

Collocation Method for Solving Problems of Generalized Fractional Bagley– Torvik Equation

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ABSTRACT

ABSTRACT: Collocation method for solving problems of Generalized fractional Bagley–Torvik equation is proposed: the boundary value problems of generalized Bagley–Torvik equation with constant coefficients is studied. Further, the fractional Cauchy-Euler equation which is a class of the initial value problem for the variable coefficients of the Bagley–Torvik equation is also considered. By using Mittag-Leffler (M-L) type function the existence and uniqueness of the solution of the boundary value problem of generalized Bagley–Torvik (BgT) equation with Caputo derivative of order $0 < \alpha < 2$ is proved. Further, a highly accurate numerical method is given by using the collocation-shooting method ($C - SM$). The collocation solution is constructed in the space $S_{m+1}^{(1)}$ as piecewise polynomials of degree at most $m+1$ and has continuous derivative on the considered domain. Furthermore, under some smoothness conditions of the exact solution, global convergence analysis are given. Numerical investigation is carried out on several examples and besides showing the justification of theoretical results comparisons with existing numerical methods are also given. Results indicate that C-SM gives highly accurate results and is efficient both in computational time and storage.

In addition, we consider a model of the fractional Cauchy-Euler type equation, where the fractional derivative operator is the Caputo with order $0 < \alpha < 2$. The problem also constitutes a class of examples of the Cauchy problem of the generalized fractional Bagley-Torvik equation with variable coefficients. For proving the existence and uniqueness of the solution of the proposed problem, the contraction mapping principle is utilized. Furthermore, a numerical method and an algorithm are developed for obtaining the approximate solution. Also, convergence analyses are

studied, and simulations on some test problems are given. It is numerically shown that the proposed method and the algorithm are easy to implement on a computer and computational complexity is given.

Keywords: Bagley–Torvik Equation, Caputo Fractional Derivative, Collocation-shooting Method, Convergence Analysis, Cauchy-Euler Equation, Existence and Uniqueness, Algorithm, Collocation Method.

ÖZ

Genelleştirilmiş kesirsell Bagley-Torvik denkleminin problemlerinin çözümü için kollokasyon yöntemi önerilir: sabit katsayılı genelleştirilmiş Bagley-Torvik denkleminin sınır değer problemleri incelenir. Ayrıca, değişken katsayılı kesirsel Bagley-Torvik denkleminin başlangıç değer problemlerinin bir sınıfı olan kesirsel Cauchy-Euler denklemi de dikkate alınır. Mittag-Leffler (M-L) tipi fonksiyon kullanılarak, $0 < \alpha < 2$ mertebesinde Caputo türevi olan genelleştirilmiş Bagley-Torvik (BgT) denkleminin sınır değer probleminin çözümünün varlığı ve tekliği kanıtlanır. Ayrıca, kollokasyon-çekim yöntemi (C-SM) kullanılarak yüksek mertebeden yakınsaklığı olan bir sayısal yöntem verilir. kollokasyon çözümü, $S_{m+1}^{(1)}$ uzayında, en fazla $m+1$ derecesinde parçalı polinomlar ve tanım kümesinde sürekli türeve sahip olarak oluşturulur. Ayrıca, kesin çözümün bazı pürüzsüzlük koşulları altında, global yakınsama analizi verilir. Sayısal inceleme birkaç örnek üzerinde yürütülür ve teorik sonuçların sağlandığını göstermenin yanı sıra mevcut sayısal yöntemlerle karşılaştırmalar da yapılır. Sonuçlar, C-SM'nin son derece doğru sonuçlar verdiğini ve hem hesaplama süresi hem de verileri depolama yönünden verimli olduğunu göstermektedir. Ek olarak, kesirli türev operatörünün $0 < \alpha < 2$ mertebesindeki Caputo olduğu fraksiyonel Cauchy-Euler tipi denklemin bir modeli incelenir. Problem aynı zamanda değişken katsayılı genelleştirilmiş kesirsel Bagley-Torvik denkleminin Cauchy probleminin bir örnek sınıfını oluşturur. Önerilen problemin çözümünün varlığını ve tekliğini kanıtlamak için büzülme dönüşümleri ilkesi kullanılır. Ayrıca, yaklaşık çözümü elde etmek için sayısal bir yöntem ve bir algoritma geliştirilir. İlaveten, yakınsama analizleri çalışılır ve bazı test problemleri üzerinde simülasyonlar verilmektedir. Önerilen yöntemin ve algoritmanın bir bilgisayarda uygulanmasının kolay olduğu sayısal olarak gösterilmiş ve hesaplama

karmaşıklığı verilmiştir.

Anahtar Kelimeler: Bagley-Torvik Denklemi, Caputo Kesirli Türev, Kollokasyon-çekim Yöntemi, Yakınsama Analizi, Cauchy-Euler Denklemi, Varlık ve Teklik, Algoritma, Kollokasyon Yöntemi.

DEDICATION

*Dedicated to My Late Parents Chin Nagher and Uishi Chin for the sacrifices
they made for my education*

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LIST OF SYMBOLS AND ABBREVIATIONS

${}_1\Psi_1$	Fox wright function
${}_p\Psi_q[z]$	Generalize Hypergeometric function
$E_{\omega,\beta}^{\eta}(x)$	Generalized Mittag-Leffler function
$(n)_k$	Pochhammer symbol
BCM	Bessel collocation method [7]
BgT	Bagley–Torvik
BVP	Boundary Value problem
C^{m+2}	Space of $(m + 2)$ times continuous differentiable function
u_h	Approximate Solution
CSM	Cubic spline method [2]
C-SM	Collocation-shooting method
CPUs	Central Processing Unit time in seconds
DCSWSGD	Discrete cubic spline weighted shifted Grunwald-Letnikov difference operator [6]
FrBVP	Fractional boundary value problem
FrC-E	Fractional Cauchy-Euler
FrDEs	Fractional differential equation
GBPM	Generalized Bessel polynomials method [8]
G-LP	Gauss-Legendre Point
GM	Galerkin method [5]
HWM	Haar wavelet method [9]
IVP	Initial value problem
M-CP	Mean-Chebyshev Points

NGLDM	Cubic spline based on nonstandard Grunwald Letnikov difference method [3]
$\mathcal{O}(h^m)$	m order of convergence
QFEM	Quadratic finite element method [4]
SSM	Spline Combine with Shooting method [1]

Chapter 1

INTRODUCTION

1.1 Motivation

Numerical methods and its applications in various fields of social sciences, science and engineering has been widely studied in recent years due to its strategic importance in various research areas/fields. Complex mathematical problems which the analytic solution may be difficult to obtain or may not exist can easily be solved using numerical methods. This ease of solution via numerical methods motivated us to give collocation method for solving problems of generalized fractional Bagley–Torvik equation. Collocation methods are powerful numerical techniques used to solve partial, fractional/ classical differential equations (initial or boundary value problems). It involves approximating the solution of a given fractional/classical, partial differential equation by constructing a system of equations (linear or non linear) using set of chosen points, known as mesh or collocation points, within the confined of the domain of interest. The collocation points are usually selected such that they satisfy the given classical/fractional, partial differential equation or its derivatives at those points [10], [11]. The main idea involved in collocation methods is to find the best-fit solution that satisfies the given differential equation at the mesh /collocation points. This is normally achieved by constructing system of algebraic equations, where the unknowns are the values of the solution at the mesh/collocation points. These equations are obtained by putting the approximate solution into the original differential equation and evaluating it at the chosen mesh/collocation points.

The Bagley Torvik equations (BgT) is an archetype of fractional differential equations (FrDE) employed in the study of the theory of visco-elasticity. It was first given by Bagley and Torvik and utilized to described the linearly damped fractional oscillator with damping term. This was contained in their paper titled "On the appearance of the fractional derivative in the behavior of real materials" [12] as an application of fractional calculus to the theory of visco-elasticity. The BgT equations has been widely studied and used to model different phenomenon in science and Engineering. For example BgT equation is used in modeling the visco-elastic behavior which exhibit both visco-elastic and elastic characteristics, These are important variables which aid scientist and engineers in understanding the deformation and recover of materials under stress conditions. In the study of fluid dynamics, the characteristics of complex fluid flow such as those inherent in Newtonian fluid are study via the BgT equation. Also in studying fluid flow in porous media, the equations are used to help understand the fluid flow in porous media which is essential in hydrogeology and petroleum engineering. In signal processing BgT equations found applications in modeling of frequency dependent damping which is crucial in signal processing applications. Furthermore, the design and analysis of Micro-electro-mechanical systems especially in instruments used in measuring viscosity in oil exploration industries, this equations played a crucial role for the design development and the analysis. In addition, in the modeling of electromagnetic phenomena , particularly in complex media , where the classic (traditional differential equations) models many not adequately be effective.

Cauchy-Euler equations refer to a class of initial value problems for ordinary differential equations (ODEs) or partial differential equations (PDEs). The Cauchy-Euler problem, also known as the Euler-Cauchy problem, is named after the

mathematicians Leonhard Euler and Augustin-Louis Cauchy, who made significant contributions to the field of differential equations. [13]. A Cauchy-type problem typically involves finding a solution to a differential equation that satisfies both a given equation and a set of initial conditions. The initial conditions specify the values of the unknown function and its derivatives at a particular point or a set of points in the domain [14]. Fractional Cauchy-type problems refer to initial value problems involving fractional differential equations with initial conditions. These problems are extensions of the classical Cauchy problems for ordinary or partial differential equations, but with fractional derivatives and initial conditions. In a fractional Cauchy-type problem, the differential equation contains fractional derivatives, which are generalizations of integer-order derivatives. The initial conditions specify the values of the unknown function and its fractional derivatives at a particular point or a set of points in the domain, [15] and [16].

The fractional Cauchy-Euler problem which is an extension of the classical Cauchy-Euler problem finds application in similar areas where the classical case is applicable and beyond, to this end, fractional Cauchy-Euler problem is used in the modeling of time-harmonic vibration of a tin elastic rod, it is also used in the modeling of annual and solid disc and wave mechanics. Furthermore, in the design of bridges, automobile and construction of buildings, these equations are employed for the analysis. Further, fractional Cauchy-Euler problem can be employ in the study of fluid dynamics to explain the dynamics of fluid, including motion of liquid and gases. In the same vein FrC-E is used in the prediction of fluid behavior or phenomena such as ocean currents, fluid flow in pipes and aerodynamics of aircraft. By providing solution to this equation, Engineers and scientist are able to predict the behavior of fluid, gives optimum design of vehicle/ automobile and pipe lines as well as getting a

better understanding of natural phenomena such as the weather patterns and ocean current circulation.

In the modeling of heat transfer, fractional Cauchy-Euler problem may be used to describe the distribution of temperature in materials gradient. The application of these equations helps scientists and engineers to have a better understanding of the heat flow through materials such as gases, liquid, visco-elastic and solids which provides a better framework for the design of heat exchangers, electronic cooling system and thermal insulation.

In the same vein, the study of dynamics system in economic, the fractional Cauchy-Euler problems are used in the modeling of economic growth, it is employed to explain the relationship between economics variables such as investments, output and accumulation of capital over a period of time.

1.2 Definition of Some Basic Terms

Under this section we defined some basic terms used in the thesis.

Definition 1.1: [15] Let $\omega = [a, b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann Liouville fractional derivative to order μ of a function $f(t)$ with respect to t , with a constant of differentiation c , is defined for a general $\mu \in \mathbb{C}$ with non-negative real part as: The left Riemann Liouville fraction derivative,

$$\begin{aligned} \left({}^{RL}D_t^\mu f \right)(t) &:= \left(\frac{d}{dt} \right)^n \left({}^{RL}I_t^{n-\mu} f \right)(t) \\ &= \frac{1}{\Gamma(n-\mu)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(x)dx}{(t-x)^{\mu-n+1}}, \quad n := \lfloor \operatorname{Re}(\mu) \rfloor + 1, \quad \operatorname{Re}(\mu) \geq 0 \end{aligned}$$

and right Riemann Liouville fraction derivative is defined as:

$$\begin{aligned} \left({}^{RL}D_t^\mu f \right)(t) &:= \left(-\frac{d^n}{dt^n} \right) \left({}^{RL}I_t^{n-\mu} f \right)(t) \\ &= \frac{1}{\Gamma(n-\mu)} \left(-\frac{d}{dt} \right)^n \int_t^b \frac{f(x)dx}{(x-t)^{\mu-n+1}}, \quad n := \lfloor \operatorname{Re}(\mu) \rfloor + 1, \quad t < b, \end{aligned}$$

where $\lfloor \operatorname{Re}(\mu) \rfloor$ means the floor of the real part.

Definition 1.2: [17] The Riemann Liouville fractional integral to order μ of a function f with respect to t , with constant of integration c , is defined for a general $\mu \in \mathbb{C}$ with positive real part is defined as:

$$\left({}^{RL}I_a^\mu f \right)(x) := \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\mu}}, \quad (x > a, \operatorname{Re}(\mu) > 0),$$

and the right Riemann Liouville fractional integral is defined as:

$$\left({}^{RL}I_b^\mu f \right)(x) := \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\mu}}, \quad (x < b, \operatorname{Re}(\mu) > 0),$$

where Γ is the gamma function.

Definition 1.3: [15] Euler gamma function $\Gamma(x)$ is defined by the Euler integral of the second kind:

$$\Gamma(s) = \int_0^\infty y^{s-1} e^{-y} dy \quad (\operatorname{Re}(s) > 0),$$

where $y^{s-1} = e^{(s-1)\log(y)}$.

Definition 1.4: [15] The gamma functions has both the upper and the lower incomplete gamma function $\gamma(v, y)$, We first define the lower incomplete gamma function: The lower incomplete gamma function $\gamma(v, y)$ is defined as:

$$\gamma(v, y) = \int_0^y e^{-s} s^{v-1} ds. \quad \operatorname{Re}(v) > 0,$$

and the upper incomplete gamma function $\Gamma(v, y)$ is defined as

$$\Gamma(v, y) = \int_y^{\infty} e^{-s} s^{v-1} ds, \quad \operatorname{Re}(v) > 0.$$

Definition 1.5: [15] The left and right Caputo fractional derivative $\left({}_a^C D_t^\mu f\right)(t)$ and $\left({}_b^C D_t^\mu f\right)(t)$ of order $\mu \in \mathbb{C}$, $\operatorname{Re}(\mu) \geq 0$ on $[a, b]$ are defined in relation to the Riemann Liouville fractional derivatives by:

$$\left({}_a^C D_t^\mu f\right)(t) := \left({}_a^{RL} D_t^\mu \left[f - \sum_{i=0}^{n-1} \frac{f^{(i)}(t)}{i!} (x-a)^i \right]\right)(t),$$

and the right Caputo fractional derivative is defined as:

$$\left({}_b^C D_t^\mu f\right)(t) := \left({}_b^{RL} D_t^\mu \left[f - \sum_{i=0}^{n-1} \frac{f^{(i)}(t)}{i!} (b-x)^i \right]\right)(t),$$

where $n := \lfloor \operatorname{Re}(\mu) \rfloor + 1$ for $\mu \notin \mathbb{N}_0 : n = \mu$ for $\mu \in \mathbb{N}_0$.

Definition 1.6: [18] Mellin transform of a function $\eta(x)$ of a real variable $x \in \mathbb{R}^+ = (0, \infty)$ is defined by

$$(\mathbb{M}\eta)(\varphi) = \mathbb{M}(\eta)(x)(\varphi) = \eta'(s) = \int_0^{\infty} x^{s-1} \eta(x) dx \quad (s \in \mathbb{C})$$

and the inverse mellin transfer is defined for $t \in \mathbb{R}$ by

$$(\mathbb{M}^{-1}h)(t) = \mathbb{M}^{-1}(h(s))(t) := \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} t^{-s} h(s) ds, \quad (\varepsilon = \operatorname{Re}(s)).$$

Definition 1.7: [19] The Gauss hypergeometric function ${}_2F_1(a_1, a_2, a_3 : z)$ is defined as

$${}_2F_1(a_1, a_2, a_3 : z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(a_3)_k k!} z^k,$$

where $|z| < 1$, $a_1, a_2 \in \mathbb{C}$, $a_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $(a_1)_k$ is the pochhammer symbol

Definition 1.8: [20] The Bessel function of the first kind $J_\nu(x)$ is defined as follows:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+1}}{k! \Gamma(\nu + k + 1)},$$

where $x \in \mathbb{C} \setminus (0, \infty)$ and $\nu \in \mathbb{C}$.

Definition 1.9: [15], [21] The generalized Mittag-Liffler function is defined for a complex $x \in \mathbb{C}$, $\varpi, \Psi, \eta, \mathbb{C}$ and $Re(\varpi) > 0$ by

$$\mathbf{E}_{\varpi, \beta}^{\eta}(x) := \sum_{k=1}^{\infty} \frac{(\eta)_k x^k}{\Gamma(\varpi k + \beta) k!} = \frac{1}{\Gamma(\eta)} {}_1\Psi_1 \left[\begin{matrix} (\eta, 1) \\ (\beta, \varpi) \end{matrix} \middle| x \right]$$

where $(\eta)_k$ is the pochhammer symbol, ${}_1\Psi_1$ defines the Fox wright function.

Definition 1.10: [22] The beta function $\beta(x, y)$ is defined as

$$\beta(x, y) = \int_0^1 s^{x-1} (1-s)^{y-1} ds, \quad Re(x) > 0, \quad Re(y) > 0.$$

Definition 1.11: [22] The incomplete beta function $\beta_i(x, y)$ is defined as

$$\beta_i(x, y) = \int_0^{\tau} s^{x-1} (1-s)^{y-1} ds, \quad Re(x) > 0, \quad Re(y) > 0.$$

Definition 1.12: [23] The fresnel integrals for sine and cosine are defined as follow:

$$c(x) = \int_0^x \cos\left(\frac{1}{2}\pi y^2\right) dy,$$

$$s(x) = \int_0^x \sin\left(\frac{1}{2}\pi y^2\right) dy,$$

where $c(x)$ and $s(x)$ are entire functions of x .

1.3 Literature Review

The modeling of numerous phenomena in diverse scientific fields leads us to consider conventional or fractional time-dependent differential equations in the modeling domain. In general, finding analytic solutions of these modeled problems is a difficult

task, or even not possible. Hence, numerical methods are needed, such as the latest approaches [24, 25].

In recent time research involving fractional calculus particularly fractional differential equations (FrDEs), have been burgeoning due to its innumerable, inexplicable applications in vast areas of science and engineering. Consequently, many researchers have deployed their energy and quality time in developing analytic and numerical methods for solving FrDEs. The famous Bagley–Torvik (BgT) equation is a core mean of FrDEs given by Bagley and Torvik [12], [26] as:

$$MD^2X(t) + 2A\sqrt{\mu\rho}\left(D_{(t)}^{3/2}X(t)\right) + KX(t) = 0. \quad (1.1)$$

Here, D^2 is the classical second order differentiation operator, $D_{(t)}^{3/2}$ is the Riemann and Liouville fractional derivative of order $3/2$. Further, ρ is the fluid density, μ is the viscosity, M and A are the mass of the thin rigid plate immersed in the viscous fluid, and the area of the rigid plate respectively, and K is the stiffness of the string and X denotes the motion of the plate. In (1.1) the fractional derivative arise inherently and describes the motion of a simple, physical system consisting of familiar mechanical and fluid components.

To emphasize the challenge and the significance of solving variant problems of BgT equation we give a brief survey of related research works. First we mention some studies on the analytical methods. Adomian decomposition method was given in [27] for solving the inhomogeneous case of BgT equation (1.1) under homogeneous initial conditions. In [28] general solutions of BgT equation (1.1) with specific fractional derivative orders $\alpha = 1/2$ or $\alpha = 3/2$ (both Caputo and Riemann and Liouville sense) were proposed by converting the FrDE into a sequential FrDEs with constant coefficients. Later, the Laplace transform methodology was employed to solve the

inhomogeneous case of BgT equation (1.1) under Caputo derivative [29]. The generalized three parameters M-L type functions were used and the solutions were expressed in terms of Wiman's functions and their derivatives. Further, in [30] the authors took only the forces, $F_s = -Ky(t)$, ($K > 0$), and $F_p = -E_0 \left({}^C_0D_t^\beta y(t) \right)$ ($E_0 > 0$) also, the external force $f(t)$ acting on the mass M . Then, the motion of the mass M along a straight line was defined by

$$MD^2y(t) + E_0 \left({}^C_0D_t^\beta y(t) \right) + Ky(t) = f(t), \quad 0 < \beta < 1. \quad (1.2)$$

Here, ${}^C_0D_t^\beta$ is the Caputo derivative of order β . The problem (1.2) for the case $\beta \in (0, 2)$ was called the generalized BgT equation and appears in the mathematical modeling of viscoelastic materials and fluid mechanical process. The existence and uniqueness of the solution to the initial value problem (IVP) of generalized BgT equation (1.2) was derived in [30]. Analytical solutions in terms of the Prabhakar function and the Wiman function were also given. In the same year, the authors of [31] proved the existence relying on an appropriate fixed point theorem in partially ordered normed linear spaces and also provided approximations to the solution of the IVP of (1.2) for $1 < \beta < 2$. Additionally, [32] gave an integral transform method for IVP of a family of non-homogeneous BgT equation with two Caputo fractional terms of orders $\beta, \alpha + 1$ for $1 < \beta < 2, 0 < \alpha < 1$. The solution was expressed in terms of generalized Green function. Recently, in [33] explicit analytical solutions were given for some classes of generalized multidimensional BgT equations for both homogeneous and non-homogeneous cases with permutable matrices and two fractional order derivatives of orders $\alpha \in (1, 2], \beta \in (0, 1]$ and $\alpha \in (1, 2], \beta \in (1, 2]$.

In general, finding analytical solutions of FrDEs is a complex task and computationally very costly, even may not be possible. However, the benefits of

advanced computers as being capable of storing and accessing vast amounts of information and saving time in processes and tasks have led to the widespread use of numerical methods for approximating the solutions. On one hand in the literature there exist many published papers on the numerical approaches for solving the IVP of families of BgT equations. We refer to an early exhaustive paper on a discrete method [34] that a reformulation of equation (1.1) for the non-homogeneous case under Caputo derivative was given as a system of fractional differential equations of order $1/2$. The authors used the Adams–Bashforth–Moulton algorithm and assuming that the non-homogeneous term $f \in C^2[0, T]$ they obtained local convergence of $\mathcal{O}(h^{3/2})$ with step size h . We mention some recent methods such as in [35] solution of IVP of BgT equation using numerical inverse Laplace transform algorithms was investigated and a method based on Gaussian quadrature formulae for numerical integration of the Bromwich’s integral was introduced. Also, a continuous method was given in [36] based upon hybrid functions approximation with block-pulse functions and Bernoulli polynomials. For more examples see the references therein.

On the other hand, in the last decade few numerical approaches are proposed for solving the boundary value problem (BVP) of families of BgT equation. Moreover, some of these methods were given for a fixed value of the fractional derivative. In the literature collocation and spline methods were given suited for this purpose. Such as, for solving generalized BgT equation (1.2) with the fixed fractional derivative order $3/2$ a method using the collocation and shooting with spline analysis [37] and Bessel collocation method [7] were given. For the generalized BgT equation (1.2) Chebyshev collocation method [38], cubic spline methods [1], [2], discrete cubic spline method [3], fractional order Fibonacci wavelet technique method [39] and discrete cubic spline weighted shifted Grünwald-Letnikov difference operator method

[6] were developed and used. An other approach is the generalized piecewise Taylor-series expansion method given in [40] for the generalized BgT equation (1.2) with variable coefficients and Riemann–Liouville type derivative of order $(0,2)$. Therein, a non-integer order integral boundary condition was used to analyze the non-local behavior of the generalized BgT equation. Recently, Galarkin and finite element methods were also applied for solving families of BgT equations. Such as for solving generalized BgT equation (1.2) a fast multiscale Galerkin algorithm [41] was given subject to zero boundary conditions and a quadratic finite element method was studied in [4]. Further, in [5] semi-analytical solutions of quadratic Riccati and Bagley-Torvik differential equations were obtained by applying the Galerkin method. Recent studies on generalized BgT equation also include the modified reproducing kernel Hilbert space treatment approach under Caputo–Fabrizio non-integer order derivative [42], also the methods via fractional powers of Laguerre polynomials [43] and generalized Bessel polynomials [8]. Besides above discussed methods, the authors of [9] gave a scheme using the Haar wavelet operational matrix method for solving linear two-point and multi-point boundary value problems of FrDEs including BgT equation.

Bagley-Torvik (BgT) equation has been studied by many researchers using different approaches \ methods such as those shown from the literature review, but none of the existing method deals with the equation similar to our approach. Furthermore, there is always a search for the best method for solving a particular problem, nevertheless, there is still a growing need for highly accurate theoretical approaches, that give stable and fast numerical algorithms for solving families of BgT equations. Motivated with these we consider the following BVP of generalized BgT equation.

$$AD^2y(t) + B\left({}_0^CD_t^\alpha y(t)\right) + Cy(t) = f(t), 0 < t < T < \infty, 0 < \alpha < 2 \quad (1.3)$$

$$y(0) = \eta_0,$$

$$y(T) = \eta_1,$$

where $A \neq 0$ and A, B, C, η_0, η_1 are known constants.

Furthermore, in recent time, the fractional Bagley–Torvik equations with variable coefficients using the Riemann–Liouville fractional operator

$$D^2\varphi(x) + p(x)D^\alpha\varphi(x) + q(x)\varphi(x) = g(x), 0 < \alpha < 2, x \in [a, b],$$

$$\varphi(a) = 0, \varphi'(a) = 0,$$

where $p(x)$, $q(x)$ and $g(x)$ are the given functions, were considered in [44]. The uniqueness of the solution was investigated by converting the above equation to a Volterra integral equation. To prove the uniqueness of the solution, a contraction operator was used. Also, a piecewise Taylor series expansion method was employed for the solution. Further, in [40] a variable coefficient generalized Bagley–Torvik equation with a fractional integral boundary condition was studied. The Riemann–Liouville fractional derivative operation was employed, and the Fredholm integral equations of the second kind were derived. For the approximate solution, a piecewise Taylor series expansion method was used.

Additionally, fractional calculus has been used in studies of transient electric circuit analysis and electrical impedance spectroscopy to the resistor–capacitor (RC) circuit as well as in many fields of sciences and engineering, including rheology, diffusive transport, electromagnetic theory, probability, and so on; see [45, 46].

Recently, some studies were conducted on the existence and uniqueness of the

solutions to models of fractional Cauchy–Euler equations (FrC-E), also known as Euler-type equations. Next, we mention some existing analytic methods for the solutions of FrC-E type equations given in the literature. Euler-type fractional differential equations were given in [47] with the left and right Liouville derivatives of the fractional order as follows:

$$\begin{aligned}\sum_{k=0}^m A_k x^{\alpha+k} \left(D_{+}^{\alpha+k} y \right)(x) &= f(x) \quad (x > 0, \alpha > 0), \\ \sum_{k=0}^m B_k x^{\alpha+k} \left(D_{-}^{\alpha+k} y \right)(x) &= f(x) \quad (x > 0, \alpha > 0),\end{aligned}$$

with real constants $A_k, B_k \in \mathbb{R}, k = 0, \dots, m$. In their method, two linear non-homogeneous ordinary differential equations were studied using the direct and inverse Mellin integral transforms (see [18] for Mellin integral transform). They gave a general approach to deduce the solution of Euler-type equations. The solution of specific cases were given in terms of the Euler psi functions, Gauss hypergeometric function, and of the generalized Wright functions.

Later, [48] proposed an analytic method for solving the homogeneous fractional differential equation of the Euler-type equation

$$x^{\alpha+2} \left(D_{0+}^{\alpha+2} y \right)(x) + \mu x^{\alpha+1} \left(D_{0+}^{\alpha+1} y \right)(x) + \lambda x^{\alpha} \left(D_{0+}^{\alpha} y \right)(x) = 0,$$

for $x > 0$ and with the fractional derivatives $D_{0+}^{\alpha+k} y$ ($k = 0, 1, 2$). The solution of the homogeneous differential equation of the Euler type was found by applying the Mellin integral transform under some conditions on the exact solution y .

In [49], the solution in closed form of the linear non-homogeneous differential equations

$$\delta x^{\alpha+2} \left(D_-^{\alpha+2} y \right) (x) + \mu x^{\alpha+1} \left(D_-^{\alpha+1} y \right) (x) + \lambda x^{\alpha} \left(D_-^{\alpha} y \right) (x) = f(x),$$

$$(x > 0, \operatorname{Re}(\alpha) \geq 0),$$

was given, with $\alpha > 0$ and complex $\delta, \mu, \lambda \in \mathbb{C}$ on a positive half-axis $\mathbb{R}_+ = (0, +\infty)$. One-dimensional direct and inverse Mellin integral transform $\mathcal{M}$ and $\mathcal{M}^{-1}$ were used with the residue theory to establish explicit solutions in terms of special cases of the generalized wright function ${}_p\Psi_q[z]$, generalized hypergeometric function ${}_pF_q[z]$, and Euler psi function; see details in [50]. An analytic solution of the Euler-type equation.

$$\sum_{k=0}^m A_k x^{\alpha+k} \left(D_+^{\alpha+k} y \right) (x) = 0, \quad (x > 0, A_m \neq 0),$$

was given in [51] with complex $A_k \in \mathbb{C}, (k = 0, 1, \dots, m)$ on the positive half axis $\mathbb{R}_+ = (0, +\infty)$. Further, general solutions were investigated using the direct and inverse Mellin transforms, the residue theory, and the properties of fractional derivatives and the Euler psi function. In addition [52], proposed an analytic variable coefficient fractional differential equations while [53], gave a numerical technique for variable fractional functional boundary value problem and [54] developed a method for approximating particular solutions for constant and variable order fractional diffusion models. In our studies, in addition to collocation method for Bagley–Torvik equation [55], we also give numerical method for the solution of the fractional Cauchy–Euler (FrC-E)-type problem, which constitutes a class of examples of the Cauchy problem of the Bagley–Torvik equation with variable coefficients as follows:

$$B_1 t^2 D^2 y(t) + B_2 t^{\alpha} \left({}^C \mathbf{D}_t^{\alpha} y(t) \right) + B_3 y(t) = f(t),$$

$$t \in (a, T), y(a) = \bar{\beta}_0, y'(a) = \bar{\beta}_1, \quad (1.4)$$

where $0 < a < T < \infty$, $\bar{\beta}_0, \bar{\beta}_1, B_1, B_2, B_3$ are the given constants, and $B_1 \neq 0$, ${}^C \mathbf{D}_t^{\alpha}$ is the Caputo fractional derivative defined as

$${}^C\mathbf{D}_t^\alpha y(t) = \frac{1}{\Gamma(\tau - \alpha)} \int_a^t (t-x)^{\tau-\alpha-1} D_x^\tau y(x) dx, \quad (1.5)$$

also, $\tau = \lceil \alpha \rceil$ and $\alpha \in (0, 2)$ and $D_x^\tau y = \frac{\partial^\tau y}{\partial x^\tau}$.

