively. Any path in the square grid starting from the cell $(0, 0)$ can be written as a sequence of mentioned movements.

$$\overbrace{-, -, -, -, \dots, -}^n$$

We know that:

$$a + b + c + d = n \tag{1}$$

$$(a - b) = i \tag{2}$$

$$(c - d) = j \tag{3}$$

From (2) we have:

$$b = a - i \tag{4}$$

From (3) we have:

$$d = c - j \tag{5}$$

From (1), (4) and (5) we have:

$$a + b + c + d = n \Rightarrow a + (a - i) + c + (c - j) = n \Rightarrow 2a + 2c - (i + j) = n$$

$$\Rightarrow 2(a + c) = n + i + j \Rightarrow a + c = \frac{n+i+j}{2} \Rightarrow$$

$$c = \frac{n+i+j}{2} - a \quad (6)$$

There are n steps in each path. We need to have at least i times i^+ steps and j times j^+ steps. The maximum for the a (i^+ movements) is $\frac{n+i-j}{2}$. Calculating the sum of combinations for all possible a, b, c and d leads us to the following formula.

$$C_n(i, j) = \begin{cases} 0 & , i + j + n \text{ is odd} \\ \sum_{a=i}^{\frac{n+i-j}{2}} \binom{n}{a} \times \binom{n-a}{b} \times \binom{n-a-b}{c} \times \binom{n-a-b-c}{d} & , i + j + n \text{ is even} \end{cases} \quad (7)$$

From (1), (4), (6) and (7) we get the answer. $\square$

Proposition 1. The diameter of Pattern P in square tiling is calculated as follow:

$$D(P) = \max\{\Delta_i, \Delta_j\} + 1, \text{ where:}$$

$$\Delta_i = \max\{|i_1 - i_2| \mid \exists j_1 \text{ and } j_2 \text{ where } (i_1, j_1) \text{ and } (i_2, j_2) \in P\} \text{ and}$$

$$\Delta_j = \max\{|j_1 - j_2| \mid \exists i_1 \text{ and } i_2 \text{ where } (i_1, j_1) \text{ and } (i_2, j_2) \in P\}.$$

The generation number in which the first copies occur is a function of the diameter of the pattern and is equal to $\min \{n = 2^k \mid n \geq D(P), k \in \mathbb{Z}\}$. For example, for the first pattern (Figure 1) $D(P)$ is 1 and the first copies occur at step 1 while for the second pattern (Figure 2) $D(P)$ is 11 and the first copies occur at step 16.

Since $S_0(i, j) \in \{-1, 1\}$, only odd values of $C_n(i, j)$ are important for $S_n(0, 0)$ as when $C_n(i, j)$ is even, $S_0(i, j)$ does not affect $S_n(0, 0)$. Figure 3 (n from 1 to 4) shows the values $C_n(i, j)$ where odd valued cells are shown with green color.

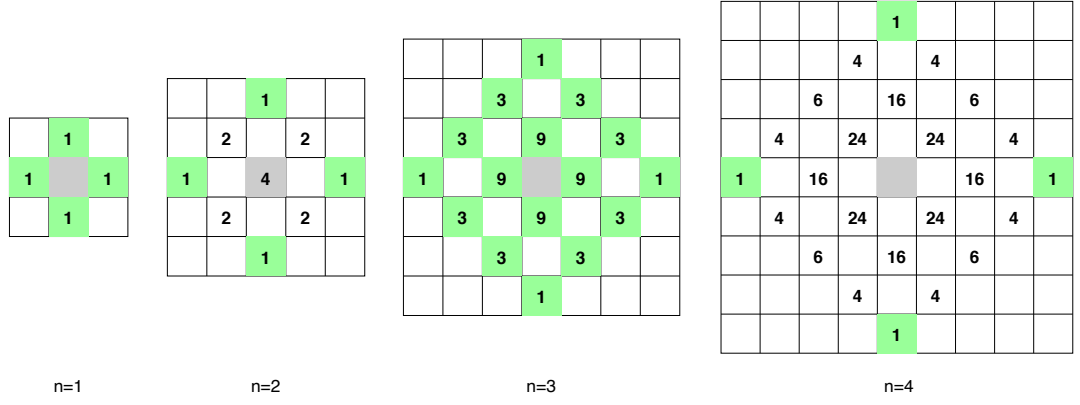


Figure 3: Number of the Paths in the Square Grid

In the general solution of $C_n(i, j)$ for square tiling (see theorem 2) it is clear that $C_n(i, j)$ can be odd if and only if all $\binom{n}{a}$, $\binom{n-a}{a-i}$ and $\binom{\frac{n-2a+i}{2}}{\frac{n+i+j}{2}-a}$ are odd and when n is a power of 2, based on theorem 1 it happens only at the cells $(n, 0), (-n, 0), (0, n), (0, -n)$. This means:

$$\forall i, j \in \mathbb{Z}, n = 2^k, k \in \mathbb{N} \rightarrow C_n(i, j) = C_0(i + n, j) \times C_0(i - n, j) \times C_0(i, j + n) \times C_0(i, j - n).$$

In step $n = 2^k, k \in \mathbb{N}$ where $n \geq D(P)$ we can separate the plain as it is shown in Figure 4.

It can be easily seen that

$$S_n(i, j) = S_0(i, j - n) \text{ if } (i, j) \in N,$$

$$S_n(i, j) = S_0(i - n, j) \text{ if } (i, j) \in E,$$

$$S_n(i, j) = S_0(i, j + n) \text{ if } (i, j) \in S,$$

$$S_n(i, j) = S_0(i + n, j) \text{ if } (i, j) \in W \text{ and}$$

$$S_n(i, j) = 1 \text{ if } (i, j) \notin (N \cup E \cup S \cup W).$$

This will lead to see copy of the pattern P in each of N, E, S and W .

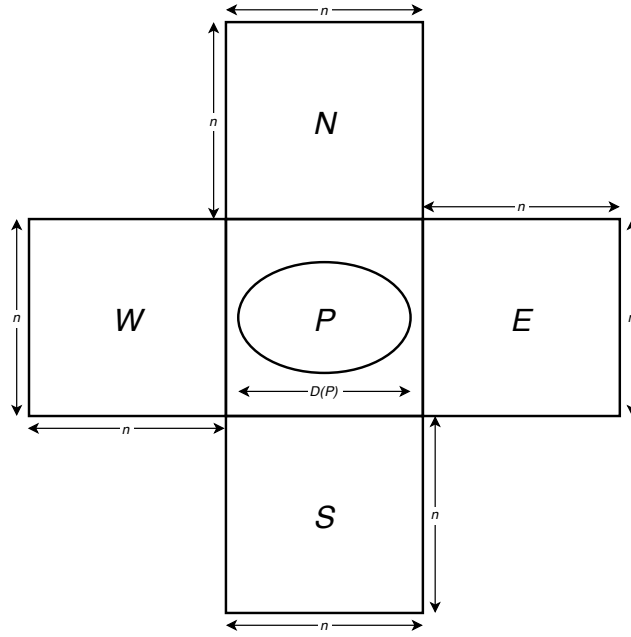


Figure 4: Square Grid Reproduction Map

3.2 Triangular Grid

A triangular tiling, also called an isometric grid, is a grid formed by tiling the plane regularly with equilateral triangles.

3.2.1 Coordinate System

We use coordinate system shown in Figure 5 to study the automaton in this grid [21]. Each cell has 3 coordinates i, j and k where $i, j, k \in \mathbb{Z}$ and $i + j + k \in \{0, 1\}$. There are 2 types of cells: type "E", even cells, where the sum of coordinates equals to 0 and type "O", odd cells, where the sum of coordinates equals to 1. Closest neighbours, also called side-neighbours are used here. Two cells are neighbours if exactly one of their coordinates differs by ± 1 . Consequently, each cell has 3 neighbours, the cells with which it shares a side. Even cells have odd neighbours and vice versa. Example of neighbours can also be seen in Figure 5 bottom left and bottom right, respectively.

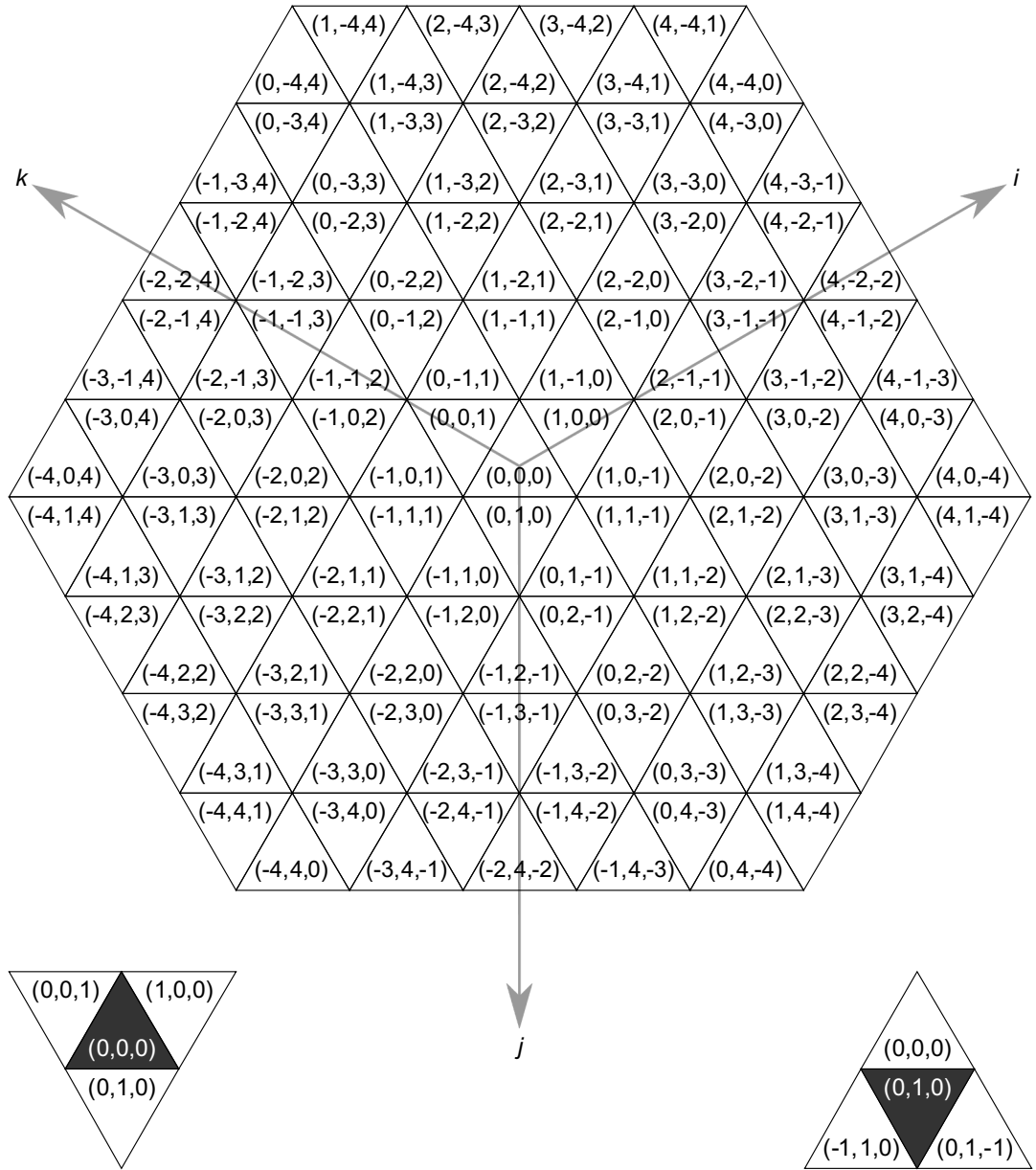


Figure 5: Triangular Grid Coordinate System

3.2.2 Automaton Study

Definition 3. CMCA on Triangular tiling is a cellular automaton on triangular tiling such that:

$$S_0(i, j, k) \in \{-1, 1\},$$

$$N(i, j, k) = \begin{cases} \{(i+1, j, k), (i, j+1, k), (i, j, k+1)\}, & i+j+k=0 \\ \{(i-1, j, k), (i, j-1, k), (i, j, k-1)\}, & i+j+k=1 \end{cases}$$

$$S_n(i, j, k) = \prod_{(i', j', k') \in N(i, j, k)} S_{n-1}(i', j', k'),$$

where $i, j, k \in \mathbb{Z}, n \in \mathbb{N}$.