Further, f is a given continuous function. The proposed problem also extends the classical Cauchy–Euler equation [13] to the Caputo fractional model problem. Additionally, we prove the existence and uniqueness of the solution of the given problem by using the contraction mapping principle. Some exhaustive studies on the existence of solutions to boundary value problems include [56, 57, 58], and related recent works on fractional model problems include [59, 60, 61]. In accordance with applications of fractional Cauchy–Euler equation, some examples are as follows:

1. [15] Engineering: They are often used in structural engineering to model the behavior of beams and columns under load, where the stiffness of the material varies with position..
2. Physics: In quantum mechanics and wave propagation, Cauchy–Euler equations can describe the behavior of systems with varying potentials or media properties.
3. Control systems: They are applied in control theory to design systems with variable parameters, enhancing the stability and response of control systems.
4. Fluid dynamics: These equations can model fluid flow where the fluid properties change with position, such as in varying temperature or pressure conditions.
5. Economics: In financial mathematics, they can be used to model economic systems with time-varying interest rates or other dynamic parameters.

These applications demonstrate the versatility and importance of Cauchy–Euler equations in modeling and solving complex, real-world problems across various disciplines. The classical Cauchy–Euler problem may be solved by using variable transformation that reduces the problem to linear differential equation with constant

coefficients. However, variable transformation for (1.4) may lead to more complicated equations because the Caputo fractional derivative does not certify the classical Leibniz rule. Moreover, the computational cost of the analytic solution is very high even if it is evaluated at a few discrete points. Therefore, numerical approximations of (1.4) are inevitable, and this motivated us to establish a numerical method for the solution. It is well known that collocation methods are continuous methods that produce approximations at discrete points, but many discrete methods cannot be used to obtain continuous approximations such as extrapolation and finite difference methods, and many Runge–Kutta methods. For this reason, they are inefficient for problems requiring globally continuous differentiable functions as approximations of the unknown solution; see [62]. Furthermore, in general, collocation methods are simple and easy to code. Some of the recent studies on collocation methods are [37], [63], [64], [65].

1.4 Achievement and Organization of the Study

In any research work, the ultimate goal is to identify a research gap and try to fill the this gap. In our research work, the gap will sink to fill is the methodological gap. Although there are some existing method for solving Bagely Torvik equation but there is growing need for more efficient and accurate method for approximating numerically the solution of constant and variable coefficient Bagely–Torvik equations.

In the study of natural phenomena or real life situations, scientist and engineers often used partial, ordinary differential and integral equations for modeling of these concepts. The solution of the constructed models are needed for a better understanding of the problem been described, however, finding the analytic solutions of the constructed mathematics models in most cases proved to be challenging to obtain. In order to overcome this problem, numerical methods becomes indispensable

tool in the arsenal of scientist and engineers. Numerical methods are used to find the approximate solution of the constructed models which help in providing insight of the problem being studied. Having understood the importance of numerical methods, we give collocation method for solving problems of generalized fractional Bagley–Torvik equation. We split the thesis into two broad objectives:

1. We study BVP for generalized fractional Bagley–Torvik equation with constant coefficients.
2. We also considered the fractional Cauchy-Euler differential equation which is also a variable coefficient Bagley–Torvik equation.

For the boundary value problems for generalized fractional Bagley–Torvik equation with constant coefficients, the existence and uniqueness of the solution of the BVP of the generalized BgT equation (1.3) is sought using M-L type function. Also, a numerical approach based on collocation method is developed and convergence analysis are given. Moreover, the method is utilized to find the solution of several constructed problems as well as some problems from the literature. The advantageous of the method are as follows: a) The method is a continuous method with high order accuracy $\mathcal{O}(h^m)$ under the assumption that the exact solution $y \in C^{m+2}[0, T]$. Here h is the step size and m is the number of collocation parameters. b) Ease of implementation in computer. c) Efficiency in computational time and storage. These theoretical and some numerical results are published in [55], [66].

Furthermore, for the Variable coefficient BTg equation , we present a collocation method for fractional Cauchy-Euler (FrC-E) problem given in (1.4) the sense of the Caputo fractional derivative of order $0 < \alpha < 2$. A collocation method for fractional Cauchy-Euler (FrC-E) problem is derived and the convergence analysis of the method

is proved. For the numerical simulation, we applied the derived method on several constructed examples. The numerical results proved to be consistent with the theoretical results and demonstrate the efficiency and applicability of the method.

The thesis is organized as follows: Chapter 2 is subdivided in Sections: Section 2.1, Section 2.2 and Section 2.3. In Section 2.1 we give the existence and uniqueness of the solution to the FrBVP of the generalized BgT equation (1.3). In Section 2.2, we derive the collocation equations and develop the collocation-shooting method and in Section 2.3 we present the convergence analysis of the collocation shooting method.

Chapter 3 is devoted to experimental investigation of the highly accurate numerical method for solving boundary value problems of generalized BgT equation (1.4). The applications of the collocation shooting method derived in Chapter 2 and numerical investigation of the method with the applications on problems from the literature and on some constructed examples are given.

Further, in Chapter 4, we present the existence and uniqueness, also a numerical approach to the solution of fractional Cauchy–Euler equation (1.4). This Chapter is also subdivided into the following subsections. Section 4.1, deals with the existence and uniqueness of the fractional Cauchy-Euler equations (FrC-E) (1.4) the derivation of the collocation method for fractional Cauchy-Euler is given in Section 4.2 and the convergence analysis of the collocation method for FrC-E is presented in Section 4.3

The experimental investigation of the collocation method for fractional Cauchy-Euler equations is given in chapter 5. The theoretical and numerical results in Chapter 4 and Chapter 5 are published in [55], [66].

Finally, the conclusion of our investigations on a highly accurate numerical method for solving boundary value problems of generalized Bagley Torvik equation and on the existence, uniqueness and a numerical approach to the solution of fractional Cauchy–Euler equation are presented in Chapter 6.

Chapter 2

A HIGHLY ACCURATE NUMERICAL METHOD FOR SOLVING BOUNDARY VALUE PROBLEM OF GENERALIZED BAGLEY–TORVIK EQUATION

2.1 Existence and Uniqueness Solution of the Boundary Value Problem

The following basic notations and results are important and will be used in this section and the proceeding sections. The classical derivative is denoted as $D_t^k y(t) = D^k y(t) = y^{(k)}(t)$, $k \in \mathbb{N}$, and for $k = 1, 2$ we also use y' and y'' which represent the first and second derivative respectively.

Definition 2.1: [33] For $\alpha, \beta > 0, \gamma \in \mathbb{R}$ the M-L type function is defined by double infinite binomial series as below:

$$\tau^{\gamma-1} E_{\alpha, \beta, \gamma}(\lambda \tau^\alpha, \mu \tau^\beta) = \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \binom{l+p}{p} \frac{\lambda^l \mu^p}{\Gamma(l\alpha + p\beta + \gamma)} \tau^{l\alpha + p\beta + \gamma - 1}.$$

Further we consider the next IVP of the generalized BgT equation

$$\begin{aligned} AD^2 y(t) + B \left({}^C_0 D_t^\alpha y(t) \right) + Cy(t) &= f(t), \quad 0 < t < T < \infty, \quad 0 < \alpha < 2, \\ y(0) &= y_0, \quad y'(0) = y'_0. \end{aligned} \tag{2.1}$$

The analytical solution to the problem (2.1) was given in [33] using M-L type function as in the next theorem:

Theorem 2.1: [33] The Cauchy problem (2.1) has a unique solution $y \in C^2[0, T]$ as

follows:

$$y(t) = \left(1 - t^2 \frac{C}{A} E_{2,2-\alpha,3} \left(-\frac{C}{A} t^2, -\frac{B}{A} t^{2-\alpha}\right)\right) y_0 + t E_{2,2-\alpha,2} \left(-\frac{C}{A} t^2, -\frac{B}{A} t^{2-\alpha}\right) y'_0 \\ + \int_0^t (t-s) E_{2,2-\alpha,2} \left(-\frac{C}{A} (t-s)^2, -\frac{B}{A} (t-s)^{2-\alpha}\right) f(s) ds, \quad t > 0. \quad (2.2)$$

Theorem 2.2: The unique solution of the problem (1.3) is,

$$y(t) = \bar{s}_h(t) + \gamma \bar{s}_p(t) \\ \gamma = \frac{\eta_1 - \bar{s}_h(T)}{\bar{s}_p(T)}.$$

Here $\bar{s}_h(t)$ is the solution of (2.1) with $y_0 = 0$, $y'_0 = \eta_0$, $f(t) = 0$ and $\bar{s}_p(t)$ corresponds to the solution of (2.1) when $y_0 = \eta_1$, $y'_0 = 0$.

Proof.

$$y(t) = c_1 \bar{s}_h(t) + c_2 \bar{s}_p(t), \quad (2.3)$$

is the general solution where c_1 and c_2 are unknown constants. Theorem 2.1 says that (2.3) is a solution of the differential equation in (1.3). Furthermore, the constants c_1 and c_2 are taken so that the boundary conditions in (1.3) are satisfied as,

$$\eta_0 = y(0) = c_1 \bar{s}_h(0) + c_2 \bar{s}_p(0) = c_1 \eta_0,$$

$$\eta_1 = y(T) = c_1 \bar{s}_h(T) + c_2 \bar{s}_p(T).$$

Therefore,

$$c_1 = 1, \quad c_2 = \frac{\eta_1 - \bar{s}_h(T)}{\bar{s}_p(T)}. \quad (2.4)$$

Inserting (2.4) into (2.3) we get the desired result. $\square$

2.2 The Collocation-Shooting Method

The Caputo fractional derivative operator ${}_0^C D_t^\alpha$ in (1.3) with order $\rho - 1 < \alpha < \rho$ where $\rho = \lceil \alpha \rceil$, (we consider $0 < \alpha < 2$) is given as [17]

$${}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(\rho - \alpha)} \int_0^t (t - \theta)^{\rho - \alpha - 1} D_\theta^\rho y(\theta) d\theta,$$

By the collocation-shooting method, the unknown derivative value $\bar{\eta}_1$ at $t = 0$ is assumed and then the BVP (1.3) is approximated as an initial value problem satisfying the below initial conditions.

$$y(0) = \bar{\eta}_0, y'(0) = \bar{\eta}_1, \quad (2.5)$$

where $\bar{\eta}_0 = \eta_0$ as given in (1.3). Let

$$I_h = \{t_k : t_k = kh, h = T/N, k = 0, 1, \dots, N\},$$

and $\sigma_n = (t_n, t_{n+1}]$, and $\bar{\sigma}_n = [t_n, t_{n+1}]$, $n = 0, 1, \dots, N - 1$. Also, let $\tilde{I}$ denote the identity operator and

$$\tilde{L} \equiv AD^2 + B({}_0^C D_t^\alpha) + C\tilde{I}. \quad (2.6)$$

For obtaining the solution of BVP of generalized BgT equation (1.3) the following set of initial value problems are considered.

$$\tilde{L}[\varphi_n] = f(t), t \in (t_n, t_{n+1}), \varphi_n(t_n) = \bar{\eta}_{2n}, \varphi_n'(t_n) = \bar{\eta}_{2n+1}, \quad (2.7)$$

for $n = 0, 1, \dots, N - 1$, where $\bar{\eta}_j$, $j = 0, 1, \dots, 2N - 1$ are the initial conditions on each subinterval $[t_n, t_{n+1}]$ for $n = 0, \dots, N - 1$ and $\bar{\eta}_0 = \eta_0$. The parameters $\bar{\eta}_j$, $j = 0, \dots, 2N - 1$ satisfy the following equations [37]

$$\begin{aligned}
\varphi_0[t_1, \bar{\eta}_1] &= \varphi_1[t_1, \bar{\eta}_2, \bar{\eta}_3] \\
\varphi'_0[t_1, \bar{\eta}_1] &= \varphi'_1[t_1, \bar{\eta}_2, \bar{\eta}_3] \\
\varphi_1[t_2, \bar{\eta}_2, \bar{\eta}_3] &= \varphi_2[t_2, \bar{\eta}_4, \bar{\eta}_5] \\
\varphi'_1[t_2, \bar{\eta}_2, \bar{\eta}_3] &= \varphi'_2[t_2, \bar{\eta}_4, \bar{\eta}_5] \\
&\vdots \quad \quad \quad \vdots \\
\varphi'_{N-2}[t_{N-1}, \bar{\eta}_{2N-4}, \bar{\eta}_{2N-3}] &= \varphi'_{N-1}[t_{N-1}, \bar{\eta}_{2N-2}, \bar{\eta}_{2N-1}] \\
\varphi_{N-1}[t_N, \bar{\eta}_{2N-2}, \bar{\eta}_{2N-1}] &= \eta_1 = y(T).
\end{aligned} \tag{2.8}$$

Let u_h be the numerical solution of (1.3) satisfying (2.5) and

$$u_h \in S_{m+1}^{(1)}(I_h), \text{ where } S_{m+1}^{(1)}(I_h) = \{P \in C^1(I) : P|_{\bar{\sigma}_n} \in \Pi_{m+1}\}.$$

Here, the space of all real polynomials of degree at most $m+1$ is denoted by Π_{m+1} .

Further, let X_h be given by

$$X_h = \{t = t_n + c_i h : 0 \leq c_1 < \dots < c_m \leq 1, n = 0, 1, \dots, N-1\}. \tag{2.9}$$

For a given mesh I_h , the parameters $\{c_i\}, i = 1, 2, \dots, m$ completely determines X_h and are called collocation parameters. We require that the collocation solution u_h satisfies the collocation problem:

$$AD^2 u_h(t) + B \left({}^C D_t^\alpha u_h(t) \right) + C u_h(t) = f(t), t \in X_h \tag{2.10}$$

$$u_h(0) = \bar{\eta}_0, u'_h(0) = \bar{\eta}_1, \tag{2.11}$$

where $\bar{\eta}_1$ satisfies (2.8). Let $t = t_n + v h \in \bar{\sigma}_n, v \in [0, 1]$ then on each subinterval $\bar{\sigma}_n$ the $u_h \in S_{m+1}^{(1)}(I_h)$ satisfying (2.10), (2.11) is given by:

$$u_h(t) = u_h(t_n + v h) = \sum_{q=0}^1 a_q^{(n)} v^q + \sum_{r=1}^m b_r^{(n)} v^{r+1}, v \in [0, 1]. \tag{2.12}$$

$$\begin{aligned}
\left[{}^C_0 D_t^\alpha u_h(t) \right]_{t=t_n+vh} &= \left(\frac{1}{\Gamma(\rho-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t-\theta)^{\rho-\alpha-1} D_\theta^\rho u_h(\theta) d\theta \right) \Big|_{t=t_n+vh} \\
&+ \left(\frac{1}{\Gamma(\rho-\alpha)} \int_{t_n}^t (t-\theta)^{\rho-\alpha-1} D_\theta^\rho u_h(\theta) d\theta \right) \Big|_{t=t_n+vh} \\
&= \sum_{k=0}^{n-1} \left[{}^{J_{t_{k+1}}}_t^\alpha u_h(t) \right]_{t=t_n+vh} + \left[{}^C_{t_n} D_t^\alpha u_h(t) \right]_{t=t_n+vh}. \tag{2.13}
\end{aligned}$$

Further, for $t = t_n + vh$ and $\theta = t_k + sh$, for $v, s \in [0, 1]$, $n = 0, 1, \dots, N-1$ we have,

$$\begin{aligned}
\sum_{k=0}^{n-1} \left[{}^{J_{t_{k+1}}}_t^\alpha u_h(t) \right]_{t=t_n+vh} &= \left(\frac{1}{\Gamma(\rho-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t-\theta)^{\rho-\alpha-1} D_\theta^\rho u_h(\theta) d\theta \right) \Big|_{t=t_n+vh} \\
&= \frac{1}{\Gamma(\rho-\alpha)} \sum_{k=0}^{n-1} \int_0^1 \frac{h^{-\rho+1} D_s^\rho u_h(t_k + sh)}{(t_n + vh - t_k - sh)^{\alpha+1-\rho}} ds \\
&= h^{-\rho+1} \sum_{k=0}^{n-1} {}^J_1^\alpha \left[u_h(t_k + sh) \right]. \tag{2.14}
\end{aligned}$$

Similarly

$$\begin{aligned}
\left[{}^C_{t_n} D_t^\alpha u_h(t) \right]_{t=t_n+vh} &= \left(\frac{1}{\Gamma(\rho-\alpha)} \int_{t_n}^t \frac{D_\theta^\rho u_h(\theta)}{(t-\theta)^{\alpha+1-\rho}} d\theta \right) \Big|_{t=t_n+vh} \\
&= \frac{1}{\Gamma(\rho-\alpha)} \int_0^v \frac{h^{-\rho+1} D_s^\rho u_h(t_n + sh)}{(vh - sh)^{\alpha+1-\rho}} ds \\
&= \frac{h^{-\alpha+\rho-1} h^{-\rho+1}}{\Gamma(\rho-\alpha)} \int_0^v \frac{D_s^\rho u_h(t_n + sh)}{(v-s)^{\alpha+1-\rho}} ds \\
&= h^{-\alpha} \left({}^C_0 D_v^\alpha \left[u_h(t_n + .h) \right] (v) \right). \tag{2.15}
\end{aligned}$$

Using (2.14) and (2.15) the equation (2.13) can be rewritten as,

$$\begin{aligned}
\left[{}^C_0 D_t^\alpha u_h(t) \right]_{t=t_n+vh} &= h^{-\rho+1} \sum_{k=0}^{n-1} {}^J_1^\alpha \left[u_h(t_k + sh) \right] \\
&+ h^{-\alpha} \left({}^C_0 D_v^\alpha \left[u_h(t_n + .h) \right] (v) \right). \tag{2.16}
\end{aligned}$$

2.2.1 Derivation of the Collocation Equations

The collocation-shooting method was also studied in [37], [67] and [68]. First we give the analogue of the collocation-shooting method by deriving the application of the operator (2.16) on the local representation of the solution u_h (2.12) on $\bar{\sigma}_n$ when $\rho - 1 < \alpha < \rho$, ($\rho = 1, 2$). We denote

$$\begin{aligned} \left[{}^C_0 D_t^\alpha u_h(t) \right]_{t=t_n+vh} &:= \left[{}^C_0 D_t^{\rho-1 < \alpha < \rho} u_h(t) \right]_{t=t_n+vh} \\ {}_0 J_1^\alpha \left[u_h(t_k + sh) \right] &:= {}_0 J_1^{\rho-1 < \alpha < \rho} \left[u_h(t_k + sh) \right] \\ {}^C_0 D_v^\alpha \left[u_h(t_n + .h) \right](v) &:= {}^C_0 D_v^{\rho-1 < \alpha < \rho} \left[u_h(t_n + .h) \right](v). \end{aligned}$$

The following are crucial for $0 < \alpha < 1$ when p is nonnegative integer.

$$\begin{aligned} & \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{D_s(s^p)}{(t_n + vh - t_k - sh)^\alpha} ds \\ &= \frac{h^{-\alpha} (n + v - k)^{-\alpha}}{\Gamma(1-\alpha)} {}_2\Phi_1^{0 < \alpha < 1} \left(\alpha, p, \frac{1}{n - k + v} \right) \end{aligned} \quad (2.17)$$

where

$${}_2\Phi_1^{0 < \alpha < 1}(\alpha, p, z) = \begin{cases} 0 & \text{if } p = 0, \\ {}_2F_1[\alpha, p, p + 1, z] & \text{if } p \geq 1 \end{cases} \quad (2.18)$$

and

$$\frac{1}{\Gamma(1-\alpha)} \int_0^v \frac{D_s(s^p)}{(v-s)^\alpha} ds = \begin{cases} 0 & \text{if } p = 0, \\ \frac{p\Gamma(p)}{\Gamma(1-\alpha+p)} v^{p-\alpha} & \text{if } p \geq 1. \end{cases} \quad (2.19)$$

Here, ${}_2F_1$ is the Gauss Hypergeometric function. Substituting (2.17) and (2.19) into (2.16) gives

$$\begin{aligned} \left[{}^C_0 D_t^{0 < \alpha < 1} u_h(t) \right]_{t=t_n+vh} &= h^{-\alpha} \sum_{k=0}^{n-1} \left(a_1^{(k)} \frac{(n + v - k)^{-\alpha}}{\Gamma(1-\alpha)} {}_2\Phi_1^{0 < \alpha < 1} \left(\alpha, 1, \frac{1}{n - k + v} \right) \right. \\ &+ \sum_{r=1}^m b_r^{(k)} \frac{(n + v - k)^{-\alpha}}{\Gamma(1-\alpha)} {}_2\Phi_1^{0 < \alpha < 1} \left(\alpha, r + 1, \frac{1}{n - k + v} \right) \Bigg) \\ &+ h^{-\alpha} \left(a_1^{(n)} \frac{v^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{r=1}^m b_r^{(n)} \left(\frac{(r+1)\Gamma(r+1)v^{r+1-\alpha}}{\Gamma(2-\alpha+r)} \right) \right). \end{aligned} \quad (2.20)$$

Also,

$$D_t^2 [u_h(t)] = h^{-2} D_v^2 [u_h(t_n + vh)] = h^{-2} \sum_{r=1}^m (1+r) r b_r^{(n)} v^{r-1}. \quad (2.21)$$

Evaluating (2.12), (2.20) and (2.21) at $t_{n,i} = t_n + c_i h$ and substituting into (2.10)

$$\begin{aligned} & A \left(\sum_{r=1}^m b_r^{(n)} (1+r) r (c_i)^{r-1} \right) \\ & + B \left(\sum_{k=0}^{n-1} \frac{h^{2-\alpha}}{\Gamma(1-\alpha)} \left(a_1^{(k)} (n+c_i-k)^{-\alpha} {}_2\Phi_1^{0<\alpha<1} \left(\alpha, 1, \frac{1}{n-k+c_i} \right) \right. \right. \\ & + \left. \sum_{r=1}^m b_r^{(k)} (n+c_i-k)^{-\alpha} {}_2\Phi_1^{0<\alpha<1} \left(\alpha, r+1, \frac{1}{n-k+c_i} \right) \right) \\ & + h^{2-\alpha} \left(a_1^{(n)} \frac{c_i^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{r=1}^m b_r^{(n)} \left(\frac{(r+1) \Gamma(r+1) c_i^{r+1-\alpha}}{\Gamma(2-\alpha+r)} \right) \right) \\ & + Ch^2 \left(\sum_{q=0}^1 a_q^{(n)} c_i^q + \sum_{r=1}^m b_r^{(n)} c_i^{r+1} \right) \\ & = h^2 f(t_{n,i}). \end{aligned} \quad (2.22)$$

The matrix representation of equation (2.22) can be written as

$$Vb^{(n)} = U_1 a^{(n)} + F_1^{(n)}, \quad (2.23)$$

where $V \in \mathbb{R}^{m \times m}$, $U_1 \in \mathbb{R}^{m \times 2}$ and $F_1^{(n)} \in \mathbb{R}^m$ with entries given as

$$(V)_{i,r} = A(1+r) r (c_i)^{r-1} + Bh^{2-\alpha} \left(\frac{(r+1) \Gamma(r+1) c_i^{r+1-\alpha}}{\Gamma(2-\alpha+r)} \right) + Ch^2 c_i^{r+1}, \quad (2.24)$$

$$(U_1)_{i,1} = -Ch^2, \quad (U_1)_{i,2} = -Bh^{2-\alpha} \frac{c_i^{1-\alpha}}{\Gamma(2-\alpha)} - Ch^2 c_i, \quad (2.25)$$

$$\begin{aligned} (F_1^{(n)})_i &= h^2 f(t_{n,i}) \\ & - \frac{h^{2-\alpha}}{\Gamma(1-\alpha)} B \left(\sum_{k=0}^{n-1} \left(a_1^{(k)} (n+c_i-k)^{-\alpha} {}_2\Phi_1^{0<\alpha<1} \left(\alpha, 1, \frac{1}{n-k+c_i} \right) \right. \right. \\ & + \left. \sum_{r=1}^m b_r^{(k)} (n+c_i-k)^{-\alpha} {}_2\Phi_1^{0<\alpha<1} \left(\alpha, r+1, \frac{1}{n-k+c_i} \right) \right) \Bigg), \end{aligned} \quad (2.26)$$

for $i = 1, \dots, m$ and $r = 1, \dots, m$. Further, when $1 < \alpha < 2$ ($\rho = 2$) analogously we use the local representation of u_h given in (2.12) on $\bar{\sigma}_n$ and the following results,

$$\begin{aligned}
& \frac{1}{\Gamma(2-\alpha)} \int_0^1 \frac{D_s^2(s^p)}{(t_n + vh - t_k - sh)^{\alpha-1}} ds \\
&= \frac{ph^{1-\alpha}(n+v-k)^{1-\alpha}}{\Gamma(2-\alpha)} {}_2\Phi_1^{1<\alpha<2} \left(-1+\alpha, -1+p, \frac{1}{n-k+v} \right)
\end{aligned} \tag{2.27}$$

$${}_2\Phi_1^{1<\alpha<2}(x, q, z) = \begin{cases} 0 & \text{if } q = -1, 0, \\ {}_2F_1[x, q, q+1, z] & \text{if } q \geq 1, \end{cases}$$

$$\frac{1}{\Gamma(2-\alpha)} \int_0^v \frac{D_s^2(s^p)}{(v-s)^{\alpha-1}} ds = \begin{cases} 0 & \text{if } p = 0, 1, \\ \frac{p(p-1)\Gamma(p-1)}{\Gamma(1-\alpha+p)} v^{p-\alpha} & \text{if } p \geq 2, \end{cases} \tag{2.28}$$

where p, q are integers and $p \geq 0$ and $q \geq -1$. Substituting (2.21), (2.27), (2.28) into (2.16) and evaluating the result at $t_{n,i} = t_n + c_i h$ that is taking $v = c_i$ gives

$$\begin{aligned}
& \left[{}^C_0 D_t^{1<\alpha<2} u_h(t) \right]_{t=t_n+c_i h} \\
&= h^{-\alpha} \sum_{k=0}^{n-1} \sum_{r=1}^m b_r^{(k)} \frac{(r+1)(n+c_i-k)^{1-\alpha}}{\Gamma(2-\alpha)} {}_2\Phi_1^{1<\alpha<2} \left(-1+\alpha, r, \frac{1}{n-k+c_i} \right) \\
&+ h^{-\alpha} \left(\sum_{r=1}^m b_r^{(n)} \frac{(r+1)\Gamma(r+1)c_i^{r+1-\alpha}}{\Gamma(2-\alpha+r)} \right).
\end{aligned} \tag{2.29}$$

Subsequently, evaluating (2.12) and (2.21), at $t_{n,i} = t_n + c_i h$, then inserting these equations with (2.29) into (2.10) we get

$$\begin{aligned}
& A \left(\sum_{r=1}^m (1+r) r (c_i)^{r-1} b_r^{(n)} \right) \\
&+ B \left(h^{2-\alpha} \sum_{k=0}^{n-1} \sum_{r=1}^m b_r^{(k)} \frac{(r+1)(n+c_i-k)^{1-\alpha}}{\Gamma(2-\alpha)} {}_2\Phi_1^{1<\alpha<2} \left(-1+\alpha, r, \frac{1}{n-k+c_i} \right) \right. \\
&+ h^{2-\alpha} \left(\sum_{r=1}^m b_r^{(n)} \frac{(r+1)\Gamma(r+1)c_i^{r+1-\alpha}}{\Gamma(2-\alpha+r)} \right) \Big) \\
&+ Ch^2 \left(\sum_{q=0}^1 a_q^{(n)} c_i^q + \sum_{r=1}^m b_r^{(n)} c_i^{r+1} \right) \\
&= h^2 f(t_{n,i} + c_i h), \quad i = 1, 2, \dots, m.
\end{aligned} \tag{2.30}$$

Analogously equation (2.30) has the form

$$Vb^{(n)} = U_2 a^{(n)} + F_2^{(n)}, \tag{2.31}$$

where V is the same matrix as given in

(2.24) and $U_2 \in \mathbb{R}^{m \times 2}$ and $F_2^{(n)} \in \mathbb{R}^m$. The entries of U_2 and $F_2^{(n)}$ are as follows:

$$(U_2)_{i,s+1} = -Ch^2c_i^s, \quad s = 0, 1, \quad (2.32)$$

$$\begin{aligned} (F_2^{(n)})_i &= h^2 f(t_n + c_i h) - \frac{h^{2-\alpha}}{\Gamma(2-\alpha)} B \left(\sum_{k=0}^{n-1} \sum_{r=1}^m b_r^{(k)} (r+1) (n+c_i-k)^{1-\alpha} \right. \\ &\quad \left. \times {}_2\Phi_1^{1<\alpha<2} \left(-1+\alpha, r, \frac{1}{n-k+c_i} \right) \right), \quad i = 1, \dots, m. \end{aligned} \quad (2.33)$$

Here, for the first interval $a^{(0)}$ is as given below

$$a^{(0)} = \begin{bmatrix} y(0) \\ hy'(0) \end{bmatrix} = \begin{bmatrix} \bar{\eta}_0 \\ h\bar{\eta}_1 \end{bmatrix}.$$

Also, $a^{(n)}$ and $b^{(n)}$ are

$$a^{(n)} = \begin{bmatrix} a_0^{(n)} & a_1^{(n)} \end{bmatrix}^T, \quad b^{(n)} = \begin{bmatrix} b_1^{(n)} & b_2^{(n)} & \dots & b_m^{(n)} \end{bmatrix}^T. \quad (2.34)$$

For $n \geq 1$, the smoothness condition on $[t_n, t_{n+1}]$ gives the relationship between $a^{(n)}$, $b^{(n)}$ and the unknown vector $a^{(n+1)}$ as

$$a^{(n+1)} = G_1 a^{(n)} + G_2 b^{(n)}, \quad (2.35)$$

where G_1 is the 2×2 upper triangular matrix and G_2 is the $2 \times m$ matrix with entries

$$G_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (G_2)_{j+1,k} = \binom{1+k}{j}, \quad j = 0, 1, k = 1, \dots, m.$$

See [69],[70],[71], [72], [73], [74] and the references therein for details.

2.3 Convergence Analysis

Theorem 2.3: Let $c_i = \frac{1+x_i}{2}$ where $x_i = 1 - \cos\left(\frac{i\pi}{2m}\right)$, $i = 1, \dots, m$ be m distinct collocation parameters. Then for sufficiently small number $\bar{h} > 0$ the linear algebraic systems (2.23) and (2.31) for $n = 0, 1, \dots, N-1$ have unique solution for $h = \frac{T}{N} \in (0, \bar{h})$. Thus, the unique collocation approximation $u_h \in S_{m+1}^{(1)}(I_h)$ exist for the local representation (2.12).