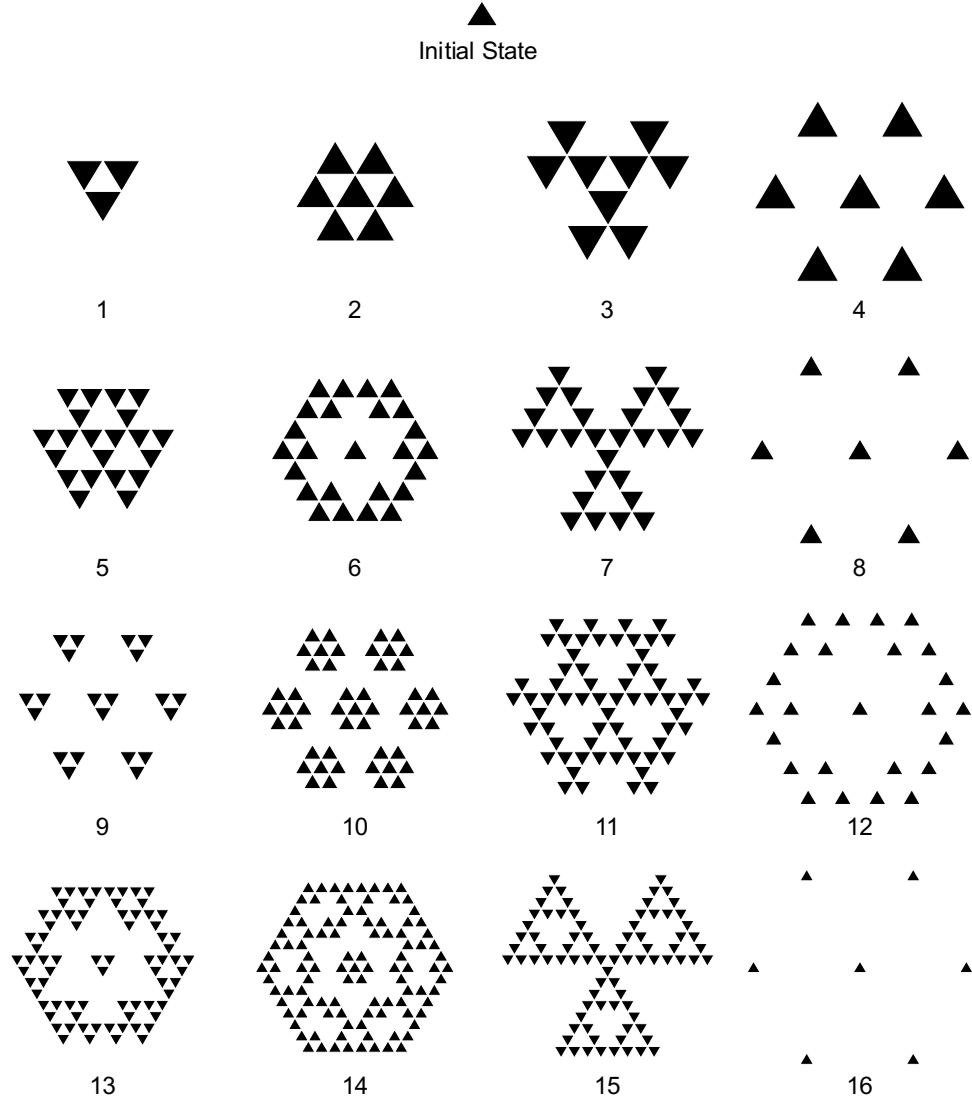


Figure 6: Triangular Grid, Sample 1

This automaton replicates the initial pattern at steps $n = 2^k$. Figure 6 and Figure 7 show two sample initial patterns and the first 16 generations of the automaton.

3.2.3 Mathematical Proof of Reproduction

To prove that the Copy Machine CA reproduces the initial pattern first we need to solve recursive equation of the automaton.

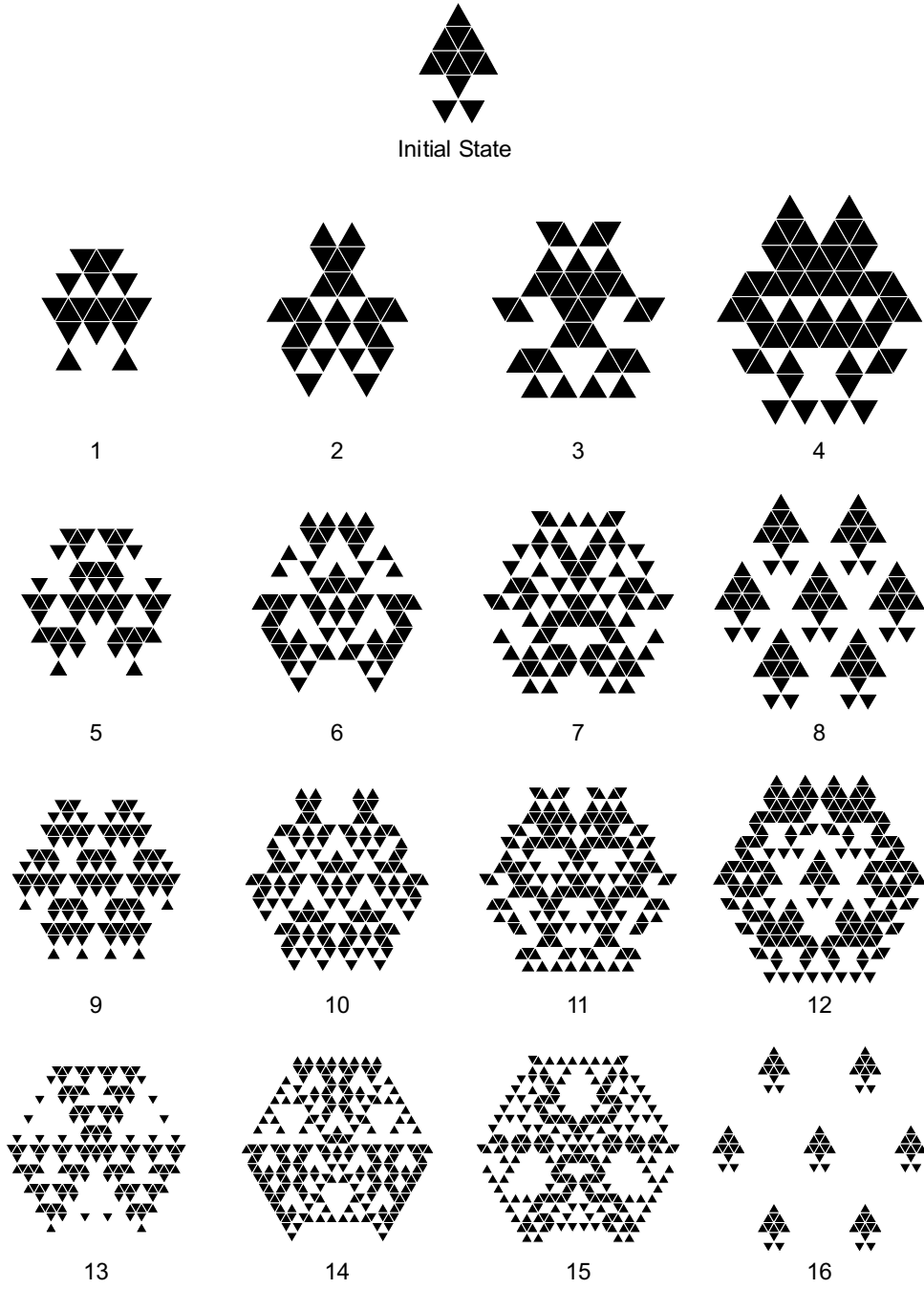


Figure 7: Triangular Grid, Sample 2

Theorem 3: Number of paths in the triangular tiling from cell $(0,0,0)$ to cell (i,j,k)

($i, j \geq 0$) with even length n is shown by $C_n(i, j, k)$ where:

$C_n(i, j, k)$

$$= \begin{cases} 0 & , \quad i + j + k = 1 \\ \sum_{a=i}^{\frac{n}{2}-j} \sum_{b=j}^{\frac{n}{2}-a} \binom{\frac{n}{2}}{a} \times \binom{\frac{n}{2}-a}{b} \times \binom{\frac{n}{2}}{a-i} \times \binom{\frac{n}{2}-(a-i)}{b-j} & , \quad i + j + k = 0 \end{cases}$$

Proof:

Case 1: $i + j + k = 1$. Since the distance between the cell and the starting cell (0,0,0) is an odd number there can't be any path with even length between them.

Case 2: $i + j + k = 0$. We show steps on Figure 8 left by i^+ , j^+ and k^+ where we increase one of the coordinates by 1 and steps on Figure 8 right by i^- , j^- and k^- where we decrease one of the coordinates by 1.

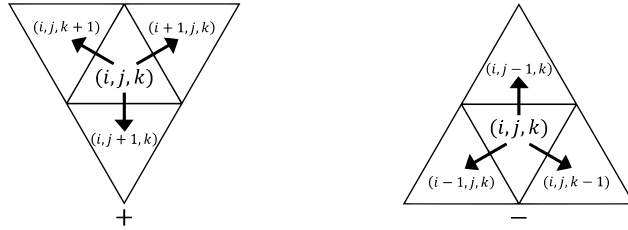


Figure 8: Triangular Grid Moving Directions

Any path in the triangular grid starting from the cell (0,0,0) can be written as a sequence of '+' and '-' steps as below.

$$\overbrace{+ \ - \ + \ - \ + \ - \ \dots \ + \ -}^n$$

There are $\frac{n}{2}$ steps in each group. In '+' group we need to have at least i times i^+ steps and j times j^+ steps. We name the number of i^+ steps as ' a ', the number of j^+ steps as ' b ' and the number of k^+ steps as ' c '.

We should also have $a - i$ times i^- steps and $b - j$ times j^- steps in ‘-’ group. In both “+” and “-“ group we only consider possible combinations for i and j movements as the rest should be filled by k movements with only 1 possible combination. Calculating the sum of combinations for all a and b leads us to the formula. $\square$

Proposition 2. Diameter of Pattern P in triangular tiling is denoted by $D(P)$ and is calculated as follow.

$$D(P) = \max\{\Delta_i, \Delta_j, \Delta_k\} + 1$$

$$\Delta_i = \max\{|j_1 - j_2| + |k_1 - k_2| \mid \exists i \text{ s. t. } (i, j_1, k_1) \text{ and } (i, j_2, k_2) \in P\},$$

$$\Delta_j = \max\{|i_1 - i_2| + |k_1 - k_2| \mid \exists j \text{ s. t. } (i_1, j, k_1) \text{ and } (i_2, j, k_2) \in P\} \text{ and}$$

$$\Delta_k = \max\{|i_1 - i_2| + |j_1 - j_2| \mid \exists k \text{ s. t. } (i_1, j_1, k) \text{ and } (i_2, j_2, k) \in P\}.$$

For triangular tiling when $n = 2^k, k \in \mathbb{N}$ the odd values of $C_n(i, j, k)$ occur at the

$$\text{cells} \quad (0, 0, 0), \left(\frac{n}{2}, \frac{-n}{2}, 0\right), \left(\frac{n}{2}, 0, \frac{-n}{2}\right), \left(0, \frac{n}{2}, \frac{-n}{2}\right), \left(\frac{-n}{2}, \frac{n}{2}, 0\right), \left(\frac{-n}{2}, 0, \frac{n}{2}\right), \left(0, \frac{-n}{2}, \frac{n}{2}\right)$$

which leads to see 6 copies and the initial pattern itself.

The generation number in which the first copies occur is a function of diameter of the pattern and is equal to $\min \{n = 2^k \mid n \geq D(P), k \in \mathbb{Z}\}$. For example, for the sample in Figure 6 $D(P)$ is 2 and the first copies occur at step 2 while for the sample in Figure 7 $D(P)$ is 6 and the first copies occur at step 8.

3.3 Hexagonal Grid

Hexagonal tiling or hexagonal tessellation is a regular tiling of the Euclidean plane, where exactly three hexagons meet at each corner.

3.3.1 Coordinate System

We use cubic coordinate system [13] shown at Figure 9 to study CMCA in this grid. Each cell has 3 coordinates i, j and k where $i, j, k \in \mathbb{Z}$ and $i + j + k = 0$. Closest neighbours, also called side-neighbours are used here. Two cells are neighbours if exactly two of their coordinates differs, one by +1 and the other one by -1. Consequently, each cell has 6 neighbours, the cells with which it shares a side.

3.3.2 Automaton Study

Definition 4. CMCA on Hexagonal tiling is a cellular automaton on hexagonal tiling such that:

$$S_0(i, j, k) \in \{-1, 1\},$$

$$N(i, j, k) = \{(i + 1, j - 1, k), (i + 1, j, k - 1), (i, j + 1, k - 1), (i - 1, j + 1, k), (i - 1, j, k + 1), (i, j - 1, k + 1)\},$$

$$S_n(i, j, k) = \prod_{(i', j', k') \in N(i, j, k)} S_{n-1}(i', j', k'),$$

where $i, j, k \in \mathbb{Z}, n \in \mathbb{N}$.

This automaton replicates the initial pattern at steps $n = 2^k$. Figure 10 and Figure 11 show two sample initial patterns and the first 16 generations of the automaton.

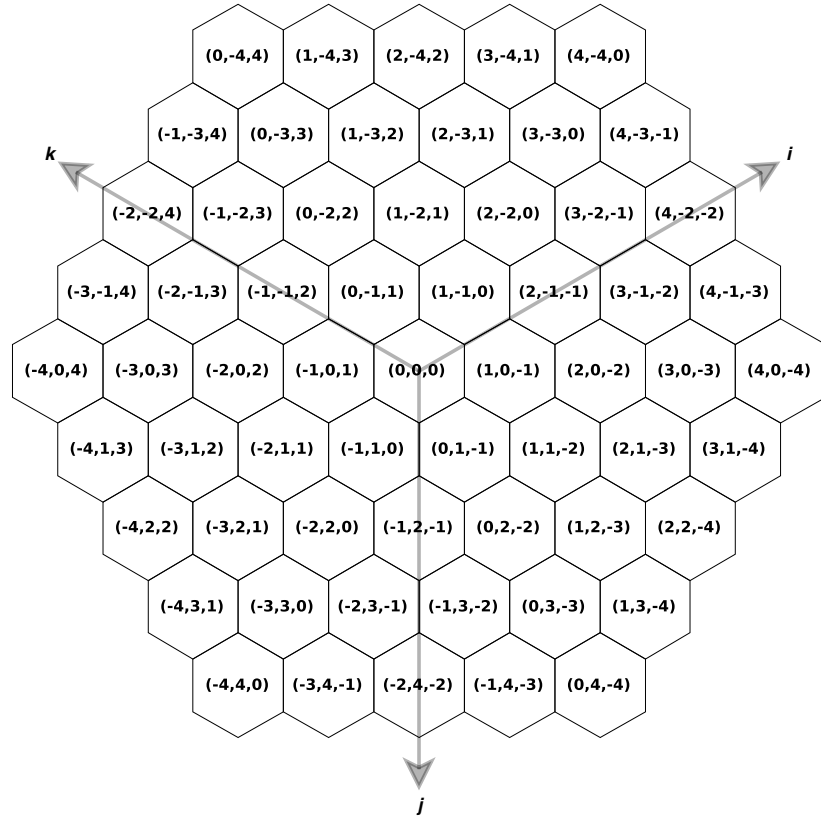


Figure 9: Hexagonal Grid Coordinate System

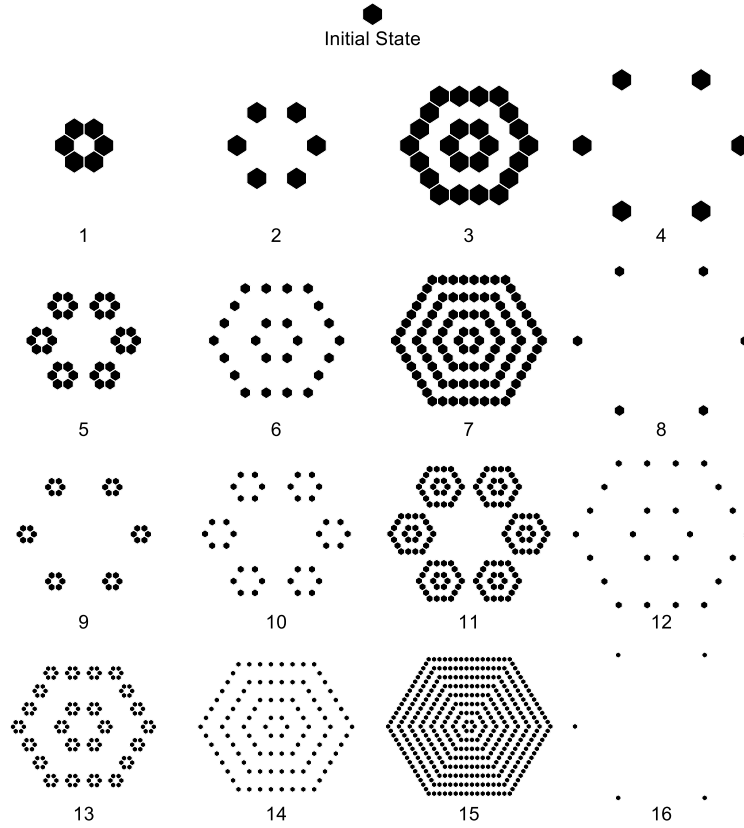


Figure 10: Hexagonal Grid, Sample 1

3.3.3 Mathematical Proof of Reproduction

To prove that CMCA reproduces the initial pattern first we need to solve recursive equation of the automaton.

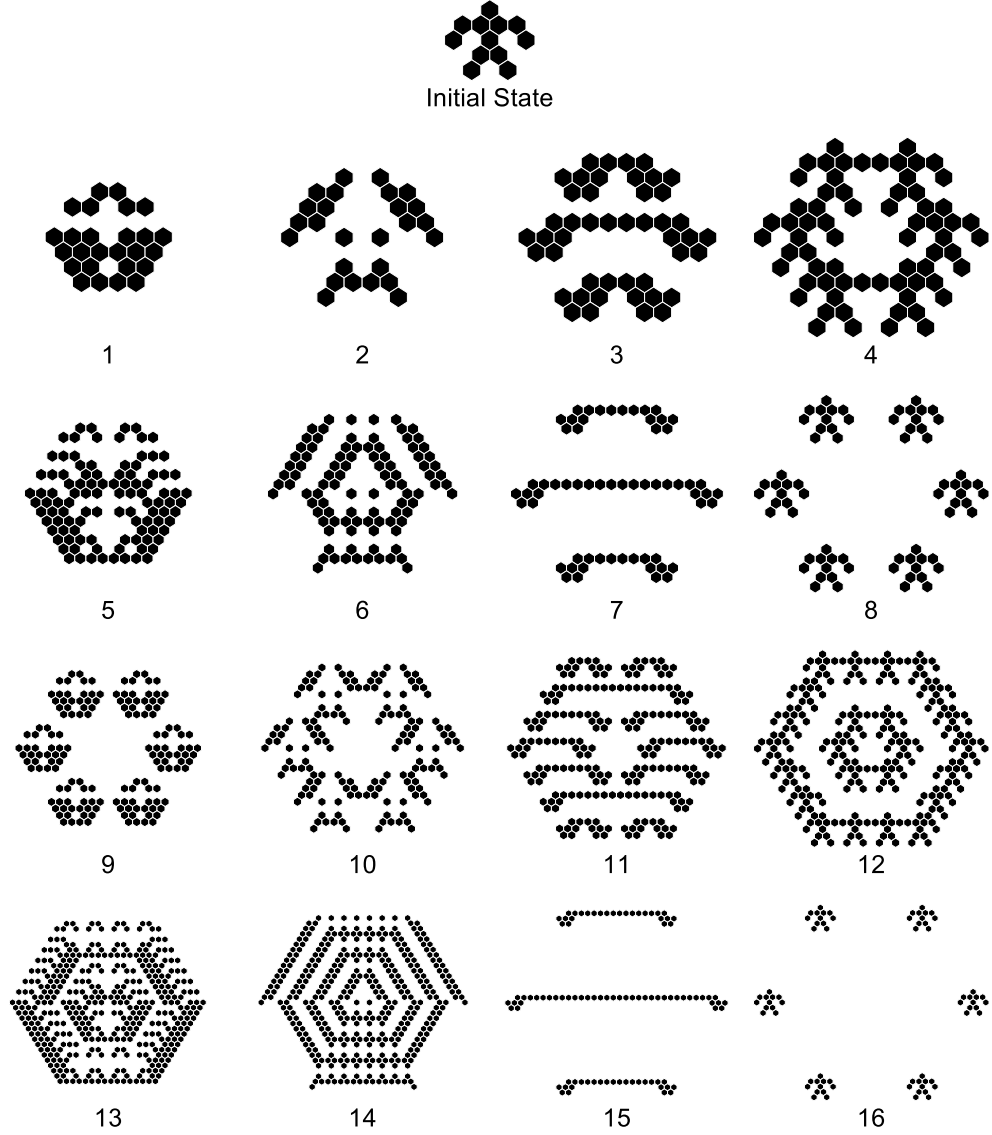


Figure 11: Hexagonal Grid, Sample 2

Theorem 4: The number of paths in the hexagonal tiling from cell $(0,0,0)$ to cell (i,j,k) (w.l.o.g. we assume $i,j \geq 0$ by symmetry) with length n is shown by $C_n(i,j,k)$ where:

$$C_n(i, j, k) = \sum_{a=i}^{n-j} \sum_{b=j}^{n-a} \sum_{t=T_1}^{T_2} \sum_{u=U_1}^{U_2} \sum_{v=V_1}^{V_2} \binom{n}{a} \binom{n-a}{b} \binom{n-a-b}{t} \binom{b}{a-i-t} \\ \binom{a}{u} \binom{b-a+i+t}{n-a-b-k-u} \binom{a-u}{v} \binom{n-a-b-t}{b-j-v},$$

$$T_1 = a - b - i, T_2 = \min(a - i, n - a - b),$$

$$U_1 = n - 2b + j - t, U_2 = \min(a, b + i + t) \text{ and}$$

$$V_1 = a + 2b + t - j - n, V_2 = \min(a, b - j).$$

Proof: In each step we may choose among directions i^+j , i^+k , j^+k , j^+i , k^+i , k^+j as it is shown in Figure 12.

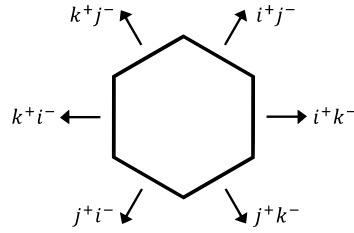


Figure 12: Hexagonal Grid Moving Directions

Each path with length n consists of n steps where each of them contains one “+” and one “-”.

$$+ \underbrace{\begin{bmatrix} \square \\ \square \end{bmatrix} \begin{bmatrix} \square \\ \square \end{bmatrix} \begin{bmatrix} \square \\ \square \end{bmatrix} \dots \begin{bmatrix} \square \\ \square \end{bmatrix}}_n$$

We start by filling the “+” parts with i , j and k where $n(i^+) = a$, $n(j^+) = b$ and $n(k^+) = c$. It is clear that $c = n - a - b$.