Proof. For any fixed m , with collocation parameters $\{c_i\}$, $i = 1, \dots, m$ as $c_i = \frac{1+x_i}{2}$, where $x_i = 1 - \cos\left(\frac{i\pi}{2m}\right)$ the entries of the matrix V in (2.24) are bounded. Further, for $j = 1, 2, \dots, m$ let

$$\begin{aligned}\gamma_j &= A(1+j)j, \\ \delta_j &= Bh^{2-\alpha} \left(\frac{(j+1)\Gamma(j+1)}{\Gamma(2-\alpha+j)} \right), \\ \eta &= Ch^2,\end{aligned}$$

then the matrix V can be put in the form

$$V = \begin{pmatrix} \gamma_1 + \delta_1 c_1^{2-\alpha} + \eta c_1^2 & \dots & \gamma_m c_1^{m-1} + \delta_m c_1^{m+1-\alpha} + \eta c_1^{m+1} \\ \gamma_1 + \delta_1 c_2^{2-\alpha} + \eta c_2^2 & \dots & \gamma_m c_2^{m-1} + \delta_m c_2^{m+1-\alpha} + \eta c_2^{m+1} \\ \vdots & \vdots & \vdots \\ \gamma_1 + \delta_1 c_m^{2-\alpha} + \eta c_m^2 & \dots & \gamma_m c_m^{m-1} + \delta_m c_m^{m+1-\alpha} + \eta c_m^{m+1} \end{pmatrix}.$$

It is obvious that for sufficiently small $h \in (0, \bar{h})$ the rows of V are linearly independent hence $\text{rank}(V) = m$. Therefore, the matrix V possesses a bounded inverse. $\square$

Theorem 2.4: Assume that

- (a) $\{c_i\}$, $i = 1, \dots, m$ are m distinct collocation parameters chosen as $c_i = \frac{1+x_i}{2}$, where $x_i = 1 - \cos\left(\frac{i\pi}{2m}\right)$ satisfying $0 < c_1 < c_2 \dots < c_m = 1$.
- (b) The exact solution y of (2.1) belongs to the space $C^{m+2}(I)$.
- (c) The approximation $u_h \in S_{m+1}^{(1)}(I_h)$ defined by (2.12) for the BVP (1.3), and the IVP (2.1) are obtained by solving (2.10) and (2.11).
- (d) $\bar{h} > 0$ is such that the linear systems (2.23), (2.31) have unique solutions for any mesh step $h \in (0, \bar{h})$. Then the estimate

$$\|y - u_h\|_\infty := \max_{t \in I} |y(t) - u_h(t)| \leq C_0 h^m, \quad (2.36)$$

holds for $h \in (0, \bar{h})$. The constant C_0 depends on the parameters c_i , $i = 1, 2, \dots, m$ and

on $D^{m+2}y(t)$ but is independent of h .

Proof. From assumption (b) we have $y \in C^{m+2}(I)$ and hence $y' \in C^{m+1}(I)$, and $y'' \in C^m(I)$. Using Peano's Theorem [62] for y on $\bar{\sigma}_n$,

$$y''(t) = \sum_{j=1}^m L_j(v) Z_{n,j} + h^m R_{m+2,n}^{(1)}(v) \quad v \in [0, 1], \quad (2.37)$$

with $Z_{n,j} = y''(t_{n,j}) = y''(t_n + c_j h)$ and the functions

$$L_j(v) := \prod_{l \neq j}^m \frac{v - c_l}{c_j - c_l} \quad (j = 1, 2, \dots, m), \quad (2.38)$$

are the Lagrange polynomials with abscissas $c_i, i = 1, 2, \dots, m$. Also, the Peano remainder term is

$$R_{m+2,n}^{(1)}(v) = \int_0^1 K_m(v, z) y^{m+2}(t_n + zh) dz, \quad (2.39)$$

and

$$K_m(t, s) = \frac{1}{(m-1)!} \left\{ (t-s)_+^{m-1} - \sum_{j=1}^m L_j(t) (c_j - s)_+^{m-1} \right\},$$

$$(t-s)_+^p = \begin{cases} 0 & \text{if } t < s, \\ (t-s)^p & \text{if } t \geq s. \end{cases}$$

Then the local Lagrange representation of $y'(t)$ on $\bar{\sigma}_n$ is,

$$y'(t_n + v h) = y'(t_n) + h \sum_{j=1}^m \beta_{1,j}(v) Z_{n,j} + h^{m+1} R_{m+2,n}(v), \quad (2.40)$$

where

$$\beta_{k,j}(v) = \int_0^v \frac{(v-s)^{1-k}}{(1-k)!} L_j(s) ds, \quad k = 0, 1,$$

$$R_{m+2,n}(v) = \int_0^v R_{m+2,n}^{(1)}(s) ds.$$

Integrating once more both sides of (2.40) from 0 to v yields

$$y(t_n + v h) = y(t_n) + h v y'(t_n) + h^2 \sum_{j=1}^m \beta_{0,j}(v) Z_{n,j} + h^{m+2} \tilde{R}_{m+2,n}(v), \quad (2.41)$$

and

$$\tilde{R}_{m+2,n}(v) = \int_0^v R_{m+2,n}(s) ds.$$

The collocation solution $u_h(t) = u_h(t_n + vh)$ satisfies the collocation equation (2.10)

and (2.11) and the local form of $u_h(t) = u_h(t_n + vh)$ is

$$u_h(t) = u_h(t_n + vh) = u_h(t_n) + hvu'_h(t_n) + h^2 \sum_{j=1}^m \beta_{0,j}(v) Y_{n,j}, \quad (2.42)$$

where $Y_{n,j} = u''_h(t_n + c_j h)$. Since $u_h \in S_{m+1}^{(1)}(I_h)$, the local function (2.42) can be also presented as (2.12). Subtracting (2.42) from (2.41) gives

$$\begin{aligned} e_h(t_n + vh) &= y(t_n) - a_0^{(n)} + v(hy'(t_n) - a_1^{(n)}) \\ &\quad + \sum_{j=1}^m (h^2 \beta_{0,j}(v) Z_{n,j} - b_j^{(n)} v^{j+1}) + h^{m+2} \tilde{R}_{m+2,n}(v), \\ &= e_h(t_n) + hve'_h(t_n) + h^2 \sum_{j=1}^m \beta_{0,j}(v) \varepsilon_{n,j} + h^{m+2} \tilde{R}_{m+2,n}(v), \end{aligned} \quad (2.43)$$

for $v \in [0, 1]$ where $\varepsilon_{n,j} = Z_{n,j} - Y_{n,j}$. Also from (2.16) and (2.43) when $1 < \alpha < 2$ we have

$$\begin{aligned} \left[{}^C D_t^{1 < \alpha < 2} e_h(t) \right]_{t=t_n+vh} &= h^{-1} \left(h^2 \sum_{k=0}^{n-1} \sum_{j=1}^m \left({}_0 J_1^{1 < \alpha < 2} [\varepsilon_{k,j} \beta_{0,j}(s)] \right) \right. \\ &\quad \left. + h^{m+2} \sum_{k=0} + h^{2-\alpha} \sum_{j=1}^m {}^C D_v^{1 < \alpha < 2} [\varepsilon_{n,j} \beta_{0,j}(s)](v) \right) \\ &\quad + h^{m+2-\alpha} \hat{R}_{m+2,n}(v). \end{aligned} \quad (2.44)$$

We define the following

$$\begin{aligned} \hat{R}_{m+2,n}(v) &= \left({}^C D_v^{1 < \alpha < 2} [\tilde{R}_{m+2,n}(s)] \right)(v), \\ \bar{\beta}_{0,j}(v) &= \left({}_0 J_1^{1 < \alpha < 2} [\beta_{0,j}(s)] \right) \\ \bar{R}_{m+2,k}(v) &= \left({}_0 J_1^{1 < \alpha < 2} [\tilde{R}_{m+2,k}(s)] \right), \\ \hat{\beta}_{0,j}(v) &= \left({}^C D_v^{1 < \alpha < 2} [\beta_{0,j}(s)] \right)(v) \end{aligned}$$

From (2.44) and differentiating (2.43) with respect to t gives

$$\begin{aligned} \left[{}^C_0 D_t^{1<\alpha<2} e_h(t) \right]_{t=t_n+vh} &= h \sum_{k=0}^{n-1} \sum_{j=1}^m \bar{\beta}_{0,j}(v) \varepsilon_{k,j} + h^{m+1} \sum_{k=0}^{n-1} \bar{R}_{m+2,k}(v) \\ &\quad + h^{2-\alpha} \sum_{j=1}^m \hat{\beta}_{0,j}(v) \varepsilon_{n,j} + h^{m+2-\alpha} \hat{R}_{m+2,n}(v) \end{aligned} \quad (2.45)$$

$$e'_h(t_n + vh) = e'_h(t_n) + h \sum_{j=1}^m \beta_{1,j}(v) \varepsilon_{n,j} + h^{m+1} R_{m+2,n}(v), \quad (2.46)$$

$$e''_h(t_n + vh) = \sum_{j=1}^m L_j(v) \varepsilon_{n,j} + h^m R_{m+2,n}^{(1)}(v). \quad (2.47)$$

At the mesh points ($v = 1$) and using recursion

$$\begin{aligned} e'_h(t_n) &= e'_h(t_{n-1}) + h \sum_{j=1}^m \beta_{1,j}(1) \varepsilon_{n-1,j} + h^{m+1} R_{m+2,n-1}(1), \\ e_h(t_n) &= e_h(t_{n-1}) + h e'_h(t_{n-1}) + h^2 \sum_{j=1}^m \beta_{0,j}(1) \varepsilon_{n-1,j} + h^{m+2} \tilde{R}_{m+2,n}(1), \end{aligned}$$

and that $e_h(0) = y(0) - u_h(0) = 0$ and assuming $e'_h(0) = y'(0) - u'_h(0) = 0$ gives

$$e'_h(t_n) = h \sum_{k=0}^{n-1} \sum_{j=1}^m \beta_{1,j}(1) \varepsilon_{k,j} + h^{m+1} \sum_{k=0}^{n-1} R_{m+2,k}(1), \quad (2.48)$$

$$\begin{aligned} e_h(t_n) &= h^2 \sum_{k=0}^{n-1} \sum_{j=1}^m \beta_{1,j}(1) \varepsilon_{k,j} + h^{m+2} \sum_{k=0}^{n-1} R_{m+2,k}(1) \\ &\quad + h^2 \sum_{k=0}^{n-1} \sum_{j=1}^m \beta_{0,j}(1) \varepsilon_{k,j} + h^{m+2} \sum_{k=0}^{n-1} \tilde{R}_{m+2,k}(1). \end{aligned} \quad (2.49)$$

Next since $e_h = y - u_h$ satisfies the following equation

$$A e''_h(t_{n,i}) + B \left({}^C_0 D_t^{1<\alpha<2} e_h \right) (t_{n,i}) + C e_h(t_{n,i}) = 0, \quad i = 1, 2, \dots, m, \quad (2.50)$$

with $t_{n,i} = t_n + c_i h$. Then evaluating (2.47) at $v = c_i$ gives

$$\varepsilon_{n,i} = e''_h(t_{n,i}) - h^m R_{m+2,n}^{(1)}(c_i). \quad (2.51)$$

Using (2.48)-(2.51) and evaluating (2.43) at $v = c_i$ one gets system of algebraic equations for $i = 1, 2, \dots, m$ that can be presented in the matrix form

$$\Omega \mathbf{E}_n = \sum_{k=0}^{n-1} h \Theta_{n,k} \mathbf{E}_k + \rho_n, \quad (2.52)$$

where $\mathbf{E}_n = [\varepsilon_{n,1}, \varepsilon_{n,2}, \dots, \varepsilon_{n,m}]^T$ and $\rho_n = [\rho_{n,1}, \rho_{n,2}, \dots, \rho_{n,m}]^T$ with the entries

$$\begin{aligned}
\rho_{n,i} = & Bh^{m+1} \sum_{k=0}^{n-1} \bar{R}_{m+2,k}(c_i) + Bh^{m+2-\alpha} \widehat{R}_{m+2,n}(c_i) + Ch^{m+2} \sum_{k=0}^{n-1} R_{m+2,k}(1) \\
& + Ch^{m+2} \sum_{k=0}^{n-1} \widetilde{R}_{m+2,k}(1) + Cc_i h^{m+2} \sum_{k=0}^{n-1} R_{m+2,k}(1) + Ch^{m+2} \widetilde{R}_{m+2,n}(c_i) \\
& + Ah^m R_{m+2,n}^{(1)}(c_i), i = 1, 2, \dots, m.
\end{aligned} \tag{2.53}$$

Since the functions $\bar{R}_{m+2,k}, R_{m+2,k}, \widehat{R}_{m+2,k}$ and $\widetilde{R}_{m+2,k}, k = 0, 1, \dots, n$ are bounded there exist a constant δ such that $\|\rho_n\|_1 \leq \delta h^m$. Further, for sufficiently small $h \in (0, \bar{h})$ the linear system (2.52) has unique solution. Next, we take $\kappa_0 = \max_{1 \leq j \leq m} \|\beta_{0,j}\|_\infty$ and $\lambda_{1,n} := \max_{v \in [0,1]} |\widetilde{R}_{m+2,n}(v)|$, $\lambda_1 = \max_{0 \leq k \leq n} \{\lambda_{1,k}\}$ and $\|\mathbf{E}_n\|_1 = \sum_{j=1}^m |\varepsilon_{n,j}|$. Also let $\kappa_1 = \max_{1 \leq j \leq m} \|\beta_{1,j}\|_\infty$, and $\lambda_{2,n} := \max_{v \in [0,1]} |R_{m+2,n}(v)|$, $\lambda_2 = \max_{0 \leq k \leq n} \{\lambda_{2,k}\}$. Then from (2.48), $|e'_h(t_n)|$ can be estimated as

$$\begin{aligned}
|e'_h(t_n)| & \leq h \sum_{k=0}^{n-1} \sum_{j=1}^m |\beta_{1,j}(1)| |\varepsilon_{k,j}| + h^{m+1} \sum_{k=0}^{n-1} |R_{m+2,k}(1)| \\
& \leq h \kappa_1 \sum_{k=0}^{n-1} \|\mathbf{E}_k\|_1 + h^m T \lambda_2.
\end{aligned} \tag{2.54}$$

Further using (2.43), (2.49) and (2.54) we get

$$\begin{aligned}
\max_{v \in [0,1]} |e_h(t_n + vh)| & \leq \kappa_1 h^2 \sum_{k=0}^{n-1} \|\mathbf{E}_k\|_1 + h^{m+1} T \lambda_2 + \kappa_0 h^2 \sum_{k=0}^{n-1} \|\mathbf{E}_k\|_1 + h^{m+1} T \lambda_1 \\
& + T \left(h \kappa_1 \sum_{k=0}^{n-1} \|\mathbf{E}_k\|_1 + h^m T \lambda_2 \right) + h^2 \kappa_0 \|\mathbf{E}_n\|_1 + h^{m+2} \lambda_1.
\end{aligned} \tag{2.55}$$

Then when $h \in (0, \bar{h})$ there exists a constant $D_0 < \infty$, so that,

$$\left\| \Omega^{-1} \right\|_1 \leq D_0 < \infty, \tag{2.56}$$

holds. Also the entries of the matrix $\Theta_{n,k}$ are bounded therefore

$\|\Omega^{-1}\|_1 \|\Theta_{n,k}\|_1 \leq D_0^* < \infty$. Then from (2.52) and using $\|\rho_n\|_1 \leq \delta h^m$ we get

$$\|\mathbf{E}_n\|_1 \leq D_0^* \sum_{k=0}^{n-1} h \|\mathbf{E}_k\|_1 + D_0 \delta h^m, \quad n = 0, 1, \dots, N-1. \tag{2.57}$$

The inequality (2.57) is called discrete Gronwall inequality. There exist a positive number μ such that

$$\|\mathbf{E}_n\|_1 \leq \mu h^m \text{ for } n = 0, 1, \dots, N-1. \quad (2.58)$$

Subsequently, from (2.55) and using (2.58) follows

$$\begin{aligned} \max_{v \in [0,1]} |e_h(t_n + vh)| &\leq \kappa_1 h^2 \sum_{k=0}^{n-1} \|\mathbf{E}_k\|_1 + h^{m+1} T \lambda_2 + \kappa_0 h^2 \sum_{k=0}^{n-1} \|\mathbf{E}_k\|_1 + h^{m+1} T \lambda_1 \\ &\quad + T \left(h \kappa_1 \sum_{k=0}^{n-1} \|\mathbf{E}_k\|_1 + h^m T \lambda_2 \right) + h^2 \kappa_0 \|\mathbf{E}_n\|_1 + h^{m+2} \lambda_1 \\ &\leq \kappa_1 T \mu h^{m+1} + h^{m+1} T \lambda_2 + \kappa_0 T \mu h^{m+1} + h^{m+1} T \lambda_1 + \kappa_1 T^2 \mu h^m \\ &\quad + T^2 h^m \lambda_2 + h^{m+2} \mu \kappa_0 + h^{m+2} \lambda_1 \\ &\leq C_0 h^m. \end{aligned}$$

Hence, the inequality (2.36) is obtained. Note also, we assumed $e'_h(0) = 0$ and symbolically solve the equation (2.8) for u_h . Equation (2.54) shows that $e'_h(0) = 0 \approx \mathcal{O}(h^m)$. Further, for $0 < \alpha < 1$ the proof is analogous. For $\alpha = 1$ the generalized BgT equation (1.3) becomes the second order ordinary differential equation and the obtained results are valid, see [62]. $\square$

Remark 2.1: In Theorem 2.4 we assumed that $y \in C^{m+2}(I)$. This is a condition stated in terms of the unknown function y . For the general initial value problem of linear differential equation with coefficients $\lambda_i \in \mathbb{R}, i = 1, \dots, n$ and the Caputo derivatives

$$\left({}^C_0 D_t^\mu y \right)(t) - \sum_{i=1}^n \lambda_i \left({}^C_0 D_t^{\mu_i} y \right)(t) = f(t), \text{ for } t > 0,$$

$$y^{(k)}(0) = c_k \in \mathbb{R}, \quad k = 0, 1, \dots, m-1 < \mu \leq m.$$

where $\mu > \mu_1 > \dots > \mu_n \geq 0$, and $m_i - 1 < \mu_i \leq m_i$, $m_i \in N_0 = N \cup \{0\}$ the sufficient conditions on the data f , for the differentiability properties of y were given in Theorem 4.1 of [75]. Furthermore, based on this theorem, later it was stated in [34] that if $f \in C^2(I)$ then the IVP of BgT equation (1.2) with Caputo derivative of order $\alpha = 3/2$ has a unique solution $y \in C^2(I)$.

Chapter 3

EXPERIMENTAL INVESTIGATION OF THE HIGHLY ACCURATE NUMERICAL METHOD FOR SOLVING BOUNDARY VALUE PROBLEM OF GENERALIZED BAGLEY TORVIK EQUATION

In this section of the thesis, we consider the implementation of the highly accurate numerical method for solving the boundary value problem of generalized Bagley-Torvik equation. For demonstrating the efficiency of the method in computation and accuracy we present some test problems. The realization of the method for the conducted examples is given by using Matlab in double precision on an HP Laptop with properties 8GB RAM INTEL i7 & 1255U 3.5GHZ and 512 GB NVME M.2SSD.

For computational efficiency in storage and time we have taken $m = 3$ or $m = 4$. This provides solving system of linear equations with same coefficient matrix of size (3×3) or (4×4) respectively, for each subinterval $n = 0, 1, \dots, N - 1$. For a fixed m and $h = T/N$ the coefficient matrix is same on each subinterval $n = 0, 1, \dots, N - 1$. Therefore, the solution of each of the algebraic linear system of equations having same coefficient matrix with different right sides is obtained by applying *LU* decomposition (*LU* decom.) to the coefficient matrix followed by forward and backward substitutions (*FB* subst.). Table 3.1 shows the computational complexity in multiplications and one can view that as $N \rightarrow \infty$ the ratio of total computational

$\text{cost}(R_{TCC})$ approaches to 2 showing that halving h will increase the total computational cost (TCC) two times. Therefore, the operational cost increases linearly with respect to N . We defined the the absolute error, Maximum absolute error and local and global error as follows:

$AE(t) = |y(t) - u_h(t)|$, $t \in I = [0, T]$ is the absolute error.

$E_2^h = \int_0^T |(y(t) - u_h(t))|^2 dt$, denotes the square L_2 norm error.

$MAE_h = \max_{t \in I_h} |y(t) - u_h(t)|$. Also, we define the local and global order of convergence by

$$R_{h, \frac{h}{2}}^* = \frac{MAE_h}{MAE_{\frac{h}{2}}}, R_{h, \frac{h}{2}} = \frac{E_2^h}{E_2^{\frac{h}{2}}}.$$

Table 3.1: Computational complexity of C-SM using LU decomposition method.

Operation	h	$h/2$
LU decom. [76]	$\frac{1}{6}m(m-1)(4m+1)$	$\frac{1}{6}m(m-1)(4m+1)$
FB subst. [76]	$Nm(2m-1)$	$2Nm(2m-1)$
Evaluating (2.35)	$(N-1)(2m+3)$	$2(N-1)(2m+3)$
TCC	$2Nm^2 + mN + 3N - 3$	$4Nm^2 + 2mN + 6N - 6$
	$+\frac{2}{3}m^3 - \frac{1}{2}m^2 - \frac{13}{6}m$	$+\frac{2}{3}m^3 - \frac{1}{2}m^2 - \frac{13}{6}m$
$R_{TCC} = \frac{TCC(h/2)}{TCC(h)}$	$1 + \frac{2Nm^2 + mN + 3N - 3}{\frac{2}{3}m^3 - \frac{1}{2}m^2 - \frac{13}{6}m + 2Nm^2 + mN + 3N - 3}$	

Example 3.1: Consider the fractional Bagley–Torvik equation

$$\frac{1}{4}D^2y(t) + \frac{2}{3}\left({}_0^C D_t^{\frac{3}{4}}y(t)\right) + y(t) = g(t), \quad t \in (0, 5),$$

subject to the boundary conditions, $y(0) = 0$, $y(5) = 125$.

where

$$g(t) = \frac{3}{2}t + \frac{2}{3}\left(\frac{\Gamma(4)}{\Gamma(\frac{17}{4})}\right)t^2 + t^3,$$

and the exact solution is $y(t) = t^3$.

In order to apply the multiple shooting method, we divide the domain $[0,5]$ into five sub-intervals, and take $\alpha = \frac{3}{4}$, and $m = 4$.

$0 = t_0 < t_1 = 1 < t_2 = 2 < t_3 = 3 < t_4 = 4 < t_5 = 5$. Consequently we split the problem into the following initial value problems.

$$\begin{aligned}\tilde{L}[\varphi_1] &= g(t), \quad 0 < t < 1, \quad \varphi_1(0:) = 0, \quad \varphi_1'(0:) = \bar{\eta}_1 \\ \tilde{L}[\varphi_2] &= g(t), \quad 1 < t < 2, \quad \varphi_2(1:) = \bar{\eta}_1, \quad \varphi_2'(1:) = \bar{\eta}_3 \\ \tilde{L}[\varphi_3] &= g(t), \quad 2 < t < 3, \quad \varphi_3(2:) = \bar{\eta}_4, \quad \varphi_3'(2:) = \bar{\eta}_5 \\ \tilde{L}[\varphi_4] &= g(t), \quad 3 < t < 4, \quad \varphi_4(3:) = \bar{\eta}_6, \quad \varphi_4'(3:) = \bar{\eta}_7 \\ \tilde{L}[\varphi_5] &= g(t), \quad 4 < t < 5, \quad \varphi_5(4:) = \bar{\eta}_8, \quad \varphi_5'(4:) = \bar{\eta}_9\end{aligned}$$

where

$$\tilde{L}[\varphi_i] = \frac{1}{4}D^2\varphi_i(t) + \frac{2}{3}\left({}_0^CD_t^\alpha\varphi_i(t)\right) + \varphi_i(t).$$

The parameter values $\bar{\eta}_i$ are determined by solving the system of algebraic equations as follows.

$$\begin{aligned}\varphi_1[1, \bar{\eta}_1] &= \varphi_1[1, \bar{\eta}_2, \bar{\eta}_3], \quad \varphi_1'[1, \bar{\eta}_1] = \varphi_2'[1, \bar{\eta}_2, \bar{\eta}_3] \\ \varphi_2[2, \bar{\eta}_2, \bar{\eta}_3] &= \varphi_3[2, \bar{\eta}_4, \bar{\eta}_5], \quad \varphi_2'[2, \bar{\eta}_2, \bar{\eta}_3] = \varphi_3'[3, \bar{\eta}_4, \bar{\eta}_5] \\ \varphi_3[3, \bar{\eta}_4, \bar{\eta}_5] &= \varphi_4[3, \bar{\eta}_6, \bar{\eta}_7], \quad \varphi_3'[3, \bar{\eta}_4, \bar{\eta}_5] = \varphi_4'[4, \bar{\eta}_6, \bar{\eta}_7] \\ \varphi_4[4, \bar{\eta}_6, \bar{\eta}_7] &= \varphi_5[4, \bar{\eta}_8, \bar{\eta}_9], \quad \varphi_4'[4, \bar{\eta}_8, \bar{\eta}_9] = \varphi_5'[5, \bar{\eta}_8, \bar{\eta}_9] \\ \varphi_5[5, \bar{\eta}_8, \bar{\eta}_9] &= 125.\end{aligned}$$

We present the unknown parameters values as follows: $\bar{\eta}_1 = 0.0000000000000000$, $\bar{\eta}_2 = 0.99999999998554634$, $\bar{\eta}_3 = 3.00000000000359499$, $\bar{\eta}_4 = 8.0000000000032809$, $\bar{\eta}_5 =$

11.99999999997389, $\bar{\eta}_6 = 26.99999999998408$, $\bar{\eta}_7 = 26.99999999998884$, $\bar{\eta}_8 = 63.9999999999730$, $\bar{\eta}_9 = 48.00000000001016$.

Further, we show the results in Table 3.2.

Table 3.2: The exact solution $y(t)$ the approximate solution $u_h(t)$ and the absolute error AE for Example 3.1.

t	y(t)	$u_h(t)$	AE
0	0.0000000000000000	0.0000000000000000	0.0000000000000000
1	1.0000000000000000	0.999999999855463	1.4453660490687525E-11
2	8.0000000000000000	8.0000000000032809	3.2809310823722626E-12
3	27.0000000000000000	26.99999999998408	1.5916157281026244E-12
4	64.0000000000000000	63.9999999999730	2.7000623958883807E-13
5	125.0000000000000000	125.00000000000000	0.0000000000000000
E_2^h	4.699191911029271E-22		

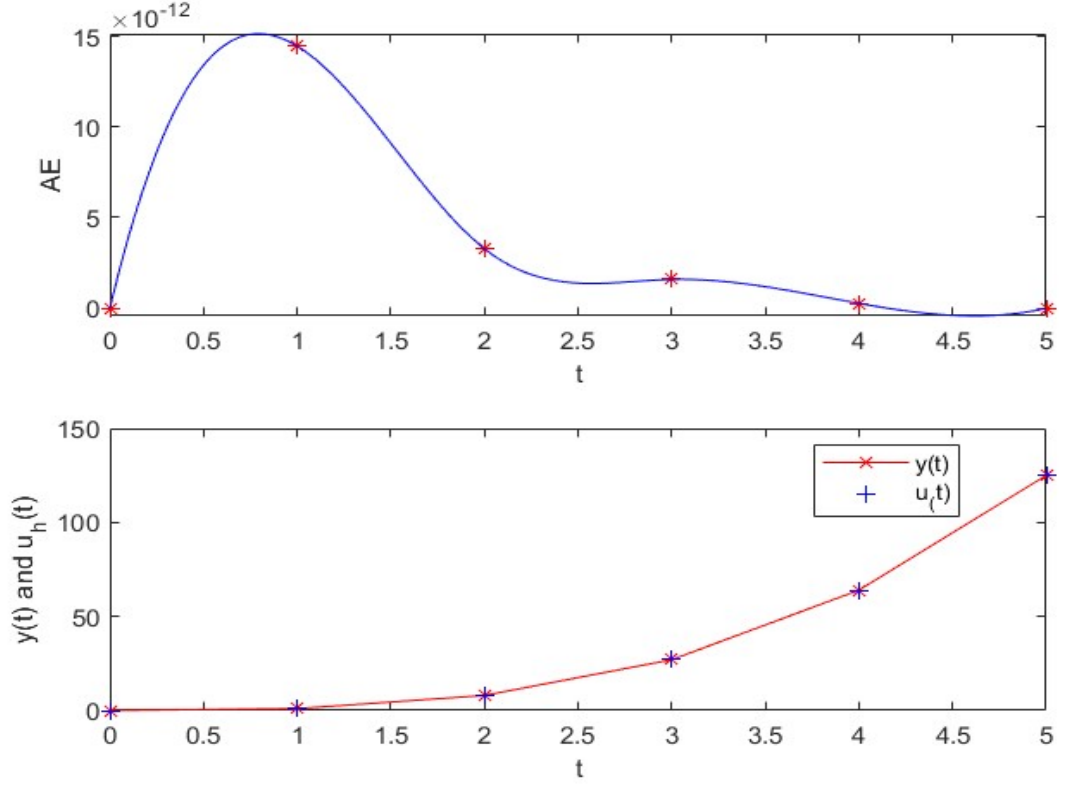


Figure 3.1: The exact solution $y(t)$ and the approximate solution $u_h(t)$ and the absolute error AE when $N = 5$ obtained by C-SM, for Example 3.1.

In Table 3.2, the exact solution is given at $t = i, i = 0, 1, 2, \dots, 5$ are on column 2, the approximate solution on column 3 and the absolute errors at $t = i, i = 0, 1, 2, \dots, 5$ are shown on column 4. Also the E_2^h norm error is given on the last row of the Table 3.2. We view from Table 3.2 that, the C-SM gives a good approximation of the exact solution. Furthermore, Figure 3.1 shows the graph of the absolute error (AE), the exact and approximate solutions. We also see from the graph of Figure 3.1 that, the method approximates the exact solution almost exact with minimal errors.

Example 3.2: Consider the fractional Bagley-Torvik equation [77]

$$D^2 y(t) + \left({}^C_0 D_t^{1/4} y(t) \right) + y(t) = f(t), \quad t \in (0, 1),$$

subject to

$$y(0) = 0, \quad y(1) = 1 + \sin(1),$$

where the exact solution is

$$y(t) = t^{5/2} + \sin(t),$$

and

$$\begin{aligned} f(t) = & \frac{15}{4}t^{1/2} + \frac{8\sqrt{\pi}}{3\Gamma(\frac{1}{4})}t^{9/4} + t^{5/2} + \frac{1}{3}\frac{4}{\Gamma(1/4)}t^{3/4}\cos(t){}_2F_1\left(\frac{1}{2}, \frac{11}{8}, \frac{3}{8}; t^{1/2}\right) \\ & + t^{7/4}\sin(2t)\Gamma\left(\frac{t}{\pi} + \frac{1}{2}\right)\Gamma\left(\frac{t}{\pi} - \frac{1}{2}\right){}_2F_1\left(\frac{7}{8}, \frac{3}{2}, \frac{15}{8}; t^{1/2}\right)\frac{1}{14\pi}. \end{aligned}$$

We take $h = 1/N$ and $N = 5, 7, 10$. Also we take collocation points $c_k = \frac{x_k+1}{2}$, $x_k = 1 - \cos\left(\frac{k\pi}{2m}\right)$, $k = 1, \dots, m$. Table 3.3 presents the $\bar{\eta}_j, j = 0, \dots, 2N - 1$ and the Table 3.4 shows the absolute error at the mesh points when $m = 3$. Furthermore, Figure 3.2 shows the exact solution $y(t)$ and the approximate solution $u_h(t)$ for $N = 10$ and the absolute error (AE) function for $N = 5, 7, 10$ on $[0, 1]$, when $m = 3$.