All possible combinations would be as follow.

$$\underbrace{\binom{n}{a}}_{i^+} \underbrace{\binom{n-a}{b}}_{j^+} \underbrace{\binom{n-a-b}{c}}_{k^+} \quad (1)$$

Now we should fill the “-” parts with i , j and k where $n(i^-) = d$, $n(j^-) = e$ and $n(k^-) = f$. It is clear that $d = a - i$, $e = b - j$ and $f = c - k$.

We start with i^- . We can have either k^+i^- or j^+i^- . Let $t = n(k^+i^-)$ so for k^+i^- we may choose between the positions where we have k^+ and for j^+i^- we may choose between the position where we have i^+ so all possible combinations would be as follows.

$$\underbrace{\binom{n-a-b}{t}}_{k^+i^-} \underbrace{\binom{b}{a-i-t}}_{j^+i^-} \quad (2)$$

We continue with k^- . We can have either i^+k^- or j^+k^- . Let $u = n(i^+k^-)$ so for i^+k^- we may choose between the positions where we have i^+ and for j^+k^- we may choose between the free positions where we have j^+ . All possible combinations would be as follows.

$$\underbrace{\binom{a}{u}}_{i^+k^-} \underbrace{\binom{b-(a-i-t)}{n-a-b-k-u}}_{j^+k^-} \quad (3)$$

We finish with j^- . We can have either i^+j^- or k^+j^- . Let $v = n(i^+j^-)$ so for i^+j^- we may choose between the free positions where we have i^+ and for k^+j^- we may choose between the free positions where we have k^+ . All possible combinations would be as follows.

$$\underbrace{\binom{a-u}{v}}_{i^+j^-} \underbrace{\binom{n-a-b-t}{b-j-v}}_{k^+j^-} \quad (4)$$

By applying the proper limits for a, b, t, u and v to the product of (1), (2), (3) and (4) we get the answer.

$$\sum_{a=i}^{n-j} \sum_{b=j}^{n-a} \sum_{t=T_1}^{T_2} \sum_{u=U_1}^{U_2} \sum_{v=V_1}^{V_2} \underbrace{\binom{n}{a}}_{i^+} \underbrace{\binom{n-a}{b}}_{j^+} \underbrace{\binom{n-a-b}{t}}_{k^+i^-} \underbrace{\binom{b}{a-i-t}}_{j^+i^-} \underbrace{\binom{a}{u}}_{i^+k^-} \underbrace{\binom{b-a+i+t}{n-a-b-k-u}}_{j^+k^-} \underbrace{\binom{a-u}{v}}_{i^+j^-} \underbrace{\binom{n-a-b-t}{b-j-v}}_{k^+j^-},$$

$$T_1 = \max\{0, a-b-i\}, T_2 = \min(a-i, n-a-b),$$

$$U_1 = \max\{0, n-2b+j-t\}, U_2 = \min(a, b+i+t) \text{ and}$$

$$V_1 = \max \{0, a + 2b + t - j - n\}, V_2 = \min(a, b - j).$$

□

Proposition 3. Diameter of Pattern P in hexagonal tiling is denoted by $D(P)$ and is calculated as follows.

$$D(P) = \max\{\Delta_i, \Delta_j, \Delta_k\} + 1$$

$$\Delta_i = \max \{|j_1 - j_2| \mid \exists i, k_1 \text{ and } k_2 \text{ s.t. } (i, j_1, k_1) \text{ and } (i, j_2, k_2) \in P\},$$

$$\Delta_j = \max \{|k_1 - k_2| \mid \exists j, i_1 \text{ and } i_2 \text{ s.t. } (i_1, j, k_1) \text{ and } (i_2, j, k_2) \in P\} \text{ and}$$

$$\Delta_k = \max\{|i_1 - i_2| \mid \exists k, j_1 \text{ and } j_2 \text{ s.t. } (i_1, j_1, k) \text{ and } (i_2, j_2, k) \in P\}.$$

For hexagonal tiling when $n = 2^k, k \in \mathbb{N}$ the odd values of $C_n(i, j, k)$ occur at the points $(n, -n, 0), (n, 0, -n), (0, n, -n), (-n, n, 0), (-n, 0, n), (0, -n, n)$ which leads to see 6 copies of initial pattern.

The generation number in which the first copies occur is a function of the diameter of the pattern and equals to $\min \{n = 2^k \mid n \geq D(P), k \in \mathbb{Z}\}$. For example, for the sample in Figure 11 $D(P)$ is 5 and the first copies occur at step 8.

Chapter 4

CMCA ON SEMI-REGULAR TESSELLATIONS

Regular tessellations of the plane by two or more convex regular polygons such that the same polygons in the same order surround each polygon vertex are called semiregular tessellations. In the plane, there are eight such tessellations. They are known since the ancient time, as we have recalled already in the Introduction.

The semi-regular tessellations may be characterized by the order in which the regular polygons occur around each vertex of the tiling. For example, 3-3-4-3-4 denotes the tiling by regular triangles and squares where the order around each vertex is two triangles, followed by a square, a triangle, and a square. Figure 5 shows the semi-regular tessellations.

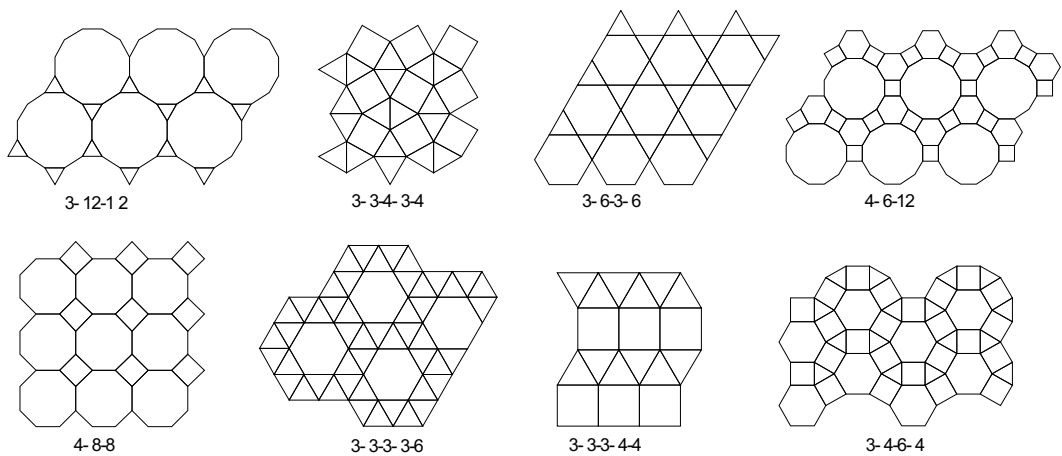


Figure 13: Semi-regular Tessellations

Whenever, a periodic tiling is given, one may consider the smallest connected regions of the grid as unit cells (or block of tiles) with the property that the tiling, the grid, is

built up by translating this unit cell by the unit vectors. For all regular and semi-regular grids, and for their duals, the unit cell always contains a relatively small finite number of tiles and we may use two of the shortest independent unit vectors to generate the whole tessellation by translations. One unit cell can be chosen to be addressed by the nullvector $(0,0)$. Every other unit cell will be addressed by the coefficients of the unit vectors in the vector connecting the (center) of the unit cell $(0,0)$ to the (center) of the given unit cell. The usual description $\mathbb{Z}^2$ of the square grid is also based on this approach, by using unit cells of the size of a tile and two orthogonal unit vectors. For the other point lattice grid, the hexagonal grid, the oblique coordinate system having axes with 60° or 120° angle between them, reflects also this method [19]. For the other grids, the unit cell contains more than one tile, thus one needs to identify the elements of the unit cell inside the cell as well. In general, we may use various letters to refer to the tiles inside a unit cell. This method gives us a coordinate system, that we call block coordinate system for the given grid. Each tile is addressed by two integer numbers (that show the position of the block in the plain) and one letter (that shows the type and the position of tile in the unit cell). Block coordinate systems can effectively be used in programming, in graphical algorithms etc., since usually the conversion to the usual Cartesian coordinates (that is needed, e.g., if one uses traditional monitor with square pixels) is very simple and the neighborhood can be described by relatively small tables.

In the following subsections, we present the Block Coordinate System on each of these grids and give some characteristic properties of CMCA.

Definition 5. Pattern P' is called an “Embryo” for Pattern P if they are different and pattern P can be created by CMCA after running the automaton for some finite steps while using pattern P' as initial state.

4.1 “3-3-3-3-6” Tiling

4.1.1 Coordinate System

The block coordinate system for “3-3-3-3-6” tiling is given in Figure 14.

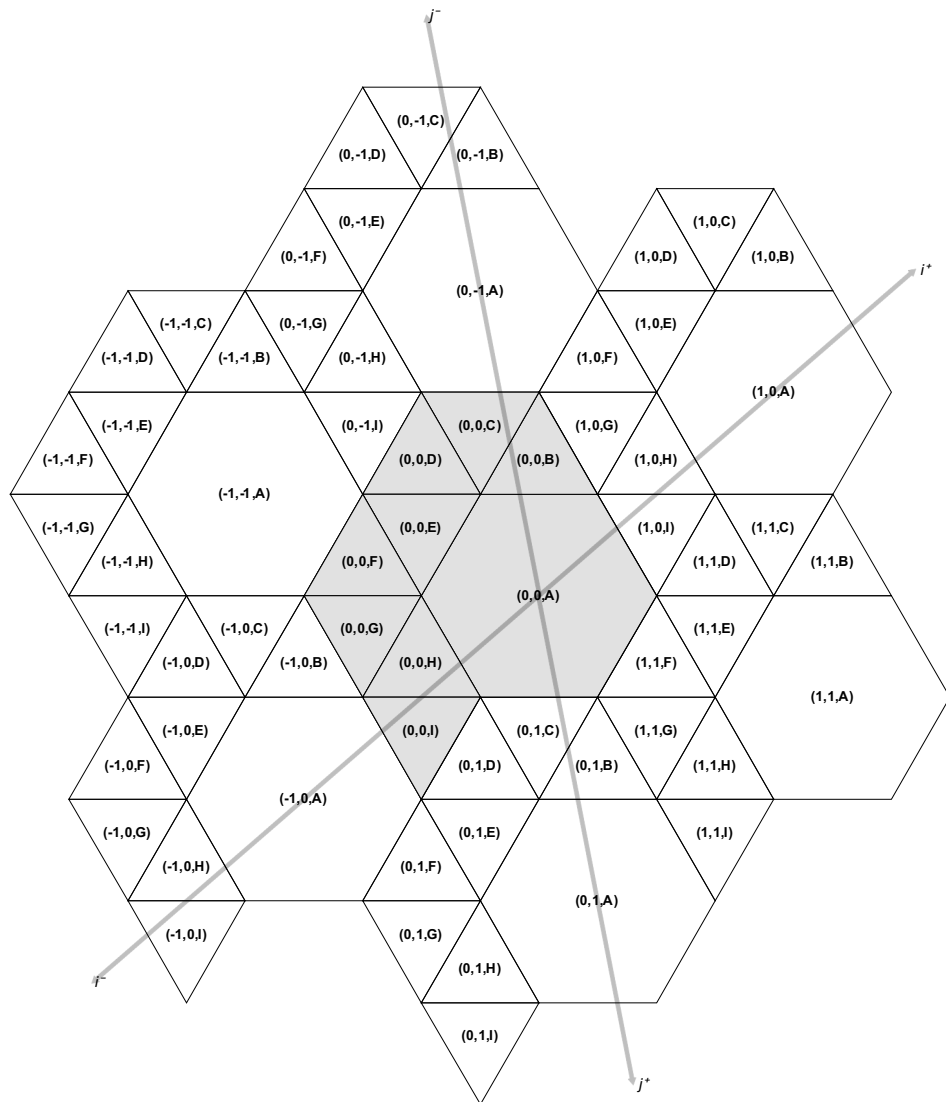


Figure 14: 3-3-3-3-6 Tiling Coordinate System

4.1.2 Observations

CMCA on “3-3-3-3-6” tiling has an interesting behavior to compare with the previous cases. When the starting pattern includes only hexagons, 6 copies can be observed in steps $3 \times 2^n, n \in \mathbb{N}$ by filtering/restricting the figure only for the hexagons, as some extra triangles also occur.