Table 3.3: Parameters vales for $N = 5, 7$ and 10 obtained by C-SM, for Example 3.2.

$N = 5$ Parameters($\bar{\eta}_i$)	$N = 7$ Parameters ($\bar{\eta}_i$)	$N = 10$ Parameters($\bar{\eta}_i$)
$\bar{\eta}_0 = 0.00000000$	$\bar{\eta}_0 = 0.00000000$	$\bar{\eta}_0 = 0.00000000$
$\bar{\eta}_1 = 0.990619713$	$\bar{\eta}_1 = 0.99430566$	$\bar{\eta}_1 = 0.99664976$
$\bar{\eta}_2 = 0.216256215$	$\bar{\eta}_2 = 0.14994741$	$\bar{\eta}_2 = 0.10293691$
$\bar{\eta}_3 = 1.203179869$	$\bar{\eta}_3 = 1.12442924$	$\bar{\eta}_3 = 1.07381452$
$\bar{\eta}_4 = 0.490314607$	$\bar{\eta}_4 = 0.32532791$	$\bar{\eta}_4 = 0.21649064$
$\bar{\eta}_5 = 1.553678551$	$\bar{\eta}_5 = 1.34124808$	$\bar{\eta}_5 = 1.20362147$
$\bar{\eta}_6 = 0.843271983$	$\bar{\eta}_6 = 0.53567753$	$\bar{\eta}_6 = 0.34474814$
$\bar{\eta}_7 = 1.987656445$	$\bar{\eta}_7 = 1.61109705$	$\bar{\eta}_7 = 1.36614474$
$\bar{\eta}_8 = 1.289669125$	$\bar{\eta}_8 = 0.78755662$	$\bar{\eta}_8 = 0.49054852$
$\bar{\eta}_9 = 2.486118468$	$\bar{\eta}_9 = 1.92122674$	$\bar{\eta}_9 = 1.55357045$
	$\bar{\eta}_{10} = 1.08620076$	$\bar{\eta}_{10} = 0.65614667$
	$\bar{\eta}_{11} = 2.26501157$	$\bar{\eta}_{11} = 1.76154490$
	$\bar{\eta}_{12} = 1.43612947$	$\bar{\eta}_{12} = 0.84345081$
	$\bar{\eta}_{13} = 2.63877256$	$\bar{\eta}_{13} = 1.987327151$
		$\bar{\eta}_{14} = 1.054145152$
		$\bar{\eta}_{15} = 2.229106256$
		$\bar{\eta}_{16} = 1.289765041$
		$\bar{\eta}_{17} = 2.485678364$
		$\bar{\eta}_{18} = 1.551748027$
		$\bar{\eta}_{19} = 2.756269183$

Table 3.4: The Absolute error AE when $N = 5, 7$ and 10 and the square L_2 norm error obtained by C-SM, for Example 3.2 .

$t,$	$N = 5, AE$	t	$N = 7, AE$	t	$N = 10, AE$
0	0.0000	0	0.0000	0	0.00000
1/5	3.0166E-04	1/7	1.3788E-04	1/10	1.1954E-05
2/5	2.9662E-04	2/7	1.4943E-04	2/10	1.4391E-05
3/5	2.2529E-04	3/7	1.3683E-04	3/10	1.4689E-05
4/5	1.2036E-04	4/7	1.1153E-04	4/10	1.3909E-05
1	0.0000	5/7	7.8288E-05	5/10	1.2433E-05
		6/7	4.0255E-05	6/10	1.0464E-05
		1	0.0000	7/10	8.1359E-06
				8/10	5.5560E-06
				9/10	2.8163E-06
				1	0.0000
E_2^h	5.1389E-08		1.18035E-08		2.4099E-09

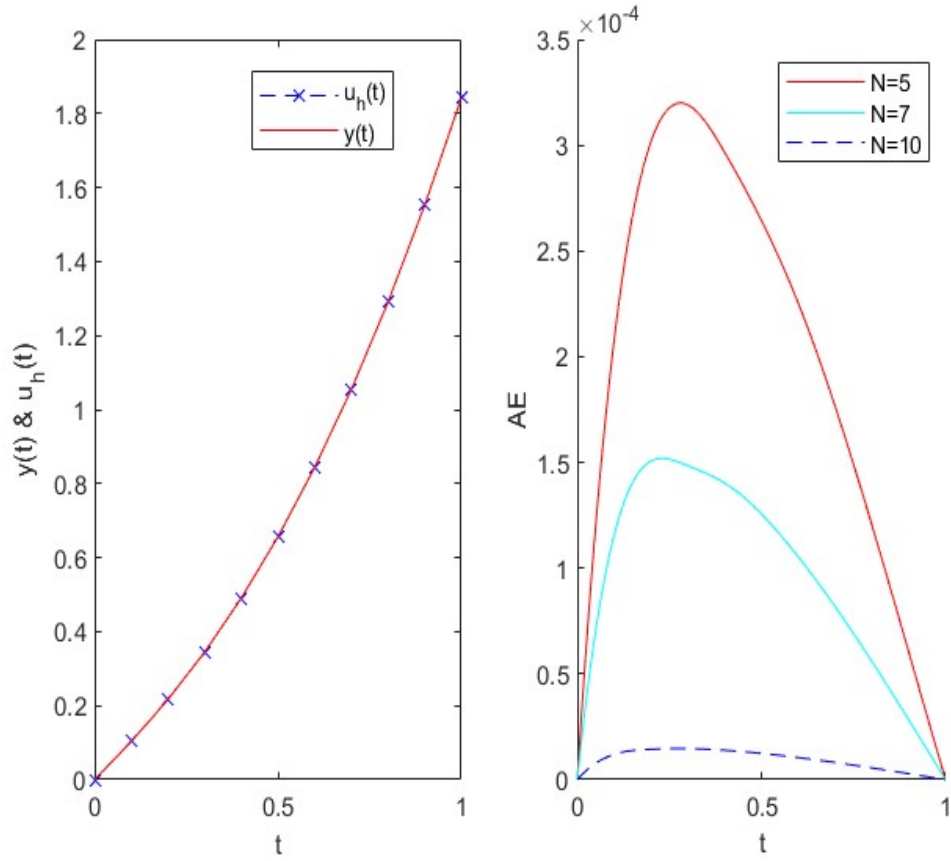


Figure 3.2: The exact solution $y(t)$ and the approximate solution $u_h(t)$ for $N = 10$ and the absolute error AE when $N = 5, 7, 10$ obtained by C-SM, for Example (3.2).

We see from the graph of the Figure 3.2 that, the collocation– shooting method gives a good approximation of the exact solution. Also the graph shows that the absolute error decreases as N increases, this result is consistent with the the theoretical results given in Chapter 2

Example 3.3: Let $\varpi \geq 6$ be an integer and $\theta > 0$ be real number that we name them as the regularity parameter and the stress factor for the following BVP of generalized BgT equation

$$D^2y(t) + \theta \left({}^C D_t^{\frac{7}{4}} y(t) \right) + y(t) = f(t), \quad t \in (0, 1)$$

$$y(0) = 0, \quad y(1) = 2,$$

where the in-homogeneous term and exact solution are

$$f(t) = \frac{\varpi}{3} \left(\frac{\varpi}{3} - 1 \right) t^{\frac{\varpi}{3}-2} + 6t + t^{\frac{\varpi}{3}} + t^3 + \theta \left(\frac{\Gamma(1 + \frac{\varpi}{3})}{\Gamma(-\frac{3}{4} + \frac{\varpi}{3})} \right) t^{-\frac{7}{4} + \frac{\varpi}{3}} + \frac{96\theta t^{\frac{5}{4}}}{5\Gamma(\frac{1}{4})},$$

$$y(t) = t^{\frac{\varpi}{3}} + t^3.$$

This example is constructed to show and analyze the behavior of the method under variations in the regularity of the solution and under variations in the stress constant. In Figure 3.3 for Example 3.3 when $m = 4$ by fixing the regularity parameter $\varpi = 7$, and the stress factor $\theta = 1$ the exact and approximate solutions are drawn for $N = 10$ and the absolute error function is illustrated for $N = 10, 20, 40, 80$. Figure 3.4 illustrates the order of convergence with respect to change in the variable θ when regularity parameter is taken large as $\varpi = 19$ and $m = 3, 4$ for Example 3.3. Further, Figure 3.5 illustrates the order of convergence with respect to change in the regularity parameter ϖ when stress factor is taken as $\theta = 1$ and $m = 3, 4$ for Example 3.3. We remark that when the exact solution is a polynomial of degree not exceeding 4 we get almost exact solution ($MAE_h \approx E - 16$) for both $m = 3$ and $m = 4$, also when $\varpi/3 = 5$ (5-th degree polynomial) the approximate solution is almost exact for $m = 4$. Therefore, in that case increasing the value of N does not improve the error much unless the precision is also increased.

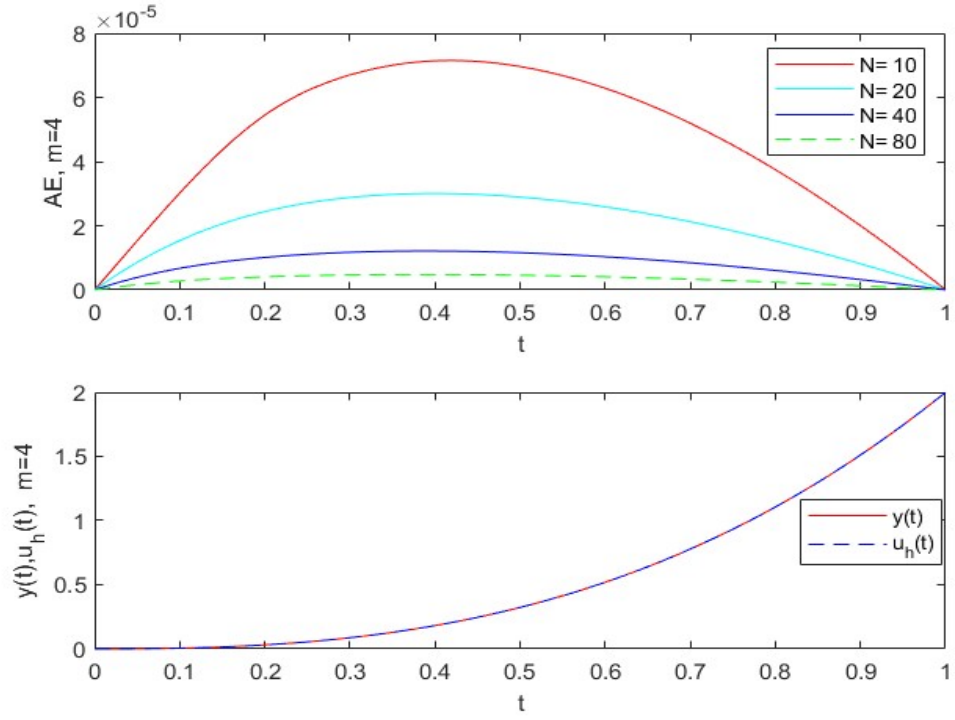


Figure 3.3: The functions $y(t)$, $u_h(t)$ for $N = 10$ and $AE(t)$ for various N , obtained for Example 3.3 when $m = 4$, $\varpi = 7$ and $\theta = 1$.

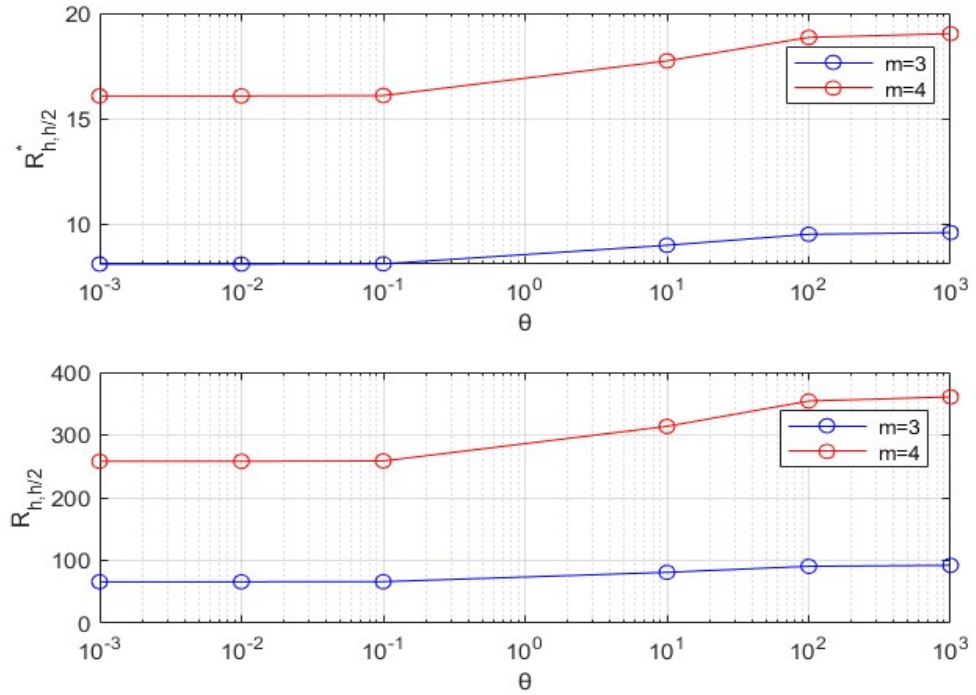


Figure 3.4: The order of convergence with respect to θ when $\varpi = 19$ and $m = 3, 4$ for Example 3.3.

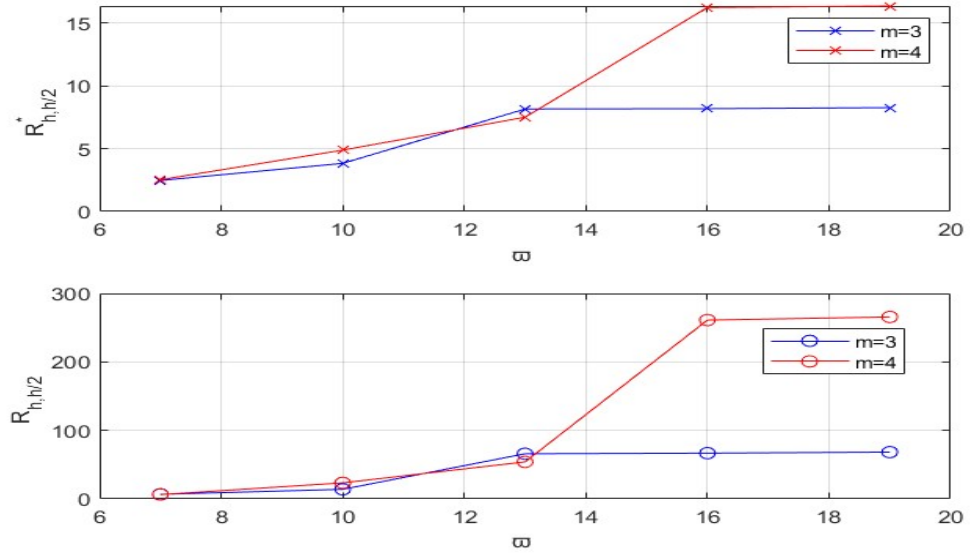


Figure 3.5: The order of convergence when $\theta = 1$, $m = 3, 4$ with respect to w for Example 3.3.

Table 3.5: The MAE_h and E_2^h errors and their ratios when $\varpi = 19$ and $m = 3, 4$ for Example 3.3.

θ	N	MAE_h	$R_{h, \frac{h}{2}}^*$	E_2^h	$R_{h, \frac{h}{2}}$	MAE_h	$R_{h, \frac{h}{2}}^*$	E_2^h	$R_{h, \frac{h}{2}}$
		$m = 3$		$m = 3$		$m = 4$		$m = 4$	
1000	10	2.57E-04		3.35E-08		5.60E-06		1.63E-11	
	20	2.60E-05	9.87	3.47E-10	96.60	2.93E-07	19.10	4.55E-14	359.29
	40	2.69E-06	9.67	3.70E-12	93.79	1.53E-08	19.05	1.25E-16	361.62
	80	2.80E-07	9.60	4.02E-14	92.19	8.09E-10	19.03	3.49E-19	360.71
10	10	3.18E-04		5.14E-08		7.28E-06		2.76E-11	
	20	3.38E-05	9.39	5.87E-10	87.67	4.02E-07	18.09	8.56E-14	323.08
	40	3.69E-06	9.14	7.01E-12	83.74	2.24E-08	17.91	2.67E-16	319.71
	80	4.11E-07	8.99	8.65E-14	81.02	1.26E-09	17.74	8.53E-19	313.77
0.1	10	9.01E-04		4.14E-07		2.56E-05		3.46E-10	
	20	1.06E-04	8.44	5.81E-09	71.22	1.57E-06	16.31	1.30E-12	264.70
	40	1.29E-05	8.23	8.57E-11	67.84	9.76E-08	16.13	5.00E-15	261.36
	80	1.59E-06	8.11	1.29E-12	66.10	6.06E-09	16.09	1.93E-17	258.60
0.001	10	9.61E-04		4.71E-07		2.77E-05		4.04E-10	
	20	1.13E-04	8.43	6.64E-09	70.98	1.70E-06	16.28	1.53E-12	264.15
	40	1.38E-05	8.21	9.82E-11	67.60	1.05E-07	16.10	5.86E-15	260.68
	80	1.71E-06	8.10	1.49E-12	65.88	6.57E-09	16.07	2.27E-17	257.95

Table 3.5 presents the MAE_h , E_2^h errors and their ratios when $m = 3, 4$ and $\varpi = 19$ for various values of θ , obtained by C-SM for Example 3.3. Subsequently, we see from Table 3.5 and Figure 3.4 that convergence order is not affected negatively for large values of stress factor θ that scales the fractional term in the differential equation. In addition, Table 3.6 shows the MAE_h and E_2^h errors and their ratios when $\theta = 1$ and

$m = 3, 4$ for various values of the regularity parameter ϖ achieved for Example 3.3. Furthermore, columns 4,6 and columns 8,10 of Table 3.5 and Table 3.6 demonstrate the local convergence order $R_{h,\frac{h}{2}}^*$ and the global convergence order $R_{h,\frac{h}{2}}$ for $m = 3$ and $m = 4$, respectively. These data justify the theoretical results given in Theorem 4.2 that when $y \in C^{m+2}(I)$, the method gives $R_{h,\frac{h}{2}} \approx 2^3$ showing $\mathcal{O}(h^3)$ accuracy for $m = 3$ and $R_{h,\frac{h}{2}} \approx 2^4$ presenting $\mathcal{O}(h^4)$ for $m = 4$. Furthermore, numerical results show that for weak regularity $\varpi = 7$, the order of convergence behaves at least linearly.

Table 3.6: The MAE_h and E_2^h and their ratios when $\theta = 1$ and $m = 3, 4$ for Example 3.3.

ϖ	N	MAE_h	$R_{h, \frac{h}{2}}^*$	E_2^h	$R_{h, \frac{h}{2}}$	MAE_h	$R_{h, \frac{h}{2}}^*$	E_2^h	$R_{h, \frac{h}{2}}$
		$m = 3$		$m = 3$		$m = 4$		$m = 4$	
7	10	9.69E-05		4.88E-09		7.13E-05		2.67E-09	
	20	4.51E-05	2.15	1.07E-09	4.56	3.00E-05	2.38	4.88E-10	5.47
	40	1.91E-05	2.36	1.95E-10	5.49	1.21E-05	2.48	7.91E-11	6.17
	80	7.75E-06	2.46	2.89E-11	6.72	4.79E-06	2.52	1.25E-11	6.30
10	10	3.86E-06		3.27E-12		2.98E-06		5.93E-12	
	20	8.31E-07	4.64	3.24E-13	10.09	7.08E-07	4.22	2.66E-13	22.43
	40	2.79E-07	2.97	3.88E-14	8.34	1.52E-07	4.65	1.23E-14	21.51
	80	7.31E-08	3.82	2.78E-15	13.98	3.12E-08	4.87	5.26E-16	23.42
13	10	2.19E-05		2.56E-10		1.88E-07		7.74E-15	
	20	2.76E-06	7.93	4.11E-12	62.30	1.96E-08	9.58	1.77E-16	43.83
	40	3.42E-07	8.08	6.33E-14	64.95	3.41E-09	5.77	5.73E-18	30.79
	80	4.19E-08	8.14	9.59E-16	65.90	4.54E-10	7.52	1.06E-19	54.03
16	10	2.05E-04		2.19E-08		1.43E-06		1.09E-12	
	20	2.47E-05	8.31	3.06E-10	71.49	9.14E-08	15.6	4.49E-15	243.73
	40	2.99E-06	8.23	4.74E-12	64.59	5.69E-09	16.04	1.75E-17	256.17
	80	3.66E-07	8.17	7.09E-14	66.76	3.51E-10	16.22	6.72E-20	261.06
19	10	6.13E-04		1.92E-07		1.61E-05		1.36E-10	
	20	7.11E-05	8.61	2.59E-09	73.95	9.75E-07	16.57	5.02E-13	272.75
	40	8.49E-06	8.38	3.69E-11	70.36	5.94E-08	16.41	1.86E-15	269.28
	80	1.02E-06	8.25	5.41E-13	68.30	3.63E-09	16.33	7.02E-18	265.63

Example 3.4: Let us consider the following BVP of generalized BgT equation considered in [7], [6], [4], [8], [9]

$$D^2u(t) + \frac{8}{17} \left({}^C D_t^\mu u(t) \right) + \frac{13}{51} u(t) = \frac{t^{-\frac{1}{2}}}{89250\sqrt{\pi}} (48f_1(t) + 7\sqrt{t}f_2(t)), \quad t \in (0, 1)$$

$$u(0) = u(1) = 0,$$

where

$$f_1(t) = 16000t^4 - 32480t^3 + 21280t^2 - 4746t + 189,$$

$$f_2(t) = 3250t^5 - 9425t^4 + 264880t^3 - 448107t^2 + 233262t - 34578,$$

$$u(t) = t^5 - \frac{29}{10}t^4 + \frac{76}{25}t^3 - \frac{339}{250}t^2 + \frac{27}{125}t.$$

Table 3.7 shows the absolute errors at grid points $t_i = i/10, i = 1, 2, \dots, 9$ and the E_2^h errors when $\mu = 1.5$ and $m = 3, 4$ for Example 3.4 obtained by C-SM. This table also compares the absolute errors when $\mu = 1.5$ with the methods BCM [7], DCSWSGD [6], QFEM [4], GBPM [8] and HWM [9]. The errors presented by BCM was obtained by using truncated Bessel series with 8 functions. Also, DCSWSGD were given for 10 grid points and the errors by HWM were obtained by using $m^* = 8$, where m^* is the number of Haar Wavelets. From Table 3.7 we conclude that our results for $m = 3$ are at least as good as the results obtained by BCM [7], DCSWSGD [6], QFEM [4], and HWM [9]. Moreover, columns 3,4 of Table 3.7 show that when $m = 4$ the proposed method gives almost exact solution in double precision. Since the 6-th order derivative of the exact solution vanishes this conclusion is not surprising and is verified by the Theorem 4.2. Furthermore, when $m = 4$ the obtained errors behave like those given by GBPM [8].

However, the advantage of the given method is that the coefficient matrix on each

subinterval is same and size does not depend on N . We present the computational efficiency in Table 3.8 by displaying the CPUs, the determinant of the coefficient matrix ($\det(V)$) and the Euclidean norm of the matrix $\|V^{-1}U_2\|_2$ given in (2.31). It can be viewed from this table that computational cost increases linearly. Finally, Figure 3.6 illustrates the functions $y(t)$, $u_h(t)$ for $N = 20$ and $AE(t)$ for $N = 15, 35, 45$ when $m = 4$.

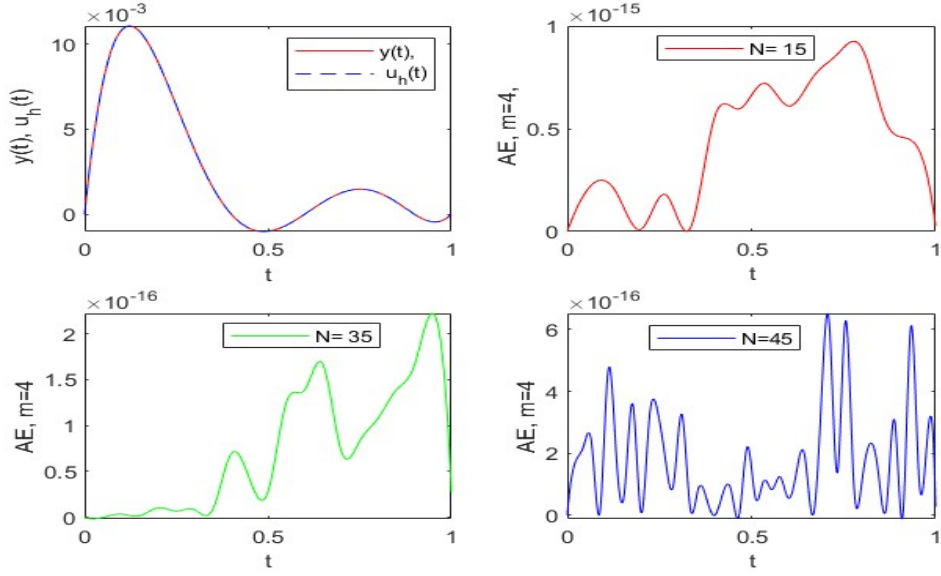


Figure 3.6: The functions $y(t)$, $u_h(t)$ and $AE(t)$ for $N = 15, 35, 45$ by C-SM for Example 3.4.

Table 3.8: The computational efficiency of the given method for Example 3.4.

N	CPU _s	CPU _s	$\det(V)$	$\det(V)$	$\ V^{-1}U_2\ _2$	$\ V^{-1}U_2\ _2$
	$m = 3$	$m = 4$	$m = 3$	$m = 4$	$m = 3$	$m = 4$
10	1.015	1.654	2.8559	0.4285	0.0051	0.0051
20	1.914	3.199	2.8538	0.4283	0.0013	0.0013
40	3.959	6.366	2.8532	0.4282	3.422E-4	3.416E-4
80	7.873	12.659	2.8529	0.4282	9.226E-5	9.439E-5

Example 3.5: Consider the BVP of the generalized BgT equation in [1], [2], [3], [4], [5]

$$D^2u(t) + \theta \left({}^C_0D_t^\mu u(t) \right) + \gamma u(t) = f(t), \quad t \in (0, 1),$$

$$u(0) = u(1) = 0,$$

where the non-homogeneous term and the exact solution are

$$f(t) = (\alpha - 1)(\alpha t - \alpha + 2)t^{\alpha-3} + \theta \frac{(\alpha - 1)!}{\Gamma(\alpha - \mu)} \left(\frac{\alpha t}{\alpha - \mu} - 1 \right) t^{\alpha-\mu-1} + \gamma t^{\alpha-1}(t - 1),$$

$$u(t) = t^{\alpha-1}(t - 1),$$

Table 3.9 shows the AE and E_2^h achieved by C-SM for $m = 3, 4$ and the AE obtained by SSM [1], CSM [2], NGLDM [3], QFEM [4], and GM [5] when $(\mu, \theta, \gamma, \alpha) = (0.3, 0.5, 1, 5)$, respectively for Example 3.5. The results presented in column 3 and column 4 of Table 3.9 demonstrate that for $m = 4$ the proposed method gives more accurate results than the methods SSM [1], CSM [2], NGLDM [3], and QFEM [4]. Moreover, the obtained errors for $m = 4, N = 8$ behave like those given by GM [5] which is a semi-analytic method. Additionally, Figure 3.7 illustrates the exact solution $y(t)$ and the approximate solution $u_h(t)$ for $N = 8$ and $AE(t)$ when $N = 8, 16, 32$ for Example 3.5 obtained by C-SM with $m = 3$.

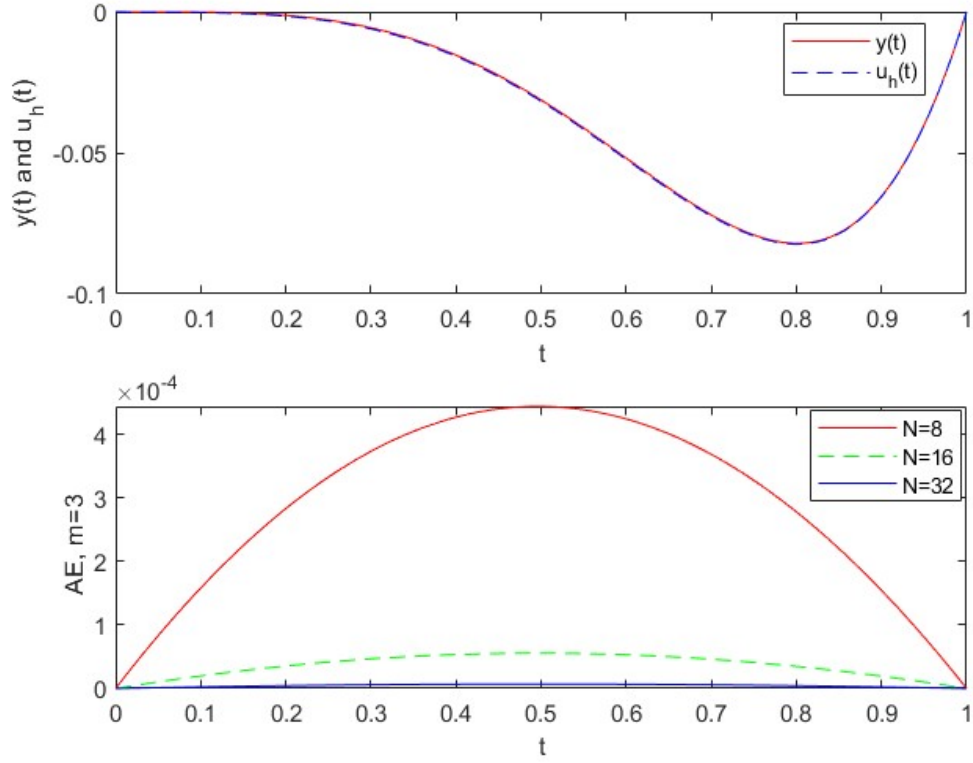


Figure 3.7: The functions $y(t)$, $u_h(t)$ when $N = 8$ and the $AE(t)$ when $N = 8, 16, 32$ for Example 3.5 obtained by C-SM with $m = 3$.