Figure 15, left, shows an initial pattern and Figure 15, right shows the CMCA in step 12 where 6 copies of pattern can be observed.

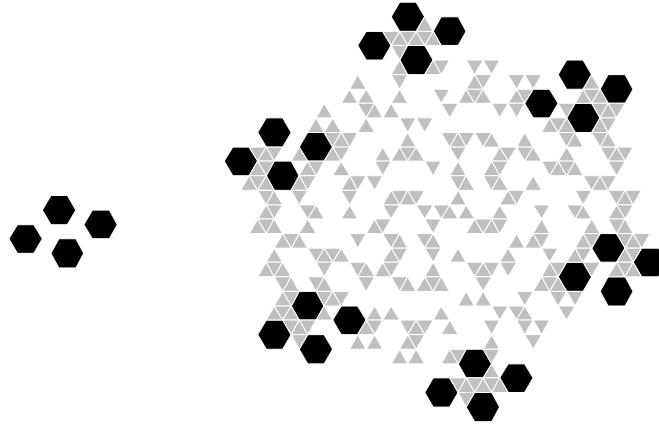


Figure 15: “3-3-3-3-6” Tiling, Sample Pattern(left), CMCA Result After 12 Steps(right)

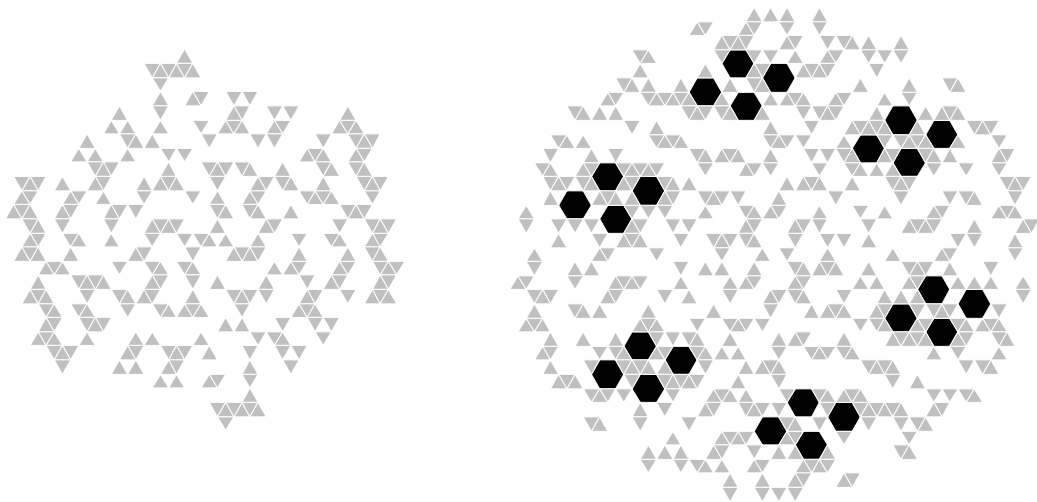


Figure 16: “3-3-3-3-6 Tiling, Embryo Sample

The extra triangles form an Embryo which can generate the original pattern after 2^m steps where $m \in \mathbb{N}$. The Embryo (Figure 16, left) which consists of extra triangles in Figure 15, right can reproduce 6 copies of original pattern after 8 steps (Figure 16, right).

4.2 “3-3-3-4-4” Tiling

4.2.1 Coordinate System

The block coordinate system for “3-3-3-4-4” tiling is given in Figure 17.

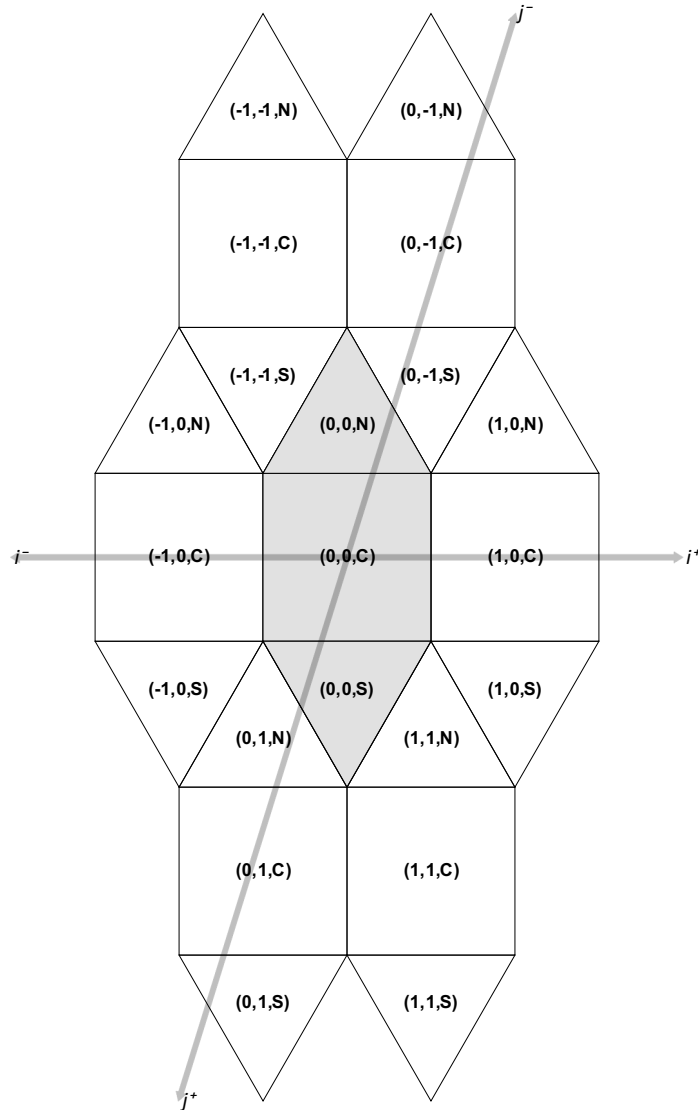


Figure 17: “3-3-3-4-4” Tiling Coordinate System

4.2.2 Observations

Considering the tiling which is built up by triangles and squares in “3-3-3-4-4” order, we can observe somewhat similar behavior as in the previous grid. For any initial pattern, 4 copies of the original pattern can be observed in steps $3 \times 2^n, n \in \mathbb{N}$ together with an extra inner pattern which is an Embryo. Embryo can generate the pattern after $2^m, m \in \mathbb{N}$ steps.

Figure 19 shows a sample initial pattern while Figure 19 shows the result of CMCA after 24 steps. As it can be observed 4 copies of the initial pattern are generated along with extra inner pattern which is the Embryo. Figure 20 shows the Embryo of the last sample and Figure 21 shows the result after running CMCA for 8 steps in which 8 copies of initial pattern are restored.



Figure 18: “3-3-3-4-4” Tessellation, Sample Pattern

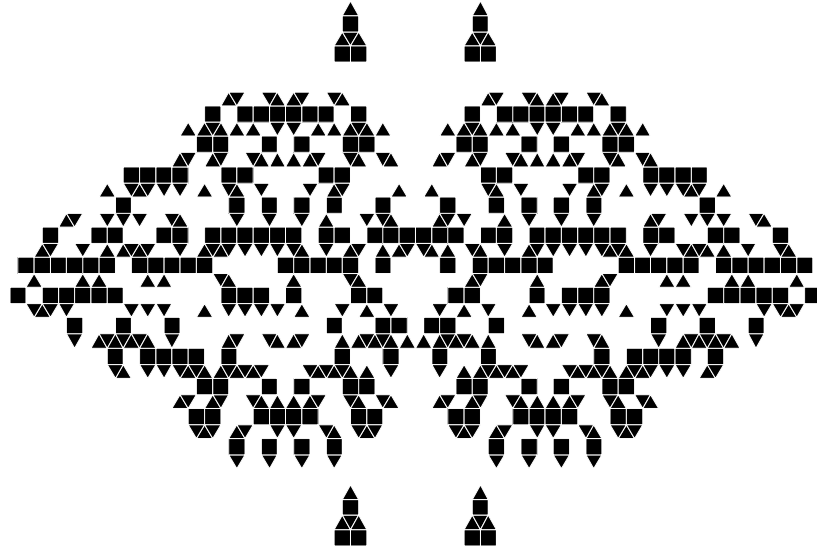


Figure 19: “3-3-3-4-4” Tessellation, CMCA Result After 24 Steps

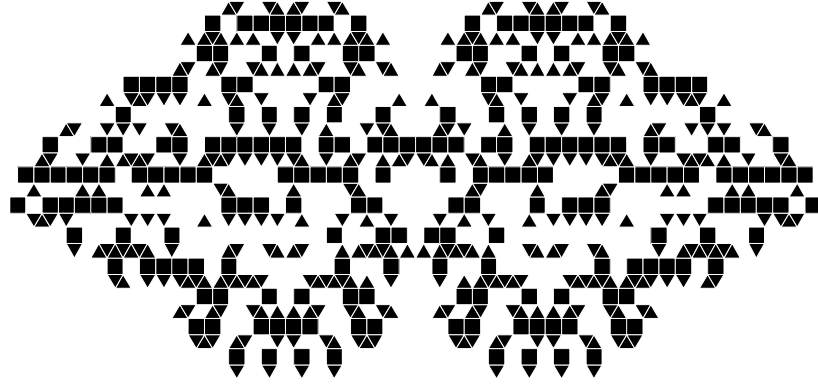


Figure 20: “3-3-3-4-4” Tessellation, Embryo Sample

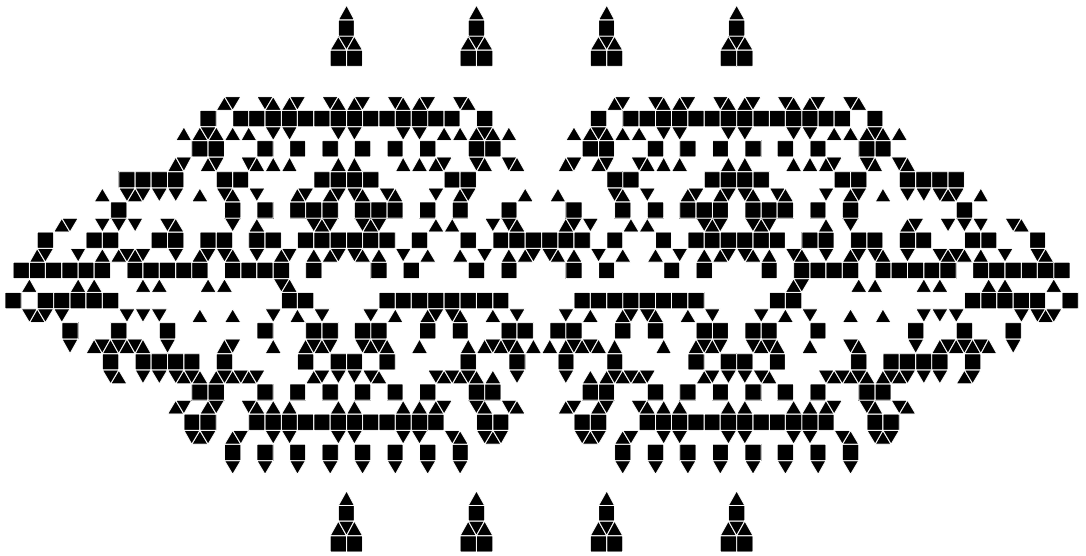


Figure 21: “3-3-3-4-4” Tessellation, CMCA Result After 8 Steps Using Embryo as Initial Pattern

4.3 “3-3-4-3-4” Tiling

4.3.1 Coordinate System

The block coordinate system for “3-3-4-3-4” tiling is given at Figure 22.