Example 3.6: Let $0 < \mu < 1$ and $\theta > 0$, $\eta \geq 0$ be real numbers. We setup the following BVP of generalized BgT equation.

$$D^2 u(t) + \theta \left({}^C_0 D_t^\mu u(t) \right) = f_1(t) + \theta t^{1-\mu} f_2(t) + f_3(t), \quad t \in (0, 1),$$

$$u(0) = 0, \quad u(1) = \eta \sin(1),$$

where

$$f_1(t) = 20t^3 - \frac{174}{5}t^2 + \frac{456}{25}t - \frac{339}{125},$$

$$f_2(t) = \frac{120}{\Gamma(6-\mu)}t^4 - \frac{348}{5\Gamma(5-\mu)}t^3 + \frac{456}{25\Gamma(4-\mu)}t^2 - \frac{339}{125\Gamma(3-\mu)}t + \frac{27}{125\Gamma(2-\mu)},$$

$$f_3(t) = \frac{\eta \theta t^{1-\mu} {}_2F_1 \left[\{1\}, \left\{1 - \frac{\mu}{2}, \frac{3}{2} - \frac{\mu}{2}\right\}, -\frac{t^2}{4} \right]}{(1-\mu)\Gamma(1-\mu)} - \eta \sin(t),$$

and the HypergeometricPFQ $[\{a_1, a_2, \dots, a_p\}, \{b_1, b_2, \dots, b_q\}, z]$ is the generalized hypergeometric function ${}_pF_q(a; b; z)$ and the exact solution is

$$u(t) = t^5 - \frac{29}{10}t^4 + \frac{76}{25}t^3 - \frac{339}{250}t^2 + \frac{27}{125}t + \eta \sin(t).$$

For $\eta = 0$ the resulting problem was considered in [3], [6], [4]. Hence, for validating and comparing the given method, we take the set of values $(N, \mu, \theta, \eta) = (10, 0.5, 1, 0)$ and present the AE, E_2^h obtained by C-SM and the AE by the methods NGLDM [3], DCSWSGD [6], and QFEM[4] in Table 3.10. We conclude from Table 3.10 that C-SM with $m = 4$ provides almost exact solution giving higher accuracy when compared with NGLDM [3], DCSWSGD [6], and QFEM [4]. Furthermore, for Example 3.6 we analyze the behavior of the given method in Figure 3.8 and Figure 3.9 by demonstrating the $AE(t)$ when $m = 3$ and $m = 4$, respectively for fixed $N = 16$ and $\eta = 1$, $\theta = 2$ and for various values of μ taken as $\mu = 0.1, 0.3, 0.5, 0.8$. Finally Table 3.11 shows the order of convergence of the given method for Example 3.6 when $\eta = 1$, $\theta = 2$ and $\mu = 0.9$.

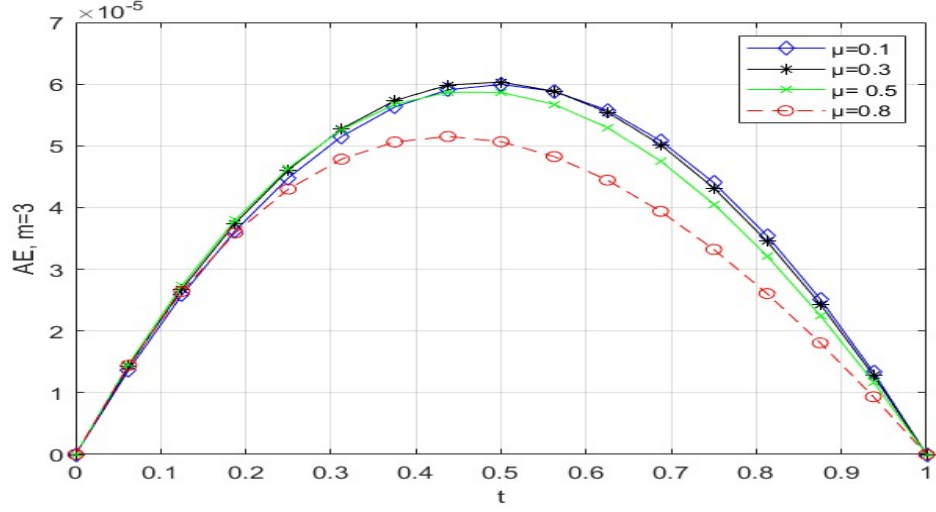


Figure 3.8: The function $AE(t)$ for $N = 16$, $\eta = 1$, $\theta = 2$ and $m = 3$ for Example 3.6 obtained by C-SM.

Table 3.10: The error comparisons of C-SM, NGLDM [3], DCSWSGD [6] and QFEM [4] for Example 3.6.

t	C-SM $m = 3$	C-SM $m = 4$	NGLDM [3] Table 4	DCSWSGD [6] Table 4.5	QFEM [4] Table 3
0.1	7.21E-04	0	2.00E-03	1.05E-05	1.40E-04
0.2	7.23E-04	9.99E-16	1.74E-03	3.21E-05	2.33E-05
0.3	6.33E-04	1.00E-15	1.53E-03	6.79E-05	1.60E-04
0.4	5.08E-04	1.19E-15	1.32E-03	7.17E-05	2.71E-04
0.5	3.91E-04	1.99E-15	1.11E-03	6.19E-05	2.80E-04
0.6	2.93E-04	1.57E-15	8.92E-04	4.75E-05	2.25E-04
0.7	2.10E-04	1.90E-15	6.53E-04	3.21E-05	1.62E-04
0.8	1.31E-04	8.99E-16	4.19E-04	1.81E-05	1.20E-04
0.9	5.71E-05	4.82E-16	2.02E-04	7.01E-06	7.75E-05
E_2^h	1.37E-08	5.32E-31			

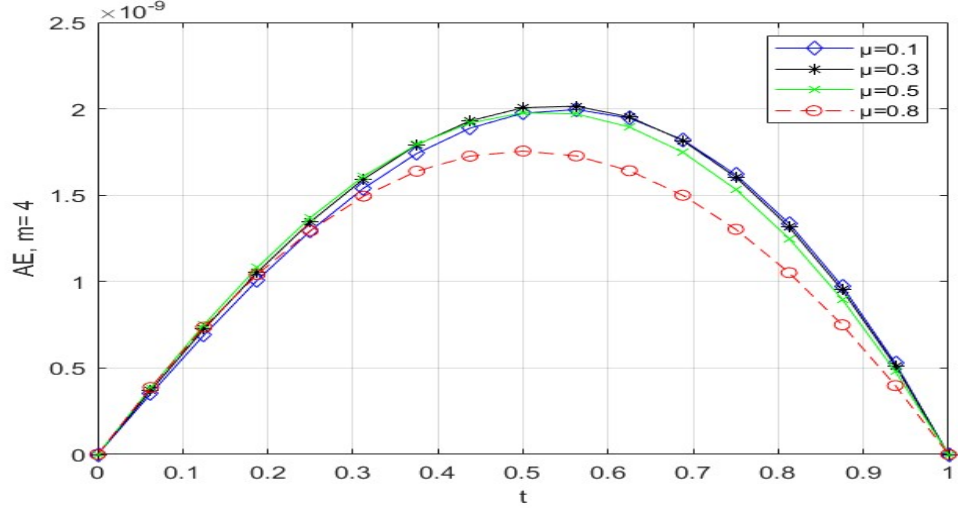


Figure 3.9: The function $AE(t)$ for $N = 16$, $\eta = 1$, $\theta = 2$ and $m = 4$ for Example 3.6 obtained by C-SM.

Table 3.11: The order of convergence of C-SM for Example 3.6 when $\eta = 1$, $\theta = 2$ and $\mu = 0.9$.

N	MAE_h	$R_{h, \frac{h}{2}}^*$	E_2^h	$R_{h, \frac{h}{2}}$	MAE_h	$R_{h, \frac{h}{2}}^*$	E_2^h	$R_{h, \frac{h}{2}}$
	$m = 3$		$m = 3$		$m = 4$		$m = 4$	
4	2.85E-03		4.76E-06		4.60E-07		1.07E-13	
8	3.72E-04	7.65	7.55E-08	63.09	2.71E-08	16.96	3.84E-16	277.3
16	4.78E-05	7.78	1.19E-09	63.29	1.64E-09	16.58	1.44E-18	277.0
32	6.01E-06	7.94	1.88E-11	63.32	1.00E-10	16.35	5.28E-21	272.4

Example 3.7: Consider the mathematical model of a Micro Electro Mechanical System (MEMS) instrument in [1] and [78]

$$D^2u(t) + \theta\sqrt{\pi}D^{\frac{3}{2}}u(t) + u(t) = 0, \quad t > 0$$

$$u(0) = 1, \quad u'(0) = 0,$$

and the exact solution is

$$u(t) = 1 - \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r (-\theta\sqrt{\pi})^j (j+r)! t^{2+2r+j/2}}{j!r!(2+2r+j/2)\Gamma(2+2r+j/2)}$$

The problem is an IVP of a real life application of (1.1) to oil industry. The analytic solution of the problem was given by [78] and a numerical solution using spline combine with shooting method (SSM) was provided in [1]. The authors of [78] state that “two extreme situations occur for the value of $\theta = 0.005$ (where the oscillations decrease so slowly in magnitude and the decade would be hard to measure) and $\theta = 1$ (where the decay is so rapid that not enough data would be collected)”. We take $t \in [0, 1]$ and Table 3.12 shows the analytic and approximate solutions for $\theta = 1, \theta = 0.2$ and $\theta = 0.005$ obtained by using C-SM when $N = 8, m = 4$ and by SSM [1]. Further, the approximate solution by C-SM is presented in Figure 3.10 while the absolute error AE function is illustrated in Figure 3.11 both as a surface plot in two parameters t and the decay coefficient θ when $m = 4$ and $N = 10$. In conclusion, Table 3.12 presents that C-SM gives better approximation to the problem when compared with the results obtained by SSM.

Table 3.12: Exact and approximate solutions for Example 3.7 obtained by using the C-MS and SSM methods.

$\theta = 1$		$\theta = 0.2$				$\theta = 0.005$			
t	Analy. Sol.	SSM [1]	C-SM	Analy. Sol.	SSM [1]	C-SM	Analy. Sol.	SSM [1]	C-SM
0.000	1	1	1	1	1	1	1	1	1
0.125	0.994370	0.993126	0.994475	0.992747	0.992391	0.992776	0.992212	0.992212	0.992213
0.250	0.979919	0.974802	0.980111	0.971922	0.970148	0.971980	0.968995	0.968983	0.968997
0.375	0.958424	0.944545	0.958689	0.938558	0.9333609	0.938644	0.930733	0.930674	0.930735
0.500	0.930957	0.904813	0.931284	0.893615	0.883958	0.893726	0.878038	0.877899	0.878041
0.625	0.898335	0.857938	0.898716	0.838087	0.822499	0.838221	0.811743	0.811497	0.811748
0.750	0.861241	0.805442	0.861669	0.773025	0.750552	0.773179	0.732892	0.732514	0.732897
0.875	0.820277	0.748795	0.820745	0.699540	0.669584	0.699713	0.642719	0.642193	0.642725
1.000	0.775989	0.688838	0.776492	0.618798	0.580978	0.618985	0.542633	0.541945	0.542639

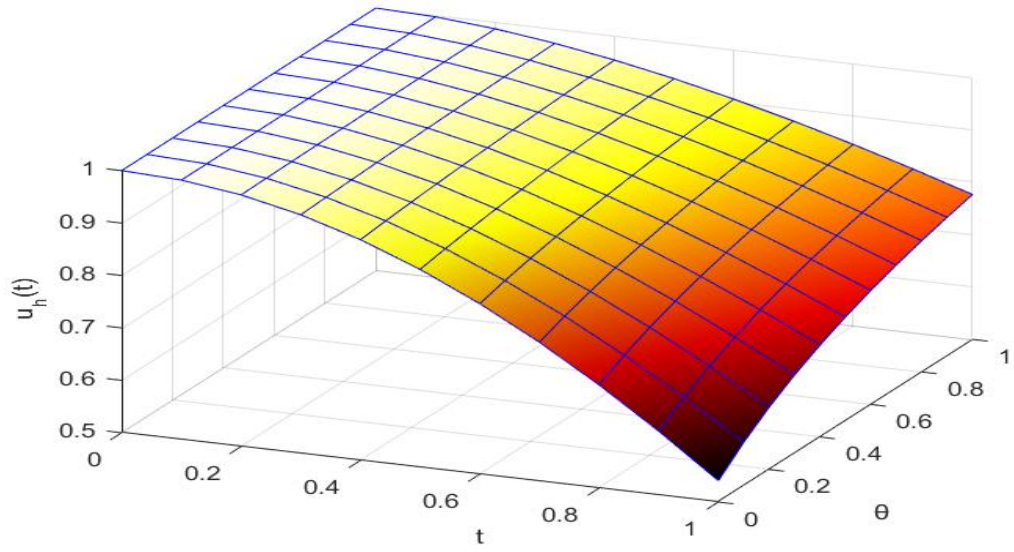


Figure 3.10: The approximate solution for Example 3.7 when $m = 4$ and $N = 10$ obtained by C-SM.

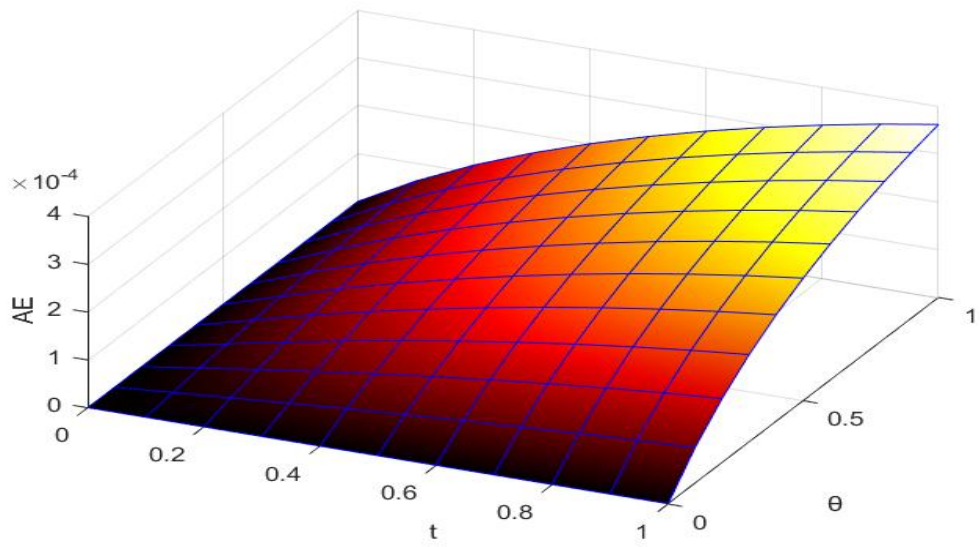


Figure 3.11: The AE for Example 3.7 when $m = 4$ and $N = 10$ obtained by C-SM.

Chapter 4

ON THE EXISTENCE, UNIQUENESS AND A NUMERICAL APPROACH TO THE SOLUTION OF FRACTIONAL CAUCHY-EULER EQUATION

The theoretical results given in this Chapter are published in [66].

4.1 Existence and Uniqueness of Solution

In order to investigate the existence and uniqueness of solution of Equation (1.4), we define a max metric d_δ containing y^α , and prove that any two solutions of Equation (1.4) are equivalent in the metric space $(C^2[a, T], d_\delta)$. Further, we show that a solution sequence $\{y_j\}_{j=1}^\infty$ of Equation (1.4) is a Cauchy sequence in the metric space. Setting $B_1 = B_2 = B_3 = 1$, we rewrite Equation (1.4) in the form

$$\begin{aligned} y''(t) &= t^{-2}f(t) - t^{\alpha-2} {}^C D_t^\alpha y(t) - t^{-2}y(t) = h(t, y(t), {}^C D_t^\alpha y(t)), \quad 0 < \alpha < 2, \\ y(a) &= \bar{\beta}_0, \quad y'(a) = \bar{\beta}_1, \end{aligned} \tag{4.1}$$

where ${}^C D_t^\alpha y(t)$ exists and is continuous, and $f(t)$ is continuous in the interval $[a, T]$. Here, $\bar{\beta}_0$ and $\bar{\beta}_1$ are real numbers. For the same value of α , the following equations are equivalent to Equation (1.4).

Lemma 4.1: [66] Let $h(t, y(t), {}^C D_t^\alpha y(t)) = t^{-2}f(t) - t^{\alpha-2} {}^C D_t^\alpha y(t) - t^{-2}y(t)$, then the initial value problem (1.4) is equivalent to the following equations, provided that ${}^C D_t^\alpha y(t)$ exists and is continuous, and $f(t)$ is continuous in the interval $[a, T]$.

(a)

$$y(t) = \int_a^t (t-s)h(s, y(s), {}^C D_t^\alpha y(s)) ds + \bar{\beta}_1(t-a) + \bar{\beta}_0, \quad t \in [a, T], \quad (4.2)$$

(b) Moreover, for $0 < \alpha < 1$,

$$\begin{aligned} {}^C D_t^\alpha y(t) &= \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-s)^{1-\alpha} h(s, y(s), {}^C D_t^\alpha y(s)) ds \\ &\quad + \frac{\bar{\beta}_1}{\Gamma(2-\alpha)} (t-a)^{1-\alpha}, \quad t \in [a, T]. \end{aligned}$$

Proof. (a) Integrating twice Equation (4.1), the result follows as in (4.2).

(b) To prove part (b), we first differentiate both sides of Equation (4.2):

$$\begin{aligned} y'(t) &= \frac{d}{dt} \int_a^t (t-s)h(s, y(s), {}^C D_t^\alpha y(s)) ds + \bar{\beta}_1 \\ &= \int_a^t h(s, y(s), {}^C D_t^\alpha y(s)) ds + \bar{\beta}_1. \end{aligned}$$

For $0 < \alpha < 1$ and $t \in [a, T]$, by applying the Caputo derivative, we obtain

$$\begin{aligned} y^{(\alpha)}(t) &= I^{1-\alpha} y'(t) = y^{(\alpha)} \left(\int_a^t h(s, y(s), {}^C D_t^\alpha y(s)) ds + \bar{\beta}_1 \right) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \left(\int_a^s h(\eta, y(\eta), {}^C D_t^\alpha y(\eta)) d\eta + \bar{\beta}_1 \right) ds \\ y^\alpha(t) &= \frac{1}{\Gamma(1-\alpha)} \int_a^t h(\eta, y(\eta), {}^C D_t^\alpha y(\eta)) d\eta \int_\eta^t (t-s)^{-\alpha} ds \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \bar{\beta}_1 ds. \end{aligned}$$

We change the order of integration in the iterative integral to obtain

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{(t-\eta)^{1-\alpha}}{(1-\alpha)} h(\eta, y(\eta), {}^C D_t^\alpha y(\eta)) d\eta - \frac{1}{\Gamma(1-\alpha)} \frac{(t-s)^{1-\alpha}}{(1-\alpha)} \Big|_a^t \bar{\beta}_1 \\ &= \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \int_a^t (t-\eta)^{1-\alpha} h(\eta, y(\eta), {}^C D_t^\alpha y(\eta)) d\eta + \frac{\bar{\beta}_1}{(1-\alpha)\Gamma(1-\alpha)} (t-a)^{1-\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-\eta)^{1-\alpha} h(\eta, y(\eta), {}^C D_t^\alpha y(\eta)) d\eta + \frac{\bar{\beta}_1}{\Gamma(2-\alpha)} (t-a)^{1-\alpha}. \end{aligned}$$

The proof is complete. □

Let $Y := C^2[a, T]$ be a set of twice continuously differentiable functions on $[a, T]$. We

consider the metric space (Y, d_δ) coupled with the max metric

$$d_\delta := \max_{t \in [a, T]} \frac{|x(t) - y(t)|}{E_\tau(\delta t^\tau)} + \max_{t \in [a, T]} \frac{|x^{(\alpha)}(t) - y^{(\alpha)}(t)|}{E_\tau(\delta t^\tau)}, \forall x, y \in Y.$$

Theorem 4.1: [66] Assume that $0 < \alpha < 2$. Equation (4.1) has only one solution $y = y(t)$ in $(C^2[a, T], d_\delta)$ defined on the interval $[a, T]$.

Proof. It is clear that

$$\begin{aligned} |h(t, w_1, u_1) - h(t, w_2, u_2)| &\leq t^{-2}|w_1 - w_2| + t^{\alpha-2}|u_1 - u_2| \\ &\leq \frac{1}{a^2}|w_1 - w_2| + \frac{1}{a^{2-\alpha}}|u_1 - u_2|. \end{aligned}$$

Define $\lambda := \max\left\{\frac{1}{a^2}, \frac{1}{a^{2-\alpha}}\right\} \left[\frac{(T-a)^2}{2} + \frac{(T-a)^{2-\alpha}}{\Gamma(1-\alpha)}\right]$ and choose T such that $\lambda < 1$. Define an operator $\Xi y : (Y, d_\delta) \rightarrow (Y, d_\delta)$ as follows:

$$\Xi y(t) = \int_a^t (t-s)h(s, y(s), {}^C D_t^\alpha y(s)) ds + \bar{\beta}_1(t-a) + \bar{\beta}_0. \quad (4.3)$$

Then,

$$\begin{aligned} |\Xi x(t) - \Xi y(t)| &\leq \int_a^t (t-s)|h(s, x(s), x^\alpha(s)) - h(s, y(s), y^\alpha(s))| ds \\ &\leq \int_a^t (t-s) \left[\frac{1}{a^2}|x(s) - y(s)| + \frac{1}{a^{2-\alpha}}|x^{(\alpha)}(s) - y^{(\alpha)}(s)| \right] ds, \end{aligned}$$

and

$$\begin{aligned} &|\Xi x(t) - \Xi y(t)| \\ &\leq E_{2-\alpha}(\delta t^{2-\alpha}) \int_a^t \frac{1}{E_{2-\alpha}(\delta s^{2-\alpha})}(t-s) \\ &\quad \times \left[\frac{1}{a^2}|x(s) - y(s)| + \frac{1}{a^{2-\alpha}}|x^{(\alpha)}(s) - y^{(\alpha)}(s)| \right] ds \\ &\leq E_{2-\alpha}(\delta t^{2-\alpha}) \max\left\{\frac{1}{a^2}, \frac{1}{a^{2-\alpha}}\right\} d_\delta(x, y) \int_a^t (t-s) ds \\ &= E_{2-\alpha}(\delta t^{2-\alpha}) \max\left\{\frac{1}{a^2}, \frac{1}{a^{2-\alpha}}\right\} \frac{(t-a)^2}{2} d_\delta(x, y). \\ &\frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})}|x(t) - y(t)| \leq \max\left\{\frac{1}{a^2}, \frac{1}{a^{2-\alpha}}\right\} \frac{(t-a)^2}{2} d_\delta(x, y). \end{aligned} \quad (4.4)$$

Similarly,

$$\begin{aligned}
& \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} |(\Xi x)^{(\alpha)}(t) - (\Xi y)^{(\alpha)}(t)| \\
& \leq \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-s)^{1-\alpha} \frac{1}{E_{2-\alpha}(\delta s^{2-\alpha})} \\
& \quad \cdot \left| h\left(s, x(s), {}^C D_t^\alpha x(s)\right) - h\left(s, y(s), {}^C D_t^\alpha y(s)\right) \right| ds \\
& \leq \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-s)^{1-\alpha} \frac{1}{E_{2-\alpha}(\delta s^{2-\alpha})} \left[M|x(s) - y(s)| + N|x^{(\alpha)}(s) - y^{(\alpha)}(s)| \right] ds \\
& \leq \frac{1}{\Gamma(2-\alpha)} \max \left\{ \frac{1}{a^2}, \frac{1}{a^{2-\alpha}} \right\} d_\delta(x, y) \int_a^t (t-s)^{1-\alpha} ds \\
& = \frac{1}{\Gamma(2-\alpha)} \frac{(t-a)^{2-\alpha}}{2-\alpha} \max \left\{ \frac{1}{a^2}, \frac{1}{a^{2-\alpha}} \right\} d_\delta(x, y) \\
& = \frac{(t-a)^{2-\alpha}}{\Gamma(1-\alpha)} \max \left\{ \frac{1}{a^2}, \frac{1}{a^{2-\alpha}} \right\} d_\delta(x, y). \tag{4.5}
\end{aligned}$$

Taking the supremum with respect to t in (4.4) and (4.5) and adding them, we obtain

$$d_\delta(\Xi x, \Xi y) \leq \max \left\{ \frac{1}{a^2}, \frac{1}{a^{2-\alpha}} \right\} \left[\frac{(T-a)^2}{2} + \frac{(T-a)^{2-\alpha}}{\Gamma(1-\alpha)} \right] d_\delta(x, y).$$

For small $T-a$, we have

$$\max \left\{ \frac{1}{a^2}, \frac{1}{a^{2-\alpha}} \right\} \left[\frac{(T-a)^2}{2} + \frac{(T-a)^{2-\alpha}}{\Gamma(1-\alpha)} \right] < 1.$$

Thus, there exist $T_1 > 0$ such that $\max \left\{ \frac{1}{a^2}, \frac{1}{a^{2-\alpha}} \right\} \left[\frac{(T_1-a)^2}{2} + \frac{(T_1-a)^{2-\alpha}}{\Gamma(1-\alpha)} \right] < 1$, and the operator Ξ is a contraction. By the Banach fixed-point theorem, Ξ has a unique fixed point in Y and consequently the Equation (4.1) has a unique solution on $[a, T_1]$. By repeating this process multiple times, we arrive at a unique solution on $[a, T]$. $\square$

4.2 The Numerical Method and Algorithm

Let $L_h = \{t_j : t_j = a + jh, h = \frac{T-a}{N}, j = 0, 1, \dots, N\}$ be a given uniform mesh on $L = [a, T]$ and set $\delta_n = (t_n, t_{n+1}]$ and $\bar{\delta}_n = [t_n, t_{n+1}]$, for $n = 0, 1, \dots, N-1$. The solution y of the problem (1.4) will be approximated by an element $u_h \in S_{m+1}^{(1)}(L_h)$, where $S_{m+1}^{(1)}(L_h) := \{P \in C^1(L) : P|_{\bar{\delta}_n} \in \Pi_{m+1}\}$. Also, Π_{m+1} presents the space of all real polynomials of a degree not exceeding $m+1$. Let

$$\tilde{X}_h := \{t = t_n + v_i h : 0 < v_1 < \cdots < v_m \leq 1, n = 0, 1, \dots, N-1\}, \quad (4.6)$$

and $v_i, i = 1, 2, \dots, m$ being the collocation parameters, the collocation solution u_h satisfies the following collocation problem:

$$B_1 t^2 D^2 u_h(t) + B_2 t^\alpha \left({}^C D_t^\alpha u_h(t) \right) + B_3 u_h(t) = f(t), \quad t \in \tilde{X}_h \quad (4.7)$$

$$u_h(a) = \bar{\beta}_0, \quad u_h'(a) = \bar{\beta}_1. \quad (4.8)$$

Let $t = t_n + wh \in \bar{\delta}_n, w \in [0, 1]$, then on each subinterval $\bar{\delta}_n$, the collocation solution $u_h \in S_{m+1}^{(1)}(L_h)$ satisfying (4.7) and (4.8) is given by

$$u_h(t) = u_h(t_n + wh) = \sum_{\bar{q}=0}^1 a_{\bar{q}}^{(n)} w^{\bar{q}} + \sum_{k=1}^m b_k^{(n)} w^{k+1}, \quad w \in [0, 1]. \quad (4.9)$$

We present ${}^C D_t^\alpha u_h$ as follows:

$$\begin{aligned} [{}^C D_t^\alpha u_h(t)]_{t=t_n+wh} &= \left(\frac{1}{\Gamma(\tau-\alpha)} \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} (t-x)^{\tau-\alpha-1} D_x^\tau u_h(x) dx \right) \Big|_{t=t_n+wh} \\ &\quad + \left(\frac{1}{\Gamma(\tau-\alpha)} \int_{t_n}^t (t-x)^{\tau-\alpha-1} D_x^\tau u_h(x) dx \right) \Big|_{t=t_n+wh} \\ &= \sum_{p=0}^{n-1} [{}^C J_{t_{p+1}}^\alpha u_h(t)]_{t=t_n+wh} + [{}^C D_{t_n}^\alpha u_h(t)]_{t=t_n+wh}. \end{aligned} \quad (4.10)$$

Also,

$$\begin{aligned} \sum_{p=0}^{n-1} [{}^C J_{t_{p+1}}^\alpha u_h(t)]_{t=t_n+wh} &= \left(\frac{1}{\Gamma(\tau-\alpha)} \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} (t-x)^{\tau-\alpha-1} D_x^\tau u_h(x) dx \right) \Big|_{t=t_n+wh} \\ &= \frac{1}{\Gamma(\tau-\alpha)} \sum_{p=0}^{n-1} \int_0^1 \frac{h^{-\tau+1} D_s^\tau u_h(t_p + sh)}{(t_n + wh - t_p - sh)^{\alpha+1-\tau}} ds \\ &= h^{-\tau+1} \sum_{p=0}^{n-1} {}^C J_1^\alpha [u_h(t_p + sh)]. \end{aligned} \quad (4.11)$$

In a similar way, we have

$$\begin{aligned}
\left[{}^C D_t^\alpha u_h(t) \right]_{t=t_n+wh} &= \left(\frac{1}{\Gamma(\tau-\alpha)} \int_{t_n}^t \frac{D_x^\tau u_h(x)}{(t-x)^{\alpha+1-\tau}} dx \right) \Big|_{t=t_n+wh} \\
&= \frac{1}{\Gamma(\tau-\alpha)} \int_0^w \frac{h^{-\tau+1} D_s^\tau u_h(t_n+sh)}{(wh-sh)^{\alpha+1-\tau}} ds \\
&= \frac{h^{-\alpha+\tau-1} h^{-\tau+1}}{\Gamma(\tau-\alpha)} \int_0^w \frac{D_s^\tau u_h(t_n+sh)}{(w-s)^{\alpha+1-\tau}} ds \\
&= h^{-\alpha} \left({}^C D_w^\alpha [u_h(t_n+.h)](w) \right). \tag{4.12}
\end{aligned}$$

By Equations (4.11) and (4.12), we write Equation (4.10) as follows:

$$\begin{aligned}
\left[{}^C D_t^\alpha u_h(t) \right]_{t=t_n+wh} &= h^{-\tau+1} \sum_{p=0}^{n-1} {}_0 J_1^\alpha [u_h(t_p+sh)] \\
&\quad + h^{-\alpha} \left({}^C D_w^\alpha [u_h(t_n+.h)](w) \right). \tag{4.13}
\end{aligned}$$

We mention that the papers [37], [79], [80], [81], [82], [83], [84], [85], [86], [87] studied the different applications of collocation methods. First, we give the analogue of the collocation method for the numerical solution of the proposed model of the fractional Cauchy–Euler (FrC-E) problem by applying the operator (4.13) on the collocation function u_h of (4.9) for $t \in \bar{\delta}_n$ when $0 < \alpha < 1$, for which we obtain

$$\begin{aligned}
&\left[{}^C D_t^{0<\alpha<1} u_h(t) \right]_{t=t_n+wh} \\
&= h^{-\alpha} \sum_{p=0}^{n-1} \left(a_1^{(p)} \frac{(n+w-p)^{-\alpha}}{\Gamma(1-\alpha)} {}_2F_1 \left[\alpha, 1, 2, \frac{1}{n-p+w} \right] \right. \\
&\quad \left. + \sum_{k=1}^m b_k^{(p)} \frac{(n+w-p)^{-\alpha}}{\Gamma(1-\alpha)} {}_2F_1 \left[\alpha, k+1, k+2, \frac{1}{n-p+w} \right] \right) \\
&\quad + h^{-\alpha} \left(a_1^{(n)} \frac{w^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{k=1}^m b_k^{(n)} \frac{(k+1)\Gamma(k+1)w^{k+1-\alpha}}{\Gamma(2-\alpha+k)} \right). \tag{4.14}
\end{aligned}$$

Here, ${}_2F_1$ is the Gauss Hypergeometric function. Also,

$$u_h''(t) = h^{-2} \left(D_w^2 u_h(t_n+wh) \right) = h^{-2} \sum_{k=1}^m (1+k) k b_k^{(n)} w^{k-1}, \quad w \in (0, 1]. \tag{4.15}$$

Evaluating (4.9), (4.14) and (4.15) at $t_{n,i} = t_n + v_i h$ and substituting into (4.7) gives

$$\begin{aligned}
& B_1(t_{n,i})^2 \left(\sum_{k=1}^m b_k^{(n)} (1+k) k (v_i)^{k-1} \right) \\
& + B_2(t_{n,i})^\alpha \left(\sum_{p=0}^{n-1} \frac{h^{2-\alpha}}{\Gamma(1-\alpha)} \left(a_1^{(p)} (n+v_i-p)^{-\alpha} {}_2F_1 \left[\alpha, 1, 2, \frac{1}{n-p+v_i} \right] \right. \right. \\
& \left. \left. + \sum_{k=1}^m b_k^{(p)} (n+v_i-p)^{-\alpha} {}_2F_1 \left[\alpha, k+1, k+2, \frac{1}{n-p+v_i} \right] \right) \right) \\
& + h^{2-\alpha} \left(a_1^{(n)} \frac{v_i^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{k=1}^m b_k^{(n)} \frac{(k+1) \Gamma(k+1) v_i^{k+1-\alpha}}{\Gamma(2-\alpha+k)} \right) \\
& + B_3 h^2 \left(\sum_{\bar{q}=0}^1 a_{\bar{q}}^{(n)} v_i^{\bar{q}} + \sum_{k=1}^m b_k^{(n)} v_i^{k+1} \right) \\
& = h^2 f(t_{n,i}).
\end{aligned} \tag{4.16}$$

Next, we rewrite Equation (4.16) in matrix representation as

$$A^{(n)} b^{(n)} = C_1^{(n)} a^{(n)} + F_1^{(n)}, \quad n = 0, 1, \dots, N-1, \tag{4.17}$$

where

$A^{(n)} \in \mathbb{R}^{m \times m}$, $C_1^{(n)} \in \mathbb{R}^{m \times 2}$ and $F_1^{(n)} \in \mathbb{R}^m$ with entries given as

$$\begin{aligned}
(A^{(n)})_{i,k} &= B_1(t_{n,i})^2 (1+k) k (v_i)^{k-1} \\
&+ B_2(t_{n,i})^\alpha h^{2-\alpha} \left(\frac{(k+1) \Gamma(k+1) v_i^{k+1-\alpha}}{\Gamma(2-\alpha+k)} \right) + B_3 h^2 v_i^{k+1}, \quad k = 1, \dots, m,
\end{aligned} \tag{4.18}$$

$$(C_1^{(n)})_{i,1} = -B_3 h^2, \quad (C_1^{(n)})_{i,2} = -B_2(t_{n,i})^\alpha h^{2-\alpha} \frac{v_i^{1-\alpha}}{\Gamma(2-\alpha)} - B_3 h^2 v_i, \tag{4.19}$$

$$\begin{aligned}
(F_1^{(n)})_i &= h^2 f(t_{n,i}) - \frac{h^{2-\alpha}}{\Gamma(1-\alpha)} B_2(t_{n,i})^\alpha \\
&\times \left(\sum_{p=0}^{n-1} \left(a_1^{(p)} (n+v_i-p)^{-\alpha} {}_2F_1 \left[\alpha, 1, 2, \frac{1}{n-p+v_i} \right] \right. \right. \\
&\left. \left. + \sum_{k=1}^m b_k^{(p)} (n+v_i-p)^{-\alpha} {}_2F_1 \left[\alpha, k+1, k+2, \frac{1}{n-p+v_i} \right] \right) \right),
\end{aligned} \tag{4.20}$$

for $i = 1, \dots, m$. We remark from the above that at $n = 0$, the vector $a^{(0)}$ is known from the initial conditions and is given as follows:

$$a^{(0)} = \begin{bmatrix} y(a), & hy'(a) \end{bmatrix}^T = \begin{bmatrix} \bar{\beta}_0, & h\bar{\beta}_1 \end{bmatrix}^T. \quad (4.21)$$

Similarly, vectors $a^{(n)}$ and $b^{(n)}$ are defined in (2.34)

For $n \geq 1$, the continuity condition on $[t_n, t_{n+1}]$, which gives the relationship between the known vectors $a^{(n)}$, $b^{(n)}$ and the unknown vector $a^{(n+1)}$, is given by (2.35) See [37, 73, 77, 88] for details and the references therein. Analogously, when $1 < \alpha < 2$, the following system is obtained from Equation (4.7) when evaluated at $t_{n,i} = t_n + v_i h$,

$$\begin{aligned} & B_1(t_{n,i})^2 \left(\sum_{k=1}^m (1+k) k (v_i)^{k-1} b_k^{(n)} \right) \\ & + B_2(t_{n,i})^\alpha \left(h^{2-\alpha} \sum_{p=0}^{n-1} \sum_{k=1}^m b_k^{(p)} \frac{(k+1)(n+v_i-p)^{1-\alpha}}{\Gamma(2-\alpha)} {}_2F_1 \left[-1+\alpha, k, k+1, \frac{1}{n-p+v_i} \right] \right. \\ & \left. + h^{2-\alpha} \left(\sum_{k=1}^m b_k^{(n)} \frac{(k+1)\Gamma(k+1)v_i^{k+1-\alpha}}{\Gamma(2-\alpha+k)} \right) \right) \\ & + B_3 h^2 \left(\sum_{\bar{q}=0}^1 a_{\bar{q}}^{(n)} v_i^{\bar{q}} + \sum_{k=1}^m b_k^{(n)} v_i^{k+1} \right) = h^2 f(t_{n,i}), \quad i = 1, 2, \dots, m. \end{aligned} \quad (4.22)$$

Writing Equation (4.22) in matrix form, we have

$$A^{(n)} b^{(n)} = C_2 a^{(n)} + F_2^{(n)}, \quad n = 0, 1, \dots, N-1, \quad (4.23)$$

where $A^{(n)}$ is the same matrix as given in (4.18) and $C_2 \in \mathbb{R}^{m \times 2}$ and $F_2^{(n)} \in \mathbb{R}^m$. The entries of C_2 and $F_2^{(n)}$ are as follows:

$$(C_2)_{i,s+1} = -B_3 h^2 v_i^s, \quad s = 0, 1, \quad (4.24)$$

$$\begin{aligned} (F_2^{(n)})_i &= h^2 f(t_{n,i}) - \frac{h^{2-\alpha}}{\Gamma(2-\alpha)} B_2(t_{n,i})^\alpha \left(\sum_{p=0}^{n-1} \sum_{k=1}^m b_k^{(p)} (k+1)(n+v_i-p)^{1-\alpha} \right. \\ & \quad \left. \times {}_2F_1 \left[-1+\alpha, k, k+1, \frac{1}{n-p+v_i} \right] \right), \end{aligned} \quad (4.25)$$

Algorithm 1 gives the described method in the Chapter 4 for solving equation (1.4).

The Algorithm considers both the case $0 < \alpha < 1$ and $1 < \alpha < 2$

Algorithm 1

Let G-LP be the Gauss points which are the zeros of the (shifted) Legendre polynomial $P_m(2s - 1)$ and also, let M-CP be the mean Chebyshev-type points, $v_i = (x_i + 1)/2$ and $x_i = 1 - \cos\left(\frac{i\pi}{2m}\right)$ for $i = 1, 2, \dots, m$.

Case $0 < \alpha < 1$: We choose the collocation parameters $v_i, i = 1, 2, \dots, m$ either as G-LP or M-CP.

Step 1: For $n = 0$

- (i) $a^{(0)}$ is computed using the initial conditions in Equation (4.21).
- (ii) We compute $F_1^{(0)}$ using Equation (4.20).
- (iii) Using Equations (4.17)–(4.20) and the result in (i) and (ii) in conjunction with the LU decomposition method, we compute $b^{(0)}$.

Step 2: Substituting the results from Step 1 into Equation (4.9), we find the solution of the initial value problem on the interval $[t_0, t_1]$.

Step 3: For $n = 1$

- (i) $a^{(1)}$ is computed using Equation (2.34) and the result in Step 1 (i) and (iii).
- (ii) Using Equations (4.17)–(4.20) and the results obtained in Step 1, we compute the vector $F_1^{(1)}$. Repeating Step 1 (iii) for $n = 1$, we find the value of $b^{(1)}$.
- (iii) Substituting the results from Step 3 (i) and (ii) into Equation (4.9) again, we find the collocated solution which presents the local solution of the initial problem on the interval $[t_1, t_2]$.

Step 4: Step 3 is repeated for $n = 2, 3, \dots, N - 1$, resulting in systems of algebraic linear equations which give the solution of the initial value problem on $[t_2, t_3]$, $[t_3, t_4], \dots, [t_{N-1}, t_N]$, respectively. Case $1 < \alpha < 2$: We choose the collocation parameters $v_i, i = 1, 2, \dots, m$ either as G-LP or M-CP. The algorithm is analogous but in Step 1 part (ii), we compute $F_2^{(0)}$ using Equation (4.25), and in Step 1 part (iii) and Step 3 part (ii), instead of Equations (4.17)–(4.20), we use (4.22)–(4.25).

The schematic diagrammatic representation of the Algorithm 1 is shown in Figure 4.1

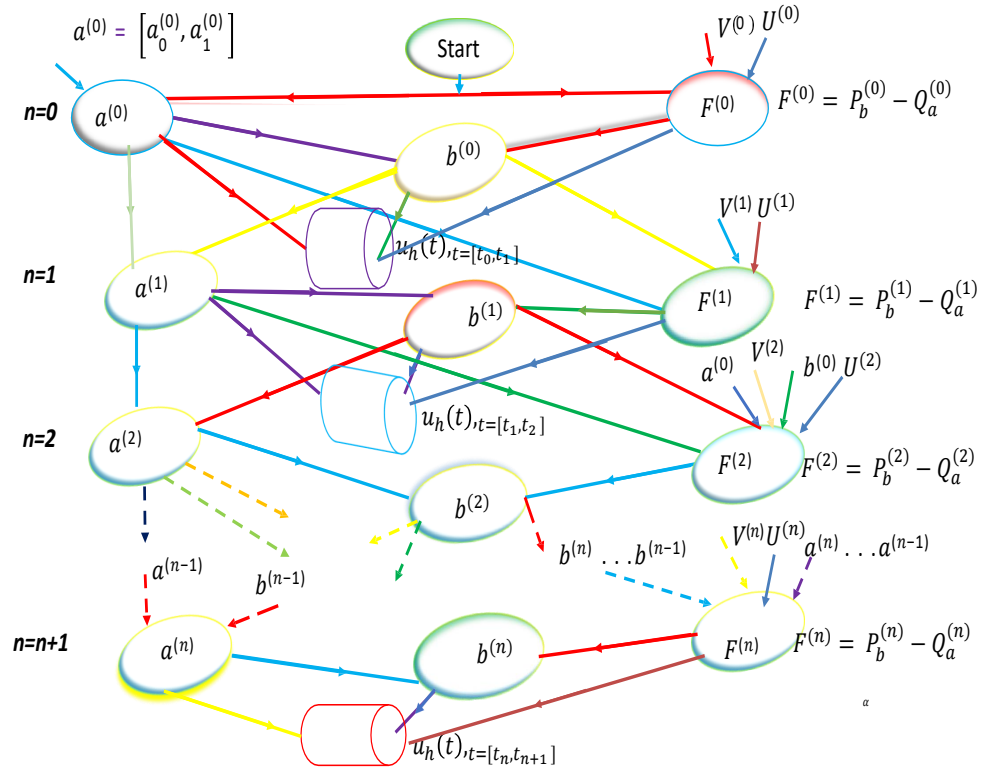


Figure 4.1: The schematic diagram for the Algorithm 1

Parameters of the schematic diagram are as follow $P_b^{(n)}$ and $Q_a^{(n)}$ are vectors obtained from equations (4.20) and (4.25) and $a^{(n)}$ are the initial conditions as defined in equation (2.34), $F^{(n)}$ is the vector obtained from (4.20) and (4.25). Also, $u_h(t)$ presents the approximate solution.

4.3 Convergence Analysis

Lemma 4.2: [66] Consider a uniform sequence of meshes for $L = [a, T]$ and let $1 < \alpha < 2$, $t_{n,j} = t_n + v_j h$. Then, for $0 \leq p \leq n-1$ ($n \leq N-1$) and for $q \in \mathbb{N}$ and for all $0 \leq v_j \leq 1$, $j = 1, 2, \dots, m$

$$\int_0^1 \left(\frac{t_{n,j} - t_p}{h} - s \right)^{1-\alpha} s^q ds < \frac{2^{\alpha-1} (n-p)^{1-\alpha}}{2-\alpha}. \quad (4.26)$$

Proof. For $p = n-1$ we have

$$\int_0^1 (1 + v_j - s)^{1-\alpha} s^q ds \leq \int_0^1 (1-s)^{1-\alpha} ds = \frac{1}{2-\alpha} < \frac{2^{\alpha-1}}{2-\alpha},$$

and the estimation (4.26) holds. For $p \leq n-2$ and from $v_j \in [0, 1]$, we obtain

$$\begin{aligned} \tilde{I}_{n,p}(\alpha) &= \int_0^1 \left(\frac{t_{n,j} - t_p}{h} - s \right)^{1-\alpha} s^q ds \leq \int_0^1 \left(\frac{t_n - t_p}{h} - s \right)^{1-\alpha} ds \\ &= \frac{1}{2-\alpha} \left(\left(\frac{t_n - t_p}{h} \right)^{2-\alpha} - \left(\frac{t_n - t_p}{h} - 1 \right)^{2-\alpha} \right) \\ &= \frac{1}{2-\alpha} \left(\frac{t_n - t_p}{h} \right)^{2-\alpha} \left(1 - \left(1 - \left(\frac{t_n - t_p}{h} \right)^{-1} \right)^{2-\alpha} \right). \end{aligned} \quad (4.27)$$

Application of the Mean Value Theorem gives

$$\left(1 - \left(\frac{t_n - t_p}{h} \right)^{-1} \right)^{2-\alpha} = 1 - (2-\alpha) \left(\frac{t_n - t_p}{h} \right)^{-1} \left(1 - \sigma_{n,p} \left(\frac{t_n - t_p}{h} \right)^{-1} \right)^{1-\alpha}, \quad (4.28)$$

and $0 < \sigma_{n,p} < 1$. Using (4.27) and (4.28), the following is obtained

$$\tilde{I}_{n,p}(\alpha) \leq \left(\frac{t_n - t_p}{h} \right)^{1-\alpha} \left(1 - \sigma_{n,p} \left(\frac{t_n - t_p}{h} \right)^{-1} \right)^{1-\alpha}. \quad (4.29)$$

Further, since $p \leq n-2$ and based on the uniform grid, we have

$$1 - \sigma_{n,p} \left(\frac{t_n - t_p}{h} \right)^{-1} = 1 - \sigma_{n,p} (n-p)^{-1} \geq \frac{1}{2}. \quad (4.30)$$

Using (4.30) in (4.29), given that $1 < \alpha < 2$, gives

$$\tilde{I}_{n,p}(\alpha) \leq 2^{\alpha-1} (n-p)^{1-\alpha} < \frac{2^{\alpha-1}}{2-\alpha} (n-p)^{1-\alpha}.$$

This complete the proof. □

Lemma 4.3: [66] Let the assumptions of Lemma 4.2 hold, then

$$\sum_{p=0}^{n-1} \int_0^1 h^{2-\alpha} \left(\frac{t_{n,j}-t_p}{h} - s \right)^{1-\alpha} ds \leq \frac{(T-a)^{2-\alpha}}{2-\alpha}, \quad 0 \leq n \leq N-1.$$

Proof.

$$\begin{aligned} & \sum_{p=0}^{n-1} \int_0^1 h^{2-\alpha} \left(\frac{t_{n,j}-t_p}{h} - s \right)^{1-\alpha} ds \leq \sum_{p=0}^{n-1} \int_0^1 h^{2-\alpha} \left(\frac{t_n-t_p}{h} - s \right)^{1-\alpha} ds \\ &= \frac{1}{2-\alpha} \sum_{p=0}^{n-1} h^{2-\alpha} \left\{ \left(\frac{t_n-t_p}{h} \right)^{2-\alpha} - \left(\frac{t_n-t_{p+1}}{h} \right)^{2-\alpha} \right\} \\ &= \frac{1}{2-\alpha} \sum_{p=0}^{n-1} \left\{ (t_n-t_p)^{2-\alpha} - (t_n-t_{p+1})^{2-\alpha} \right\} \\ &= \frac{(t_n-a)^{2-\alpha}}{2-\alpha} \leq \frac{(T-a)^{2-\alpha}}{2-\alpha}. \end{aligned}$$

□

Theorem 4.2: [66] Assume that

- (a) $\{v_i\}$, $i = 1, \dots, m$ are $m > 1$ distinct collocation parameters (which may be chosen as M-CP or G-LP as given in Algorithm 1) satisfying $0 < v_1 < v_2 < \dots < v_m \leq 1$.
- (b) The exact solution y of (1.4) satisfies $y \in C^{m+2}(L)$.
- (c) The collocation solution $u_h \in S_{m+1}^{(1)}(L_h)$ for the FrC-E problem (1.4) corresponding to the collocation points $\tilde{X}_h$ is defined by (4.9).
- (d) $\bar{h} > 0$ is such that, for any $h \in (0, \bar{h})$, each of the linear systems (4.17) and (4.23) has a unique solution. Then, the estimate

$$\|y - u_h\|_\infty = \max_{t \in L} |y(t) - u_h(t)| \leq \bar{C}_0 h^m, \quad (4.31)$$

holds true for $h \in (0, \bar{h})$. The constant $\bar{C}_0$ depends on the collocation parameters v_i and on $D_t^{m+2}y(t)$ but are independent of h .

Proof. From assumption (b), we have $y \in C^{m+2}(L)$ and hence $D_t^1 y(t) \in C^{m+1}(L)$, and

$D_t^2 y(t) \in C^m(L)$. For simplicity, we will use the notation $D_t^1 y(t) = y'(t)$ and $D_t^2 y(t) = y''(t)$. Thus, we have, using Peano's Theorem [62] for y'' on $\bar{\delta}_n$,

$$y''(t) = \sum_{j=1}^m L_j(w) Z_{n,j} + h^m R_{m+2,n}^{(1)}(w), \quad w \in [0, 1], \quad (4.32)$$

with $Z_{n,j} = y''(t_{n,j}) = y''(t_n + v_j h)$, and the polynomials

$$L_j(w) = \prod_{l \neq j}^m \frac{w - v_l}{v_j - v_l} \quad (j = 1, 2, \dots, m), \quad (4.33)$$

denote the Lagrange fundamental polynomials with respect to the (distinct) collocation parameters $v_j, j = 1, 2, \dots, m$. Also, the Peano remainder term is

$$R_{m+2,n}^{(1)}(w) = \int_0^1 K_m(w, z) y^{m+2}(t_n + zh) dz, \quad (4.34)$$

and

$$K_m(t, s) = \frac{1}{(m-1)!} \left\{ (t-s)_+^{m-1} - \sum_{j=1}^m L_j(t) (v_j - s)_+^{m-1} \right\}.$$

Here, $(t-s)_+^p := 0$ if $t < s$ and $(t-s)_+^p := (t-s)^p$ for $t \geq s$. Then, the local Lagrange representation of $y'(t)$ on $\bar{\delta}_n$ is

$$y'(t_n + wh) = y'(t_n) + h \sum_{j=1}^m \beta_{1,j}(w) Z_{n,j} + h^{m+1} R_{m+2,n}(w), \quad (4.35)$$

where

$$\beta_{r,j}(w) = \int_0^w \frac{(w-s)^{1-r}}{(1-r)!} L_j(s) ds, \quad r = 0, 1,$$

and

$$R_{m+2,n}(w) = \int_0^w R_{m+2,n}^{(1)}(s) ds.$$

Integrating once more both sides of (4.35) from 0 to w yields

$$y(t_n + wh) = y(t_n) + h w y'(t_n) + h^2 \sum_{j=1}^m \beta_{0,j}(w) Z_{n,j} + h^{m+2} \tilde{R}_{m+2,n}(w), \quad (4.36)$$

and

$$\tilde{R}_{m+2,n}(w) = \int_0^w R_{m+2,n}(s) ds.$$

The collocation solution $u_h(t) = u_h(t_n + wh)$ satisfies the collocation Equations (4.7)

and (4.8), and the local representation of $u_h(t) = u_h(t_n + wh)$ is

$$u_h(t) = u_h(t_n + wh) = u_h(t_n) + h w u'_h(t_n) + h^2 \sum_{j=1}^m \beta_{0,j}(w) Y_{n,j}, \quad (4.37)$$

where $Y_{n,j} = u''_h(t_n + v_j h)$. The local function (4.37) can also be presented as (4.9).

Let the collocation error be $e_h := y - u_h$ on $\bar{\delta}_n$. Subtracting (4.37) from (4.36) gives

$$\begin{aligned} e_h(t_n + wh) &= y(t_n) - a_0^{(n)} + w \left(h y'(t_n) - a_1^{(n)} \right) \\ &\quad + \sum_{j=1}^m \left(h^2 \beta_{0,j}(w) Z_{n,j} - b_j^{(n)} w^{j+1} \right) + h^{m+2} \tilde{R}_{m+2,n}(w) \\ &= e_h(t_n) + h w e'_h(t_n) + h^2 \sum_{j=1}^m \beta_{0,j}(w) \varepsilon_{n,j} + h^{m+2} \tilde{R}_{m+2,n}(w), \end{aligned} \quad (4.38)$$

for $w \in [0, 1]$ where $\varepsilon_{n,j} = Z_{n,j} - Y_{n,j}$. Further,

$$\begin{aligned} \left[{}^C D_t^{1 < \alpha < 2} e_h(t) \right]_{t=t_n+wh} &= h \sum_{p=0}^{n-1} \sum_{j=1}^m \bar{\beta}_{0,j}(w) \varepsilon_{p,j} + h^{m+1} \sum_{p=0}^{n-1} \bar{R}_{m+2,p}(w) \\ &\quad + h^{2-\alpha} \sum_{j=1}^m \hat{\beta}_{0,j}(w) \varepsilon_{n,j} + h^{m+2-\alpha} \hat{R}_{m+2,n}(w), \end{aligned} \quad (4.39)$$

$$e'_h(t_n + wh) = e'_h(t_n) + h \sum_{j=1}^m \beta_{1,j}(w) \varepsilon_{n,j} + h^{m+1} R_{m+2,n}(w), \quad (4.40)$$

$$e''_h(t_n + wh) = \sum_{j=1}^m L_j(w) \varepsilon_{n,j} + h^m R_{m+2,n}^{(1)}(w). \quad (4.41)$$

where

$$\begin{aligned} \hat{R}_{m+2,n}(w) &= \left({}^C D_w^{1 < \alpha < 2} \left[\tilde{R}_{m+2,n}(s) \right] \right)(w), \quad n = 0, 1, \dots, N-1, \\ \bar{R}_{m+2,p}(w) &= \left({}^C J_1^{1 < \alpha < 2} \left[\tilde{R}_{m+2,p}(s) \right] \right), \quad p = 0, 1, \dots, n-1, \\ \bar{\beta}_{0,j}(w) &= \left({}^C J_1^{1 < \alpha < 2} \left[\beta_{0,j}(s) \right] \right), \quad j = 1, 2, \dots, m, \\ \hat{\beta}_{0,j}(w) &= \left({}^C D_w^{1 < \alpha < 2} \left[\beta_{0,j}(s) \right] \right)(w), \quad j = 1, 2, \dots, m. \end{aligned} \quad (4.42)$$

Using that $e_h(0) = y(0) - u_h(0) = 0$ and $e'_h(0) = y'(0) - u'_h(0) = 0$ and that e_h, e'_h are

continuous applying recursion and evaluating at $w = 1$ gives

$$e'_h(t_n) = h \sum_{p=0}^{n-1} \sum_{j=1}^m \beta_{1,j}(1) \varepsilon_{p,j} + h^{m+1} \sum_{p=0}^{n-1} R_{m+2,p}(1) \quad (4.43)$$

$$\begin{aligned} e_h(t_n) &= h^2 \sum_{p=0}^{n-1} \sum_{j=1}^m \beta_{1,j}(1) \varepsilon_{p,j} + h^{m+2} \sum_{p=0}^{n-1} R_{m+2,p}(1) \\ &\quad + h^2 \sum_{p=0}^{n-1} \sum_{j=1}^m \beta_{0,j}(1) \varepsilon_{p,j} + h^{m+2} \sum_{p=0}^{n-1} \tilde{R}_{m+2,p}(1) \end{aligned} \quad (4.44)$$

Next, $e_h := y - u_h$ satisfies the following equation at $t = t_{n,i} = t_n + v_i h$

$$B_1 t_{n,i}^2 e''_h(t_{n,i}) + B_2 t_{n,i}^\alpha \left({}^C D_t^{1 < \alpha < 2} e_h \right)(t_{n,i}) + B_3 e_h(t_{n,i}) = 0, \quad i = 1, 2, \dots, m. \quad (4.45)$$

Then, evaluating (4.41) at $w = v_i$ gives

$$\varepsilon_{n,i} = e''_h(t_{n,i}) - h^m R_{m+2,n}^{(1)}(v_i). \quad (4.46)$$

Further, evaluating (4.38) and (4.39) at $w = v_i$ and using (4.43)–(4.46), we obtain the following algebraic system of equations for $i = 1, 2, \dots, m$:

$$\begin{aligned} & -B_1 t_{n,i}^2 \varepsilon_{n,i} - B_2 t_{n,i}^\alpha h^{2-\alpha} \sum_{j=1}^m \widehat{\beta}_{0,j}(v_i) \varepsilon_{n,j} - B_3 h^2 \sum_{j=1}^m \beta_{0,j}(v_i) \varepsilon_{n,j} \\ &= B_2 t_{n,i}^\alpha \left(h \sum_{p=0}^{n-1} \sum_{j=1}^m \bar{\beta}_{0,j}(v_i) \varepsilon_{p,j} + h^{m+1} \sum_{p=0}^{n-1} \bar{R}_{m+2,p}(v_i) + h^{m+2-\alpha} \widehat{R}_{m+2,n}(v_i) \right) \\ &+ B_3 \left(h^2 \sum_{p=0}^{n-1} \sum_{j=1}^m \beta_{1,j}(1) \varepsilon_{p,j} + h^{m+2} \sum_{p=0}^{n-1} R_{m+2,p}(1) \right. \\ &+ h^2 \sum_{p=0}^{n-1} \sum_{j=1}^m \beta_{0,j}(1) \varepsilon_{p,j} + h^{m+2} \sum_{p=0}^{n-1} \tilde{R}_{m+2,p}(1) \\ &+ v_i \left(h^2 \sum_{p=0}^{n-1} \sum_{j=1}^m \beta_{1,j}(1) \varepsilon_{p,j} + h^{m+2} \sum_{p=0}^{n-1} R_{m+2,p}(1) \right) \\ &\left. + h^{m+2} \tilde{R}_{m+2,n}(v_i) \right) + B_1 t_{n,i}^2 h^m R_{m+2,n}^{(1)}(v_i). \end{aligned} \quad (4.47)$$

The system of Equation (4.47) can be presented in the matrix form

$$\left[\Omega_n^1 + h^{2-\alpha} \Omega_n^2 + h^2 \Omega_n^3 \right] \mathbf{E}_n = \sum_{p=0}^{n-1} h \Psi_{n,p} \mathbf{E}_p + \rho_n \quad (4.48)$$

where $\Omega_n^1, \Omega_n^2, \Omega_n^3, \Psi_{n,p} \in \mathbb{R}^{m \times m}$ and Ω_n^1 is an invertible diagonal matrix, and Ω_n^2, Ω_n^3

have bounded entries and $\mathbf{E}_n = [\varepsilon_{n,1}, \varepsilon_{n,2}, \dots, \varepsilon_{n,m}]^T$ and $\rho_n = [\rho_{n,1}, \rho_{n,2}, \dots, \rho_{n,m}]^T$

with

the entries

$$\begin{aligned}
(\rho_n)_i &= B_2 t_{n,i}^\alpha h^{m+1} \sum_{p=0}^{n-1} \bar{R}_{m+2,p}(t_{n,i}) + B_2 t_{n,i}^\alpha h^{m+2-\alpha} \widehat{R}_{m+2,n}(v_i) \\
&\quad + B_3 t_{n,i}^\alpha h^{m+2} \sum_{p=0}^{n-1} R_{m+2,p}(1) + B_3 h^{m+2} \sum_{p=0}^{n-1} \widetilde{R}_{m+2,p}(1) \\
&\quad + B_3 v_i h^{m+2} \sum_{p=0}^{n-1} R_{m+2,p}(1) + B_3 h^{m+2} \widetilde{R}_{m+2,n}(v_i) \\
&\quad + B_1 t_{n,i}^2 h^m R_{m+2,n}^{(1)}(v_i), i = 1, 2, \dots, m.
\end{aligned} \tag{4.49}$$

Since $t_{n,i} \in L$, $n = 0, 1, \dots, N-1$, and using (4.42) and Lemmas 4.2 and 4.3, there exists a constant $\bar{C}$ such that $\|\rho_n\|_1 \leq \bar{C}h^m$. Further, for sufficiently small $h \in (0, \bar{h})$, the linear system (4.48) has a unique solution. Furthermore, we take $\|\mathbf{E}_n\|_1 = \sum_{j=1}^m |\varepsilon_{n,j}|$. Then, whenever $h \in (0, \bar{h})$, for some $\bar{h} > 0$, there exists a constant $\bar{C}_1 < \infty$ so that the uniform bound

$$\left\| \left[\Omega_n^1 + h^{2-\alpha} \Omega_n^2 + h^2 \Omega_n^3 \right]^{-1} \right\|_1 \leq \bar{C}_1 < \infty, \tag{4.50}$$

holds. Also, from Lemma 4.3 there exist a positive constant $\bar{C}_2$ that $\sum_{p=0}^{n-1} \|h\Psi_{n,p}\|_1 \leq \bar{C}_2$. Then, from (4.48) and using $\|\rho_n\|_1 \leq \bar{C}h^m$, we obtain

$$\|\mathbf{E}_n\|_1 \leq \bar{C}_1 \sum_{p=0}^{n-1} \|h\Psi_{n,p}\|_1 \|\mathbf{E}_p\|_1 + \bar{C}_1 \bar{C} h^m, \quad n = 0, 1, \dots, N-1. \tag{4.51}$$

and it follows that

$$\|\mathbf{E}_n\|_1 = \mathcal{O}(h^m) \text{ for } n = 0, 1, \dots, N-1. \tag{4.52}$$

Subsequently, using (4.38), (4.43), (4.44) and (4.52) yields

$$\max_{w \in [0,1]} |e_h(t_n + wh)| \leq \bar{C}_0 h^m.$$

Hence, the inequality (4.31) is obtained. Further, for $0 < \alpha < 1$, the proof is analogous.