4.3.2 Observations

Considering the tiling which is also built up by triangles and squares, we have the following observations.

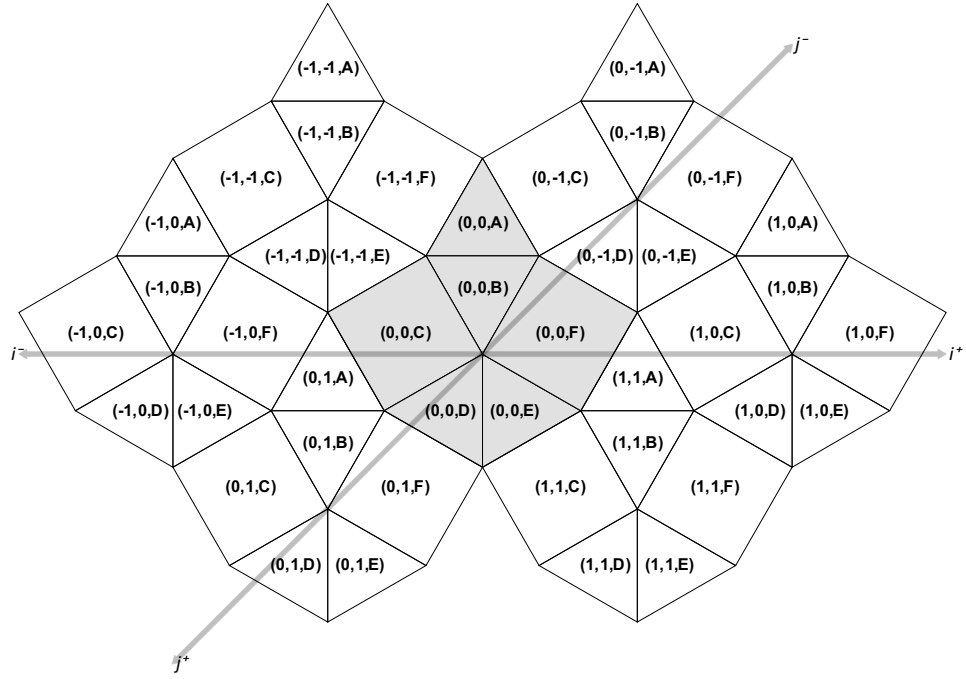


Figure 22: “3-3-4-3-4” Tiling Coordinate System

For any initial pattern, 4 copies of the original pattern can be observed in steps $3 \times 2^n, n \in \mathbb{N}$ together with an extra inner pattern which is an Embryo. Embryo can generate the pattern after $2^m, m \in \mathbb{N}$ steps.

Figure 23 shows a sample initial pattern while Figure 24 shows the result of CMCA after 24 steps. As it can be seen, 4 copies of the initial pattern are generated along with extra inner pattern which is the Embryo. Figure 25 shows the Embryo of the last sample and Figure 26 shows the result after running CMCA for 16 steps in which 8 copies of initial pattern are restored.



Figure 23: “3-3-4-3-4” Tessellation, Sample Pattern

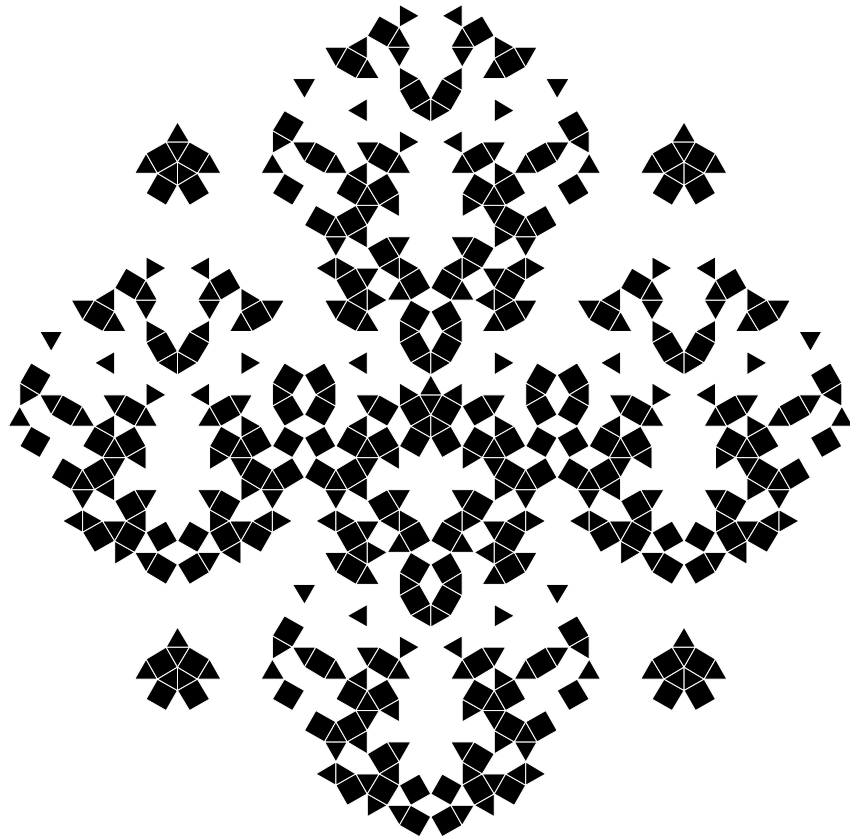


Figure 24: “3-3-4-3-4” Tessellation, CMCA Result After 24 Steps

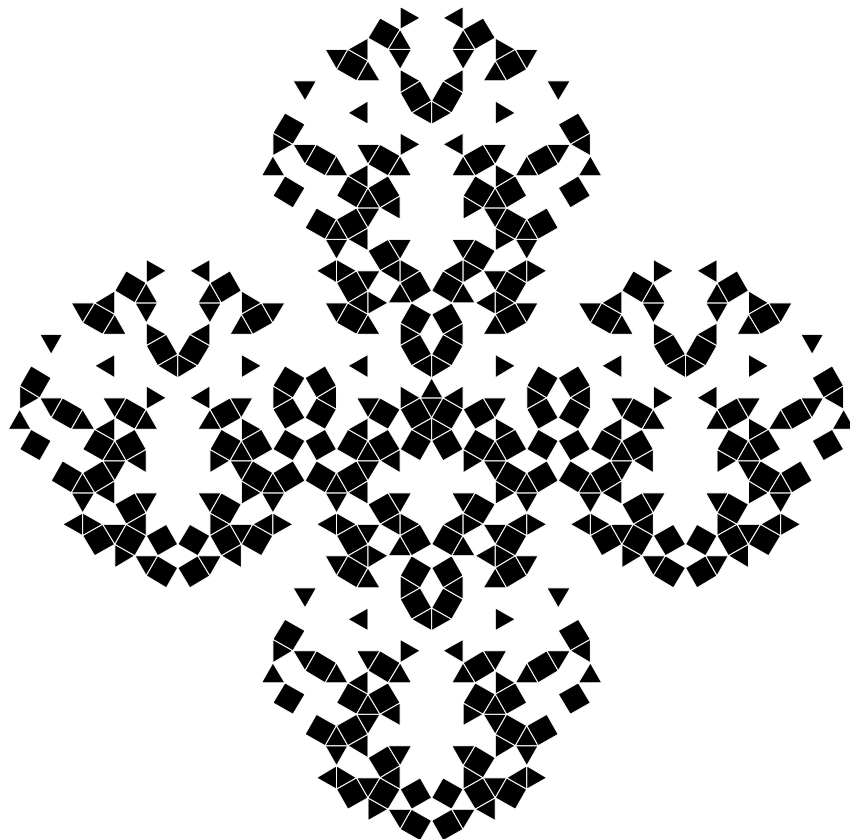


Figure 25: “3-3-4-3-4” Tessellation, Embryo Sample

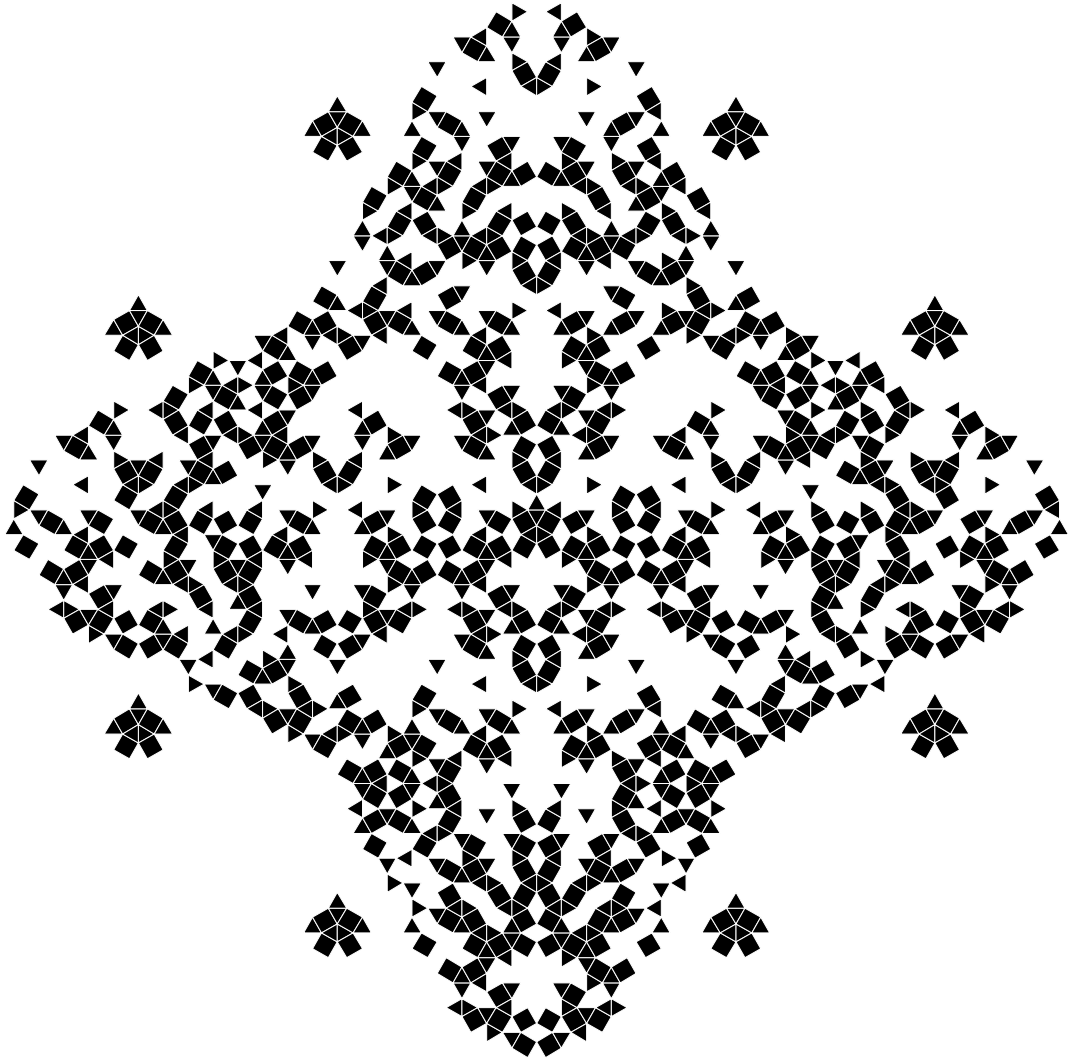


Figure 26: “3-3-3-4-4” Tessellation, CMCA Result After 16 Steps Using Embryo as Initial Pattern

4.4 “3-4-6-4” Tiling

4.4.1 Coordinate System

The block coordinate system for “3-4-6-4” tiling is given at Figure 27.