□

Chapter 5

EXPERIMENTAL INVESTIGATION OF THE EXISTENCE, UNIQUENESS AND A NUMERICAL APPROACH TO THE SOLUTION OF FRACTIONAL CAUCHY-EULER EQUATION

In this section we constructed examples of fractional Cauchy-Euler (FrC-E) problem to demonstrate efficiency and accuracy of the method presented in the previous sections. All the computations are carried out in Wolfram Mathematica and Matlab using double precision and are presented in tables and figures. Some of these results are published in [66]

All the computations in this section are carried out on an HP Laptop (notebook) with properties 8GB RAM INTEL 17 & 1255U 3.5GHZ and 512 GB NVME M.2SSD and using Wolfram Mathematica and Matlab in double precision. For the ease of implementation we have taken the number of collocation parameters $m = 3, 4$. The following notations are used in tables and figures:

The Gauss hypergeometric function ${}_2F_1[p, r, s, z]$ is defined in the unit disk as the sum of the hypergeometric series [15]

$${}_2F_1[p, r, s, z] = \sum_{i=0}^{\infty} \frac{(p)_i (r)_i}{(s)_i} \frac{z^i}{i!}.$$

where $|z| < 1$; $p, r \in \mathbb{C}$; $s \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $(p)_i$ is the pochhammer symbol. The Error

function is defined as [23]

$$Erf[z] = \frac{2}{\pi} \int_0^z e^{-x^2} dx.$$

G-LP denotes the Gauss-Legendre points, defined to be the roots of the shifted Legendre polynomials, where $P_m(2s - 1)$

$$P_m(x) = \frac{1}{2^m} \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{m}{k} \binom{2m-2k}{n} x^{m-2k}.$$

M-CP denotes the Mean-Chebyshev type points, $v_i = (x_i + 1)/2$ and $x_i = 1 - \cos\left(\frac{i\pi}{2m}\right)$ for $i = 1, 2, \dots, m$.

$$AE_h(t) = |y(t) - u_h(t)|, \quad t \in L = [a, T],$$

where y is the exact solution and u_h is the approximate solution defines the absolute error function.

$$E_2^h = \int_a^T (AE_h(t))^2 dt, \quad MAE_h = \max_{t \in L_h} AE_h(t),$$

are the square L_2 norm error and the absolute maximum error, respectively. Also, the local and global order of convergence are defined by

$$R_{h, \frac{h}{2}}^* = \frac{MAE_h}{MAE_{\frac{h}{2}}}, \quad R_{h, \frac{h}{2}} = \frac{E_2^h}{E_2^{\frac{h}{2}}}.$$

Example 5.1: The fractional Cauchy-Euler (FrC-E) problem is such that:

$$t^2 D^2 y(t) + t^{\frac{3}{2}} \left({}_{1/2}^C D_t^{\frac{3}{2}} y(t) \right) + y(t) = f(t), \quad \frac{1}{2} < t < 1,$$

$$y\left(\frac{1}{2}\right) = \frac{1}{32}$$

$$y'\left(\frac{1}{2}\right) = \frac{1}{2},$$

where the non-homogeneous term and the exact solution are:

$$f(t) = \frac{1}{5} \sqrt{\frac{2}{\pi}} t \left(3\sqrt{t(-1+2t)} + 8\sqrt{t^3(-1+2t)} + 32\sqrt{t^5(-1+2t)} \right) + 13t^4.$$

$$y(t) = t^4.$$

Table 5.1: The exact solution $y(t)$, the approximate solution $u_h(t)$, absolute error AE_h and square L_2 norm error (E_2^h), for Example 5.1.

t	$y(t)$	$u_h(t)$	AE_h
0.50	0.0625000000000000	0.0625000000000000	0
0.55	0.0915062500000000	0.09150625000000001	9.7145E-17
0.60	0.1296000000000000	0.12960000000000002	1.9429E-16
0.65	0.1785062500000000	0.17850625000000004	3.8858E-16
0.70	0.2400999999999999	0.24010000000000006	6.9389E-16
0.75	0.3164062500000000	0.31640625000000008	7.7716E-16
0.80	0.40960000000000001	0.40960000000000010	9.4369E-16
0.85	0.5220062499999999	0.52200625000000013	1.4433E-15
0.90	0.65610000000000001	0.65610000000000017	1.5543E-15
0.95	0.8145062499999999	0.81450625000000021	2.2204E-15
1.00	1.0000000000000000	1.00000000000000027	2.6645E-15
E_2^h	2.1234E-22		

In Table 5.1, we used the M-CP and take $m = 4$. Table 5.1 shows the exact solution on column 2, the approximate solution on column 3, the absolute errors on column 4 and the square L_2 norm errors.

The results of Example 5.1 presented in Table 5.1 indicated that, the collocation method for FrC-E equation gives a good approximation to the exact solution. This

result is consistent with the Theorem 4.2

Furthermore, we present the graph of the absolute errors at mesh points, the exact solution and approximate solution in Figure 5.1. The graph of the absolute errors shows a graduate increase of the absolute error, this is because of the propagation of the errors within the sub-intervals from the initial conditions, however the over all performance of the developed method is good in approximating the exact solution of FrC-E.

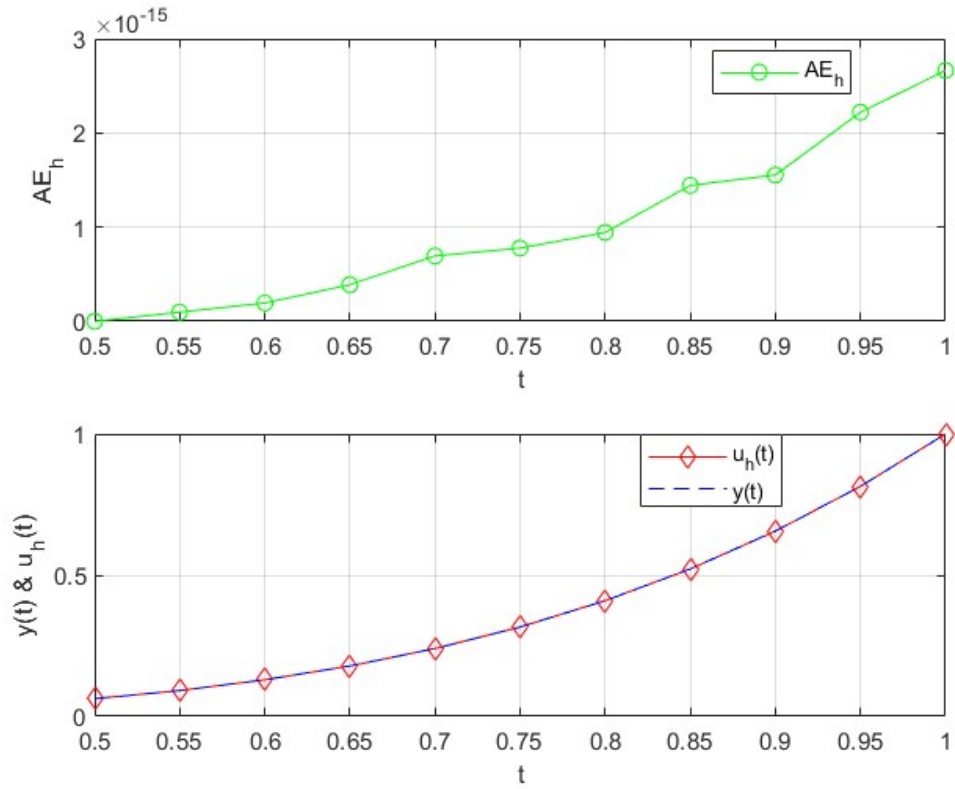


Figure 5.1: The absolute error function, the exact solution and the approximate solution when $N = 10$ for Example 5.1.

Example 5.2: Consider the fractional Cauchy-Euler (FrC-E) problem [88]:

$$\begin{aligned}
t^2 D^2 y(t) + t^{\frac{1}{2}} \left({}_1^C D_t^{\frac{1}{2}} y(t) \right) + y(t) &= f(t), \quad \frac{1}{2} < t < 1, \\
y\left(\frac{1}{2}\right) &= \frac{1}{64} + \sin\left(\frac{1}{2}\right) + \cos\left(\frac{1}{2}\right), \\
y'\left(\frac{1}{2}\right) &= \frac{3}{16} + \cos\left(\frac{1}{2}\right) - \sin\left(\frac{1}{2}\right),
\end{aligned}$$

where the exact solution and non-homogeneous term are

$$\begin{aligned}
y(t) &= t^6 + \sin(t) + \cos(t), \\
f(t) &= t^2 \left(30t^4 - \sin(t) + \cos(t) \right) + t^{\frac{1}{2}} \left[\frac{12}{\Gamma(\frac{1}{2})} \left((5t^4(t-1/2))/2 + t(t-1/2)^{5/2} \right. \right. \\
&\quad \left. \left. + (t-1/2)^{1/2} + (5t^2(t-1/2)^2)/2 + (10t^3(t-1/2)^{3/2})/3 + t^5(t-1/2)^{1/2} \right) \right. \\
&\quad \left. + \frac{2}{\Gamma(\frac{1}{2})} \left(2^{1/2} \pi^{1/2} \text{fresnelc}((2^{1/2}(t-1/2)^{1/2})/\pi^{1/2}) \cos(t)/2 \right. \right. \\
&\quad \left. \left. + (2^{1/2} \pi^{1/2} \text{fresnels}((2^{1/2}(t-1/2)^{1/2})/\pi^{1/2}) \sin(t))/2 \right) \right. \\
&\quad \left. + \frac{2}{\Gamma(\frac{1}{2})} \left((2^{1/2} \pi^{1/2} \text{fresnelc}((2^{1/2}(t-1/2)^{1/2})/\pi^{1/2}) \sin(t))/2 \right. \right. \\
&\quad \left. \left. - (2^{1/2} \pi^{1/2} \text{fresnels}((2^{1/2}(t-1/2)^{1/2})/\pi^{1/2}) \cos(t))/2 \right) \right] + t^6 + \sin(t) + \cos(t)
\end{aligned}$$

respectively, and the fresnel integrals are $\text{fresnels}(z) = s(z) = \int_0^z \sin(\frac{1}{2} \pi t^2) dt$ and

$$\text{fresnelc}(z) = c(z) = \int_0^z \cos(\frac{1}{2} \pi t^2) dt, [23].$$

Table 5.2: The absolute errors (AE_h) and square L_2 norm error (E_2^h), for Example 5.2.

t	AE_h	AE_h	AE_h
	$N = 8$	$N = 16$	$N = 32$
0.5000	0.000000	0.00000	0.0000
0.5625	3.9726E-06	3.8805E-07	4.1880E-08
0.6250	1.2859E-05	1.3814E-06	1.5882E-07
0.6875	2.6922E-05	3.0150E-06	3.5532E-07
0.7500	4.6406E-05	5.3206E-06	6.3544E-07
0.8125	7.1557E-05	8.3294E-06	1.0031E-06
0.8750	1.0263E-04	1.2073E-05	1.4624E-06
0.9375	1.3988E-04	1.6586E-05	2.0174E-06
1.0000	1.8358E-04	2.1901E-05	2.6724E-06
E_2^h	1.9529E-09	2.8739E-11	3.9843E-13

In Example 5.2 we consider the M-CP and take $m = 3$. Table 5.2 presents the absolute errors at some mesh points and the square L_2 norm errors E_2^h . Table 5.3 shows the ratios of the maximum absolute errors and the ratios of square of L_2 norm errors. We see from Table 5.3 that the order of convergence is at least third order given in column 3 and in column 5. Also, the determinant of the matrix $A^{(0)}$ with respect to h is shown in column 6 of Table 5.3. Further, the total computational time in seconds (CPUs) is given in the last column. Figure 5.2 illustrates the absolute error function for $N = 4, 8, 16, 32$.

Table 5.3: The order of convergence and computational efficiency, for Example 5.2.

N	MAE_h	$R_{h,\frac{h}{2}}^*$	E_2^h	$R_{h,\frac{h}{2}}$	$det(A^{(0)})$	CPUs
4	1.6033E-03		1.7657E-07		0.1312	0.2887
8	1.8358E-04	8.7335	1.9530E-09	90.4098	0.0780	0.4599
16	2.1901E-05	8.3824	2.8739E-11	67.9562	0.0593	0.8553
32	2.6724E-06	8.1951	3.9843E-13	72.1305	0.0514	1.6493

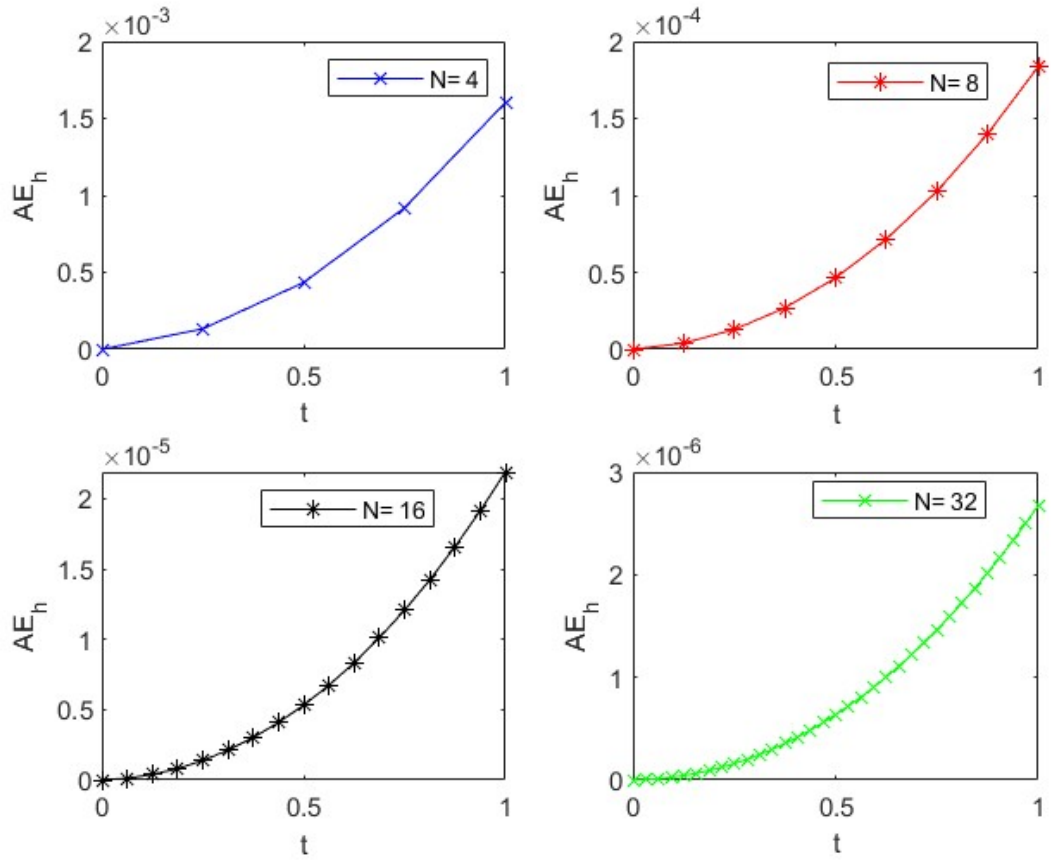


Figure 5.2: The absolute error function for $N = 4, 8, 16, 32$, for Example 5.2.

Example 5.3: [66] An Example which the regularity parameter $\varpi > 2$, being a fractional number is not fixed. We consider:

$$t^2 D^2 y(t) + t^{\frac{1}{2}} \left({}^C D_t^{\frac{1}{2}} y(t) \right) + y(t) = f(t), \quad t \in \left(\frac{1}{4}, 1 \right)$$

$$y\left(\frac{1}{4}\right) = e^{\frac{1}{2}} + \frac{1}{256}, \quad y'\left(\frac{1}{4}\right) = 2e^{\frac{1}{2}} + \frac{1}{16},$$

with the exact solution and non-homogeneous term as given below

$$y(t) = \left(t - \frac{1}{4} \right)^{\varpi} + e^{2t} + t^4,$$

$$f(t) = e^{2t} + \left(t - \frac{1}{4} \right)^{\varpi} + t^4 + t^2 \left(4e^{2t} + 12t^2 + \left(t - \frac{1}{4} \right)^{\varpi-2} (\varpi-1)\varpi \right)$$

$$+ \sqrt{t} \left(\frac{\sqrt{-1+4t} (5 + 8t (3 + 16t(1+8t)))}{560\sqrt{\pi}} + \sqrt{2}e^{2t} \text{Erf} \left(\sqrt{-\frac{1}{2} + 2t} \right) \right.$$

$$\left. + \frac{2^{1-2\varpi} \varpi (4t-1)^{\varpi-\frac{1}{2}} \Gamma(\varpi)}{\Gamma(\frac{1}{2} + \varpi)} \right).$$

Figure 5.3 shows the approximate solution u_h , and Figure 5.4 illustrates the absolute error function AE_h for Example 5.4 when $N = 40, m = 3$ with respect to ϖ using the M-CP as collocation parameters.

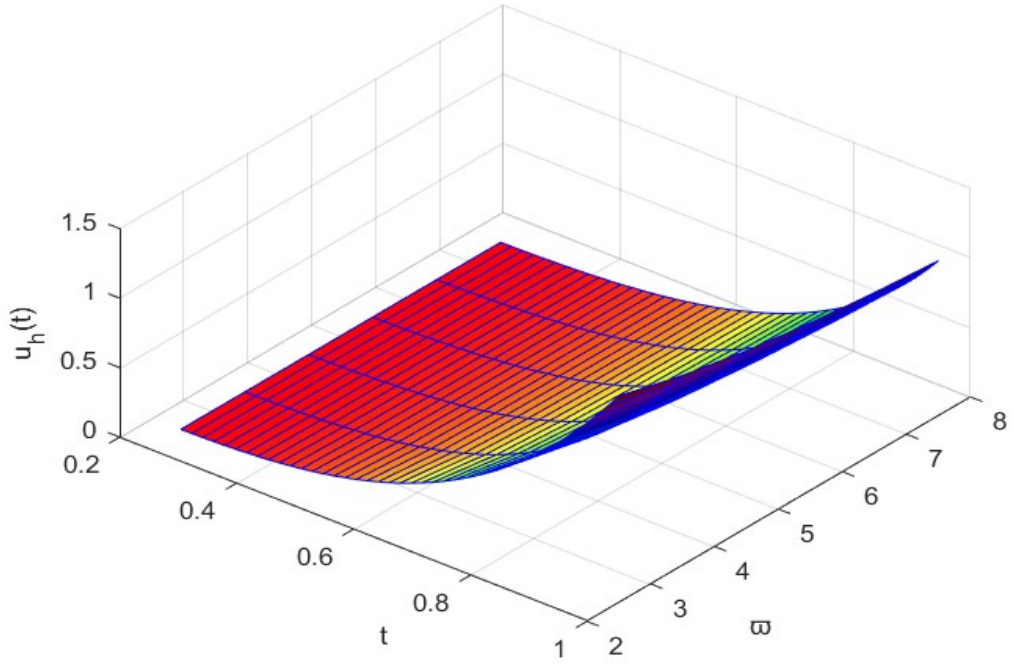


Figure 5.3: The graph of the approximate solution u_h for Example 5.4 when $N = 40, m = 3$ with respect to ϖ using the M-CP.

Example 5.4: [66] This is an example where the regularity parameter $\varpi > 2$, being a fractional number, is not fixed. We consider

$$t^2 D^2 y(t) + t^{\frac{1}{2}} {}^C D_t^{\frac{1}{4}} y(t) + y(t) = f(t), \quad t \in \left(\frac{1}{4}, 1\right),$$

$$y\left(\frac{1}{4}\right) = \eta e^{\frac{1}{2}} + \frac{1}{256}, \quad y'\left(\frac{1}{4}\right) = 2\eta e^{\frac{1}{2}} + \frac{1}{16},$$

with the exact solution and non-homogeneous term as given below:

$$y(t) = \left(t - \frac{1}{4}\right)^{\varpi} + \eta e^{2t} + t^4,$$

$$f(t) = \eta e^{2t} + \left(t - \frac{1}{4}\right)^{\varpi} + t^4 + t^2 \left(4\eta e^{2t} + 12t^2 + \left(t - \frac{1}{4}\right)^{\varpi-2} (\varpi - 1)\varpi \right)$$

$$+ \sqrt{t} \left(\frac{\sqrt{-1+4t}(5+8t(3+16t(1+8t)))}{560\sqrt{\pi}} + \sqrt{2}\eta^2 e^{2t} \text{Erf}\left(\sqrt{-\frac{1}{2}+2t}\right) \right.$$

$$\left. + \frac{2^{1-2\varpi}\varpi(4t-1)^{\varpi-\frac{1}{2}}\Gamma(\varpi)}{\Gamma(\frac{1}{2}+\varpi)} \right).$$

In this problem, for the application of the given Algorithm 1, we take $\eta = 0, m = 3$

and the regularity parameter $\varpi = 5/2, 7/2, 9/2, 11/2, 13/2, 15/2$. Table 5.4 shows the maximum absolute error (MAE_h) and square L_2 norm error (E_2^h) and the order of convergence $R_{h,\frac{h}{2}}^*$ and $R_{h,\frac{h}{2}}$ by using the mean Chebyshev points (M-CP) for Example 5.4 with respect to several values of N . Additionally, Table 5.5 presents the analogous quantities for the same values of the parameters N, m, ϖ obtained by using the Gauss–Legendre points (G-LP). Furthermore, the last columns of both Tables 5.4 and 5.5 show the determinant of the matrix $A^{(0)}$. These tables justify that the local and global orders of convergence are at least $\mathcal{O}(h^3)$ when the exact solution $y \in C^5 [\frac{1}{4}, 1]$.

Table 5.4: The error norms, convergence orders, and $\det A^{(0)}$ for Example 5.4 obtained by using the M-CP.

ϖ	N	MAE_h	$R_{h,\frac{h}{2}}^*$	E_2^h	$R_{h,\frac{h}{2}}$	$\det A^{(0)}$
5/2	10	6.4830×10^{-4}		1.7791×10^{-7}		0.00250
	20	2.2697×10^{-4}	2.8563	2.2636×10^{-8}	7.8595	0.00140
	40	7.9439×10^{-5}	2.8572	2.8375×10^{-9}	7.9776	0.00098
	80	2.7878×10^{-5}	2.8495	3.5424×10^{-10}	8.0101	0.00083
7/2	10	7.2600×10^{-5}		1.8369×10^{-9}		0.00250
	20	1.3398×10^{-5}	5.4187	6.6275×10^{-11}	27.7158	0.00140
	40	2.4216×10^{-6}	5.5327	2.2789×10^{-12}	29.0824	0.00098
	80	4.3270×10^{-7}	5.5965	7.5928×10^{-14}	30.0135	0.00083
9/2	10	7.1755×10^{-5}		1.1338×10^{-9}		0.00250
	20	9.3254×10^{-6}	7.6946	1.9186×10^{-11}	59.0941	0.00140
	40	1.1977×10^{-6}	7.7861	3.1837×10^{-13}	60.2626	0.00098
	80	1.5260×10^{-7}	7.8486	5.2023×10^{-15}	61.1989	0.00083
11/2	10	1.9365×10^{-4}		5.1845×10^{-9}		0.00250
	20	2.3288×10^{-5}	8.3154	7.3108×10^{-11}	70.9154	0.00140
	40	2.8515×10^{-6}	8.1669	1.0826×10^{-12}	67.5310	0.00098
	80	3.5266×10^{-7}	8.0857	1.6457×10^{-14}	65.7822	0.00083
13/2	10	3.1570×10^{-4}		1.0053×10^{-8}		0.00250
	20	3.6854×10^{-5}	8.5662	1.3324×10^{-10}	75.4497	0.00140
	40	4.4454×10^{-6}	8.2904	1.9140×10^{-12}	69.6133	0.00098
	80	5.4566×10^{-7}	8.1468	2.8661×10^{-14}	66.7796	0.00083
15/2	10	4.3498×10^{-4}		1.5195×10^{-8}		0.00250
	20	4.9527×10^{-5}	8.7827	1.9079×10^{-10}	79.6447	0.00140
	40	5.8977×10^{-6}	8.3977	2.6686×10^{-12}	71.4950	0.00098
	80	7.1922×10^{-7}	8.2001	3.9435×10^{-14}	67.6704	0.00083

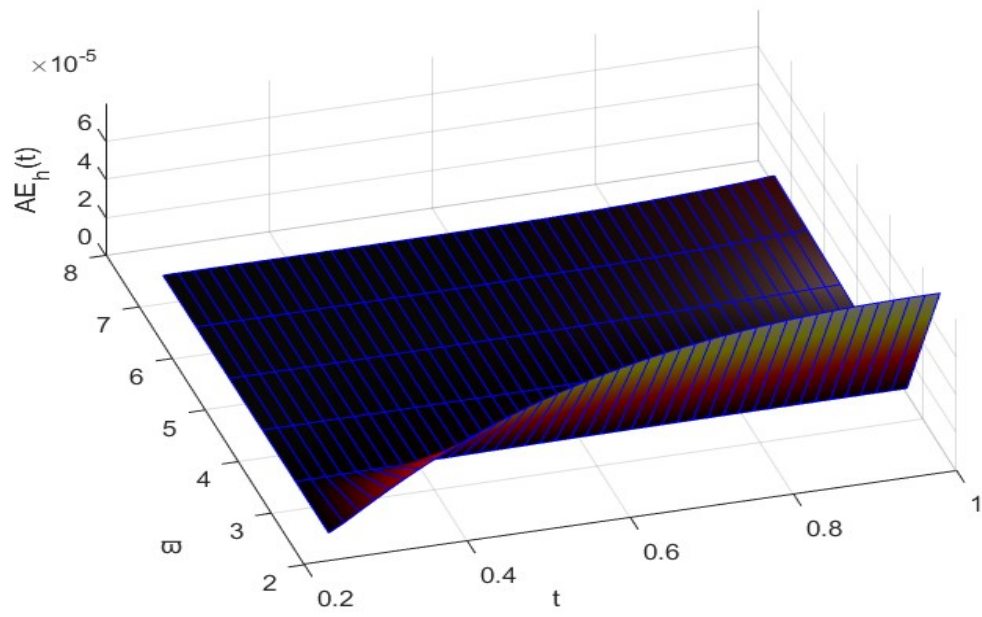


Figure 5.4: The graph of the AE_h for Example 5.4 when $N = 40, m = 3$ with respect to ω using the M-CP.

Table 5.5: The error norms, convergence orders, and $\det A^{(0)}$ for Example 5.4 obtained by using the G-LP.

ϖ	N	MAE_h	$R_{h,\frac{h}{2}}^*$	E_2^h	$R_{h,\frac{h}{2}}$	$\det A^{(0)}$
5/2	10	5.3795×10^{-5}		1.3718×10^{-9}		0.0095
	20	1.9239×10^{-5}	2.7961	1.7319×10^{-10}	7.9205	0.0063
	40	6.8404×10^{-6}	2.8126	2.1751×10^{-11}	7.9624	0.0051
	80	2.4252×10^{-6}	2.8206	2.7252×10^{-12}	7.9815	0.0046
7/2	10	6.7614×10^{-7}		2.2650×10^{-13}		0.0095
	20	1.2348×10^{-7}	5.4757	7.3007×10^{-15}	31.0244	0.0063
	40	2.2169×10^{-8}	5.5699	2.3119×10^{-16}	31.5788	0.0051
	80	3.9491×10^{-9}	5.6137	7.2699×10^{-18}	31.8001	0.0046
9/2	10	1.8556×10^{-8}		2.2893×10^{-16}		0.0095
	20	1.7731×10^{-9}	10.4653	1.6024×10^{-18}	142.8669	0.0063
	40	1.6030×10^{-10}	11.0611	1.2336×10^{-20}	129.8962	0.0051
	80	1.4286×10^{-11}	11.2208	9.5980×10^{-23}	128.5268	0.0046
11/2	10	2.6778×10^{-9}		4.1769×10^{-16}		0.0095
	20	1.0480×10^{-10}	25.5515	4.0885×10^{-19}	1021.6216	0.0063
	40	4.1229×10^{-12}	25.4190	4.0042×10^{-22}	1021.0529	0.0051
	80	1.6698×10^{-13}	24.6910	3.9265×10^{-25}	1019.7886	0.0046
13/2	10	5.1669×10^{-9}		2.1949×10^{-15}		0.0095
	20	1.9575×10^{-10}	26.3954	2.1545×10^{-18}	1018.7515	0.0063
	40	7.9070×10^{-12}	24.7565	2.1110×10^{-21}	1020.6281	0.0051
	80	3.3107×10^{-13}	23.8832	2.0694×10^{-24}	1020.0808	0.0046
15/2	10	7.6511×10^{-9}		7.3628×10^{-15}		0.0095
	20	2.5028×10^{-10}	30.5702	7.1430×10^{-18}	1030.7724	0.0063
	40	9.4049×10^{-12}	26.6117	7.1077×10^{-21}	1004.9599	0.0051
	80	3.8303×10^{-13}	24.5540	6.9541×10^{-24}	1022.0890	0.0046

Figure 5.5 gives the graph of the approximate solution u_h depending on ϖ , while Figure 5.6 presents the graph of AE_h with respect to ϖ , both by using the G-LP and $N = 40, m = 3$ for Example 5.4. We see from Figures 5.4 and 5.6 that the AE_h is higher when the regularity parameter is taken as $5/2$ and $7/2$.

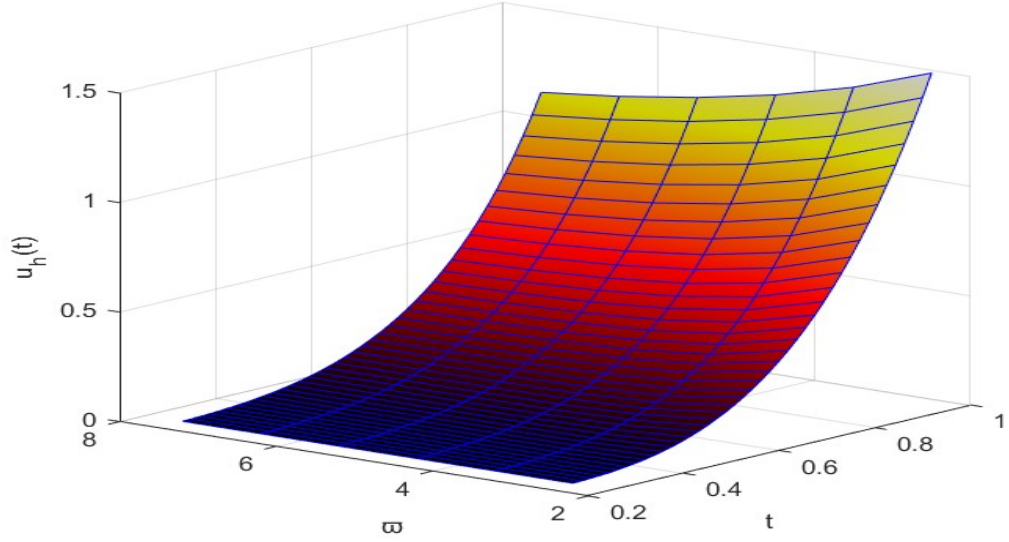


Figure 5.5: The graph of approximate solution u_h for Example 5.4 when $N = 40, m = 3$ with respect to ϖ using the G-LP.