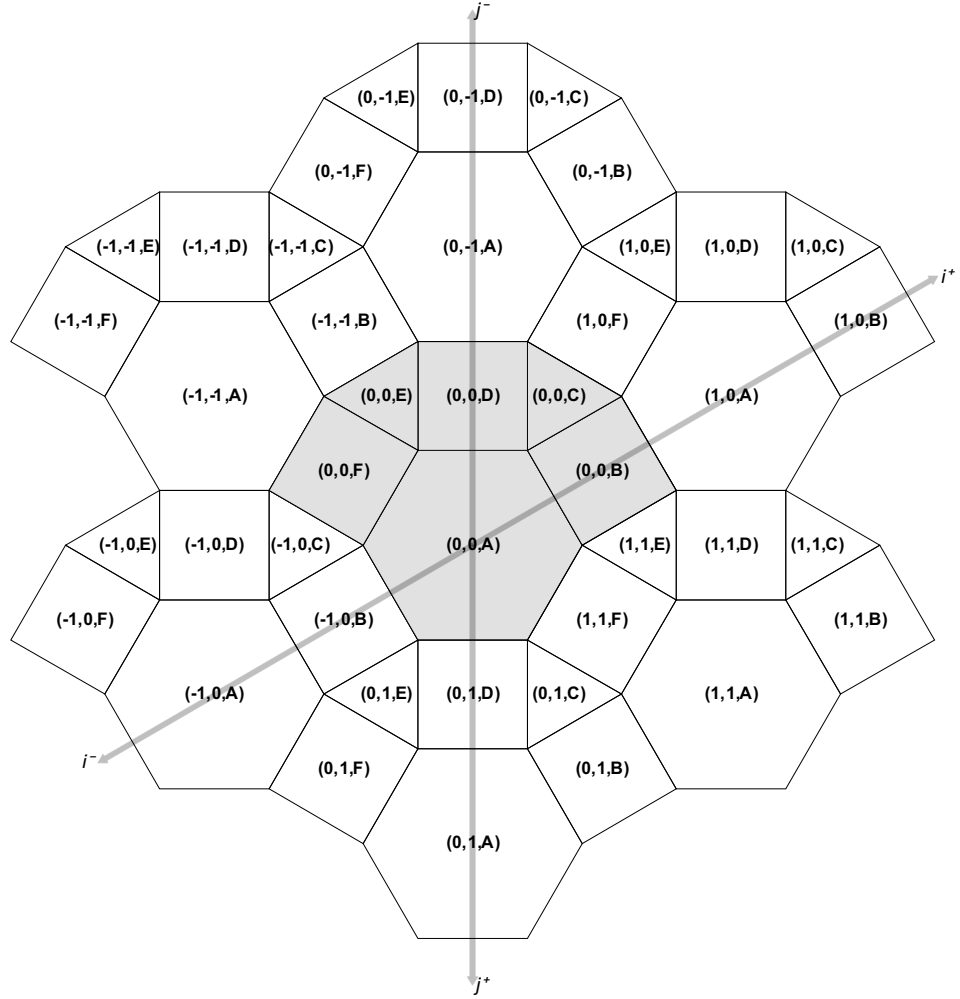


Figure 27: “3-4-6-4” Tiling Coordinate System

4.4.2 Observations

Considering the tiling which is built up by triangles, squares and hexagons, we have the following observation.

If the initial pattern consists of either only hexagons or only triangles, then six copies are obtained in the steps $2^n, n \in \mathbb{N}$.

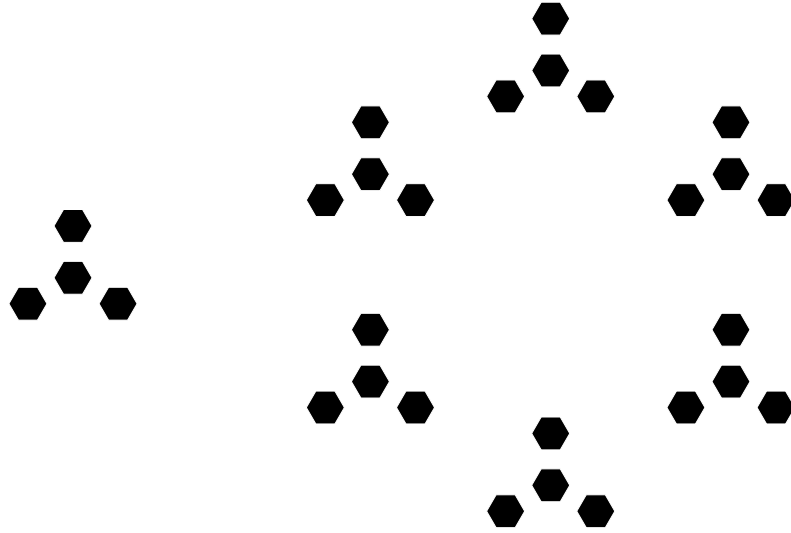


Figure 28: “3-4-6-4” Tessellation, Sample Pattern Consists Only of Hexagons(left),
CMCA Result After 8 Steps(right)

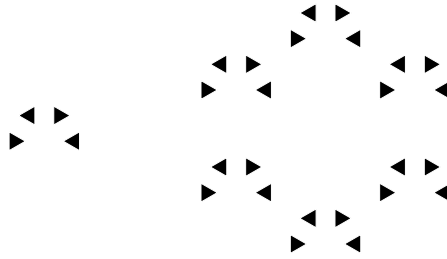


Figure 29: “3-4-6-4” Tessellation, Sample Pattern Consists Only of Triangles(left),
CMCA Result After 8 Steps(right)

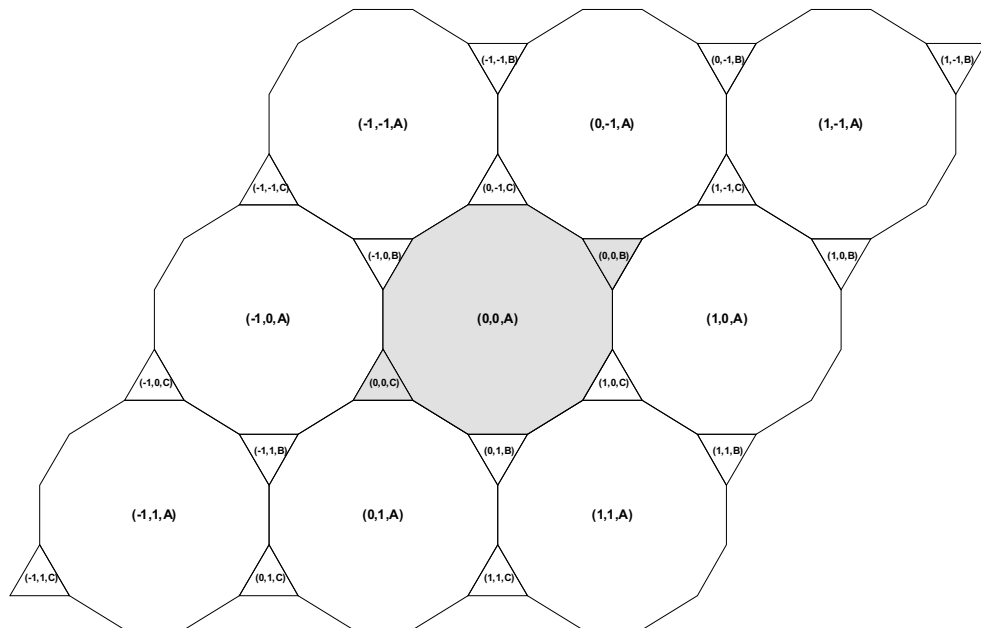


Figure 30: “3-12-12” Tiling Coordinate System

4.5 “3-12-12” Tiling

4.5.1 Coordinate System

The block coordinate system for “3-12-12” tiling is given in Figure 30.

4.5.2 Observations

Considering the tiling which is built up by triangles and dodecagons, we have the following observation.

If the starting pattern includes only dodecagons, the copy can be observed in steps $2^n, n \in \mathbb{N}$ with extra triangles. The extra triangles do not create an Embryo in this case. Figure 31, left shows a sample initial pattern and Figure 31, right shows the result of CMCA after 4 steps where 6 copies can be observed along with extra triangles.

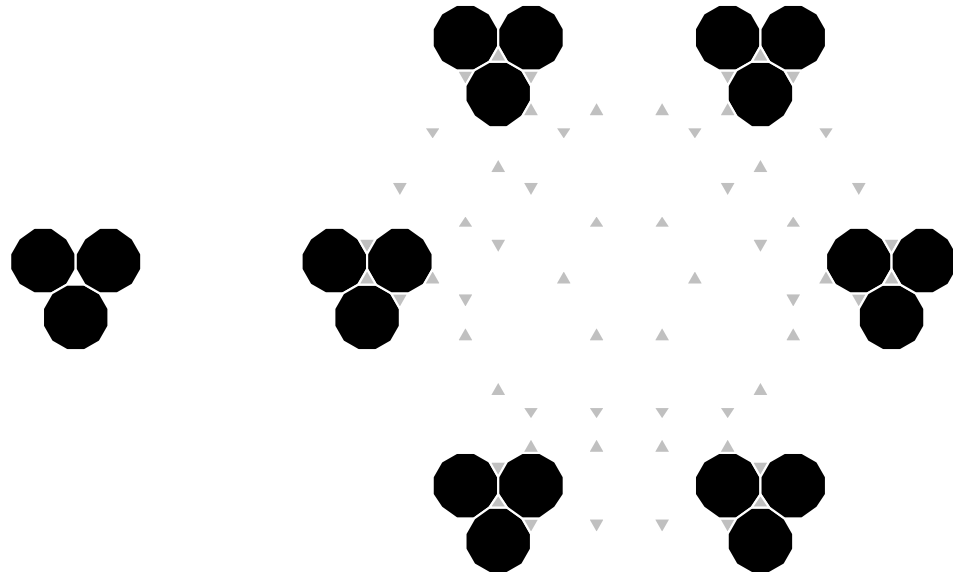


Figure 31: “3-12-12” Tessellation, Sample Pattern Consisting Only Dodecagons (left), CMCA Result After 4 Steps (right)

4.6 “4-6-12” Tiling

4.6.1 Coordinate System

The block coordinate system for “4-6-12” tiling is given in Figure 32.

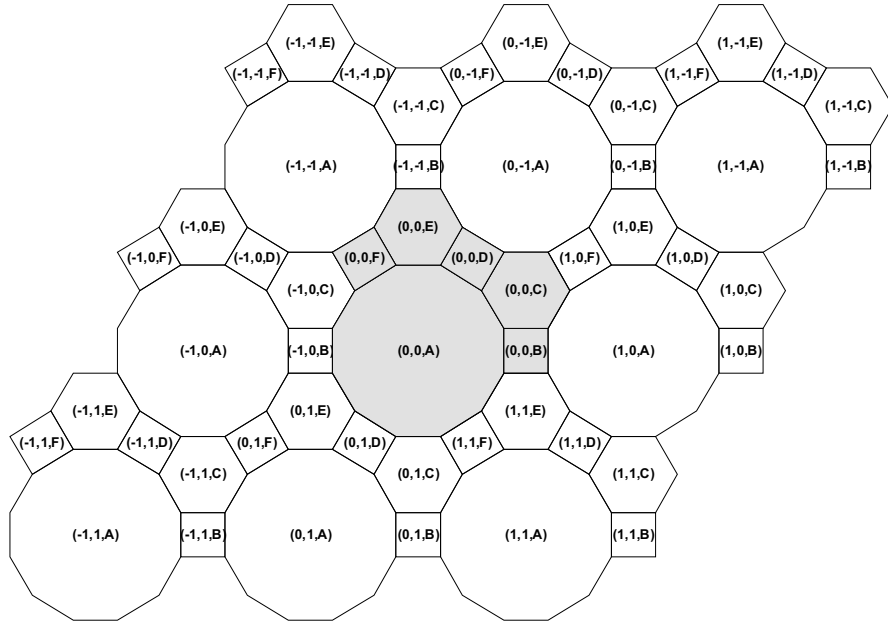


Figure 32: “4-6-12” Tiling Coordinate System

4.6.2 Observations

Considering the tiling which is built up by squares, hexagons and dodecagons, we have the following observation.