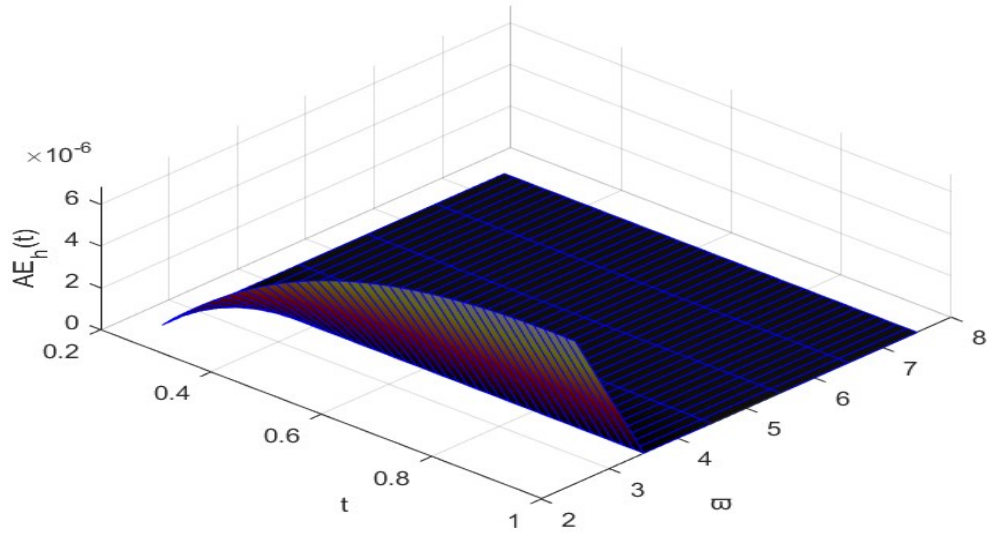


Figure 5.6: The graph of the AE_h for Example 5.4 when $N = 40, m = 3$ depending on ϖ using the G-LP.

Example 5.5: [66] This is an example where the fractional order α is not fixed, i.e.,

$1 < \alpha < 2$, and the exact solution y is smooth:

$$t^2 D^2 y(t) + t^\alpha \left({}^C D_t^\alpha y(t) \right) + y(t) = f(t), \quad t \in \left(\frac{1}{2}, 1 \right),$$

is subject to

$$y\left(\frac{1}{2}\right) = \frac{1}{2^{\frac{13}{2}}} + \frac{65}{32}, \quad y'\left(\frac{1}{2}\right) = \frac{13}{2(2^{\frac{11}{2}})} + \frac{37}{16},$$

where the exact solution and the non-homogeneous term are

$$\begin{aligned} y(t) &= t^{\frac{13}{2}} + t^5 + 2t + 1, \\ f(t) &= \frac{1}{4} t^5 \left(80 + 143 t^{\frac{3}{2}} \right) + t^{\frac{13}{2}} + t^5 + 2t + 1 \\ &\quad + \frac{1}{128 \Gamma(2 - \alpha)} t^\alpha \left(\frac{1}{(-5 + \alpha)(-4 + \alpha)(-3 + \alpha)(-2 + \alpha)} \right. \\ &\quad \times 5(2^{4+\alpha})(2t - 1)^{2-\alpha} \left(24 - 26\alpha - \alpha^3 - 6\alpha t(5 + 4t) \right. \\ &\quad \left. \left. + \alpha^2(9 + 6t) + 12t(3 + 4t(1 + t)) \right) \right. \\ &\quad \left. \left. + 13t^{-\alpha} \left(\frac{10395\sqrt{\pi} t^{\frac{13}{2}} \Gamma(2 - \alpha)}{\Gamma(\frac{15}{2} - \alpha)} - \sqrt{2} t {}_2F_1 \left[\frac{11}{2}, -1 + \alpha, \frac{13}{2}, \frac{1}{2t} \right] \right) \right) \right). \end{aligned}$$

In this example, the proposed Algorithm 1 is also applied for $m = 3$. Table 5.6 gives the MAE_h , E_2^h , and the ratios $R_{h, \frac{h}{2}}^*$, $R_{h, \frac{h}{2}}$ and the determinant of matrix $A^{(0)}$ for various values of N and α when M-CP is used. A close look at the ratios indicates that the numerical method gives third-order convergence $\mathcal{O}(h^3)$, remarking that $R_{h, \frac{h}{2}}^* \approx 2^3$ and $R_{h, \frac{h}{2}} \approx (2^3)^2$. The analogous quantities are presented in Table 5.7 for G-LP.

Table 5.6: The error norms, convergence orders, and $\det A^{(0)}$ for Example 5.5 obtained by using the M-CP.

α	N	MAE_h	$R_{h,\frac{h}{2}}^*$	E_2^h	$R_{h,\frac{h}{2}}$	$\det A^{(0)}$
1.5	10	1.2612×10^{-4}		1.5442×10^{-9}		0.118010
	20	1.5152×10^{-5}	8.3237	2.1882×10^{-11}	70.5694	0.082311
	40	1.8573×10^{-6}	8.1581	3.2595×10^{-13}	67.1330	0.066086
	80	2.3000×10^{-7}	8.0752	4.9500×10^{-15}	65.8485	0.057684
1.3	10	1.3570×10^{-4}		1.7942×10^{-9}		0.089907
	20	1.6443×10^{-5}	8.2528	2.5876×10^{-11}	69.3384	0.065772
	40	2.0252×10^{-6}	8.1192	3.8923×10^{-13}	66.4800	0.055313
	80	2.5143×10^{-7}	8.0547	5.9420×10^{-15}	65.5049	0.050279
1.2	10	1.4165×10^{-4}		1.9556×10^{-9}		0.082713
	20	1.7164×10^{-5}	8.2527	2.8206×10^{-11}	69.3328	0.061921
	40	2.1129×10^{-6}	8.1234	4.2386×10^{-13}	66.5456	0.053035
	80	2.6218×10^{-7}	8.0590	6.4633×10^{-15}	65.5795	0.048861
1.1	10	1.4759×10^{-4}		2.1221×10^{-9}		0.078016
	20	1.7860×10^{-5}	8.2637	3.0525×10^{-11}	69.5201	0.059551
	40	2.1963×10^{-6}	8.1319	4.5767×10^{-13}	66.6965	0.051716
	80	2.7231×10^{-7}	8.0654	6.9675×10^{-15}	65.6864	0.048091
1.001	10	1.5317×10^{-4}		2.2826×10^{-9}		0.074978
	20	1.8503×10^{-5}	8.2781	3.2713×10^{-11}	69.7765	0.058109
	40	2.2727×10^{-6}	8.1414	4.8929×10^{-13}	66.8581	0.050963
	80	2.8158×10^{-7}	8.0712	7.4374×10^{-15}	65.7878	0.047679

Table 5.7: The error norms, convergence orders, and $\det A^{(0)}$ for Example 5.5 obtained using G-LP.

α	N	MAE_h	$R_{h,\frac{h}{2}}^*$	E_2^h	$R_{h,\frac{h}{2}}$	$\det(A^0)$
1.5	10	1.5523×10^{-7}		2.8048×10^{-15}		0.520922
	20	1.3358×10^{-8}	11.6208	1.8408×10^{-17}	152.3685	0.405034
	40	1.1620×10^{-9}	11.4957	1.3516×10^{-19}	136.1941	0.347087
	80	1.0184×10^{-10}	11.4101	1.0229×10^{-21}	132.1341	0.315164
1.3	10	4.5573×10^{-8}		4.0353×10^{-16}		0.417067
	20	3.3904×10^{-9}	13.4418	1.3828×10^{-18}	291.8209	0.338744
	40	2.5551×10^{-10}	13.2691	6.8916×10^{-21}	200.6501	0.301922
	80	1.9437×10^{-11}	13.1455	3.8457×10^{-23}	179.2027	0.283354
1.2	10	2.0151×10^{-8}		2.0179×10^{-16}		0.392175
	20	1.3926×10^{-9}	14.4701	3.6027×10^{-19}	560.1077	0.324243
	40	9.7632×10^{-11}	14.2638	1.1446×10^{-21}	314.7562	0.292946
	80	6.9171×10^{-12}	14.1146	5.0524×10^{-24}	226.5458	0.277638
1.1	10	6.5025×10^{-9}		1.5170×10^{-16}		0.376448
	20	4.1659×10^{-10}	15.6089	1.6210×10^{-19}	935.8421	0.315603
	40	2.7153×10^{-11}	15.3423	2.1777×10^{-22}	744.3633	0.287919
	80	1.7906×10^{-12}	15.1642	4.6867×10^{-25}	464.6553	0.274636
1.001	10	7.7479×10^{-11}		1.4279×10^{-16}		0.366557
	20	3.0056×10^{-12}	25.7782	1.3932×10^{-19}	1024.9067	0.310488
	40	1.5632×10^{-13}	19.2272	1.3602×10^{-22}	1024.2611	0.285126
	80	8.8818×10^{-15}	17.6000	1.3247×10^{-25}	1026.7985	0.273074

Figure 5.7 shows the graph of the approximate solution u_h , and Figure 5.8 demonstrates the absolute error AE_h both with respect to α for Example 5.5 when $N = 80, m = 3$ using the M-CP. Further, the functions u_h and AE_h with respect to α for the same parameters are shown in Figures 5.9 and 5.10, respectively, using the G-LP. We view from Figures 5.8 and 5.10 that the absolute error is higher near the corner $(t, \alpha) = (1, 2)$. These figures justify a good approximation of the exact solution.

Example 5.6: This is an example in which the exact solution is unknown. Consider the following problem:

$$t^2 D^2 y(t) + t^\alpha \left({}^C D_t^\alpha y(t) \right) + y(t) = f(t), \quad t \in \left(\frac{1}{2}, 1 \right),$$

$$y\left(\frac{1}{2}\right) = \frac{1}{2^{\frac{13}{2}}} + \frac{65}{32} \quad y'\left(\frac{1}{2}\right) = \frac{13}{2(2^{\frac{11}{2}})} + \frac{37}{16},$$

where $0 < \alpha < 2$, and the exact solution y is unknown, and

$$f(t) = \begin{cases} 26t^5 - \frac{173}{4}t^{\frac{13}{2}} + 4t + 1, & \text{if } 0 < \alpha \leq 1 \\ 26t^5 - \frac{173}{4}t^{\frac{13}{2}} + t \left(4 - y'\left(\frac{1}{2}\right) \right) + 1, & \text{if } 1 < \alpha < 2. \end{cases}$$

The problem in Example 5.6 reduces to the classical Cauchy–Euler problem for $\alpha = 1$ with the exact solution $y(t) = t^{13/2} + t^5 + 2t + 1$. Table 5.8 shows the approximate solution u_h of the problem in Example 5.6 at some mesh points obtained by using the G-LP when $N = 80, m = 4$, for $0 < \alpha < 1$ and $1 < \alpha < 2$. Table 5.8 also presents the exact solution when $\alpha = 1$. Further, Table 5.9 gives the approximate solution u_h , for $h = \frac{1}{40}, \frac{1}{80}, \frac{1}{160}$ and the convergence order when $\alpha = 1.0001$ and $\alpha = 0.9999$ for Example 5.6 for $m = 4$ by using G-LP for collocation. In Table 5.9, na means the order is not available at that point since the error is 0. Additionally, Table 5.10 presents the total computational time required for applying the algorithm for Example 5.6 when G-LP and M-CP points are used as collocation parameters. It can be concluded from this

table that the total computational time varies linearly with respect to N . Finally, Figure 5.11 illustrates the graph of the approximate solution u_h for $N = 80, m = 4, 0 < \alpha < 2$

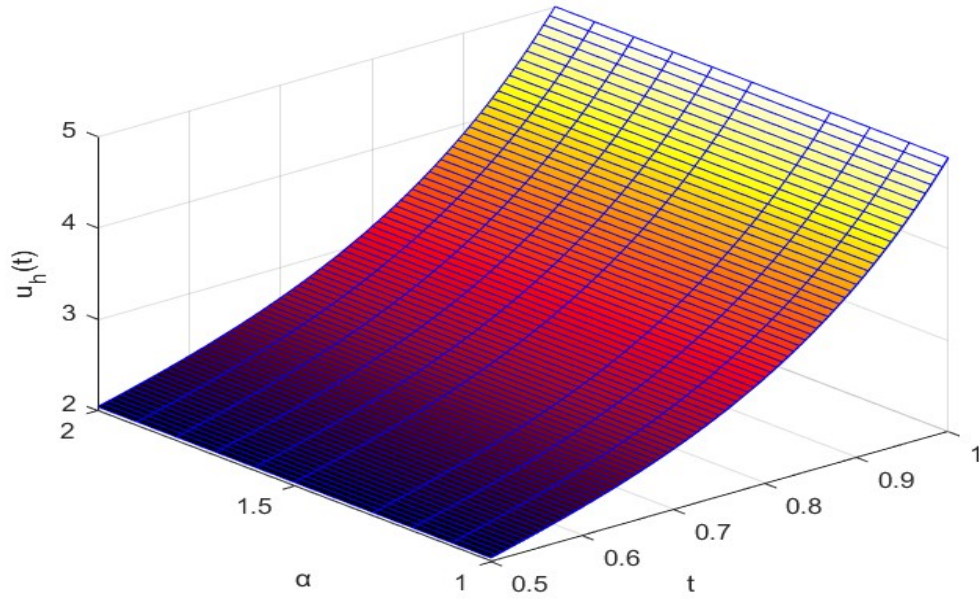


Figure 5.7: The graph of the approximate solution u_h for $N = 80, m = 3$ with respect to α using the M-CP for Example 5.5.

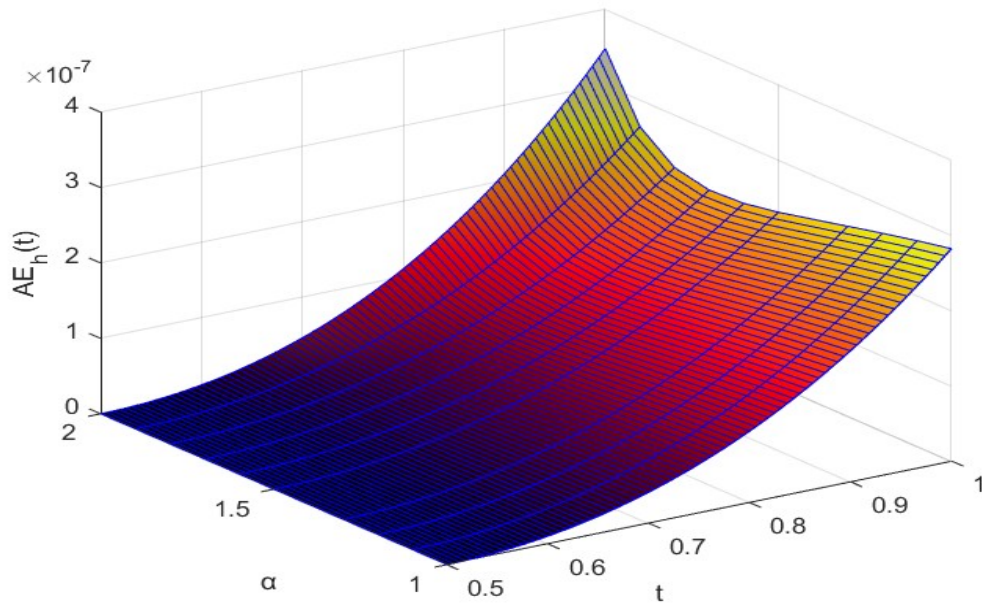


Figure 5.8: The graph of the AE_h for $N = 80, m = 3$ with respect to α using the M-CP for Example 5.5.

Table 5.8: The approximate solution u_h obtained by using the G-LP when $0 < \alpha < 1$, $1 < \alpha < 2$ and $N = 80, m = 4$, and the exact solution when $\alpha = 1$ for Example 5.6.

t	$\alpha = 0.000001$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 0.9999$	$\alpha = 1$
0.50	2.0422985435	2.0422985435	2.0422985435	2.0422985435	2.0422985435	2.0422985435	2.0422985435
0.55	2.1767805121	2.1764037955	2.1758232163	2.1746903204	2.1725380227	2.1708588887	2.1708569500
0.60	2.3369024513	2.3347716925	2.3319658673	2.3271986993	2.3193731879	2.3139056592	2.3138995822
0.65	2.5274156028	2.5216636520	2.5147425376	2.5038899232	2.4875408103	2.4768453377	2.4768337160
0.70	2.7549968510	2.7434643682	2.7304274618	2.7110949449	2.6836631424	2.6665205545	2.6665022155
0.75	3.0284929630	3.0087827046	2.9875216050	2.9572869381	2.9162563165	2.8914648279	2.8914386034
0.80	3.3592046620	3.3286793244	3.2969387426	3.2532481862	3.1959458560	3.1621841421	3.1621487216
0.85	3.7612101952	3.7169586037	3.6722763258	3.6123322973	3.5357501179	3.4914661707	3.4914199851
0.90	4.2517280048	4.1905099616	4.1301429778	4.0507848212	3.9514084005	3.8947180844	3.8946592006
0.95	4.8515182766	4.7696929559	4.6905325451	4.5881098490	4.4617466989	4.3903338681	4.3902598857
1.00	5.5853233461	5.4787636267	5.3772403414	5.2474775023	5.0890786630	5.0000920558	5.0000000000
t	$\alpha = 1.9$	$\alpha = 1.7$	$\alpha = 1.5$	$\alpha = 1.3$	$\alpha = 1.1$	$\alpha = 1.01$	$\alpha = 1.00001$
0.50	2.0422985435	2.0422985435	2.0422985435	2.0422985435	2.0422985435	2.0422985435	2.0422985435
0.55	2.1685531044	2.1694376642	2.1701085260	2.1705384260	2.1707822586	2.1708505494	2.1708569437
0.60	2.3034979414	2.3069732352	2.3098697446	2.3120010835	2.3134119334	2.3138562759	2.3138995394
0.65	2.4506782624	2.4587657163	2.4658149461	2.4713707428	2.4753607944	2.4767003017	2.4768335840
0.70	2.6146277713	2.6298871983	2.6435414918	2.6547572542	2.6632359905	2.6662024934	2.6665019185
0.75	2.8009911854	2.8266990317	2.8500965497	2.8698446529	2.8853034382	2.8908703942	2.8914380399
0.80	3.0166988503	3.0570373680	3.0941795395	3.1261217059	3.1517532229	3.1611793495	3.1621477597
0.85	3.2701646359	3.3304340391	3.3863863632	3.4351515048	3.4749961111	3.4898805355	3.4914184567
0.90	3.5715038824	3.6583631732	3.7394835794	3.8108721430	3.8699896105	3.8923376437	3.8946568947
0.95	3.9327709570	4.0545114009	4.1687095689	4.2699264243	4.3545935706	4.3868930719	4.3902565406
1.00	4.3682166429	4.5350713106	4.6921020473	4.8320202532	4.9499557991	4.9952644911	4.9999952937

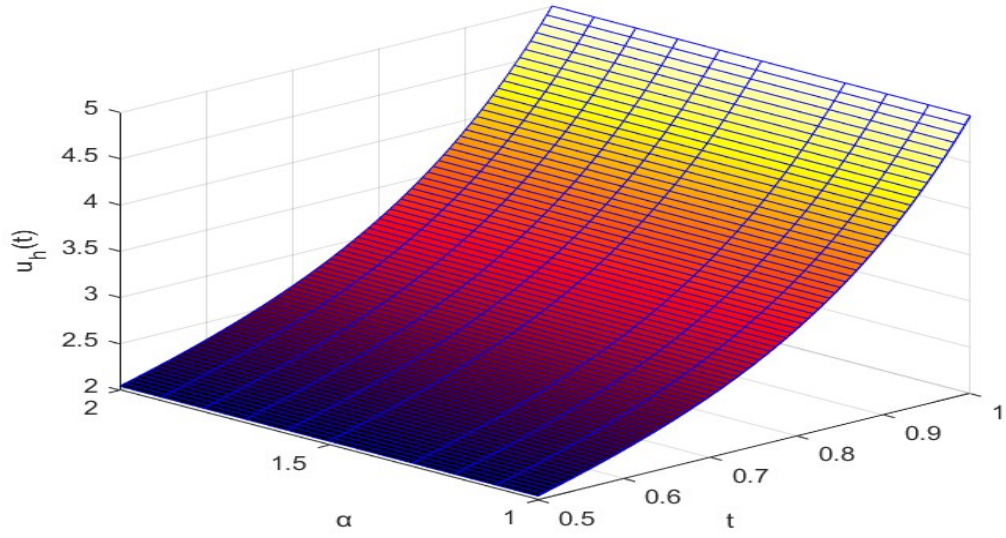


Figure 5.9: The graph of the approximate solution u_h for $N = 80, m = 3$ with respect to α using the G-LP for Example 5.5.

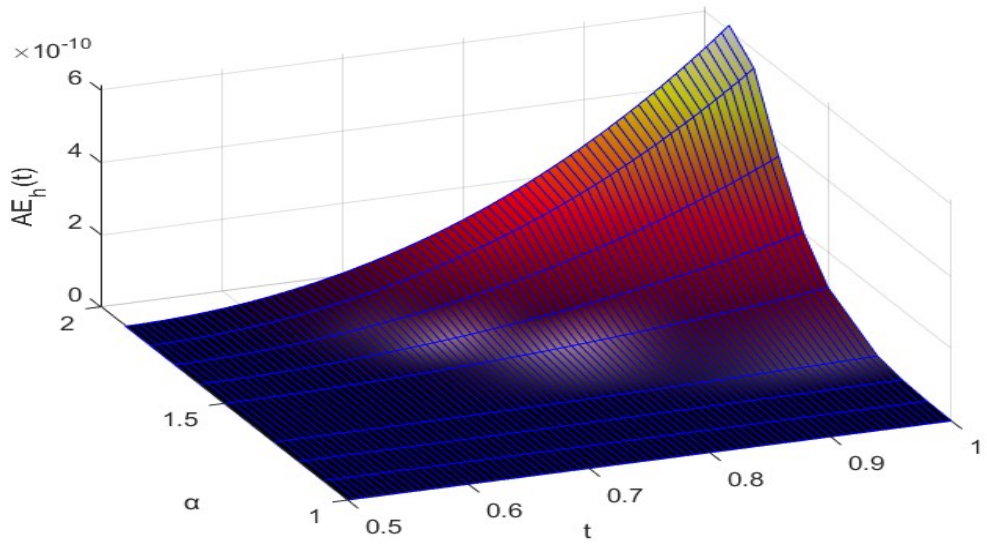


Figure 5.10: The graph of the AE_h with respect to α when $N = 80, m = 3$ using the G-LP for Example 5.5.

Table 5.9: The approximate solution u_h , for $h = \frac{1}{40}, \frac{1}{80}, \frac{1}{160}$ obtained by using the G-LP when $\alpha = 1.0001$, $\alpha = 0.9999$ and $N = 80, m = 4$, and the exact solution when $\alpha = 1$ for Example 5.6.

$\alpha = 1.0001$									
$\mathbf{t}$	$u_{\frac{1}{40}}$	$u_{\frac{1}{80}}$	$u_{\frac{1}{160}}$	$ u_{\frac{1}{80}} - u_{\frac{1}{40}} $	$ u_{\frac{1}{160}} - u_{\frac{1}{80}} $	$ u_{\frac{1}{160}} - u_{\frac{1}{40}} $	$\log_2 \frac{ u_{\frac{1}{160}} - u_{\frac{1}{40}} }{\frac{ u_{\frac{1}{160}} - u_{\frac{1}{80}} }{\frac{ u_{\frac{1}{80}} - u_{\frac{1}{40}} }}}$	na	na
0.5	2.042298543456040	2.042298543456040	2.042298543456040	0	0	0	na	na	na
0.6	2.313899148354915	2.313899154236738	2.313899154719453	5.88182×10^{-9}	4.82715×10^{-10}	4.82715×10^{-10}	12.18488	3.60702	3.60702
0.7	2.666499225421245	2.666499245233377	2.666499246682199	1.98121×10^{-8}	1.44882×10^{-9}	1.44882×10^{-9}	13.67465	3.77343	3.77343
0.8	3.162139059025395	3.162139100430150	3.162139103305870	4.14048×10^{-8}	2.87572×10^{-9}	2.87572×10^{-9}	14.39805	3.84780	3.84780
0.9	3.894636067629396	3.894636138137670	3.894636142895072	7.05083×10^{-8}	4.75740×10^{-9}	4.75740×10^{-9}	14.82075	3.88955	3.88955
1.0	4.999952824239302	4.999952931336956	4.999952938431949	1.07098×10^{-7}	7.09499×10^{-9}	7.09499×10^{-9}	15.09482	3.91598	3.91598

$\alpha = 0.9999$									
$\mathbf{t}$	$u_{\frac{1}{40}}$	$u_{\frac{1}{80}}$	$u_{\frac{1}{160}}$	$ u_{\frac{1}{80}} - u_{\frac{1}{40}} $	$ u_{\frac{1}{160}} - u_{\frac{1}{80}} $	$ u_{\frac{1}{160}} - u_{\frac{1}{40}} $	$\log_2 \frac{ u_{\frac{1}{160}} - u_{\frac{1}{40}} }{\frac{ u_{\frac{1}{160}} - u_{\frac{1}{80}} }{\frac{ u_{\frac{1}{80}} - u_{\frac{1}{40}} }}}$	na	na
0.5	2.042298543456040	2.042298543456040	2.042298543456040	0	0	0	na	na	na
0.6	2.336382834484979	2.336382890186138	2.336382904938831	5.57012×10^{-8}	1.47527×10^{-8}	1.47527×10^{-8}	3.77566	1.90917	1.90917
0.7	2.751974647483427	2.751974754449280	2.751974782928411	1.06966×10^{-7}	2.84791×10^{-8}	2.84791×10^{-8}	3.75594	1.90644	1.90644
0.8	3.350897309586328	3.350897463143591	3.350897504104982	1.53557×10^{-7}	4.09614×10^{-8}	4.09614×10^{-8}	3.74883	1.90494	1.90494
0.9	4.234669738339650	4.234669933390302	4.234669985474072	1.95051×10^{-7}	5.20838×10^{-8}	5.20838×10^{-8}	3.74494	1.90395	1.90395
1.0	5.555168946791253	5.555169178255524	5.555169240105265	2.31464×10^{-7}	6.18497×10^{-8}	6.18497×10^{-8}	3.74236	1.90917	1.90917

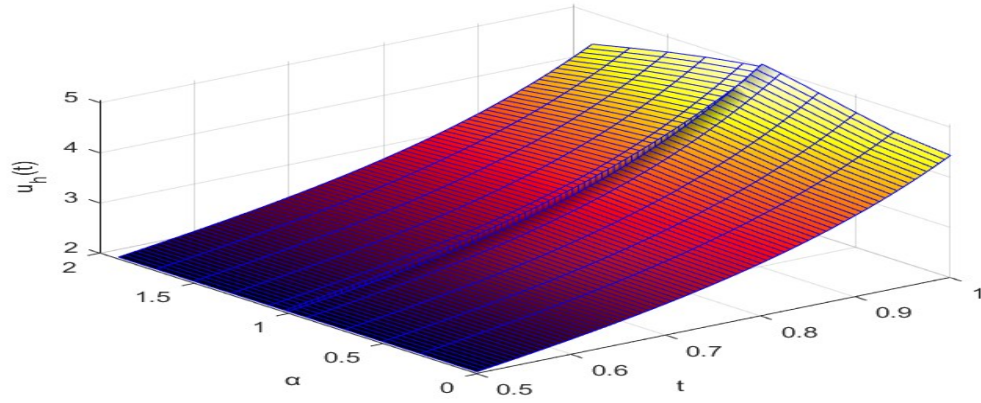


Figure 5.11: The graph of the approximate solution $u_h(t)$ for $N = 80, m = 4$ and $0 < \alpha < 2$ for Example 5.6 using the G-LP for collocation.

Table 5.10: The total computational time (CPU(s)) with respect to N obtained when G-LP and M-CP are used.

N	10	20	40	80	160	320
G-LP	1.19	2.12	3.17	6.25	13.33	27.63
M-CP	1.06	2.03	3.94	7.96	13.50	28.21

Chapter 6

CONCLUSION AND FINAL REMARKS

Under this section we present the summary of our findings and drawn conclusion based on the results of our findings. The unique analytic solution of BVP of generalized BgT equation (1.3) is proved using M-L type function. A highly accurate numerical method using collocation-shooting method is given. The local and global convergence analysis of the proposed method is proved and several applications from constructed and existing works from the literature are provided justifying the theoretical results. The comparison shows that, the collocation shooting method perform extremely well, under some smoothness conditions of the exact solution. Numerical results verified the applicability, stability and the efficiency of the method when compared with the existing other methods.

Further, a fractional Cauchy–Euler problem in the Caputo sense is studied. The existence and uniqueness of the solution are investigated using the contracting mapping principle. Further, a collocation method and an algorithm is given for the numerical solution. Additionally, convergence analyses are provided, and the developed method is applied on some constructed fractional Cauchy–Euler (FrC-E) problems. The simulations justify the theoretical results. Additionally, in Example 5.6, the numerical results given in Table 5.9 show that if $\alpha \rightarrow 1$, the solution approaches the exact solution of the classical differential equation ($\alpha = 1$)

In future research, the authors may consider discussing the potential extension of their

method, specifically the application of the proposed numerical approach to higher-dimensional Caputo fractional Cauchy–Euler problems. This could be advantageous for addressing more complex systems. The authors can also conduct an error analysis to pinpoint possible sources of numerical inaccuracies and enhance the algorithm for improved precision and stability.

Moreover, they will consider the effectiveness of their method using different types of fractional derivatives operators, such as the Riemanns–Liouville or Grunwald–Letnikov derivatives.

Remark 6.1: When the solution of BVP of generalized BgT equation (1.3) has weak regularity e.g $y \in C^3(0, T]$ as in the case of Example 3.3 for the value of the regularity parameter $\varpi = 7$, a numerical investigation of the method may be given by using graded meshes. Also the method can be used for solving various families of generalized BgT equation with variable coefficients.

Remark 6.2: When the solution of fractional Cauchy-Euler (FrC-E) problem (1.4) has weak regularity e.g $y \in C^3(0, T]$ as in the case of Example 5.4 for the value of the regularity parameter $\varpi > 2$, a numerical treatment of the given method may be given by using graded meshes.

6.1 Suggestion for Further Work

A collocation method for nonlinear fractional differential equation in the sense of Reiman Louville Hadmand fractional operator with integral boundary condition with graded mesh may be considered and higher order Bagley–Torvik equation with integral boundary condition using graded mesh is another area where possible further research work may conducted.

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