For few special initial patterns, the copy can be observed in steps $2^n, n \in \mathbb{N}$ only in one direction. Figure 33, left shows a sample initial pattern and Figure 33, right shows the result of CMCA after 4 steps, where 2 copies can be observed in horizontal direction.

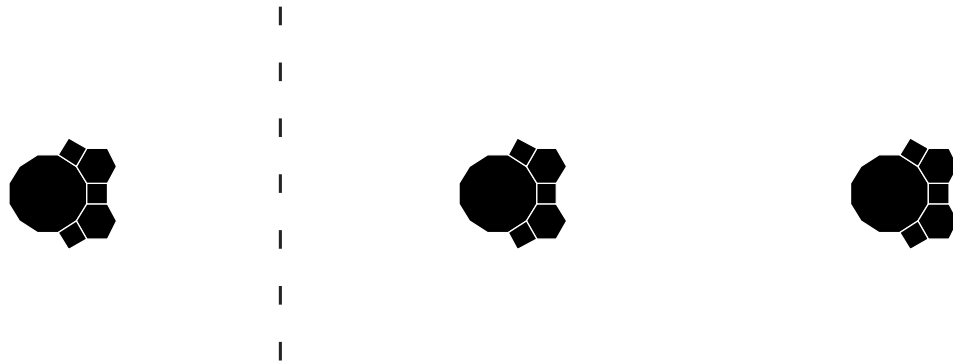


Figure 33: “4-6-12” Tessellation, Sample Pattern(left), CMCA Result After 4 Steps(right)

4.7 “4-8-8” Tiling

4.7.1 Coordinate System

The block coordinate system for “4-8-8” tiling is given in Figure 34.

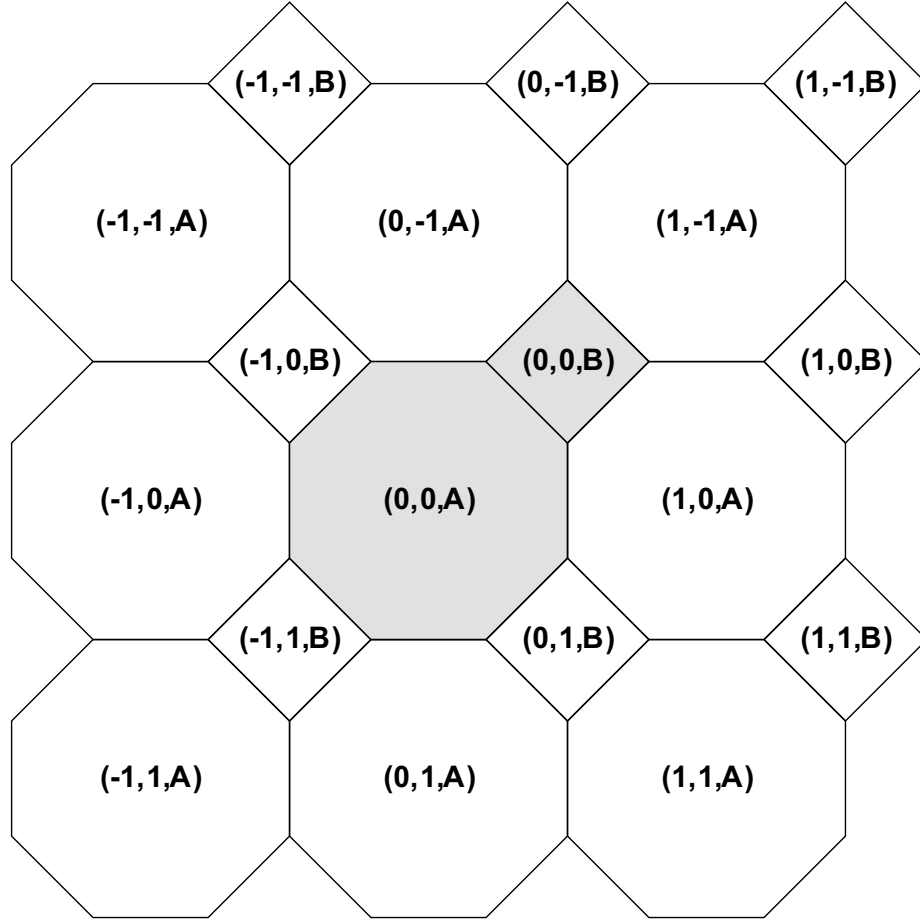


Figure 34: “4-8-8” Tiling Coordinate System

4.7.2 Observations

Considering the tiling which is built up by squares and octagons in “4-8-8” order, we can observe somewhat similar behavior as in the previous grid.

For few special initial patterns, the copy can be observed in steps $2^n, n \in \mathbb{N}$ only in one direction. Figure 35, left shows a sample initial pattern and Figure 35, right shows

the result of CMCA after 4 steps where 2 copies can be observed in horizontal direction.

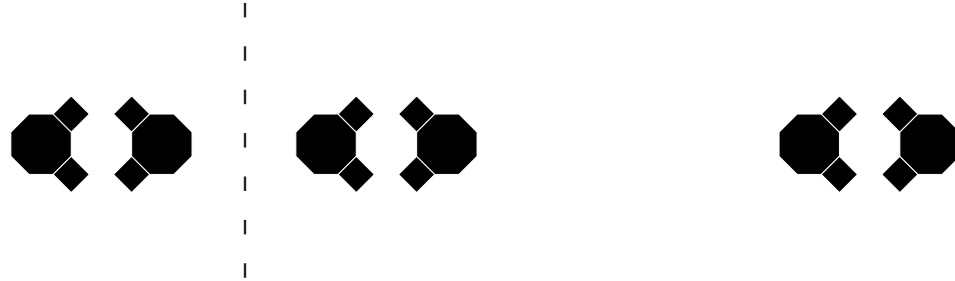


Figure 35: “4-8-8” Tessellation, Sample Pattern(left), CMCA Result After 4 Steps(right)

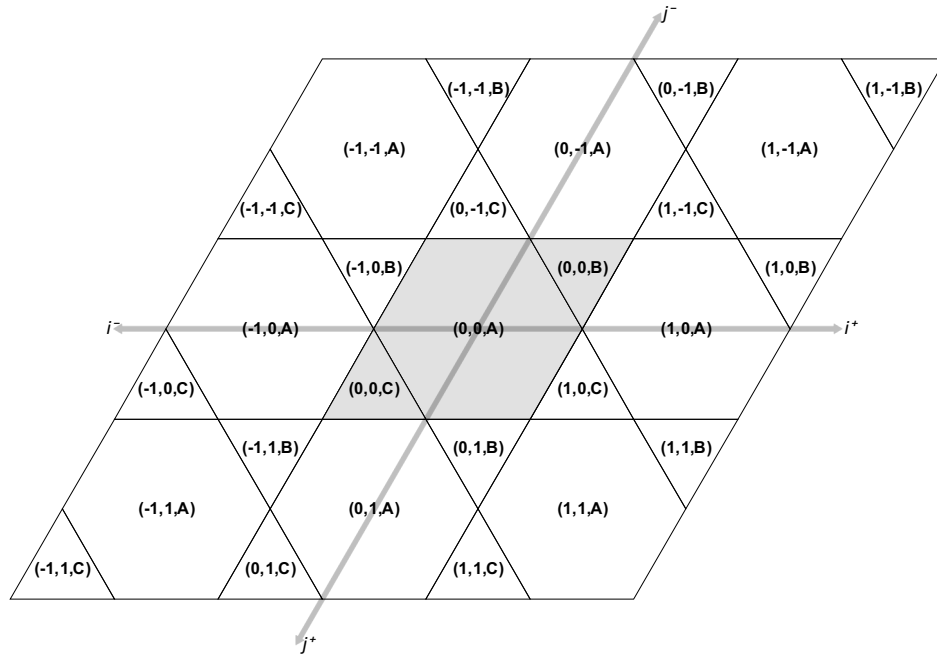


Figure 36: “3-6-3-6” Tiling Coordinate System

4.8 “3-6-3-6” Tiling

4.8.1 Coordinate System

The block coordinate system for “3-6-3-6” tiling is given in Figure 36.

4.8.2 Observations

Considering the tiling which is built up by triangles and hexagons in “3-6-3-6” order, we have the following observation.

No copy can be observed by running the CMCA, and any initial state will completely disappear in step 3.

Chapter 5

DISCUSSION AND EXTENSION

Although here we used only two states to have the simplest automata that are able to do self-reproduction, our results can be easily transferred to other cases, e.g., from binary to color pictures. In honor of John von Neumann and John Horton Conway we applied CMCA on a 256×256 pixels image that was presented in 24 bits bitmap format (see Figure 37).

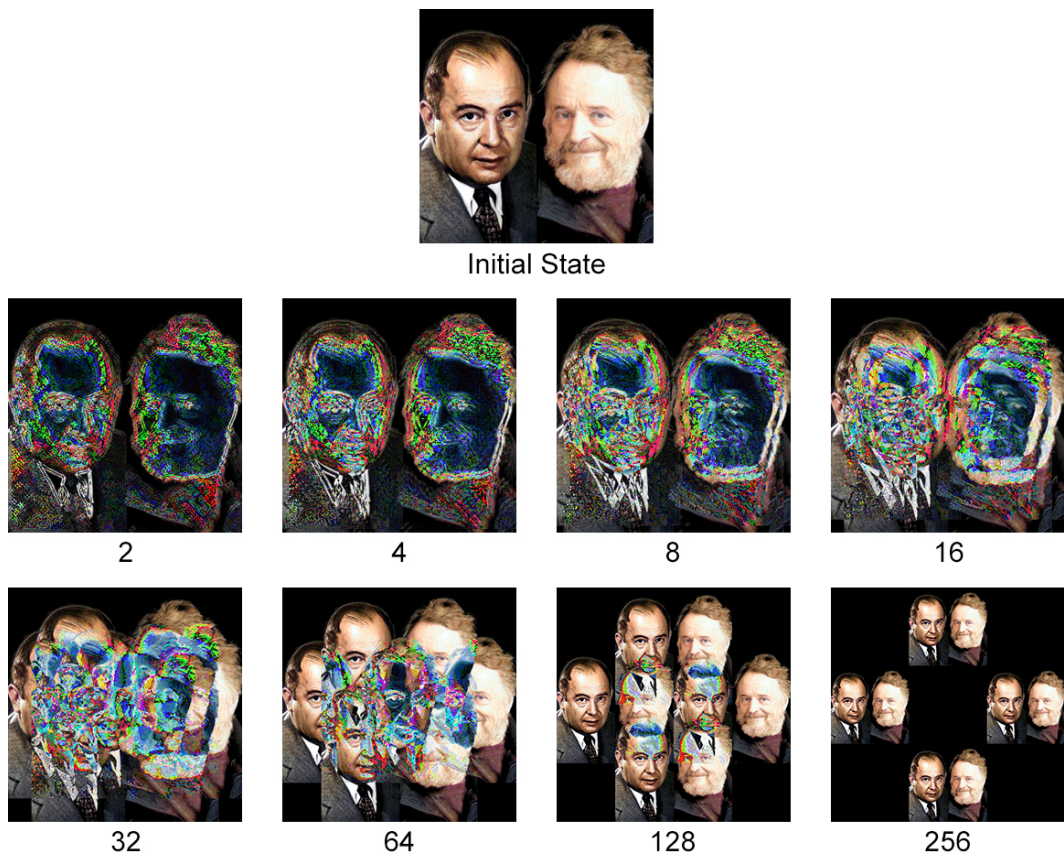


Figure 37: Extension of CMCA on a 256 x 256 Pixel Image

As bitmaps are defined for each pixel to contain a color value. In 24 bits bitmap format there are 8 bits allocated to each of the red, green, and blue component. In each component the value of 0 refers to no contribution of that color, 255 refers to fully saturated contribution of that color. The CMCA automaton is applied to each bit of the bitmap independently, like 24 parallel binary CMCA and the color image is made by putting together to 24 bits for each pixel.

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